

Spectrum of Blackbody

Sudheendra B.R.

Suhas P.K.

Shahsank K.K.

Guided by V.H. Belvadi
Yuvaraja's College, Mysuru

Part 1

Introduction to the Black Body problem

Charged oscillators

Rayleigh's intensity function

The Ultra-Violet catastrophe

Plank's empirical formula

Part 2

Intensity of a Black Body spectrum and the problem

Einstein's work on Black Bodies

Part 3

Part 1

An idealised object that can **absorb** and subsequently **emit**, all the radiation incident on it is called a **Blackbody**.

- Consider gaseous medium, in which we have a charged oscillator.
- Due to random motion of the gaseous atoms, it often collides with the oscillator.

- Consider gaseous medium, in which we have a charged oscillator.
- Due to random motion of the gaseous atoms, it often collides with the oscillator.
- As a result the gaseous atoms imparts some of it's kinetic energy to the oscillator.

- Mean kinetic energy acquired by the oscillator is, $\frac{1}{2} kT$.
- And the total kinetic energy will be kT .

- Since the charge is radiating energy **the system cannot attain equilibrium.**

- How do we tackle this situation?

- How do we tackle this situation?
- We confine the system.

- Inner walls of the box is totally reflective.
- After some time the system will attains **thermal equilibrium**

So, the energy radiated by the oscillator per second can be given by,

$$\frac{1}{Q} = \frac{\left(\frac{dW}{dt}\right)}{\omega_0 W}$$

Q is the radiation reaction.

ω_0 is the natural frequency of vibration.

W is the total energy content of the oscillator.

Also, we have

$$\frac{1}{Q} = \frac{\gamma}{\omega_0}$$

Where γ is the damping constant.

Remember the mean energy of the oscillator is kT ? for three degrees of freedom, it is $3kT$.

Average energy loss due to radiation is given by,

$$\left\langle \frac{dW}{dt} \right\rangle = \gamma kT$$

- Let, $I(\omega)d\omega$ be the incident light energy with in a frequency range $d\omega$.
- Assuming all the radiation which falls on a *cross section* ' σ_s ' is absorbed completely,
- Scattered light must be the product of $I(\omega)d\omega$ and σ_s .

Expression for the cross section is given by,

$$\sigma_s = \frac{8\pi r_0^2}{3} \left[\frac{\omega^4}{(\omega^2 - \omega_0^2)^2 + \gamma^2 \omega^2} \right]$$

When $\omega = \omega_0$,

$$\sigma_s = \frac{2\pi r_0^2 3\omega_0^2}{3(\omega - \omega_0)^2 + \frac{\gamma^2}{4}}$$

Rayleigh's law

$$\frac{dW_s}{dt} = \int_0^\infty \mathbf{I}(\omega) \sigma_s d\omega = \int_0^\infty \frac{2\pi r_0^2 \omega_0^2 \mathbf{I}(\omega) d\omega}{3 \left[(\omega - \omega_0)^2 + \frac{\gamma^2}{4} \right]}$$

since we're considering the equilibrium condition ($\omega = \omega_0$), equation reduces to,

$$\frac{dW_s}{dt} = \frac{2}{3} \pi r_0^2 \omega_0^2 \mathbf{I}(\omega_0) \int_{-\infty}^\infty \frac{d\omega}{(\omega - \omega_0)^2 + \left(\frac{\gamma}{2}\right)^2}$$

Since the integral reduces to arctan function we get, $\frac{2\pi}{\gamma}$. and as

$$\frac{dW_s}{dt} = \frac{dW}{dt}$$

,

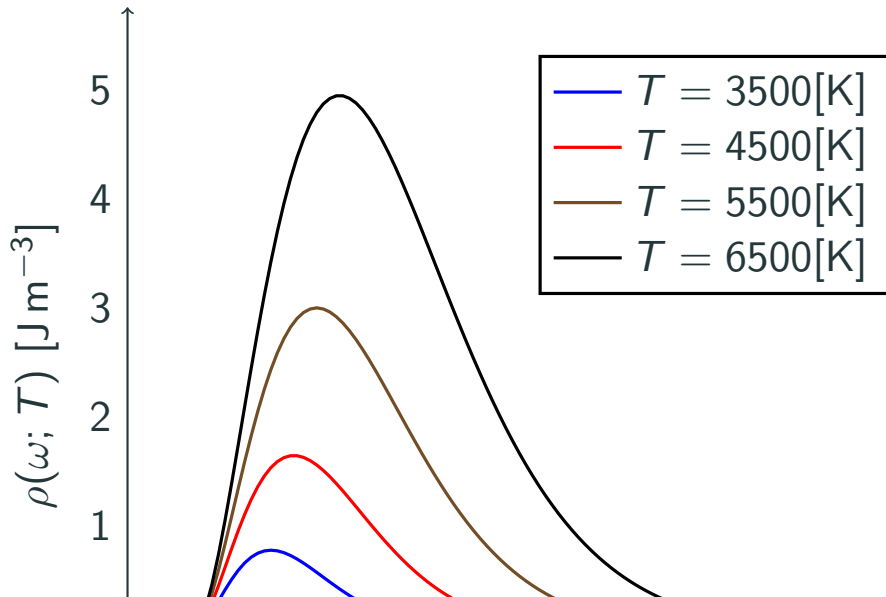
$$I(\omega) = \frac{9\gamma^2 kT}{4\pi^2 r_0^2 \omega_0^2}$$

