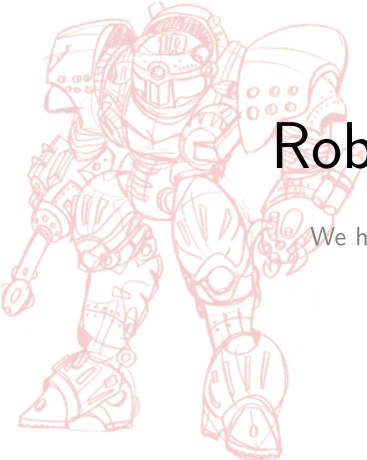


Robot Velocity

We have the need for speed

TECHNICAL UNIVERSITY OF CLUJ-NAPOCA ROMANIA

Last update: October 28, 2021

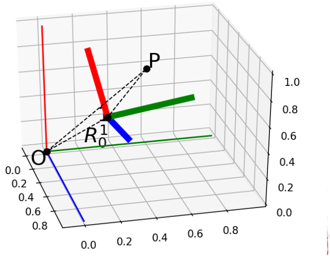
Agenda

- Background
- Linear and angular velocity
- The Jacobian
- Inverting the Jacobian - Singularities
- Velocity eclipse

Recap

What do we know already?

$$P_0 = R_0^1 * P_1$$



Recap

What we know already?

Definition

A transformation matrix that calculates the pose of the robot's end effector in terms of the joint coordinates $q_1, q_2, \ldots, q_n$

$$\left[\begin{array}{ccc|c} X_X & Y_X & Z_X & P_x \\ X_Y & Y_Y & Z_Y & P_y \\ X_Z & Y_Z & Z_Z & P_z \\ \hline 0 & 0 & 0 & 1 \end{array}\right]$$

Robot velocity

Background

A robot is a mechanism which consists of joints and links.

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By controlling the position of the joints, we can control the position of the end-effector.

Robot velocity

Background

A robot is a mechanism which consists of joints and links.

By controlling the position of the joints, we can control the position of the end-effector.

Can we do this for velocities as well?

Robot velocity

Background

We define a matrix called the 'Jacobian' that shows us how can we calculate the end-effector velocity if we know the joint velocities

$$\xi = J\dot{q}$$

Robot velocity

The Jacobian

$$\begin{bmatrix} \dot{x} \\ \dot{y} \\ \dot{z} \\ \omega_x \\ \omega_y \\ \omega_z \end{bmatrix} = \xi = J\dot{q} = J \begin{bmatrix} \dot{q}_1 \\ \dot{q}_2 \\ \vdots \\ \dot{q}_n \end{bmatrix}$$

By vector ξ we denote a vector that contains 6 velocities, 3 linear and 3 angular. By vector $\dot{q}$ we denote a vector containing all the n joint velocities.

Robot velocity

The Jacobian

$$\begin{bmatrix} \dot{x} \\ \dot{y} \\ \dot{z} \\ \omega_x \\ \omega_y \\ \omega_z \end{bmatrix} = \xi = J\dot{q} = J \begin{bmatrix} \dot{q}_1 \\ \dot{q}_2 \\ \vdots \\ \dot{q}_n \end{bmatrix}$$

By vector ξ we denote a vector that contains 6 velocities, 3 linear and 3 angular. By vector $\dot{q}$ we denote a vector containing all the n joint velocities.

What is the size of the Jacobian matrix J?

Robot velocity

The Jacobian

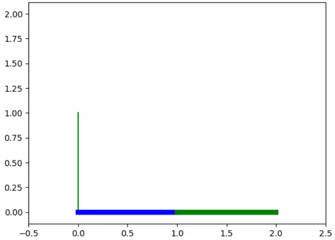
The Jacobian is a matrix of $6 \times n$ (six rows and n columns). The first three rows, related to the linear velocities u, the last three to the angular velocities ω.

