MB5042 Probability DCU PDMT

Tutorial 1
Detiled Solutions

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30 January 2025

Contents

Problem 1: Integers Between 0 and 999 Inclusive

Consider the integers between 0 and 999 inclusive.

(a) In how many of these numbers does the digit 9 occur at least once?

(b) In how many of these numbers does the digit 9 occur exactly once?

(c) How many of these numbers contain both the digits 8 and 9 at least once?

Solution

Understanding the Problem

Before diving into calculations, it's crucial to comprehend what each part of the problem is asking:

- **Part (a):** We need to find out how many numbers between 0 and 999 contain the digit **9** at least once.
- **Part (b):** Here, we're interested in numbers where the digit **9** appears **exactly once**.
- **Part (c):** This part requires us to determine how many numbers include **both** the digits **8** and **9** at least once.

To tackle these questions, we'll consider the range from 000 to 999, treating each number as a three-digit number by adding leading zeros where necessary. This approach simplifies the analysis since each digit position (hundreds, tens, units) can be treated uniformly.

Part (a): Numbers with at Least One Digit 9

Objective: Count all numbers between 0 and 999 that have the digit 9 appearing at least once.
Approach:

1. **Total Numbers:** First, recognize that there are 1000 numbers in total (from 000 to 999).
2. **Complementary Counting:** Instead of directly counting numbers with at least one 9, it's easier to count the numbers that **do not** have any 9s and subtract this from the total.
3. **Counting Numbers Without Any 9s:**
   - **Hundreds Place:** Can be any digit from 0 to 8 (9 options).
   - **Tens Place:** Similarly, 0 to 8 (9 options).
   - **Units Place:** Again, 0 to 8 (9 options).

   Therefore, the total number of numbers without any 9s is $9 \times 9 \times 9 = 729$.
4. **Calculating Numbers with At Least One 9:**

$$\text{Total numbers} - \text{Numbers without any 9s} = 1000 - 729 = 271$$

Answer: There are **271** numbers between 0 and 999 that contain the digit 9 at least once.

Part (b): Numbers with Exactly One Digit 9

Objective: Determine how many numbers between 0 and 999 have the digit 9 appearing exactly once.
Approach:

1. **Identify Positions for the Digit 9:** The digit 9 can appear in any one of the three positions (hundreds, tens, or units).
2. **Calculate for Each Position:**
   - **Case 1:** 9 is in the **hundreds place**.
     - **Hundreds Place:** 9 (1 option).

  - **Tens Place:** 0 to 8 (9 options).
  - **Units Place:** 0 to 8 (9 options).
  - **Total for this case:** $1 \times 9 \times 9 = 81$.
- **Case 2:** 9 is in the **tens place**.
  - **Hundreds Place:** 0 to 8 (9 options).
  - **Tens Place:** 9 (1 option).
  - **Units Place:** 0 to 8 (9 options).
  - **Total for this case:** $9 \times 1 \times 9 = 81$.
- **Case 3:** 9 is in the **units place**.
  - **Hundreds Place:** 0 to 8 (9 options).
  - **Tens Place:** 0 to 8 (9 options).
  - **Units Place:** 9 (1 option).
  - **Total for this case:** $9 \times 9 \times 1 = 81$.

3. **Summing All Cases:**

$$81(\text{Case 1}) + 81(\text{Case 2}) + 81(\text{Case 3}) = 243$$

Answer: There are **243** numbers between 0 and 999 that contain the digit 9 exactly once.

Part (c): Numbers Containing Both Digits 8 and 9 at Least Once

Objective: Find how many numbers between 0 and 999 include both the digits 8 and 9 at least once.
Approach:

1. **Total Numbers:** As before, there are 1000 numbers in total.
2. **Use Inclusion-Exclusion Principle:** To find numbers that contain both 8 and 9 at least once, we can use the principle of inclusion-exclusion.

$$\begin{aligned}\text{Numbers with at least one 8 and at least one 9} &= \text{Numbers with at least one 8}\\ &+ \text{Numbers with at least one 9}\\ &- \text{Numbers with at least one 8 or 9}\end{aligned}$$

However, a more straightforward approach is to calculate:

$$\begin{aligned}\text{Numbers with at least one 8 and at least one 9} &= \text{Total numbers}\\ &- \text{Numbers without any 8}\\ &- \text{Numbers without any 9}\\ &+ \text{Numbers without any 8 or 9}\end{aligned}$$

3. **Calculate Each Component:**
   - **Numbers without any 8:**
     - **Hundreds Place:** 0-7,9 (9 options).
     - **Tens Place:** 0-7,9 (9 options).
     - **Units Place:** 0-7,9 (9 options).
     - **Total:** $9 \times 9 \times 9 = 729$.
   - **Numbers without any 9:**
     - As calculated in part (a), this is also 729.
   - **Numbers without any 8 or 9:**
     - **Hundreds Place:** 0-7 (8 options).
     - **Tens Place:** 0-7 (8 options).
     - **Units Place:** 0-7 (8 options).
     - **Total:** $8 \times 8 \times 8 = 512$.

