Understanding the spectrum of a blackbody

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Overview

Part 1

- The Black Body problem
- Charged oscillators
- Rayleigh’s intensity function
- The Ultra-Violet catastropy
- Plank’s emphirical formula

Part 2

- Intensity of a Black Body spectrum and the problem
- Einstein’s work on Black Bodies

Overview

Part 3

Amplitude-based descriptiom of identical particles

n-Boson systems

Probability based emission and absorption of Black Bodies

Concluding remarks

Part 1

An idealised object that can **absorb**, and subsequently **emit**, all the radiation incident on it is called a blackbody.

Rayleigh's law

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- Consider gaseous medium, in which we have a charged oscilator.
- Due to random motion of the gaseous atoms, it often collides with the oscillator.
- As a result the gaseous atoms imparts some of it's kinetic energy to the oscillator.

Rayleigh's law

- Mean kinetic energy acquired by the oscillator is, $\frac{1}{2}\, kT$.
- And the total kinetic energy will be kT.

Rayleigh’s law

- Since the charge is radiating energy the system cannot attain equilibrium.

Rayleigh’s law

- How do we tackle this situation?

Rayleigh’s law

- How do we tackle this situation?
- We confine the system.

Rayleigh’s law

- Inner walls of the box is totally reflective.
- After some time the system will attains thermal equilibrium

Rayleigh's law

So, the energy radiated by the oscillator per second can be given by,

$$\frac{1}{Q} = \frac{\left(\frac{dW}{dt}\right)}{\omega_0 W}$$

Q is the radiation reaction.

ω_0 is the natural frequency of vibration.

W is the total energy content of the oscillator.

Rayleigh's law

Also, we have

$$\frac{1}{Q} = \frac{\gamma}{\omega_0}$$

Where γ is the dampping constant.

Remember the mean energy of the oscillator is kT? for three degrees of freedom, it is $3kT$.

Rayleigh's law

Average energy loss due to radiation is given by,

$$\boxed{\left\langle \frac{dW}{dt} \right\rangle = \gamma kT}$$

Rayleigh's law

- Let, $\mathbf{I}(\omega)d\omega$ be the incident light energy with in a frequeny range $d\omega$.
- Assuming all the radiation which falls on a *cross section* 'σ_S' is absorbed completely,
- Scattered light must be the product of $\mathbf{I}(\omega)d\omega$ and σ_S.

Expression for the cross section is given by,

$$\sigma_s = \frac{8\pi {r_0}^2}{3}\left[\frac{\omega^4}{(\omega^2 - {\omega_0}^2)^2 + \gamma^2\omega^2}\right]$$

When $\omega = \omega_0$,

$$\sigma_s = \frac{2\pi {r_0}^2 3\omega_0^2}{3(\omega - \omega_0)^2 + \frac{\gamma^2}{4}}$$

Rayleigh's law

$$\frac{dW_s}{dt} = \int_0^\infty \mathbf{I}(\omega)\,\sigma_s\,d\omega = \int_0^\infty \frac{2\pi {r_0}^2 \omega_0^2\,\mathbf{I}(\omega)\,d\omega}{3\left[(\omega-\omega_0)^2 + \frac{\gamma^2}{4}\right]}$$

since we're considering the equilibrium condition ($\omega = \omega_0$), equation reduces to,

$$\frac{dW_s}{dt} = \frac{2}{3}\pi {r_0}^2 \omega_0^2\,\mathbf{I}(\omega_0) \int_{-\infty}^{\infty} \frac{d\omega}{(\omega-\omega_0)^2 + \left(\frac{\gamma}{2}\right)^2}$$

Since the integral reduces to arctan function we get, $2\pi/\gamma$. and as

$$\frac{dW_s}{dt} = \frac{dW}{dt} = 3\gamma kT$$

$$\therefore \mathbf{I}(\omega) = \frac{9\gamma^2 kT}{4\pi^2 r_0^2 \omega_0^2}$$

$$\implies \gamma = \frac{2}{3}\frac{r_0\omega_0^2}{c}$$

Rayleigh's law

$$\mathbf{I}(\omega) = \frac{\omega^2\, kT}{\pi^2 c^2}$$

Graph

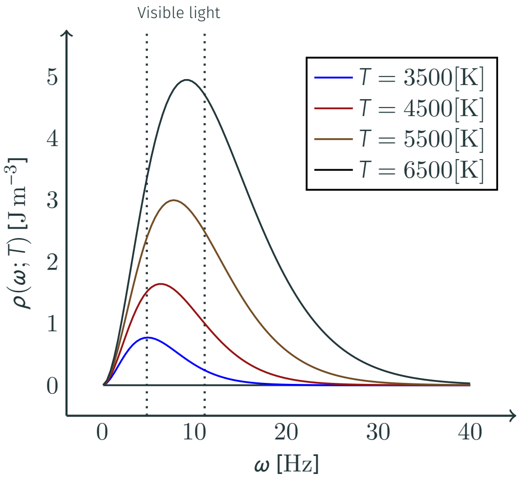

Figure 1: Example spectrum of a blackbody.

