Università degli Studi di Milano

Department of Mathematics “Federigo Enriques"
Doctoral School in Mathematical Sciences
Ph.D. in Mathematics and Statistic for the Computational Sciences

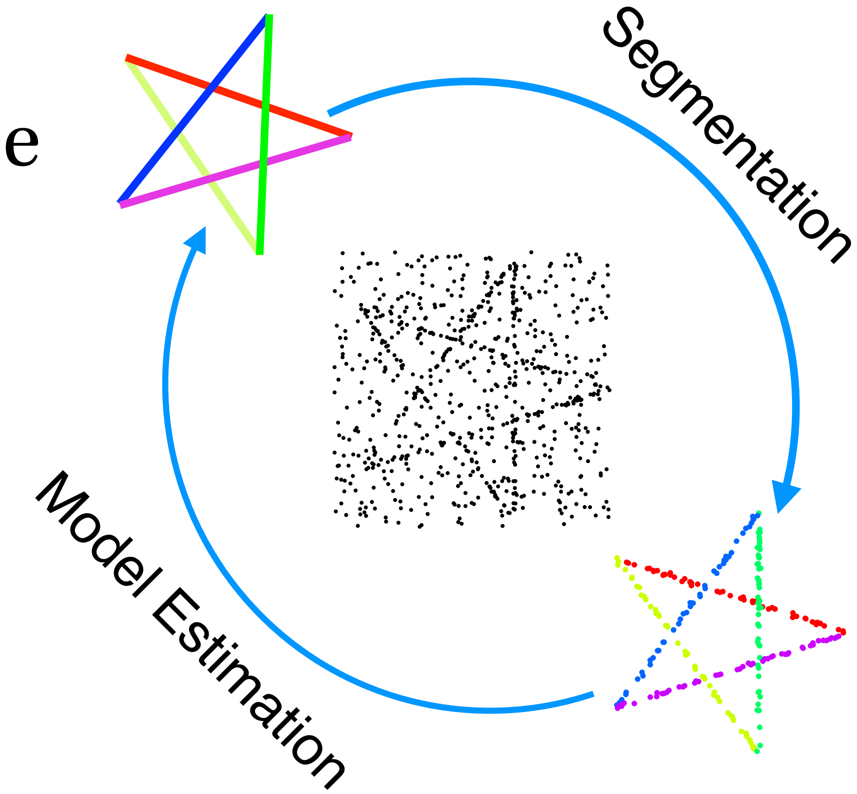
Multiple structures recovery via preference analysis in conceptual space

Luca Magri

Advisor: Prof. Andrea Fusiello
Coordinator: Prof. Giovanni Naldi
XVII cycle, 2015

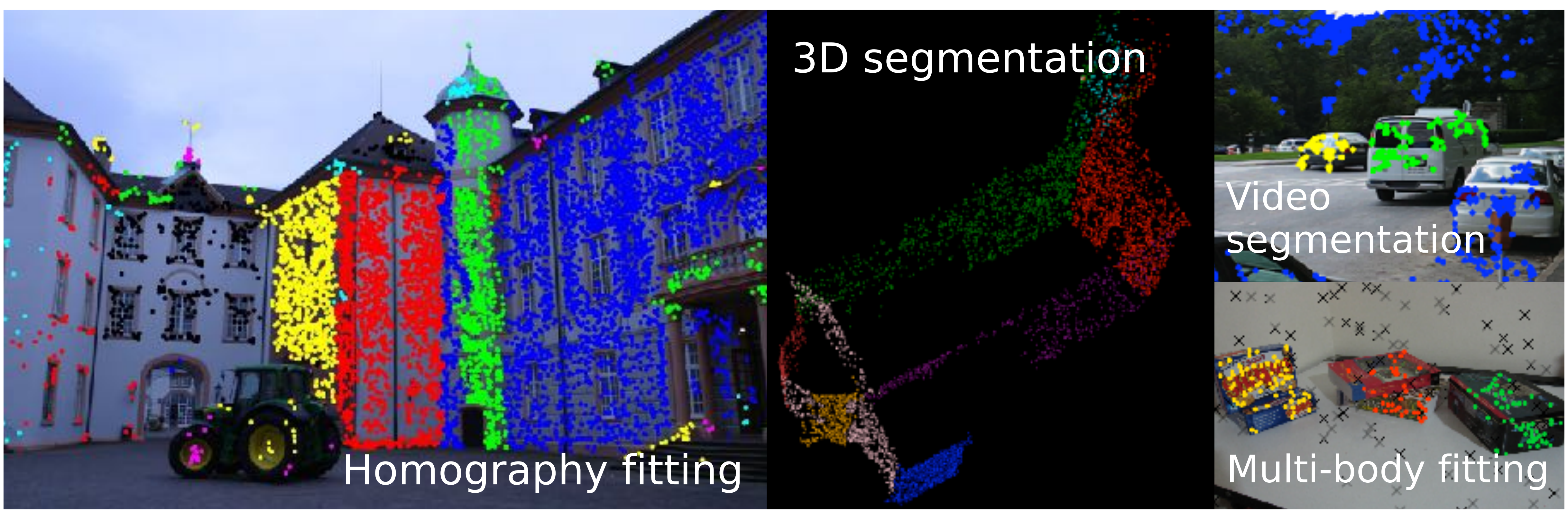
The challenges of multiple structure estimation

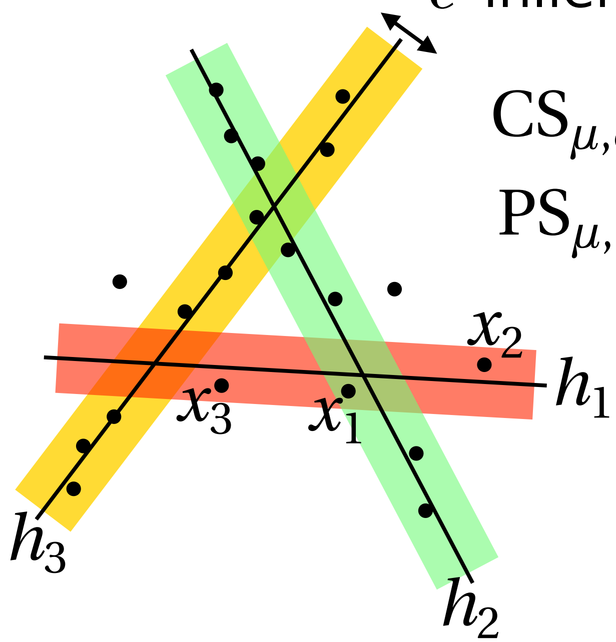
Fitting multiple instances of a mathematical model – also called "structures" – to measured data, which is invariably contaminated by noise and outliers.



Challenges:

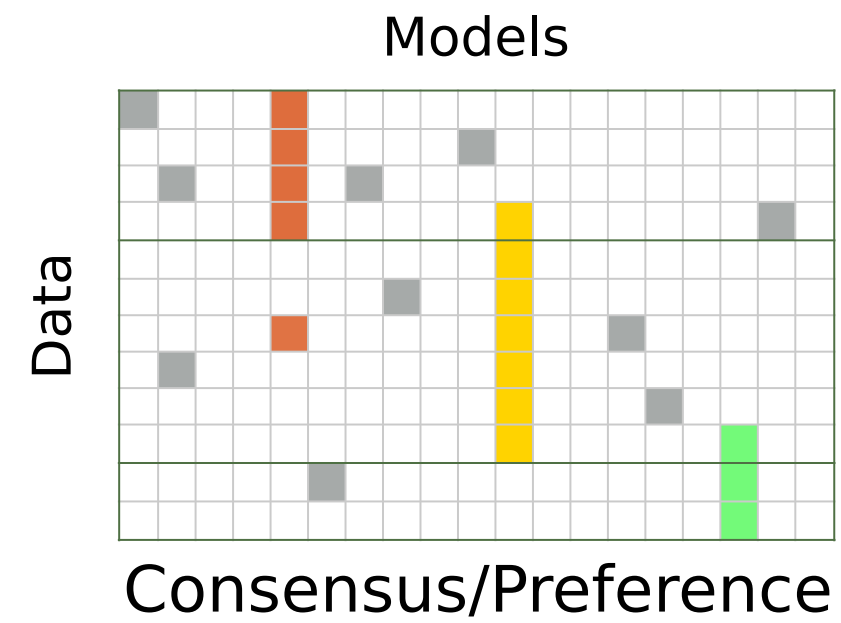
- outliers
- pseudo-outliers
- chicken-&-egg-dilemma
- ill-posed





$\mathrm{CS}_{\mu,\epsilon}(h_1) = \{x_1, x_2, x_3\}$ consensus

$\mathrm{PS}_{\mu,\epsilon}(x_1) = \{h_1, h_2\}$ preference



Consensus Analysis

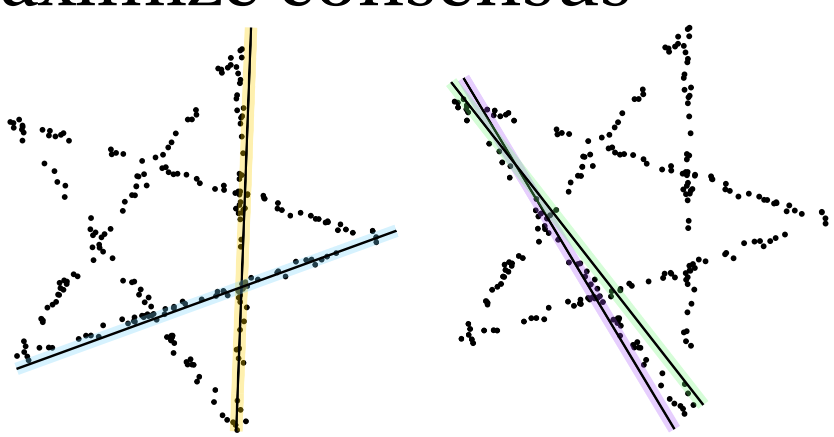
- Ransac & variants
- Sequential Ransac
- Multi-Ransac
- Hough Transform

1. represent models (disjoint)
2. voting
3.maximize consensus

Preference Analysis

- Residual Histogram Analysis
- J-Linkage
- Kernel Methods
- Higher order clustering

1. voting
2. represent points
3. clustering



Disambiguate between redundant models and genuine ones

Outline and contributions

Preference space

- integrate M-estimators
- density analysis

L. Magri and A. Fusiello. Density based analysis in Tanimoto space for multi-model fitting. In International Conf. on Image Analysis and Processing, 2015.

Linkage

- T-Linkage
- Automatic threshold

L.Magri and A. Fusiello. T-Linkage: a continuous relaxation of J-Linkage for multi model fitting. In *Conf. on Computer Vision and Pattern Recognition*, pp. 3954–3961, 2014.