4. **Apply Inclusion-Exclusion:**

   Numbers with at least one 8 and at least one $9 = 1000 - 729 - 729 + 512 = 1000 - 1458 + 512 = 54$

Answer: There are **54** numbers between 0 and 999 that contain both the digits 8 and 9 at least once.

Verification and Reflection

After performing these calculations, it's essential to verify the results to ensure accuracy.

- **Part (a):** The complementary counting method is reliable, and the subtraction yields 271 numbers with at least one 9.
- **Part (b):** By considering each position separately and summing the possibilities, we arrive at 243 numbers with exactly one 9.
- **Part (c):** Applying the inclusion-exclusion principle correctly accounts for overlaps, resulting in 54 numbers containing both 8 and 9.

These methods provide a systematic approach to solving combinatorial problems involving digit constraints. Breaking down each part and verifying each step ensures that the solutions are both accurate and understandable.

Final Answers

- **(a) 271** numbers contain the digit 9 at least once.
- **(b) 243** numbers contain the digit 9 exactly once.
- **(c) 54** numbers contain both the digits 8 and 9 at least once.

Problem 2: Four-Digit and Three-Digit Numbers

(a) How many different four-digit whole numbers can be formed from the digits 1, 2, 3, and 4 if repeated digits are allowed?

(b) What is the answer if no repeated digits are allowed?

(c) What is the answer if the digit 4 may be repeated but the digits 1, 2, and 3 are allowed to occur at most once?

(d) How many integers between 100 and 999 contain three different digits? (Remember, 0 can be a digit, but it cannot be the first digit.)

Solution

Understanding the Problem

We are given a series of questions related to forming four-digit whole numbers using the digits 1, 2, 3, and 4 under different constraints. Additionally, there's a question about three-digit integers with unique digits. Let's break down each part:

- **Part (a):** Form four-digit numbers using digits 1, 2, 3, and 4 **with repetition allowed**.
- **Part (b):** Form four-digit numbers using digits 1, 2, 3, and 4 **without repetition**.
- **Part (c):** Form four-digit numbers where the digit **4 can be repeated**, but digits **1, 2, and 3 can occur at most once**.
- **Part (d):** Determine how many three-digit integers (from 100 to 999) have **three different digits**, noting that **0 cannot be the first digit**.

Part (a): Four-Digit Numbers with Repeated Digits Allowed

Objective: Find how many different four-digit numbers can be formed using digits 1, 2, 3, and 4 with repetition allowed.

Approach:

1. **Digits Available:** 1, 2, 3, 4.
2. **Number of Positions:** Four (thousands, hundreds, tens, units).
3. **Repetition Allowed:** Each digit can be used more than once.
4. **Calculating Possibilities:**
   - **Thousands Place:** Can be any of the 4 digits.
   - **Hundreds Place:** Can be any of the 4 digits.
   - **Tens Place:** Can be any of the 4 digits.
   - **Units Place:** Can be any of the 4 digits.

   Therefore, total numbers $= 4 \times 4 \times 4 \times 4 = 4^4 = 256$.

Answer: 256 different four-digit numbers can be formed with repetition allowed.

Part (b): Four-Digit Numbers without Repeated Digits

Objective: Determine how many different four-digit numbers can be formed using digits 1, 2, 3, and 4 without repeating any digit.

Approach:

1. **Digits Available:** 1, 2, 3, 4.
2. **Number of Positions:** Four.

3. **No Repetition:** Each digit must be unique in the number.

4. **Calculating Possibilities:**

   - **Thousands Place:** 4 choices (1, 2, 3, 4).
   - **Hundreds Place:** 3 remaining choices.
   - **Tens Place:** 2 remaining choices.
   - **Units Place:** 1 remaining choice.

   Therefore, total numbers $= 4 \times 3 \times 2 \times 1 = 24$.

Answer: 24 different four-digit numbers can be formed without repeating any digit.

Part (c): Four-Digit Numbers with 4 Repeatable and 1, 2, 3 Non-Repeatable

Objective: Find how many different four-digit numbers can be formed where the digit **4 can be repeated**, but digits **1, 2, and 3 can occur at most once**.

