A PRIMER ON

GAUGE TRANSFORMATIONS

IN CLASSICAL ELECTRODYNAMICS

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These notes are based on a standalone lecture given as an accompaniment to the course on classical electrodynamics to postgraduate physics students at YCM, University of Mysore, in winter 2016–2017.

Please e-mail `hello@vhbelvadi.com` with your thoughts or suggestions, or if you spot any errors. These notes are, and will probably always remain, a work in progress. They may be updated from time to time and will be available for interested readers to download from `http://vhbelvadi.com/notes/` for free.

NB Footnotes are marked by superscript numbers ^{x}, equations by parentheses (x).

1 Scalar and vector potentials

The study of electromagnetism deals, generally, with predicting the electric and magnetic fields, $\mathbf{E}$ and $\mathbf{B}$, generated by charges under various circumstances and understanding their implications thereafter. If the charges involved are at rest, discussions fall under electrostatics; if the charges are in motion, they fall under electrodynamics. Of course these statements call for better descriptions of rest and motion and so on, but, in spirit, they will do for now. These electric and magnetic fields are best described by the equations first published by J.C. Maxwell [1] in the mid-19th century.

1. $\boldsymbol{\nabla} \cdot \mathbf{E} = 4\pi\rho$
2. $\boldsymbol{\nabla} \cdot \mathbf{B} = 0$
3. $\boldsymbol{\nabla} \times \mathbf{E} = -\frac{1}{c}\frac{\mathrm{d}\mathbf{B}}{\mathrm{d}t}$
4. $\boldsymbol{\nabla} \times \mathbf{B} = \frac{1}{c}\left(4\pi\mathbf{J} + \frac{\partial \mathbf{E}}{\partial t}\right)$

where ρ and $\mathbf{J}$ are the charge and current densities respectively.

The divergence of the curl of a vector is zero [2], so Maxwell's second equation implies that $\mathbf{B}$ may be written as the curl of some vector

$$\mathbf{B} = \boldsymbol{\nabla} \times \mathbf{A} \tag{1}$$

so that, on re-substitution, $\boldsymbol{\nabla} \cdot (\boldsymbol{\nabla} \times \mathbf{A}) = 0$ is satisfied exactly as we expected.

Likewise, we can now claim that

$$\boldsymbol{\nabla} \times \mathbf{E} = -\frac{1}{c}\frac{\mathrm{d}}{\mathrm{d}t}(\boldsymbol{\nabla} \times \mathbf{A}) \Rightarrow \boldsymbol{\nabla} \times \left(\mathbf{E} - \frac{1}{c}\frac{\partial \mathbf{A}}{\partial t}\right) = 0$$

means the term in parentheses may be written as the gradient of some scalar, V, since the curl of the gradient of a scalar is zero [2] leaving us with

$$\mathbf{E} = -\nabla V - \frac{1}{c}\frac{\partial \mathbf{A}}{\partial t} \tag{2}$$

These two newly introduced quantities, $\mathbf{A}$ and V, are known as the vector and scalar potentials. It is important to realise that these are *not* physical entities, but merely mathamtical constructs that help us define our fields of interest, $\mathbf{E}$ and $\mathbf{B}$, more elegantly. We could go further and simply substitute (1) and (2) in Maxwell's first and fourth results to arrive at

$$\Delta V + \frac{1}{c}\frac{\partial}{\partial t}(\boldsymbol{\nabla} \cdot \mathbf{A}) = -4\pi\rho \tag{3}$$

$$\text{and} \quad \left(\Delta \mathbf{A} - \frac{1}{c^2}\frac{\partial^2 \mathbf{A}}{\partial t^2}\right) - \boldsymbol{\nabla}\left(\boldsymbol{\nabla} \cdot \mathbf{A} + \frac{1}{c}\frac{\partial V}{\partial t}\right) = -\frac{4\pi}{c}\mathbf{J} \tag{4}$$

which condense Maxwell's four equations into just two *inhomogeneous equations*. But they bring with them a certain problem: we know that for a given circumstance (a given magnet, a given charge etc.) the $\mathbf{E}$ and $\mathbf{B}$ fields can only have a certain characteristic. But if they are defined in terms of the derivatives of some potentials, $\mathbf{A}$ and V, does it not mean that our field values itself may change based on our choice of potentials?

