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Pohyb klasické struny

Diplomová práce

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Abstrakt

V této diplomové práci se věnujeme pohybu klasické relativistické struny v zakřiveném časoprostoru. Nejprve předkládáme nástroje potřebné pro odvození a zjednodušení pohybových rovnic struny v zakřiveném časoprostoru. Dále tyto nástroje používáme pro hledání pohybu strun v plochém prostoru, rozpínajícím se vesmíru a také v časoprostoru s pozadím gravitačních vln. Stručně studujeme struny v plochém prostoru, což poté využíváme k vytvoření a ucelení představy o pohybu takovéto struny. V druhém případě ukazujeme, že chování velmi malé kruhové struny se v rozpínajícím se vesmíru příliš neliší od chování strun v plochém prostoru. Pro struny s dostatečně velkou klidovou energií se zde objevuje zajímavý jev. Ukazuje se, že tyto struny se vždy rozpínají přes všechny meze. Ve třetím případě, kde struna interaguje s gravitační vlnou, objevujeme resonanční frekvence, pro které struna získává velké množství energie z gravitační vlny. Všechna uvedená teoretická odvození jsou doprovázena názornými obrázky.

Abstract

In this thesis we study the motion of a classical relativistic string in curved spacetime. We develop tools for acquiring and simplifying the equations of motion of a string in a general curved spacetime. Furthermore, we apply these tools in finding the motion of strings in flat spacetime, expanding universe and gravitational wave background. We study the strings in flat spacetime only briefly, mostly to get the idea of how the string normally behaves. In the second case, we show that in the expanding universe small circular strings feel little change compared to the flat spacetime. Moreover, we find that a string, that has rest energy higher than some critical value, will always expand beyond all limits. In the third case, where the string interacts with the gravitational wave, we find that there exist resonant frequencies for which the string receives large amounts of energy from the gravitational wave. All these theoretical analyses are supported by illustrative visualizations.

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A string moving in spacetime traces out a generalization of a geodesic: a surface of extremal area. In this diploma work we will study the equations that describe the classical motion of strings and membranes and find explicit solutions of classical strings in various spacetimes.

Literatura:
ZWIEBACH, Barton. *A first course in string theory*. 2nd ed. Cambridge: Cambridge University Press, 2009. xxii, 673. ISBN 9780521880329.

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Poděkování

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Prohlášení

Prohlašuji, že jsem svoji diplomovou práci vypracoval samostatně s využitím informačních zdrojů, které jsou v práci citovány.

Brno 15. května 2019 ....

Vojtěch Liška

Contents

Introduction

In this thesis, we will study the motion of classical relativistic strings on various backgrounds. The superstring theory is an attempt to explain all of the particles and fundamental forces in nature in one theory. It is a very complicated theory that requires plenty of mathematical prowess. The superstring theory pursues the strings as quantized, which complicates the calculations. We chose to study classical strings which still provide a good insight and intuition into the matters of the string theory and many mathematically interesting topics.

In the first chapter, we will show the differences between the action principle for particles and for strings in the general relativity. Furthermore, we will find a convenient form of the equations of motion and the boundary conditions, that must be imposed on the string's endpoints.

In the second chapter, we will briefly look at the behavior of strings in flat spacetime. We will also show, how the choice of parameterization of such a string can help in solving its equations of motion.

In the third chapter, we will study a circular string in a constantly expanding universe. First, we will find a potential for static strings, which indicates that there are some critical points for which the string has a very interesting behavior. The equations of motion and the conservation of energy will help us find all trajectories in phase space and compare them to trajectories in flat spacetime.

The fourth chapter will be concerning strings on the plane gravitational wave background. Firstly, we will find a suitable parameterization in which we can solve the equations of motion. Then we will choose two types of gravitational waves, namely periodic gravitational waves and a burst of gravitational waves with Gaussian envelope. In the former case we cannot interpret the results as easily as in the later case, where we have flat space-time before and after the gravitational wave burst. In the former case, we will only look at the stability of solutions, which will hint towards some resonance of the string with the gravitational wave for specific frequencies. In the later case, we will compare the motion of the string before and after the gravitational wave burst.

Chapter 1

Classical string motion on a general background

In this chapter, we will develop some fundamental tools for acquiring the equations of motion of a classical relativistic string. Let us first summarize the motion of a single classical relativistic particle, which is described in detail in [1].

1.1 Motion of classical relativistic particle

In classical relativistic mechanics a particle moves along a curve in spacetime called a world-line. This curve can be parametrized in many ways, but the physics have to be invariant of the choice of parameterization. When we want to find this world-line, we usually use the action principle. The action of a world-line is proportional to its Lorentz invariant "proper length", which in turn is equal to the proper time associated with this world-line times the factor $c = 1$. The infinitesimal proper time $\mathrm{d}\lambda$ for a particle with mass takes the form

$$-\,\mathrm{d}\lambda^2 = \mathrm{d}s^2 = g_{MN}(X)\,\mathrm{d}X^M\,\mathrm{d}X^N \tag{1.1}$$

where g_{MN} is the metric of the spacetime.

Since the proper time has the units of time, we need an additional multiplicative factor to get the units of action, which is energy $\times$ time. The energy of a static particle would be $E = mc^2$, but we put $c = 1$, therefore the multiplicative factor will be m. Also, since the proper time is always positive, we will add a $-$ sign so that the extrema of the action is a minimum. This does not change anything from the mathematical point of view, but it is a convention in physics. The action of such a particle is then

$$S = -m\int \mathrm{d}\lambda = -\int \sqrt{-\,\mathrm{d}s^2} \tag{1.2}$$

If we choose a specific parameter τ

$$X^M = X^M(\tau),$$

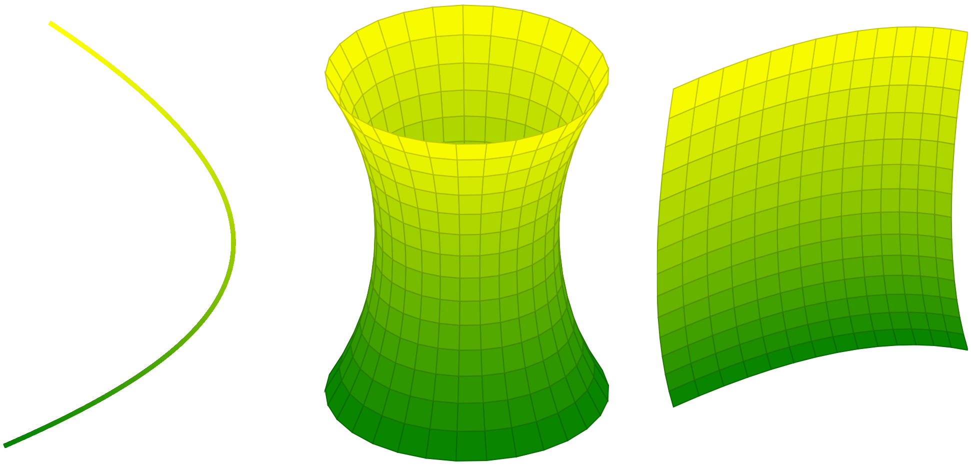

Figure 1.1: Trajectory of a particle (left) vs a closed string (middle) vs an open string (right).

we can express the action in the following form

$$S = -m \int_{\tau_i}^{\tau_f} \sqrt{-g_{MN}(X) \frac{\mathrm{d}X^M}{\mathrm{d}\tau} \frac{\mathrm{d}X^N}{\mathrm{d}\tau}}\, \mathrm{d}\tau \tag{1.3}$$

From the variation of this action, we acquire the geodesic equation (equations of motion) for a particle in a space with metric g_{MN}.

1.2 String action

Let us now turn towards the classical relativistic strings and find the action associated with them. Just as a particle traces out a curve in spacetime called a world-line, a string traces out a two dimensional surface called a world-sheet. There can be two types of strings. A closed string that traces out a tube and an open string that traces out a strip as seen in fig. 1.1.

In the previous section, we found that the action of a particle is proportional to its Lorentz invariant "proper length". In a similar way, we will define a Lorentz invariant "proper area" of a world-sheet. The action of a classical relativistic string, called the Nambu–Goto action, is proportional to this two dimensional proper area $\mathrm{d}A$.

$$S = -T_0 \int \mathrm{d}A \tag{1.4}$$

where T_0 is again a multiplicative factor to get the units of action and has the meaning of tension of the string and its units are energy / length. The infinitesimal area dA has units of length $\times$ time, so the units of the action are, again, energy $\times$ time.

In order to parameterize the world-sheet, we require two parameters ξ^1 and ξ^2. The surface is described by the collection of functions

$$X^M = X^M(\xi^1, \xi^2) \tag{1.5}$$

Also, we will call the *target space* the space where the strings propagate. We can define an induced metric on the world-sheet as

$$\gamma_{\alpha\beta} = g_{MN} \frac{\mathrm{d}X^M}{\mathrm{d}\xi^\alpha} \frac{\mathrm{d}X^N}{\mathrm{d}\xi^\beta} = g_{MN}\, \partial_\alpha X^M\, \partial_\beta X^N \tag{1.6}$$

The area element dA can be defined as a square root of the metric. Since we want this element to be expressed in terms of the tangent vectors to the world-sheet $\frac{\partial}{\partial \xi^1}$ and $\frac{\partial}{\partial \xi^2}$, we will use the induced metric.

$$\mathrm{d}A = \sqrt{-\det \gamma}\, \mathrm{d}\xi^1\, \mathrm{d}\xi^2 \tag{1.7}$$

where the $-$ sign corresponds to the fact, that the determinant of the induced metric is always negative. Furthermore, we will name the parameters $\xi^1 = \tau$ and $\xi^2 = \sigma$ and denote $\det \gamma$ as γ written without indices. The Nambu–Goto action of a classical string is then given by

$$\begin{aligned} S &= -T_0 \int_{\tau_i}^{\tau_f} \mathrm{d}\tau \int_0^{\sigma_1} \mathrm{d}\sigma\, \sqrt{-\gamma} = -T_0 \int_{\tau_i}^{\tau_f} \mathrm{d}\tau \int_0^{\sigma_1} \mathrm{d}\sigma\, \sqrt{\gamma_{\tau\sigma}\gamma_{\sigma\tau} - \gamma_{\tau\tau}\gamma_{\sigma\sigma}} \\ &= -T_0 \int_{\tau_i}^{\tau_f} \mathrm{d}\tau \int_0^{\sigma_1} \mathrm{d}\sigma\, \sqrt{\left(g_{MN}\partial_\tau X^M \partial_\sigma X^N\right)^2 - \left(g_{MN}\partial_\tau X^M \partial_\tau X^N\right)\left(g_{KL}\partial_\sigma X^K \partial_\sigma X^L\right)} \end{aligned} \tag{1.8}$$

1.3 Equations of motion

To get the equations of motion, we have to find the minima of the action. At the minima, the first variation of the action has to be zero. We could immediately insert a metric and choose a coordinate system in which we would get the equations of motion after the variation. We will not do that, instead, we will further develop the γ notation and obtain a more convenient general form of the equations of motion. The variation of this action takes the form

$$\delta S = -T_0 \int_{\tau_i}^{\tau_f} \mathrm{d}\tau \int_0^{\sigma_1} \mathrm{d}\sigma\, \delta\sqrt{-\gamma} = -T_0 \int_{\tau_i}^{\tau_f} \mathrm{d}\tau \int_0^{\sigma_1} \mathrm{d}\sigma\, \frac{-\delta\gamma}{2\sqrt{-\gamma}} \tag{1.9}$$

The variation of the determinant is given by

$$
\begin{aligned}
\delta(\det\gamma) &= \delta\left(\frac{1}{n!}\varepsilon^{\alpha_1\ldots\alpha_n}\varepsilon^{\beta_1\ldots\beta_n}\gamma_{\alpha_1\beta_1}\cdots\gamma_{\alpha_n\beta_n}\right)\\
&= \left(\frac{1}{(n-1)!}\varepsilon^{\alpha_1\ldots\alpha_n}\varepsilon^{\beta_1\ldots\beta_n}\gamma_{\alpha_1\beta_1}\cdots\gamma_{\alpha_{i-1}\beta_{i-1}}\gamma_{\alpha_{i+1}b_{i+1}}\cdots\gamma_{\alpha_n\beta_n}\right)\delta\gamma_{\alpha_i\beta_i}\\
&= \det(\gamma)\gamma^{\alpha\beta}\delta\gamma_{\alpha\beta}
\end{aligned}
\tag{1.10}
$$

We can now rewrite the variation of the action as

$$
\delta S = T_0\int_{\tau_i}^{\tau_f}\mathrm{d}\tau\int_0^{\sigma_1}\mathrm{d}\sigma\,\frac{1}{2}\sqrt{-\gamma}\,\gamma^{\alpha\beta}\,\delta\gamma_{\alpha\beta}
\tag{1.11}
$$

Using eq. (1.6) and the fact that the metric components g_{MN} depend only on X^K, the variation of the induced metric $\gamma_{\alpha\beta}$ takes the form

$$
\begin{aligned}
\delta\gamma_{\alpha\beta} = \delta\left(g_{MN}\,\partial_\alpha X^M\,\partial_\beta X^N\right) &= \frac{\partial g_{MN}}{\partial X^K}\delta X^K\,\partial_\alpha X^M\partial_\beta X^N\\
+g_{MN}\,\delta\left(\partial_\alpha X^M\right)\partial_\beta X^N &+ g_{MN}\,\partial_\alpha X^M\,\delta\left(\partial_\beta X^N\right)
\end{aligned}
\tag{1.12}
$$

Because of the symmetry of both metrics $g_{MN} = g_{NM}$, $\gamma_{\alpha\beta} = \gamma_{\beta\alpha}$ we can show that the second and third term in eq. (1.12) is the same.

$$
\begin{aligned}
\gamma^{\alpha\beta}g_{MN}\,\delta\left(\partial_\alpha X^M\right)\partial_\beta X^N = \left|\begin{matrix}\alpha\leftrightarrow\beta\\ M\leftrightarrow N\end{matrix}\right| &= \gamma^{\beta\alpha}g_{NM}\,\delta\left(\partial_\beta X^N\right)\partial_\alpha X^M\\
&= \gamma^{\alpha\beta}g_{MN}\,\partial_\alpha X^M\,\delta\left(\partial_\beta X^N\right)
\end{aligned}
\tag{1.13}
$$

Substituting eq. (1.12) into eq. (1.11) yields:

$$
\delta S = -\frac{T_0}{2}\int_{\tau_i}^{\tau_f}\mathrm{d}\tau\int_0^{\sigma_1}\mathrm{d}\sigma\,\sqrt{-\gamma}\,\gamma^{\alpha\beta}\left(\frac{\partial g_{MN}}{\partial X^K}\,\partial_\alpha X^M\partial_\beta X^N\,\delta X^K + 2\,g_{MN}\,\partial_\alpha X^M\partial_\beta\delta X^N\right)
\tag{1.14}
$$

If we want to factor out δX^K, we need to integrate by parts the second term in eq. (1.14)

$$
\begin{aligned}
&-T_0\int_{\tau_i}^{\tau_f}\mathrm{d}\tau\int_0^{\sigma_1}\mathrm{d}\sigma\,\sqrt{-\gamma}\,\gamma^{\alpha\beta}g_{MN}\,\partial_\alpha X^M\partial_\beta\left(\delta X^N\right)\\
&= -T_0\int_{\tau_i}^{\tau_f}\mathrm{d}\tau\int_0^{\sigma_1}\mathrm{d}\sigma\,\Big[\partial_\beta\left(\sqrt{-\gamma}\,\gamma^{\alpha\beta}g_{MN}\,\partial_\alpha X^M\,\delta X^N\right)\\
&\qquad -\partial_\beta\left(\sqrt{-\gamma}\,\gamma^{\alpha\alpha}g_{MN}\,\partial_\beta X^M\right)\delta X^N\Big]
\end{aligned}
\tag{1.15}
$$

The first term is a total derivative. This implies, that it needs to vanish at the boundary. For now, we will assume, that they do, but we will touch on this topic in more detail in section 1.4.

