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Strings in Gravitational Shock Wave Backgrounds

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In this paper and the subsequent one, the quantum scattering of particles by a gravitational shock wave (GSW) is computed within the framework of string theory. The exact string equations of motion and constraints in the GSW background are reviewed for the light-cone gauge. In addition, the ambiguity in the solution for the longitudinal coordinate is solved. The light-cone solution is generalized to an arbitrary covariant gauge. This general solution turns out to be completely explicit and not much more involved than the solution for the light-cone gauge. The extension of the string solution to the case of a GSW generated by an arbitrary ultrarelativistic source is also presented. © 1991 Academic Press, Inc.

I. INTRODUCTION

At particle energies of the order or larger than the Planck mass, the curved space-time geometry created by the particles dominate their collision process. In such situation, the description of fields or strings in flat space-time is no longer valid. The dynamics of the quantum fields or strings is then governed by their equations of motion in the classical background geometry.

This has been the motivation to investigate string propagation in relevant background geometries. In Ref. [1] a systematic approach to quantize strings in curved space-times was proposed. It has been applied to cosmological space-times [2], black-hole geometries [3], and more general ones [4]. In addition, the string equations of motion turned out to be exactly soluble, in closed form, for some interesting geometries, like gravitational shock waves [5, 6] and conical space-time [7] (the geometry around a straight cosmic string).

The purpose of this paper and of the subsequent one [8] is to investigate the scattering of particles by a gravitational shock wave in the framework of string theory. As it was stressed in Ref. [9], the shock wave described by the Aichelburg–Sexl metric (that is the gravitational field of a neutral spinless ultrarelativistic particle) is relevant to particle scattering at Planck energy. We then choose to investigate specifically the scattering of a string (in one of its stationary

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states) by a particle with an energy of the order or larger than the Planck mass. The string is considered here as a test string, in other words its energy must be much smaller than the energy carried by the shock wave.

To compute N-point particle amplitudes in a curved, but asymptotically flat, space-time we start from the following generalization of the usual flat space-time formula [6]:

$$A_N(k_1, ..., k_N) = \int \prod_{i=1}^{N} [d\sigma_i\, d\tau_i] \langle O_< | \prod_{j=1}^{N} : \Psi(k_j, X(\sigma_j, \tau_j)) \colon |O_< \rangle. \tag{1.1}$$

Here $\Psi(k, X(\sigma, \tau))$ represents the vertex operator for a particle of asymptotic momentum k in curved space-time. It is a solution of the corresponding wave equation in the given geometry, i.e., the Klein–Gordon equation for a scalar particle [6]. Furthermore, the string coordinates $X^\mu(\sigma, \tau)$ fulfil the propagation equations in the choosen curved geometry,

$$\partial_A [G_{\lambda\mu}(X)\, \partial^A X^\mu] - \tfrac{1}{2}[\partial_\lambda G_{\mu\nu}(X)](\partial_A X^\mu)(\partial^A X^\nu) = 0,$$

where $G_{\mu\nu}(X)$ is the space-time metric ($\mu, \nu = 0, 1, ..., D-1$) and we use the orthonormal gauge for the world-sheet. Hence, the string interaction with the geometry shows in two different places: the functional form of $\Psi(k, X)$ and the solution for $X^\mu(\sigma, \tau)$.

The vertex operators $\Psi(k_j, X(\sigma_j, \tau_j))$ pinch the world-sheet at N different points. These pinches describe the ingoing and outgoing particles intervening in the process. Of course, the integration in Eq. (1.1) must cover the whole string world-sheet.

The aim of the present articles is to compute the two-point amplitude, $A_2(k_2, k_1)$, for the scattering of a scalar particle (the tachyon in a bosonic string) by the shock wave. We start by solving exactly the string equations of motion and the constraint equations for a shock wave space-time, in the light-cone gauge [5, 6]. We recall that the string obeys the flat equations of motion in one side ($<$) and the other ($>$) of the shock wave. There is a non-trivial matching between both flat space-time string solutions, which is reviewed and completed in Section II. The ambiguity in the longitudinal coordinate is solved explicitly. We find that the constraints are satisfied if and only if we choose the mean-value prescription (2.18). This string solution will be used as the starting point for the computation of the scattering amplitude in shock wave space-time presented in the companion paper [8].

In order to complete the study of the string solutions in shock wave space-times, in Section III we generalize the light-cone solution of Section II to an arbitrary covariant gauge. The general solution to the string matching problem is still completely explicit and not much more involved than the solution for the light-cone gauge. This happens in spite of the fact that now the intersection between the string world-sheet and the shock wave becomes a generic curve depending on the reparametrisation functions (it is simply $\tau = 0$ in the light-cone gauge).

It can be noticed that the transverse string center of mass coordinates are discontinuous in the covariant gauge, while they are continuous in the light-cone gauge, as well as for the coordinates of a point particle propagating through the shock wave [10]. On the contrary, the discontinuities of the string center of mass momenta turn out to be gauge independent.