Approach:

1. **Digits Available:** 1, 2, 3, 4.

2. **Constraints:**

   - Digit **4** can be used multiple times.
   - Digits **1, 2, 3** can each be used at most once.

3. **Possible Scenarios:**

   - **Case 1:** The number contains **no 4s**.
   - **Case 2:** The number contains **one 4**.
   - **Case 3:** The number contains **two 4s**.
   - **Case 4:** The number contains **three 4s**.
   - **Case 5:** The number contains **four 4s**.

4. **Calculating Each Case:**

   - **Case 1: No 4s**
     - Only digits 1, 2, 3 are used, each at most once.
     - This is equivalent to forming a four-digit number with digits 1, 2, 3 without repetition.
     - However, we only have three digits, and we need four digits. This is impossible.
     - **Total for Case 1:** 0.
   - **Case 2: One 4**
     - Choose 1 position out of 4 for the digit 4.
     - The remaining 3 positions are filled with digits 1, 2, 3 without repetition.
     - Number of ways:
       * Choose position for 4: $\binom{4}{1} = 4$.
       * Arrange 1, 2, 3 in the remaining 3 positions: $3! = 6$.
     - **Total for Case 2:** $4 \times 6 = 24$.
   - **Case 3: Two 4s**
     - Choose 2 positions out of 4 for the digit 4.
     - The remaining 2 positions are filled with digits 1, 2, 3 without repetition.
     - Number of ways:
       * Choose positions for 4s: $\binom{4}{2} = 6$.
       * Arrange 1, 2, 3 in the remaining 2 positions: $P(3,2) = 3 \times 2 = 6$.
     - **Total for Case 3:** $6 \times 6 = 36$.
   - **Case 4: Three 4s**

    - Choose 3 positions out of 4 for the digit 4.
    - The remaining 1 position is filled with one of the digits 1, 2, 3.
    - Number of ways:
      * Choose positions for 4s: $\binom{4}{3} = 4$.
      * Choose digit for the remaining position: 3 choices.
    - **Total for Case 4:** $4 \times 3 = 12$.

  - **Case 5: Four 4s**
    - All four positions are filled with the digit 4.
    - Only 1 possibility: 4444.
    - **Total for Case 5:** 1.

5. **Summing All Cases:**

$$0(\text{Case 1}) + 24(\text{Case 2}) + 36(\text{Case 3}) + 12(\text{Case 4}) + 1(\text{Case 5}) = 73$$

Answer: **73** different four-digit numbers can be formed where the digit 4 may be repeated, but digits 1, 2, and 3 occur at most once.

Part (d): Three-Digit Integers with Three Different Digits

Objective: Determine how many integers between 100 and 999 have three different digits, noting that 0 cannot be the first digit.

Approach:

1. **Range:** 100 to 999 (all three-digit numbers).
2. **Digits Available:** 0-9, with the constraint that the first digit cannot be 0.
3. **Constraint:** All three digits must be different.
4. **Calculating Possibilities:**
   - **Hundreds Place (First Digit):** Cannot be 0, so 9 choices (1-9).
   - **Tens Place (Second Digit):** Can be any digit except the one used in the hundreds place, so 9 choices (0-9, excluding the first digit).
   - **Units Place (Third Digit):** Can be any digit except the ones used in the hundreds and tens places, so 8 choices.

   Therefore, total numbers $= 9 \times 9 \times 8 = 648$.

Answer: **648** three-digit integers between 100 and 999 contain three different digits.

Verification and Reflection

To ensure the accuracy of these solutions, let's briefly verify each part:

- **Part (a):** With repetition allowed, each of the four positions has 4 choices, leading to $4^4 = 256$ numbers.
- **Part (b):** Without repetition, the number of choices decreases by one for each subsequent position, resulting in $4 \times 3 \times 2 \times 1 = 24$ numbers.
- **Part (c):** By considering the number of 4s and arranging the remaining digits, we systematically account for all valid combinations, totaling 73 numbers.
- **Part (d):** Ensuring the first digit is not 0 and all digits are unique, the calculation $9 \times 9 \times 8 = 648$ correctly counts the valid three-digit numbers.

These methods demonstrate a structured approach to combinatorial problems, emphasizing the importance of understanding constraints and systematically exploring all possible scenarios.

Final Answers

- **(a) 256** different four-digit numbers can be formed with repeated digits allowed.
- **(b) 24** different four-digit numbers can be formed without repeated digits.
- **(c) 73** different four-digit numbers can be formed where the digit 4 may be repeated, but digits 1, 2, and 3 occur at most once.
- **(d) 648** three-digit integers between 100 and 999 contain three different digits.

