Assignment # 3

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1.

Theorem. *For any two sets A and B, $A \subseteq B$ if and only if $\mathcal{P}(A) \subseteq \mathcal{P}(B)$.*

Proof. Assume $A \subseteq B$ and $x \in \mathcal{P}(A)$. We will show that $x \in \mathcal{P}(B)$. Since $x \in \mathcal{P}(A)$, $x \subseteq A$. Because $x \subseteq A, x \subseteq B$. Given that $x \subseteq B, x \in \mathcal{P}(B)$. Therefore $\mathcal{P}(A) \subseteq \mathcal{P}(B)$.

Assume $\mathcal{P}(A) \subseteq \mathcal{P}(B)$ and $x \in A$. We will show that $x \in B$. Since $\{x\} \in \mathcal{P}(A)$, $\{x\} \in \mathcal{P}(B)$. Therefore $x \in B$. □

2.

Theorem. *For any family of sets $\{A_i\}_{i\in I}$ and any set B, $B \times (\bigcup_{i\in I} A_i) = \bigcup_{i\in I}(B \times A_i)$.*

Proof. Assume $x \in B \times (\bigcup_{i\in I} A_i)$. We will show that $x \in \bigcup_{i\in I}(B \times A_i)$.

Following $x \in B \times (\bigcup_{i\in I} A_i)$, $x = (b, a)$ such that $b \in B$ and $a \in \bigcup_{i\in I} A_i$. Therefore $a \in A_i$ for some $i \in I$. Consequently $(b, a) \in B \times A_i$. Therefore $x \in B \times A_i$. Given that $x \in B \times A_i$ for some $i \in I$, $x \in \bigcup_{i\in I}(B \times A_i)$.

The reverse follows analogously. □

3.

Theorem. *For any two sets* A*,* $B \subseteq \mathbb{R}$*,* $A = B$ *if and only if* $\mathcal{X}_A = \mathcal{X}_B$*.*

Proof. Assume $A = B$ and $x \in A$. We show $\mathcal{X}_A = \mathcal{X}_B$.

Note that by definition both $\mathcal{X}_A$ and $\mathcal{X}_B$ have domain and co-domain $\mathbb{R}$, verifying that the domain and co-domain of $\mathcal{X}_A$ and $\mathcal{X}_B$ are equal.

Case 1: Consider $x \in A$. Since $x \in A$, $\mathcal{X}_A(x) = 1$. Because $x \in A$ and $A = B$, $x \in B$. Given that $x \in B$, $\mathcal{X}_B(x) = 1$. Therefore $\mathcal{X}_A(x) = \mathcal{X}_B(x)$.

Case 2: Consider $x \notin A$. Since $x \notin A$, $\mathcal{X}_A(x) = 0$. Because $x \notin A$ and $A = B$, $x \notin B$. Given that $x \notin B$, $\mathcal{X}_B(x) = 0$. Therefore $\mathcal{X}_A = \mathcal{X}_B$.

Assume $\mathcal{X}_A = \mathcal{X}_B$ and $x \in A$.

We will show that $x \in B$. Since $x \in A, \mathcal{X}_A(x) = 1$. Given that $\mathcal{X}_A = \mathcal{X}_B, \mathcal{X}_B(x) = 1$. Therefore since $\mathcal{X}(x) = 1,\ x \in B$.

The reverse follows analogously. $\square$