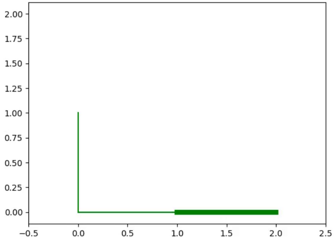
$$\begin{bmatrix} u \\ \omega \end{bmatrix} = \begin{bmatrix} J_u \\ J_\omega \end{bmatrix} \dot{q}$$

Linear and angular velocity

What is the difference?

Each of the robot segments can be moving with a linear, angular velocity, or a combination of the two.

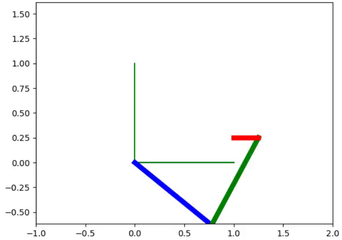


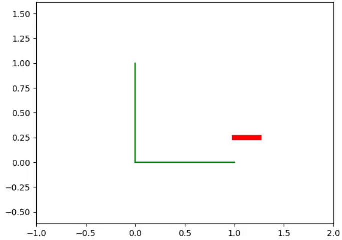


Linear and angular velocity

What is the difference?

Each of the robot segments can be moving with a linear, angular or a complex velocity.





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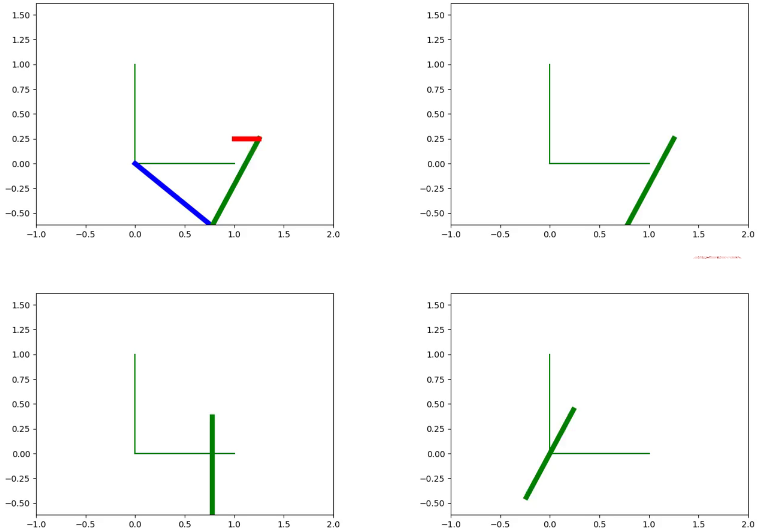

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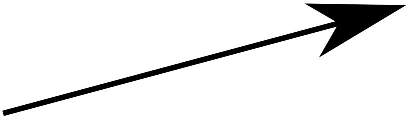

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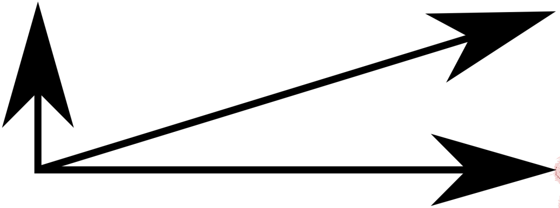
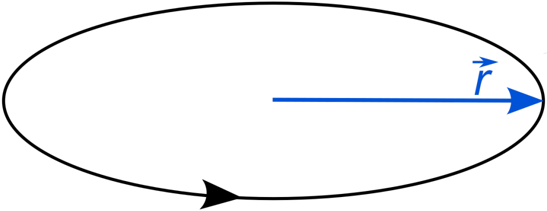


What is angular velocity



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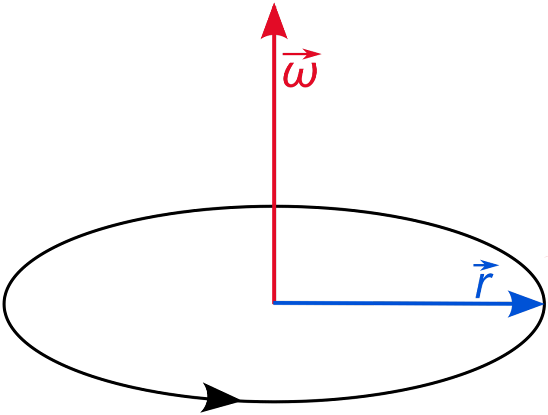