Problem 3: Choosing Four Different Even Numbers

Problem: In how many ways can you choose four different even numbers from the numbers 1, 2, 3, ..., 20?

Solution

Understanding the Problem

We are tasked with determining the number of ways to choose four different even numbers from the set of numbers 1 through 20. To approach this, we need to:

1. Identify the even numbers within the range 1 to 20.
2. Determine how many ways we can select four distinct numbers from this subset of even numbers.

Step 1: Identifying Even Numbers from 1 to 20

First, let's list out all the even numbers between 1 and 20. Even numbers are integers divisible by 2.

$$\text{Even numbers from 1 to 20} = \{2, 4, 6, 8, 10, 12, 14, 16, 18, 20\}$$

Counting these, we find there are **10 even numbers** in this range.

Step 2: Choosing Four Different Even Numbers

Now, we need to calculate how many ways we can choose 4 different numbers from these 10 even numbers. This is a classic combination problem where the order of selection does not matter.

The formula for combinations is:

$$C(n, k) = \frac{n!}{k!(n-k)!}$$

Where:

- $n = 10$ (total even numbers)
- $k = 4$ (numbers to choose)
- ! denotes factorial, which is the product of all positive integers up to that number.

So,

$$C(10, 4) = \frac{10!}{4!(10-4)!} = \frac{10!}{4! \times 6!}$$

Step 3: Calculating the Combination

Let's compute the value step by step.

First, recall that:

$$10! = 10 \times 9 \times 8 \times 7 \times 6!$$

This allows us to simplify the equation:

$$C(10, 4) = \frac{10 \times 9 \times 8 \times 7 \times 6!}{4! \times 6!} = \frac{10 \times 9 \times 8 \times 7}{4!}$$

Now, calculate 4!:

$$4! = 4 \times 3 \times 2 \times 1 = 24$$

So,

$$C(10, 4) = \frac{10 \times 9 \times 8 \times 7}{24}$$

Next, compute the numerator:

$$10 \times 9 = 9090 \times 8 = 720720 \times 7 = 5040$$

Now, divide by the denominator:

$$\frac{5040}{24} = 210$$

Verification of the Calculation

To ensure the accuracy of our calculation, let's verify the steps:

- **Total even numbers between 1 and 20:** Correctly identified as 10.
- **Combination formula application:** Correctly applied $C(10, 4)$.
- **Factorial simplification:** Properly simplified 10! to $10 \times 9 \times 8 \times 7 \times 6!$, canceling out 6!.
- **Calculation of** $4!$**:** Correctly computed as 24.
- **Final multiplication and division:** Accurately performed, resulting in 210.

Conclusion

After systematically breaking down the problem and applying combinatorial mathematics, we've determined that there are **210** ways to choose four different even numbers from the numbers 1 through 20.

Final Answer: There are **210** ways to choose four different even numbers from the numbers 1 to 20.

Problem 4: Total Number of Games in a Football League

Problem: In a football league, there are twenty teams, and each team plays each other team twice, once at home and once away. What is the total number of games played in the course of this league's season?

Solution

Understanding the Problem

We have a football league consisting of **20 teams**. Each team plays against every other team **twice**—once at **home** and once **away**. We need to determine the **total number of games** played throughout the league's season.

Breaking Down the Problem

- **Total Teams:** 20
- **Games per Pair of Teams:** 2 (home and away)

Our goal is to find out how many unique pairs of teams exist and then multiply by the number of games each pair plays.

Step 1: Calculating the Number of Unique Pairs

First, we need to determine how many unique pairs of teams can be formed from 20 teams. This is a combination problem where the order of selection doesn't matter (i.e., Team A vs. Team B is the same as Team B vs. Team A).

The formula for combinations is:

$$C(n,k) = \frac{n!}{k!(n-k)!}$$

Where:

- $n = 20$ (total teams)
- $k = 2$ (since a game involves 2 teams)
- ! denotes factorial.

So,

$$C(20,2) = \frac{20!}{2!(20-2)!} = \frac{20!}{2! \times 18!}$$

Step 2: Simplifying the Combination

We can simplify the equation by canceling out 18! in the numerator and denominator:

$$C(20,2) = \frac{20 \times 19 \times 18!}{2! \times 18!} = \frac{20 \times 19}{2!}$$

Now, calculate 2!:

$$2! = 2 \times 1 = 2$$

So,

$$C(20,2) = \frac{20 \times 19}{2} = \frac{380}{2} = 190$$

This means there are **190 unique pairs** of teams.