The general covariant solution of the string equations in the shock wave space-time has the advantage of being linear on the oscillator modes also for the longitudinal coodinates and should in principle be preferred for accomplishing exact computations of scattering amplitudes or other relevant physical quantities.

Finally, in Section IV we generalize the string solutions for the case of a gravitational shock wave generated by an arbitrary ultrarelativistic source.

II. Exact Solution of the String in D-dimensional AS Geometry (Light-Cone Gauge)

The Aichelburg–Sexl (AS) metric in D dimensions is given by [11]

$$ds^2 = dU\,dV - (dX^i)^2 + f_D(\rho)\,\delta(U)\,dU^2, \tag{2.1}$$

where U and V are null coordinates,

$$U \equiv X^0 - X^{D-1}, \qquad V \equiv X^0 + X^{D-1}, \tag{2.2}$$

and

$$X^i, \qquad i = 1, 2, ..., D-2, \tag{2.3}$$

are transverse spatial coordinates. Here $\rho = \sqrt{\sum_{i=1}^{D-2} (X^i)^2}$. The function $f_D(\rho)$ obeys

$$\nabla_i^2 f_D(\rho) = 16\pi G\tilde{p}\,\delta^{(D-2)}(X^i), \tag{2.4}$$

whose solution can be written as

$$f_D(\rho) = \begin{cases} k\rho^{4-D}, & D > 4 \\ 8G\tilde{p}\ln\rho, & D = 4, \end{cases} \tag{2.5}$$

where

$$K \equiv -\frac{8\pi^{2-D/2}}{D-4}\,\Gamma\left(\frac{D}{2}-1\right) G\tilde{p} \tag{2.6}$$

and G is the gravitational constant. This metric describes the gravitational field of an ultrarelativistic particle moving along the X^{D-1} axis with momentum $\tilde{p}$. Notice that the space-time is everywhere flat except on the plane $U = X^0 - X^{D-1} = 0$, where a shock wave in located.

The Riemann tensor follows from the metric (2.1), with the result

$$R_{UiUj}=\tfrac{1}{2}\delta(U)\,\partial_i\,\partial_j f_D(\rho). \tag{2.7}$$

The other components either vanish or follow from Eq. (2.7) by symmetries.

Let us consider a test string propagating in the D-dimensional AS geometry. Its equations of motion are

$$\partial_A\,[G_{\lambda\mu}(X)\,\partial^A X^\mu]-\tfrac{1}{2}[\partial_\lambda G_{\mu\nu}(X)](\partial_A X^\mu)(\partial^A X^\nu)=0, \tag{2.8}$$

in orthonormal gauge. Here $\mu, \nu=U, V, i$ $(1\leqslant i\leqslant D-2)$, $A=0, 1$, and the metric tensor $G_{\mu\nu}(X)$ can be read from Eq. (2.1). The world-sheet coordinates are $\eta^0=\tau$, $\eta^1=\sigma$, with $0\leqslant\sigma\leqslant 2\pi$ being the spatial coordinate. Inserting the AS metric (2.1) in (2.8) yields [5, 6]:

$$U''-\ddot{U}=0, \tag{2.9a}$$

$$V''-\ddot{V}+f_D(\rho)[U'^2-\dot{U}^2]\frac{\partial}{\partial U}\delta(U)+2\partial_i f_D(\rho)[U'X'^i-\dot{U}\dot{X}^i]\,\delta(U)=0, \tag{2.9b}$$

$$X''^i-\ddot{X}^i+\tfrac{1}{2}\partial_i f_D(\rho)[U'^2-\dot{U}^2]\,\delta(U)=0, \tag{2.9c}$$

where $'$ and $\dot{}$ stand for $\partial/\partial\sigma$ and $\partial/\partial\tau$, respectively.

Since $U(\sigma,\tau)$ obeys the d'Alembert equation, we can always make a reparametrization,

$$\begin{aligned}\sigma+\tau&\rightarrow\varphi_1(\sigma+\tau),\\ \sigma-\tau&\rightarrow\varphi_2(\sigma-\tau),\end{aligned} \tag{2.10}$$

such that

$$U=2\alpha' p^U\tau, \tag{2.11}$$

where p^U is constant. Therefore, the equations of motion become

$$V''-\ddot{V}-f_D(\rho(\sigma,0))\,\dot{\delta}(\tau)-\dot{f}_D(\rho(\sigma,\tau))\,\delta(\tau)=0, \tag{2.12a}$$

$$X''^i-\ddot{X}^i-\alpha' p^U\,\partial_i f_D(\rho(\sigma,0))\,\delta(\tau)=0, \tag{2.12b}$$

where we used the property:

$$f(\tau)\,\dot{\delta}(\tau)=f(0)\,\dot{\delta}(\tau)-\dot{f}(\tau)\,\delta(\tau).$$