Defining the Jacobian

Angular velocities

We are looking for a relationship between joint velocities and angular velocity of the end-effector.

$$\omega = J_\omega \dot{q}$$

Defining the Jacobian

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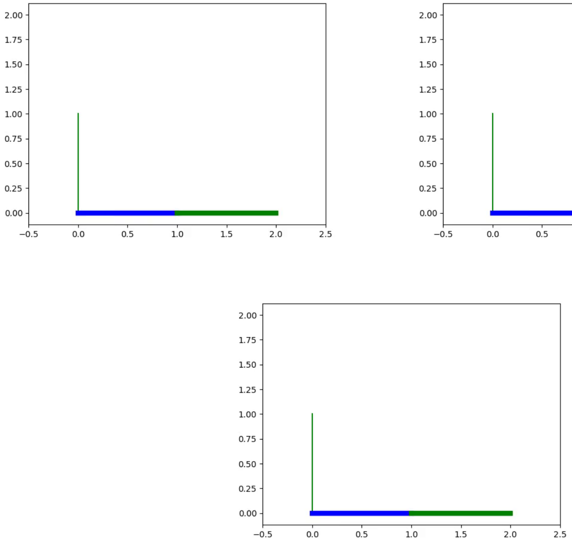
What is the dimension of J_ω?

Defining the Jacobian

Addition of Angular velocities

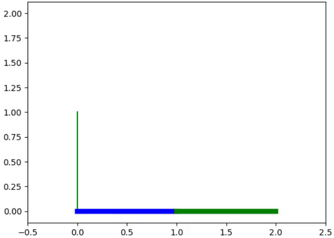


Defining the Jacobian

Angular velocities

We can add the angular velocities of each segment to calculate the angular velocity of the end effector.

$$\omega_0^n = \dot{q_1} + \dot{q_2} + \ldots + \dot{q_n}$$



Defining the Jacobian

Angular velocities

What happens if we have motion in $\mathbb{R}^3$?

Defining the Jacobian

Angular velocities

What happens if we have motion in $\mathbb{R}^3$?

In the general case, we need to express/transform the angular velocity of each segment to the base coordinate frame.

$$\omega_0^n = \rho_1 R_0^1 \dot{q}_1 + \rho_2 R_0^2 k \dot{q}_2 + \ldots + \rho_n R_0^n k \dot{q}_n$$

Defining the Jacobian

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Where k is the unit coordinate vector $(0, 0, 1)^T$

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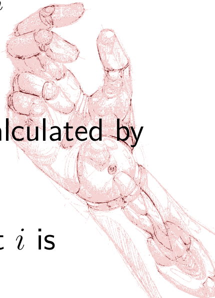
Where k is the unit coordinate vector $(0, 0, 1)^T$

R_0^n is the **rotation** matrix from base to joint n, as calculated by the DH convention.

And ρ_i is equal to 1 if joint i is revolute and 0 if joint i is prismatic. Why?



Defining the Jacobian

Angular velocities

What is the result of:

$$R_0^n k$$

Defining the Jacobian

Angular velocities

What is the result of:

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$$\begin{bmatrix} X_X & Y_X & Z_X \\ X_Y & Y_Y & Z_Y \\ X_Z & Y_Z & Z_Z \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} Z_X \\ Z_Y \\ Z_Z \end{bmatrix} = z_n$$

What does z_n represent in DH convention?

Defining the Jacobian

Angular velocities

We have therefore defined a relationship that shows us how the joint coordinates q relate to the angular velocity of the end-effector ω

$$\omega = \left[\rho_1 z_1, \rho_2 z_2, \ldots, \rho_n z_n\right] \dot{q}$$

Defining the Jacobian

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Therefore, the Jacobian for the angular velocities is:

$$J_\omega = \left[\rho_1 z_1, \rho_2 z_2, \ldots \rho_n z_n\right]$$

Defining the Jacobian

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What dimension does it have?

