Understanding the spectrum of a blackbody

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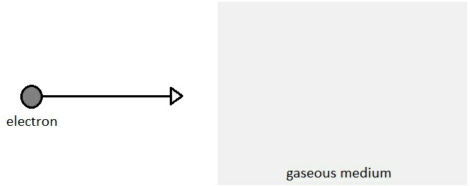
Overview

Overview

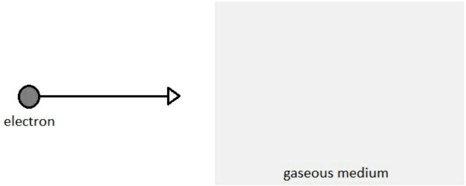
Part 1

An idealised object that can **absorb**, and subsequently **emit**, all the radiation incident on it is called a blackbody.

Rayleigh’s law

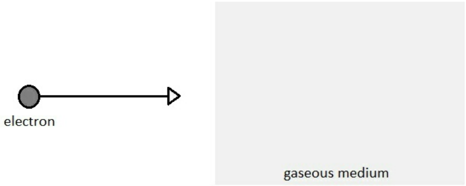


Rayleigh's law



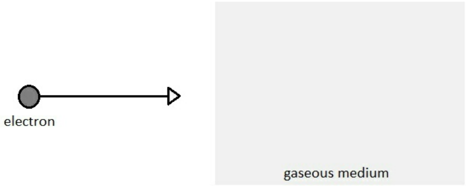
- Consider gaseous medium, in which we have a charged oscilator (Say an electron).

Rayleigh's law



- Consider gaseous medium, in which we have a charged oscillator (Say an electron).
- Due to random motion of the gaseous atoms, it often collides with the oscillator.

Rayleigh's law



- Consider gaseous medium, in which we have a charged oscillator (Say an electron).
- Due to random motion of the gaseous atoms, it often collides with the oscillator.
- As a result the gaseous atoms imparts some of it's kinetic energy to the oscillator.

Rayleigh's law

- Kinetic energy of the oscillator is, $\frac{1}{2}\,kT$.
- And the total kinetic energy will be kT.

Rayleigh's law

- Since the charge is radiating energy
  the system cannot attain equilibrium

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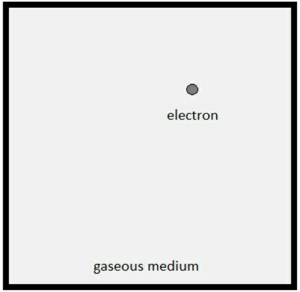
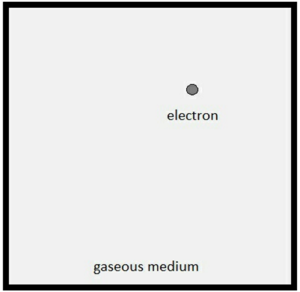


Rayleigh's law

- Since the charge is radiating energy **the system cannot attain equilibrium**

- Inner walls of the box is totally reflective.



Rayleigh's law

So, the energy radiated by the oscillator per second can be given by,

$$\frac{1}{Q} = \frac{\left(\frac{dW}{dt}\right)}{\omega_0 W}$$

Q is the radiation reaction.

ω_0 is the natural frequency of vibration.

W is the total energy content of the oscillator.

Rayleigh's law

Also, we have

$$\frac{1}{Q} = \frac{\gamma}{\omega_0}$$

Where γ is the dampping constant.

Average energy loss due to radiation is given by,

$$\left\langle \frac{dW}{dt} \right\rangle = \gamma kT$$

Since the degree of freedom is 3,

$$\left\langle \frac{dW}{dt} \right\rangle = 3\gamma kT$$

Rayleigh's law

- $\mathbf{I}(\omega)d\omega$ is the incident light energy within a frequeny range $d\omega$.
- **Asumption:** The radiation which falls on a *cross section* 'σ_S' is absorbed completely,
- Scattered light is the product of $\mathbf{I}(\omega)d\omega$ and σ_S.