Step 3: Calculating Total Number of Games

Each unique pair of teams plays **2 games** (home and away). Therefore, to find the total number of games, we multiply the number of unique pairs by 2:

$$\text{Total Games} = \text{Number of Unique Pairs} \times \text{Games per Pair} = 190 \times 2 = 380$$

Verification of the Calculation

To ensure the accuracy of our calculation, let's verify the steps:

- **Total Teams:** Correctly identified as 20.
- **Combination Formula Application:** Correctly applied $C(20, 2)$.
- **Factorial Simplification:** Properly simplified 20! to $20 \times 19 \times 18!$, canceling out 18!.
- **Calculation of 2!:** Correctly computed as 2.
- **Final Multiplication:** Accurately performed, resulting in 190 unique pairs.
- **Total Games Calculation:** Correctly multiplied by 2 to account for home and away games, resulting in 380 total games.

Conclusion

After systematically breaking down the problem and applying combinatorial mathematics, we've determined that there are **380** total games played in the league's season.

Final Answer: The total number of games played in the course of the league's season is **380**.

Problem 5: Forming a Committee

Problem: A committee of four people is to be chosen from six men and six women.

(a) In how many ways can this be done?

(b) In how many of the committees so formed are there more men members than women?

Solution

Understanding the Problem

We are tasked with two main questions:

- **Part (a):** Determine how many ways a committee of four people can be chosen from six men and six women.
- **Part (b):** Find out how many of these committees have more men than women.

To approach this, we'll use combinatorial mathematics, specifically combinations, since the order of selection doesn't matter in forming a committee.

Part (a): Total Number of Committees

Objective: Calculate the total number of ways to choose a committee of four people from six men and six women.

Approach:

1. **Total People Available:** 6 men + 6 women = 12 people.
2. **Committee Size:** 4 people.
3. **Combination Formula:** The number of ways to choose 4 people from 12 is given by the combination formula:
$$C(n, k) = \frac{n!}{k!(n-k)!}$$
Where:
   - $n = 12$ (total people)
   - $k = 4$ (committee size)

   So,
$$C(12, 4) = \frac{12!}{4! \times 8!}$$
4. **Calculating the Combination:**
   - Simplify 12! as $12 \times 11 \times 10 \times 9 \times 8!$.
   - Cancel 8! in the numerator and denominator:
$$C(12, 4) = \frac{12 \times 11 \times 10 \times 9}{4!}$$
   - Calculate 4!:
$$4! = 4 \times 3 \times 2 \times 1 = 24$$
   - Compute the numerator:
$$12 \times 11 = 132132 \times 10 = 13201320 \times 9 = 11880$$
   - Divide by the denominator:
$$\frac{11880}{24} = 495$$

Answer: There are **495** total ways to choose a committee of four people from six men and six women.

Part (b): Committees with More Men than Women

Objective: Determine how many of the 495 committees have more men than women.

Approach:

For a committee to have more men than women, given that the committee size is 4, the possible distributions are:

- **3 Men and 1 Woman**
- **4 Men and 0 Women**

We'll calculate the number of ways for each scenario and then sum them up.

1. **Case 1: 3 Men and 1 Woman**
   - **Choosing 3 Men from 6:**
   $$C(6,3) = \frac{6!}{3! \times 3!} = \frac{6 \times 5 \times 4}{3 \times 2 \times 1} = 20$$
   - **Choosing 1 Woman from 6:**
   $$C(6,1) = \frac{6!}{1! \times 5!} = 6$$
   - **Total Ways for This Case:**
   $$20 \times 6 = 120$$
2. **Case 2: 4 Men and 0 Women**
   - **Choosing 4 Men from 6:**
   $$C(6,4) = \frac{6!}{4! \times 2!} = \frac{6 \times 5}{2 \times 1} = 15$$
   - **Choosing 0 Women from 6:**
   $$C(6,0) = 1$$
   - **Total Ways for This Case:**
   $$15 \times 1 = 15$$
3. **Total Committees with More Men than Women:**
   $$120(\text{Case 1}) + 15(\text{Case 2}) = 135$$

Answer: There are **135** committees with more men than women.

Verification and Reflection

To ensure the accuracy of our solutions, let's briefly verify each part:

- **Part (a):** The combination $C(12, 4) = 495$ correctly calculates the total number of ways to form a committee of four from twelve people.
- **Part (b):** By considering the two valid scenarios (3 men & 1 woman, and 4 men & 0 women) and summing their possibilities, we accurately determine that there are 135 committees with more men than women.

