Assignment # 5

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1.

Theorem. *The equivalence class* $[h(x)] = \{f \in P \mid f(x) = ax$ *for some* $a \in \mathbb{N}\}$

Proof. Part 1. Assume $g = ax^1$ for some $a \in \mathbb{N}$. We show $g \sim h$ by existence proof. Let $c = a$. Then $x \leq x \leq a^2x$. Therefore, $\frac{1}{c}ax \leq x \leq cax$ for some $c \in \mathbb{N}$.

Part 2. Proof by contrapositive. Assume $g = ax^n$ and $n > 1$. We show $g \not\sim h$ by existence proof. Let $x = 2c$. Therefore, $\frac{1}{c}(2c)^{n-1} > 1$ and $2^{n-1}ac^{n-1} > 1$ since $n \in N$. Therefore, $\frac{1}{c}ax^n > x$ or $x > cax^n$ for some $x \in N$ and $\frac{1}{c}ax^{n-1} > 1$. $\square$

2.

Theorem. *The relation $\approx$ is an equivalence relation on the set of all groups.*

Proof. Part 1. We show $\approx$ is symmetric.

We show if $(G, *) \approx (H, \diamond)$ then $(H, \diamond) \approx (G, *)$. Assume $(G, *) \approx (H, \diamond)$. So there exists a function $f : G \to H$ such that $f(a*b) = f(a) \diamond f(b)\ \forall\, a, b \in G$ and f is a bijection. Since f is a bijection, f has an inverse $f^{-1} : H \to G$ and f^{-1} is a bijection. Therefore, $f^{-1}(f(a*b)) = f^{-1}(f(a) \diamond f(b))$ and $a*b = f^{-1}(f(a) \diamond f(b))$. Let $f(a) = c$ and $f(b) = d$. Therefore $f^{-}1(c) = a$ and $f^{-}1(b) = d$. Then, $f^{-}1(c \diamond d) = f^{-1}(c) * f^{-1}(d)$. Therefore, $f^{-1}(c \diamond d) = f^{-1}(c) * f^{-1}(d)$.

Part 2. Show $\approx$ is reflexive. We prove $(G, *) \approx (G, *)$.

Let $f(x) = x$. Substituting we get, $a * b = a * b.$. Therefore the relation is reflexive, and $f(a * b) = a * b$ for all a, b.

Part 3. Assume $(G, *) \approx (H, \diamond)$ and $(H, \diamond) \approx (K, \oplus)$. We now show if $(G, *) \approx (H, \diamond)$ and $(H, \diamond) \approx (K, \oplus)$, then $(G, *) \approx (K, \oplus)$.

We show $\approx$ is transitive. We prove $(G, *) \approx (H, \diamond) and (H, \diamond) \approx (K, \oplus)$.

There exists $f : G \to H$ such that $f(a * b) = f(d) \diamond f(b)\ \forall\, a, b \in G$ and f is a bijection. Additionally, there exists $j : H \to K$ such that $f(c \diamond d) = j(c) \oplus j(d)$ for all $c, d \in H$ and j is a bijection. Therefore $j \circ f : G \to H$ is bijective.

We now show $j \circ f(a * b) = j \circ f(a) \oplus j \circ f(b)$, for all a, b. Simplifying, $j(f(a * b)) = f(f(a) \diamond f(b)) = j(f(a) \diamond f(b)) = j(f(a)) \oplus j(f(b)) = j(c) \oplus j(d)$. □