The value of γ was found to be equal to,

$$\frac{23}{3} \frac{r_0 \omega_0^2}{c}$$

$$I(\omega) = \frac{\omega^2 kT}{\pi^2 c^2}$$

Graph



- Max Planck came up with an idea.
- We had the experimental result, and what we needed was a theoretical explanation for the curve.
- Planck studied the curve and came up with a mathematical formula.

- He made an assumption that the energy level of the harmonic oscillator is quantized.
- The harmonic oscillator can take up energies only in the multiples of $\hbar\omega$.
- The probability of occupying an energy level \mathbf{E} is $\mathbf{P}(\mathbf{E}) = \alpha e^{\frac{-\hbar\omega}{kT}}$.

Part 2

- The intensity of radiation for frequency ω is given by

$$I(\omega)d\omega = \frac{\hbar\omega^3 d\omega}{\pi^2 c^2 \left(e^{\frac{\hbar\omega}{kT}} - 1 \right)}$$

- Planck considered that the matter was quantized and not light.
- Einstein came up with an alternate idea to solve Black body problem.

- Einstein considered that even the light was quantized, by taking that light is actually photons with energy $\hbar\omega$
- a pic is needed.

- Einstein assumed that Planck's formula was right.
- Consider two energy levels , say , the n^{th} level and m^{th} level.
- A light of certain frequency is incident on atom and by absorbing a photon, transition from state n to state m occurs.

- The probability of the transition is proportional to intensity of the light.
- The proportionality constant is B_{nm} and mathematically can be written as,

$$R_{n \rightarrow m} = N_n B_{nm} I(\omega) \quad (1)$$

- Einstein considers that there are two types of emissions that take place in an atom.
- Spontaneous emission.
- Stimulated emission.

- The combined mathematical expression is,

$$R_{m \rightarrow n} = N_m[A_{mn} + B_{mn}I(\omega)] \quad (2)$$

- At thermal equilibrium, the number of atoms going to higher energy level must be equal to the number of atoms coming to lower energy state.
- The ratio of N_m to N_n is given by $e^{\frac{-\hbar\omega}{kT}}$

- At thermal equilibrium both $R_{n \rightarrow m}$ and $R_{m \rightarrow n}$ are equal.
- $N_n B_{nm} I(\omega) = N_m [A_{mn} + B_{mn} I(\omega)]$
- On dividing the previous equation by N_m we get,

- $B_{nm}I(\omega)e^{\frac{\hbar\omega}{kT}} = A_{mn} + B_{mn}I(\omega)$
- So the equation that Einstein got for intensity of radiation for frequency is

$$I(\omega) = \frac{A_{mn}}{B_{nm}e^{\frac{\hbar\omega}{kT}} - B_{mn}} \quad (3)$$

- Since Einstein considered that the formula given by Planck was correct, we should get ,
- $B_{nm} = B_{mn}$ and also

$$\frac{A_{mn}}{B_{nm}} = \frac{\hbar\omega^3 d\omega}{\pi^2 c^2}$$

This is how Einstein rederived Planck's formula from a quantum mechanical viewpoint.

Part 3

- **P** = probability
- ϕ = Probability amplitude
- **P** = $\left| \phi \right|^2$

- $\phi = \phi_1 + \phi_2$

- $\mathbf{P} = \left| \phi_1 + \phi_2 \right|^2$

- Detector 1 is set to detect only α particles and Detector 2 is set to detect only oxygen atoms.
- Probability amplitude of the scattering is given as $f(\theta)$ when they are at an angle θ .
- The probability of this event = $\left| f(\theta) \right|^2$

Case B

- Set up the detectors such that the detectors would detect either α particle or oxygen atom.
- We will not distinguish which particle is which entering the detector.
- This means that if oxygen atom in position θ , then α particle on the opposite side is at an angle $\pi - \theta$.

- Probability amplitude of oxygen atom = $f(\pi - \theta)$
- Probability amplitude of α particle = $f(\theta)$
- The probability of a particle being detected at detector 1 = $\left| f(\theta) \right|^2 + \left| f(\pi - \theta) \right|^2$

- Consider if both are α particles,
- Then we would not know which particle entered the detector, so the total probability changes to,
- The probability of a α particle being detected at detector 1 = $\left| f(\theta) + f(\pi - \theta) \right|^2$

- If $\theta = \frac{\pi}{2}$, then applying this to the expression $\left| f(\theta) + f(\pi - \theta) \right|^2$ we get,
- Probability = $4 \left| f\left(\frac{\pi}{2}\right) \right|^2$, if the particles are indistinguishable.