Rayleigh's law

Expression for the cross section is given by,

$$\sigma_S = \frac{8\pi {r_0}^2}{3}\left[\frac{\omega^4}{(\omega^2 - {\omega_0}^2)^2 + \gamma^2\omega^2}\right]$$

When $\omega = \omega_0$,

$$\sigma_S = \frac{2\pi {r_0}^2 3\omega_0^2}{3(\omega - \omega_0)^2 + \frac{\gamma^2}{4}}$$

Rayleigh's law

$$\frac{dW_s}{dt} = \int_0^\infty \mathbf{I}(\omega)\,\sigma_s\,d\omega = \int_0^\infty \frac{2\pi {r_0}^2 \omega_0^2\,\mathbf{I}(\omega)\,d\omega}{3\left[(\omega-\omega_0)^2 + \frac{\gamma^2}{4}\right]}$$

$\because (\omega = \omega_0)$,

$$\frac{dW_s}{dt} = \frac{2}{3}\pi {r_0}^2 \omega_0^2\,\mathbf{I}(\omega_0) \int_{-\infty}^\infty \frac{d\omega}{(\omega-\omega_0)^2 + \left(\frac{\gamma}{2}\right)^2}$$

Rayleigh's law

The integral reduces to arctan function. We get, $2\pi/\gamma$.

$$\frac{dW_s}{dt} = \frac{dW}{dt} = 3\gamma kT$$

$$\therefore \mathbf{I}(\omega) = \frac{9\gamma^2 kT}{4\pi^2 r_0^2 \omega_0^2}$$

$$\implies \gamma = \frac{2}{3}\frac{r_0\omega_0^2}{c}$$

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The intensity distribution function $\mathbf{I}(\omega)$ is,

$$\mathbf{I}(\omega) = \frac{\omega^2\, kT}{\pi^2 c^2}$$

The experimental observation

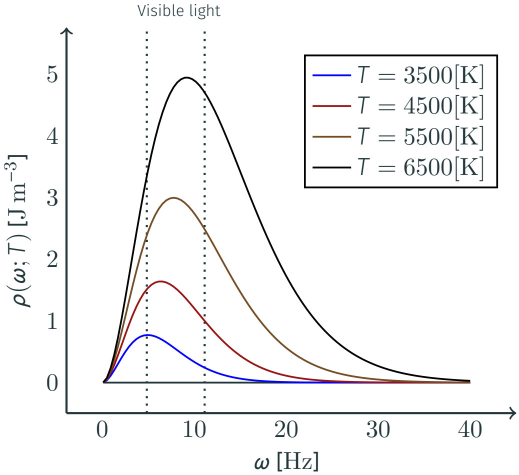
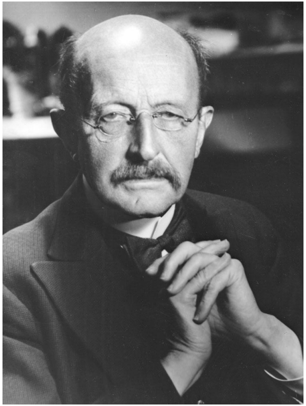


Figure 1: Spectrum of a blackbody.

A theoretical explanation for the obtained experimental results were needed.



- The harmonic oscillator can take up energies only in the multiples of $\hbar\omega$.

- The probability of occupying an energy level **E** is $\mathbf{P}(\mathbf{E}) = \alpha e^{-\hbar\omega/kT}$.

- The number of oscillators in ground state is N_0 .
- The number of oscillator in first state is $N_1 = N_0 e^{-\hbar\omega/kT}$.
- Then $N_n = N_0 x^n$, where $x = e^{-\hbar\omega/kT}$

- The energy in the ground state is zero.
- The energy in the first state is $N_1 \hbar\omega$ or $N_0 \hbar\omega x$
- Total energy is given by

$$E_{(total)} = N_0 \hbar\omega (0 + x + 2x^2 + \cdots)$$

- The total number of oscillators is

$$N_{(total)} = N_0(1 + x + x^2 + \cdots)$$

- The average energy is

$$\langle E \rangle = \frac{E_{(total)}}{N_{(total)}} = \frac{N_0 \hbar\omega(0 + x + 2x^2 + \cdots)}{N_0(1 + x + x^2 + \cdots)}$$

