Classical string motion

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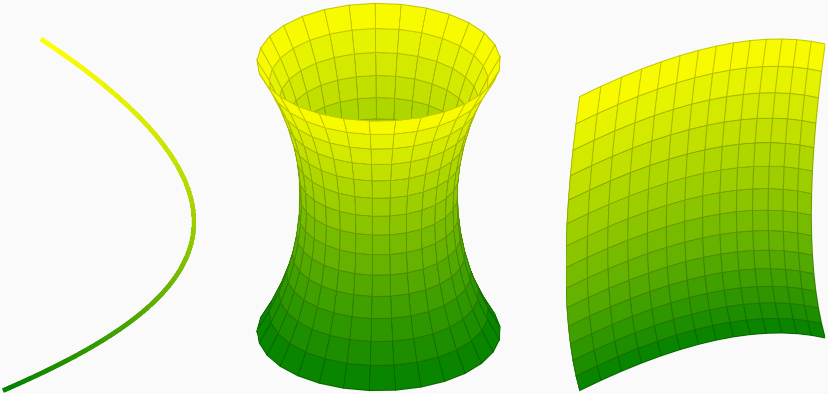
Objectives

Find explicit solutions of classical relativistic strings on various backgrounds, specifically:

- constantly expanding universe – de Sitter spacetime
- gravitational wave background
- interaction with gravitational wave pulse

Strings vs. particles



Nambu-Goto action for strings

- Action in terms of induced metric on string worldsheet:

$$S = -T \int_0^{\sigma_1} \int_{\tau_i}^{\tau_f} \mathrm{d}\sigma \, \mathrm{d}\tau \sqrt{-\det \gamma}$$

  where τ and σ are parametric coordinates on the worldsheet.

- γ in spacetime coordinates X:

$$\gamma_{\alpha\beta} = g_{MN} \, \partial_\alpha X^M \, \partial_\beta X^N$$

- Good choice of parameterization can greatly simplify the equations of motion.

Flat spacetime

Flat spacetime

- Minkowski spacetime – with the right choice of parameterization

$$\gamma_{\tau\tau} + \gamma_{\sigma\sigma} = 0, \qquad \gamma_{\tau\sigma} = \gamma_{\sigma\tau} = 0,$$

  the equations of motion become the wave equation:

$$\partial_\tau^2 X^M - \partial_\sigma^2 X^N = 0$$

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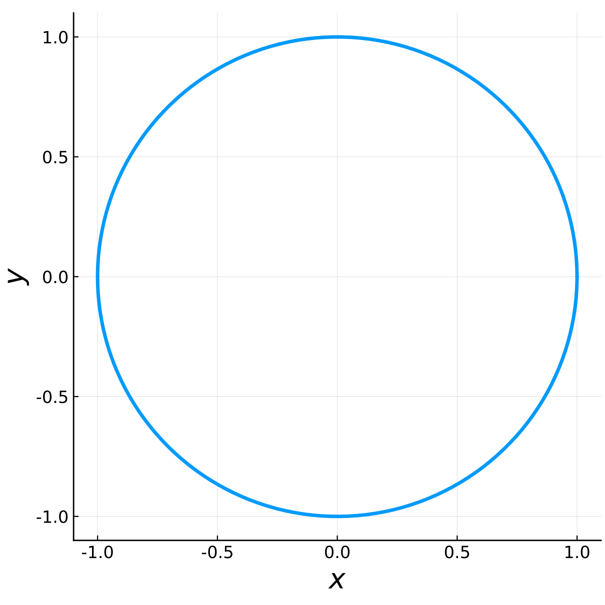
- Closed strings have to be σ_1 periodic in σ – solutions can be expressed as a superposition of modes (eigenstates):

$$X^M(\tau, \sigma) = \sum_{k \in \mathbb{Z}} X_k^M(\tau) \mathrm{e}^{2\pi \mathrm{i} k \sigma / \sigma_1},$$

where the evolution in τ of each mode is just

$$X_k^M(\tau) = A^M \mathrm{e}^{2\pi \mathrm{i} k \tau / \sigma_1} + B^M \mathrm{e}^{-2\pi \mathrm{i} k \tau / \sigma_1}$$

Flat spacetime



Constantly expanding universe – de Sitter spacetime

Constantly expanding universe – de Sitter spacetime

- Metric takes the form:

$$ds^2 = -\,dt^2 + e^{2Ht}\left(dr^2 + r^2\,d\theta^2 + dz^2\right)$$

where H is the Hubble expansion rate.

