

Signal Inpainting from Fourier magnitudes: An Almost Uniqueness Result

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Abstract

This document takes interest in signal inpainting from Fourier magnitudes. This task consists in reconstructing consecutive missing samples in a finite discrete 1D signal, while assuming the magnitudes of its Fourier transform are known. In this report, we theoretically show that for almost all signals of length L , this problem admits a unique solution if at most $(L - 1)/3$ samples are missing.

1 Introduction and problem setting

Signal inpainting [1] is an inverse problem that consists in restoring signals degraded by sample loss. This problem typically arises as a result of degradation during signal transmission, digitization of physically degraded media, or degradation so heavy that the information about the samples can be considered lost [5, 2, 3]. Let $\mathbf{x} \in \mathbb{R}^L$ be a signal. Let $\bar{v} \subset \{0, \dots, L - 1\}$ denote a set of consecutive indices corresponding to missing samples in \mathbf{x} and v denote its complement, *i.e.*, the set of indices corresponding to observed samples. We denote by $\mathbf{x}_{\bar{v}} \in \mathbb{R}^d$ the sub-signal of \mathbf{x} restricted to missing samples and $\mathbf{x}_v \in \mathbb{R}^{L-d}$ the sub-signal restricted to observed samples. We denote by $\mathbf{b} \in \mathbb{R}_+^L$ the magnitudes of the discrete Fourier transform (DFT) of \mathbf{x} , *i.e.*, $\mathbf{b} = |\Phi \mathbf{x}|$, where $\Phi \in \mathbb{C}^{L \times L}$ is the DFT matrix. For a given observed signal $\mathbf{y} \in \mathbb{R}^{L-d}$ and Fourier magnitudes \mathbf{b} , the task of signal inpainting from Fourier magnitudes can then be stated as:

$$\text{Find } \mathbf{u} \in \mathbb{R}^d \text{ such that (s.t.) } |\Phi \mathbf{x}| = \mathbf{b} \text{ with } \mathbf{x}_{\bar{v}} = \mathbf{u} \text{ and } \mathbf{x}_v = \mathbf{y}. \quad (1)$$

We focus on the situation where the given vector \mathbf{b} corresponds to the true magnitudes of the Fourier transform of a completed signal \mathbf{x} . Hence, the existence of at least one solution of (1) is guaranteed. In this document, we will show that when $d < (L - 1)/3$, this solution is unique for *almost all* signals $\mathbf{x} \in \mathbb{R}^L$. We use a dimension counting argument, similar in spirit to the one employed by [4] in the context of sparse phase retrieval. Specifically, we show that signals \mathbf{x} for which more than one solution exists, referred to hereinafter as *counter examples*, necessary lie on a manifold of \mathbb{R}^L with strictly less than L degrees of freedom. They hence form a set of measure zero, *i.e.*, a null set.

2 Almost uniqueness: statement and proof

We will assume throughout this section that the missing samples are placed at the beginning of \mathbf{x} , allowing us to write $\mathbf{x} = [\mathbf{u}; \mathbf{y}]$ where $[\cdot; \cdot]$ denotes vertical concatenation. This comes without loss of generality, because for any counter-example signal $\tilde{\mathbf{x}}$ with missing samples placed anywhere in the signal, one can construct a counter example with samples placed at the beginning of the signal (and reciprocally) by a simple circular shift of $\tilde{\mathbf{x}}$, since a circular shift does not affect DFT magnitudes. We prove the following theorem:

Theorem 1. *Let $\mathcal{E} = \{\mathbf{x} = [\mathbf{u}; \mathbf{y}] \in \mathbb{R}^L \mid \exists \mathbf{v} \in \mathbb{R}^d, \mathbf{v} \neq \mathbf{u}, \text{ s.t. } |\Phi[\mathbf{u}; \mathbf{y}]| = |\Phi[\mathbf{v}; \mathbf{y}]]\}$. For $d < (L - 1)/3$, \mathcal{E} is a manifold of \mathbb{R}^L with strictly less than L degrees of freedom.*

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In other words, the set of counter examples to the unicity of (1) has strictly less than L degrees of freedom, and is hence of measure zero.