Defining the Jacobian

Linear velocities

We are looking for a relationship between joint velocities and linear velocity of the end-effector.

$$u = J_u \dot{q}$$

What is the dimension of J_u ?

Defining the Jacobian

Linear velocities

We can visualise that the linear velocity of the end-effector is equal to the linear velocity of the joint for **Prismatic** joints.

Defining the Jacobian

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Defining the Jacobian

Linear velocities

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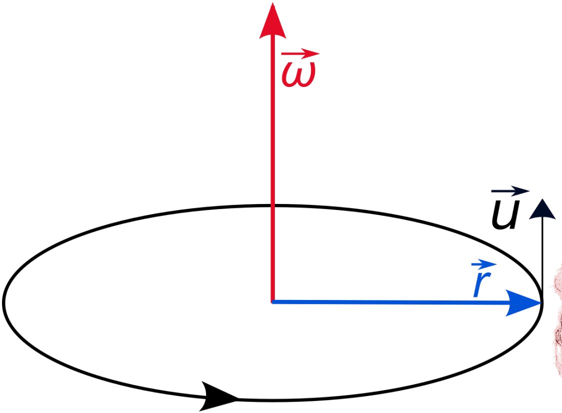
What is z_i?

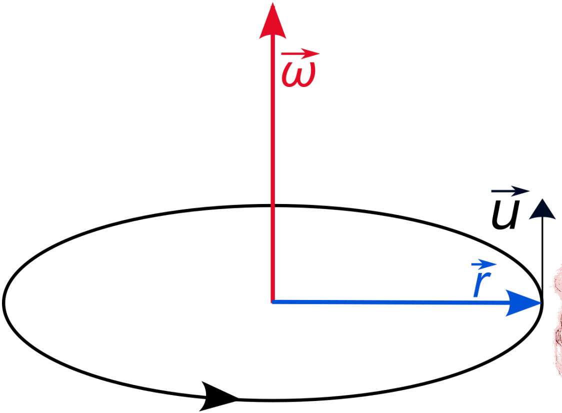
Defining the Jacobian

Linear velocity of a rotating body



Defining the Jacobian

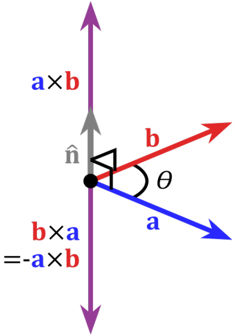
Linear velocity of a rotating body



$$\vec{u} = \vec{\omega} \times \vec{r}$$

Defining the Jacobian

What is the cross product?



If we have two vectors a and b with coordinates $[a_i, a_2, a_3]$ and $[b_1, b_2, b_3]$ respectively then, the cross product is defined as:

$$a \times b = (a_2 b_3 - a_3 b_2)i + (a_3 b_1 - a_1 b_3)j + (a_1 b_2 - a_2 b_1)k$$

Defining the Jacobian

Linear velocities

For a revolute joint, the column of the linear Jacobian for that joint is equal to the cross product of the axis of the joint and the vector connecting the end-effector with the joint

$$J_{u_i} = z_i \times (o_{n+1} - o_i)$$

Defining the Jacobian

Combining angular and linear velocities

We can calculate each column of the Jacobian matrix individually. Each column represents one joint. If joint i is revolute, then:

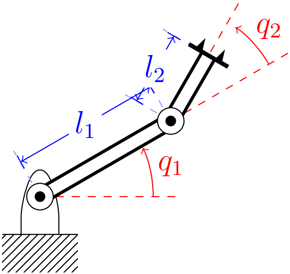
$$J_i = \begin{bmatrix} z_i \times (o_{n+1} - o_i) \\ z_i \end{bmatrix}$$