$$Y = 1 + 2x + 3x^2 + \cdots$$

$$Yx = x + 2x^2 + 3x^3 + \cdots$$

$$Y(1 - x) = 1 + x + x^2 + \cdots$$

$$Y = \frac{1}{(1-x)^2}$$

$$\langle E \rangle = \frac{N_0 \hbar\omega x/(1-x)^2}{N_0/(1-x)}$$

$$\langle E\rangle = \frac{\hbar\omega e^{-\hbar\omega/kT}}{1 - e^{-\hbar\omega/kT}} \qquad \because x = e^{-\hbar\omega/kT}$$

$$\boxed{\langle E\rangle = \frac{\hbar\omega}{e^{\hbar\omega/kT} - 1}} \tag{1}$$

Hence The intensity of radiation for frequency ω is given by,

$$I(\omega)d\omega = \frac{\hbar\omega^3 d\omega}{\pi^2 c^2 \left(e^{\hbar\omega/kT} - 1\right)} \tag{2}$$

Graph

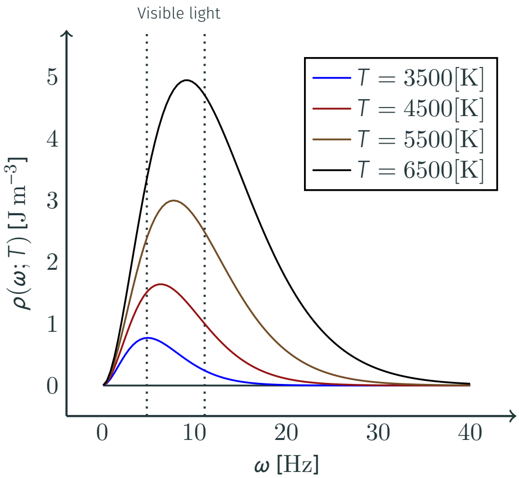


Figure 2: Spectrum of a blackbody.

Part 2

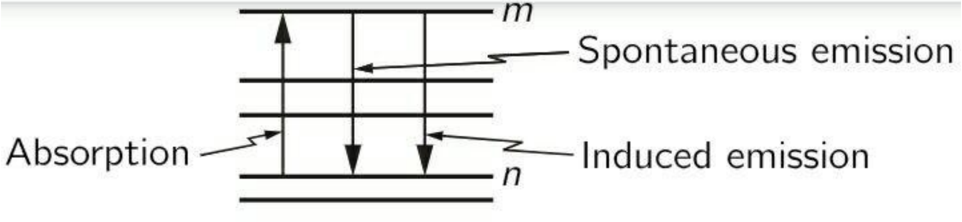
- The intensity of radiation for frequency ω is given by

$$I(\omega)d\omega = \frac{\hbar\omega^3 d\omega}{\pi^2 c^2 \left(e^{\hbar\omega/kT} - 1\right)}$$

- Planck considered that the matter was quantized and not light.

- Einstein came up with an alternate idea to solve Black body problem.

- Einstein considered that even the light was quantized, by taking that light is actually photons with energy $\hbar\omega$



- Einstein assumed that Planck's formula was right.
- Consider two energy levels , say , the n^{th} level and m^{th} level.

- The probability of the transition is proportional to intensity of the light.

$$R_{n \to m} = N_n B_{nm} I(\omega) \tag{3}$$

- Einstein considers that there are two types of emissions that take place in an atom.

- Spontaneous emission.

- Stimulated emission.

- The combined mathematical expression is,

$$R_{m \to n} = N_m [A_{mn} + B_{mn} I(\omega)] \tag{4}$$

- At thermal equilibrium, the number of atoms going to higher energy level ,ust be equal to the number of atoms coming to lower energy state.

- The ratio of N_m to N_n is given by $e^{-\hbar\omega/kT}$

- At thermal equilibrium both $R_{n \to m}$ and $R_{m \to n}$ are equal.