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  where H is the Hubble expansion rate.

- We study only circular strings in static gauge $t = \tau$:

$$X^M = \Big(t = \tau, \quad r(\tau), \quad \theta = \sigma, \quad z = 0\Big)^M$$

- With the change of coordinates $R = r\mathrm{e}^{Ht}$, we can construct a Lagrangian

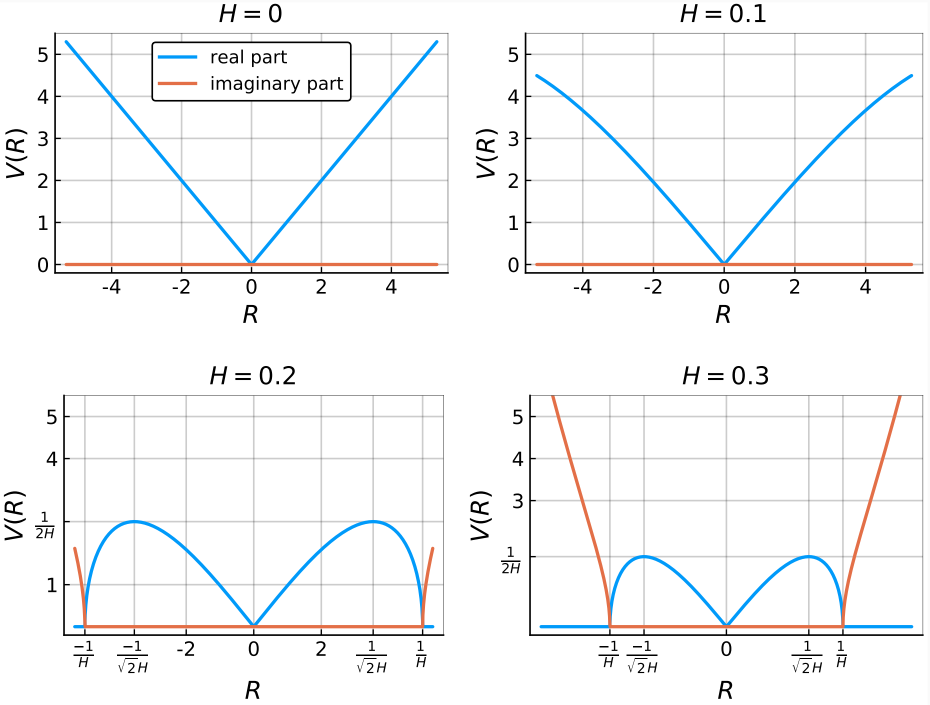
$$L = -2\pi T_0 R\sqrt{1 - \left(\partial_t R - HR\right)^2}$$

