Signal Inpainting from Fourier magnitudes: An Almost Uniqueness Result

Supplementary Material for an ICASSP 2023 Submission

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Abstract

This document takes interest in signal inpainting from Fourier magnitudes. This task consists in reconstructing consecutive missing samples in a finite discrete 1D signal, while assuming the magnitudes of its Fourier transform are known. In this report, we theoretically show that for almost all signals of length L, this problem admits a unique solution if at most $(L-1)/3$ samples are missing. *Note: in the submitted ICASSP manuscript, the result is formulated for at most $L/4$ missing samples, which is strictly weaker. We have improved the bound since then.*

1 Introduction and problem setting

Signal inpainting [1] is an inverse problem that consists in restoring signals degraded by sample loss. This problem typically arises as a result of degradation during signal transmission, digitization of physically degraded media, or degradation so heavy that the information about the samples can be considered lost [5, 2, 3]. Let $\boldsymbol{x} \in \mathbb{R}^L$ be a signal. Let $\bar{v} \subset \{0, \dots, L-1\}$ denote a set of consecutive indices corresponding to missing samples in $\boldsymbol{x}$ and v denote its complement, *i.e.*, the set of indices corresponding to observed samples. We denote by $\boldsymbol{x}_{\bar{v}} \in \mathbb{R}^d$ the sub-signal of $\boldsymbol{x}$ restricted to missing samples and $\boldsymbol{x}_v \in \mathbb{R}^{L-d}$ the sub-signal restricted to observed samples. We denote by $\boldsymbol{b} \in \mathbb{R}_+^L$ the magnitudes of the discrete Fourier transform (DFT) of $\boldsymbol{x}$, *i.e.*, $\boldsymbol{b} = |\boldsymbol{\Phi x}|$, where $\boldsymbol{\Phi} \in \mathbb{C}^{L \times L}$ is the DFT matrix. For a given observed signal $\boldsymbol{y} \in \mathbb{R}^{L-d}$ and Fourier magnitudes $\boldsymbol{b}$, the task of signal inpainting from Fourier magnitudes can then be stated as:

$$\text{Find } \boldsymbol{u} \in \mathbb{R}^d \text{ such that (s.t.) } |\boldsymbol{\Phi x}| = \boldsymbol{b} \text{ with } \boldsymbol{x}_{\bar{v}} = \boldsymbol{u} \text{ and } \boldsymbol{x}_v = \boldsymbol{y}. \tag{1}$$

We focus on the situation where the given vector $\boldsymbol{b}$ corresponds to the true magnitudes of the Fourier transform of a completed signal $\boldsymbol{x}$. Hence, the existence of at least one solution of (1) is guaranteed. In this document, we will show that when $d < (L-1)/3$, this solution is unique for *almost all* signals $\boldsymbol{x} \in \mathbb{R}^L$. We use a dimension counting argument, similar in spirit to the one employed by [4] in the context of sparse phase retrieval. Specifically, we show that signals $\boldsymbol{x}$ for which more than one solution exists, referred to hereinafter as *counter examples*, necessary lie on a manifold of $\mathbb{R}^L$ with strictly less than L degrees of freedom. They hence form a set of measure zero, *i.e.*, a null set.

2 Almost uniqueness: statement and proof

We will assume throughout this section that the missing samples are placed at the beginning of $\boldsymbol{x}$, allowing us to write $\boldsymbol{x} = [\boldsymbol{u}; \boldsymbol{y}]$ where $[\cdot;\cdot]$ denotes vertical concatenation. This comes without loss of generality, because for any counter-example signal $\widetilde{\boldsymbol{x}}$ with missing samples placed anywhere in the signal, one can construct a counter example with samples placed at the beginning of the signal (and reciprocally) by a simple circular shift of $\widetilde{\boldsymbol{x}}$, since a circular shift does not affect DFT magnitudes. We prove the following theorem:

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Theorem 1. *Let* $\mathcal{E} = \{\boldsymbol{x} = [\boldsymbol{u}; \boldsymbol{y}] \in \mathbb{R}^L \mid \exists \boldsymbol{v} \in \mathbb{R}^d, \boldsymbol{v} \neq \boldsymbol{u},\ s.t.\ |\boldsymbol{\Phi}[\boldsymbol{u}; \boldsymbol{y}]| = |\boldsymbol{\Phi}[\boldsymbol{v}; \boldsymbol{y}]|\}$*. For* $d < (L-1)/3$*,* $\mathcal{E}$ *is a manifold of* $\mathbb{R}^L$ *with strictly less than* L *degrees of freedom.*

In other words, the set of counter examples to the unicity of (1) has strictly less than L degrees of freedom, and is hence of measure zero.

