Adjoint of Matrices

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2023-05-28

Introduction

In linear algebra, the adjoint of a matrix plays an important role in various applications. The adjoint matrix, also known as the adjugate or classical adjoint, is closely related to the concept of the inverse of a matrix. In this lecture, we will explore the definition and properties of the adjoint of a matrix, with a specific focus on 3x3 matrices.

Definition

Let A be a square matrix of order n. The adjoint of A, denoted as $\text{adj}(A)$ or A^*, is defined as the transpose of the cofactor matrix of A. The cofactor of an element a_{ij} in A is given by $C_{ij} = (-1)^{i+j}\det(M_{ij})$, where M_{ij} is the $(n-1)\times(n-1)$ submatrix obtained by deleting the i-th row and j-th column of A.

The adjoint matrix can be expressed as $\text{adj}(A) = [C_{ij}]^T$. In other words, the (i,j)-th entry of the adjoint matrix is the cofactor of the (j,i)-th entry of the original matrix.

Illustration 2 x 2 matrices

Let's find the adjoint of the matrix $A = \begin{bmatrix} 2 & 3 \\ 1 & 4 \end{bmatrix}$.

Step 1: Calculate the Cofactors

The cofactors of matrix A are obtained by taking the determinants of the 1x1 matrices formed by each element of A. Thus:

$$\begin{aligned} C_{11} &= (-1)^{1+1}\cdot \begin{vmatrix} 4 \end{vmatrix} = 4 \\ C_{12} &= (-1)^{1+2}\cdot \begin{vmatrix} 1 \end{vmatrix} = -1 \\ C_{21} &= (-1)^{2+1}\cdot \begin{vmatrix} 3 \end{vmatrix} = -3 \\ C_{22} &= (-1)^{2+2}\cdot \begin{vmatrix} 2 \end{vmatrix} = 2 \end{aligned}$$

Thus, the cofactor matrix $\text{cof}(A)$ is:

$$\begin{bmatrix} 4 & -1 \\ -3 & 2 \end{bmatrix}$$

Step 2: Transpose the Cofactor Matrix

The adjoint of matrix A is obtained by taking the transpose of the cofactor matrix $\text{cof}(A)$. Thus, the adjoint A^* is:

$$A^* = \begin{bmatrix} 4 & -3 \\ -1 & 2 \end{bmatrix}$$

Illustration 3 x 3 matrices

Let's consider a 3x3 matrix A:

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

We can find the cofactor matrix of A by computing the cofactors of each element:

$$\text{cof}(A) = \begin{bmatrix} C_{11} & C_{12} & C_{13} \\ C_{21} & C_{22} & C_{23} \\ C_{31} & C_{32} & C_{33} \end{bmatrix}$$

where,

$$C_{ij} = (-1)^{i+j} \det(M_{ij})$$

and M_{ij} is the (2×2) submatrix obtained by deleting the i-th row and j-th column of A.

Finally, we take the transpose of the cofactor matrix to obtain the adjoint of A:

$$\text{adj}(A) = \begin{bmatrix} C_{11} & C_{21} & C_{31} \\ C_{12} & C_{22} & C_{32} \\ C_{13} & C_{23} & C_{33} \end{bmatrix}^T$$

Example

Let's find the adjoint of the following matrix:

$$A = \begin{bmatrix} 2 & 3 & -1 \\ 0 & 4 & 2 \\ 1 & -3 & 5 \end{bmatrix}$$

To find the cofactor matrix, we calculate the cofactor for each entry of matrix A:

$$C_{11} = (-1)^{1+1} \begin{vmatrix} 4 & 2 \\ -3 & 5 \end{vmatrix} = 26$$

$$C_{12} = (-1)^{1+2} \begin{vmatrix} 0 & 2 \\ 1 & 5 \end{vmatrix} = 2$$

$$C_{13} = (-1)^{1+3} \begin{vmatrix} 0 & 4 \\ 1 & -3 \end{vmatrix} = -4$$

$$C_{21} = (-1)^{2+1} \begin{vmatrix} 3 & -1 \\ -3 & 5 \end{vmatrix} = -12$$

$$C_{22} = (-1)^{2+2} \begin{vmatrix} 2 & -1 \\ 1 & 5 \end{vmatrix} = 11$$

$$C_{23} = (-1)^{2+3} \begin{vmatrix} 2 & 3 \\ 1 & -3 \end{vmatrix} = 9$$

$$C_{31} = (-1)^{3+1} \begin{vmatrix} 3 & -1 \\ 4 & 2 \end{vmatrix} = 10$$

$$C_{32} = (-1)^{3+2} \begin{vmatrix} 2 & -1 \\ 0 & 2 \end{vmatrix} = -4$$

$$C_{33} = (-1)^{3+3} \begin{vmatrix} 2 & 3 \\ 0 & 4 \end{vmatrix} = 8$$

The cofactor matrix is given by:

$$\text{cof}(A) = \begin{bmatrix} 26 & 2 & -4 \\ -12 & 11 & 9 \\ 10 & -4 & 8 \end{bmatrix}$$

Finally, we take the transpose of the cofactor matrix to obtain the adjoint of matrix A:

$$\text{adj}(A) = \begin{bmatrix} 26 & -12 & 10 \\ 2 & 11 & -4 \\ -4 & 9 & 8 \end{bmatrix}$$

Properties of the Adjoint

The adjoint of a matrix has several properties:

1. If A is an invertible matrix, then $\text{adj}(A)$ is also invertible, and $(\text{adj}(A))^{-1} = \frac{1}{\det(A)} A$.
2. If A is a symmetric matrix, then $\text{adj}(A) = A$.
3. If A and B are matrices of the same size, then $\text{adj}(AB) = \text{adj}(B) \cdot \text{adj}(A)$.

Conclusion

The adjoint of a matrix provides valuable information about the original matrix, such as its invertibility and the relationship between the matrix and its determinant. In this lecture, we focused on the definition and properties of the adjoint of 2x2 and 3x3 matrices. However, the concept of the adjoint can be extended to matrices of any order. Further exploration of this topic will deepen your understanding of linear algebra and its applications.