Constantly expanding universe – de Sitter spacetime

- Potential $V(R)$ for static strings

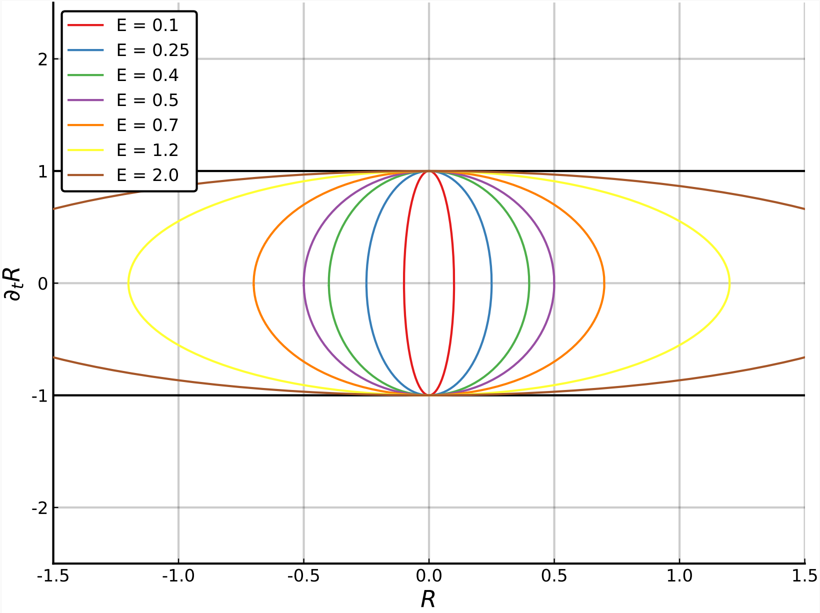
$$V(R) = 2\pi R\sqrt{1-(HR)^2}$$

- Critical points at $R = 1/\sqrt{2}H$ and $R = 1/H$

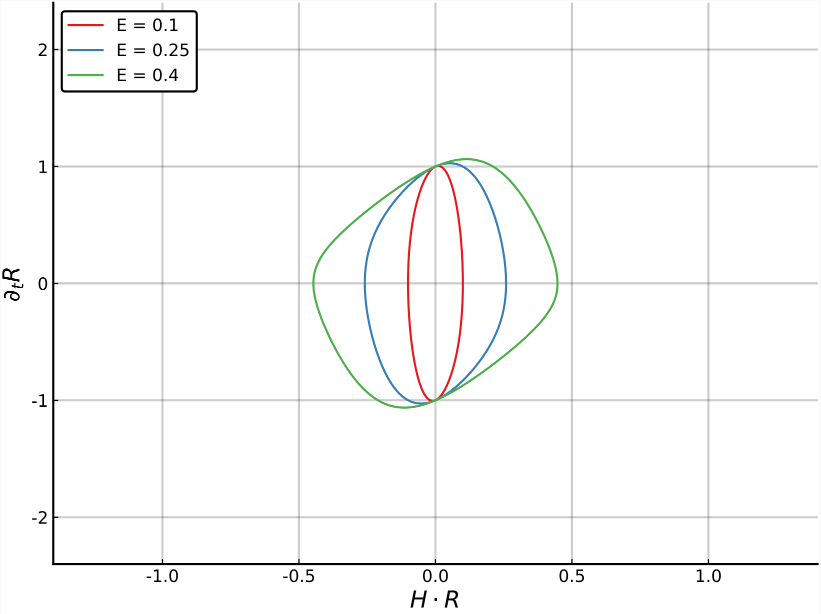


H = 0
H = 0.1
H = 0.2
H = 0.3
real part
imaginary part
V(R)
R
5
4
3
2
1
0
-4
-2
0
2
4
1/2H
-1/H
-1/√2H
1/√2H
1/H

Flat spacetime



Constantly expanding universe – de Sitter spacetime



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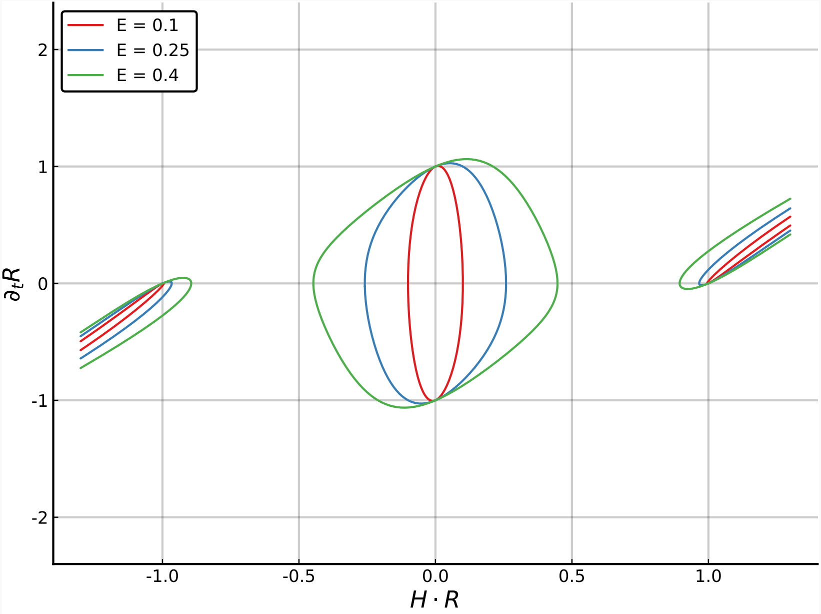