These methods demonstrate a structured approach to combinatorial problems, emphasizing the importance of understanding constraints and systematically exploring all possible scenarios.

Final Answers

- **(a) 495** ways to choose a committee of four from six men and six women.
- **(b) 135** committees have more men than women.

Problem 6: Combinatorial Identities and Equations

(a) Show that
$$\frac{n!}{(n-2)!} + \frac{(n-1)!}{n!} = \frac{n^3 - n^2 + 1}{n}.$$

(b) Let $n \geq 3$ be a whole number. If
$$3\binom{n}{3} = 5\binom{n}{2},$$
find n.

(c) Let $r \geq 1$ be a whole number. If
$$\binom{15}{r} = \binom{15}{2r},$$
find r.

Solution

Problem (a): Proving the Identity

Objective: Show that
$$\frac{n!}{(n-2)!} + \frac{(n-1)!}{n!} = \frac{n^3 - n^2 + 1}{n}.$$

Approach:

1. **Simplify Each Term:**
   - **First Term:** $\frac{n!}{(n-2)!}$
     - Recall that $n! = n \times (n-1) \times (n-2)!$.
     - So, $\frac{n!}{(n-2)!} = n \times (n-1) = n^2 - n$.
   - **Second Term:** $\frac{(n-1)!}{n!}$
     - Recall that $n! = n \times (n-1)!$.
     - So, $\frac{(n-1)!}{n!} = \frac{1}{n}$.
2. **Combine the Simplified Terms:**
$$\frac{n!}{(n-2)!} + \frac{(n-1)!}{n!} = (n^2 - n) + \frac{1}{n}$$
3. **Express with a Common Denominator:**
   - To combine $n^2 - n$ and $\frac{1}{n}$, express $n^2 - n$ as $\frac{n^3-n^2}{n}$.
   - So,
$$\frac{n^3 - n^2}{n} + \frac{1}{n} = \frac{n^3 - n^2 + 1}{n}$$
4. **Conclusion:**
$$\frac{n!}{(n-2)!} + \frac{(n-1)!}{n!} = \frac{n^3 - n^2 + 1}{n}$$

Answer: The equation holds true as shown above.

Problem (b): Solving for n

Objective: Let $n \geq 3$ be a whole number. If

$$3\binom{n}{3} = 5\binom{n}{2},$$

find n.

Approach:

1. **Express the Combinations:**
   - Recall that:
   $$\binom{n}{k} = \frac{n!}{k!(n-k)!}$$
   - So,
   $$\binom{n}{3} = \frac{n!}{3!(n-3)!} = \frac{n(n-1)(n-2)}{6}$$
   $$\binom{n}{2} = \frac{n!}{2!(n-2)!} = \frac{n(n-1)}{2}$$

2. **Substitute into the Given Equation:**
   $$3 \cdot \frac{n(n-1)(n-2)}{6} = 5 \cdot \frac{n(n-1)}{2}$$

3. **Simplify Both Sides:**
   - Left Side:
   $$3 \cdot \frac{n(n-1)(n-2)}{6} = \frac{n(n-1)(n-2)}{2}$$
   - Right Side:
   $$5 \cdot \frac{n(n-1)}{2} = \frac{5n(n-1)}{2}$$

4. **Set the Simplified Expressions Equal:**
   $$\frac{n(n-1)(n-2)}{2} = \frac{5n(n-1)}{2}$$

5. **Cancel Common Terms:**
   - Since $n \geq 3$, $n(n-1) \neq 0$, so we can divide both sides by $\frac{n(n-1)}{2}$:
   $$n - 2 = 5$$

6. **Solve for n:**
   $$n = 5 + 2 = 7$$

Answer: $n = 7$.

Problem (c): Solving for r

Objective: Let $r \geq 1$ be a whole number. If

$$\binom{15}{r} = \binom{15}{2r},$$

find r.

Approach:

1. **Understand the Property of Combinations:**

- Recall that:
$$\binom{n}{k} = \binom{n}{n-k}$$
- So, $\binom{15}{r} = \binom{15}{2r}$ implies either:
  - $r = 2r$, or
  - $r = 15 - 2r$.

2. **Solve the Two Cases:**
   - **Case 1:** $r = 2r$
   $$r = 2r \implies r = 0$$
     - But $r \geq 1$, so this case is invalid.
   - **Case 2:** $r = 15 - 2r$
   $$r = 15 - 2r \implies 3r = 15 \implies r = 5$$
3. **Verify the Solution:**
   - Check $\binom{15}{5} = \binom{15}{10}$:
     - By the property of combinations, $\binom{15}{5} = \binom{15}{10}$, so $r = 5$ is valid.

