Tutorial 1 – Solutions

Probability MB5042

1. Consider the integers between 0 and 999 inclusive.

   (a) In how many of these numbers does the digit 9 occur at least once?

   (b) In how many of these numbers does the digit 9 occur exactly once?

   (c) How many of these numbers contain both the digits 8 and 9 at least once?

Question 1 answers.

(a) In how many integers between 0 and 999 inclusive does the digit 9 occur at least once.

If we use the 10 digits 0 to 9, and allow numbers such as 089 to stand for 89, we have 10^3 possible numbers, including 0. Those that do not contain any 9 are formed from the 9 digits 0 to 8, and there are 9^3 of them, including 0. The difference in these numbers, $10^3 - 9^3 = 271$ is the number of integers that contain at least one 9.

(b) In how many of these numbers does the digit 9 occur exactly once?

If 9 is the first digit, we have 9×9 choices for the other two digits. The same holds if 9 is the second digit, or if 9 is the third digit. So we have $3 \times 9 \times 9 = 243$ numbers that contain the digit 9 exactly once.

(c) How many of these numbers contain both the digits 8 and 9 at least once?

We first count how many numbers contain both the digits 8 and 9 exactly once. We have 8 choices for a third different digit. Given 8, 9 and another digit, we have 3!=6 ways of arranging these digits into a number. So, we have $8 \times 6 = 48$ numbers that contain each of 8 and 9 exactly once. Finally we have the numbers 889, 898, 988, 899, 989, 998 which contain only 8's and 9's. The total is $48 + 6 = 54$.

2. (a) How many different four-digit whole numbers can be formed from the digits 1, 2, 3 and 4 if repeated digits are allowed?
   (b) What is the answer if no repeated digits are allowed?
   (c) What is the answer if the digit 4 may be repeated but the digits 1, 2 and 3 are allowed to occur at most once?
   (d) How many integers between 100 and 999 contain three different digits?

Question 2 answers.

(a) How many different four-digit whole numbers can be formed from the digits 1, 2, 3 and 4 if repeated digits are allowed?

The answer is $4^4 = 256$.

(b) What is the answer if no repeated digits are allowed?

The answer here is $4! = 24$, since in this case we are just permuting the digits 1, 2, 3, 4.

(c) What is the answer if the digit 4 may be repeated but the digits 1, 2 and 3 are allowed to occur at most once?

We have 24 numbers where all the digits are different. If we have exactly two 4's in the digits, we can choose two more digits from 1, 2, and 3 in 3 ways. For each choice of these two digits from 1, 2, and 3, we have

$$\frac{4!}{2!} = 12$$

ways to arrange these digits with the two 4's. This gives $3 \times 12 = 36$ numbers with exactly two 4's.

If we use exactly three 4's, we have 3 choices for one additional digit from 1, 2, and 3. We can then arrange the three 4's and one different digit in

$$\frac{4!}{3!} = 4$$

ways. This gives $3 \times 4 = 12$ numbers with exactly three 4's.

Finally, there is just one number with exactly four 4's. The total is

$$24 + 36 + 12 + 1 = 73.$$

(d) How many integers between 100 and 999 contain three different digits?

Let's first count how many integers between 0 and 999 contain three different digits. We have 10 digits to use, and 10 choices for the first position, 9 for the second, and 8 for the third. This gives $10 \times 9 \times 8 = 720$ integers. Now let's count how many begin with a 0. If an integer begins with a zero and has two other different digits, there are 9 choices for the second, and 8 for the third, giving $9 \times 8 = 72$ integers. When we subtract this number from the first, we get the required total of $720 - 72 = 648$.

3. (a) In how many ways can you choose four different *even* numbers from the numbers $1, 2, 3, \ldots, 20$?

Question 3 answer.

(a) In how many ways can you choose four different *even* numbers from the numbers $1, 2, 3, \ldots, 20$?

There are exactly 10 even numbers among $1, 2, 3, \ldots, 20$? Thus the answer is

$$\binom{10}{4} = 210$$

4. (a) In a football league, there are twenty teams and each team plays each other team twice, once at home and once away. What is the total number of games played in the course of this league's season?

Question 4 answer.

(a) We may think of the games as ordered 2-subsets of 20 teams. The subsets are ordered, because there is a home team and away team for each choice of two teams. The number of games is $20 \times 19 = 380$.

5. (a) A committee of four people is to be chosen from six men and six women. In how many ways can this be done? In how many of the committees so formed are there more men members than women.

Question 5 answers.

(a) In the first case, we form a 4-subset from 12 people. We can do this in

$$\binom{12}{4} = 495$$

ways.

If there are more men than women on the committee, we either have 3 men or 4 men. If we have 3 men,we can choose them in $\binom{6}{3} = 20$ ways and then we have 6 choices for one woman to join them. This gives $20 \times 6 = 120$ possibilities with exactly 3 men. If we have 4 men, we can choose them in $\binom{6}{4} = 15$ ways. The total number is $120 + 15 = 135$.