2 Gauge freedoms, conditions and transformations

The idea of gauge freedom is a somewhat touchy subject since it has several inconsistent interpretations depending on the context. For one, it could mean dealing with redundant degrees of freedom, an idea analogous to degeneracy, that says all degrees of freedom but one are redundant because they all describe the same system or effect and you can easily keep one of them and discard the rest since you cannot, empirically, differentiate between them anyway. This is a crude way to put it, but it makes the point. A gauge invariance will then rid us of all these so-called "unphysical", purely mathematical quantities and preserve those which are physically meaningful.

An excellent, consice description of gauges, transformations and invariance was provided by T. Tao [3] in an article published online—

> A gauge is nothing more than a coordinate system that varies depending on one's location with respect to some 'base space' or 'parameter space', a gauge transform is a change of coordinates applied to each such location, and [gauge invariance is when] all *physically meaningful* quantities are left unchanged (or transform naturally) under gauge transformations.

Specifically in electromagnetism, we notice from (1) and (2) that any number of different values for $\mathbf{A}$ and V can satisfy these equations and result in various $\mathbf{E}$ and $\mathbf{B}$ fields. A gauge will then be a mathematical operation that provides us a form of $\mathbf{A}$ and V that leaves $\mathbf{E}$ and $\mathbf{B}$ physically unchanged [4]. Gauge freedom is our choice to pick a convenient gauge. Gauge invariance here is simply the fact that a number of scalar and vector potentials, all related by gauge transformations, may be used to describe the same electric and magnetic fields.

To put it in an even simpler manner, this means we can come up with an $\mathbf{A}'$ related to $\mathbf{A}$ by what is called a gauge transformation such that $\mathbf{A}'$ will give rise to the same fields as $\mathbf{A}$. It is precisely because of this possibility of there being several values of $\mathbf{A}$ and V that they cannot have a consistent physical picture associated with them.

To understand this, let us consider the transformation $\mathbf{A} \to \mathbf{A} + \boldsymbol{\nabla} S$ and see if this changes our magnetic field as given by (1).

$$\mathbf{B} = \boldsymbol{\nabla} \times \mathbf{A} \to \boldsymbol{\nabla} \times (\mathbf{A} + \boldsymbol{\nabla} S) = \boldsymbol{\nabla} \times \mathbf{A} + \boldsymbol{\nabla} \times \boldsymbol{\nabla} S = \boldsymbol{\nabla} \times \mathbf{A} = \mathbf{B}$$

This is good, since such a transformation leaves our field untouched, which means we can use $\mathbf{A} + \nabla S$ as our vector field just like we use $\mathbf{A}$. Next we need to ensure that this also works for the electric field as given by (2).

$$\mathbf{E} = -\nabla V - \frac{1}{c}\frac{\partial \mathbf{A}}{\partial t} \to -\nabla V - \frac{1}{c}\frac{\partial}{\partial t}(\mathbf{A} + \boldsymbol{\nabla} S) = -\nabla V - \frac{1}{c}\frac{\partial \mathbf{A}}{\partial t} - \nabla \frac{1}{c}\frac{\partial S}{\partial t} \neq \mathbf{E}$$

So that clearly does not work. But what if we also transform $V \to V - \frac{1}{c}\frac{\partial S}{\partial t}$ and re-calculate the electric field?

$$\mathbf{E} = -\nabla V - \frac{1}{c}\frac{\partial \mathbf{A}}{\partial t} \to -\nabla\left(V - \frac{1}{c}\frac{\partial S}{\partial t}\right) - \frac{1}{c}\frac{\partial}{\partial t}(\mathbf{A} + \boldsymbol{\nabla} S) = -\nabla V - \frac{1}{c}\frac{\partial \mathbf{A}}{\partial t} = \mathbf{E}$$

Thus the transformation $\mathbf{A} \to \mathbf{A} + \boldsymbol{\nabla} S$ and $V \to V - \frac{1}{c}\frac{\partial S}{\partial t}$ constitutes a gauge transformation giving us scalar and vector potentials that ensure our $\mathbf{E}$ and $\mathbf{B}$ fields are invariant. This is a good example, but a generic one that holds no special meaning.

3 Gauge conditions in electromagnetism

Solving second order differential equations like (3) and (4) is not really necessary when we invoke the idea of gauges. Knowing that we have several solutions to these equations, all of which will yield the same fields, we can impose restrictions upon ourselves to arrive at more specific and, possibly, more physically meaningful solutions. These are called gauge conditions and, while we can come up with them as we please, there are a few invaluable ones worth knowing.