Answer: $r = 5$.

Summary of Answers

- **(a)** The equation holds as shown.
- **(b)** $n = 7$.
- **(c)** $r = 5$.

Problem 7: Binomial Expansion and Coefficient Extraction

(a) Expand

$$\left(x^2 - \frac{1}{x}\right)^6$$

using the binomial theorem.

(b) Find the coefficient of x^3 in the expansion of

$$(x+1)(x-2)^5.$$

Solution

Problem (a): Expand $\left(x^2 - \frac{1}{x}\right)^6$ *using the binomial theorem.*

Objective: Expand $\left(x^2 - \frac{1}{x}\right)^6$ using the binomial theorem.
Approach:

1. **Recall the Binomial Theorem:**

$$(a+b)^n = \sum_{k=0}^{n} \binom{n}{k} a^{n-k} b^k$$

2. **Identify a, b, and n:**
   - $a = x^2$
   - $b = -\frac{1}{x}$
   - $n = 6$

3. **Apply the Binomial Theorem:**

$$\left(x^2 - \frac{1}{x}\right)^6 = \sum_{k=0}^{6} \binom{6}{k} (x^2)^{6-k} \left(-\frac{1}{x}\right)^k$$

4. **Simplify Each Term:**
   - $(x^2)^{6-k} = x^{12-2k}$
   - $\left(-\frac{1}{x}\right)^k = (-1)^k \cdot x^{-k}$

   So, each term becomes:

$$\binom{6}{k} \cdot (-1)^k \cdot x^{12-3k}$$

5. **Write Out the Expansion:**

$$\left(x^2 - \frac{1}{x}\right)^6 = \sum_{k=0}^{6} \binom{6}{k} (-1)^k x^{12-3k}$$

6. **Calculate Each Term Explicitly:**
   - For $k = 0$:

$$\binom{6}{0} (-1)^0 x^{12-0} = 1 \cdot 1 \cdot x^{12} = x^{12}$$

   - For $k = 1$:

$$\binom{6}{1} (-1)^1 x^{12-3} = 6 \cdot (-1) \cdot x^9 = -6x^9$$

   - For $k = 2$:

$$\binom{6}{2} (-1)^2 x^{12-6} = 15 \cdot 1 \cdot x^6 = 15x^6$$

- For $k = 3$:
$$\binom{6}{3}(-1)^3 x^{12-9} = 20 \cdot (-1) \cdot x^3 = -20x^3$$
- For $k = 4$:
$$\binom{6}{4}(-1)^4 x^{12-12} = 15 \cdot 1 \cdot x^0 = 15$$
- For $k = 5$:
$$\binom{6}{5}(-1)^5 x^{12-15} = 6 \cdot (-1) \cdot x^{-3} = -6x^{-3}$$
- For $k = 6$:
$$\binom{6}{6}(-1)^6 x^{12-18} = 1 \cdot 1 \cdot x^{-6} = x^{-6}$$

7. **Combine All Terms:**
$$\left(x^2 - \frac{1}{x}\right)^6 = x^{12} - 6x^9 + 15x^6 - 20x^3 + 15 - 6x^{-3} + x^{-6}$$

Answer:
$$\left(x^2 - \frac{1}{x}\right)^6 = x^{12} - 6x^9 + 15x^6 - 20x^3 + 15 - 6x^{-3} + x^{-6}$$

Problem (b): Find the coefficient of x^3 in the expansion of $(x+1)(x-2)^5$.

Objective: Find the coefficient of x^3 in the expansion of $(x+1)(x-2)^5$.
Approach:

1. **Expand $(x-2)^5$ Using the Binomial Theorem:**
$$(x-2)^5 = \sum_{k=0}^{5} \binom{5}{k} x^{5-k}(-2)^k$$
2. **Multiply by $(x+1)$:**
$$(x+1)(x-2)^5 = (x+1)\sum_{k=0}^{5} \binom{5}{k} x^{5-k}(-2)^k$$
3. **Distribute $(x+1)$:**
$$\begin{aligned} &= x\sum_{k=0}^{5} \binom{5}{k} x^{5-k}(-2)^k + 1\sum_{k=0}^{5} \binom{5}{k} x^{5-k}(-2)^k \\ &= \sum_{k=0}^{5} \binom{5}{k}(-2)^k x^{6-k} + \sum_{k=0}^{5} \binom{5}{k}(-2)^k x^{5-k} \end{aligned}$$
4. **Identify Terms Contributing to x^3:**
   - In the first sum (x^{6-k}), x^3 occurs when $6 - k = 3 \Rightarrow k = 3$.
   - In the second sum (x^{5-k}), x^3 occurs when $5 - k = 3 \Rightarrow k = 2$.
5. **Calculate the Corresponding Coefficients:**
   - For $k = 3$ in the first sum:
$$\binom{5}{3}(-2)^3 = 10 \cdot (-8) = -80$$
   - For $k = 2$ in the second sum:
$$\binom{5}{2}(-2)^2 = 10 \cdot 4 = 40$$
6. **Combine the Coefficients:**
$$\text{Coefficient of } x^3 = -80 + 40 = -40$$