The string coordinates satisfy the d'Alembert equation for all (σ,τ) except at $\tau=0$, where $\dot{X}^i(\sigma,\tau)$, $V(\sigma,\tau)$, and $\dot{V}(\sigma,\tau)$ are discontinuous. One finds from (2.12) [6]:

$$[\dot{X}^i_>(\sigma,\tau)-\dot{X}^i_<(\sigma,\tau)]_{\tau=0}=-\alpha' p^U\,\partial_i f_D(\rho(\sigma,0)), \tag{2.13a}$$

$$X^i_>(\sigma,0)-X^i_<(\sigma,0)=0, \tag{2.13b}$$

$$[\dot{V}_>(\sigma,\tau)-\dot{V}_<(\sigma,\tau)]_{\tau=0}=-\partial_\tau f_D(\rho(\sigma,\tau))|_{\tau=0}, \tag{2.13c}$$

$$V_>(\sigma,0)-V_<(\sigma,0)=-f_D(\rho(\sigma,0)). \tag{2.13d}$$

Here $X^\mu_<(\sigma,\tau)$ and $X^\mu_>(\sigma,\tau)$ stand for the string coordinates before ($\tau<0$) and after ($\tau>0$) the collision with the shock wave. Notice that $U_> = U_< = 2\alpha' p^U \tau$ is continuous at $\tau=0$. The r.h.s. term in Eq. (2.13c) is indeed ambiguous due to the discontinuity of $\dot{X}^i(\sigma,\tau)$ at $\tau=0$. We shall give below (see Eq. (2.18)) the solution to this ambiguity.

Since $X^\mu_<(\sigma,\tau)$ and $X^\mu_>(\sigma,\tau)$ obey the flat space-time (d'Alembert) equations,

$$\begin{aligned} &\ddot{X}^\mu_<(\sigma,\tau) - X''^\mu_<(\sigma,\tau) = 0 \qquad (\tau<0),\\ &\ddot{X}^\mu_>(\sigma,\tau) - X''^\mu_>(\sigma,\tau) = 0 \qquad (\tau>0), \end{aligned} \tag{2.14}$$

one can write the string solution as

$$X^i_{\gtrless}(\sigma,\tau) = \vec{X}^i_{\gtrless}(\sigma-\tau) + \bar{X}^i_{\gtrless}(\sigma+\tau), \tag{2.15a}$$

$$V_{\gtrless}(\sigma,\tau) = \vec{V}_{\gtrless}(\sigma-\tau) + \bar{V}_{\gtrless}(\sigma+\tau). \tag{2.15b}$$

Inserting Eqs. (2.15) in the discontinuity relations (2.13) yields

$$\vec{X}^i_>(\theta_-) - \vec{X}^i_<(\theta_-) = \frac{\alpha' p^U}{2}\int_{\sigma^i}^{\theta_-} d\sigma'\, \partial_i f_D(\rho(\sigma',0)), \tag{2.16a}$$

$$\bar{X}^i_>(\theta_+) - \bar{X}^i_<(\theta_+) = -\frac{\alpha' p^U}{2}\int_{\sigma^i}^{\theta_+} d\sigma'\, \partial_i f_D(\rho(\sigma',0)), \tag{2.16b}$$

$$\vec{V}_>(\theta_-) - \vec{V}_<(\theta_-) = -\frac{1}{2} f_D(\rho((\theta_-,0)) + \frac{1}{2}\int_{\sigma^0}^{\theta_-} d\sigma'\, \partial_{\tau'} f_D(\rho(\sigma',\tau'))|_{\tau'=0}, \tag{2.17a}$$

$$\bar{V}_>(\theta_+) - \bar{V}_<(\theta_+) = -\frac{1}{2} f_D(\rho(\theta_+,0)) - \frac{1}{2}\int_{\sigma^0}^{\theta_+} d\sigma'\, \partial_{\tau'} f_D(\rho(\sigma',\tau'))|_{\tau'=0}, \tag{2.17b}$$

where $\theta_\pm \equiv \sigma \pm \tau$ and σ^0 and σ^i are parameters that will be determined later (see Eqs. (2.27) and (2.28)).

The last terms in Eqs. (2.17) need special care since $\dot{X}^i(\sigma,\tau)$ and then $\dot{\rho}(\sigma,\tau)$ are discontinuous at $\tau=0$. We found that one must define:

$$\partial_\tau f_D(\rho(\sigma,\tau))|_{\tau=0} \equiv \tfrac{1}{2}\partial_i f_D(\rho(\sigma,0))[\dot{X}^i_<(\sigma,\tau) + \dot{X}^i_>(\sigma,\tau)]_{\tau=0}. \tag{2.18}$$