- $N_n B_{nm} I(\omega) = N_m [A_{mn} + B_{mn} I(\omega)]$

- On dividing the previous equation by N_m we get,

- $B_{nm} I(\omega) e^{\hbar\omega/kT} = A_{mn} + B_{mn} I(\omega)$

- The equation that Einsteim got for the intensity of radiation for frequency is

$$I(\omega) = \frac{A_{mn}}{B_{nm} e^{\hbar\omega/kT} - B_{mn}} \tag{5}$$

- Since Einstein considered that the formula given by Planck was correct, we should get ,
- $B_{nm} = B_{mn}$ and also

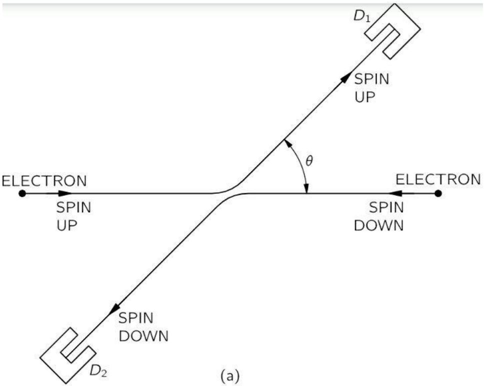
$$\frac{A_{mn}}{B_{nm}} = \frac{\hbar\omega^3 d\omega}{\pi^2 c^2}$$

This is how Einstein rederived Planck’s formula from a quantum mechanical viewpoint.

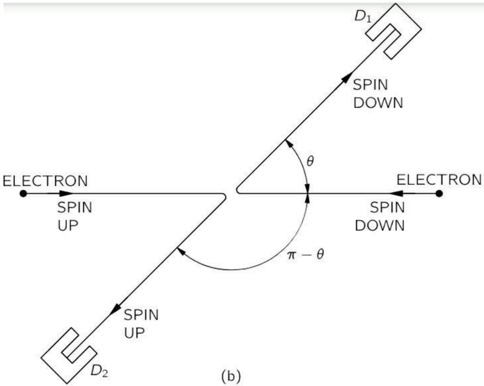
Part 3

- **P** = probability
- ϕ = Probability amplitude
- **P** = $\left|\phi\right|^2$

- $\phi = \phi_1 + \phi_2$
- **P** = $\left|\phi_1 + \phi_2\right|^2$


D1
SPIN
UP
ELECTRON
θ
ELECTRON
SPIN
UP
SPIN
DOWN
SPIN
DOWN
D2

(a)


D1
SPIN
DOWN
θ
ELECTRON
ELECTRON
SPIN
UP
SPIN
DOWN
π − θ
SPIN
UP
D2

(b)

- Detector 1 is set to detect only α particles and Dectector 2 is set to detect only oxygen atoms.

- Probability amplitude of the scattering is given by $f(\theta)$ when they are at an angle θ.

- The probability of this event = $\left|f(\theta)\right|^2$

- Set up the dectectors such that the detectors would dectect either α particle or oxygen atom.

- We will not distinguish which particle is which entering the detector.

- This means that if oxygen atom in position θ , then α particle on the opposite side is at an angle $\pi - \theta$.

- Probability amplitude of oxygen atom = $f(\pi - \theta)$
- Probability amplitude of α particle = $f(\theta)$
- The probability of a particle being detected at detector 1 = $\left|f(\theta)\right|^2$ + $\left|f(\pi - \theta)\right|^2$

- Consider if both are α particles,
- Then we would not know which particle entered the detector, so the total probability changes to,
- The probability of a α particle being detected at detector 1 = $\left|f(\theta) + f(\pi - \theta)\right|^2$

- If $\theta = \dfrac{\pi}{2}$, then applying this to the expression $\left|f(\theta) + f(\pi - \theta)\right|^2$ we get,
- Probability = $4\left|f\left(\dfrac{\pi}{2}\right)\right|^2$, if the particles are indistinguishible.

- Suppose the particles were distinguishible, then the probability for $\theta = \dfrac{\pi}{2}$ when applied for $\left|f(\theta)\right|^2$ + $\left|f(\pi - \theta)\right|^2$ is given by,
- Probability = $2\left|f\left(\dfrac{\pi}{2}\right)\right|^2$
- This shows that the probability gets doubled for indistinguishible particles.

- *Can we apply the same logic to the electron-electron scattering ?*

- *OBSERVATION* : ” When we have situation in which the identity of the electron which is arriving at a point is echanged with another one , the new amplitude interfere with old one with an opposite phase.”