We will now focus our attention on the remaining terms and we arrive at a convenient form of the variation of action:

$$\begin{aligned} \delta S = -\frac{T_0}{2} \int_{\tau_i}^{\tau_f} \mathrm{d}\tau \int_0^{\sigma_1} \mathrm{d}\sigma \Big[& \sqrt{-\gamma}\, \gamma^{\alpha\beta}\, \partial_K g_{MN}\, \partial_\alpha X^M\, \partial_\beta X^N \\ & -2\partial_\alpha \left(\sqrt{-\gamma}\, \gamma^{\alpha\beta} g_{KN}\, \partial_\beta X^N \right) \Big]\, \delta X^K \end{aligned} \tag{1.16}$$

This first variation of the action must be zero for any δX^K for a minimal trajectory. This implies, that the term on the inside of the square brackets in eq. (1.16) must be zero:

$$2\, \partial_\alpha \left(\sqrt{-\gamma}\, \gamma^{\alpha\beta} g_{KN}\, \partial_\beta X^N \right) - \sqrt{-\gamma}\, \gamma^{\alpha\beta}\, \frac{\partial g_{MN}}{\partial X^K}\, \partial_\alpha X^M\, \partial_\beta X^N = 0 \tag{1.17}$$

Together with the boundary conditions, these equations, that are the equivalent of the Euler–Lagrange equations from classical mechanics, fully specify the motion of a classical relativistic string.

1.4 Boundary conditions

In this section, we will be interested in terms, that must vanish at the boundary. Specifically, it is the first term in eq. (1.15) that has to vanish

$$-T_0 \int_{\tau_i}^{\tau_f} \mathrm{d}\tau \int_0^{\sigma_1} \mathrm{d}\sigma\, \partial_\beta \left(\sqrt{-\gamma}\, \gamma^{\alpha\beta} g_{MN}\, \partial_\alpha X^M\, \delta X^N \right) = 0 \tag{1.18}$$

To better understand what this equation means, we will use Stokes' theorem. If we denote

$$\omega = \sqrt{-\gamma}\, \gamma^{\alpha\beta} g_{MN}\, \partial_\alpha X^M\, \delta X^N \tag{1.19}$$

We can take an outer derivative

$$\mathrm{d}\omega = \partial_\beta \left(\sqrt{-\gamma}\, \gamma^{\alpha\beta} g_{MN}\, \partial_\alpha X^M\, \delta X^N \right) \mathrm{d}y^\beta \tag{1.20}$$

where $\mathrm{d}y^\beta$ is a coordinate basis vector in the tangent space of the manifold over which we integrate. This manifold corresponds to our world-sheet. Using the Stokes' theorem

$$\int_V \mathrm{d}\omega = \int_{\partial V} \omega \tag{1.21}$$

we conclude, that if we have an integral over some part of a manifold, like the one in eq. (1.15), we can transform it to an integral over the boundary of this manifold. We can always choose τ and σ to be perpendicular. When this is true, we can split the integral over the boundary into four integrals.

$$\begin{aligned}\int_{\partial V} \omega = &\int_{\tau_i}^{\tau_f} \mathrm{d}\tau \left[\sqrt{-\gamma}\,\gamma^{\alpha\sigma} g_{MN}\,\partial_\alpha X^M\,\delta X^N\right]_{\sigma=0}\\ &+\int_{0}^{\sigma_1} \mathrm{d}\sigma \left[\sqrt{-\gamma}\,\gamma^{\alpha\tau} g_{MN}\,\partial_\alpha X^M\,\delta X^N\right]_{\tau=\tau_f}\\ &+\int_{\tau_f}^{\tau_i} \mathrm{d}\tau \left[\sqrt{-\gamma}\,\gamma^{\alpha\sigma} g_{MN}\,\partial_\alpha X^M\,\delta X^N\right]_{\sigma=\sigma_1}\\ &+\int_{\sigma_1}^{0} \mathrm{d}\sigma \left[\sqrt{-\gamma}\,\gamma^{\alpha\tau} g_{MN}\,\partial_\alpha X^M\,\delta X^N\right]_{\tau=\tau_i}\end{aligned} \tag{1.22}$$

For easier manipulation, we will denote part of the inside of these integrals as

$$\mathcal{P}^\tau_N = \frac{\partial\mathcal{L}}{\partial(\partial_\tau X^N)} = T_0\sqrt{-\gamma}\,\gamma^{\alpha\tau} g_{MN}\,\partial_\alpha X^M \tag{1.23}$$

$$\mathcal{P}^\sigma_N = \frac{\partial\mathcal{L}}{\partial(\partial_\sigma X^N)} = T_0\sqrt{-\gamma}\,\gamma^{\alpha\sigma} g_{MN}\,\partial_\alpha X^M \tag{1.24}$$

Also, we will restrict ourselves to variations with fixed endpoints in proper time, such that $\delta X^M(\tau_i,\sigma) = \delta X^M(\tau_f,\sigma) = 0$. In that case, the second and fourth term do not contribute. Therefore, the boundary conditions impose

$$T_0\int_{\partial V} \omega = -\int_{\tau_i}^{\tau_f} \mathrm{d}\tau \left[\mathcal{P}^\tau_N\,\delta X^N\right]_0^{\sigma_1} = 0 \tag{1.25}$$

These terms must vanish at every boundary $\sigma_b \in \{0,\sigma_1\}$. This can be ensured when either $\mathcal{P}^\tau_N$ or δX^N are zero at the boundary. The former case is called the free endpoint boundary condition, because it does not impose any restriction to $\delta X^N(\tau,\sigma_b)$. The later case is called the Dirichlet boundary condition. This means, that the string endpoint is fixed throughout the motion. This can also be expressed as $\partial_\tau X^N(\tau,\sigma_b)$ being equal to zero.

$$\text{Dirichlet boundary condition:} \qquad \partial_\tau X^N(\tau,\sigma_b) = 0 \tag{1.26}$$

$$\text{free endpoint boundary condition:} \qquad \mathcal{P}^\tau_N(\tau,\sigma_b) = 0 \tag{1.27}$$

Open strings have to satisfy at least one condition for every component N. We must be careful with the choice of boundary conditions. For example, we cannot choose a Dirichlet boundary condition for the time-like coordinate component, because the time-like coordinate has to change with proper time.

For closed strings, the terms in eq. (1.25) always cancel out, because $X^N(\tau, 0) = X^N(\tau, \sigma_1)$.

Chapter 2

Flat spacetime

In this chapter, we will study the motion of strings in a four dimensional flat spacetime. This will help both in showing how the equations of motion (1.17) are used and finding a reference solution, which can be used to compare results on different backgrounds. This chapter follows the approach in [1].

2.1 Equations of motion and parameterization

First, we will choose such a coordinate system, that $X^0 = t$, $X^1 = x$, $X^2 = y$ and $X^3 = z$. In flat spacetime, the metric is that of Minkowski

$$\mathrm{d}s^2 = -\,\mathrm{d}t^2 + \mathrm{d}x^2 + \mathrm{d}y^2 + \mathrm{d}x^2 \tag{2.1}$$

We will choose a static gauge, which means that $t = \tau$. This greatly simplifies the derivatives

$$\partial_\tau X^M = \begin{pmatrix} 1 & \dot{x} & \dot{y} & \dot{z} \end{pmatrix}^M \qquad \partial_\sigma X^M = \begin{pmatrix} 0 & x' & y' & z' \end{pmatrix}^M \tag{2.2}$$

where we denoted $\partial_\tau x = \dot{x}$ and $\partial_\sigma x = x'$. Also, since we have the freedom of choice of the σ parameterization, we want the lines of constant σ be perpendicular to lines of constant τ

$$g_{MN}\, \partial_\tau X^M\, \partial_\sigma X^N = 0 \tag{2.3}$$

To be able to use the equations of motion (1.17), we must first calculate the $\gamma_{\alpha\beta}$ from eq. (1.6).

$$\gamma_{\tau\tau} = g_{MN}\, \partial_\tau X^M\, \partial_\tau X^N = -1 + \dot{x}^2 + \dot{y}^2 + \dot{z}^2 \tag{2.4}$$

$$\gamma_{\tau\sigma} = \gamma_{\sigma\tau} = g_{MN}\, \partial_\tau X^M\, \partial_\sigma X^N = 0 \tag{2.5}$$

$$\gamma_{\sigma\sigma} = g_{MN}\, \partial_\sigma X^M\, \partial_\sigma X^N = x'^2 + y'^2 + z'^2 \tag{2.6}$$

Using this, we can calculate the determinant γ and the inverse $\gamma^{\alpha\beta}$

$$\gamma = \det \gamma = \gamma_{\tau\tau}\gamma_{\sigma\sigma} - \gamma_{\sigma\tau}\gamma_{\tau\sigma} = \left(-1 + \dot{x}^2 + \dot{y}^2 + \dot{z}^2\right)\left(x'^2 + y'^2 + z'^2\right) \tag{2.7}$$

$$\gamma^{\tau\tau} = (\gamma_{\tau\tau})^{-1} = \frac{1}{-1 + \dot{x}^2 + \dot{y}^2 + \dot{z}^2} \tag{2.8}$$

$$\gamma^{\sigma\sigma} = (\gamma_{\sigma\sigma})^{-1} = \frac{1}{x'^2 + y'^2 + z'^2} \tag{2.9}$$

Now, we are ready to insert these calculations into the equations of motion (1.17). But first, lets simplify them a little bit.

$$\begin{gathered} 2\,\partial_\tau\left(\sqrt{-\gamma}\,\gamma^{\tau\tau} g_{KN}\,\partial_\tau X^N\right) + 2\,\partial_\sigma\left(\sqrt{-\gamma}\,\gamma^{\sigma\sigma} g_{KN}\,\partial_\sigma X^N\right) - \\ - \sqrt{-\gamma}\,\gamma^{\tau\tau}\frac{\partial g_{MN}}{\partial X^K}\,\partial_\tau X^M\,\partial_\tau X^N - \sqrt{-\gamma}\,\gamma^{\sigma\sigma}\frac{\partial g_{MN}}{\partial X^K}\,\partial_\sigma X^M\,\partial_\sigma X^N = 0 \end{gathered} \tag{2.10}$$

First, lets take a look at the equation of motion for $K = 0$

$$2\,\partial_\tau\left(\sqrt{\frac{x'^2 + y'^2 + z'^2}{1 - \dot{x}^2 - \dot{y}^2 - \dot{z}^2}}\right) = 0 \tag{2.11}$$

Therefore, the term in parentheses must be constant in τ.

$$C(\sigma) = \sqrt{\frac{x'^2 + y'^2 + z'^2}{1 - \dot{x}^2 - \dot{y}^2 - \dot{z}^2}} \tag{2.12}$$

This is convenient, because this constant only corresponds to a choice of σ parameterization. Writing the rest of the equations of motion for $K = i \in \{1, 2, 3\}$ gives us

$$\begin{aligned} -2\,\partial_\tau\left(\sqrt{\frac{x'^2 + y'^2 + z'^2}{1 - \dot{x}^2 - \dot{y}^2 - \dot{z}^2}}\,\partial_\tau X^i\right) + 2\,\partial_\sigma\left(\sqrt{\frac{1 - \dot{x}^2 - \dot{y}^2 - \dot{z}^2}{x'^2 + y'^2 + z'^2}}\,\partial_\sigma X^i\right) &= \\ -2\,\partial_\tau\left(C(\sigma)\,\partial_\tau X^i\right) + 2\,\partial_\sigma\left(\frac{1}{C(\sigma)}\,\partial_\sigma X^i\right) &= 0. \end{aligned} \tag{2.13}$$

We can choose σ in such a way, that $C(\sigma) = 1$. This results in the equations of motion taking the form of the well known and expected wave equation

$$\partial_\sigma^2 X^i - \partial_\tau^2 X^i = 0 \tag{2.14}$$

We can also rewrite the σ parameterization condition into a more convenient form

$$\begin{gathered} \left[x'^2 + y'^2 + z'^2\right] = -\left[-1 + \dot{x}^2 + \dot{y}^2 + \dot{z}^2\right] \\ \Downarrow \\ \gamma_{\sigma\sigma} = -\gamma_{\tau\tau} \end{gathered} \tag{2.15}$$

The fact, that with this choice of parameterization, the equations of motion are in its simplest form makes sense, since we just found out, that the γ matrix is in a diagonal form.

$$\gamma_{\alpha\beta} = \begin{pmatrix} -\gamma_{\sigma\sigma} & 0 \\ 0 & \gamma_{\sigma\sigma} \end{pmatrix}_{\alpha\beta} \tag{2.16}$$

Choosing this parameterization is a very good method, not just in flat spacetime, to make the equations of motion much easier to solve.

To find the motion of a relativistic string in flat spacetime, we need to solve four equations, which are collected here:

$$\text{equation of motion:} \quad \partial_\sigma^2 X^i - \partial_\tau^2 X^i = 0 \tag{2.17}$$

$$\text{parametrization condition:} \quad \gamma_{\sigma\sigma} = -\gamma_{\tau\tau} \tag{2.18}$$

$$\text{parametrization condition:} \quad \gamma_{\sigma\tau} = \gamma_{\tau\sigma} = 0 \tag{2.19}$$

$$\text{boundary condition:} \quad \partial_\sigma X^i \Big|_{\sigma=0} = \partial_\sigma X^i \Big|_{\sigma=\sigma_1} = 0 \tag{2.20}$$

2.2 Rest energy

We would like to find the rest energy m of the string. This is given by

$$m^2 = -p_M p^M, \tag{2.21}$$

where p_M is given by [1]

$$p_M = \int_0^{\sigma_1} \mathcal{P}_M^\tau \, \mathrm{d}\sigma = \int_0^{\sigma_1} T_0 \sqrt{-\gamma} \gamma^{\tau\alpha} g_{MN} \partial_\alpha X^N \, \mathrm{d}\sigma \tag{2.22}$$

We can notice, that the inside of the integral is constant because of eq. (2.15). It then reduces to

$$p_M = -\int_0^{\sigma_1} T_0 g_{MN} \partial_\tau X^N \, \mathrm{d}\sigma \tag{2.23}$$

From this, we can calculate the rest energy of the string at any time, even in curved space. Also, the total energy of the string is given by

$$E = p^t = -p_t = T_0 \int_0^{\sigma_1} \mathrm{d}\sigma = T_0 \sigma_1 \tag{2.24}$$

This is a consequence of eq. (2.15) and it means, that σ_1 gets fixed by this condition and is always $\sigma_1 = E/T_0$.

Chapter 3

Strings in expanding universe

In this chapter, we will discuss the motion of a classical string in expanding universe [1]. First, we assume that the metric has the form of the Friedmann-Lemaître-Robertson-Walker (FLWR) metric

$$\mathrm{d}s^2 = -\,\mathrm{d}t^2 + a(t)^2 \left[\frac{\mathrm{d}r^2}{1-kr^2} + r^2\left(\mathrm{d}\theta^2 + \sin^2\theta\,\mathrm{d}\phi^2\right)\right] \tag{3.1}$$

This metric describes a homogeneous, isotropic and expanding universe with the "scaling" of spatial coordinates $a(t)$. The constant $k \in \{-1, 0, +1\}$ represents the curvature of the space, but we will consider $k = 0$. When we solve the Einstein field equations, we get

$$\left(\frac{\dot{a}}{a}\right)^2 + \frac{kc^2}{a^2} = \frac{8\pi G}{3}\rho \tag{3.2}$$

$$2\,\frac{\ddot{a}}{a} + \left(\frac{\dot{a}}{a}\right)^2 + \frac{kc^2}{a^2} = -\frac{8\pi G}{c^2}P \tag{3.3}$$

where ρ is the total energy density, P is pressure and G is Newton's gravitational constant. Also, we will introduce the Hubble parameter of expansion of the universe as $H = \dot{a}/a$. The standard course of action would now be to consider contributions to the energy density ρ and pressure P from different sources, for example dust, radiation, cosmological constant. A given proportion of these types of energy in the total energy density and pressure would give us the evolution of of this Hubble parameter H.