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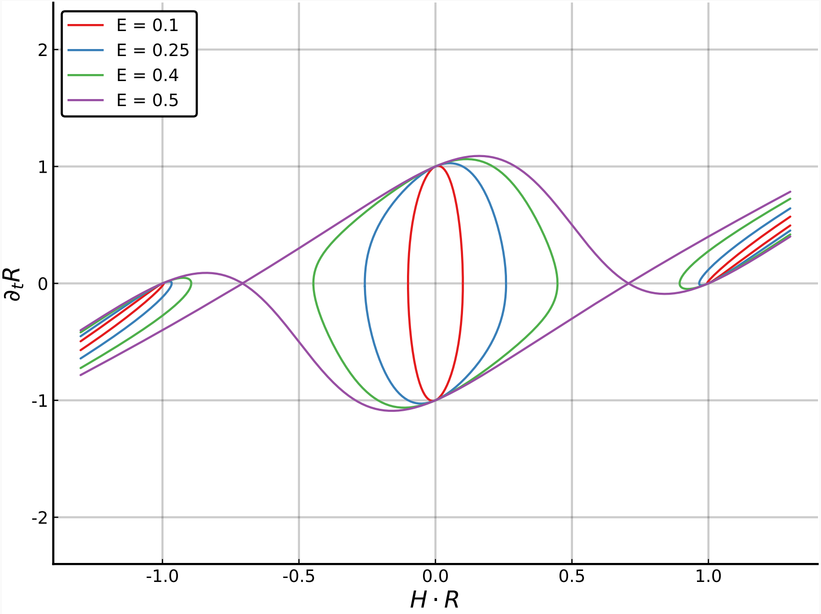

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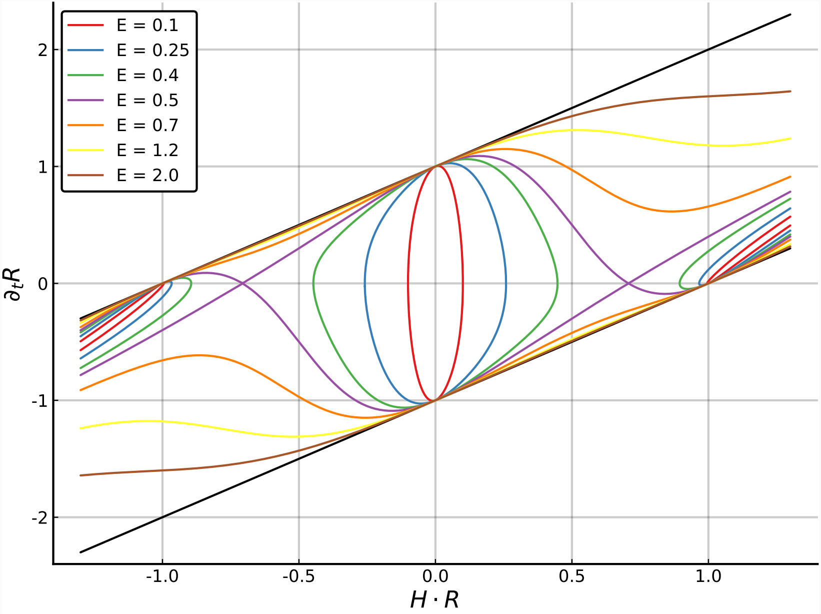