- In electrons case , the interfering amplitude for exchange interfere with a negative sign. Probability for electron = $\left| f(\theta) - f(\pi - \theta) \right|^2$

Scattering of unpolarized spin one-half particles

Fraction of cases	Spin of particle 1	Spin of particle 2	Spin at D_1	Spin at D_2	Probability
$\frac{1}{4}$	up	up	up	up	$\lvert f(\theta) - f(\pi - \theta)\rvert^2$
$\frac{1}{4}$	down	down	down	down	$\lvert f(\theta) - f(\pi - \theta)\rvert^2$
$\frac{1}{4}$	up	down	up	down	$\lvert f(\theta)\rvert^2$
			down	up	$\lvert f(\pi - \theta)\rvert^2$
$\frac{1}{4}$	down	up	up	down	$\lvert f(\pi - \theta)\rvert^2$
			down	up	$\lvert f(\theta)\rvert^2$

Total probability $= \frac{1}{2}\lvert f(\theta) - f(\pi - \theta)\rvert^2 + \frac{1}{2}\lvert f(\theta)\rvert^2 + \frac{1}{2}\lvert f(\pi - \theta)\rvert^2$

Identical particles, also called **indistinguishable particle**, are particles that cannot be distinguished from one another.

Identical - Indistinguishable Particles

- Consider particle ’a’ and particle ’b’.
- Let the two particle collide and get sacttered in two different directions say ’1’ and ’2’ over a surface element ds_1 and ds_2 of the detector respectively.
- If the particles are indistinguishable then the amplitudes of these process will add up.
- Probability that the two particles arrive at ds_1 and ds_2 is
$$\Big| < 1|a >< 2|b > + < 2|a >< 1|b > \Big|^2 ds_1 ds_2$$

- Integrating over the area of the detector , if we let ds_1 and ds_2 range over the whole area $(\vartriangle S)$, we could count each part of the area twice since the expression $\Big| <1|a><2|b> + <2|a><1|b> \Big|^2 ds_1 ds_2$ contains everything that can happen with any pair of surface elements ds_1 and ds_2.

- $Probability_{BOSE} = \dfrac{\left(4\Big|a\Big|^2\Big|b\Big|^2\right)}{2}(\vartriangle S)$

- This is just twice what we got the probability for distinguished particles.

State with n Bose particle

- Consider n particles say a, b, c… scattered in n direction say 1, 2, 3 …
- Probability that each particle acting alone would go into an element of the surface ds of the detector is $\left| <> \right|^2 ds$.

- **Assumption** :All particles are distinguishable.

- Probability that n particles will be counted together in n different surface elements = $\left|a_1 b_2 c_3 ...\right|^2 ds_1 ds_2 ...$

- If the amplitude does not depend on where ds is located in the detector , then the $Probability = \left(\left|a\right|^2 \left|b\right|^2 ...\right)(ds_1 ds_2 ...)$

- Integrating each dS over the surface $\triangle S$ of the dectector

$$(P_n)_{different} = \left(\left|a\right|^2 \left|b\right|^2 ... \right) (\triangle S)^n$$

- Now suppose that all the particle are Bose particles.
- For n particles, there are n! different , but indistinguishable possibilities for which we must add the amplitudes.

- Probability that n particles will be counted on the n surface elements is given by
  $Probability = \left(\left| a_1 b_2 c_3 ... + a_1 b_3 c_2 ... \right|^2 \right) (ds_1 ds_2 ...)$
- $Probability = \left(\left| n! abc... \right|^2 \right) (ds_1 ds_2 ...)$
- Integrate each ds over the area $\triangle S$ of the detector
  $(P_n)_{BOSE} = n! \left(\left| abc... \right|^2 \right) (\triangle S)^n$

- Comparing the probability when the particles are distinguishable and indistinguishable

$$(P_n)_{BOSE} = n!\left(\left| abc... \right|^2 \right) (\triangle S)^n$$

$$(P_n)_{different} = \left(\left| a \right|^2 \left| b \right|^2 ... \right) (\triangle S)^n$$

- $(P_n)_{BOSE} = n!(P_n)_{different}$

- What is the probability that a Bose particle will go into particular state when there are already n particles present ?