Proof. We denote by $\mathcal{E}'$ the set of triplets $(\boldsymbol{y}, \boldsymbol{u}, \boldsymbol{v}) \in \mathbb{R}^{L-d} \times \mathbb{R}^d \times \mathbb{R}^d$ forming a counter example, namely:

$$\mathcal{E}' = \{(\boldsymbol{y}, \boldsymbol{u}, \boldsymbol{v}) \in \mathbb{R}^{L-d} \times \mathbb{R}^d \times \mathbb{R}^d \mid \boldsymbol{v} \neq \boldsymbol{u},\ |\boldsymbol{\Phi}[\boldsymbol{u}; \boldsymbol{y}]| = |\boldsymbol{\Phi}[\boldsymbol{v}; \boldsymbol{y}]|\}. \tag{2}$$

We will show that this manifold of $\mathbb{R}^{L-d} \times \mathbb{R}^d \times \mathbb{R}^d$ has strictly less that L degrees of freedom; This implies that its projection $\mathcal{E}$ also has strictly less that L degrees of freedom. We first prove the following:

Lemma 1. *There is a linear bijection between* $\mathcal{E}'$ *and the following set:*

$$\mathcal{E}'' = \left\{(\boldsymbol{a}, \boldsymbol{w}) \in \mathbb{R}^d \times \mathbb{R}^L \mid \boldsymbol{a} \neq \boldsymbol{0}_d,\ \mathcal{R}\left(\overline{\boldsymbol{\Phi}[\boldsymbol{a}; \boldsymbol{0}_{L-d}]} \odot \boldsymbol{\Phi}\boldsymbol{w}\right) = \boldsymbol{0}_L\right\} \tag{3}$$

where, $\mathcal{R}(\cdot)$ *denotes the real part of a vector and* $\odot$ *denotes element-wise product.*

Proof. We horizontally split the DFT matrix as $\boldsymbol{\Phi} = [\boldsymbol{\Phi}^{(1)}, \boldsymbol{\Phi}^{(2)}]$ where $\boldsymbol{\Phi}^{(1)} \in \mathbb{R}^{L \times d}$ and $\boldsymbol{\Phi}^{(2)} \in \mathbb{R}^{L \times L-d}$ We have the following chain of equivalences (where $\boldsymbol{v} \neq \boldsymbol{u}$ is kept implicit):

$$(\boldsymbol{y}, \boldsymbol{u}, \boldsymbol{v}) \in \mathcal{E}' \tag{4}$$

$$\Leftrightarrow |\boldsymbol{\Phi}[\boldsymbol{u}; \boldsymbol{y}]|^2 = |\boldsymbol{\Phi}[\boldsymbol{v}; \boldsymbol{y}]|^2 \tag{5}$$

$$\Leftrightarrow \left|\boldsymbol{\Phi}^{(1)}\boldsymbol{u} + \boldsymbol{\Phi}^{(2)}\boldsymbol{y}\right|^2 = \left|\boldsymbol{\Phi}^{(1)}\boldsymbol{v} + \boldsymbol{\Phi}^{(2)}\boldsymbol{y}\right|^2 \tag{6}$$

$$\Leftrightarrow \left|\boldsymbol{\Phi}^{(1)}\boldsymbol{u}\right|^2 + 2\mathcal{R}\left(\overline{\boldsymbol{\Phi}^{(1)}\boldsymbol{u}} \odot \boldsymbol{\Phi}^{(2)}\boldsymbol{y}\right) + \left|\boldsymbol{\Phi}^{(2)}\boldsymbol{y}\right|^2 = \left|\boldsymbol{\Phi}^{(1)}\boldsymbol{v}\right|^2 + 2\mathcal{R}\left(\overline{\boldsymbol{\Phi}^{(1)}\boldsymbol{v}} \odot \boldsymbol{\Phi}^{(2)}\boldsymbol{y}\right) + \left|\boldsymbol{\Phi}^{(2)}\boldsymbol{y}\right|^2 \tag{7}$$

$$\Leftrightarrow \left|\boldsymbol{\Phi}^{(1)}\boldsymbol{u}\right|^2 - \left|\boldsymbol{\Phi}^{(1)}\boldsymbol{v}\right|^2 + 2\mathcal{R}\left(\overline{\boldsymbol{\Phi}^{(1)}(\boldsymbol{u}-\boldsymbol{v})} \odot \boldsymbol{\Phi}^{(2)}\boldsymbol{y}\right) = \boldsymbol{0}_L \tag{8}$$

$$\Leftrightarrow \mathcal{R}\left((\overline{\boldsymbol{\Phi}^{(1)}\boldsymbol{u}} - \overline{\boldsymbol{\Phi}^{(1)}\boldsymbol{v}}) \odot (\boldsymbol{\Phi}^{(1)}\boldsymbol{u} + \boldsymbol{\Phi}^{(1)}\boldsymbol{v})\right) + 2\mathcal{R}\left(\overline{\boldsymbol{\Phi}^{(1)}(\boldsymbol{u}-\boldsymbol{v})} \odot \boldsymbol{\Phi}^{(2)}\boldsymbol{y}\right) = \boldsymbol{0}_L \tag{9}$$

