Assignment # 2

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1.

Theorem. *For all real numbers x, there is exactly one real number y for which $x = y^3 + 1$.*

Proof. We prove the theorem by existence and uniqueness.

We first prove that y exists. Consider $y = \sqrt[3]{x-1}$. Since,

$$x = \sqrt[3]{x-1}$$

$$x = x$$

there exists y satisfying the equation, $x = y^3 + 1$.

We now prove that y is unique. Suppose there are two real numbers y_1 and y_2 such that $x = y_1^3 + 1$ and $x = y_2^3 + 1$. Therefore since,

$$y_1^3 + 1 = y_2^3 + 1$$

$$y_1^3 = y_2^3$$

$$y_1 = y_2$$

y is unique.

$\square$

2.

Theorem. *Given a positive integer $n \geq 2$, if for all integers a and b, $n|ab$ implies $n|a$ or $n|b$, then n is prime.*

Proof. We prove the theorem by contrapositive. Assume n is composite. Therefore there exists integers a and b such that $n \nmid ab$ and $n \nmid a$.

Since n is composite there exists an integer k such that $1 < k < n$ and $k|n$.

Considering $k|n$, there exists an integer j such that $k(j) = n$.

Let $a = k$ and $b = j$.

Since $n = kj,\ n|kj$ and therefore $n|ab$.

Since $k < n,\ n \nmid kj$ and therefore $n \nmid a$.

Since $kj = n,\ k > 1,\ j < n,$ and $n \nmid j,\ n \nmid b$.

By contrapositive, if for all integers a and b, $n|ab$ implies $n|a$ or $n|b$, then n is prime. $\square$

3.

Theorem. *For any natural number m,* $3|f_{4m}$.

Proof. We prove the theorem by induction.

Base case: Assume $m = 1$. Since

$$3(1) = 3, \text{ then } 3|3.$$

Therefore $3|f_{4(1)}$.

Induction: Assume $3|f_{4k}$. We want to show $3|f_{4(k+1)}$.

$$\begin{aligned} f_{4k+2} &= f_{4k+3} + f_{4k+2} \\ &= f_{4k+2} + f_{4k+1} + f_{4k+1} + f_{4k} \\ &= f_{4k+2} + 2f_{4k+1} + f_{4k} \\ &= f_{4k+1} + f_{4k} + 2f_{4k+1} + f_{4k} \\ &= 2f_{4k} + 3f_{4k+1} \end{aligned}$$

Given that $3|3f_{4k+1}$ and $3|2f_{4k},\ 3|f_{4k}$. □