Emission and Absorption of photons

- When the light is emitted, a photon is ”created”.
- Consider that there are some atom emitting n photons.
- *OBSERVATION* : The probability that an atom will wmit a photon into a particular final state is increased by the factor (n+1) if there are already n photons in that state.

- In quantum mechanics we can show that
$< \chi|\phi >=< \phi|\chi >^*$

- The amplitude to get from any condition ϕ to any other condition χ.

- We have that amplitude that a photon will be added to some state, say j, when there are already n photons present we can express this conditon as
$< n+1|n >= (\sqrt{n+1})a$
$< n|n+1 >= \left(\sqrt{n+1}\right) a^*$
where $a =< j|a >$ is the amplitude when there are no other photons are present.

- The amplitude to absorb a phot when there are n photons present is given by $< n - 1|n >= (\sqrt{n})a^*$
- $< n + 1|n >= (\sqrt{n + 1})a$
  $< n|n + 1 >= \left(\sqrt{n + 1}\right) a^*$
- The above two equation shows that they are symmetric in nature.

- *Thought Experiment*: Lets us consider that there are n photons that are created in the same state, that of same frequency but they cannot be distinguished .

- The probability that an atom can emit another poton into same state is
$(Probability)_{emit} = (n+1)\left|a\right|^2$

- The probability that an atom absorbs a photon into the same state is
$(Probability)_{absorb} = (n)\left|a\right|^2$

- Rate at which an atom will make a transition to downwards has two parts.

- Probability that it will make a spontaneous transition $\left|a\right|^2$ is proportional to the number of photons.

- The co-efficient of absorption, of induced emission and spontaneous emission are all equal and are related to the probability of spontaneous emission.

The Blackbody Spectrum

- For each light frequency ω, there are certain N number of atoms which have two energy states separated, given by the equation $E = \omega\hbar$.

- Let N_e and N_g be the average numbers of atoms that are in excited state and ground state.

- In thermal equilibrium at temperature T, from statistical mechanics

$$\frac{N_e}{N_g} = e^{\left(\frac{-\Delta E}{\omega\hbar}\right)}$$

- NOTE: Each atom in the ground stae can absorb a photon and go into the excited state and each atom in the excited state can emit a photon and go to the ground state.

- At equilibrium, the rate of these two process must be equal.
- Rate is proportional to the probability of the event and the number of atoms present.
- $\overline{n}$ is the average number of photons present in a given state with the frequency ω

- The absorption rate from the state is $N_g \overline{n} \Big| a \Big|^2$, and the emission rate into that state is $N_e(\overline{n}+1) \Big| a \Big|^2$.
- At equilibrium $N_g \overline{n} \Big| a \Big|^2 = N_e(\overline{n}+1) \Big| a \Big|^2$

- Solving for the average number of photons present in a given state with the frequency ω
  $\overline{n} = \dfrac{1}{e^{\hbar\omega/k_B T - 1}}$

- The energy of each photon is given by $\dfrac{\hbar\omega}{e^{\hbar\omega}/k_B t - 1}$

- For any harmonic oscillator, the quantum mechanical energy levels are equally spaced with a seperation $\hbar\omega$.

- The energy levels are equally spaced and the n^{th} energy level is the the mean enegry of the oscillator.

$$(E)_{mean} = \frac{\hbar\omega}{e^{\hbar\omega}/k_B t - 1}$$

Considering the bose particle which donot interact with each other, and in that state the whole system of particles behaves (for all quantum mechanical purpose) exactly like an harmonic oscillator.

- Analysing the Electro-magnetic field in a box, it show the properties of an harmonic oscillation.
- Thus, the number of photons in a particular state in a box, can be equated to the number of energy levels associated with the particular modes of oscillation of the electromagetic fields.

- Mean energy in any particular modes in a box at a temperature T is given by

$$(E)_{mean} = \frac{\hbar\omega}{e^{\hbar\omega}/k_B t - 1}$$

ASSUMPTIONS

- For every mode there are some atoms in the box, which have energy levels that can radiate into that mode so that each mode can get into thermal equilibrium.
- The assumtion of Blackbody radiation law.

- There will be billions of modes in the box and there will be many small frequency intervals $\Delta\omega$.