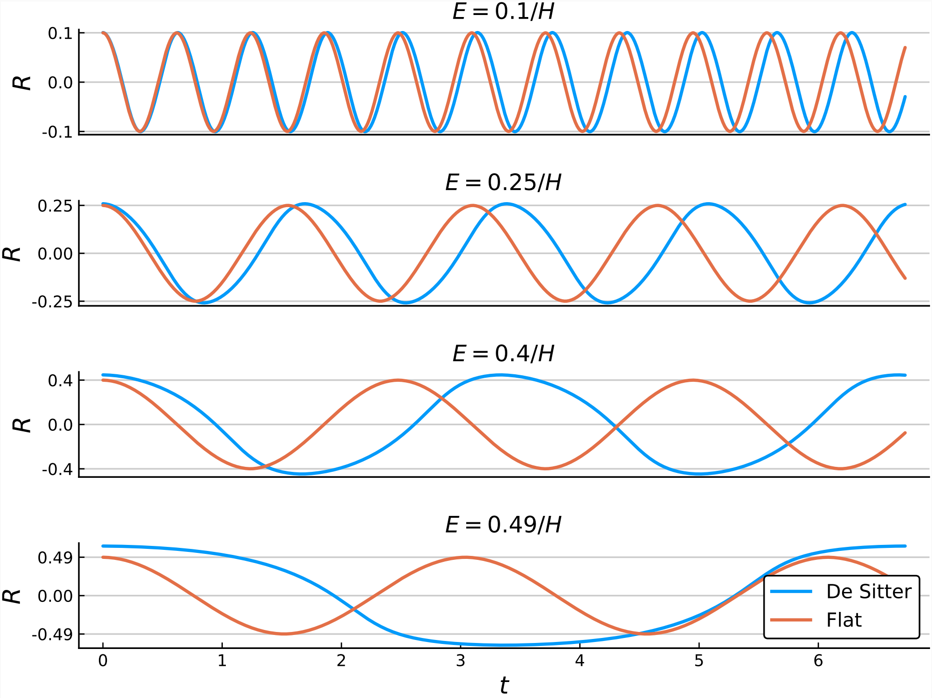

$E = 0.1/H$
$E = 0.25/H$
$E = 0.4/H$
$E = 0.49/H$
R
0.1
0.0
-0.1
0.25
0.00
-0.25
0.4
0.0
-0.4
0.49
0.00
-0.49
0
1
2
3
4
5
6
t
De Sitter
Flat

Gravitational wave background

Gravitational wave background

- Metric takes the form:

$$\mathrm{d}s^2 = H\,\mathrm{d}u^2 + 2\,\mathrm{d}u\,\mathrm{d}v + \mathrm{d}x^2 + \mathrm{d}y^2$$

  where $u = (z+t)/\sqrt{2}$ and $v = (z-t)/\sqrt{2}$
- $H = H(u,x,y)$ satisfies

$$\partial_x^2 H + \partial_y^2 H = 0$$

Gravitational wave background

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where $u = (z+t)/\sqrt{2}$ and $v = (z-t)/\sqrt{2}$

- $H = H(u, x, y)$ satisfies

$$\partial_x^2 H + \partial_y^2 H = 0$$

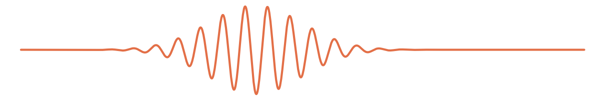
- Same parameterization conditions as in flat spacetime

$$\gamma_{\tau\tau} + \gamma_{\sigma\sigma} = 0, \qquad \gamma_{\tau\sigma} = \gamma_{\sigma\tau} = 0$$

Interaction with gravitational wave burst

- We choose the gravitational wave function H to be a gaussian burst with $+$ polarization, frequency ω and amplitude A:

$$H = (x^2 - y^2)h(\tau)$$



- Space is flat in regions long before and long after the gravitational wave burst

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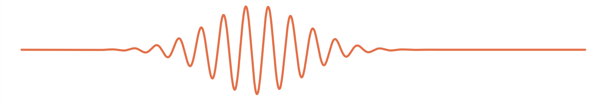

- Space is flat in regions long before and long after the gravitational wave burst
- Similarly to flat spacetime, we can expand the σ dependance into modes:

$$X^i(\sigma,\tau) = \sum_{k_i \in \mathbb{Z}} X^i_{k_i}(\tau) \mathrm{e}^{2\pi \mathrm{i} k_i \sigma/\sigma_1} \qquad i \in \{v, x, y\}$$

Interaction with gravitational wave burst

- The τ evolution of each mode is given by its equation of motion

$$
\begin{aligned}
\left(\partial_\tau^2 + k_x^2 - h(\tau)\right) x_{k_x} &= 0 \\
\left(\partial_\tau^2 + k_y^2 + h(\tau)\right) y_{k_y} &= 0 \\
\left(\partial_\tau^2 + k_v^2\right) v_{k_v} &= \sum_{k_x} x_{k_x} x_{k_v - k_x} \partial_\tau h(\tau)/2 - 2 x_{k_x} \partial_\tau x_{k_v - k_x} h(\tau) \\
&+ \sum_{k_y} y_{k_y} y_{k_v - k_y} \partial_\tau h(\tau)/2 - 2 y_{k_y} \partial_\tau y_{k_v - k_y} h(\tau)
\end{aligned}
$$

Interaction with gravitational wave burst

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$$\begin{aligned}
\left(\partial_\tau^2 + k_x^2 - h(\tau)\right) x_{k_x} &= 0 \\
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\left(\partial_\tau^2 + k_v^2\right) v_{k_v} &= \sum_{k_x} x_{k_x} x_{k_v - k_x} \partial_\tau h(\tau)/2 - 2 x_{k_x} \partial_\tau x_{k_v - k_x} h(\tau) \\
&+ \sum_{k_y} y_{k_y} y_{k_v - k_y} \partial_\tau h(\tau)/2 - 2 y_{k_y} \partial_\tau y_{k_v - k_y} h(\tau)
\end{aligned}$$