6. (a) Show that

$$\frac{n!}{(n-2)!} + \frac{(n-1)!}{n!} = \frac{n^3 - n^2 + 1}{n}.$$

(b) Let $n \geq 3$ be a whole number. If

$$3\binom{n}{3} = 5\binom{n}{2},$$

find n.

(c) Let $r \geq 1$ be a whole number. If

$$\binom{15}{r} = \binom{15}{2r},$$

find r.

Question 6 answers.

(a) Show that

$$\frac{n!}{(n-2)!} + \frac{(n-1)!}{n!} = \frac{n^3 - n^2 + 1}{n}.$$

Now we know that $n! = n \times (n-1) \times (n-2)!$. Thus

$$\frac{n!}{(n-2)!} = n(n-1).$$

Similarly, as $n! = n \times (n-1)!$, we have

$$\frac{(n-1)!}{n!} = \frac{1}{n}$$

Hence

$$\frac{n!}{(n-2)!} + \frac{(n-1)!}{n!} = n(n-1) + \frac{1}{n} = \frac{n^3 - n^2 + 1}{n}.$$

(b) We have seen elsewhere that

$$\binom{n}{2} = \frac{n(n-1)}{2}, \quad \binom{n}{3} = \frac{n(n-1)(n-2)}{6}$$

Thus our equation

$$3\binom{n}{3} = 5\binom{n}{2},$$

translates into

$$\frac{3n(n-1)(n-2)}{6} = \frac{5n(n-1)}{2}.$$

We cancel the terms n and $n-1$, which are irrelevant, and get

$$\frac{(n-2)}{2} = \frac{5}{2}.$$

This leads to $n = 7$.

(c) Let $r \geq 1$ be a whole number. If

$$\binom{15}{r} = \binom{15}{2r},$$

find r.

We can have the solution $r = 2r$, which leads to $r = 0$. Although technically correct, we have assumed at the start that $r \geq 1$ and so ignore this solution. Since

$$\binom{15}{2r} = \binom{15}{15-2r},$$

another solution occurs when

$$r = 15 - 2r.$$

This gives $r = 5$ and we do indeed have the solution

$$\binom{15}{5} = \binom{15}{10}.$$

7. (a) Expand

$$\left(x^2 - \frac{1}{x}\right)^6$$

using the binomial theorem

(b) Find the coefficient of x^3 in the expansion of $(x+1)(x-2)^5$.

Question 7 answers.

(a) Using the binomial theorem,

$$\begin{aligned}\left(x^2 - \frac{1}{x}\right)^6 &= (x^2)^6 + \binom{6}{1}(x^2)^5\left(-\frac{1}{x}\right) + \binom{6}{2}(x^2)^4\left(-\frac{1}{x}\right)^2 \\ &+ \binom{6}{3}(x^2)^3\left(-\frac{1}{x}\right)^3 + \binom{6}{4}(x^2)^2\left(-\frac{1}{x}\right)^4 \\ &+ \binom{6}{5}(x^2)\left(-\frac{1}{x}\right)^5 + \left(-\frac{1}{x}\right)^6\end{aligned}$$

This works out to be

$$x^{12} - 6x^9 + 15x^6 - 20x^3 + 15 - \frac{6}{x^3} + \frac{1}{x^6}.$$

(b) Using the binomial theorem,

$$\begin{aligned}(x-2)^5 &= x^5 + \binom{5}{1}x^4(-2) + \binom{5}{2}x^3(-2)^2 + \binom{5}{3}x^2(-2)^3 \\ &+ \binom{5}{4}x(-2)^4 + (-2)^5 \\ &= x^5 - 10x^4 + 40x^3 - 80x^2 + 80x - 32\end{aligned}$$

We now form the product

$$(x+1)(x^5 - 10x^4 + 40x^3 - 80x^2 + 80x - 32).$$

The term x^3 in this product arises in two ways: as $1 \times 40x^3$ and as $x \times (-80x^2)$. The coefficient of x^3 is $40 - 80 = -40$.

8. (a) In how many ways may 12 white draughts pieces be placed on the 32 black squares of a draughts board if no more than one can be placed on each black square?

   (b) In how many ways may 12 white draughts pieces be placed on the 32 black squares of a draughts board if any number from 0 to 12 may be placed on a black square.? (Draughts pieces of the same colour are taken to be indistinguishable. The squares are all distinct.)

Question 8 answers.

(a) We just choose 12 squares out of 32 available and get

$$\binom{32}{12},$$

a very large number!

(b) In this case, we distribute 12 identical objects to the 32 squares available, and get

$$\binom{43}{12},$$

an even larger number!