However, in our case we will consider the Hubble parameter H to be constant. Currently, the latest measurement of the Hubble constant by the Fermi-LAT is $H = 68.0 \pm 4.2\ \mathrm{kms^{-1}Mpc^{-1}}$ [2]. If the Hubble parameter is constant, then the metric in eq. (3.1) becomes

$$\mathrm{d}s^2 = -\,\mathrm{d}t^2 + \mathrm{e}^{2Ht}\left[\mathrm{d}r^2 + r^2\left(\mathrm{d}\theta^2 + \sin^2\theta\,\mathrm{d}\phi^2\right)\right] \tag{3.4}$$

For the motion of strings in this space, we will restrict ourselves to closed strings, that are always circular. The metric in cylindrical coordinates then takes the form

$$\mathrm{d}s^2 = -\,\mathrm{d}t^2 + \mathrm{e}^{2Ht}\left(\mathrm{d}r^2 + r^2\,\mathrm{d}\theta^2 + \mathrm{d}z^2\right) \tag{3.5}$$

A natural parameterization for a circular string is $\theta = \sigma$, while r does not depend on σ, so $r = r\,(\tau)$. Also, because of the circularity of the string, we can rotate and translate the coordinate system such that $z = 0$. For the τ parameterization, we will choose the static gauge $t = \tau$. The vector of the string coordinates in the target space then takes the form

$$X^M = \left(t = \tau,\quad r(\tau),\quad \theta = \sigma,\quad 0\right). \tag{3.6}$$

The derivatives with respect to the world-sheet parameters are given by

$$\partial_\tau X^M = \left(1,\quad \dot{r},\quad 0,\quad 0\right) \tag{3.7}$$

$$\partial_\sigma X^M = \left(0,\quad 0,\quad 1,\quad 0\right) \tag{3.8}$$

Since we have already chosen our parameterization, we will first calculate the action and use the Euler-Lagrange equations. This might look like too much simplification compared to the derivation in section 1.3, but we will show that we are allowed to use this approach in section 3.4.

3.1 Lagrange approach

In this section, we will look at the Lagrangian and try to find information about the system. Then, we will calculate the equations of motion and find the solutions.

3.1.1 Lagrangian and potential

We start with inserting eqs. (3.6) to (3.8) into eq. (1.8)

$$\begin{aligned} S &= -T_0 \int_{\tau_i}^{\tau_f} \mathrm{d}\tau \int_0^{\sigma_1} \mathrm{d}\sigma \sqrt{-\left(-1 + \dot{r}^2\,\mathrm{e}^{2Ht}\right)\left(r^2\,\mathrm{e}^{2Ht}\right)} \\ &= -T_0 \int_{t_i}^{t_f} \mathrm{d}\tau \int_0^{\sigma_1} \mathrm{d}\sigma\, \left(\left|r\,\mathrm{e}^{Ht}\right| \sqrt{1 - \dot{r}^2\,\mathrm{e}^{2Ht}}\right) \end{aligned} \tag{3.9}$$

Since the Lagrangian density, the term within the integral, does not depend on σ, we can integrate over it and receive only a factor of σ_1. Because θ is 2π periodic, the choice of the parameterization $\theta = \sigma$ infers that $\sigma_1 = 2\pi$. Moreover, we will perform a change of coordinates $R = r\,\mathrm{e}^{Ht}$, which will lead to great simplification in calculation of the equations of motion. The action then becomes

$$S = -2\pi T_0 \int_{t_i}^{t_f} \mathrm{d}\tau \, |R| \, \sqrt{1 - \left(\dot{R} - HR\right)^2} \tag{3.10}$$

Just from this Lagrangian we can extract a lot of information about the solution. We can, for example, look at strings with constant R and find the corresponding potential $V(R)$ from the Lagrangian [1]

$$L(t, R, \dot{R}) = -2\pi T_0 \, |R| \, \sqrt{1 - (\dot{R} - HR)^2} \tag{3.11}$$

$$\Downarrow \quad V(t, R) = -L(t, R, 0)$$

$$V(R) = 2\pi T_0 \, |R| \, \sqrt{1 - (HR)^2} \tag{3.12}$$

This potential for static strings is plotted in fig. 3.1.

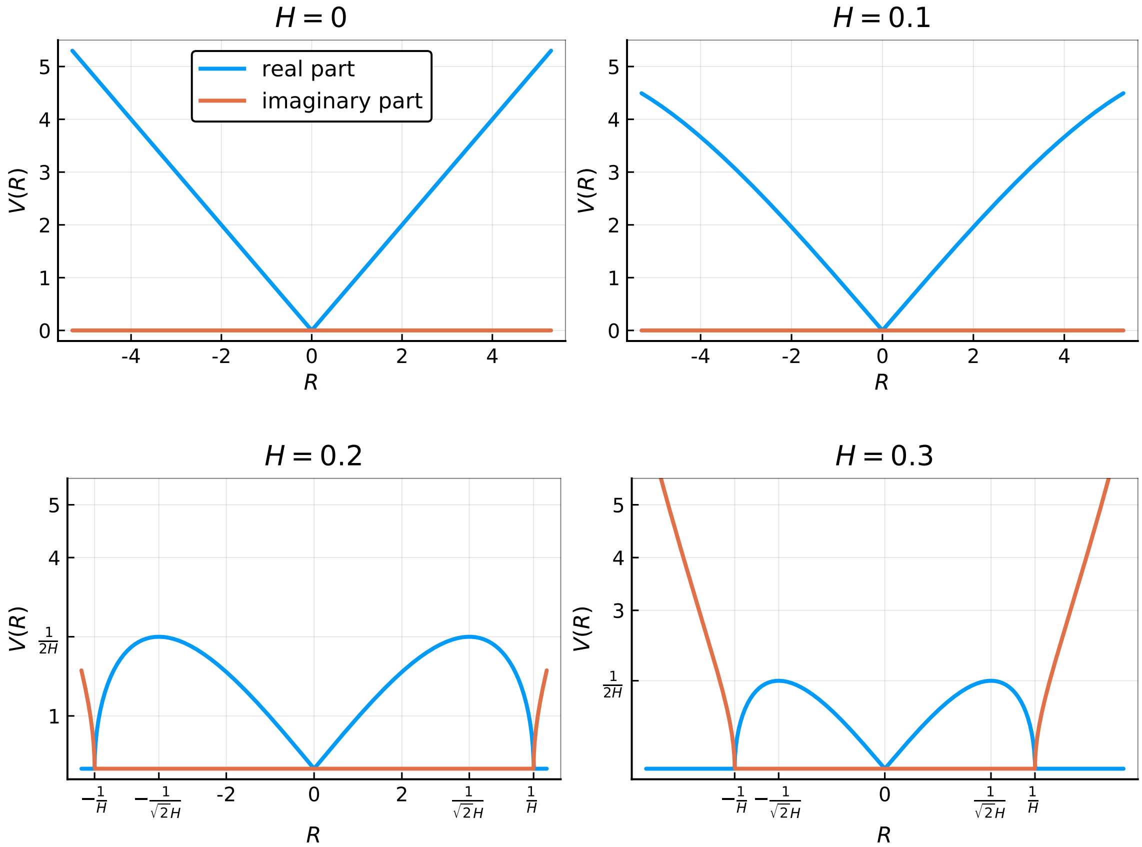

Figure 3.1: Potential for statics strings.

We can immediately see, that the potential is well defined only for $R \leq 1/H$ and that there is a potential well with minimum at $R = 0$ and maximum at

$R = \pm 1/\sqrt{2}H$. This hints towards the fact, that every string that goes through the point $\dot{R} = 0$, $|R| < 1/\sqrt{2}H$ will have a closed trajectory in phase space. On the other hand, strings that pass through the point ($\dot{R} = 0$, $1/\sqrt{2}H < |R| < 1/H$) expand infinitely or until they break.

3.1.2 Equations of motion and critical points

A more complex analysis requires us to solve the equations of motion. The variation of action (3.10) gives the same result as in classical mechanics, so we can use the Euler-Lagrange equations

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{R}}\right) - \frac{\partial L}{\partial R} = 0 \tag{3.13}$$

After some calculations, we will get the equations of motion in this form:

$$\frac{R\ddot{R} - \dot{R}^2 + 1 + 3HR\left(\dot{R} - HR\right) - 2HR\left(\dot{R} - HR\right)^3}{\left[1 - \left(\dot{R} - HR\right)^2\right]^{3/2}} = 0 \tag{3.14}$$

We will split this differential equation of second order into two differential equations of first order by denoting $S = \dot{R}$. This leads to a system of differential equations in the form:

$$\begin{aligned} \dot{S} &= \frac{S^2 - 1 - 3HR\left(S - HR\right) + 2HR\left(S - HR\right)^3}{R} \\ \dot{R} &= S \end{aligned} \tag{3.15}$$

We already mentioned, that there are four interesting points in the potential, namely $S = 0$ with $R = \pm 1/\sqrt{2}H$ and $R = \pm 1/H$. The former correspond to a maximum of the potential, so these should be unstable nodes. But we cannot say much about the latter just from potential. However if we look at the equations of motion, these points can be studied in more detail.

Let us first take a look at the first one, $R = \pm 1/\sqrt{2}H$. Since the equations are symmetric around the origin, we can drop the $\pm$ sign and the results apply to both points. When we insert the values of S and R, we get

$$\begin{aligned} \dot{S} &= \left[-1 - \frac{3}{\sqrt{2}}\left(-\frac{1}{\sqrt{2}}\right) + \frac{2}{\sqrt{2}}\left(\frac{1}{\sqrt{2}}\right)^3\right]\sqrt{2}H = \left[-1 + \frac{3}{2} - \frac{1}{2}\right]\sqrt{2}H = 0 \\ \dot{R} &= 0 \end{aligned} \tag{3.16}$$

This is a critical point in phase space, because the velocity vector field is zero at this point. For further study, we will linearize eq. (3.15) in the neighborhood of this point with δS and δR being the distance from this point. We arrive at a linear system of differential equations:

$$\begin{pmatrix} \delta\dot{S} \\ \delta\dot{R} \end{pmatrix} = \mathbf{A} \cdot \begin{pmatrix} \delta S \\ \delta R \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} \delta S \\ \delta R \end{pmatrix}, \tag{3.17}$$

where A is a matrix.

If we calculate the eigenvalues and eigenvectors of $\mathbf{A}$, we can classify the critical points. There are three types of critical points depending on the eigenvalues:

- Both eigenvalues are positive $\implies$ unstable (repulsive) node – all velocity vectors point away from the critical point
- Both eigenvalues are negative $\implies$ stable (attractive) node – all velocity vectors point towards the critical point
- Eigenvalues are of opposite sign $\implies$ saddle – eigenvectors divide regions of similar flow of velocity vector field

In the case of the critical point at $R = 1/\sqrt{2}H$, we will first expand eq. (3.15) in the neighbourhood such that $R = 1/\sqrt{2}H + \delta R$, $S = 0 + \delta S$ and then consider only the terms of the first order in both δR and δS to be of significance. After some calculations we get:

$$\begin{aligned}
\delta\dot{S} &= \frac{\delta S^2 - 1 - 3\left(\frac{1}{\sqrt{2}} + H\delta R\right)\left(\delta S - \frac{1}{\sqrt{2}} - H\delta R\right) + 2\left(\frac{1}{\sqrt{2}} + H\delta R\right)\left(\delta S - \frac{1}{\sqrt{2}} - H\delta R\right)^3}{\frac{1}{\sqrt{2}H} + \delta R} \\
&= 2H^2\delta R + \mathcal{O}(\delta S^2, \delta R^2, \delta S\delta R) \\
\delta\dot{R} &= \delta S
\end{aligned} \tag{3.18}$$

Rewritten into matrix equation

$$\begin{pmatrix} \delta\dot{S} \\ \delta\dot{R} \end{pmatrix} = \begin{pmatrix} 0 & 2H^2 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} \delta S \\ \delta R \end{pmatrix}. \tag{3.19}$$

From the characteristic equation, we arrive at the eigenvalues λ:

$$\det \begin{pmatrix} -\lambda & 2H^2 \\ 1 & -\lambda \end{pmatrix} = \lambda^2 - 2H^2 = 0, \qquad \lambda_{1,2} = \pm\sqrt{2}H \tag{3.20}$$

The eigenvalues are of opposite sign, which means, that we are looking at a saddle point. Finding the eigenvectors will help us understand the flow of the vector field around the saddle point. Eigenvectors correspond to the direction in which the velocity has the same direction as the distance vector from the saddle point. So strings that are displaced from the critical point in the direction of the eigenvector will move steadily in that direction either inwards or outwards, depending on the associated eigenvalue.

$$\begin{pmatrix} \sqrt{2}H & 2H^2 \\ 1 & \sqrt{2}H \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \implies v_1 = \begin{pmatrix} -\sqrt{2}H \\ 1 \end{pmatrix} \tag{3.21}$$

$$\begin{pmatrix} -\sqrt{2}H & 2H^2 \\ 1 & -\sqrt{2}H \end{pmatrix} \begin{pmatrix} c \\ d \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \implies v_2 = \begin{pmatrix} \sqrt{2}H \\ 1 \end{pmatrix} \tag{3.22}$$

The eigenvectors and the flow of the velocity vector field are depicted in fig. 3.2.

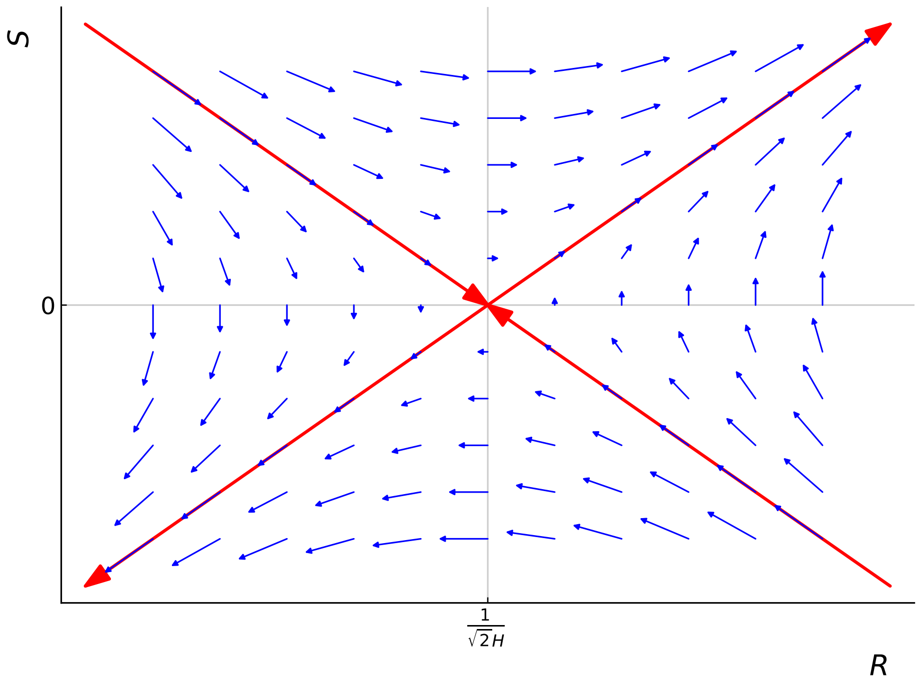

Figure 3.2: Eigenvectors (red) and flow of the velocity vector field (blue) around the saddle point $(R, S) = (1/\sqrt{2}H, 0)$.