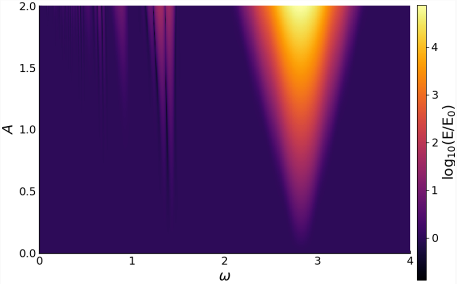
- Strong interaction between string and gravitational wave for resonant frequencies

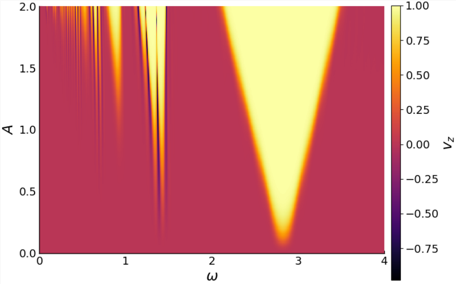
$$\omega \approx \frac{2\sqrt{2}}{n}, \qquad n \in \mathbb{Z}$$

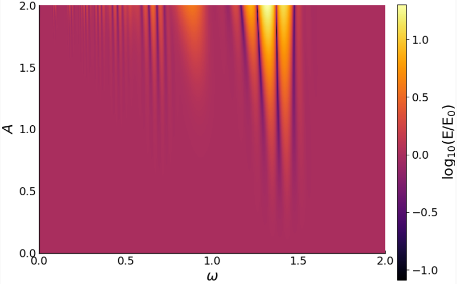
Interaction with gravitational wave burst

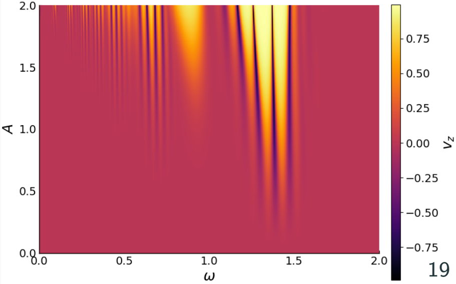
Energy

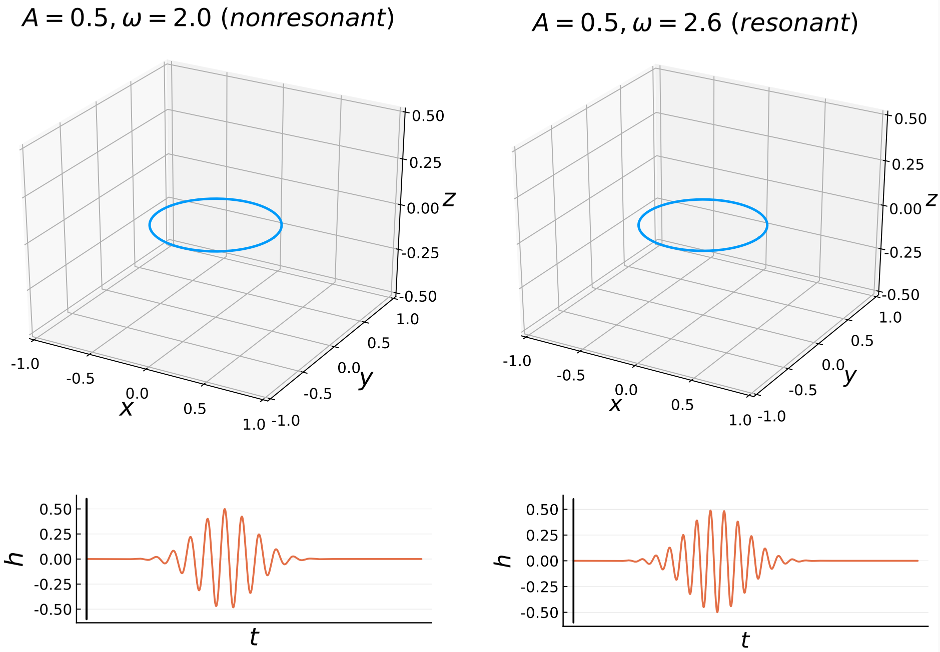
Velocity v_z










$A = 0.5, \omega = 2.0$ (*nonresonant*)
$A = 0.5, \omega = 2.6$ (*resonant*)
x
y
z
h
t
0.50
0.25
0.00
-0.25
-0.50
-1.0
-0.5
0.0
0.5
1.0

Summary

- Understanding of the gauge choices and some tricks to simplify equations.
- Small circular strings in an expanding universe feel little difference from flat spacetime.
- Circular strings of cosmological scale in an expanding universe expand to infinity or until they break.
- General solution for gravitational wave background.
- Interaction of circular string with gravitational wave pulse.
- Resonant frequencies between strings and gravitational waves.