We show below that one must choose this definition in order to fulfil the constraint equations. Using Eq. (2.18) for $\partial_{\tau'} f_D(\rho(\sigma',\tau'))|_{\tau'=0}$, we find from Eq. (2.17),

$$\vec{V}_>(\theta_-) - \vec{V}_<(\theta_-) = -\frac{1}{2} f_D(\rho(\sigma^0,0)) - \int_{\sigma^0}^{\theta_-} d\sigma'\, \partial_i f_D(\rho(\sigma',0))\left[\dot{\vec{X}}^i_<(\sigma') + \frac{\alpha' p^U}{4}\partial_i f_D(\rho(\sigma',0))\right], \tag{2.19a}$$

$$\tilde{V}_>(\theta_+) - \tilde{V}_<(\theta_+) = -\frac{1}{2} f_D(\rho(\sigma^0, 0))$$
$$- \int_{\sigma^0}^{\theta_+} d\sigma' \, \partial_i f_D(\rho(\sigma', 0)) \left[\dot{\tilde{X}}^i_<(\sigma') - \frac{\alpha' p^U}{4} \partial_i f_D(\rho(\sigma', 0)) \right], \tag{2.19b}$$

where the integrals have been recast entirely in terms of light-cone variables. Here and in what follows, the point ($\dot{}$) stands for the derivative with respect to the argument.

Using Eq. (2.5) for $D > 4$ and the definitions,

$$C^i(\sigma) \equiv X^i_<(\sigma, 0) = X^i_>(\sigma, 0), \qquad C(\sigma) \equiv \sqrt{\sum_{i=1}^{D-2} [C^i(\sigma)]^2},$$
$$D^i(\sigma) \equiv C^i(\sigma)\, C^{2-D}(\sigma), \qquad \tilde{K} \equiv \frac{D-4}{2} K, \tag{2.20}$$

the discontinuity equations (2.16) and (2.19) can be summarized as

$$\vec{X}^i_>(\theta_-) - \vec{X}^i_<(\theta_-) = -\alpha' p^U \tilde{K} \int_{\sigma'}^{\theta_-} d\sigma \, D^i(\sigma), \tag{2.21a}$$

$$\tilde{X}^i_>(\theta_+) - \tilde{X}^i_<(\theta_+) = \alpha' p^U \tilde{K} \int_{\sigma^i}^{\theta_+} d\sigma \, D^i(\sigma), \tag{2.21b}$$

$$\vec{V}_>(\theta_-) - \vec{V}_<(\theta_-) = -\frac{K}{2} C^{4-D}(\sigma^0)$$
$$+ 2\tilde{K} \int_{\sigma^0}^{\theta_-} d\sigma \, D^i(\sigma) \left[\dot{\vec{X}}^i_<(\sigma) - \frac{\alpha' p^U \tilde{K}}{2} D^i(\sigma) \right], \tag{2.21c}$$

$$\tilde{V}_>(\theta_+) - \tilde{V}_<(\theta_+) = -\frac{K}{2} C^{4-D}(\sigma^0)$$
$$+ 2\tilde{K} \int_{\sigma^0}^{\theta_+} d\sigma \, D^i(\sigma) \left[\dot{\tilde{X}}^i_<(\sigma) + \frac{\alpha' p^U \tilde{K}}{2} D^i(\sigma) \right]. \tag{2.21d}$$

The string solutions $X^\mu_<(\sigma, \tau)$ and $X^\mu_>(\sigma, \tau)$ can be Fourier expanded as in flat space-time (see Eq. (2.14)), i.e.,

$$\vec{X}^\mu_{\gtrless}(\theta_-) = \frac{1}{2} q^\mu_{\gtrless} - \alpha' p^\mu_{\gtrless} \theta_- + i\sqrt{\alpha'} \sum_{n \neq 0} \frac{1}{n} \tilde{\alpha}^\mu_{n\gtrless} e^{in\theta_-}, \tag{2.22}$$

$$\tilde{X}^\mu_{\gtrless}(\theta_+) = \frac{1}{2} q^\mu_{\gtrless} + \alpha' p^\mu_{\gtrless} \theta_+ + i\sqrt{\alpha'} \sum_{n \neq 0} \frac{1}{n} \alpha^\mu_{n\gtrless} e^{-in\theta_+}. \tag{2.23}$$

Notice that in the light-cone gauge, $q^U = \alpha_n^U = \tilde{\alpha}_n^U = 0$. Equations (2.13) and (2.21) allow to relate the $<$ and $>$ coefficients in Eqs. (2.22) and (2.23) [6]:

$$q^i_> - q^i_< = 0, \tag{2.24a}$$

$$p^i_> - p^i_< = \frac{p^U \tilde{K}}{2\pi} \int_0^{2\pi} d\sigma\, D^i(\sigma), \tag{2.24b}$$

$$\alpha^i_{n>} - \alpha^i_{n<} = \frac{\sqrt{\alpha'}\, p^U \tilde{K}}{2\pi} \int_0^{2\pi} d\sigma\, D^i(\sigma)\, e^{in\sigma}, \tag{2.24c}$$