We will now focus our attention on the second critical point at $S = 0$, $R = \pm 1/H$. When we again look at the velocity vector field directly at this point, we get

$$\dot{S} = \frac{-1 - 3(-1) + 2(-1)^3}{\frac{1}{H}} = (3 - 1 - 2)H = 0 \tag{3.23}$$

We can see, that this is another critical point and we will proceed with the same method as above, to determine the type of this critical point. Substituting into similar coordinates in the neighborhood of the point $S = \delta S$, $R = 1/H + \delta R$ and taking only the terms up to first order in δS and δR, we get:

$$\begin{aligned}\delta\dot{S} &= \frac{\delta S^2 - 1 - 3(1+H\delta R)(\delta S - 1 - H\delta R) + 2(1+H\delta R)(\delta S - 1 - H\delta R)^3}{\frac{1}{H} + \delta R} \\ &= 3H\delta S - 2H^2\delta R + \mathcal{O}(\delta S^2, \delta R^2, \delta S\delta R) \\ \delta\dot{R} &= \delta S\end{aligned} \tag{3.24}$$

In matrix notation, we arrive at:

$$\begin{pmatrix}\delta\dot{S} \\ \delta\dot{R}\end{pmatrix} = \begin{pmatrix}3H & -2H^2 \\ 1 & 0\end{pmatrix}\begin{pmatrix}\delta S \\ \delta R\end{pmatrix} \tag{3.25}$$

Solving the characteristic equation for eigenvalues λ then gives us:

$$\det\begin{pmatrix}3H-\lambda & -2H^2 \\ 1 & -\lambda\end{pmatrix} = \lambda^2 - 3H\lambda + 2H^2, \qquad \lambda_{1,2} = \frac{3H \pm H}{2} \tag{3.26}$$

In conclusion, we have found a repulsive point, because both eigenvalues are positive. All velocity vectors in this vector field aim out of this critical point as it is depicted in fig. 3.3. Eigenvectors can, again, tell us more about the flow of the velocity vector field

$$\begin{pmatrix}2H & -2H^2 \\ 1 & -H\end{pmatrix}\begin{pmatrix}a \\ b\end{pmatrix} = \begin{pmatrix}0 \\ 0\end{pmatrix} \implies v_1 = \begin{pmatrix}H \\ 1\end{pmatrix} \tag{3.27}$$

$$\begin{pmatrix}H & -2H^2 \\ 1 & -2H\end{pmatrix}\begin{pmatrix}c \\ d\end{pmatrix} = \begin{pmatrix}0 \\ 0\end{pmatrix} \implies v_2 = \begin{pmatrix}2H \\ 1\end{pmatrix} \tag{3.28}$$

These vectors are depicted in fig. 3.3 together with the flow of the velocity vector field.

Now, we would like to find explicit solutions. This gets a little tricky, because we cannot solve these equations analytically, so we will proceed with finding numerical solutions. However, as it turns out, most trajectories pass through the axis $R = 0$. But at this point, the eq. (3.15) contains a singularity

$$\lim_{R\to 0}\dot{S} = \lim_{R\to 0}\frac{S^2-1}{R} = \lim_{R\to 0}\frac{(S-1)(S+1)}{R}\begin{cases}\pm\infty, & S \neq \pm 1 \\ \frac{0}{0}, & S = \pm 1\end{cases} \tag{3.29}$$

In other words, $\dot{S}$ does not diverge only if S goes to ± 1 faster, than R goes to 0. If a string passes through the $R = 0$ axis, it must do so at the speed of light $S = \pm 1$. As it turns out in the numerical solutions, this happens to be the case for every trajectory.

From this, however, arises another problem with this set of equations of motion. Even if $\dot{S}$ does not diverge, the numerical accuracy around this point has to be very high and almost unreachable for standard numerical solvers. We will therefore try to find a different approach in which this singularity disappears and allows us to solve the equations with better numerical precision.

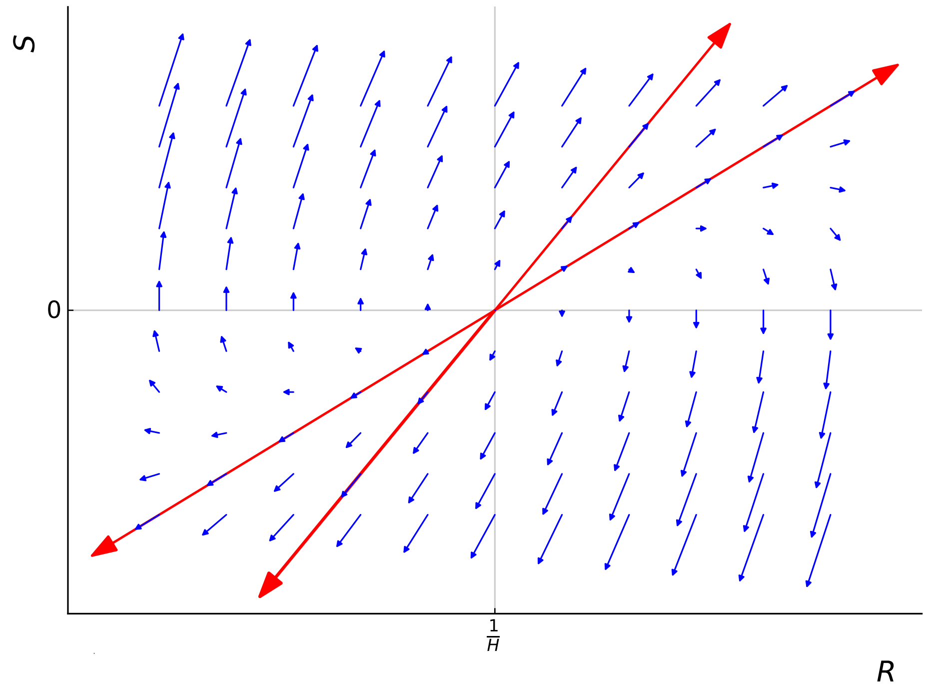

Figure 3.3: Eigenvectors (red) and flow of the velocity vector field (blue) around the unstable (repulsive) point (R, S) = (1/H, 0).

3.2 Conservation of energy

As we can see in eq. (3.11), the Lagrangian does not depend explicitly on time t. This means, that the Hamiltonian and therefore the energy is conserved. We will denote the Hamiltonian as $\mathcal{E}$ to prevent ambiguity. Also, we will not perform the full Legendre transform, but we will express the Hamiltonian in terms of $\dot{R}$ and R

$$\mathcal{E}(t, R, \dot{R}) = 2\pi T_0 E(t, R, \dot{R}) = \frac{\partial L}{\partial \dot{R}}\dot{R} - L(t, R, \dot{R}) =$$
$$2\pi T_0|R|\left[\frac{\dot{R}^2 + \dot{R}HR}{\sqrt{1-(\dot{R}-HR)^2}} + \sqrt{1-(\dot{R}-HR)^2}\right] = 2\pi T_0|R|\frac{1+\dot{R}HR - H^2R^2}{\sqrt{1-(\dot{R}-HR)^2}} \tag{3.30}$$

where we used $\mathcal{E} = 2\pi T_0 E$ to get a more convenient form. As we already mentioned, both the Lagrangian and Hamiltonian do not depend explicitly on time t and therefore, the total energy $\mathcal{E}$ is conserved, E = const. Furthermore, we will express $\dot{R}$ as a function of R, H and E. First, we square eq. (3.30) and then solve the quadratic equation for $\dot{R}$

$$S = \dot{R}(R, H, E) = \frac{\sqrt{E^2 + H^2R^4 - R^2}\left(HR\sqrt{E^2+H^2R^4-R^2} \pm E\right)}{E^2 + H^2R^4} \tag{3.31}$$

where we denoted $S = \dot{R}$ as in previous section. This is much nicer function compared to eq. (3.15). First of all, it is a first order differential equation. Second, there is no singularity at $R = 0$, and third, the critical points remain (as they should).

3.2.1 Domain of definition

Because of the square root in eq. (3.31) and the fact that we do not want to expand our solutions to the complex plane, we can expect some conditions to arise for the domain of definition. These will arise from the condition that

$$E^2 + H^2R^4 - R^2 \geq 0 \tag{3.32}$$

For a fixed and positive H and E, we are going to look when this expression is satisfied. First, we look for the roots in R.

$$R^2 = \frac{1 \pm \sqrt{1 - 4E^2H^2}}{2H^2} \tag{3.33}$$

We can sort the condition into two cases. In the first case, we have $E > 1/2H$. For this energy, there is no real value of R, for which the expression in eq. (3.32) is equal to zero. That means that for such values of E, this condition is either always or never satisfied. Inserting any value larger, for example $E = 1/H > 1/2H$ and $R = 1/H$, we get

$$\frac{1}{H^2} + \frac{1}{H^2} - \frac{1}{H^2} = \frac{1}{H^2} > 0$$

This is great, because the conditions is satisfied for every $R \in \mathbb{R}$. In other words, the string can have any radius R in the interval $\{-\infty, \infty\}$.

On the other hand, if we look at energies $E < 1/2H$, we get two roots for R^2 and four roots for R. This splits R into five regions that are separated by the roots of eq. (3.33). These regions are depicted in fig. 3.4, where we denote

$$\begin{aligned} R^+ &= \sqrt{\frac{1 + \sqrt{1 - 4E^2H^2}}{2H^2}} \\ R^- &= \sqrt{\frac{1 - \sqrt{1 - 4E^2H^2}}{2H^2}} \end{aligned} \tag{3.34}$$

Each of these regions must be examined separately, but we can still see the symmetry around $R = 0$, so we will only need to examine three of them. As is shown in fig. 3.4, we can group the intervals $R \in [-\infty, -R^+]$ and $R \in [R^+, \infty]$ as the red segments, and the intervals $R \in [-R^+, -R^-]$ and $R \in [R^-, R^+]$ as the the green segments. To find out where the condition eq. (3.32) is satisfied, we need to evaluate it in every region. Let us start with the red interval $R \in [R^+, \infty]$:

$$R = \frac{2}{H} > R^+ \implies E^2 + \frac{16}{H^2} - \frac{4}{H^2} = E^2 + \frac{12}{H^2} > 0 \tag{3.35}$$

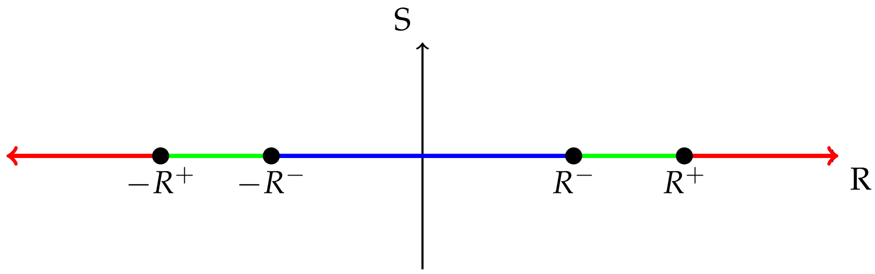

Figure 3.4: Highlighted regions on R axis, that are split by the roots of eq. (3.32).

In this region, strings with any energy can exist. Analogously for the green interval $R \in [R^-, R^+]$:

$$R^+ > R = \frac{1}{\sqrt{2}H} > R^- \implies E^2 + \frac{1}{4H^2} - \frac{1}{2H^2} = E^2 - \frac{1}{4H^2} < 0 \tag{3.36}$$

The last inequality follows from the fact, that we are considering only $E < 1/2H$. Strings with such energy cannot exist in the green intervals.

The final interval is the blue interval $R \in [-R^-, R^-]$:

$$R^- > R = 0 > -R^- \implies E^2 > 0 \tag{3.37}$$

Again, this is a region, where strings with all energies can exist.

In conclusion, we have two types of strings. First, we have strings with energy $E > 1/2H$ that have the domain of definition all of $R \in \mathbb{R}$. Second, strings with energy $E < 1/2H$ have limitations. They can only exist in the red and blue regions depicted in fig. 3.4. We avoided the string with energy exactly $E = 1/2H$ for which the green regions vanish. This is not unexpected, because this string goes through the saddle point $S = 0$, $R = 1/\sqrt{2}H$.

All this can also be interpreted physically from the potential in fig. 3.1. There we have a potential well centered around $R = 0$ with maximum at $R = \pm 1/\sqrt{2}H$ and the corresponding value $V = 1/2H$. This implies that strings with energy lower than the maximum of this potential ($E < 1/2H$) are stuck inside this potential well. On the other hand if this string finds itself already outside of the potential well, it will expand to infinity. The strings that have energy above the potential barrier ($E > 1/2H$) can move over and therefore will never have closed solutions.

3.3 Summary of results

This brings us to the conclusion where we will present the phase space plots of the numerical solutions. First, we start with closed trajectories depicted in fig. 3.5. These can happen only for strings with $E < 1/2H$ and they are also the most likely to be observed, because of the fact, that our universe's expansion rate H is relatively small, and we would need strings with very high energies to overcome the potential well. Strings that are not of cosmological magnitude therefore do experience little change. In fig. 3.10 we show the explicit time dependence of radius R on time, which was acquired by numerical integration of eq. (3.31). If we compare it to the flat spacetime, where it is a sine function, we can confirm, that for smaller energies, the expansion of space has little effect, but with increasing energy the trajectory gets more deformed and has increased frequency.

On the other hand, the existence of the expansion of the universe can have some unforeseen consequences. It is surprising, that even with a really small expansion rate, there can exist a string with much larger radius than its counterpart stuck in closed trajectory. In fact, even a string with $E = 0$ can have very large radius. This is due to the repulsive node in $R = 1/H$, where a string with any energy can be created and then expand to infinity. This is shown in fig. 3.6.

This brings us to the special case where $E = 1/2H$. This is depicted as the purple line in fig. 3.7. It would seem that when this string arrives at $R = 1/\sqrt{2}H$, it can decide whether to continue to the closed trajectory part or expand to infinity. However this dilemma solves itself, because it would take the string an infinite amount of proper time to reach this saddle point.

The last case are strings with $E > 1/2H$ which is depicted in fig. 3.8. These strings can start as static strings at $R = 1/H$, which is an unstable node. If these strings are then shifted by a small amount, they either start to shrink, reach a minimal radius and then expand or they expand to infinity right away.

All these solutions are constrained by the black lines and fill the whole of this space. For comparison, we add solutions in flat spacetime, that are depicted in fig. 3.9.

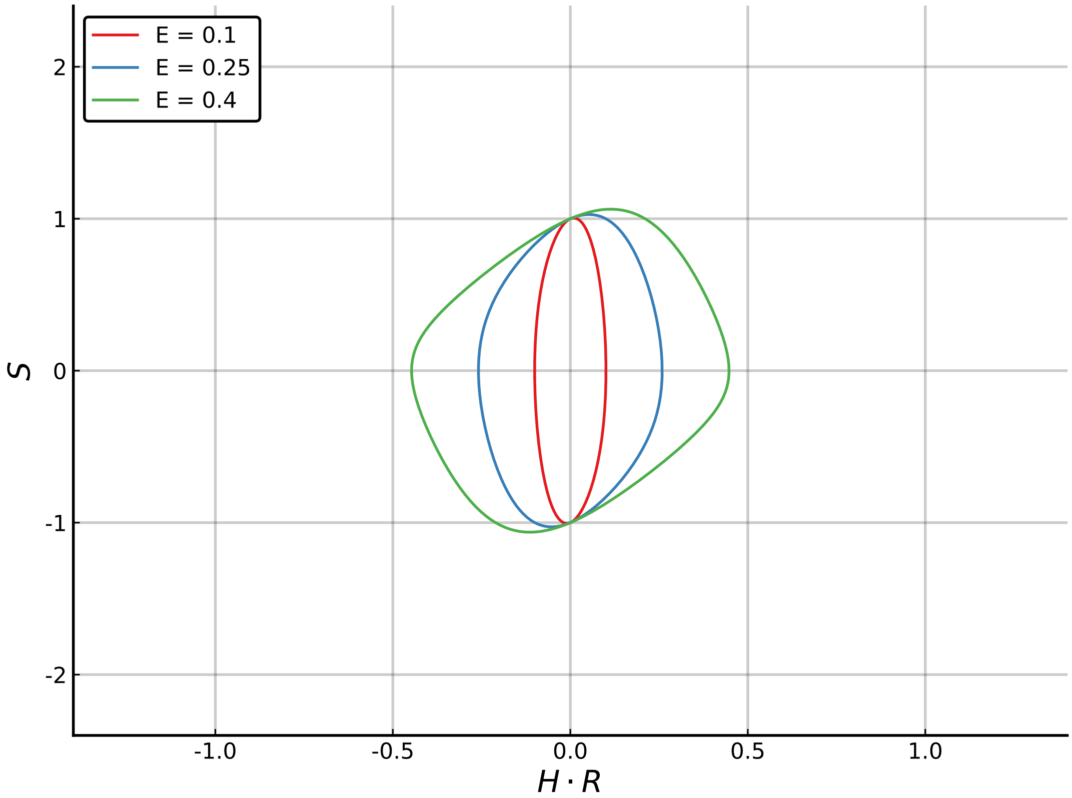

Figure 3.5: Closed trajectories in de Sitter space for $E < 1/2H$.