Proof. We denote by \mathcal{E}' the set of triplets $(\mathbf{y}, \mathbf{u}, \mathbf{v}) \in \mathbb{R}^{L-d} \times \mathbb{R}^d \times \mathbb{R}^d$ forming a counter example, namely:

$$\mathcal{E}' = \{(\mathbf{y}, \mathbf{u}, \mathbf{v}) \in \mathbb{R}^{L-d} \times \mathbb{R}^d \times \mathbb{R}^d \mid \mathbf{v} \neq \mathbf{u}, |\Phi[\mathbf{u}; \mathbf{y}]| = |\Phi[\mathbf{v}; \mathbf{y}]|\}. \quad (2)$$

We will show that this manifold of $\mathbb{R}^{L-d} \times \mathbb{R}^d \times \mathbb{R}^d$ has strictly less than L degrees of freedom; This implies that its projection \mathcal{E} also has strictly less than L degrees of freedom. We first prove the following:

Lemma 1. *There is a linear bijection between \mathcal{E}' and the following set:*

$$\mathcal{E}'' = \{(\mathbf{a}, \mathbf{w}) \in \mathbb{R}^d \times \mathbb{R}^L \mid \mathbf{a} \neq \mathbf{0}_d, \mathcal{R}(\overline{\Phi[\mathbf{a}; \mathbf{0}_{L-d}]} \odot \Phi \mathbf{w}) = \mathbf{0}_L\} \quad (3)$$

where, $\mathcal{R}(\cdot)$ denotes the real part of a vector and \odot denotes element-wise product.

Proof. We horizontally split the DFT matrix as $\Phi = [\Phi^{(1)}, \Phi^{(2)}]$ where $\Phi^{(1)} \in \mathbb{R}^{L \times d}$ and $\Phi^{(2)} \in \mathbb{R}^{L \times L-d}$. We have the following chain of equivalences (where $\mathbf{v} \neq \mathbf{u}$ is kept implicit):

$$(\mathbf{y}, \mathbf{u}, \mathbf{v}) \in \mathcal{E}' \quad (4)$$

$$\Leftrightarrow |\Phi[\mathbf{u}; \mathbf{y}]|^2 = |\Phi[\mathbf{v}; \mathbf{y}]|^2 \quad (5)$$

$$\Leftrightarrow \left| \Phi^{(1)} \mathbf{u} + \Phi^{(2)} \mathbf{y} \right|^2 = \left| \Phi^{(1)} \mathbf{v} + \Phi^{(2)} \mathbf{y} \right|^2 \quad (6)$$

$$\Leftrightarrow \left| \Phi^{(1)} \mathbf{u} \right|^2 + 2\mathcal{R}(\overline{\Phi^{(1)} \mathbf{u}} \odot \Phi^{(2)} \mathbf{y}) + \left| \Phi^{(2)} \mathbf{y} \right|^2 = \left| \Phi^{(1)} \mathbf{v} \right|^2 + 2\mathcal{R}(\overline{\Phi^{(1)} \mathbf{v}} \odot \Phi^{(2)} \mathbf{y}) + \left| \Phi^{(2)} \mathbf{y} \right|^2 \quad (7)$$

$$\Leftrightarrow \left| \Phi^{(1)} \mathbf{u} \right|^2 - \left| \Phi^{(1)} \mathbf{v} \right|^2 + 2\mathcal{R}(\overline{\Phi^{(1)}(\mathbf{u} - \mathbf{v})} \odot \Phi^{(2)} \mathbf{y}) = \mathbf{0}_L \quad (8)$$

$$\Leftrightarrow \mathcal{R}(\overline{(\Phi^{(1)} \mathbf{u} - \Phi^{(1)} \mathbf{v})} \odot (\Phi^{(1)} \mathbf{u} + \Phi^{(1)} \mathbf{v})) + 2\mathcal{R}(\overline{\Phi^{(1)}(\mathbf{u} - \mathbf{v})} \odot \Phi^{(2)} \mathbf{y}) = \mathbf{0}_L \quad (9)$$

$$\Leftrightarrow \mathcal{R}(\overline{\Phi^{(1)}(\mathbf{u} - \mathbf{v})} \odot (\Phi^{(1)} \mathbf{u} + \Phi^{(1)} \mathbf{v} + 2\Phi^{(2)} \mathbf{y})) = \mathbf{0}_L \quad (10)$$