- Max Planck came up with an idea.
- We had the experimental result, and what we needed was a theoretical explanation for the curve.
- Planck studied the curve and came up with a mathematical formula.

- He made an assumption that the energy level of the harmonic oscillator is quantized.
- The harmonic oscillator can take up energies only in the multiples of $\mathbf{\hbar}\omega$.
- The probability of occupying an energy level $\mathbf{E}$ is $\mathbf{P}(\mathbf{E}) = \alpha e^{-\hbar\omega/kT}$.

Part 2

- The intensity of radiation for frequency ω is given by

$$I(\omega)d\omega = \frac{\hbar\omega^3 d\omega}{\pi^2 c^2 \left(e^{\frac{\hbar\omega}{kT}} - 1\right)}$$

- Planck considered that the matter was quantized and not light.

- Einstein came up with an alternate idea to solve Black body problem.

- Einstein considered that even the light was quantized, by taking that light is actually photons with energy $\hbar\omega$

- a pic is needed.

- Einstein assumed that Planck's formula was right.
- Consider two energy levels , say , the n^{th} level and m^{th} level.
- A light of certain frequency is incident on atom and by absorbing a photon, transition from state n to state m occurs.

- The probability of the transition is proportional to intensity of the light.
- The proportionality constant is B_{nm} and mathematically can be written as,

$$R_{n\to m} = N_n B_{nm} I(\omega) \tag{1}$$

- Einstein considers that there are two types of emissions that take place in an atom.

- Spontaneous emission.

- Stimulated emission.

- The combined mathematical expression is,

$$R_{m\to n} = N_m[A_{mn} + B_{mn}I(\omega)] \tag{2}$$

- At thermal equilibrium, the number of atoms going to higher energy level must be equal to the number of atoms coming to lower energy state.

- The ratio of N_m to N_n is given by $e^{\frac{-\hbar\omega}{kT}}$

- At thermal equilibrium both $R_{n\rightarrow m}$ and $R_{m\rightarrow n}$ are equal.

- $N_n B_{nm} I(\omega) = N_m [A_{mn} + B_{mn} I(\omega)]$

- On dividing the previous equation by N_m we get,

- $B_{nm}I(\omega)e^{\frac{\hbar\omega}{kT}} = A_{mn} + B_{mn}I(\omega)$

- So the equation that Einsteim got for intensity of radiation for frequency is

$$I(\omega) = \frac{A_{mn}}{B_{nm}e^{(\hbar\omega/kT)} - B_{mn}} \tag{3}$$

- Since Einstein considered that the formula given by Planck was correct, we should get ,

- $B_{nm} = B_{mn}$ and also

$$\frac{A_{mn}}{B_{nm}} = \frac{\hbar\omega^3 d\omega}{\pi^2 c^2}$$

This is how Einstein rederived Planck’s formula from a quantum mechanical viewpoint.

Part 3

- **P** = probability
- ϕ = Probability amplitude
- **P** = $\left|\phi\right|^2$

- $\phi = \phi_1 + \phi_2$
- **P** = $\left|\phi_1 + \phi_2\right|^2$

Case A

- Detector 1 is set to detect only α particles and Dectector 2 is set to detect only oxygen atoms.
- Probability amplitude of the scattering is given as $f(\theta)$ when they are at an angle θ.
- The probability of this event = $\left|f(\theta)\right|^2$

Case B

- Set up the dectectors such that the detectors would dectect either α particle or oxygen atom.

- We will not distinguish which particle is which entering the detector.

- This means that if oxygen atom in position θ , then α particle on the opposite side is at an angle $\pi - \theta$.

- Probability amplitude of oxygen atom = $f(\pi - \theta)$

- Probability amplitude of α particle = $f(\theta)$

- The probability of a particle being detected at detector 1 = $\left|f(\theta)\right|^2$ + $\left|f(\pi - \theta)\right|^2$

- Consider if both are α particles,
- Then we would not know which particle entered the detector, so the total probability changes to,
- The probability of a α particle being detected at detector 1 = $\left|f(\theta) + f(\pi - \theta)\right|^2$

- If $\theta = \dfrac{\pi}{2}$, then applying this to the expression $\left| f(\theta) + f(\pi - \theta) \right|^2$ we get,

- Probability = $4 \left| f\left(\dfrac{\pi}{2}\right) \right|^2$, if the particles are indistinguishible.

- Suppose the particles were distinguishible, then the probability for $\theta = \dfrac{\pi}{2}$ when applied for $\Big|f(\theta)\Big|^2$ + $\Big|f(\pi - \theta)\Big|^2$ is given by,

- Probability = $2\left|f\left(\dfrac{\pi}{2}\right)\right|^2$

- This shows that the probability gets doubled for indistinguishible particles.