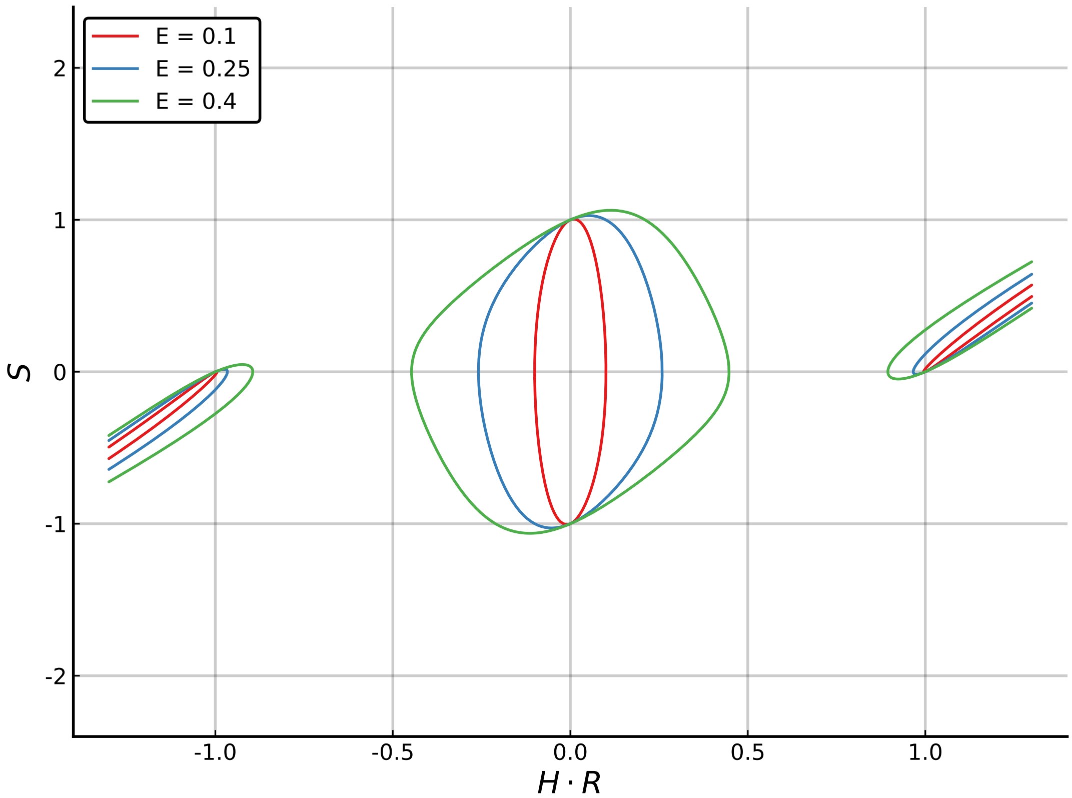

Figure 3.6: All trajectories in de Sitter space for $E < 1/2H$.

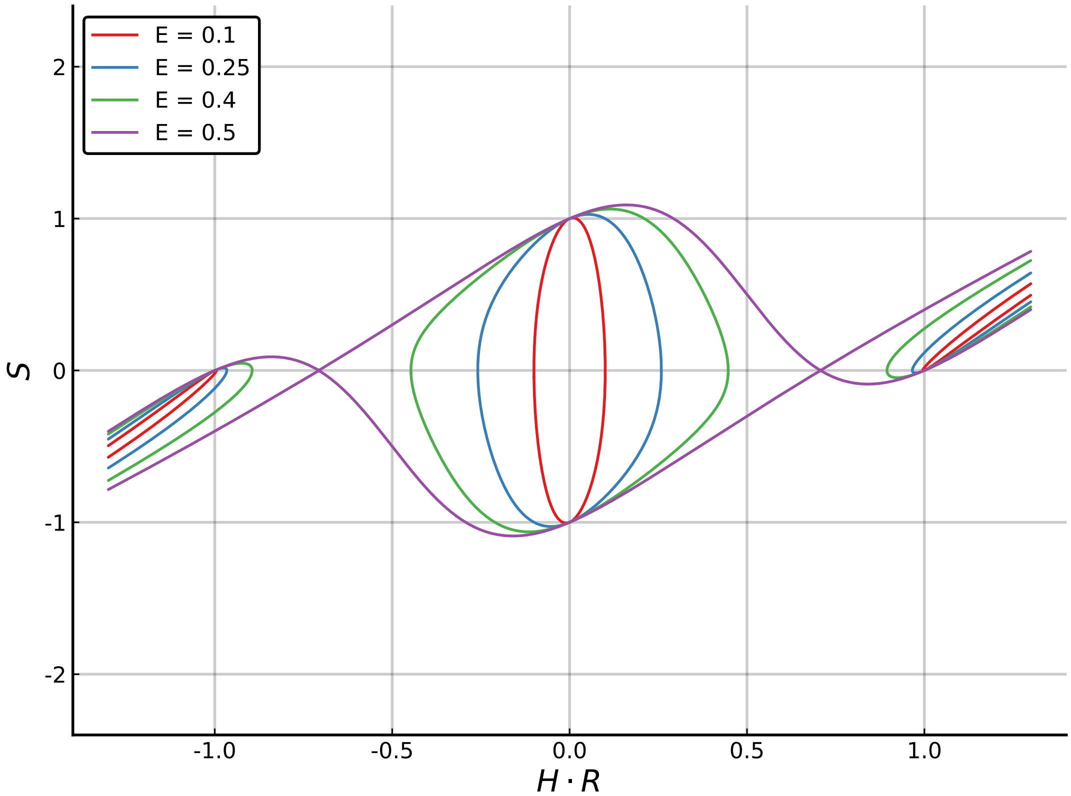

Figure 3.7: All trajectories in de Sitter space for $E \leq 1/2H$.

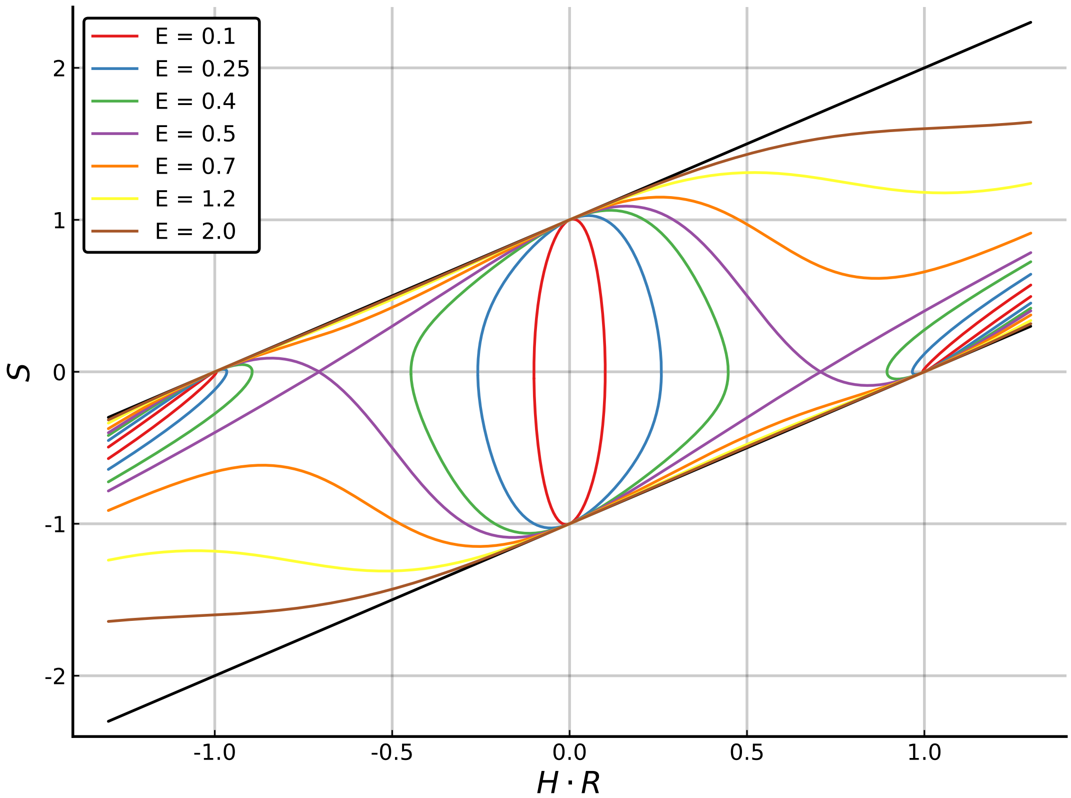

Figure 3.8: Trajectories for all energies E in de Sitter space.

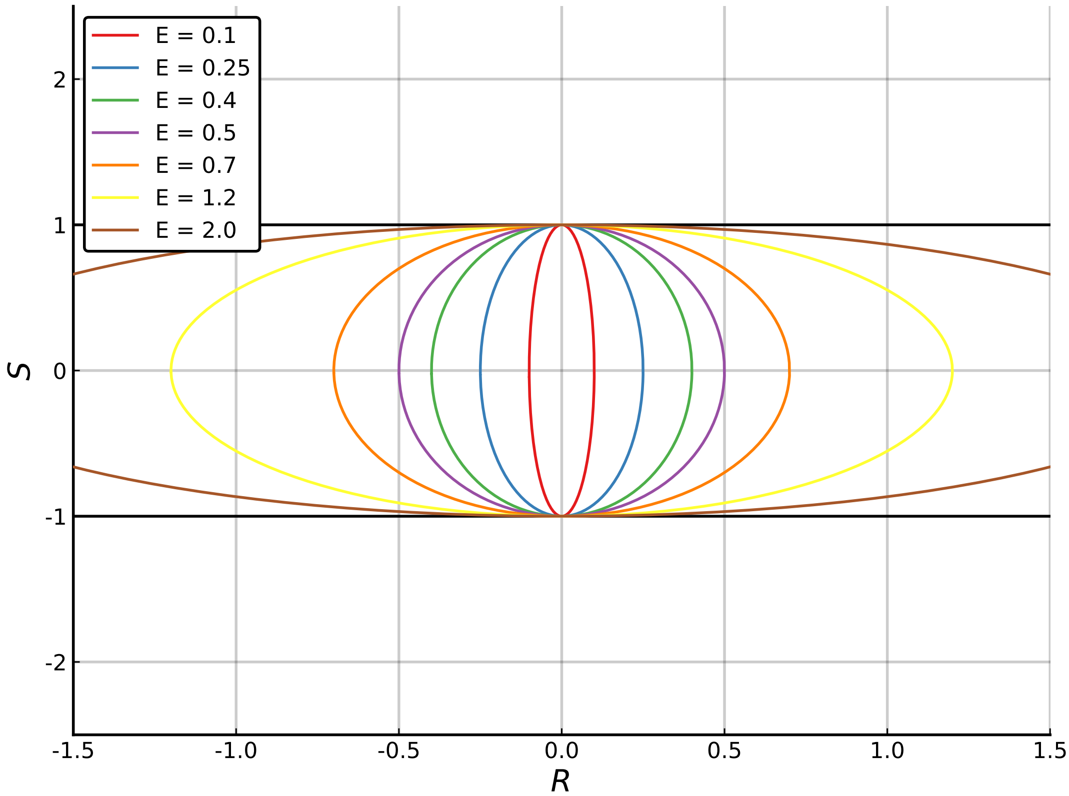

Figure 3.9: Trajectories for all energies E in flat space.

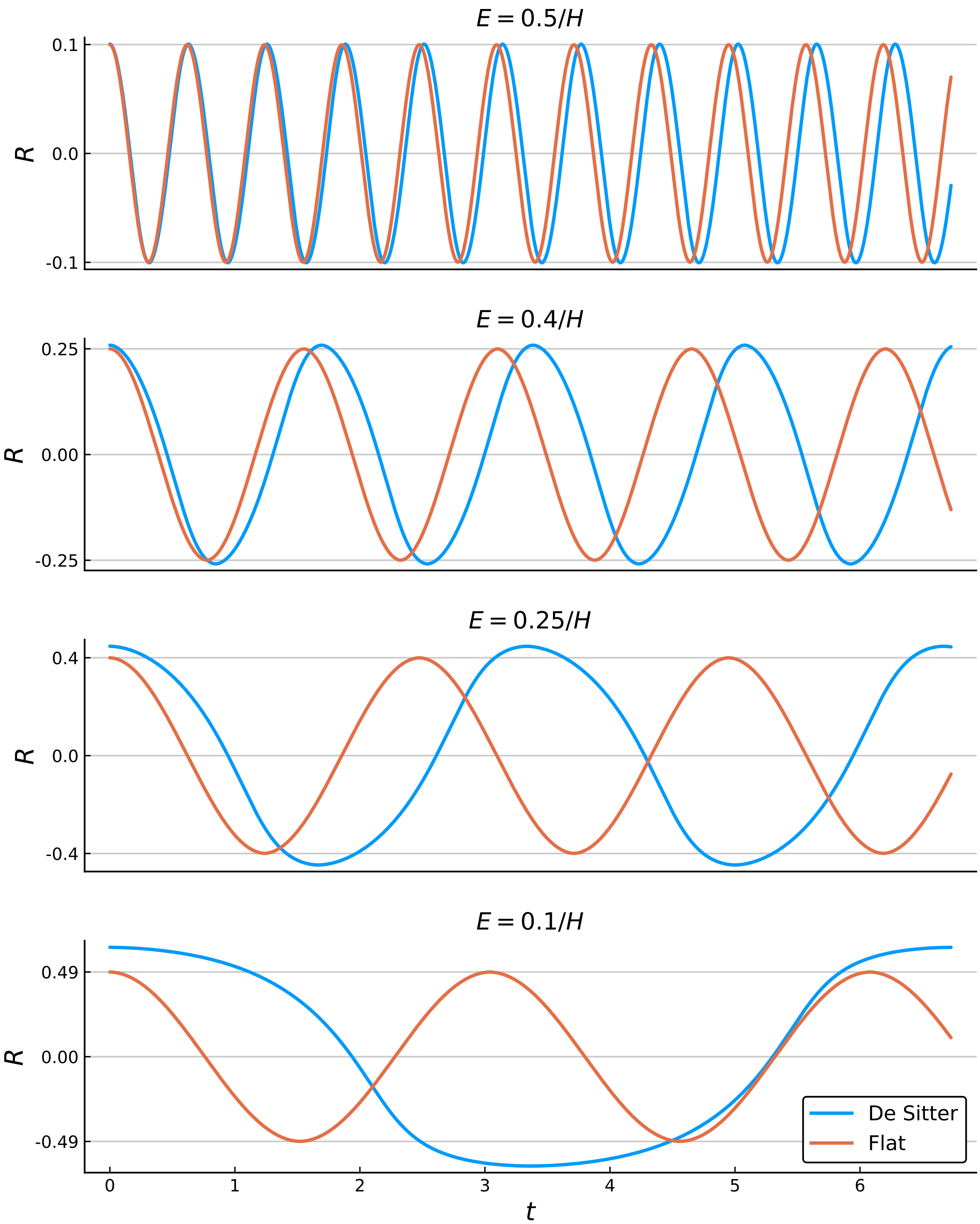

Figure 3.10: Explicit time evolution of closed strings with energies $E < 1/2H$

3.4 Verification with general equations of motion

In the beginning of this chapter we started with the Nambu–Goto action, and we immediately fixed the parameterization to $t = \tau$ and $\theta = \sigma$ and inserted a change of variables $R = re^{Ht}$. This way we could extract more information about the system, specifically the potential, and the Hamiltonian helped us find crucial features of this equation and allowed us to get around the bad numerical precision of solving eq. (3.15). However, in 1.3 we derived the equations of motion (1.17) for a general background where we get four equations instead of one. One might ask if the other three equations could add some restrictions to the system. For this reason, we will solve these equations with the same choice of parameterization. First, we will calculate the components of induced metric γ from eq. (1.6)

$$\begin{aligned}\gamma_{\tau\tau} &= -1 + \dot{r}^2 e^{2Ht} \\ \gamma_{\sigma\sigma} &= r^2 e^{2Ht} \\ \gamma_{\tau\sigma} &= \gamma_{\sigma\tau} = 0\end{aligned} \tag{3.38}$$

From (1.17), we obtain the equations of motion

$$t: \quad 2\partial_\tau \left(\sqrt{\frac{r^2 e^{2Ht}}{1 - \dot{r}^2 e^{2Ht}}} \right) + 2H\dot{r}^2 e^{2Ht} \sqrt{\frac{r^2 e^{2Ht}}{1 - \dot{r}^2 e^{2Ht}}} - 2Hr^2 e^{2Ht} \sqrt{\frac{1 - \dot{r}^2 e^{2Ht}}{r^2 e^{2Ht}}} = 0 \tag{3.39}$$

$$r: \quad 2\partial_\tau \left(\dot{r} e^{2Ht} \sqrt{\frac{r^2 e^{2Ht}}{1 - \dot{r}^2 e^{2Ht}}} \right) - 2re^{2Ht} \sqrt{\frac{1 - \dot{r}^2 e^{2Ht}}{r^2 e^{2Ht}}} = 0 \tag{3.40}$$

$$\theta,\ z: \quad 0 = 0 \tag{3.41}$$

As we can see, the equations of motion in θ and z are trivial. First, we focus on solving the t equation (3.39). After some modification, we arrive at

$$\frac{r^2 \dot{r}\ddot{r} e^{4Ht} - r\dot{r}^3 e^{4Ht} + r\dot{r} e^{2Ht} + 3Hr^2 \dot{r}^2 e^{4Ht} - 2Hr^2 \dot{r}^4 e^{6Ht}}{\sqrt{r^2 e^{2Ht}} \left(1 - \dot{r}^2 e^{2Ht}\right)^{\frac{3}{2}}} = 0 \tag{3.42}$$

If we now perform the change of variables $R = re^{Ht}$ and simplify, we get to the equation

$$\frac{R\ddot{R} - \dot{R}^2 + 1 + 3HR\left(\dot{R} - HR\right) - 2HR\left(\dot{R} - HR\right)^3}{\left[1 - \left(\dot{R} - HR\right)^2\right]^{3/2}} = 0 \tag{3.43}$$

which is the same as eq. (3.14). Lets look at the r equation (3.40):

$$\frac{r^2 \ddot{r} e^{4Ht} - r\dot{r}^2 e^{4Ht} + re^{2Ht} + 3Hr^2 \dot{r} e^{4Ht} - 2Hr^2 \dot{r}^3 e^{6Ht}}{\sqrt{r^2 e^{2Ht}} \left(1 - \dot{r}^2 e^{2Ht}\right)^{\frac{3}{2}}} = 0 \tag{3.44}$$

It is the same as eq. (3.42). This means that there really is only one equation of motion and it is the same as the one we used in section 3.1.2.

