Assignment # 4

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1.

Theorem. *The function* $f : A \to B$ *is injective if and only if* $C = f^{-1}(f(C))$ *for all subsets* $C \subseteq A$*.*

Proof. Assume $f : A \to B$ is injective. We show $C = f^{-1}(f(C))$ for all subsets $C \subseteq A$. Let $C \subseteq A$. Assume $x \in C$. We show $x \in f^{-1}(f(C))$. Since $x \in C$, by definition of image, $f(x) \in f(C)$. By definition of inverse image, $x \in f^{-1}(f(C))$. We assume $x \in f^{-1}(f(C))$. We show $x \in C$. Since $x \in f^{-1}(f(C))$, by definition of inverse image $f(x) \in f(C)$. By definition of image, $f(x) = f(a)$ for $a \in C$. Since f is injective, by definition of injective, $x = a$. Therefore since $a \in C$, $x \in C$.

Assume $C = f^{-1}(f(C))$ for all subsets $C \subseteq A$. We show f is injective. Let $a_1 \in A$ and $a_2 \in A$. Assume $f(a_1) = f(a_2)$, we show $a_1 = a_2$. Let $C = \{a_1\}$. By definition of image $f(a_1) \in f(C)$. Since $f(a_1) = f(a_2)$, $f(a_2) \in f(C)$. Because $f(a_2) \in f(C)$, $a_2 \in f^{-1}(f(C))$. Since $C = f^{-1}(f(C))$, $f^{-1}(f(C)) = \{a_1\}$. Seeing that $a_2 \in f^{-1}(f(C))$ and $f^{-1}(f(C)) = \{a_1\}$, $a_1 = a_2$. Therefore f is injective. $\square$

2.

Theorem. *The function* $f : A \to B$ *is surjective if and only if* $D = f(f^{-1}(D))$ *for all subsets* $D \subseteq B$.

Proof. Assume $f : A \to B$ is surjective. Show $D = f(f^{-1}(D))$ for all subsets $D \subseteq B$. Let $D \subseteq B$. Assume $x \in D$. We show $x \in f(f^{-1}(D))$. Since $x \in D$ and f is surjective there exists some $a \in A$ such that $f(a) = x$. Furthermore, $f(a) \in D$. Therefore $a \in f^{-1}(D)$. Since $a \in f^{-1}(D)$, $f(a) \in f(f^{-1}(D))$. Consequently, $x \in f(f^{-1}(D))$.
Assume $x \in f(f^{-1}(D))$ for all subsets $D \subseteq B$. We show $x \in D$.
Because $x \in f(f^{-1}(D))$, by definition of image, $f(a) = x$ for some $a \in f^{-1}(D)$. By definition of inverse image, $f(a) \in D$. Therefore $x \in D$.
Assume $D = f(f^{-1}(D))$ for all subsets $D \subseteq B$. We show f is surjective.
Let $b \in B$. We produce $a \in A$ such that $f(a) = b$ and $D = \{b\}$. Because $D = f(f^{-1}(D))$, by definition of the inverse image there exists an $a \in C$ such that $f(a) = b$. Therefore f is surjective. $\square$