Questions

Oponent's question no. 1

- Q: In chapter one it is said that to find the equations of motion one has to minimize the action. In general one only looks for extrema of the action, which could be a minimum, a maximum or a saddle point. Is it clear that in the present context only minima arise?
- A: All types of extrema are possible and physically correct.

Oponent's question no. 2

- Q: In eq. (1.10) the last step is not very obvious to me.

$$\delta(\det \gamma) = \det \gamma \gamma^{\alpha\beta} \delta\gamma_{\alpha\beta}$$

- A: Maybe it is better to use the identity

$$\det\left(\mathrm{e}^{A}\right) = \mathrm{e}^{\mathrm{tr}(A)}$$

then we can write

$$\begin{aligned}\delta \det \gamma = \delta \det\left(\mathrm{e}^{\ln \gamma}\right) = \delta \mathrm{e}^{\mathrm{tr}\ln \gamma} = \det \gamma \,\mathrm{tr}\left(\gamma^{-1}\delta\gamma\right) \\ = \det \gamma \gamma^{\alpha\beta} \delta\gamma_{\alpha\beta}\end{aligned}$$

Oponent's question no. 3

- Q: Eqs. (1.19) and (1.20) are incorrect

$$\omega = \sqrt{-\gamma}\gamma^{\alpha\beta} g_{MN} \partial_\alpha X^M \delta X^N \tag{1.19}$$

$$\mathrm{d}\omega = \partial_\beta \Big(\sqrt{-\gamma}\gamma^{\alpha\beta} g_{MN} \partial_\alpha X^M \delta X^N \Big)\, \mathrm{d}y^\beta \tag{1.20}$$

- A: The integral for which we use the Stoke's theorem is

$$\int_{\tau_i}^{\tau_f} \int_0^{\sigma_1} \mathrm{d}\tau\, \mathrm{d}\sigma\ \partial_\beta \Big(\mathcal{P}_N^\beta \delta X^N \Big) \qquad \mathcal{P}_N^\beta = \sqrt{-\gamma}\gamma^{\alpha\beta} g_{MN} \partial_\alpha X^M$$

the corresponding 2-form is

$$\mathrm{d}\omega = \Big[\partial_\tau \Big(\mathcal{P}_N^\tau \delta X^N \Big) + \partial_\sigma \Big(\mathcal{P}_N^\sigma \delta X^N \Big) \Big]\, \mathrm{d}\tau \wedge \mathrm{d}\sigma$$

we can see, that this is an exterior derivative of

$$\mathrm{d}\omega = \mathrm{d}\Big[\mathcal{P}_N^\tau \delta X^N\, \mathrm{d}\sigma - \mathcal{P}_N^\sigma \delta X^N\, \mathrm{d}\tau \Big]$$

Oponent's question no. 4

- Q: In sec. 2.1 the freedom of reparameterizing the string worldsheet is used to take $t = \tau$ and impose eq. (2.3) and also $C(\sigma) = 1$. This is three conditions while the freedom to make reparameterizations involves two free functions. A comment about why this is okay would be useful.
- A: Imposing eq. (2.3)

$$g_{MN}\partial_\tau X^M \partial_\sigma X^N = \gamma_{\tau\sigma} = 0 \tag{2.3}$$

does not fully specify the parameterization (only that it is perpendicular to the direction of constant lines of τ). And because the quantity

$$C(\sigma) = \sqrt{\frac{\gamma_{\sigma\sigma}}{\gamma_{\tau\tau}}} = \text{const.}$$

we can choose the "length" of $\partial_\sigma X^M$, which corresponds to choosing a specific value for the const.

Oponent's question no. 5

- Q: In eq. (2.20) the boundary conditions given are for an open string, without comment. While the rest of the thesis seems to be about closed strings, or were open strings also considered?
- A: The conditions for open strings were for a complete picture. We only studied closed strings, that automatically satisfy these conditions.