Chapter 4

Gravitational wave background

In this chapter, we will be investigating the effect of gravitational waves on classical strings. We will be working with the non-linear theory of gravitational waves, where the line element has the form [3]

$$\mathrm{d}s^2 = H(u,x,y)\,\mathrm{d}u^2 + 2\,\mathrm{d}u\,\mathrm{d}v + \mathrm{d}x^2 + \mathrm{d}y^2 \tag{4.1}$$

We are using the coordinates $u = (z+t)/\sqrt{2}$, $v = (z-t)/\sqrt{2}$. This corresponds to a gravitational wave moving in the direction of the z coordinate. Also, to avoid confusion, we note that the function H which fully specifies the gravitational wave is different from the Hubble parameter H used in previous chapter. We will now study the conditions on H so that it is a solution to the Einstein vacuum field equations, which can be rewritten as

$$R_{\mu\nu} = 0, \tag{4.2}$$

where $R_{\mu\nu}$ is Ricci curvature, which is defined as

$$R_{\mu\nu} = R^{\rho}_{\mu\rho\nu} = \partial_\rho \Gamma^{\rho}_{\mu\nu} - \partial_\nu \Gamma^{\rho}_{\mu\rho} + \Gamma^{\rho}_{\rho\lambda}\Gamma^{\lambda}_{\mu\nu} - \Gamma^{\rho}_{\lambda\nu}\Gamma^{\lambda}_{\mu\rho} \tag{4.3}$$

and $\Gamma^{\rho}_{\mu\nu}$ are Christoffel symbols defined as

$$\Gamma^{\rho}_{\mu\nu} = \frac{g^{\rho\sigma}}{2}\left(\partial_\nu g_{\sigma\mu} + \partial_\mu g_{\sigma\nu} - \partial_\sigma g_{\mu\nu}\right) \tag{4.4}$$

The only non-zero components are

$$\begin{gathered} \Gamma^{v}_{uu} = \frac{\partial_u H}{2} \qquad \Gamma^{x}_{uu} = -\frac{\partial_x H}{2} \qquad \Gamma^{y}_{uu} = -\frac{\partial_y H}{2} \\ \Gamma^{v}_{ux} = \Gamma^{v}_{xu} = \frac{\partial_x H}{2} \qquad \Gamma^{v}_{uy} = \Gamma^{v}_{yu} = \frac{\partial_y H}{2} \end{gathered} \tag{4.5}$$

From this we can calculate the components of the Ricci curvature, but as it turns out only one components is non-zero and that is

$$R_{uu} = -\frac{\partial_x^2 H + \partial_y^2 H}{2} \tag{4.6}$$

This has to be zero in order for this to be a solution to the Einstein vacuum field equations (4.2). The function $H(x, y, u)$ must therefore be a solution to Laplace's equation

$$\partial_x^2 H + \partial_y^2 H = 0 \tag{4.7}$$

Now we know what the metric is, so we can focus on the motion of a string in this background. In order to get the equations of motion we will use the method described in chapter 1, specifically eq. (1.17). The first step is to choose a parameterization. We will choose it in such a way, that we get, in theory, the simplest equations or at least equations most similar to wave equations. This will be true if the induced metric has the form:

$$g_{MN}\partial_\alpha X \partial_\beta X = \gamma_{\alpha\beta} = \begin{pmatrix} \gamma_{\tau\tau} & 0 \\ 0 & -\gamma_{\tau\tau} \end{pmatrix} \tag{4.8}$$

This can also be written as two conditions

$$\begin{aligned} \gamma_{\tau\sigma} &= \gamma_{\sigma\tau} = 0 \\ \gamma_{\tau\tau} &= -\gamma_{\sigma\sigma}. \end{aligned} \tag{4.9}$$

The equations of motion from eq. (1.17) written in components then take the form

$$u: \quad \partial_\tau(H\,\partial_\tau u + \partial_\tau v) - \partial_\sigma(H\,\partial_\sigma u + \partial_\sigma v) - \frac{\partial_u H}{2}\left[(\partial_\tau u)^2 - (\partial_\sigma u)^2\right] = 0 \tag{4.10}$$

$$v: \quad \partial_\tau^2 u - \partial_\sigma^2 u = 0 \tag{4.11}$$

$$x: \quad \partial_\tau^2 x - \partial_\sigma^2 x - \frac{\partial_x H}{2}\left[(\partial_\tau u)^2 - (\partial_\sigma u)^2\right] = 0 \tag{4.12}$$

$$y: \quad \partial_\tau^2 y - \partial_\sigma^2 y - \frac{\partial_y H}{2}\left[(\partial_\tau u)^2 - (\partial_\sigma u)^2\right] = 0 \tag{4.13}$$

We will now choose a τ parameterization in accordance with eq. (4.9). Since $u(\tau, \sigma)$ obeys the d'Alambert equation eq. (4.11), we can make a reparameterization [4, 5]

$$\begin{aligned} \sigma + \tau &\to \varphi_1(\sigma + \tau) \\ \sigma - \tau &\to \varphi_2(\sigma - \tau) \end{aligned} \tag{4.14}$$

such that

$$u = \lambda\tau \tag{4.15}$$

where λ is constant. The equations of motion then reduce from four to just three equations

$$\partial_\tau^2 v - \partial_\sigma^2 v + \left(\frac{\partial_u H}{2}\lambda + \partial_x H\,\partial_\tau x + \partial_y H\,\partial_\tau y\right)\lambda = 0 \tag{4.16}$$

$$\partial_\tau^2 x - \partial_\sigma^2 x - \frac{\partial_x H}{2}\lambda^2 = 0 \tag{4.17}$$

$$\partial_\tau^2 y - \partial_\sigma^2 y - \frac{\partial_y H}{2}\lambda^2 = 0 \tag{4.18}$$

We now need to specify the function $H(u, x, y)$, which describes the gravitational wave. This function needs to satisfy eq. (4.7). The two most simple nontrivial solutions are

$$H = (x^2 - y^2) f(u) \tag{4.19}$$
$$H = xyf(u) \tag{4.20}$$

where $f(u)$ is an arbitrary function of u. These correspond to basic polarizations of the gravitational wave. First, we will use the former polarization. The last thing we need is the function $f(u)$ and we will analyze few choices of this function in the following sections.

4.1 Periodic gravitational wave

We will start with a simple cosine function with frequency ω. The function H then takes the form

$$H(u, x, y) = A(x^2 - y^2)\cos(\omega u) = A(x^2 - y^2)\cos(\omega\lambda\tau) \tag{4.21}$$

The equations of motion given by eq. (1.17) take the form

$$\begin{aligned}\partial_\tau^2 v - \partial_\sigma^2 v - \frac{1}{2}A\omega\lambda^2(x^2 - y^2)\sin(\omega\lambda\tau) + 2A\lambda\,(x\partial_\tau x - y\partial_\tau y)\,cos(\omega\lambda\tau) &= 0\\ \partial_\tau^2 x - \partial_\sigma^2 x - A\lambda^2 x\cos(\omega\lambda\tau) &= 0\\ \partial_\tau^2 y - \partial_\sigma^2 y + A\lambda^2 y\cos(\omega\lambda\tau) &= 0.\end{aligned} \tag{4.22}$$

We will now focus on the equation for x. We can separate this partial differential equation of second order such that $x(\tau, \sigma) = T(\tau)\cdot S(\sigma)$. This leads to

$$T''(\tau)S(\sigma) - T(\tau)S''(\sigma) - T(\tau)S(\sigma)A\lambda^2\cos(\omega\lambda\tau) = 0 \tag{4.23}$$

$$\Downarrow$$

$$\frac{T''}{T}(\tau) - \lambda^2\cos(\omega\lambda\tau) = \frac{S''}{S}(\sigma) \tag{4.24}$$

Because each side of this equation is a function of independent parameters, it can only be equal to a constant that we will write as $-k^2$

$$\partial_\tau^2 T = -\left(k^2 - A\lambda^2 \cos(\omega\lambda\tau)\right) T \tag{4.25}$$

$$\partial_\sigma^2 S = -k^2 S. \tag{4.26}$$

The second eq. (4.26) is just a Helmholtz equation. Its solution for a given k can be expressed in the following way:

$$S_k(\sigma) = c_1 \mathrm{e}^{\mathrm{i}k\sigma} + c_2 \mathrm{e}^{-\mathrm{i}k\sigma}, \qquad k \in \mathbb{Z} \tag{4.27}$$

On the other hand, eq. (4.25) is similar to Mathieu equation, which has the form:

$$\frac{\partial^2 y}{\partial x^2} + [a - 2q\cos(2x)]\, y = 0 \tag{4.28}$$

Substituting $x = \omega\lambda\tau/2$ gives us the canonical form of Mathieu differential equation:

$$\frac{\partial^2 T}{\partial x^2} + \left[\left(\frac{2k}{\omega\lambda}\right)^2 - A\left(\frac{2}{\omega}\right)^2 \cos(2x)\right] T = 0 \tag{4.29}$$

where we can assign

$$a = \left(\frac{2k}{\omega\lambda}\right)^2 \tag{4.30}$$

$$q = \frac{2A}{\omega^2} \tag{4.31}$$

The solutions of the Mathieu equation are described in detail in [6]. For us, a qualitative study will suffice. Figure 4.1 depicts the stability of solutions based on its parameters a and q. We can see that if

$$\left(\frac{2k}{\omega\lambda}\right)^2 \approx n^2, \qquad n \in \mathbb{Z} \tag{4.32}$$

the string becomes unstable even for small amplitudes A. Physically, this corresponds to some kind of resonance between the natural frequency of the string k (if there was no gravitational wave) and the frequency of the gravitational wave ω. Also, if we fix the frequency ω, the change of the amplitude A will correspond to horizontal lines in fig. 4.1. On the other hand, fixing the amplitude A to a specific value and letting the frequency ω vary, will results in lines that intersect the origin with slope $2k^2/(A\lambda^2)$. Also, due to the inverse dependence on ω^2, the higher the frequency, the closer we get to zero.

The same applies to the y component, because the instability regions of the solutions of Mathieu equation are symmetric with respect to the sign of q. On the

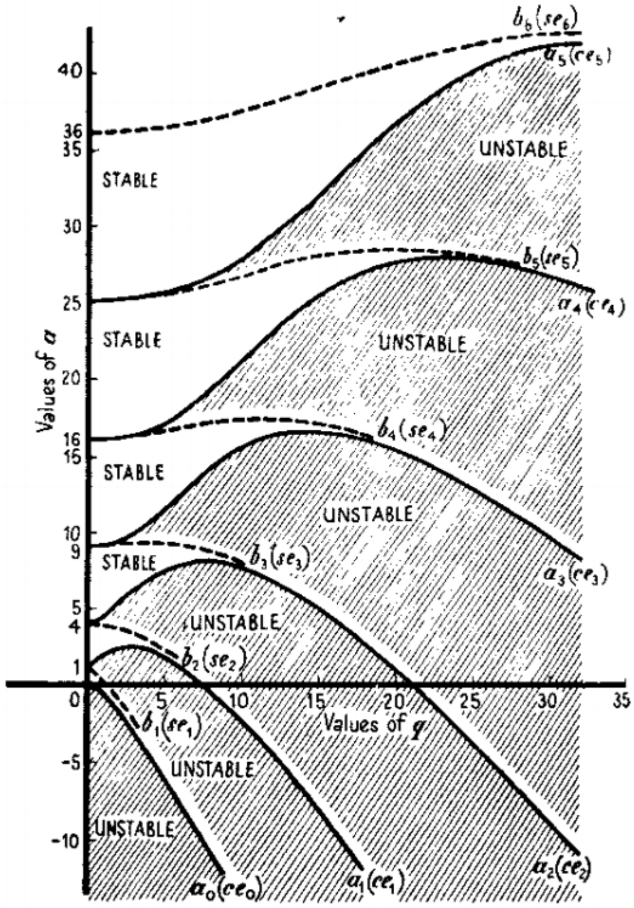

Figure 4.1: Stability of solutions to Mathieu equation (4.28) taken from [6].

other hand, the v component is more complicated and hard to interpret. Also, it would be hard to find a good way to measure the energy of the string in this background, because the vector $\frac{\partial}{\partial\tau}$ can change from time-like to space-like during the motion. We will therefore turn to the next section, where we hope to solve this problem with different profile of gravitational waves.

4.2 Gravitational wave burst with Gaussian profile

In this section, we will study how a burst of gravitational waves affects the motion of a string in flat spacetime. We will choose $f(u)$ in eq. (4.19) to be a cosine function multiplied by the a Gaussian envelope

$$\begin{aligned} H(u,x,y) &= (x^2-y^2)\cos(\omega u)\exp\left(-\frac{(u-u_0)^2}{2\rho^2}\right) \\ &= (x^2-y^2)\cos(\omega\lambda\tau)\exp\left(-\frac{\lambda^2(\tau-\tau_0)^2}{2\rho^2}\right) \end{aligned} \tag{4.33}$$

with frequency ω and the size of the Gaussian envelope ρ, which corresponds to the duration of the gravitational wave burst that hits its peak at proper time τ_0.

The string is initially in a flat space, because H is almost zero. Then comes the gravitational wave burst and after some time it fades away. During this time it is not easy to identify what is a time-like coordinate and how the string really moves but after the burst ends, the metric returns again to the form of flat spacetime. This allows us to evaluate how the string changed its behaviour, specifically, to calculate the energy of the string before and after the gravitational wave burst hits it. We want to create a graph showing the amount of energy transfered between the string and the gravitational wave burst and compare it with fig. 4.1.