$$\Leftrightarrow \mathcal{R}\left(\overline{\boldsymbol{\Phi}^{(1)}(\boldsymbol{u}-\boldsymbol{v})} \odot (\boldsymbol{\Phi}^{(1)}\boldsymbol{u} + \boldsymbol{\Phi}^{(1)}\boldsymbol{v} + 2\boldsymbol{\Phi}^{(2)}\boldsymbol{y}\right) = \boldsymbol{0}_L \tag{10}$$

$$\Leftrightarrow \mathcal{R}\left(\overline{\boldsymbol{\Phi}[\boldsymbol{u}-\boldsymbol{v}; \boldsymbol{0}_{L-d}]} \odot (\boldsymbol{\Phi}[\boldsymbol{u}+\boldsymbol{v}; 2\boldsymbol{y}]\right) = \boldsymbol{0}_L \tag{11}$$

$$\Leftrightarrow \mathcal{R}\left(\overline{\boldsymbol{\Phi}[\boldsymbol{a}; \boldsymbol{0}_{L-d}]} \odot \boldsymbol{\Phi}\boldsymbol{w}\right) = \boldsymbol{0}_L \text{ where } \boldsymbol{a} = \boldsymbol{u}-\boldsymbol{v} \neq \boldsymbol{0}_d \text{ and } \boldsymbol{w} = [\boldsymbol{u}+\boldsymbol{v}; 2\boldsymbol{y}] \in \mathbb{R}^L. \tag{12}$$

Since the transformation from $(\boldsymbol{y}, \boldsymbol{u}, \boldsymbol{v})$ to $(\boldsymbol{a}, \boldsymbol{w})$ is linear and bijective, this concludes the proof. □

Based on Lemma 1, it is sufficient to show that $\mathcal{E}''$ has strictly less than L degrees of freedom. Since the non-zero signal $\boldsymbol{a} \in \mathbb{R}^d$ in (3) can be chosen arbitrarily (d degrees of freedom), it remains to show that for a fixed $\boldsymbol{a} \neq \boldsymbol{0}_d$, the set of $\boldsymbol{w} \in \mathbb{R}^L$ such that $(\boldsymbol{a}, \boldsymbol{w}) \in \mathcal{E}''$ has strictly less than $L-d$ degrees of freedom. For conciseness, we will only treat here the case where L is even, as the odd case only requires minor adjustments.

Let $\hat{\boldsymbol{a}} = \boldsymbol{\Phi}[\boldsymbol{a}; \boldsymbol{0}_{L-d}]$ and $\hat{\boldsymbol{w}} = \boldsymbol{\Phi}\boldsymbol{w}$ be the DFTs of $[\boldsymbol{a}; \boldsymbol{0}_{L-d}]$ and $\boldsymbol{w}$, indexed by the L discrete frequency numbers $f \in \{-L/2+1, \ldots, L/2\}$. Since the signals $\boldsymbol{a}$ and $\boldsymbol{w}$ are real-valued, their DFTs are fully determined by their values at non-negative frequencies, two of which are real (at $f = 0$ and $f = L/2$), the rest being complex. For every frequency number $f \in \{1, \cdots, L/2-1\}$ such that $\hat{a}(f) \neq 0$, the constraint $\mathcal{R}(\overline{\hat{\boldsymbol{a}}} \odot \hat{\boldsymbol{w}}) = \boldsymbol{0}_L$ fixes the phase of $\hat{w}(f)$ up to $\pm\pi/2$, reducing the degrees of freedom of $\boldsymbol{w}$ by 1 (from a total of L). Let us now count for how many distinct $f \in \{1, \cdots, L/2-1\}$ we can have $\hat{a}(f) \neq 0$. The z-transform of $[\boldsymbol{a}; \boldsymbol{0}_{L-d}]$ is a polynomial of degree at most $d-1$. Hence, this polynomial admits at most $d-1$ roots, and since $\boldsymbol{a}$ is real-valued, these roots are either real or come in conjugate pairs. This implies that $\hat{a}(f)$ can be 0 for at most $\lfloor (d-1)/2 \rfloor$ distinct f in $\{1, \cdots, L/2-1\}$. Hence, the constraint$\mathcal{R}(\overline{\hat{\boldsymbol{a}}} \odot \hat{\boldsymbol{w}}) = \boldsymbol{0}_L$ enforces at least $L/2 - 1 - \lfloor (d-1)/2 \rfloor$ phase constraints on $\hat{\boldsymbol{w}}$. Subtracting these constraints from L, we get that $\boldsymbol{w}$ has at most $P = L/2 + 1 + \lfloor (d-1)/2 \rfloor$ degrees of freedom. By hypothesis, $d < L/3 - 1$, which implies $P < L - d$ and concludes the proof. □

3 Conclusion

We have conducted a theoretical study on the solutions to the problem of signal inpainting from Fourier magnitudes. We have shown that if the number of missing samples d is strictly less than $(L-1)/3$, where L is the total signal length, then almost all signals containing a subset of d consecutive missing values are uniquely determined from the magnitudes of their Fourier transform.

References

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