If joint i is prismatic, then:

$$J_i = \begin{bmatrix} z_i \\ 0 \end{bmatrix}$$

Defining the Jacobian

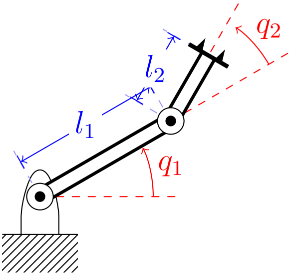
Example in $\mathbb{R}^2$



$$R_0^1 = \begin{bmatrix} c_1 & -s_1 & 0 & 0 \\ s_1 & c_1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Defining the Jacobian

Example in $\mathbb{R}^2$



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$$R_0^2 = \begin{bmatrix} c_{1,2} & -s_{1,2} & 0 & l_1 c_1 \\ s_{1,2} & c_{1,2} & 0 & l_1 s_1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Defining the Jacobian

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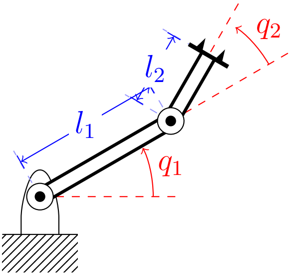

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$$R_0^3 = \begin{bmatrix} c_{1,2} & -s_{1,2} & 0 & l_2 c_{1,2} + l_1 c_1 \\ s_{1,2} & c_{1,2} & 0 & l_2 s_{1,2} + l_1 s_1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Defining the Jacobian

2 link planar manipulator

$$R_0^1 = \begin{bmatrix} c_1 & -s_1 & 0 & 0 \\ s_1 & c_1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} R_0^2 = \begin{bmatrix} c_{1,2} & -s_{1,2} & 0 & l_1 c_1 \\ s_{1,2} & c_{1,2} & 0 & l_1 s_1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

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$$J = \begin{bmatrix} z_1 \times (o_3 - o_1) & z_2 \times (o_3 - o_2) \\ z_1 & z_2 \end{bmatrix}$$

Defining the Jacobian

2 link planar manipulator

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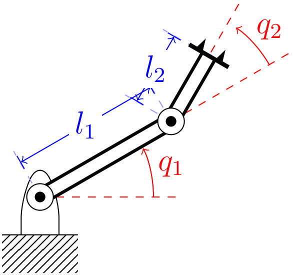
where:

$$o_1 = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, o_2 = \begin{bmatrix} l_1 c_1 \\ l_1 s_1 \\ 0 \end{bmatrix}, o_3 = \begin{bmatrix} l_1 c_1 + l_2 c_{1,2} \\ l_1 s_1 + l_2 s_{1,2} \\ 0 \end{bmatrix}, z_1 = z_2 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

Defining the Jacobian

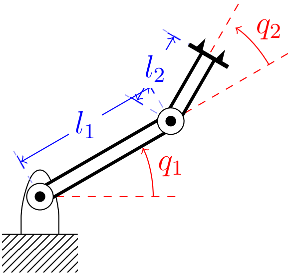
2 link planar manipulator



$$J = \begin{bmatrix} -l_1 s_1 - l_2 s_{1,2} & -l_2 s_{1,2} \\ l_1 c_1 + l_2 c_{1,2} & l_2 c_{1,2} \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 1 & 1 \end{bmatrix}$$

Defining the Jacobian

2 link planar manipulator



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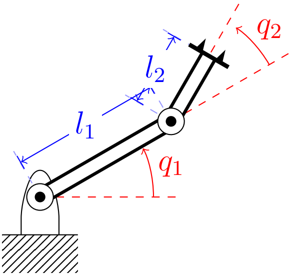
The Jacobian is a function of joint coordinates!

Defining the Jacobian

2 link planar manipulator



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How do we 'use' the Jacobian?

Jacobian

Inverse velocity

We now have a method to define the end-effector velocity (angular and linear) based on the joint velocities

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$$J^{-1}\xi = \dot{q}$$

Jacobian

Inverting the velocity

Is J always inversible?