Calculation of the equations of motion from eq. (1.17) gives us

$$\begin{aligned}\partial_\tau^2 v - \partial_\sigma^2 v - A\lambda \exp\left(-\frac{\lambda^2(\tau-\tau_0)^2}{2\rho^2}\right)\left[\frac{x^2-y^2}{2}\omega\lambda\sin(\omega\lambda\tau)\right.\\ \left.+\frac{x^2-y^2}{2}\frac{\lambda^2(\tau-\tau_0)}{\rho^2}\cos(\omega\lambda\tau) - 2\left(x\partial_\tau x - y\partial_\tau y\right)\cos(\omega\lambda\tau)\right] = 0\end{aligned} \tag{4.34}$$

$$\partial_\tau^2 x - \partial_\sigma^2 x - A\lambda^2 x\cos(\omega\lambda\tau)\exp\left(-\frac{\lambda^2(\tau-\tau_0)^2}{2\rho^2}\right) = 0 \tag{4.35}$$

$$\partial_\tau^2 y - \partial_\sigma^2 y + A\lambda^2 y\cos(\omega\lambda\tau)\exp\left(-\frac{\lambda^2(\tau-\tau_0)^2}{2\rho^2}\right) = 0 \tag{4.36}$$

Since v, x and y are σ_1 periodic in σ, we can expand them into Fourier series

$$v(\sigma,\tau) = \sum_{k_v\in\mathbb{Z}} v_{k_v}(\tau)\mathrm{e}^{2\pi \mathrm{i} k_v\sigma/\sigma_1} \qquad v_{k_v}(\tau) = \frac{1}{\sigma_1}\int_0^{\sigma_1} v(\sigma,\tau)\mathrm{e}^{-2\pi \mathrm{i} k_v\sigma/\sigma_1} \tag{4.37}$$

$$x(\sigma,\tau) = \sum_{k_x\in\mathbb{Z}} x_{k_x}(\tau)\mathrm{e}^{2\pi \mathrm{i} k_x\sigma/\sigma_1} \qquad x_{k_x}(\tau) = \frac{1}{\sigma_1}\int_0^{\sigma_1} x(\sigma,\tau)\mathrm{e}^{-2\pi \mathrm{i} k_x\sigma/\sigma_1} \tag{4.38}$$

$$y(\sigma,\tau) = \sum_{k_y\in\mathbb{Z}} y_{k_y}(\tau)\mathrm{e}^{2\pi \mathrm{i} k_y\sigma/\sigma_1} \qquad y_{k_y}(\tau) = \frac{1}{\sigma_1}\int_0^{\sigma_1} y(\sigma,\tau)\mathrm{e}^{-2\pi \mathrm{i} k_y\sigma/\sigma_1} \tag{4.39}$$

The equations for x and y then take the form

$$\begin{aligned}&\left[\partial_\tau^2 - \partial_\sigma^2 - A\lambda^2\cos(\omega\lambda\tau)\exp\left(-\frac{\lambda^2(\tau-\tau_0)^2}{2\rho^2}\right)\right]\sum_{k_x} x_{k_x}\mathrm{e}^{2\pi\mathrm{i}k_x\sigma/\sigma_1}\\ &\quad=\sum_{k_x}\left[\partial_\tau^2 + k_x^2 - A\lambda^2\cos(\omega\lambda\tau)\exp\left(-\frac{\lambda^2(\tau-\tau_0)^2}{2\rho^2}\right)\right] x_{k_x}\mathrm{e}^{2\pi\mathrm{i}k_x\sigma/\sigma_1} = 0\end{aligned} \tag{4.40}$$

$$\begin{aligned}&\left[\partial_\tau^2 - \partial_\sigma^2 + A\lambda^2\cos(\omega\lambda\tau)\exp\left(-\frac{\lambda^2(\tau-\tau_0)^2}{2\rho^2}\right)\right]\sum_{k_y} y_{k_y}\mathrm{e}^{2\pi\mathrm{i}k_y\sigma/\sigma_1}\\ &\quad\sum_{k_y}\left[\partial_\tau^2 + k^2 + A\lambda^2\cos(\omega\lambda\tau)\exp\left(-\frac{\lambda^2(\tau-\tau_0)^2}{2\rho^2}\right)\right] y_{k_y}\mathrm{e}^{2\pi\mathrm{i}k_y\sigma/\sigma_1} = 0\end{aligned} \tag{4.41}$$

In order for this to be equal to zero, every term in the sum must vanish. The equation for v is a little bit more complicated, so for more clear view of the equations, we will denote

$$h(\tau) = \frac{A\lambda}{2}\exp\left(-\frac{\lambda^2(\tau-\tau_0)^2}{2\rho^2}\right)\left[\omega\lambda\sin(\omega\lambda\tau) + \frac{\lambda^2(\tau-\tau_0)}{\rho^2}\cos(\omega\lambda\tau)\right] \tag{4.42}$$

$$b(\tau) = 2A\lambda\exp\left(-\frac{\lambda^2(\tau-\tau_0)^2}{2\rho^2}\right)\cos(\omega\lambda\tau) \tag{4.43}$$

If we multiply the equation by $\mathrm{e}^{-2\pi\mathrm{i}k_v\sigma/\sigma_1}$ and integrate over σ, we get

$$\begin{aligned}&\int_0^{\sigma_1}\mathrm{e}^{-2\pi\mathrm{i}k_v\sigma/\sigma_1}\left(\partial_\tau^2 - \partial_\sigma^2\right) v\,\mathrm{d}\sigma =\\ &\quad=\int_0^{\sigma_1}\mathrm{e}^{-2\pi\mathrm{i}k_v\sigma/\sigma_1}\left[\left(x^2-y^2\right)h(\tau) - \left(x\,\partial_\tau x - y\,\partial_\tau y\right)b(\tau)\right]\mathrm{d}\sigma\end{aligned} \tag{4.44}$$

We will modify each side of the equation independently. The left side can be written as

$$\begin{aligned}&\int_0^{\sigma_1}\mathrm{e}^{-2\pi\mathrm{i}k_v\sigma/\sigma_1}\left(\partial_\tau^2 - \partial_\sigma^2\right)\sum_{k'_v} v_{k'_v}\mathrm{e}^{\mathrm{i}k'_v\sigma}\,\mathrm{d}\sigma = \sum_{k'_v}\int_0^{\sigma_1}\mathrm{e}^{-2\pi\mathrm{i}(k_v-k'_v)\sigma/\sigma_1}\,\mathrm{d}\sigma\left(\partial_\tau^2 + {k'_v}^2\right)v_{k'_v}\\ &\quad=\sum_{k'_v}\sigma_1\delta_{k_v}^{k'_v}\left(\partial_\tau^2 + {k'_v}^2\right)v_{k'_v} = \sigma_1\left(\partial_\tau^2 + {k_v}^2\right)v_{k_v}\end{aligned} \tag{4.45}$$

Simplifying the first term on the right side of eq. (4.44) gives us

$$\begin{aligned}
&\int_0^{\sigma_1} e^{-2\pi i k_v \sigma/\sigma_1} \left(\sum_{k_x} \sum_{k'_x} x_{k_x} x_{k'_x} e^{2\pi i (k_x + k'_x)\sigma/\sigma_1} - \sum_{k_y} \sum_{k'_y} y_{k_y} y_{k'_y} e^{2\pi i (k_y + k'_y)\sigma/\sigma_1} \right) h(\tau)\, d\sigma \\
&= \left(\sum_{k_x k'_x} x_{k_x} x_{k'_x} \int_0^{\sigma_1} e^{2\pi i (k_x + k'_x - k_v)\sigma/\sigma_1}\, d\sigma - \sum_{k_y k'_y} y_{k_y} y_{k'_y} \int_0^{\sigma_1} e^{2\pi i (k_y + k'_y - k_v)\sigma/\sigma_1}\, d\sigma \right) h(\tau) \\
&= \sigma_1 \left(\sum_{k_x k'_x} x_{k_x} x_{k'_x} \delta^{k'_x}_{k_v - k_x} - \sum_{k_y k'_y} y_{k_y} y_{k'_y} \delta^{k'_y}_{k_v - k_y} \right) h(\tau) \\
&= \sigma_1 \left(\sum_{k_x} x_{k_x} x_{k_v - k_x} - \sum_{k_y} y_{k_y} y_{k_v - k_y} \right) h(\tau)
\end{aligned} \tag{4.46}$$

The second term on the right side of eq. (4.44) will end up similarly to the first term

$$\begin{aligned}
&\int_0^{\sigma_1} e^{-2\pi i k_v \sigma/\sigma_1} \left(\sum_{k_x k'_x} x_{k_x} \partial_\tau x_{k'_x} e^{2\pi i (k_x + k'_x)\sigma/\sigma_1} - \sum_{k_y k'_y} y_{k_y} \partial_\tau y_{k'_y} e^{2\pi i (k_y + k'_y)\sigma/\sigma_1} \right) b(\tau)\, d\sigma \\
&= \sigma_1 \left(\sum_{k_x k'_x} x_{k_x} \partial_\tau x_{k'_x} \delta^{k'_x}_{k_v - k_x} - \sum_{k_y k'_y} y_{k_y} \partial_\tau y_{k'_y} \delta^{k'_y}_{k_v - k_y} \right) b(\tau) \\
&= \sigma_1 \left(\sum_{k_x} x_{k_x} \partial_\tau x_{k_v - k_x} - \sum_{k_y} y_{k_y} \partial_\tau y_{k_v - k_y} \right) b(\tau)
\end{aligned} \tag{4.47}$$

Combining eq. (4.45) with eq. (4.46) and eq. (4.47), we arrive at equation for v_{k_v}

$$\begin{aligned}
\left(\partial_\tau^2 + {k_v}^2 \right) v_{k_v} = \Bigg(&\sum_{k_x} x_{k_x} x_{k_v - k_x} h(\tau) - x_{k_x} \partial_\tau x_{k_v - k_x} b(\tau) \\
&+ \sum_{k_y} y_{k_y} \partial_\tau y_{k_v - k_y} b(\tau) - y_{k_y} y_{k_v - k_y} h(\tau) \Bigg)
\end{aligned} \tag{4.48}$$

Equations (4.40), (4.41) and (4.48) together with eq. (4.15) describe the motion of the string affected by the gravitational wave burst. We can notice, that the equations for x_{k_x} (4.40) and y_{k_y} (4.41) depend only on the k_x or k_y modes respectively. Therefore, we can say that every mode that is zero at the beginning will also be zero throughout the motion. However, the k_v-th mode of v is affected by the modes of x and y.

We will study a string that is initially circular in the x, y plane. The components x_{k_x} and y_{k_y} are nonzero only for $k_{x/y} = \pm 1$, specifically

$$\begin{aligned} x_1 &= \frac{r_0}{2}, & x_{-1} &= \frac{r_0}{2} \\ y_1 &= \frac{r_0}{2\mathrm{i}}, & y_{-1} &= -\frac{r_0}{2\mathrm{i}} \end{aligned} \tag{4.49}$$

Furthermore, we will set the $k_v = 0$ mode to be $v_0 = \varepsilon\tau$. The contributions from the x and y modes will affect only $k_v = \{-2, 0, 2\}$ modes. We want to look at a string that is initially in rest with respect to the observer and is in static gauge. This can be satisfied if

$$\partial_\tau t(\tau = 0) = \partial_\tau \frac{\sqrt{2}(u - v)}{2} = \frac{\sqrt{2}(\lambda - \varepsilon)}{2} = 1 \tag{4.50}$$

$$\partial_\tau z(\tau = 0) = \partial_\tau \frac{\sqrt{2}(u + v)}{2} = \frac{\sqrt{2}(\lambda + \varepsilon)}{2} = 0 \tag{4.51}$$

This gives us the initial parameterization of the string

$$\lambda = -\varepsilon = \frac{1}{\sqrt{2}} \tag{4.52}$$

Now we can solve the eqs. (4.40), (4.41) and (4.48) numerically with these initial values. Since the behaviour of the strings varies a lot, we will measure the rest energy of the string after its interaction with the gravitational wave burst. This energy is given by eq. (2.21). After the gravitational wave burst has passed, the spacetime is flat again. We can therefore make the same arguments as in section 2.2 and arrive at the fact that the only contribution to the four-momenta will come from the 0-th modes. The only non-zero components of the four-momenta are given by

$$p_u = \int_0^{\sigma_1} \partial_\tau v \, \mathrm{d}\sigma = \phi\sigma_1 \tag{4.53}$$

$$p_v = \int_0^{\sigma_1} \partial_\tau u \, \mathrm{d}\sigma = \lambda\sigma_1 \tag{4.54}$$

where we denoted the part of v that is proportional to τ as ϕ. This follows from the fact, that only 0-th k_v mode is not zero when integrated over $\{0, 2\pi\}$ and the constant term vanishes with the τ derivative. The rest energy of the string is then calculated as

$$m = \sigma_1 \sqrt{-2\lambda\partial_\tau v} \tag{4.55}$$

We performed a series of numerical solutions of eq. (4.48) while varying the parameters of frequency ω and amplitude A of the gravitational wave. Furthermore, we calculated the energy before and after the gravitational wave burst and plotted their fraction E/E_0 in fig. 4.2 and fig. 4.3. Note, that we chose the logarithmic scale for E/E_0, because otherwise, only very bright spots would be visible in the whole graph. In this figure, we can see, that there are some resonance frequencies for which the energy recieved by the string is high even for small amplitudes of the gravitational wave. In fact, these resonant frequencies perfectly correspond with fig. 4.1 and eq. (4.32), where we have regions of instability of the solutions. We chose $k = 1$ and $\lambda = 1/\sqrt{2}$. Therefore for frequencies

$$\omega \approx \frac{2\sqrt{2}}{n}, \qquad n \in \mathbb{Z} \tag{4.56}$$

the interaction of gravitational wave with the string affects the string much more than for other frequencies even if it has small amplitude. The rest energy is not the only thing the string acquires from the gravitational wave. The string also gets pushed in the direction of the propagation of the gravitational wave z. The velocity of the centre of mass of the string in the j spatial direction is given by

$$v^j = \frac{p^j}{p^t} \tag{4.57}$$

The velocity in the z direction is plotted for various frequencies ω and amplitudes A of the gravitational wave in fig. 4.4 fig. 4.5. All in all, the string can receive or loose both rest energy and velocity in the direction of the propagation of the gravitational wave during the interaction with the gravitational wave. But for some resonant frequencies, the effect of the gravitational wave is of few orders higher, than the rest.

Also, because we want the reader to get a notion of how the evolution of the string looks like, we add two plots of the motion of the string. One for a resonant frequency in fig. 4.6 and one for non-resonant frequency in fig. 4.7.

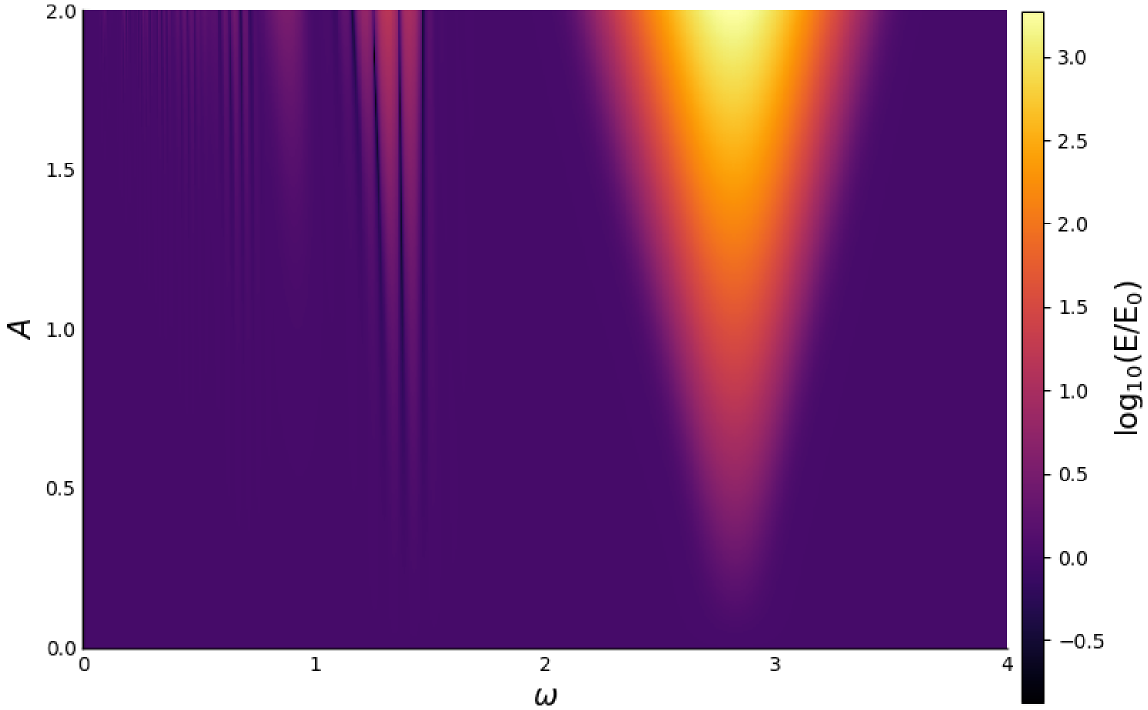

Figure 4.2: Rest energy received or taken away by the gravitational wave burst based on its frequency and amplitude.