- Suppose the particles were distinguishable, then the probability for $\theta = \frac{\pi}{2}$ when applied for $\left| f(\theta) \right|^2 + \left| f(\pi - \theta) \right|^2$ is given by,
- Probability = $2 \left| f\left(\frac{\pi}{2}\right) \right|^2$
- This shows that the probability gets doubled for indistinguishable particles.

- *Can we apply the same logic to the electron-electron scattering ?*
- *OBSERVATION : " When we have situation in which the identity of the electron which is arriving at a point is exchanged with another one , the new amplitude interfere with old one with an opposite phase."*
- In electrons case , the interfering amplitude for exchange interfere with a negative sign.

$$\text{Probability for electron} = \left| f(\theta) - f(\pi - \theta) \right|^2$$

SPIN PROBABILITY TABLE

Identical particles , also called **indistinguishable particle** , are particles that cannot be distinguished from one another.

Identical - Indistinguishable Particles

- Consider particle 'a' and particle 'b'.
- Let the two particles collide and get scattered in two different directions say '1' and '2' over a surface element ds_1 and ds_2 of the detector respectively.
- If the particles are indistinguishable then the amplitudes of these processes will add up.
- Probability that the two particles arrive at ds_1 and ds_2 is
$$\left| \langle 1|a \rangle \langle 2|b \rangle + \langle 2|a \rangle \langle 1|b \rangle \right|^2 ds_1 ds_2$$

- Integrating over the area of the detector , if we let ds_1 and ds_2 rangr over the whole area (ΔS) , we could count each part of the area twice since the expression $\left| \langle 1|a \rangle \langle 2|b \rangle + \langle 2|a \rangle \langle 1|b \rangle \right|^2 ds_1 ds_2$ contains everything that can happen with any pair of surface elements ds_1 and ds_2 .

- $Probability_{BOSE} = \frac{\left(4 |a|^2 |b|^2 \right)}{2} (\Delta S)$

- This is just twice what we got the probability for distinguished particles.

State with n Bose particle

- Consider n particles say a, b, c... scattered in n direction say 1, 2, 3 ...
- Probability that each particle acting alone would go into an element of the surface ds of the detector is $\left| \langle \rangle \right|^2 ds$.

- **Assumption** :All particles are distinguishable.
- Probability that n particles will be counted together in n different surface elements

$$= \left| a_1 b_2 c_3 \dots \right|^2 ds_1 ds_2 \dots$$
- If the amplitude does not depend on where ds is located in the detector , then the

$$Probability = \left(\left| a \right|^2 \left| b \right|^2 \dots \right) (ds_1 ds_2 \dots)$$

- Integrating each dS over the surface ΔS of the dectector

$$(P_n)_{different} = \left(|a|^2 |b|^2 \dots \right) (\Delta S)^n$$

- Now suppose that all the particle are Bose particles.
- For n particles, there are $n!$ different , but indistinguishable possibilities for which we must add the amplitudes.

- Probability that n particles will be counted on the n surface elements is given by

$$Probability = \left(\left| a_1 b_2 c_3 \dots + a_1 b_3 c_2 \dots \right|^2 \right) (ds_1 ds_2 \dots)$$

- $Probability = \left(\left| n! abc \dots \right|^2 \right) (ds_1 ds_2 \dots)$

- Integrate each ds over the area ΔS of the detector

$$(P_n)_{BOSE} = n! \left(\left| abc \dots \right|^2 \right) (\Delta S)^n$$

- Comparing the probability when the particles are distinguishable and indistinguishable

$$(P_n)_{BOSE} = n! \left(|abc\dots|^2 \right) (\Delta S)^n$$

$$(P_n)_{different} = \left(|a|^2 |b|^2 \dots \right) (\Delta S)^n$$

- $(P_n)_{BOSE} = n! (P_n)_{different}$

What is the probability that a Bose particle will go into particular state when there are already n particles present ?

Emission and Absorption of photons

- When the light is emitted, a photon is "created".
- Consider that there are some atom emitting n photons.
- *OBSERVATION* : The probability that an atom will emit a photon into a particular final state is increased by the factor $(n+1)$ if there are already n photons in that state.

The Blackbody Spectrum

The Blackbody Spectrum

- For each light frequency ω , there are certain N number of atoms which have two energy states separated, given by the equation $E = \omega\hbar$.
- Let N_e and N_g be the average numbers of atoms that are in excited state and ground state.
- In thermal equilibrium at temperature T , from statistical mechanics
- NOTE: Each atom in the ground state can absorb a photon and go into the excited state and each atom in the excited state can emit a photon and go to the ground state.

- At equilibrium, the rate of these two process must be equal.
- Rate is proportional to the probability of the event and the number of atoms present.
- \bar{n} is the average number of photons present in a given state with the frequency ω

- The absorption rate from the state is $N_g \bar{n} |a|^2$, and the emission rate into that state is $N_e(\bar{n} + 1) |a|^2$.
- At equilibrium $N_g \bar{n} |a|^2 = N_e(\bar{n} + 1) |a|^2$