$$\tilde{\alpha}^i_{n>} - \tilde{\alpha}^i_{n<} = \frac{\sqrt{\alpha'}\, p^U \tilde{K}}{2\pi} \int_0^{2\pi} d\sigma\, D^i(\sigma)\, e^{-in\sigma}, \tag{2.24d}$$

$$q^V_> - q^V_< = -\frac{K}{2\pi} \int_0^{2\pi} d\sigma\, C^{4-D}(\sigma), \tag{2.25a}$$

$$\begin{aligned} p^V_> - p^V_< &= -\frac{\tilde{K}}{\pi\alpha'} \int_0^{2\pi} d\sigma\, D^i(\sigma) \left[\dot{X}^i_<(\sigma) - \frac{\alpha' p^U \tilde{K}}{2} D^i(\sigma) \right] \\ &= \frac{\tilde{K}}{\pi\alpha'} \int_0^{2\pi} d\sigma\, D^i(\sigma) \left[\dot{X}^i_<(\sigma) + \frac{\alpha' p^U \tilde{K}}{2} D^i(\sigma) \right], \end{aligned} \tag{2.25b}$$

$$\alpha^V_{n>} - \alpha^V_{n<} = \frac{\tilde{K}}{\pi\sqrt{\alpha'}} \int_0^{2\pi} d\sigma\, D^i(\sigma) \left[\dot{X}^i_<(\sigma) + \frac{\alpha' p^U \tilde{K}}{2} D^i(\sigma) \right] e^{in\sigma}, \tag{2.25c}$$

$$\tilde{\alpha}^V_{n>} - \tilde{\alpha}^V_{n<} = -\frac{\tilde{K}}{\pi\sqrt{\alpha'}} \int_0^{2\pi} d\sigma\, D^i(\sigma) \left[\dot{X}^i_<(\sigma) - \frac{\alpha' p^U \tilde{K}}{2} D^i(\sigma) \right] e^{-in\sigma}, \tag{2.25d}$$

where

$$C^i(\sigma) = q^i_< + i\sqrt{\alpha'} \sum_{n \neq 0} \frac{1}{n} [\alpha^i_{n<} - \tilde{\alpha}^i_{-n<}]\, e^{in\sigma}. \tag{2.26}$$

The identity between the second members of Eq. (2.25b) follows from the periodicity condition for the closed string, i.e.,

$$\int_0^{2\pi} d\sigma\, D^i(\sigma)[\dot{X}^i_<(\sigma) + \dot{X}^i_<(\sigma)] = -2\tilde{K}[C^{4-D}(2\pi) - C^{4-D}(0)] = 0.$$

The parameters σ^0 and σ^i appearing in Eqs. (2.21) can now be determined by computing

$$\int_0^{2\pi} d\sigma [X^\mu_>(\sigma, 0) - X^\mu_<(\sigma, 0)]$$

from Eqs. (2.21), on one hand, and from Eqs. (2.24)–(2.25), on the other hand. Equating both results yields [6]

$$\int_0^{2\pi} d\sigma \left\{ D^i(\sigma) - \int_{\sigma'}^{\sigma} \frac{d\sigma'}{\pi} D^i(\sigma') \right\} = 0, \tag{2.27}$$

$$\int_0^{2\pi} d\sigma \left\{ D^i(\sigma) \left[\dot{X}^i_<(\sigma) - \frac{\alpha' p^U \tilde{K}}{2} D^i(\sigma) \right] - \int_{\sigma^0}^{\sigma} \frac{d\sigma'}{\pi} D^i(\sigma') \left[\dot{X}^i_<(\sigma') - \frac{\alpha' p^U \tilde{K}}{2} D^i(\sigma') \right] \right\}$$
$$= \int_0^{2\pi} \frac{d\sigma}{2\pi(D-4)} \left[C^{4-D}(\sigma) - C^{4-D}(\sigma^0) \right]. \tag{2.28}$$

The constraint equations in light-cone gauge,

$$T_{\pm\pm} = G_{\mu\nu}(X)(\partial_\pm X^\mu)(\partial_\pm X^\nu) = 0$$

(where $x_\pm = \sigma \pm \tau$ and $\partial_\pm = \frac{1}{2}(\partial_\sigma \pm \partial_\tau)$) simply read

$$\pm \partial_\pm V_< = \frac{1}{\alpha' p^U} (\partial_\pm X^i_<)^2 \tag{2.29}$$

for $\tau < 0$ and

$$\pm \partial_\pm V_> = \frac{1}{\alpha' p^U} (\partial_\pm X^i_>)^2 \tag{2.30}$$

for $\tau > 0$. Inserting Eqs. (2.16)–(2.17) for $V_>(\sigma, \tau)$ and $X^i_>(\sigma, \tau)$ in Eq. (2.30), we reproduce Eq. (2.29) *if and only if* Eq. (2.18) holds. That is, the ambiguity in Eq. (2.12a) is eliminated, through Eq. (2.18), by the requirement of conservation of the constraints.