L. Magri and A. Fusiello. Scale estimation in multiple models fitting via Consensus Clustering. In *International Conf. on Computer Analysis of Images and Patterns*, 2015.

Spectral

- Low Rank & sparse
- break the chicken & egg dilemma

L.Magri and A.Fusiello. Robust Multiple Model Fitting with Preference Analysis and Low-rank Approximation. In *British Machine Vision Conf.*, 2015.

Coverage

- well founded
- intersecting models

Outline and contributions

Preference space

- integrate M-estimators
- density analysis

Linkage

- T-Linkage
- Automatic threshold

Spectral

- Low Rank & sparse
- break the chicken & egg dilemma

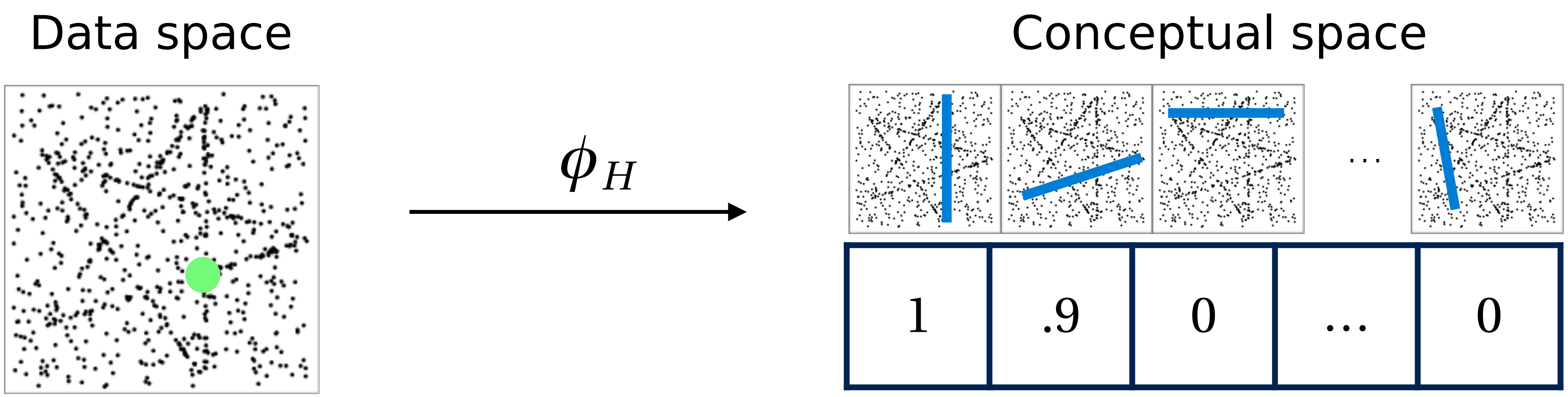
Coverage

- well founded
- intersecting models

A lift to Tanimoto Space

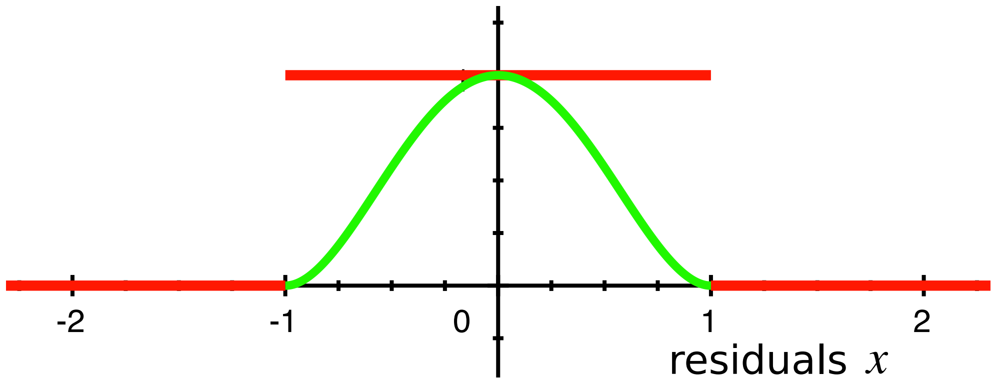
Unitary m-dimensional cube endowed with the Tanimoto distances:

$$d_{\mathcal{T}}(p,q) = 1 - \frac{\langle p,q\rangle}{\|p\|^2 + \|q\|^2 - \langle p,q\rangle}$$



Use M-Estimators w_c to depict preferences of points, e.g.

$$w_{\text{step}}(u) = \begin{cases} 1 & \text{if } |u| \le 1 \\ 0 & \text{otherwise,} \end{cases} \quad w_{\text{tukey}}(u) = \begin{cases} (1-u^2)^2 & \text{if } |u| \le 1 \\ 0 & \text{otherwise.} \end{cases}$$

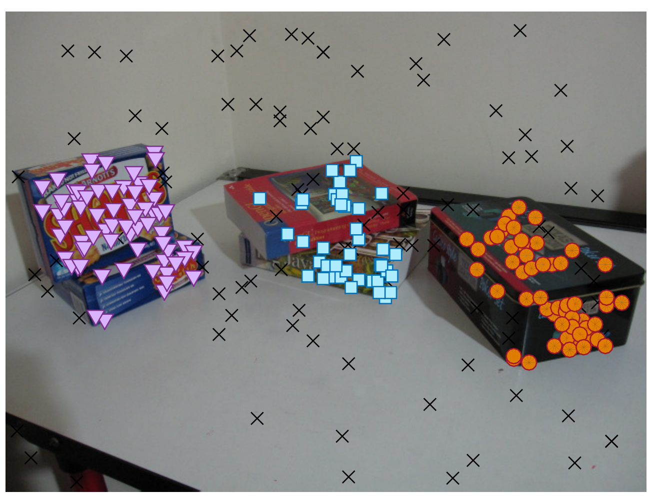


By defining a map $\phi : X \times H \to [0,1]^m$ such that:

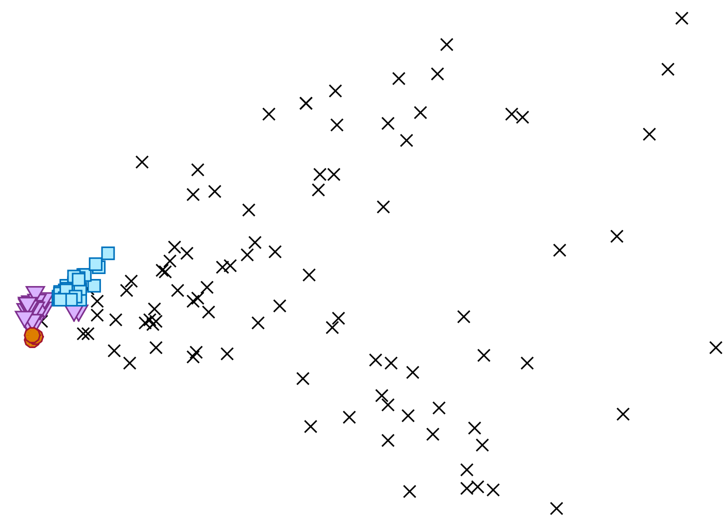
$$\phi(x_i, h_j) = w_c\left(\frac{\text{err}_\mu(x_i, h_j)}{\tau\sigma_n}\right)$$

Density Analysis

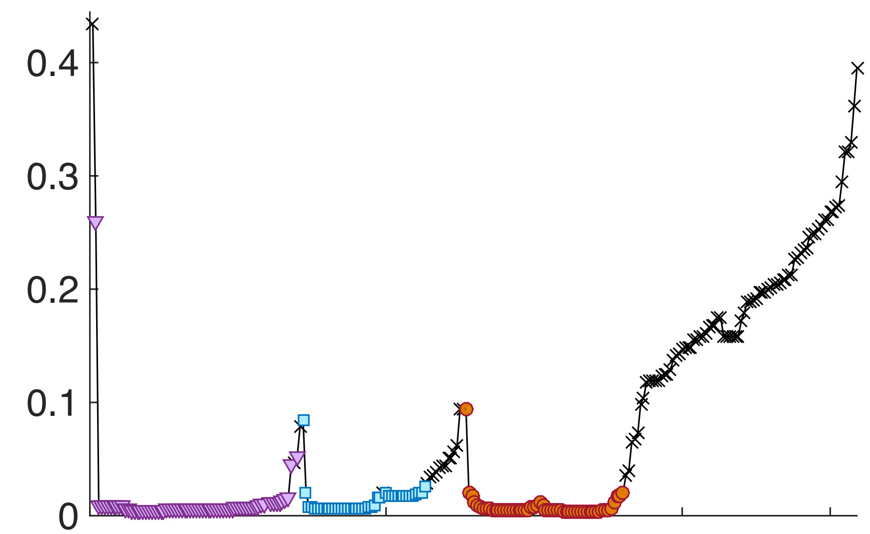
A geometric analysis of the Tanimoto space can confirm the intuition that points sharing the same preference are grouped together in the conceptual space.



Data

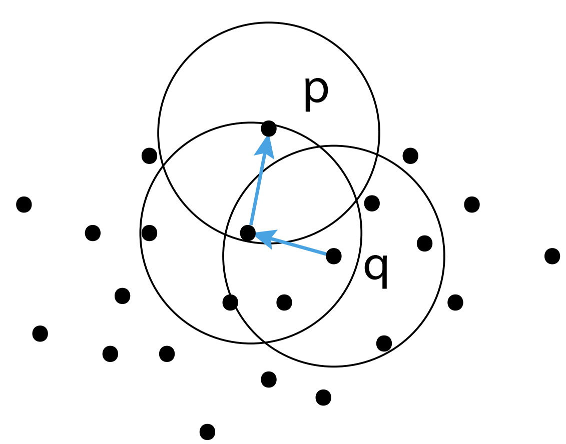


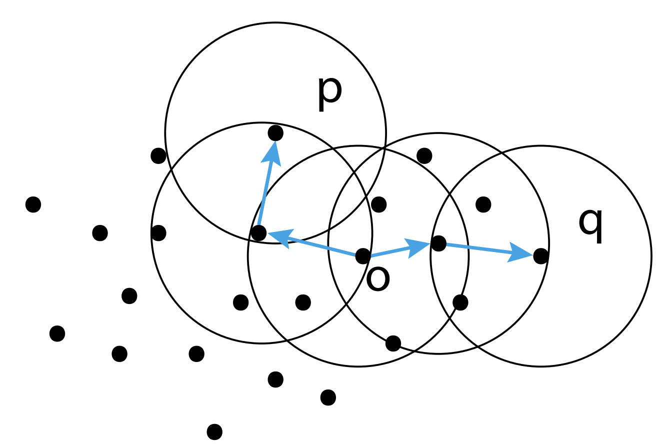
MDS of the Tanimoto space



Reachability plot

Reachability plot: multiscale approach that frames the local density of points in a special kind of dendrogram.