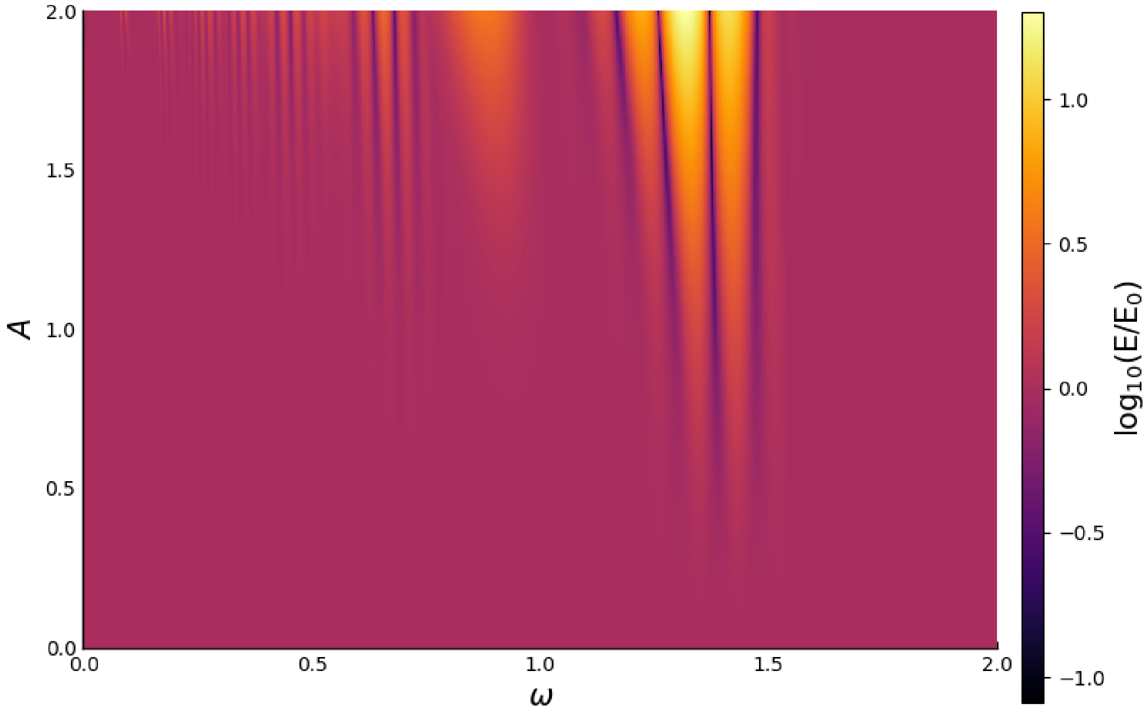

Figure 4.3: Energy received or taken away by the gravitational wave burst based on its frequency and amplitude with detail on smaller frequencies.

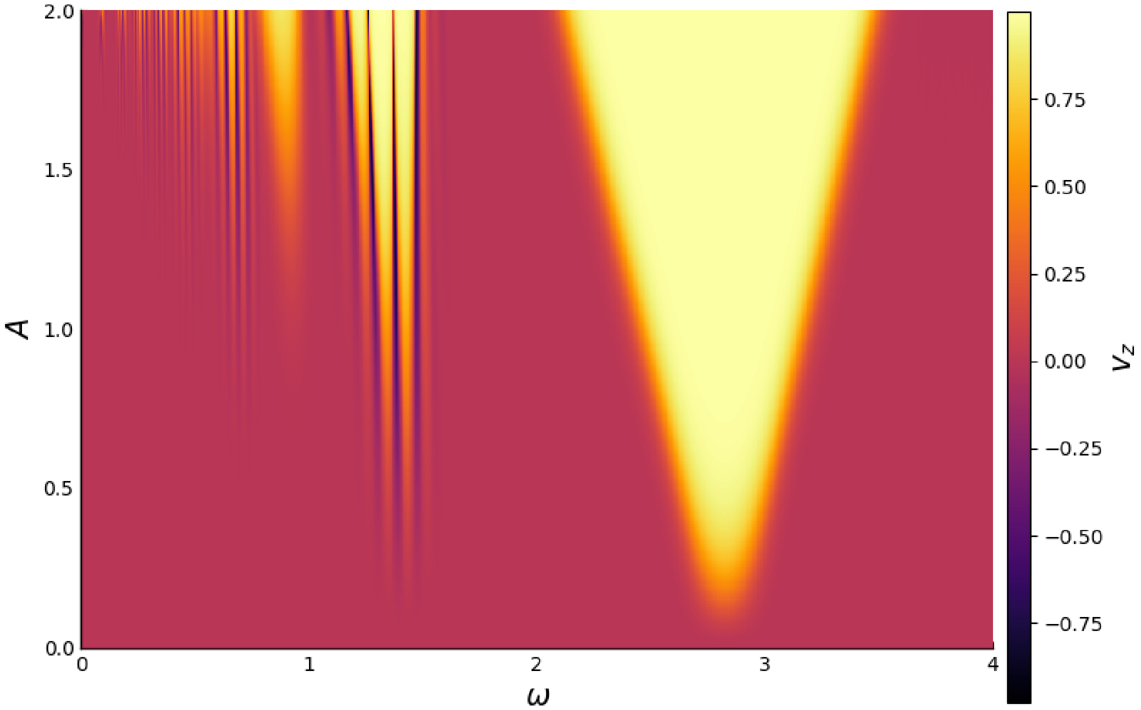

Figure 4.4: Velocity of the string in z direction after the gravitational wave burst with frequency ω and amplitude A.

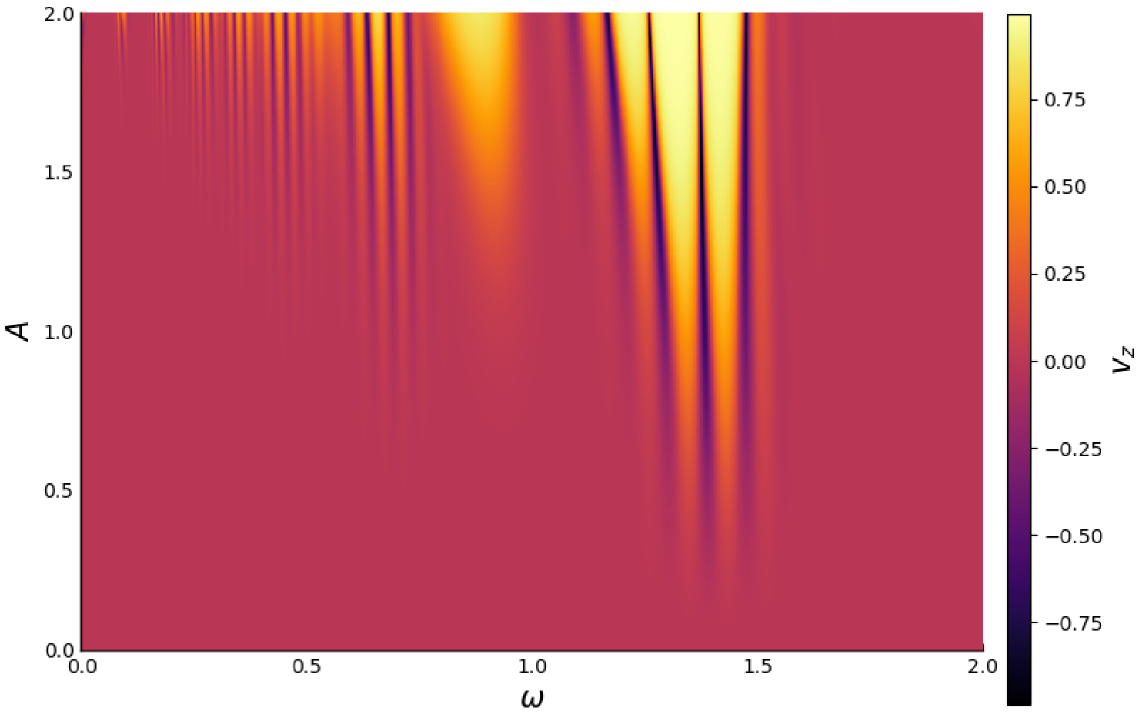

Figure 4.5: Velocity of the string in z direction after the gravitational wave burst with frequency ω and amplitude A with detail on smaller frequencies.

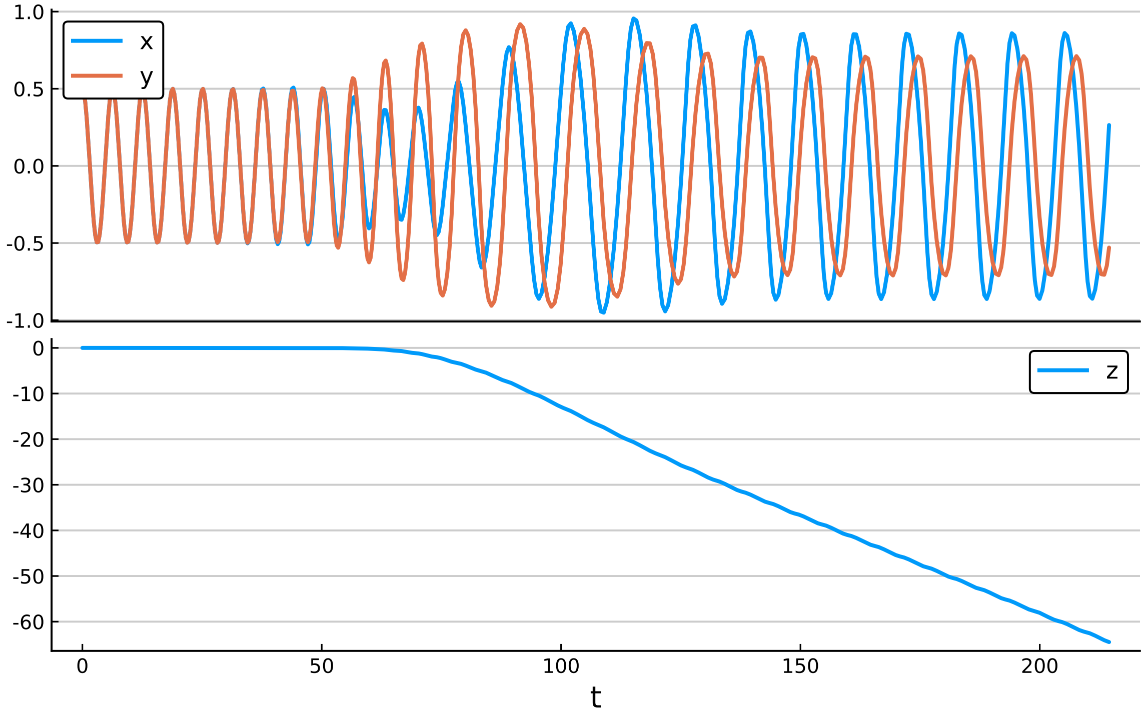

Figure 4.6: The evolution of a string while interacting with the gravitational wave burst, that has resonant frequency $\omega = 2\sqrt{2} - 0.2$ and amplitude $A = 0.5$.

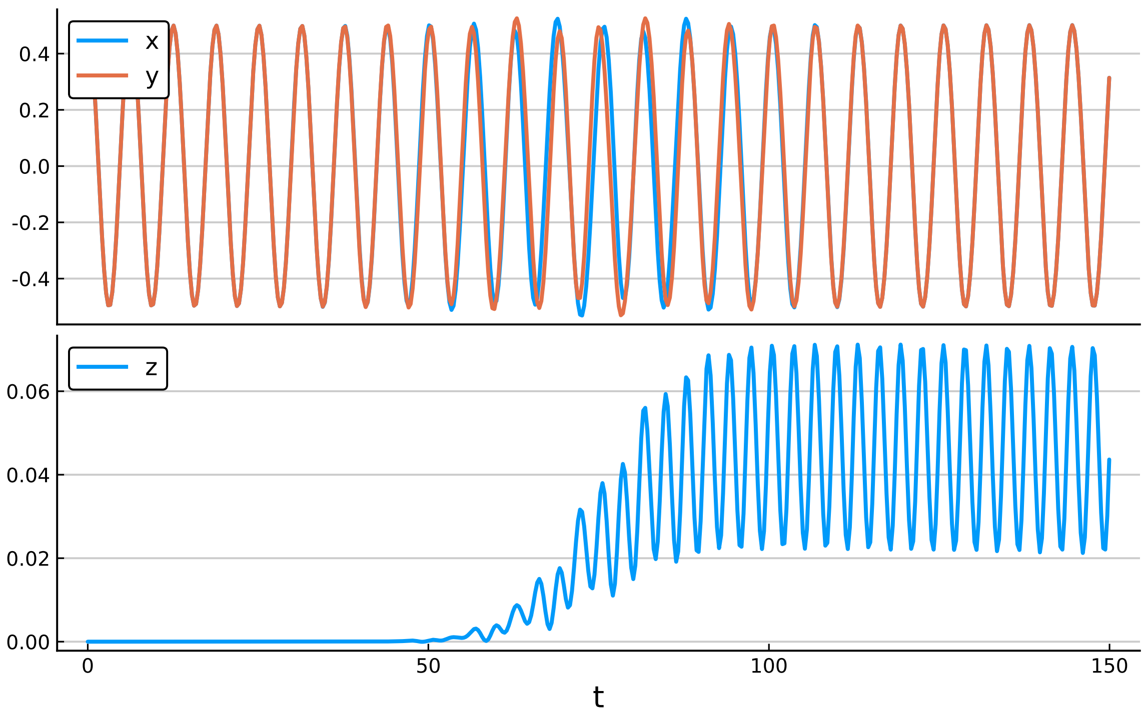

Figure 4.7: The evolution of a string while interacting with the gravitational wave burst, that has non-resonant frequency $\omega = 0.5$ and amplitude $A = 0.5$.

Conclusion

In this thesis our main goal was to study the motion of classical relativistic strings in a curved spacetime, specifically in a constantly expanding universe and in flat spacetime with gravitational wave background. Throughout the way, we also derived very useful tools for solving the equations of motion of a classical string in curved spacetime.

In the first chapter, we explained the difference between solving equations of motion for a particle and solving them for a string. We have intuitively shown how the action looks like and derived the equations of motion together with the boundary conditions.

In the second chapter, we briefly studied a classical string in flat spacetime. We have demonstrated how the choice of parameterization can simplify the equations of motion. Also, we have shown how to calculate the rest energy of a string, which is used in the fourth chapter.

In the third chapter we took a look at circular strings in a constantly expanding universe. We have briefly described why the metric of an expanding universe has this form. Moreover, we used the action principle developed in the first chapter and derived the equations of motion. Because of the complicated nature of these equations, we resorted to different approach such as calculating a potential and using the conservation of energy. This allowed us to find all solutions and describe how the string is affected by this expansion of the universe. We found, that the expansion of the universe has a very small effect on circulars strings with small radii, but can be devastating for large strings with the rest energy greater than some critical energy. In that case, the string expand beyond all limits or until they break.

In the final, fourth chapter, we studied relativistic strings in flat spacetime, but with gravitational wave background. First, we have shown that the used form of the metric is a solution to the vacuum Einstein equations and what it implies on its components. Second, with the right choice of parameterization, we arrived at equations of motion on periodic gravitational wave background. Furthermore, we have rewritten two of these equations into the form of the Mathieu equation and discussed the regions of stability and instability of the solutions. Last but not least, we looked at how the string is affected by a gravitational wave burst. This allowed us to properly evaluate the effect of this burst on an initially circular string. We found, that the effect of gravitational waves on strings is amplified, when the frequency of the string and the gravitational wave are in or close to resonance. The string then receives large amounts of energy and momentum in the direction of propagation of the gravitational wave.

Overall this thesis achieved the goals of finding solutions for strings in various curved spacetimes. It also allowed us to learn about the technical and mathematical difficulties of general relativity and string theory and how to overcome them.

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