III. The String Solution in the Covariant Gauge

We derive in this section the solution of the string equations of motion in the covariant gauge. We do that by the following reparametrization of the world-sheet coordinates (σ, τ) into (σ', τ'),

$$\begin{aligned} \theta_+ &= \sigma + \tau = \frac{1}{\alpha' p^U} \vec{U}(s_+), \\ \theta_- &= \sigma - \tau = \frac{1}{\alpha' p^U} \vec{U}(s_-), \end{aligned} \tag{3.1}$$

with $s_\pm \equiv \sigma' \pm \tau'$. Here $\vec{U}(s_+)$ and $\vec{U}(s_-)$ are arbitrary functions, besides the fact that they must satisfy the quasi-periodicity conditions:

$$\vec{U}(s_+ + 2\pi) - \vec{U}(s_+) = 2\pi\alpha' p^U,$$
$$\vec{U}(s_- + 2\pi) - \vec{U}(s_-) = 2\pi\alpha' p^U.$$

In this way $0 \leqslant \sigma' \leqslant 2\pi$, as is the case for σ. From Eqs. (2.11) and (3.1) we find

$$U(\sigma', \tau') = \vec{U}(s_+) - \vec{U}(s_-).$$

Since $U(\sigma', \tau')$ is now a general solution of the string equation (2.9a), the (σ', τ') gauge is convariant.

In the covariant gauge, the intersection of the string world-sheet with the shock wave becomes

$$\vec{U}(s_+) = \vec{U}(s_-). \tag{3.2}$$

It is therefore a generic curve on the world-sheet. (In light-cone gauge, it was the straight line $\tau = 0$).

We have free string propagation in the regions $<(\vec{U}(s_+) < \vec{U}(s_-))$ and $>(\vec{U}(s_+) > \vec{U}(s_-))$. The matching relations at $U(\sigma', \tau') = 0$ for the covariant gauge coordinates $X^\mu(\sigma', \tau')$ are obtained by reparametrizing Eqs. (2.21) with the help of Eq. (3.1). We find

$$\vec{X}^i_>(s_-) - \vec{X}^i_<(s_-) = -\tilde{K} \int_{\lambda'_-}^{s_-} ds'_-\, \dot{\vec{U}}(s'_-)\, D^i_-(s'_-), \tag{3.3}$$

$$\vec{X}^i_>(s_+) - \vec{X}^i_<(s_+) = \tilde{K} \int_{\lambda'_+}^{s_+} ds'_+\, \dot{\vec{U}}(s'_+)\, D^i_+(s'_+), \tag{3.4}$$

$$\begin{aligned} \vec{V}_>(s_-) - \vec{V}_<(s_-) = {} & \frac{K}{2} C_-^{4-D}(\lambda^0_-) \\ & + 2\tilde{K} \int_{\lambda^0_-}^{s_-} ds'_-\, D^i_-(s'_-) \left[\dot{\vec{X}}^i_<(s'_-) - \frac{\tilde{K}}{2} \dot{\vec{U}}(s'_-)\, D^i_-(s'_-) \right], \end{aligned} \tag{3.5}$$

$$\begin{aligned} \vec{V}_>(s_+) - \vec{V}_<(s_+) = {} & \frac{K}{2} C_+^{4-D}(\lambda^0_+) \\ & + 2\tilde{K} \int_{\lambda^0_+}^{s_+} ds'_+\, D^i_+(s'_+) \left[\dot{\vec{X}}^i_<(s'_+) + \frac{\tilde{K}}{2} \dot{\vec{U}}(s'_+)\, D^i_+(s'_+) \right], \end{aligned} \tag{3.6}$$

where

$$C^i_-(s_-) \equiv C^i\left(\frac{\vec{U}(s_-)}{\alpha' p^U}\right), \qquad C^i_+(s_+) \equiv C^i\left(\frac{\vec{U}(s_+)}{\alpha' p^U}\right),$$

$$C_\pm(s_\pm) \equiv \sqrt{\sum_{i=1}^{D-2} [C^i_\pm(s_\pm)]^2}, \qquad D^i_\pm(s_\pm) \equiv C^i_\pm(s_\pm) C_\pm^{2-D}(s_\pm).$$

In Eqs. (3.3)–(3.6), the parameters $\lambda^i_\pm$, $\lambda^0_\pm$ follow by reparametrization of σ^i and σ^0 through $\bar{U}/\alpha' p^U$ and $\vec{U}/\alpha' p^U$, respectively, i.e.,

$$\begin{aligned}\sigma^i &= \frac{1}{\alpha' p^U} \bar{U}(\lambda^i_+) = \frac{1}{\alpha' p^U} \vec{U}(\lambda^i_-),\\ \sigma^0 &= \frac{1}{\alpha' p^U} \bar{U}(\lambda^0_+) = \frac{1}{\alpha' p^U} \vec{U}(\lambda^0_-).\end{aligned} \tag{3.7}$$