Outline and contributions

Preference space

- integrate M-estimators
- density analysis

Linkage

- T-Linkage
- Automatic threshold

Spectral

- Low Rank & sparse
- break the chicken & egg dilemma

Coverage

- well founded
- intersecting models

Outline and contributions

Preference space

- integrate M-estimators
- density analysis

Linkage

- T-Linkage
- Automatic threshold

Spectral

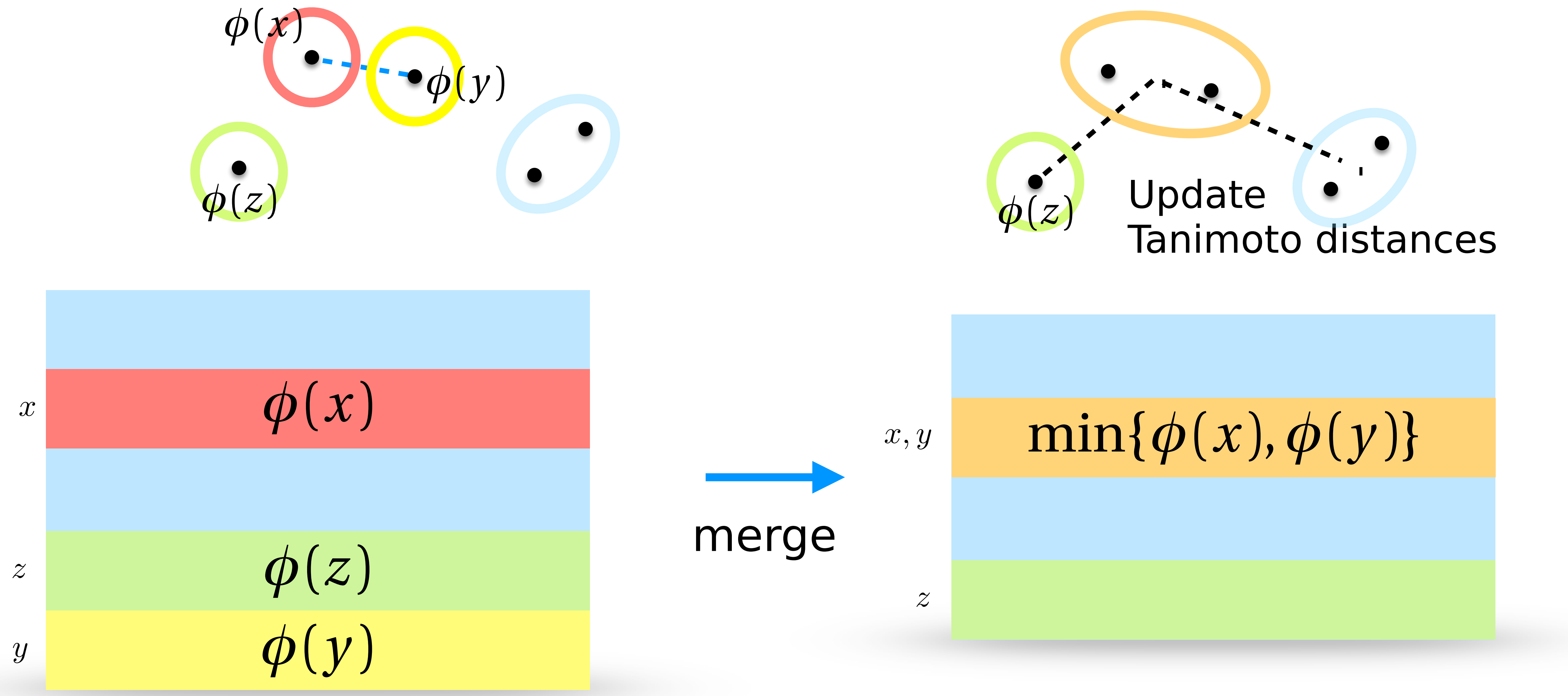
- Low Rank & sparse
- break the chicken & egg dilemma

Coverage

- well founded
- intersecting models

Linkage formulation

- Hierarchical clustering in the Tanimoto space (average linkage style)
- Preferences of clusters are merged until they are orthogonal
- The number of clusters is automatically detected



Linkage formulation

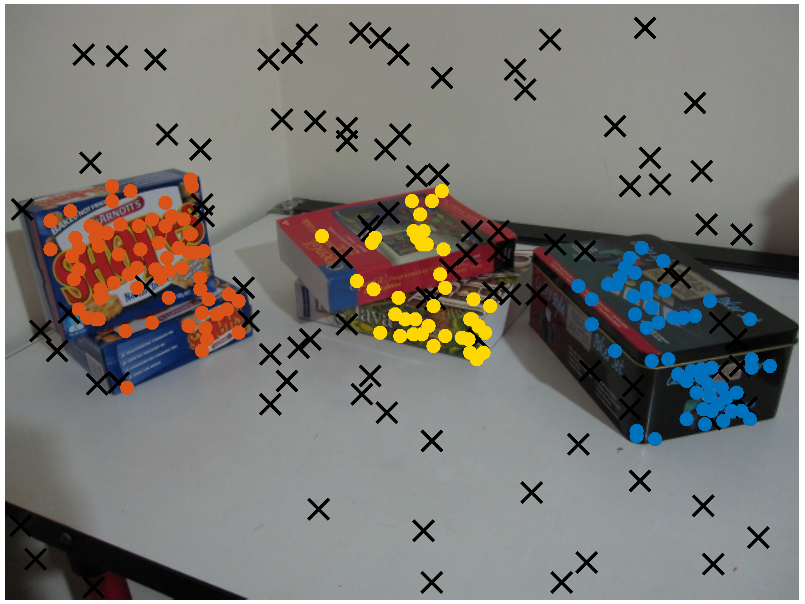
Outliers tend to emerge as micro-clusters

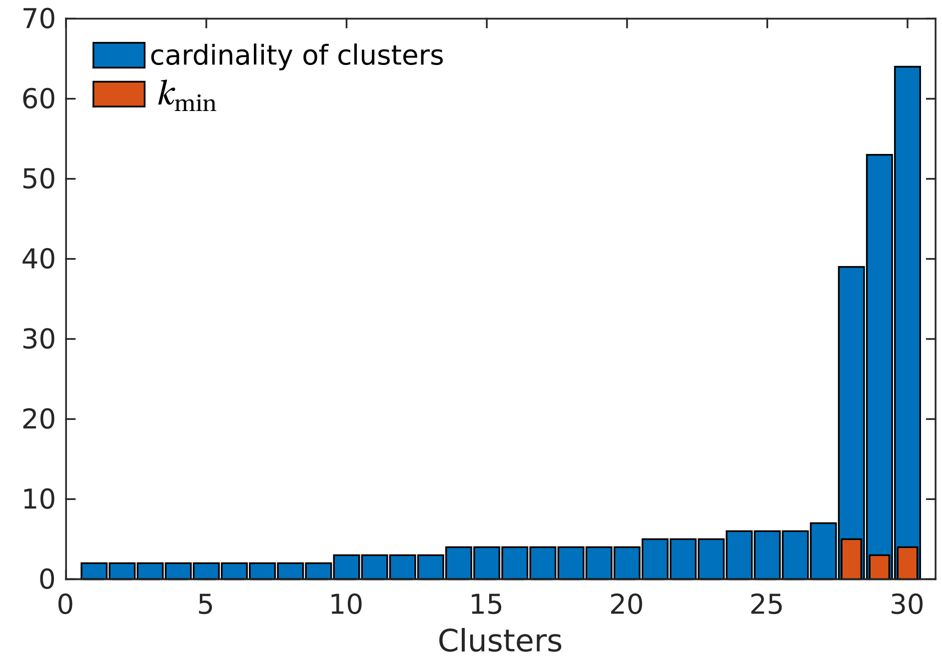
For each model:

- compute the cardinality of its consensus set
- compute the probability p that an outlier falls in its inlier band
- compute the probability that k outliers fall in its inlier band as

$$\alpha(k) = 1 - F(n, k, p)$$

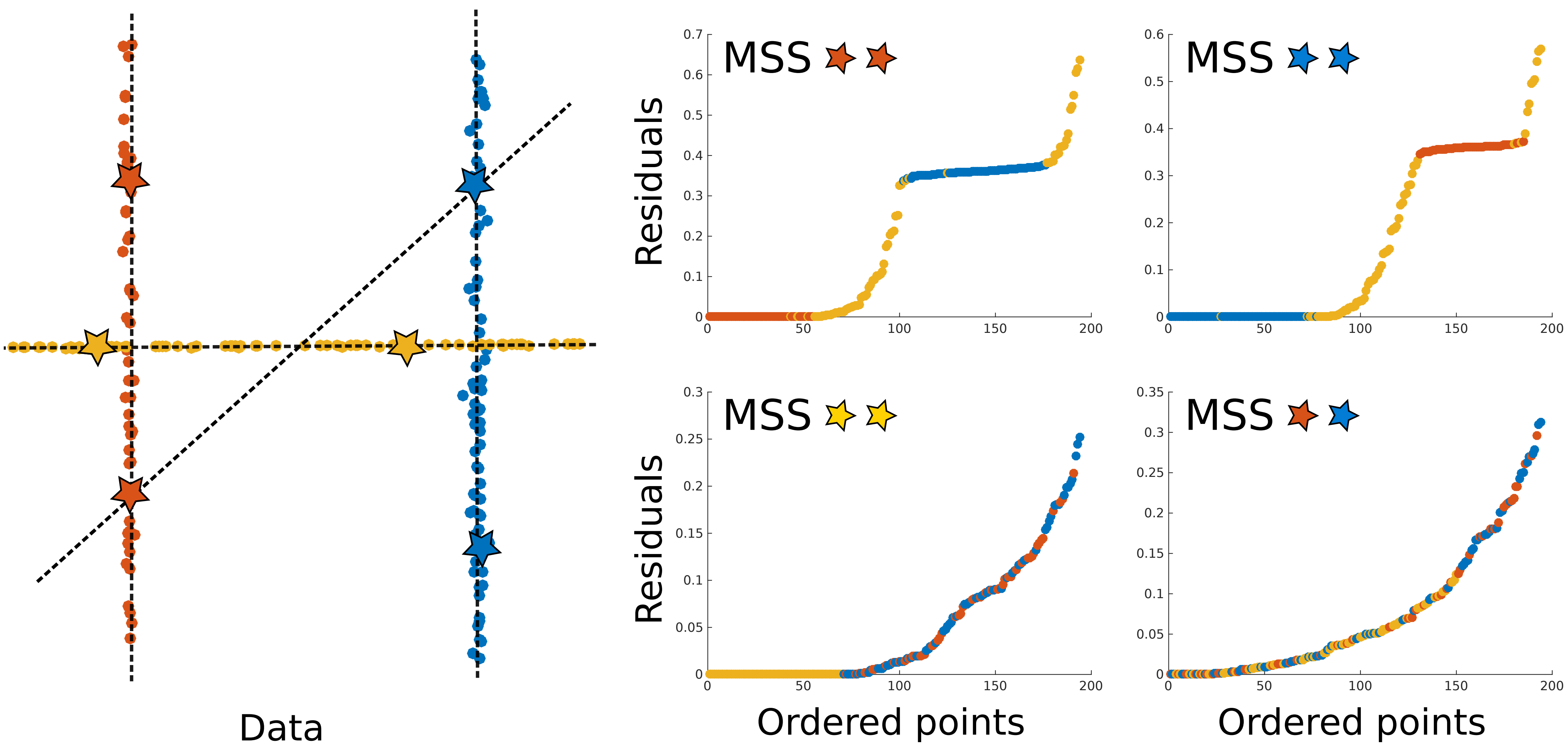
- compute $\alpha^{-1}(0.01)$ the minimum cardinality necessary to be not considered mere coincidence
- if the considered model is supported by less points it is rejected as outlier.