3.1 The Coulomb gauge

Let us define our gauge using an as yet undetermined *gauge condition*, χ, with our gauge transforming $\mathbf{A} \to \mathbf{A}'$ and $V \to V'$ as

$$\mathbf{A} = \mathbf{A}' + \boldsymbol{\nabla}\chi \qquad \text{and} \qquad V = V' - \frac{1}{c}\frac{\partial \chi}{\partial t}$$

both of which, it is worth observing, are definitions we have intelligently modeled on our transformation $\mathbf{A} \to \mathbf{A} + \boldsymbol{\nabla} S$ from §2. It is now up to us to put a condition on our solutions. Say we want all solutions which satisfy the magnetostatic condition that

$$\boldsymbol{\nabla} \cdot \mathbf{A} = 0 \tag{5}$$

That is, we want to find a χ such that the above condition gives us a valid gauge. We can always simply take the divergence of $\mathbf{A}$ and equate it to zero.

$$\boldsymbol{\nabla} \cdot \mathbf{A} = \boldsymbol{\nabla} \cdot \mathbf{A}' + \Delta\chi = 0 \quad \Rightarrow \quad \boldsymbol{\nabla} \cdot \mathbf{A}' = -\Delta\chi$$

which is a Poisson equation. What we have done is simple: we exploited the fact that a Poisson equation is familiar and easy to solve, and, in order to arrive at a Poisson equation, we used the fact that adding any gradient to $\mathbf{A}$ and any constant to V leaves our fields unchanged. This proves another key point: since a Poisson equation has a solution, our pick of χ is clearly valid.

Recall that this arose due to our setting $\boldsymbol{\nabla} \cdot \mathbf{A} = 0$ in (5). The idea of choosing the value of $\boldsymbol{\nabla} \cdot \mathbf{A}$ is called *gauge fixing*, and this particular choice is called the *Coulomb gauge.*

This is all well and good, but why did we pick $\boldsymbol{\nabla} \cdot \mathbf{A} = 0$ in the first place? While the choice of the Coulomb gauge may not be immediately apparent in electrodynamics, it becomes more obvious in the domain of magnetostatics whence it came.

On the one hand, we pick what makes our equations simple so long as it is valid, and, on the other, note that when the condition $\boldsymbol{\nabla} \cdot \mathbf{A}' = a \neq 0$ is applied to our transformation $\mathbf{A} \to \mathbf{A}' + \boldsymbol{\nabla}\chi$ too, we end up with $\boldsymbol{\nabla} \cdot \mathbf{A}' + \Delta\chi = 0$ or $\Delta\chi = -a$ if $\boldsymbol{\nabla} \cdot \mathbf{A} = 0$. In other words, we have another Poisson equation which we know has a solution.

Physically, if we are defining the magnetic field in terms of some vector, it must be one that is divergence free. This is something we know from magnetostatics. Consider Gauss's theorem with no net magnetic flux:

$$\iiint\limits_{\text{volume}} \mathrm{d}\mathbf{V}\, \boldsymbol{\nabla} \cdot \mathbf{B} = \iint\limits_{\text{surface}} \mathrm{d}\mathbf{S}\, \mathbf{B} = 0$$

This simply means that no magnetic monopoles exist, since all field lines going out through any surface are coming back in as well, to wit, the divergence must be zero and hence our choice to fix the gauge at $\boldsymbol{\nabla} \cdot \mathbf{A} = 0$ is not all that arbitrary.

At the end of the day, though, the Coulomb gauge is not all that convenient in all situations. So far we considered the electrostatic case (3) where $\mathbf{E}$ and $\mathbf{B}$ were varying only spatially and not temporally and the gauge proved to be useful. This is not surprising now since we just saw that the condition defining the Coulomb gauge itself comes from magnetostatics. Suppose we turn to the electrodynamic case where the fields vary temporally as well, i.e. suppose we invoke the second inhomogeneous Maxwell equation (4), is the Coulomb gauge then still meaningful?

Plugging $\boldsymbol{\nabla} \cdot \mathbf{A} = 0$ into (4) we have

$$\left(\Delta \mathbf{A} - \frac{1}{c} \frac{\partial^2 \mathbf{A}}{\partial t^2} \right) - \boldsymbol{\nabla} \left(\frac{1}{c} \frac{\partial V}{\partial t} \right) = -\frac{4\pi}{c} \mathbf{J}$$

which is really nothing much, to say the least. It is certainly not a Poisson equation, and it certainly is no easier to solve than it was in its original form. (However, there *is* a way to solve and interpret this; we will return to this in §5.) A gauge lets us restrict ourselves to a certain subset of all possible solutions for $\mathbf{A}$ and V that define a given field. But the Coulomb gauge, as well as it does this for the electrostatic and magnetostatic case, does not seem to be an effective approach in the electrodynamic case.

