# Paper review: Log-det heuristic for matrix rank minimization with applications to Hankel and Euclidean distance matrices

## Brief Summary:

The general Rank Minimization Problem (RMP) has been known to be computationally difficult to solve since the *rank* function for a matrix tends to be discontinuous and non-convex [1]. Unlike previous heuristics developed to handle such problems, this paper proposes a *log-det heuristic* that can handle *rank minimization* for general matrices without the need for a user-specified initial point.

In applying the heuristic to the case of general matrices, the authors introduce a *semi-definite embedding lemma* to map a general (non-square) matrix (X) to a positive semi-definite (PSD) matrix whose rank is shown to be exactly twice the rank of the original non-square matrix. Furthermore, the authors also prove a refined result of the *lemma* in the case where X is chosen to have a *block diagonal structure* and where X is chosen to be *symmetric.*

In defining the *log-det heuristic*, the authors convexify the Rank Minimization Problem (RMP) by first using the function *log det* $(X+\delta I)$ as a *smooth surrogate* for Rank (X) ,where X is assumed to be PSD, and then minimizing this function by representing it as a first order Taylor series expansion. This minimization of the first-order Taylor series expansion of *log det*$(X+\delta I)$ leads to an iterative Trace minimization which is convex in nature and can thus lead to a minimum. The heuristic is then extended to the general matrix Rank minimization case by first using *lemma 1* to convert the general matrix to a PSD matrix and then applying the *log-det heuristic.* When this heuristic is applied to a special case where the matrix X is diagonal, the derivation leads to an iterative $l_1$ *norm* minimization problem.

The authors also prove the application of the heuristic in signal processing to design a minimum order system with time-domain constraints and by finding the lowest dimension embedding of points in a Euclidean space.

## Paper Contributions:

The main contributions of the paper are threefold:

1. *The semi-embedding lemma (Lemma 1):* The authors prove that a given non-square and non-PSD matrix X can be associated to a PSD matrix whose rank is twice the rank of the original matrix X.
2. *The log-det heuristic*: This is the most important contribution and helps convert the computationally difficult RMP into a convex iterative Trace minimization problem that can be solved within a few iterations
3. *Expression of problems from different fields as RMPs*: In providing the applications of the heuristic, the authors successfully express problems involving Hankel and Euclidean Distance matrices into RMPs. Furthermore, through numerical examples, the authors are also able to show that the heuristic can help obtain a minimal solution within 4-6 iterations.

## Additional Comments:

While explaining the semidefinite embedding lemma, the authors prove how the RMP for a general matrix can be converted to an RMP for a PSD matrix in *eq (3)*. However, when using this *lemma* for applying the *log-det heuristic* to the general case (*eq-(8)*), the authors ignore the '$\frac{1}{2}$' factor from *eq (3)* making the application of *lemma 1* unclear. Also, since the Trace function is a convex envelope of the Rank function over the set of matrices with norm less than 1 [2], a minimum found over the Trace function may be lower than the optimal minimum in some cases.

# References

[1] J. M. X Shen, “A penalty method for rank minimization problems in symmetric matrices,” *Computational Optimization and Applications* , vol. 71, no. 2, pp. 353-380, 2018.

[2] H. H. S. B. M Fazel, “A rank minimization heuristic with application to minimum order system approximation,” *Proceedings of the 2001 American Control Conference. (Cat. No.01CH37148),* vol. 6, pp. 4734-4739, 2001.