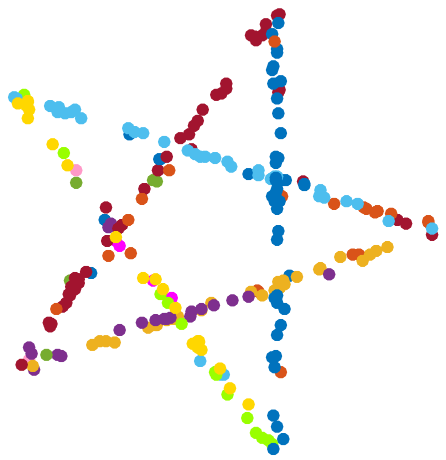
Scale estimation

Tuning the inlier threshold separately per each tentative model is not a viable solution.

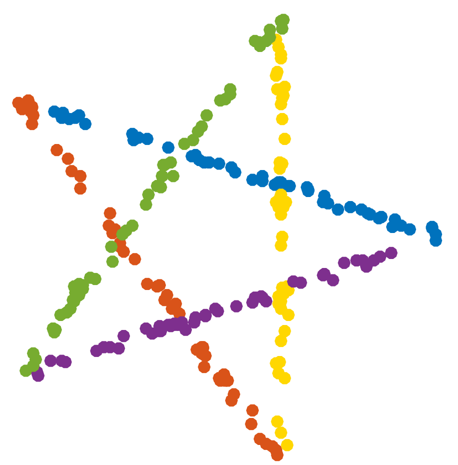


Scale estimation by Consensus Clustering

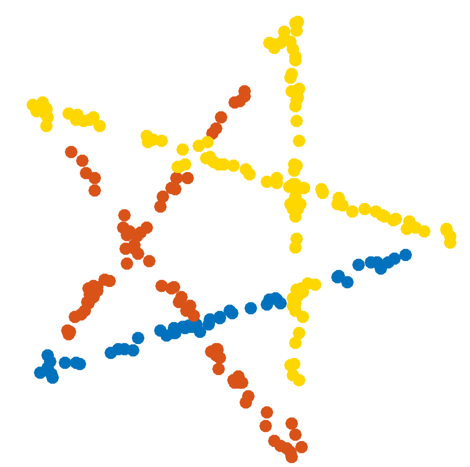
Under small perturbations of the input (resampling of the hypothesis space, equivalent of adding noise in the conceptual space), perform multiple T-linkage and use cluster stability to select the correct scale



Unstable output



Stable output



Unstable output

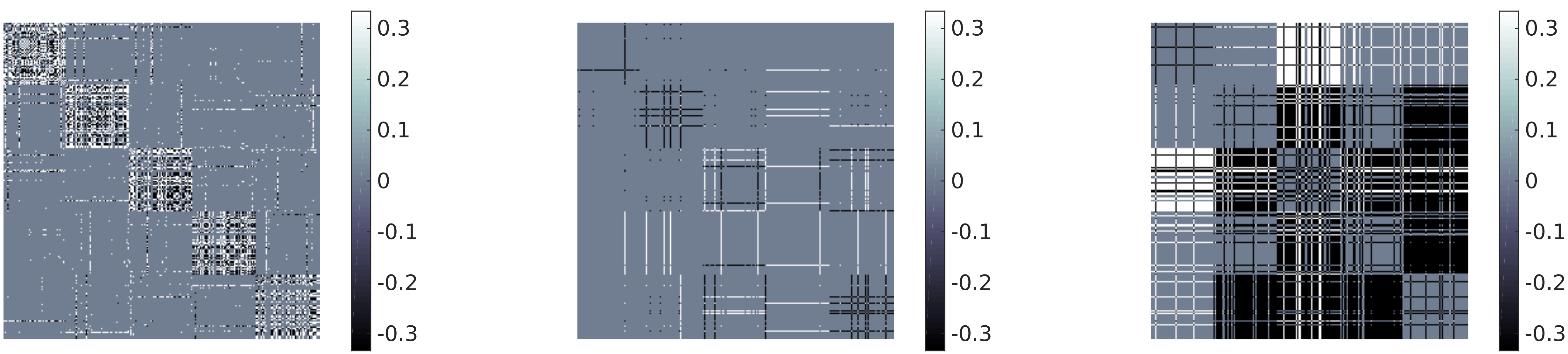
Scale estimation by Consensus Clustering

To measure stability: A matrix is defined as follows: the element stores the number of times points i and j are assigned to the same cluster divided by the total number of times the clustering is executed (4 times in our exp)

Entries are then mapped from [0,1] to [.5,.5] via:

$$F(x) = \begin{cases} x & \text{if } x < 0.5 \\ x - 1 & \text{if } x \geq 0.5. \end{cases}$$

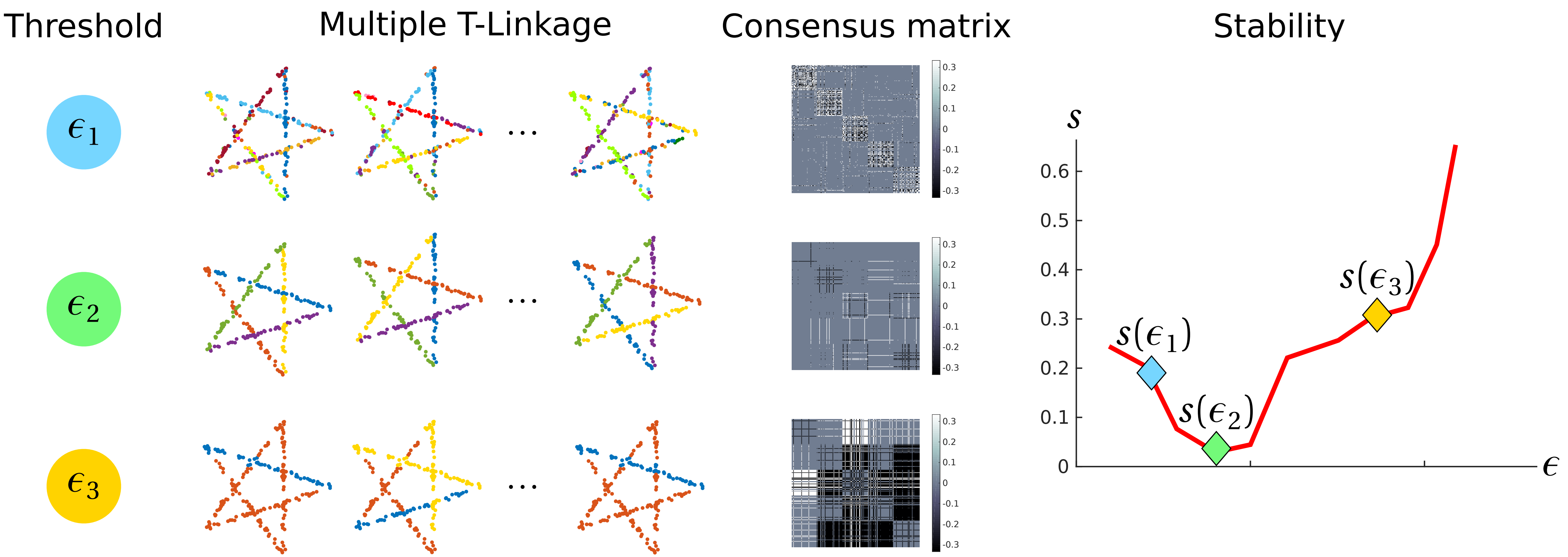
In this way the null matrix corresponds to perfect consensus.



Scale estimation by Consensus Clustering

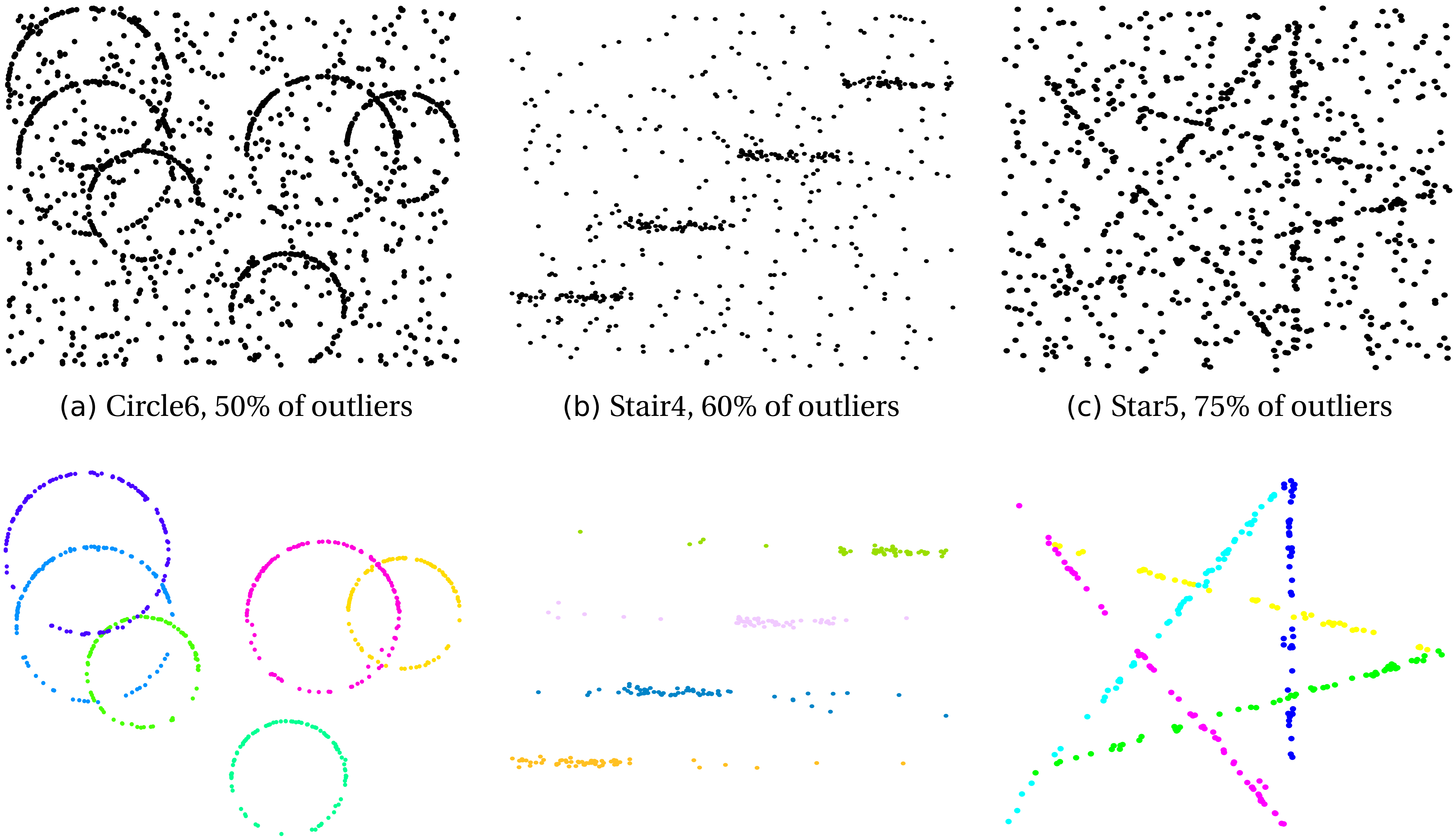
The information encapsulated in the consensus matrix is captured by a stability index:

$$s(\epsilon) = \mathrm{Var}(\mathrm{vech}(F(M_\epsilon))),$$



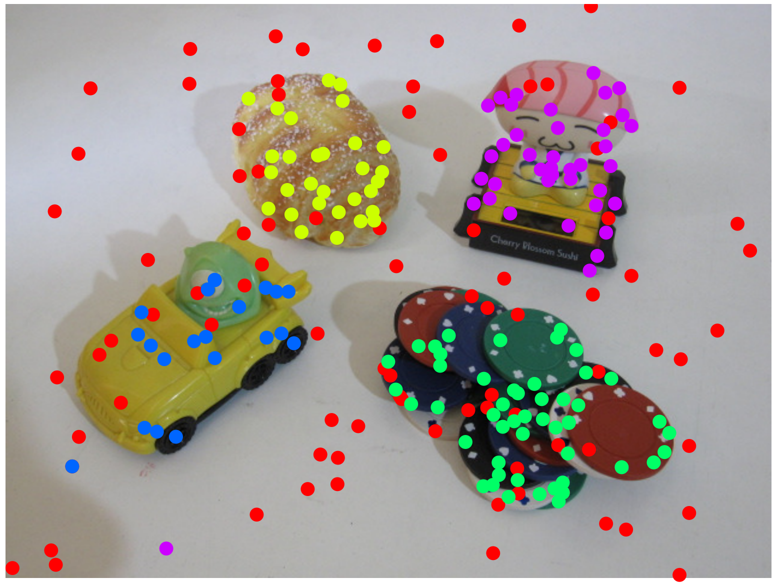
Minimizing the stability index yields the correct threshold.

Experiments

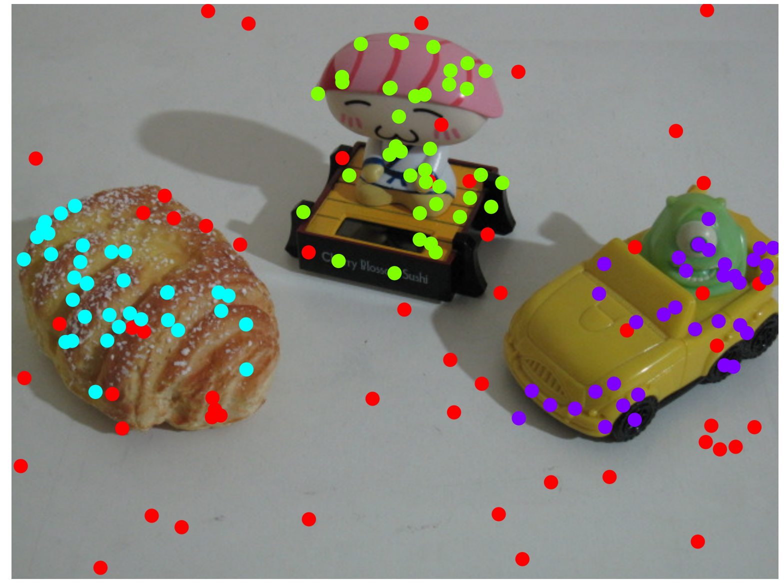


(a) Circle6, 50% of outliers
(b) Stair4, 60% of outliers
(c) Star5, 75% of outliers
(d) Estimated models
(e) Estimated models
(f) Estimated models

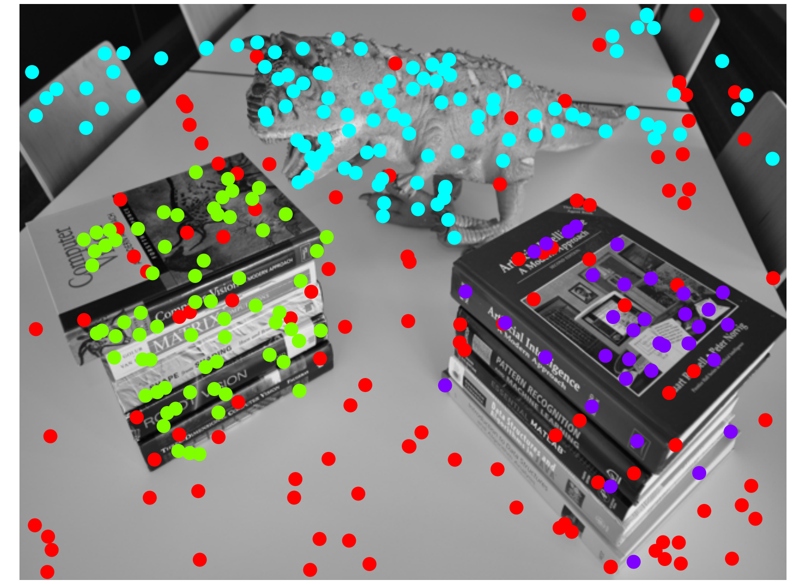
Experiments



(a) breadcartoychips



(b) breadtoycar



(c) dinobooks

	Two term model selection					Stability	
	PEARL	QP-MF	FLOSS	ARJMC	SA-RCM	TLCC	T-Link*
biscuitbookbox	4.25	9.27	8.88	8.49	7.04	**2.71**	0.39
breadcartoychips	5.91	10.55	11.81	10.97	**4.81**	5.19	5.19
breadcubechips	4.78	9.13	10.00	7.83	7.85	**2.17**	2.17
breadtoycar	6.63	11.45	10.84	9.64	**3.82**	4.27	4.27
carchipscube	11.82	7.58	11.52	11.82	11.75	**1.22**	1.22
cubebreadtoychips	4.89	9.79	11.47	6.42	5.93	**4.46**	3.50
dinobooks	14.72	19.44	17.64	18.61	**8.03**	13.86	13.86
toycubecar	9.5	12.5	11.25	15.5	7.32	**3.03**	3.03
Mean	7.81	11.21	11.68	11.16	7.07	**4.62**	

Outline and contributions

Preference space

- integrate M-estimators
- density analysis

Linkage

- T-Linkage
- Automatic threshold

Spectral

- Low Rank & sparse
- break the chicken & egg dilemma

Coverage

- well founded
- intersecting models

Outline and contributions

Preference space

- integrate M-estimators
- density analysis

Linkage

- T-Linkage
- Automatic threshold

Spectral

- Low Rank & sparse
- break the chicken & egg dilemma

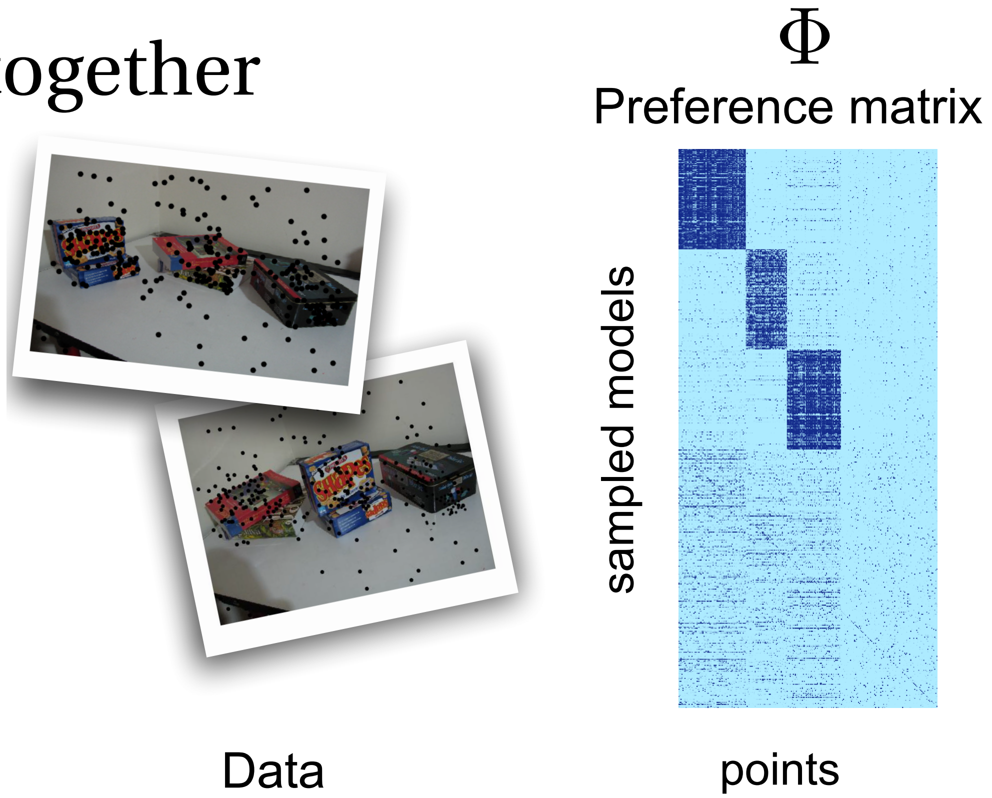
Coverage

- well founded
- intersecting models

Robust preference analysis

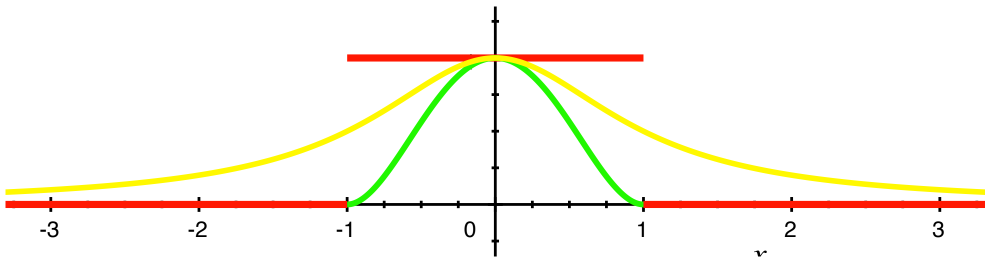
We want to integrate the preference approach together with consensus considerations.