Oponent's question no. 6

- Q: In sec. 3.1 a coordinate $R = r\mathrm{e}^{Ht}$ is introduced which is positive by definition, but then negative values of R are allowed, e.g. Fig. 3.1. What is the reason for this?
- A: It is similar to harmonic oscilator – when the string goes to the point $r = 0$ it reaches maximum velocity $\partial_t r$. The passage through this point would mean either that $r \to -r$ or $\theta \to \theta + \pi$

Oponent's question no. 7

- Q: Figures 3.5-10 mention de Sitter space but it is not explained anywhere what this space is and how it's related to the expanding FLRW universe considered.
- A: De Sitter space (in 4D) is defined as a submanifold in Minkowski spacetime

$$\mathrm{d}s^2 = -\mathrm{d}t^2 + \mathrm{d}x^2 + \mathrm{d}y^2 + \mathrm{d}z^2$$

that satisfies

$$-t^2 + x^2 + y^2 + z^2 = \alpha^2$$

Oponent's question no. 7

It can be shown, that with the coordinate change

$$t = \alpha \sinh\left(\frac{t'}{\alpha}\right) + (y'^2 + z'^2)\frac{\mathrm{e}^{t'/\alpha}}{2\alpha}$$

$$x = \alpha \cosh\left(\frac{t'}{\alpha}\right) - (y'^2 + z'^2)\frac{\mathrm{e}^{t'/\alpha}}{2\alpha}$$

$$y = \mathrm{e}^{t'/\alpha} y' \qquad z = \mathrm{e}^{t'/\alpha} z'$$

the metric becomes

$$\mathrm{d}s^2 = -\,\mathrm{d}t'^2 + \mathrm{e}^{2t'/\alpha}\left(\mathrm{d}x'^2 + \mathrm{d}y'^2 + \mathrm{d}z'^2\right)$$

This is the same form as the FLWR metric with only the cosmological constant

Oponent's question no. 8

- Q: Related to point 4. Why are you allowed to impose the condition $u = \lambda\tau$ in (4.15) when you have already imposed two conditions on γ in (4.9)?

$$\gamma_{\tau\tau} + \gamma_{\sigma\sigma} = 0 \qquad \gamma_{\tau\sigma} = \gamma_{\sigma\tau} = 0 \tag{4.15}$$

- A: In order for the transformations

$$\tau' = \varphi_1(\tau, \sigma), \qquad \sigma' = \varphi_2(\tau, \sigma)$$

to preserve the conditions (4.15), they must satisfy

$$(\partial_\tau^2 - \partial_\sigma^2)\tau' = 0, \qquad (\partial_\tau^2 - \partial_\sigma^2)\sigma' = 0$$

Moreover, the second equation of motion reads

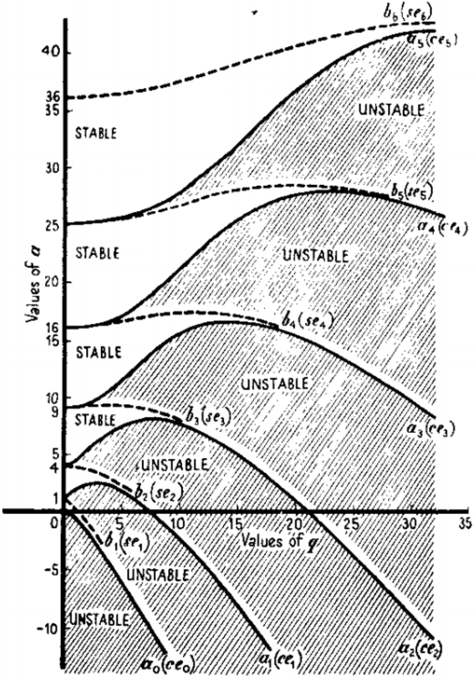
$$(\partial_\tau^2 - \partial_\sigma^2)u = 0,$$

so we can choose $\tau' = u/\lambda$.

Oponent's question no. 9

- Q: In chapter 4 it would be nice to indicate the duration of the gravitational wave in fig 4.6-7 and to have an explanation in words of what happens to the string when the gravitational wave hits and contrasting this with what happens to a point-like particle.
- A: The ρ parameter that specifies the duration of the pulse had the value $\rho = 10$ in the units of time. The position of the strong interactions for resonant frequencies does not change while varying the length of interaction. The animations shown in this presentation should answer the second part of this question.

Bonus slides



Taken from N.W. McLachlan. *Theory and Application of Mathieu Functions*. **Oxford: The clarendon Press, 1951**