3.2 The Lorenz gauge

One of our problems with the Coulomb gauge in electrodynamics was the fact that it did not simplify derivatives by any measure. What if we could solve the equation either for $\mathbf{A}$ or for V exclusively?

It is actually much easier to see where the Lorenz gauge comes from than where the Coulomb gauge came from. We said already that, in electromagnetism, gauge fixing refers to our choice of a value for $\boldsymbol{\nabla} \cdot \mathbf{A}$, since this is the term common to both of Maxwell's equations. With this in mind, if we refer to (4) we quickly realise that getting rid of the second term on the left hand side leaves us with an equation in terms of $\mathbf{A}$ alone. Driven by this, we can choose to define our gauge condition as

$$\boldsymbol{\nabla} \cdot \mathbf{A} = -\frac{1}{c} \frac{\partial V}{\partial t} \tag{6}$$

which is known as the *Lorenz gauge*. It is among the most popular choices for a gauge in electromagnetism and with good reason, as we will now see.

In the Lorenz gauge, here is what happens to Maxwell's two inhomogeneous equations:

$$\Delta V - \frac{1}{c^2} \frac{\partial^2 V}{\partial t^2} = -4\pi\rho \tag{7}$$

$$\text{and} \qquad \Delta \mathbf{A} - \frac{1}{c^2} \frac{\partial^2 \mathbf{A}}{\partial t^2} = -\frac{4\pi}{c} \mathbf{J} \tag{8}$$

It is extremely important to understand, here, that our use of the Lorenz gauge has not removed the time-depenence of Maxwell's equations: they are still electrodynamic. The equations now, though, are much simpler to solve and, right on the surface, it is easy to see how similar they look. Indeed we can write them with a common operator as

$$\Delta - \frac{1}{c^2}\frac{\partial^2}{\partial t^2} \begin{cases} V &= 4\pi\rho \\ \\ \mathbf{A} &= -\frac{4\pi}{c}\mathbf{J} \end{cases} \tag{7, 8}$$

The Lorenz gauge, therefore, gives us a pair of equations that treat both V and $\mathbf{A}$ alike and serve as an excellent starting point for further electrodynamic studies. However, it does not explicitly show $\mathbf{A}$ as solenoidal (i.e. $\boldsymbol{\nabla} \cdot \mathbf{A} = 0$ is not clear). It is worth mentioning, at this point, that the operator in (7, 8) has a special representation:

$$\Box^2 = \Delta - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}$$

It is called the *d'Alembert operator*, or the *d'Alembertian*, and sees considerable use, especially in the field of special relativity.

Unlike the Coulomb gauge that was only applicable in time-independent, electrostatic and magnetostatic cases, the Lorenz gauge is applicable in time-dependent, electrodynamic cases and, as a result, sees a lot of use in such discussions as retarded potentials, Liénard-Wiechert fields and so on.

We have established that the Lorenz gauge is elegant and useful, and brings a nice symmetry to Maxwell's inhomogeneous equations, bringing the complex-looking equations (3, 4) down to simpler, solvable forms (7, 8), and, rather conveniently, treats both our potentials symmetrically, using a common operator. But a more fundamental question remains: does the Lorenz gauge condition result in a guage at all?

To prove this, we can try to substitute the transformations, $\mathbf{A} \to \mathbf{A}' + \nabla\chi$ and $V \to V' - \frac{1}{c}\frac{\partial\chi}{\partial t}$ that we previously defined, into the Lorenz gauge condition.

$$\boldsymbol{\nabla} \cdot \mathbf{A} + \frac{1}{c}\frac{\partial V}{\partial t} \to \boldsymbol{\nabla} \cdot \mathbf{A}' + \Delta\chi + \frac{1}{c}\frac{\partial V'}{\partial t} - \frac{1}{c^2}\frac{\partial^2\chi}{\partial t^2} = 0$$

which leaves us with the gauge condition untouched.

$$\boldsymbol{\nabla} \cdot \mathbf{A}' + \frac{1}{c}\frac{\partial V'}{\partial t} = -\left(\Delta\chi - \frac{1}{c^2}\frac{\partial^2\chi}{\partial t}\right) = f(x,t)$$

which is a solvable equation, since we work on the understanding that any familiar or otherwise recognisably well-behaved function can be reasonably expected to yield solutions under most physically meaningful conditions.