The main intuition is that the preference trick can be seen as a method to translate a general multi-model fitting problem in a subspace clustering task



Cauchy weighting function is used to depict points preferences (no cutoff):

$$\phi_H(x) = \left[\frac{1}{1+\left(\frac{\mathrm{err}_\mu(x,h_1)}{\tau\sigma_n}\right)^2}, \ldots, \frac{1}{1+\left(\frac{\mathrm{err}_\mu(x,h_m)}{\tau\sigma_n}\right)^2} \right]$$



Tanimoto distances are "kernelized":

$$K(i,j) = \exp\left(-\tau(i,j)^2\right) \quad \text{where} \quad \tau(i,j) = 1 - \frac{\left\langle P_i^\top, P_j^\top \right\rangle}{\|P_i^\top\|^2 + \|P_j^\top\|^2 - \left\langle P_i^\top, P_j^\top \right\rangle}.$$

Robust preference analysis

Partitioning the data in k segments starting from the positive semi-definite affinity matrix K is equivalent to approximating K in a least square sense by means of an ideal affinity co-membership matrix F.

Let $F = UU^\top$

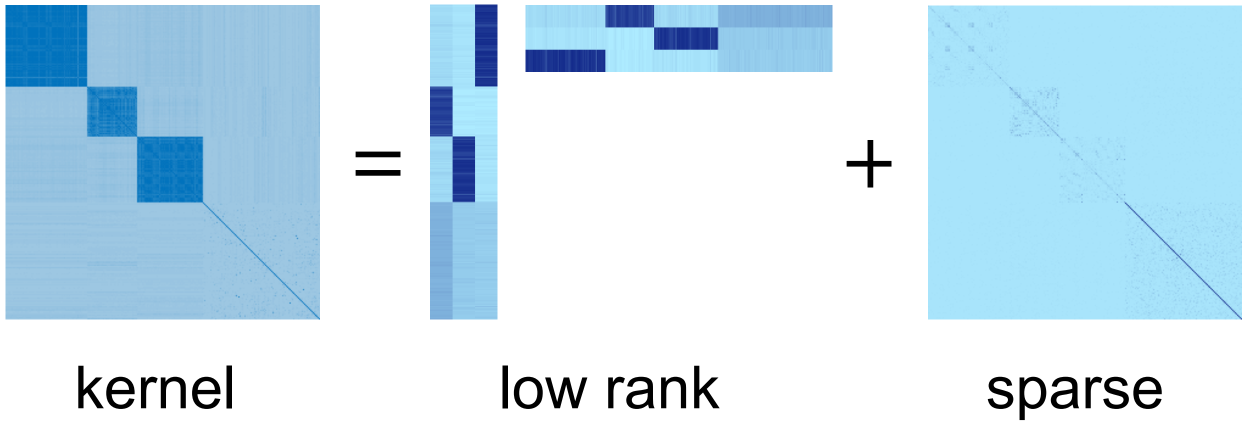
$$
\begin{aligned}
& \min \|K - UU^\top\|_F^2 \\
\Longleftrightarrow\ & \min \operatorname{trace}[(K - UU^\top)^\top (K - UU^\top)] \\
\Longleftrightarrow\ & \min \operatorname{trace}(K^\top K) - 2\operatorname{trace}(U^\top K U) + \operatorname{trace}(I) \\
\Longleftrightarrow\ & \max \operatorname{trace}(U^\top K U).
\end{aligned}
$$

(**SymNMF**)

Since outliers are present in the data,
only the low rank part of K is considered

$$\arg\min \|L\|_* + \lambda \|S\|_1 \qquad \text{s.t} \qquad K = L + S$$

(**Robust PCA**)

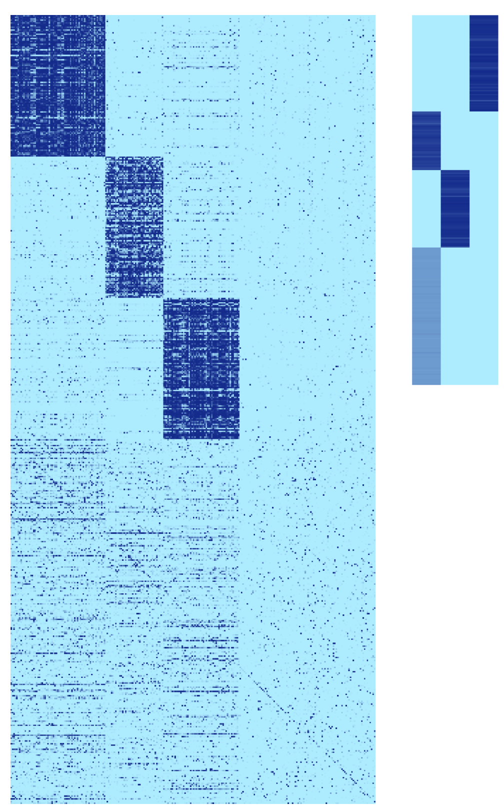


Robust preference analysis: consensus into play

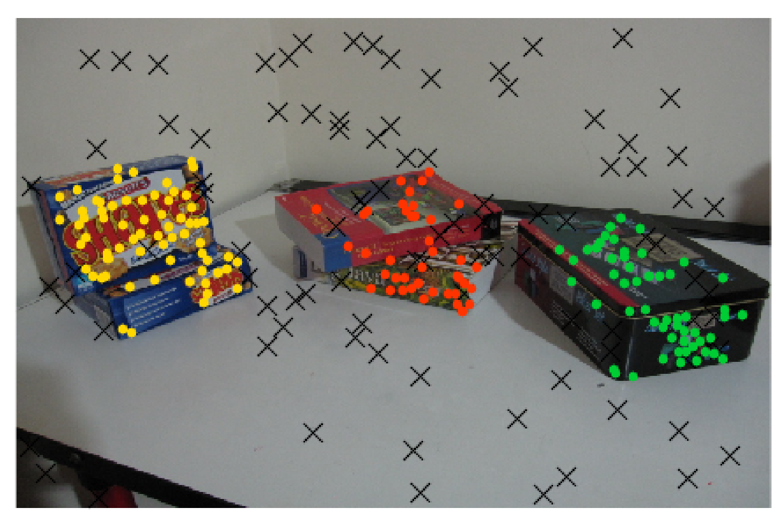
Let us first observe that $P^\top \mathbf{1}$ (where $\mathbf{1}$ is a vector of ones) is the sum of the preference vectors of all the points in the preference matrix, so its entries are the votes obtained by each model.

Hence finding the maximal entry of $P^\top \mathbf{1}$ is equivalent to doing a sort of RANSAC with the Cauchy weighting function.

Along the same line: $\max\limits_{\text{cols}}(\Phi(U \odot B))$ = MSS with higher consensus



final segmentation (refined by **robust statistic**)

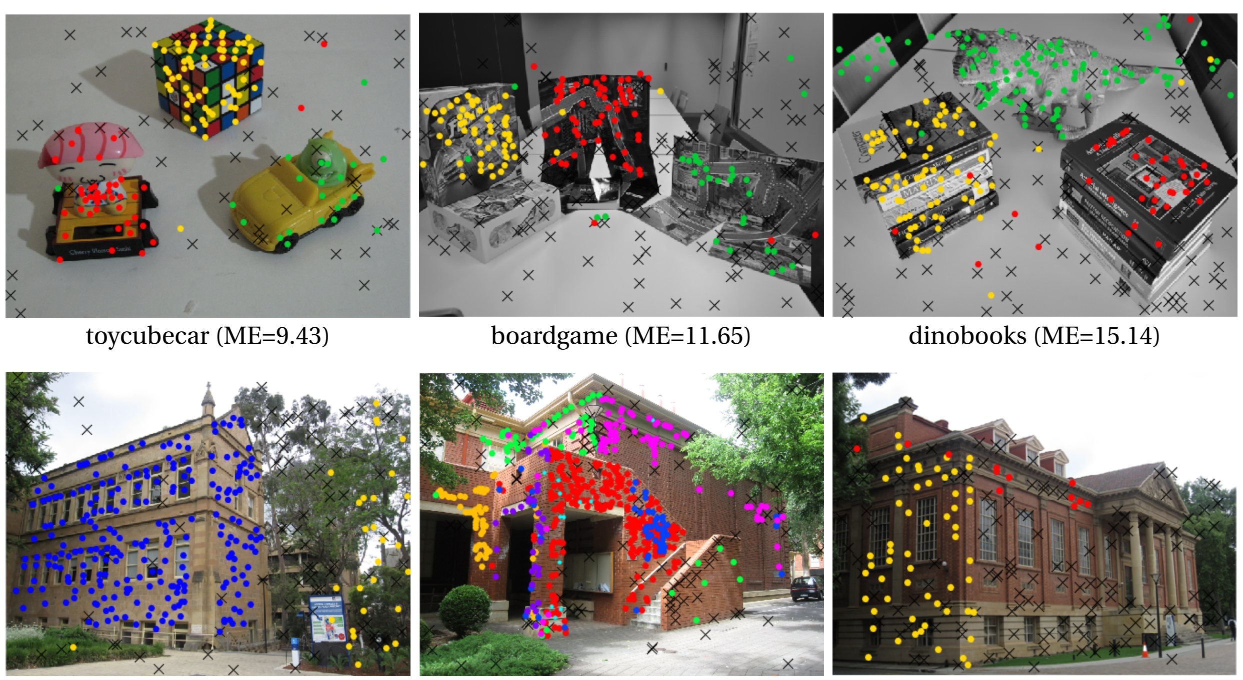


Robust preference analysis

- three levels of protection against outliers:
  1. vote with M-estimators
  2. low rank decomposition
  3. robust statistic on individual models

- use soft descenders: the inlier threshold is a more educated guess

- disentangle the chicken and egg dilemma



toycubecar (ME=9.43) | boardgame (ME=11.65) | dinobooks (ME=15.14)

oldclassicswing (ME=25.25) | jhonsonb (ME=26.76) | library (ME=31.29)