Answer: The coefficient of x^3 in the expansion is -40.

Summary of Answers

- **(a)** The expansion is:

$$\left(x^2 - \frac{1}{x}\right)^6 = x^{12} - 6x^9 + 15x^6 - 20x^3 + 15 - 6x^{-3} + x^{-6}$$

- **(b)** The coefficient of x^3 in $(x+1)(x-2)^5$ is -40.

Problem 8: Placing Draughts Pieces on a Board

(a) In how many ways may 12 white draughts pieces be placed on the 32 black squares of a draughts board if no more than one can be placed on each black square?

(b) In how many ways may 12 white draughts pieces be placed on the 32 black squares of a draughts board if any number of pieces from 0 to 12 may be placed on a black square? (Draughts pieces of the same color are taken to be indistinguishable. The squares are all distinct.)

Solution

Problem (a): Placing 12 White Draughts Pieces with No More Than One per Square

Objective: Determine the number of ways to place 12 indistinguishable white draughts pieces on 32 distinct black squares, with no more than one piece per square.

Approach:

1. **Understand the Problem:**
   - **Total Squares:** 32 (all distinct).
   - **Pieces to Place:** 12 (indistinguishable).
   - **Constraint:** No more than one piece per square.
2. **Identify the Type of Problem:**
   - This is a combination problem where we are choosing 12 squares out of 32 to place the pieces, with no repetition.
3. **Use the Combination Formula:**

   $$C(n,k) = \frac{n!}{k!(n-k)!}$$

   Where:
   - $n = 32$ (total squares)
   - $k = 12$ (pieces to place)
4. **Calculate the Combination:**

   $$C(32,12) = \frac{32!}{12! \times 20!}$$
5. **Simplify the Calculation:**
   - While the exact numerical value can be computed using a calculator or software, the expression represents the number of ways to choose 12 squares out of 32.

Answer: The number of ways to place 12 white draughts pieces on 32 black squares with no more than one per square is:

$$C(32,12) = \frac{32!}{12! \times 20!}$$

Problem (b): Placing 12 White Draughts Pieces with Any Number of Pieces per Square

Objective: Determine the number of ways to place 12 indistinguishable white draughts pieces on 32 distinct black squares, allowing any number of pieces (from 0 to 12) on each square.

Approach:

1. **Understand the Problem:**
   - **Total Squares:** 32 (all distinct).
   - **Pieces to Place:** 12 (indistinguishable).
   - **Constraint:** Any number of pieces can be placed on each square.

2. **Identify the Type of Problem:**
   - This is a "stars and bars" problem in combinatorics, where we distribute indistinguishable items (pieces) into distinct bins (squares).
3. **Use the Stars and Bars Theorem:**
   - The formula for distributing k indistinguishable items into n distinct bins is:
   $$\binom{k+n-1}{n-1}$$
   - Here:
     - $k = 12$ (pieces)
     - $n = 32$ (squares)
4. **Apply the Formula:**
   $$\binom{12+32-1}{32-1} = \binom{43}{31}$$
5. **Simplify the Combination:**
   - Recall that $\binom{n}{k} = \binom{n}{n-k}$, so:
   $$\binom{43}{31} = \binom{43}{12}$$
6. **Calculate the Combination:**
   $$\binom{43}{12} = \frac{43!}{12! \times 31!}$$

Answer: The number of ways to place 12 white draughts pieces on 32 black squares with any number of pieces per square is:

$$\binom{43}{12} = \frac{43!}{12! \times 31!}$$

Summary of Answers

- **(a)** The number of ways to place 12 white draughts pieces on 32 black squares with no more than one per square is:
  $$C(32, 12) = \frac{32!}{12! \times 20!}$$
- **(b)** The number of ways to place 12 white draughts pieces on 32 black squares with any number of pieces per square is:
  $$C(43, 12) = \frac{43!}{12! \times 31!}$$