- *Can we apply the same logicto the electron-electron scattering ?*

- *OBSERVATION* : ” When we have situation in which the identity of the electron which is arriving at a point is echanged with anothe one , the new amplitude interfere with old one with an opposite phase.”

- In electrons case , the interfering amplitude for exchange interfere with a negative sign. Probability for electron = $\left| f(\theta) - f(\pi - \theta) \right|^2$

SPIN PROBABILITY TABLE

Identical Particles

Identical particles, also called **indistinguishable particle**, are particles that cannot be distinguished from one another.

Identical - Indistinguishable Particles

- Consider particle ’a’ and particle ’b’.
- Let the two particle collide and get sactttered in two different directions say ’1’ and ’2’ over a surface element ds_1 and ds_2 of the detector respectively.
- If the particles are indistinguishable then the amplitudes of these process will add up.
- Probability that the two particles arrive at ds_1 and ds_2 is $\Big| < 1|a >< 2|b > + < 2|a >< 1|b > \Big|^2 ds_1 ds_2$

- Integrating over the area of the detector , if we let ds_1 and ds_2 rangr over the whole area $(\triangle\ S)$, we could count each part of the area twice since the expression $\Big| < 1|a >< 2|b > + < 2|a >< 1|b > \Big|^2 ds_1 ds_2$ contains everything that can happen with any pair of surface elements ds_1 and ds_2.

- $Probability_{BOSE} = \dfrac{\left(4\Big|a\Big|^2\Big|b\Big|^2\right)}{2}(\triangle\ S)$

- This is just twice what we got the probability for distinguished particles.

State with n Bosons

- Consider n particles say a, b, c... scattered in n direction say 1, 2, 3 ...
- Probability that each particle acting alone would go into an element of the surface ds of the detector is $\left| <> \right|^2 ds$.

- **Assumption** :All particles are distinguishable.

- Probability that n particles will be counted together in n different surface elements = $\left|a_1 b_2 c_3 ...\right|^2 ds_1 ds_2 ...$

- If the amplitude does not depend on where ds is located in the detector , then the $Probability = \left(\left|a\right|^2 \left|b\right|^2 ...\right)(ds_1 ds_2 ...)$

- Integrating each dS over the surface $\triangle S$ of the dectector
  $$(P_n)_{different} = \left(\left|a\right|^2 \left|b\right|^2 ... \right) (\triangle S)^n$$
- Now suppose that all the particle are Bose particles.
- For n particles, there are n! different , but indistinguishable possibilities for which we must add the amplitudes.

- Probability that n particles will be counted on the n surface elements is given by
  $$Probability = \left(\left| a_1 b_2 c_3 ... + a_1 b_3 c_2 ... \right|^2 \right) (ds_1 ds_2 ...)$$
- $$Probability = \left(\left| n! abc ... \right|^2 \right) (ds_1 ds_2 ...)$$
- Integrate each ds over the area $\triangle S$ of the detector
  $$(P_n)_{BOSE} = n! \left(\left| abc ... \right|^2 \right) (\triangle S)^n$$

- Compareing the probability when the particles are distinguishable and indistinguishable

$$(P_n)_{BOSE} = n!\left(\left|abc...\right|^2\right)(\triangle S)^n$$

$$(P_n)_{different} = \left(\left|a\right|^2\left|b\right|^2...\right)(\triangle S)^n$$

- $(P_n)_{BOSE} = n!(P_n)_{different}$

What is the probability that a Bose particle will go into particular state when there are already n particles present ?

Emission and Absorption of photons

- When the light is emitted, a photon is ”created”.
- Consider that there are some atom emitting n photons.
- *OBSERVATION* : The probability that an atom will wmit a photon into a particular final state is increased by the factor (n+1) if there are already n photons in that state.

The Blackbody Spectrum

- For each light frequency ω, there are certain N number of atoms which have two energy states separated, given by the equation $E = \omega\hbar$.
- Let N_e and N_g be the average numbers of atoms that are in excited state and ground state.
- In thermal equilibrium at temperature T, from statistical mechanics

$$\frac{N_e}{N_g} = e^{(-\Delta E/\hbar\omega)}$$

- NOTE: Each atom in the ground state can absorb a photon and go into the excited state and each atom in the excited state can emit a photon and go to the ground state.

- At equilibrium, the rate of these two process must be equal.
- Rate is proportional to the probability of the event and the number of atoms present.
- $\overline{n}$ is the average number of photons present in a given state with the frequency ω

- The absorption rate from the state is $N_g \overline{n} \left| a \right|^2$, and the emission rate into that state is $N_e(\overline{n}+1) \left| a \right|^2$.
- At equilibrium $N_g \overline{n} \left| a \right|^2 = N_e(\overline{n}+1) \left| a \right|^2$

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