Since the string coordinates $X^\mu_<(\sigma', \tau')$ and $X^\mu_>(\sigma', \tau')$ obey the d'Alembert equation for $U(\sigma', \tau') < 0$ and $U(\sigma', \tau') > 0$, respectively, we can Fourier expand them as

$$\begin{aligned}\vec{X}^\mu_{\gtrless}(s_-) &= \frac{1}{2} q^\mu_{\gtrless} - \alpha' p^\mu_{\gtrless} s_- + i\sqrt{\alpha'} \sum_{n \neq 0} \frac{1}{n} \tilde{\alpha}^\mu_{n\gtrless} e^{ins_-}\\ \bar{X}^\mu_{\gtrless}(s_+) &= \frac{1}{2} q^\mu_{\gtrless} + \alpha' p^\mu_{\gtrless} s_+ + i\sqrt{\alpha'} \sum_{n \neq 0} \frac{1}{n} \alpha^\mu_{n\gtrless} e^{-ins_+}.\end{aligned} \tag{3.8}$$

The operators q^μ, p^μ, α^μ_n, and $\tilde{\alpha}^\mu_n$, here defined, correspond to the covariant gauge; please notice that they are *different* from the operators with the same names used in Section II for the light-cone gauge. Inserting these Fourier expansions in the matching relations (3.3)–(3.6), we find

$$q^i_> - q^i_< = \frac{\tilde{K}}{2\pi} \int_0^{2\pi} ds \int_s^{\omega(s)} ds' \dot{\vec{U}}(s')\, D^i_-(s'), \tag{3.9}$$

$$q^U_> - q^U_< = 0, \tag{3.10}$$

$$\begin{aligned}q^V_> - q^V_< = &-\frac{K}{2\pi} \int_0^{2\pi} ds\, C^{4-D}(s)\\ &- \frac{K}{2\pi} \int_0^{2\pi} ds \int_s^{\omega(s)} ds'\, D^i_-(s') \left[\dot{\vec{X}}^i_<(s') - \frac{\tilde{K}}{2} \dot{\vec{U}}(s')\, D^i_-(s') \right].\end{aligned} \tag{3.11}$$

In the course of the derivation of these relations, the following identity was used:

$$\int_0^{2\pi} ds \left[\int_s^{\bar{U}(s)/\alpha' p^U} ds'\, D^i(s') + \int_s^{\vec{U}(s)/\alpha' p^U} ds'\, D^i(s') \right] = 0.$$

The function $\omega = \omega(s)$ is defined by the functional equation

$$\vec{U}(\omega) = \bar{U}(s).$$

Here, $s_- = \omega(s_+)$ describes the intersection curve of the string world-sheet with the shock wave in the covariant gauge (it is $\tau = 0$ in the light-cone gauge).

Similarly, for the components of the center of mass momentum we obtain

$$p^i_> - p^i_< = \frac{\tilde{K}}{2\pi\alpha'} \int_0^{2\pi} ds\, \dot{\vec{U}}(s)\, D^i_-(s) = \frac{\tilde{K}}{2\pi\alpha'} \int_0^{2\pi} ds\, \dot{\vec{U}}(s)\, D^i_+(s), \tag{3.12}$$

$$p^U_> - p^U_< = p^U, \tag{3.13}$$

$$\begin{aligned} p^V_> - p^V_< &= -\frac{\tilde{K}}{\pi\alpha'} \int_0^{2\pi} ds\, D^i_-(s) \left[\dot{X}^i_<(s) - \frac{\tilde{K}}{2} \dot{\vec{U}}(s)\, D^i_-(s) \right] \\ &= \frac{\tilde{K}}{\pi\alpha'} \int_0^{2\pi} ds\, D^i_+(s) \left[\dot{X}^i_<(s) + \frac{\tilde{K}}{2} \dot{\vec{U}}(s)\, D^i_+(s) \right]. \end{aligned} \tag{3.14}$$

The equality between the second members of Eq. (3.14), as well as that between the second members of Eq. (3.12), is a consequence of the periodicity condition for the closed string.

Analogously, for the oscillator modes we find

$$\alpha^i_{n>} - \alpha^i_{n<} = \frac{\tilde{K}}{2\pi\sqrt{\alpha'}} \int_0^{2\pi} ds\, \dot{\vec{U}}(s)\, D^i_+(s)\, e^{ins} \tag{3.15}$$

$$\tilde{\alpha}^i_{n>} - \tilde{\alpha}^i_{n<} = \frac{\tilde{K}}{2\pi\sqrt{\alpha'}} \int_0^{2\pi} ds\, \dot{\vec{U}}(s)\, D^i_-(s)\, e^{-ins} \tag{3.16}$$

$$\alpha^U_{n>} - \alpha^U_{n<} = 0, \qquad \tilde{\alpha}^U_{n>} - \tilde{\alpha}^U_{n<} = 0 \tag{3.17}$$

