16MSPAH101 — ADVANCED CLASSICAL MECHANICS

INTRODUCTION TO HAMILTONIAN MECHANICS

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1 THE LEGENDRE TRANSFORMATION

WE BEGIN OUR discussion of Hamiltonian mechanics with the a quick look at a mathematical technique known as the Legendre transformation. The Legendre transformation for some function $f(u_1, u_2, \ldots, u_n)$ generates another function $f'(u'_1, u'_2, \ldots, u'_n)$ where u' and u are related as

$$u'_i = \frac{\partial f}{\partial u_i}$$

and may be done using

$$f' = u_i u'_i - f \tag{1}$$

which is quite straightforward. Firstly this tells us that f' is only a function of u_i and not, interestingly, of u'_i. To see this consider the differential of the transformed function:

These transformations are the work of the XVIII century french mathematician Adrien-Marie Legendre.

$$\begin{aligned} \mathrm{d}f' &= u_i \mathrm{d}u'_i + u'_i \mathrm{d}u_i - \frac{\partial f}{\partial u_i}\mathrm{d}u_i \\ &= u_i \mathrm{d}u'_i + \left(u'_i - \frac{\partial f}{\partial u_i}\right)\mathrm{d}u_i \end{aligned}$$

However, since $u'_i = \partial f / \partial u_i$ as stated above we have simply

$$\mathrm{d}f' = u_i \mathrm{d}u'_i$$

which means f' is not a function of u', just of u.

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Secondly, we ask what would happen if f' was in fact a function of u'? Its differential would then be given by

$$\mathrm{d}f' = \frac{\partial f'}{\partial u'_i}\,\mathrm{d}u'_i$$

Keep in mind that everywhere you see repeating indices, i.e. to instances of the same subscript, summation is implied. So, for example, eq. (1) is actually written as

$$f' = \sum_i^n u_i u'_i - f$$

but we omit the $\sum$ as agreed upon in Einstein's summation convention.

Compare the two equations we have for $\mathrm{d}f'$ now and it is easy to see that

$$u_i = \frac{\partial f'}{\partial u'_i}$$

In other words, this brings us back to the first equation we wrote down above and, consequently, a symmetric result follows eq. (1) giving us

$$f = u_i u'_i - f' \tag{2}$$

a sort of 'reverse' Legendre transformation.

Lastly, we need to account for the possibility that $f \equiv f(u,v)$ where the us transform but the vs do not. In this case, as we did before, consider the differential of the transformed function:

$$\begin{aligned}\mathrm{d}f' &= (u_i\mathrm{d}u'_i + u'_i\mathrm{d}u_i) - \mathrm{d}f(u,v)\\ &= (u_i\mathrm{d}u'_i + u'_i\mathrm{d}u_i) - \frac{\partial f}{\partial u_i}\mathrm{d}u_i - \frac{\partial f}{\partial v_i}\mathrm{d}v_i\\ \therefore\ \mathrm{d}f' &= u_i\mathrm{d}u'_i - \frac{\partial f}{\partial v_i}\mathrm{d}v_i\end{aligned}$$

where the two terms get cancelled in the last step for the same reasons as they did earlier, telling us that f' is a function of u and v both. Once again like we did before we now consider what would happen in f' was in fact a function of u' too. Its differential would then become

$$\mathrm{d}f' = \frac{\partial f'}{\partial u_i}\mathrm{d}u_i + \frac{\partial f'}{\partial v_i}\mathrm{d}v_i$$

Compare the two equations we now have for $\mathrm{d}f'$ and you will see that

$$\frac{\partial f}{\partial v_i} = -\frac{\partial f'}{\partial v'_i} \tag{3}$$

This is the relation between any non-transforming variable, here v, and its transformed functions, $f \leftrightarrow f'$ undergoing a Legendre transformation.

2 THE HAMILTONIAN

So far the Legendre transformation has largely been mathematical; it is now time for us to give it a more physical form. To start with, consider the Lagrangian of some system: $\mathscr{L}(q_k, \dot{q}_k, t)$. Following our discussion on the Legendre transformation we can transform $\mathscr{L}$ into some $\mathscr{H}$ using a transformation that involves the variable $\dot{q}_k$ alone. So our non-participants are q_k and t and our transformed function then is

$$\mathscr{H} \equiv \mathscr{H}(q_k, \partial\mathscr{L}/\partial\dot{q}_k, t)$$

However we know that $\partial\mathscr{L}/\partial\dot{q}_k$ is nothing but the generalised momentum p_k which means the function is actually given by

$$\mathscr{H} \equiv \mathscr{H}(q_k, p_k, t)$$

So if $\mathscr{L} \rightarrow \mathscr{H}$ we have, from eq. (1), the transformation

$$\mathscr{H}(q_k, p_k, t) = p_k\dot{q}_k - \mathscr{L}(q_k, \dot{q}_k, t) \tag{4}$$

where the function $\mathscr{H}$ is known as the **Hamiltonian**.