	κ	%out	T-lnkg	RCMSA	RPA
biscuitbookbox	3	37.21	**3.10**	16.92	3.88
breadcartoychips	4	35.20	14.29	25.69	**7.50**
breadcubechips	3	35.22	**3.48**	8.12	5.07
breadtoycar	3	34.15	9.15	18.29	**7.52**
carchipscube	3	36.59	**4.27**	18.90	6.50
cubebreadtoychips	4	28.03	9.24	13.27	**4.99**
dinobooks	3	44.54	20.94	23.50	**15.14**
toycubecar	3	36.36	15.66	13.81	**9.43**
biscuit	1	57.68	16.93	14.00	**1.15**
biscuitbook	2	47.51	**3.23**	8.41	**3.23**
boardgame	1	42.48	21.43	19.80	**11.65**
book	1	44.32	3.24	4.32	**2.88**
breadcube	2	32.19	19.31	9.87	**4.58**
breadtoy	2	37.41	5.40	3.96	**2.76**
cube	1	69.49	7.80	8.14	**3.28**
cubetoy	2	41.42	**3.77**	5.86	4.04
game	1	73.48	**1.30**	5.07	3.62
gamebiscuit	2	51.54	9.26	9.37	**2.57**
cubechips	2	51.62	6.14	7.70	**4.57**
mean			9.36	12.37	**5.49**
median			7.80	9.87	**4.57**

	κ	%out	T-lnkg	RCMSA	RPA
unionhouse	5	18.78	48.99	**2.64**	10.87
bonython	1	75.13	**11.92**	17.79	15.89
physics	1	46.60	29.13	48.87	**0.00**
elderhalla	2	60.75	10.75	29.28	**0.93**
ladysymon	2	33.48	**24.67**	39.50	**24.67**
library	2	56.13	**24.53**	40.72	31.29
nese	2	30.29	7.05	46.34	**0.83**
sene	2	44.49	7.63	20.20	**0.42**
napiera	2	64.73	28.08	31.16	**9.25**
hartley	2	62.22	21.90	37.78	**17.78**
oldclassicswing	2	32.23	**20.66**	21.30	25.25
barrsmith	2	69.79	49.79	**20.14**	36.31
neem	3	37.83	25.65	41.45	**19.86**
elderhallb	3	49.80	31.02	35.78	**17.82**
napierb	3	37.13	**13.50**	29.40	31.22
johnsona	4	21.25	34.28	36.73	**10.76**
johnsonb	7	12.02	24.04	**16.46**	26.76
unihouse	5	18.78	33.13	**2.56**	5.21
bonhall	6	6.43	21.84	**19.69**	41.67
mean			24.66	28.30	**17.20**
median			23.38	29.40	**17.53**

Outline and contributions

Preference space

- integrate M-estimators
- density analysis

Linkage

- T-Linkage
- Automatic threshold

Spectral

- Low Rank & sparse
- break the chicken & egg dilemma

Coverage

- well founded
- intersecting models

Outline and contributions

Preference space

- integrate M-estimators
- density analysis

Linkage

- T-Linkage
- Automatic threshold

Spectral

- Low Rank & sparse
- break the chicken & egg dilemma

Coverage

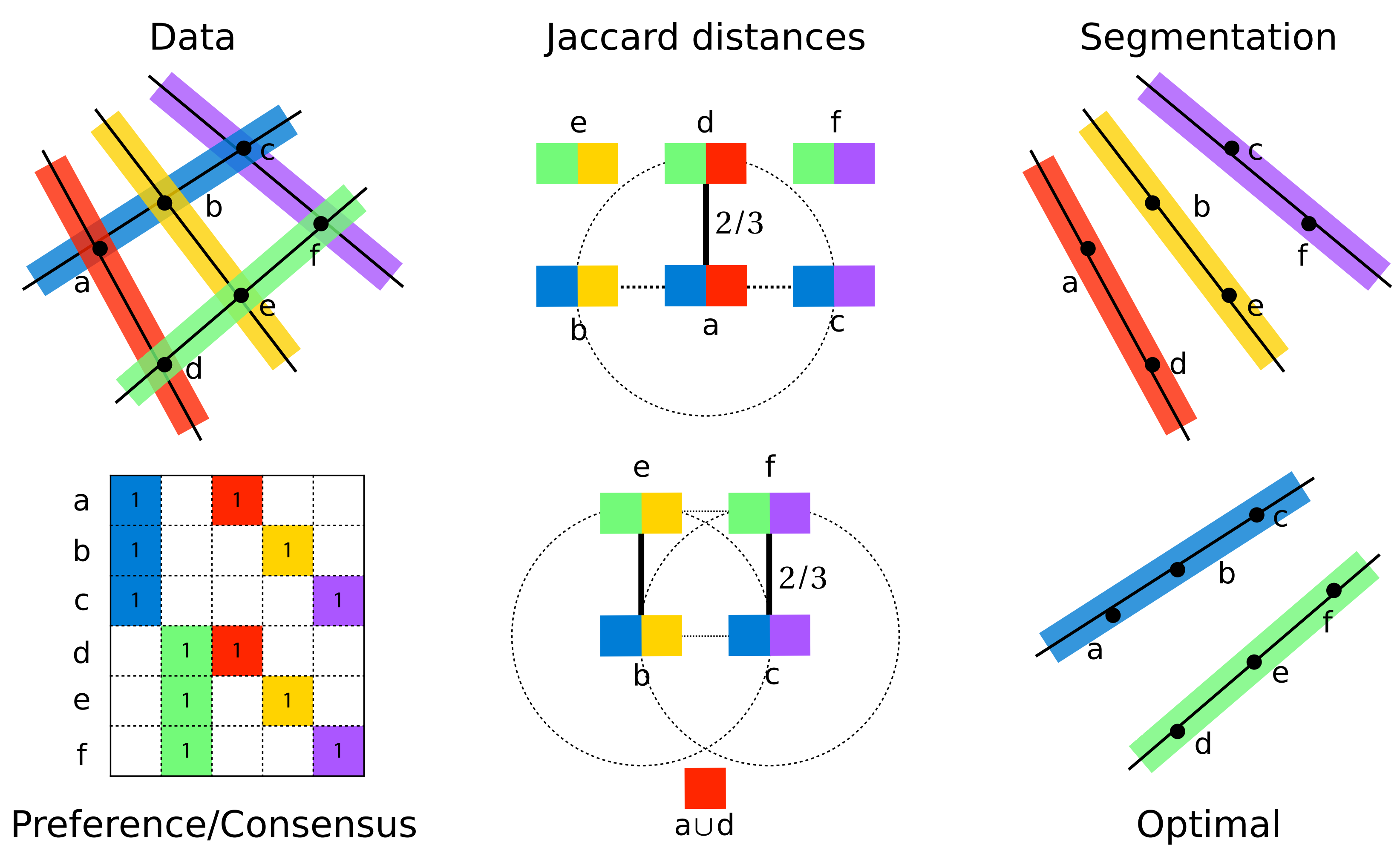
- well founded
- intersecting models

Fraughts and ambiguity of clustering

Shortcoming of preference analysis:

- bias towards the segmentation side of the problem.
- “No free lunch” theorem for clustering (Wolpert-Macready ‘97)
- Outliers are not properly handled
- Partitions do not take into account intersecting models

e.g. the greediness of J-Linkage



Coverages: back to consensus

Relaxing the notion of partitions to covers we address the problem of intersecting models and we can return back to maximize consensus

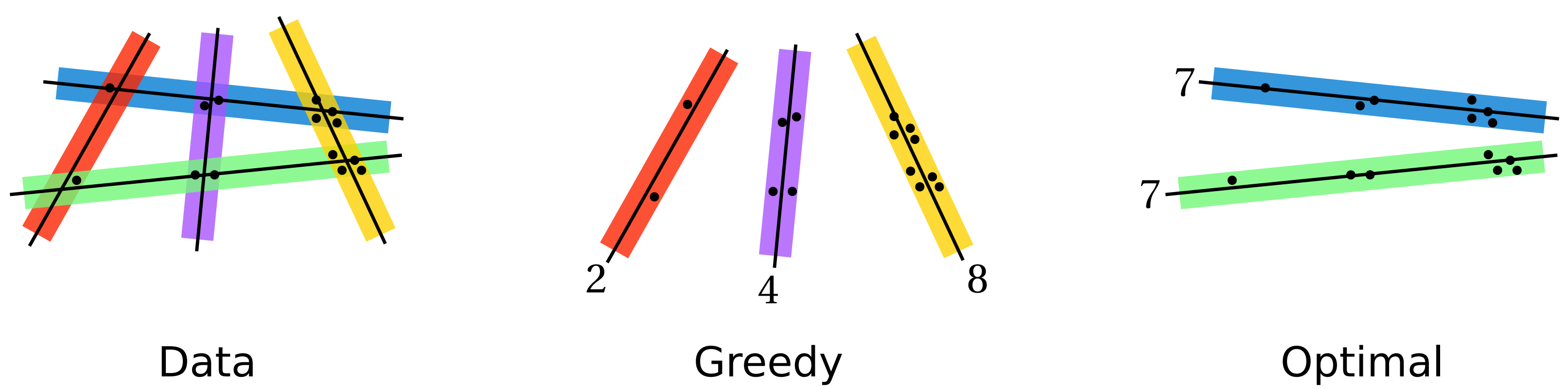
Set cover problem: Given a ground set X and a cover $F = \{S_1, \ldots, S_m\}$, the goal of set cover is to find a minimum subfamily in F that also covers X.

$$\min \sum_{j=1}^{m} z_j \quad \text{subject to} \quad Pz \geq \mathbf{1}.$$

With the convention than if S_j is selected in the solution then $z_j = 1$, otherwise $z_j = 0$.

ILP program

(still NP-hard but among the oldest, most studied and widespread problem.)