Something even more interesting in the above result is that the gauge function, χ, is itself the solution to a wave equation. In the Lorenz gauge, in other words, it is not hard to see how solutions might often end up as wave equations (hence electromagnetic waves). Further, the transformation that χ allows yields solutions $\mathbf{A}'$ and V' that satisfy the Lorenz gauge condition themselves.

This means we can go on and on with a χ for every solution $\mathbf{A}$ and V and further still for every $\mathbf{A}'$ and V' and so on to find an infinite number of solutions, all within the Lorenz gauge itself.

4 Green's function for Poisson's equation

A Poisson equation has the form $\Delta\varphi(\mathbf{r}) = -4\pi f(\mathbf{r})$ where φ is called the *potential* and f is the *source function.*

Suppose there exists some $G(\mathbf{r}, \mathbf{r}')$ which satisfies the Poisson equation in its Dirac delta form,

$$\Delta G(\mathbf{r}, \mathbf{r}') = -4\pi\delta(\mathbf{r} - \mathbf{r}')$$

for some source point, $\mathbf{r}'$ in three dimensions. Then, we can expect our general solution for the roots of the Poisson equation to look like

$$\varphi(\mathbf{r}) = \varphi_0(\mathbf{r}) + \int \mathrm{d}^3\mathbf{r}'\, G(\mathbf{r}, \mathbf{r}')f(\mathbf{r}') \tag{9}$$

where $\varphi_0(\mathbf{r})$ is such that it solves the Laplace equation $\Delta\varphi_0(\mathbf{r}) = 0$. (All of this can, of course, be proven by a simple re-substitution of $\varphi(\mathbf{r})$ into $\Delta\varphi(\mathbf{r}) = -4\pi f(\mathbf{r})$.)

The Fourier transform of $G(\mathbf{r}, \mathbf{r}')$ is given in terms of some function $H(\mathbf{y})$ in reciprocal space such that $H(\mathbf{y}) = 4\pi/y^2$ by definition. The Fourier transform is

$$G(\mathbf{r}, \mathbf{r}') = \frac{1}{(2\pi)^3}\int \mathrm{d}^3y\, H(\mathbf{y})\, e^{i\mathbf{y}\cdot(\mathbf{r}-\mathbf{r}')}$$

where the delta function, as usual, is defined as

$$\delta(\mathbf{r} - \mathbf{r}') = \frac{1}{(2\pi)^3}\int \mathrm{d}^3y\, e^{i\mathbf{y}\cdot(\mathbf{r}-\mathbf{r}')}$$

Using the definition (above) of $H(\mathbf{y})$ we get

$$G(\mathbf{r}, \mathbf{r}') = \frac{4\pi}{(2\pi)^3}\int \mathrm{d}^3y\, \frac{H(\mathbf{y})\, e^{i\mathbf{y}\cdot(\mathbf{r}-\mathbf{r}')}}{y^2}$$

This integral may be solved [10] in spherical polar coördinates, assuming an angle θ between $\mathbf{k}$ and $(\mathbf{r} - \mathbf{r}')$ so that $\mathbf{k}\cdot(\mathbf{r} - \mathbf{r}') = k\,|\mathbf{r} - \mathbf{r}'|\cos\theta$ to get

$$G(\mathbf{r}, \mathbf{r}') = \frac{1}{|\mathbf{r} - \mathbf{r}'|}$$

Substituting this back into (9) with $\varphi_0(\mathbf{x}) = 0$ (this is assumed for similarity in §5) we arrive at

$$\varphi(\mathbf{r}) = \int \mathrm{d}^3\mathbf{r}'\, \frac{f(\mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|} \tag{10}$$

5 Further notes on gauges in electromagnetism

There are a number of other gauges one can talk about: the Weyl gauge, the Poincaré gauge, the axial gauge and so on. The most extensively used ones are the Coulomb and Lorenz gauges, which is why we can restrict ourselves to those for now.