How did we choose u_i and u_i' in eq. (4)? Recall that u_i and u_i' are related by $u_i' = \partial f/\partial u_i$ which, here, refers to $p_k = \partial\mathscr{L}/\partial\dot{q}_k$ making $u_i \equiv \dot{q}_k$ and $u_i' \equiv p_k$. As before, summation is implicit.

3 THE CANONICAL EQUATIONS

The reversible nature of the Legendre transformation described earlier allows us to use the relation

$$p_k = \frac{\partial\mathscr{L}}{\partial\dot{q}_k} \tag{5}$$

to similarly transform from $\dot{q}_k$ to p_k, but this time via $\mathscr{H}$, as

$$\dot{q}_k = \frac{\partial\mathscr{H}}{\partial p_k} \tag{5}$$

At this point let us not forget that the Lagrangian is made up of two other non-participating variables q_k and t. Their relations to the transformed function is given by eq. (3) as

$$\frac{\partial\mathscr{H}}{\partial q_k} = -\frac{\partial\mathscr{L}}{\partial q_k} \qquad \text{and} \qquad \frac{\partial\mathscr{H}}{\partial t} = -\frac{\partial\mathscr{L}}{\partial t} \tag{6}$$

Now recall the Euler-Lagrange equation

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial\mathscr{L}}{\partial\dot{q}_k}\right) = \frac{\partial\mathscr{L}}{\partial q_k}$$

We can combine eq. (5) and (6) with the Euler-Lagrange equation to arrive at two key equations. The first of these arises from the Euler-Lagrange equation given above where the parenthetical term is replaced by the generalised momentum and written using eq. (3) as follows:

$$\dot{p}_k = \frac{\partial \mathscr{L}}{\partial q_k}$$

$$\Rightarrow \dot{p}_k = -\frac{\partial \mathscr{H}}{\partial q_k}$$

As an exercise try deriving these canonical equations from Hamilton's principle.

and the second equation, for $\dot{q}_k$, is as given by eq. (5) above. Together these simple but powerful transformation equations

$$\dot{q}_k = \frac{\partial \mathscr{H}}{\partial p_k} \quad \text{and} \quad \dot{p}_k = -\frac{\partial \mathscr{H}}{\partial q_k} \tag{7}$$

are known as **Hamilton's canonical equations**.

All of this ties up nicely if we make one final observation: use the fact that so long as the Lagrangian is time-independent i.e. $\partial \mathscr{L}/\partial t = 0$ in a conservative system we can write

$$2\mathrm{T} = \left(\frac{\partial \mathscr{L}}{\partial \dot{q}_k}\right)\dot{q}_k$$

The intermediate result used here is easy to see if we consider a Cartesian system where $q_k \equiv x$ as follows:

$$\mathscr{L}(q_k, \dot{q}_k) \equiv \mathscr{L}(x, \dot{x}) = \frac{m\dot{x}^2}{2}$$

$$\Rightarrow \frac{\partial \mathscr{L}}{\partial \dot{q}_k} = m\dot{x}$$

$$\therefore \dot{q}_k\left(\frac{\partial \mathscr{L}}{\partial \dot{q}_k}\right) = m\dot{x}^2 = 2\mathrm{T}$$

so that eq. (4) reduces to

$$\begin{aligned}\mathscr{H} &= p_k\dot{q}_k - \mathscr{L} \\ &= \left(\frac{\partial \mathscr{L}}{\partial \dot{q}_k}\right)\dot{q}_k - \mathscr{L} \\ &= 2\mathrm{T} - (\mathrm{T} - \mathrm{V}) \\ \therefore \mathscr{H} &= \mathrm{T} + \mathrm{V} = \mathrm{E}\end{aligned} \tag{8}$$

In other words, for conservative systems the Hamiltonian gives us the total energy of a system. This fact becomes particularly useful in the domain of quantum mechanics where, often, we simply call the total energy of a system the 'Hamiltonian' of that system.

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