Sequential Ransac as a greedy heuristic to solve set cover

Weighted maximum coverage: preference into play

If outliers are present in the data:

Weighted maximum coverage: Given a ground set X, an integer k and a collection of subsets F. Non negative weights c_i are associated to the elements of X, the aim is to select at most k sets from F so as to maximize the overall weight of covered points.

$$\max \sum_{i=1}^{n} c_i y_i . \qquad \text{such that:} \quad \sum_{j=1}^{m} z_j \le k$$

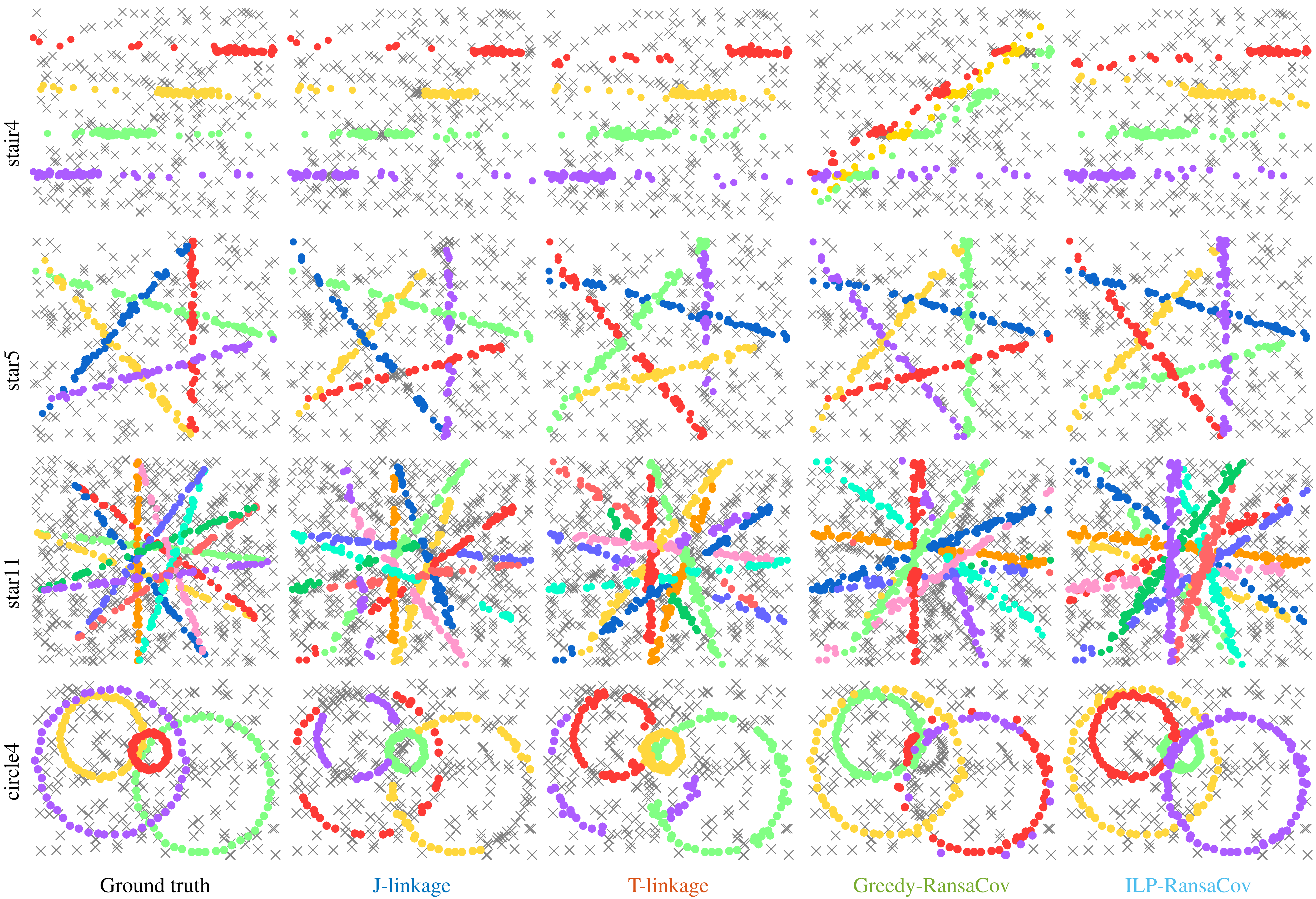
$$\sum_{j : x_i \in S_j} z_j \ge y_i \quad \forall x_i \in X$$

$$0 \le y_i \le 1 \quad \text{points covered}$$

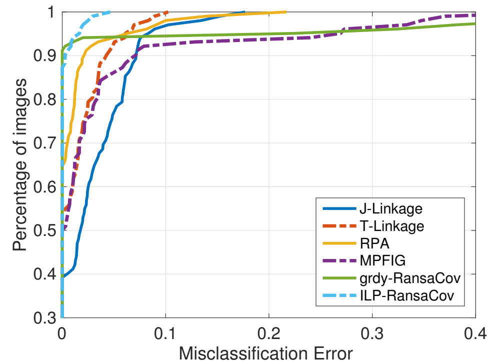
$$z_j \in \{0, 1\}. \quad \text{models selected}$$

Several measures can be used in order to down weights outliers, we use a simple score based on the reachability distance

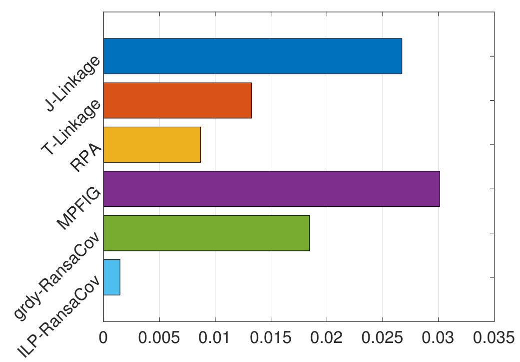
$$c_i = 1 - \mathrm{rd}(x_i) \in [0, 1]$$


stair4
star5
star11
circle4
Ground truth
J-linkage
T-linkage
Greedy-RansaCov
ILP-RansaCov

Vanishing point detection



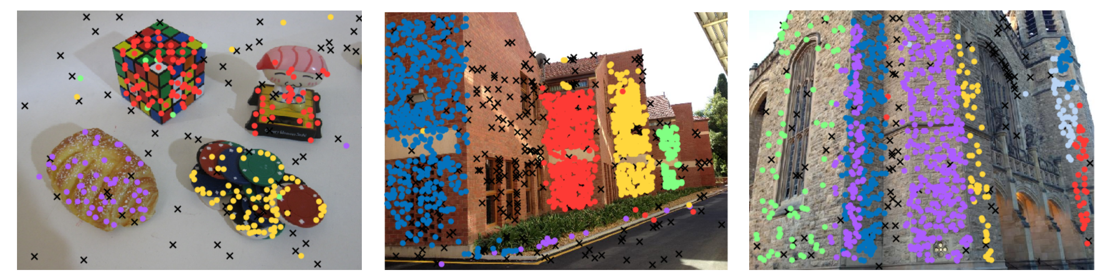
(a) Cumulative ME

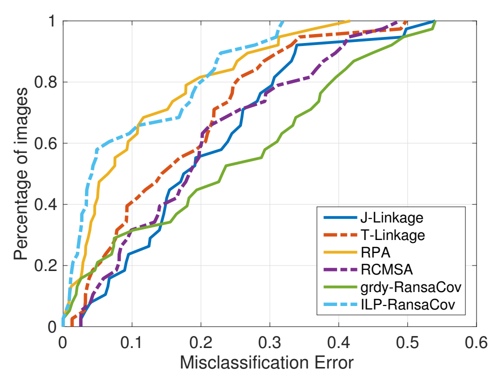


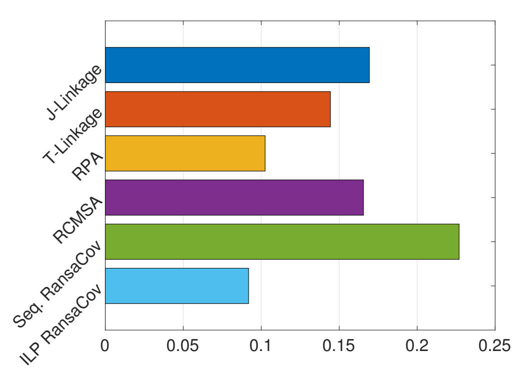
(b) Area under the cumulative ME

	J-Lnkg	T-Lnkg	RPA	MFIGP	Grdy-RansaCov	ILP-RansaCov
Mean	2.85	1.44	1.08	3.51	2.38	**0.19**
Med	1.80	**0.00**	**0.00**	0.16	0.00	**0.00**

Two views segmentation



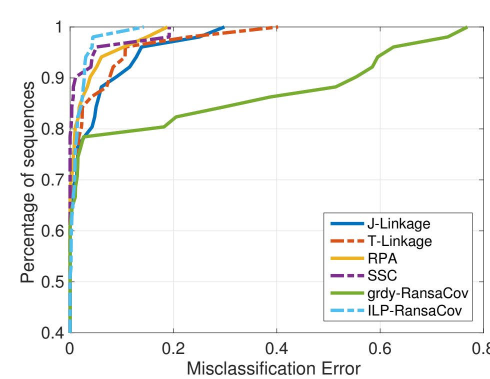




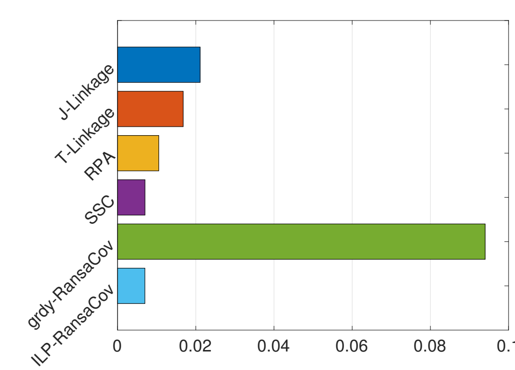
		J-Lnkg	T-Lnkg	RPA	RCMSA	Grdy-RansaCov	ILP-RansaCov
F	Mean	16.43	9.37	**5.49**	12.37	17.08	6.04
	Med	14.29	7.80	4.57	9.87	21.65	**4.27**
H	Mean	25.50	24.66	17.20	28.30	26.85	**12.91**
	Med	24.48	24.53	17.78	29.40	28.77	**12.34**

Video segmentation

		J-Lnkg	T-Lnkg	RPA	SSC	Grdy-RansaCov	ILP-RansaCov
traffic 3	Mean	1.58	0.48	**0.19**	0.76	28.65	0.35
	Med	0.34	0.19	**0.00**	**0.00**	1.53	0.19
traffic 2	Mean	1.75	1.31	0.14	**0.06**	7.48	0.54
	Med	**0.00**	**0.00**	**0.00**	**0.00**	**0.00**	**0.00**
others 3	Mean	6.91	5.32	9.11	2.13	14.89	**2.13**
	Med	6.91	5.32	9.11	2.13	14.89	**2.13**
others 2	Mean	5.32	6.47	4.41	3.95	8.57	**2.40**
	Med	1.30	2.38	2.44	**0.00**	0.20	1.30
All	Mean	2.70	2.47	1.42	1.08	10.91	**0.98**
	Med	**0.00**	**0.00**	**0.00**	**0.00**	**0.00**	**0,00**



(a) Cumulative ME



(b) Area under the cumulative ME

Outline and contributions

Preference space

- integrate M-estimators
- density analysis

Linkage

- T-Linkage
- Automatic threshold

Spectral

- Low Rank & sparse
- break the chicken & egg dilemma

Coverage

- well founded
- intersecting models