If we look back at the scalar potential result afforded by the first inhomogeneous Maxwell equation (3) in the Coulomb gauge (5), we find

$$\Delta V = 4\pi\rho$$

which is a Poisson equation and, using Green's function (see [5] or (10)), has its general solution (again, see [6] or (10)) as

$$V = \int \mathrm{d}^3 r' \, \frac{\rho(\mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|} \tag{11}$$

This means our electric potential now has a troubling inplication. If there is any change in the charge density, ρ, the above equation suggests that we will see a change in the potential, V, everywhere, instantaneouosly. This is another reason why the Coulomb gauge does not work well in discussions of time-dependent fields like electodynamics (and, once again, when we speak of retarded potentials etc.) where it violates causality.

As for the vector potential, $\mathbf{A}$, we can show a result that is equally interesting, but not half as troubling. The current density vector, $\mathbf{J}$, can be decomposed into a transverse and a longitudinal current [7] given as $\mathbf{J} = \mathbf{J}_t + \mathbf{J}_l$ where the subscripts t and l refer to the transverse and longitudinal components respectively. This means (4) has to be slightly modified, in the Coulomb gauge, into

$$\left(\Delta \mathbf{A} - \frac{1}{c^2} \frac{\partial^2 \mathbf{A}}{\partial t^2} \right) - \left(\boldsymbol{\nabla} \, \frac{1}{c} \, \frac{\partial V}{\partial t} \right) = - \frac{4\pi}{c} \mathbf{J}_t - \frac{4\pi}{c} \mathbf{J}_l$$

The transverse current is also known as the *solenoidal current*, a term which (you may recall from section §3.2) means that our vector is not divergent: $\boldsymbol{\nabla} \cdot \mathbf{J}_t = 0$. The longitudinal current is described by $\boldsymbol{\nabla} \times \mathbf{J}_l = 0$, and, as a result, is sometimes also known as the *irrotational current*.

Given that $\boldsymbol{\nabla} \times (\boldsymbol{\nabla} \times \mathbf{J}) = \boldsymbol{\nabla}(\boldsymbol{\nabla} \cdot \mathbf{J}) - \nabla^2 \mathbf{J}$, we can use the electrodynamic continuity equation[1] and the scalar potential solution (11) to show that [8] the transverse and longitudinal currents are

$$\mathbf{J}_t = \boldsymbol{\nabla} \times \boldsymbol{\nabla} \times \int \mathrm{d}^3 \mathbf{r}' \, \frac{\mathbf{J}}{|\mathbf{r} - \mathbf{r}'|} \qquad \text{and} \qquad \mathbf{J}_l = \boldsymbol{\nabla} \int \mathrm{d}^3 \mathbf{r}' \, \frac{\boldsymbol{\nabla}' \cdot \mathbf{J}}{|\mathbf{r} - \mathbf{r}'|}$$

as a result of which

$$\frac{4\pi}{c} \, \mathbf{J}_l = \frac{1}{c} \, \boldsymbol{\nabla} \, \frac{\partial V}{\partial t}$$

and (4) becomes

$$\Delta \mathbf{A} - \frac{1}{c^2} \frac{\partial^2 \mathbf{A}}{\partial t^2} = - \frac{4\pi}{c} \mathbf{J}_t \tag{12}$$

This may look exactly like the result of working in the Lorenz gauge (8) but the physical interpretation of having only $\mathbf{J}_t$ on the right hand side, instead of $\mathbf{J}$ in its entirety, plays an important role in studying the far field radiation zones and is worth discussing.

Whereas the scalar potential solution (11) for the Coulomb gauge told us that information travels instantaneously, we now realise this only occurs in the near field region. Better yet, thanks to this, the Coulomb gauge is often applied when the source point is nowhere near the field point, or, mathematically, when $V = 0$ can safely be assumed.

As we move farther from the source point, information is transmitted by the transverse current (12), which alone gives rise to the vector potential. Not only does $\mathbf{J}_t$ make

[1] The equation that arises due to the fundamental law of charge conservation: $\partial \rho / \partial t + \boldsymbol{\nabla} \cdot \mathbf{J} = 0$.

up for the instantaneous nature of interactions as described by (11) but it is also present in the far field zone even if, due to a source, **J** itself is localised [9]. Further, (12) is time-dependent, satisfies the wave equation, ensures a finite speed for c, is generally extremely well-behaved, and corresponds to the radiation fields observed in the far field zone.

The fact that the scalar potential solution is simply the Coulomb field (11) is what gives the Coulomb gauge its name. Additionally, the fact that this gauge tells us that the far field zones observe transverse, wavelike radiation fields, which is a consequence of transverse currents and *not* the scalar potential (which, in the far field zone goes to zero anyway), has led to its being known also as the *transverse gauge* or the *radiation gauge*.

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