Jacobian

Inverting the velocity

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Conditions for Jacobian inversibility

- The Jacobian must be square
- The rank of the Jacobian must be equal to its size

Jacobian

Inverting the velocity

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Conditions for Jacobian inversibility

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For achieving any velocity in $\mathbb{R}^3$, the Jacobian must be 6×6.
What do we need for such a Jacobian?

Jacobian

The pseudoinverse

In the cases we cannot invert the Jacobian (e.g. we have redundant joints), we can calculate the *pseudoinverse.*

Jacobian

The pseudoinverse

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$$\textit{For } J \in \mathbb{R}^{m\times n}, \textit{ if } m < n, \textit{ then } (JJ^T)^{-1} exists.$$

Jacobian

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$$
\begin{gathered}
(JJ^T)(JJ^T)^{-1} = I \\
J[J^T(JJ^T)^{-1}] = I \\
JJ^+ = I
\end{gathered}
$$

where:

$$J^+ = J^T(JJ^T)^{-1}$$

Jacobian

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$$J[J^T(JJ^T)^{-1}] = I$$
$$JJ^+ = I$$

where:

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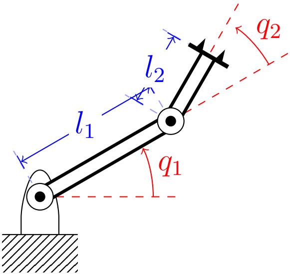
therefore:

$$\dot{q} = J^+\xi$$

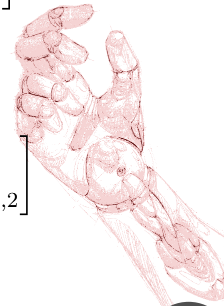
Jacobian inverse

2 link planar manipulator



$$J_u = \begin{bmatrix} -l_1 s_1 - l_2 s_{1,2} & -l_2 s_{1,2} \\ l_1 c_1 + l_2 c_{1,2} & l_2 c_{1,2} \end{bmatrix}$$

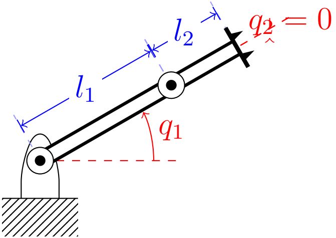
$$J_u^{-1} = \frac{1}{l_1 l_2 s_2} \begin{bmatrix} l_2 c_{1,2} & l_2 s_{1,2} \\ -l_1 c_1 - l_2 c_{1,2} & -l_1 s_1 - l_2 s_{1,2} \end{bmatrix}$$



Jacobian inverse

2 link planar manipulator

What happens when $q_2 = 0$?



Jacobian inverse

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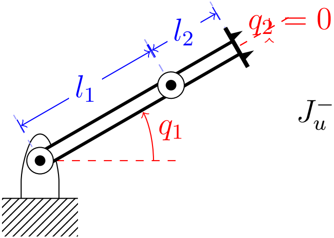

$$J_u^{-1} = \frac{1}{l_1 l_2 s_2} \begin{bmatrix} l_2 c_{1,2} & l_2 s_{1,2} \\ -l_1 c_1 - l_2 c_{1,2} & -l_1 s_1 - l_2 s_{1,2} \end{bmatrix}$$

Jacobian

Singularities

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In some cases, the Jacobian might lose rank, or might become non-invertible, or its determinant might become zero (which is practically the same thing)

In such cases, the robot loses dexterity, or even a degree of freedom.

Robot manipulability

Why does this all matter?

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Robot manipulability

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hint: yes!

Robot manipulability

Velocity ellipse

We model our robot as an input-output system (input is joint velocities, output is end-effector velocities). If we consider unit inputs, then we have:

$$q^T q = 1$$

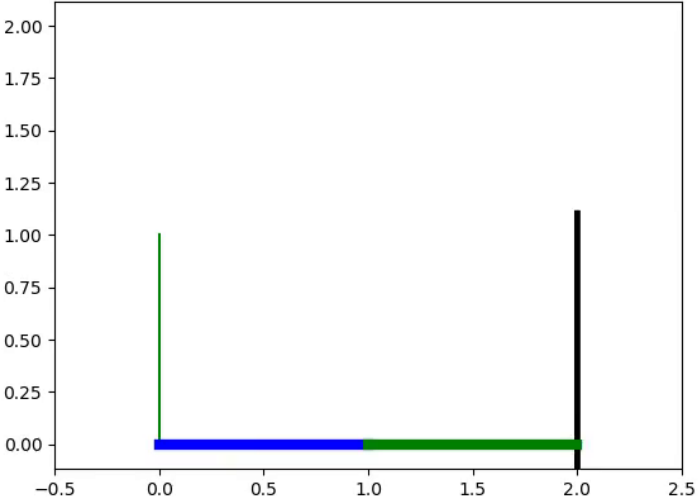
which we can write as:

$$\xi^T (JJ^T)^{-1} \xi = 1$$

which is the equation of an elipse

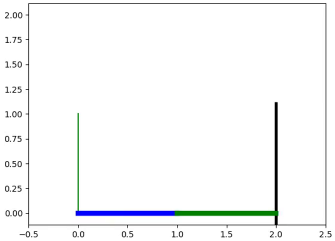
Robot manipulability

Velocity ellipse



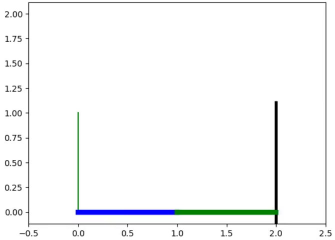
Robot manipulability

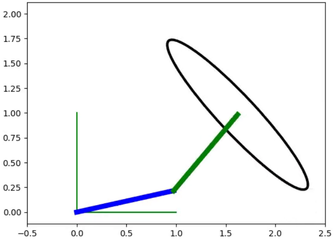
Velocity ellipse



Robot manipulability

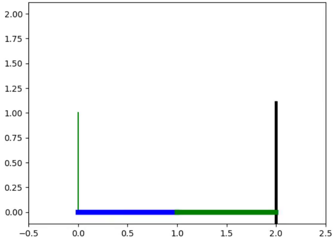
Velocity ellipse

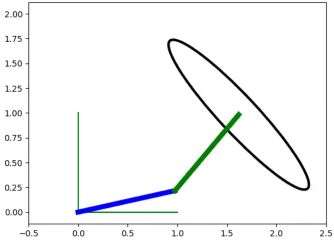


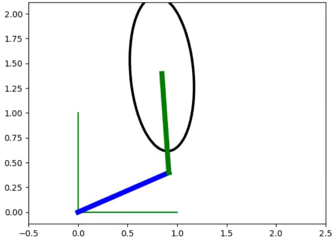


Robot manipulability

Velocity ellipse

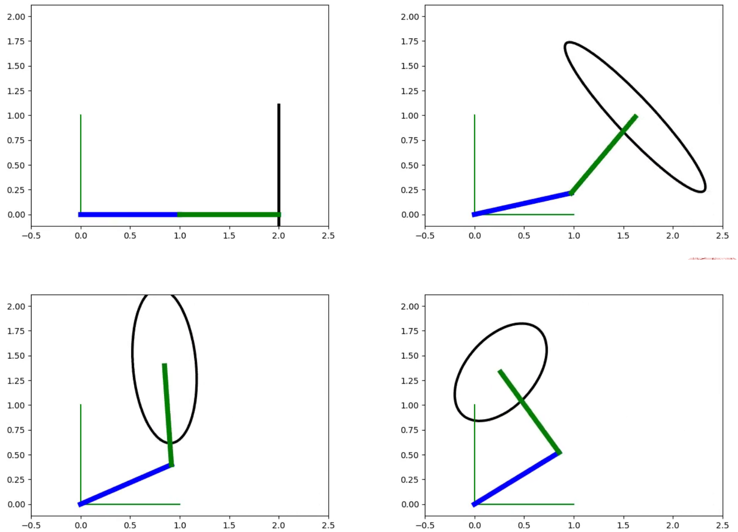






Robot manipulability

Velocity ellipse



Questions?