$$\Leftrightarrow \mathcal{R}(\overline{\Phi[\mathbf{u} - \mathbf{v}, \mathbf{0}_{L-d}]} \odot (\Phi[\mathbf{u} + \mathbf{v}, 2\mathbf{y}])) = \mathbf{0}_L \quad (11)$$

$$\Leftrightarrow \mathcal{R}(\overline{\Phi[\mathbf{a}; \mathbf{0}_{L-d}]} \odot \Phi \mathbf{w}) = \mathbf{0}_L \text{ where } \mathbf{a} = \mathbf{u} - \mathbf{v} \neq \mathbf{0}_d \text{ and } \mathbf{w} = [\mathbf{u} + \mathbf{v}, 2\mathbf{y}] \in \mathbb{R}^L. \quad (12)$$

Since the transformation from $(\mathbf{y}, \mathbf{u}, \mathbf{v})$ to (\mathbf{a}, \mathbf{w}) is linear and bijective, this concludes the proof. \square

Based on Lemma 1, it is sufficient to show that \mathcal{E}'' has strictly less than L degrees of freedom. Since the non-zero signal $\mathbf{a} \in \mathbb{R}^d$ in (3) can be chosen arbitrarily (d degrees of freedom), it remains to show that for a fixed $\mathbf{a} \neq \mathbf{0}_d$, the set of $\mathbf{w} \in \mathbb{R}^L$ such that $(\mathbf{a}, \mathbf{w}) \in \mathcal{E}''$ has strictly less than $L - d$ degrees of freedom. For conciseness, we will only treat here the case where L is even, as the odd case only requires minor adjustments.

Let $\hat{\mathbf{a}} = \Phi[\mathbf{a}; \mathbf{0}_{L-d}]$ and $\hat{\mathbf{w}} = \Phi \mathbf{w}$ be the DFTs of $[\mathbf{a}; \mathbf{0}_{L-d}]$ and \mathbf{w} , indexed by the L discrete frequency numbers $f \in \{-L/2 + 1, \dots, L/2\}$. Since the signals \mathbf{a} and \mathbf{w} are real-valued, their DFTs are fully determined by their values at non-negative frequencies, two of which are real (at $f = 0$ and $f = L/2$), the rest being complex. For every frequency number $f \in \{1, \dots, L/2 - 1\}$ such that $\hat{a}(f) \neq 0$, the constraint $\mathcal{R}(\hat{\mathbf{a}} \odot \hat{\mathbf{w}}) = \mathbf{0}_L$ fixes the phase of $\hat{w}(f)$ up to $\pm\pi/2$, reducing the degrees of freedom of \mathbf{w} by 1 (from a total of L). Let us now count for how many distinct $f \in \{1, \dots, L/2 - 1\}$ we can have $\hat{a}(f) \neq 0$. The z-transform of $[\mathbf{a}; \mathbf{0}_{L-d}]$ is a polynomial of degree at most $d - 1$. Hence, this polynomial admits at most $d - 1$ roots, and since \mathbf{a} is real-valued, these roots are either real or come in conjugate pairs. This implies that $\hat{a}(f)$ can be 0 for at most $\lfloor (d - 1)/2 \rfloor$ distinct f in $\{1, \dots, L/2 - 1\}$. Hence, the constraint $\mathcal{R}(\hat{\mathbf{a}} \odot \hat{\mathbf{w}}) = \mathbf{0}_L$ enforces at least $L/2 - 1 - \lfloor (d - 1)/2 \rfloor$ phase constraints on $\hat{\mathbf{w}}$. Subtracting these constraints from L , we get that \mathbf{w} has at most $P = L/2 + 1 + \lfloor (d - 1)/2 \rfloor$ degrees of freedom. By hypothesis, $d < L/3 - 1$, which implies $P < L - d$ and concludes the proof. \square

3 Conclusion

We have conducted a theoretical study of the solutions to the signal inpainting from Fourier magnitudes problem. We have shown that if the number of missing samples d is strictly less than $(L - 1)/3$, where L is the total signal length, then almost all signals containing a subset of d missing values are uniquely determined from the magnitudes of their Fourier transform.

References

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