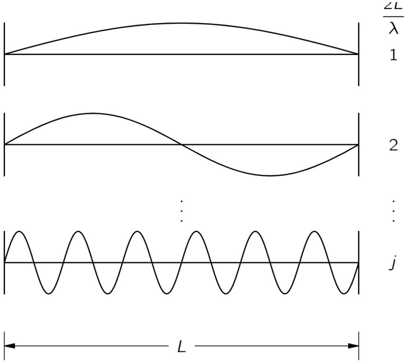


Fig. 4–8. The standing wave modes on a line.

- The wave number k is given by $k = \dfrac{t\pi}{\lambda}$.
- The δk between successive modes is given by

$$\delta k = k_{j+1} - k_j = \frac{\pi}{L}$$

- An asuumption is made that kL is large that in small interval Δk, there are many modes.

- ΔW is the number of modes in the interval Δk.

- This is given by $\Delta W = \dfrac{\Delta k}{\delta k}$

  and $\delta k = \dfrac{j\pi}{L}$.

  Thus, $\Delta W = \dfrac{L(\Delta k)}{\pi}$

  $$\boxed{\Delta W = \frac{L(\Delta k)}{2\pi}}$$

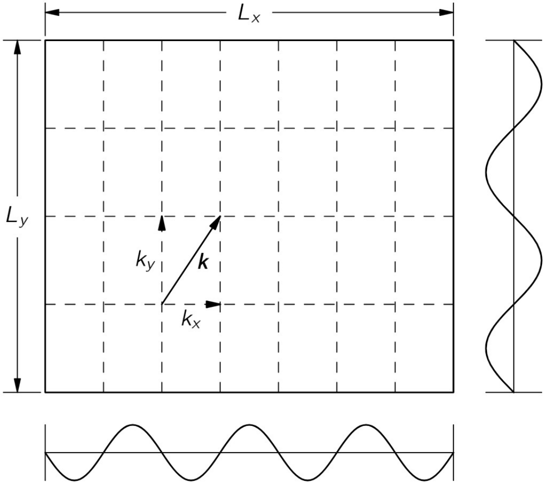


Fig. 4–9. Standing wave modes in two dimensions.

- A standing wave in a rectangular box must have an integral number of half waves along each axis.
- Thus, ΔW the number of modes for a vector wave number **k** between the axes components $k and k + \Delta k$ is

$$\Delta W = \frac{L_x L_y L_z}{(2\pi)^3}(\Delta k_x \Delta k_y \Delta k_z)$$

$$dW(K) = V\frac{d^3 k}{(2\pi)^3}$$

- Applying the above result to find number of photon modes for photons with frequencies in the range Δk.
- In vaccum the magnitude of **k** is related to the frequency by

$$\left|k\right| = \frac{\omega}{c}.$$

- In the frequency interval $\Delta\omega$, these are all the modes which correspond to k's with magnitude between k and $k + \Delta k$, independent of the direction.
- The "volume in the k-space" between k and $k + \Delta k$ is a spherical shell of volume $4\pi(k^2)\Delta k$.

- The number of modes is then ,

$$\Delta W = \frac{V4\pi(k^2)\Delta k}{(2\pi)^3}.$$

substitute $k = \frac{\omega}{c}$

$$\Delta W(\omega) = \frac{V4\pi(\omega^3)\Delta\omega}{(2\pi c)^3}$$

- These modes are independent, we must for double the number of modes.
- This is given by,

$$\Delta W(\omega) = \frac{V\pi(\omega^3)\Delta\omega}{(\pi c)^3} \text{ (for light).}$$

- We have shown that each mode has an average the energy
$\overline{n}\hbar\omega = \dfrac{\hbar\omega}{e^{\hbar\omega/k_B T} - 1}$

- Multiplying This by the number of modes, we get the energy ΔE in the modes that lie in the interval $\Delta\omega$:

$$\Delta E = (\frac{\hbar\omega}{e^{\hbar\omega/k_B T} - 1})(\frac{V\pi(\omega^3)\Delta\omega}{(\pi^2)(c^3)})$$

- The photons are the Bose particles,which have tendency to try to get to all into the same state.

Thank you