$$\alpha^V_{n>} - \alpha^V_{n<} = \frac{\tilde{K}}{\pi\sqrt{\alpha'}} \int_0^{2\pi} ds\, D^i_+(s) \left[\dot{X}^i_<(s) + \frac{\tilde{K}}{2} \dot{\vec{U}}(s)\, D^i_+(s) \right] e^{ins} \tag{3.18}$$

$$\tilde{\alpha}^V_{n>} - \tilde{\alpha}^V_{n<} = -\frac{\tilde{K}}{\pi\sqrt{\alpha'}} \int_0^{2\pi} ds\, D^i_-(s) \left[\dot{X}^i_<(s) - \frac{\tilde{K}}{2} \dot{\vec{U}}(s)\, D^i_-(s) \right] e^{-ins}. \tag{3.19}$$

Please, note from Eq. (3.9) that the center of mass transverse coordinates are continuous at the shock wave ($q^i_> = q^i_<$) if and only if $\vec{U}(s) = \tilde{U}(s)$ ($\omega(s) = s$). This happens, for instance, in the light-cone gauge (see Eq. (2.24a)). In general covariant gauges we see that the dynamical variables appear through the function $\omega(s)$ on the integration bounds of Eqs. (3.9) and (3.11). This complication is absent in the light-cone gauge.

Furthermore, the discontinuities $p^\mu_> - p^\mu_<$ happen to be gauge invariant, as we see by comparing Eqs. (3.12)–(3.14) with Eq. (2.24b) and (2.25b) and using Eq. (3.1). On the contrary, the oscillator modes' discontinuities do depend on the gauge.

IV. Generalized Gravitational Shock Waves

In the previous sections, we have considered the source of the gravitational shock-wave as point-like (see Eq. (2.4)). In this last section, we show how our results can be easily extended to arbitrary ultrarelativistic sources. In such a case, instead of Eq. (2.4) we have

$$\nabla_i^2 f_D(\rho) = 16\pi G\tilde{p}\mu(x^i), \tag{4.1}$$

where $\mu(x^i)$ stands for the mass density as a function of the transverse coordinates.

Equation (4.1) can be solved easily by Fourier transformation, yielding to

$$f_D(\vec{X}) = -16\pi G\tilde{p} \int \frac{d^{D-2}k}{\vec{k}^2}\, e^{i\vec{k}\cdot\vec{X}}\tilde{\mu}(\vec{k}), \tag{4.2}$$

where $\vec{X}$ stands now for $D-2$ transverse coordinates and

$$\tilde{\mu}(\vec{k}) = \int \frac{d^{D-2}X}{(2\pi)^{D-2}}\, e^{-i\vec{k}\cdot\vec{X}}\mu(\vec{X}). \tag{4.3}$$

Repeating the same steps from Eq. (2.9) to Eq. (2.28), we arrive at the same matching equations (2.25), but now with

$$D^i(\sigma) = -\frac{2i\pi^{D/2-1}}{\Gamma(D/2-1)} \int \frac{d^{D-2}k}{\vec{k}^2}\, k^i\, e^{i\vec{k}\cdot\vec{C}(\sigma)}\tilde{\mu}(\vec{k}) \tag{4.4}$$

and

$$\frac{4\pi^{D/2-1}}{\Gamma(D/2-1)} \int \frac{d^{D-2}k}{\vec{k}^2}\, e^{i\vec{k}\cdot\vec{C}(\sigma)}\tilde{\mu}(\vec{k}), \tag{4.5}$$

instead of $[C(\sigma)]^{4-D}$.

Analogous changes generalize the results of Section III.

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References

1. H. J. de Vega and N. Sánchez, *Phys. Lett. B* **197** (1987), 320.
2. N. Sánchez and G. Veneziano, *Nucl. Phys. B* **333** (1990), 253.
3. H. J. de Vega and N. Sánchez, *Nucl. Phys. B* **309** (1988), 552, 577.
4. H. J. de Vega, M. R. Medrano, and N. Sánchez, *Nucl. Phys. B* **351** (1991), 277.
5. D. Amati and C. Klimčik, *Phys. Lett. B* **210** (1988), 92.
6. H. J. de Vega and N. Sánchez, *Nucl. Phys. B* **317** (1989), 706.
7. H. J. de Vega and N. Sánchez, *Phys. Rev. D* **42** (1990), 3969.
8. M. E. V. Costa and H. J. de Vega, *Ann. Phys.* **211**, 235–248.
9. G. 't Hooft, *Phys. Lett. B* **198** (1987), 61.
10. H. J. de Vega and N. Sánchez, *Nucl. Phys. B* **317** (1989), 731.
11. P. C. Aichelburg and R. U. Sexl, *Gen. Rel. Grav.* **2** (1971), 303; T. Dray and G. 't Hooft, *Nucl. Phys. B* **253** (1985), 173.