Lecture notes

Semiconductors in a magnetic field: Hall effect and cyclotron frequency

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These notes are based on a series of lectures given to graduate physics students at YCM, University of Mysore, as part of the PHYS404 specialisation course in solid state physics.

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A note on references: Footnotes are marked by superscript numbers ^{x}, equations by parentheses (x) and references to the appendix by square brackets [x].

Contents

1 Effects of a magnetic field

Charge carriers in a semiconductor have a momentum given by $m\mathbf{v}$, which is related to the wave vector, $\mathbf{k}$ as $m\mathbf{v} = \hbar\mathbf{k}$. This can be equated to the Lorentz force law

$$\mathbf{F}_L = m\dot{\mathbf{v}} = \hbar\dot{\mathbf{k}} = -e\left[\mathbf{E} + \frac{\mathbf{v}\times\mathbf{B}}{c}\right] \tag{1}$$

which gives us the force on a charge carrier (the electron according to this equation) due to an electric and a magnetic field. The same equation can be written in SI units setting $c = 1$; this applies to all equations in this paper.

Over time, say from 0 to t, and under a constant magnetic field, we can integrate the above equation to arrive at

$$\int_0^t \hbar\,\mathrm{d}\mathbf{k} = -\int_0^t \mathrm{d}t\; e\mathbf{E}$$

$$\mathbf{k}(t) - \mathbf{k}(0) = \frac{-et\mathbf{E}}{\hbar} = \delta\mathbf{k}\ \text{(say)}$$

All of this applies not just to a particular type of charge but to a **Fermi sphere** itself, i.e. a region of study in reciprocal space. This means the displacement of $\delta\mathbf{k}$ applies not to a single electron but to the Fermi surface, which means all electrons in the region under consideration are displaced by this amount.

Given a displacement of $\delta\mathbf{k}$, we can write from $\mathbf{F}_L = \hbar\,\delta\dot{\mathbf{k}}$ and $\hbar\,\delta\dot{\mathbf{k}} = m\mathbf{v}$ that

$$\mathbf{F}_L = \hbar\left(\frac{\mathrm{d}}{\mathrm{d}t} + \frac{1}{\tau}\right)\delta\mathbf{k} = m\left(\frac{\mathrm{d}}{\mathrm{d}t} + \frac{1}{\tau}\right)\mathbf{v} \tag{2}$$

where the first term on the right-hand side arises because of eq. (1) and the second term, $\mathbf{k}/\tau$ is the velocity imparted due to friction arising from collisions. τ^{-1} itself represents the rate of collisions. Note that term, like the first one inside the parentheses, too has dimensions of velocity.

From eq. (1) and (2) we have

$$m\left(\frac{\mathrm{d}}{\mathrm{d}t} + \frac{1}{\tau}\right)\mathbf{v} = -e\left[\mathbf{E} + \frac{\mathbf{v}\times\mathbf{B}}{c}\right]$$

which can be split into its components under the assumption that our constant magnetic field is in, say, the z direction, i.e. $\mathbf{B} \equiv B_z$, to get

$$m\left(\frac{\mathrm{d}}{\mathrm{d}t} + \frac{1}{\tau}\right)v_x = -e\left[E_x + \frac{v_y B_z}{c}\right]$$

$$m\left(\frac{\mathrm{d}}{\mathrm{d}t} + \frac{1}{\tau}\right)v_y = -e\left[E_y + \frac{v_x B_z}{c}\right]$$

$$m\left(\frac{\mathrm{d}}{\mathrm{d}t} + \frac{1}{\tau}\right)v_z = -eE_z$$

The drift velocity components, v_x, v_y and v_z, in the static case, i.e. when the time derivatives are zeroes, can now be written as

$$v_x = \frac{-e\tau}{m}E_x - \frac{eB_z}{mc}v_y\tau$$

giving us the following three equations:

$$v_x = \frac{-e\tau}{m}E_x - \omega_c v_y \tau$$

$$v_y = \frac{-e\tau}{m}E_y - \omega_c v_x \tau$$

$$v_z = \frac{-e\tau}{m}E_z$$

where $\omega_c = eB/mc$ is called the **cyclotron frequency** of the semiconductor. We will discuss the reason for this naming and the nature of this phenomenon in detail later on, after we discuss another important effect semiconductors experience when placed in a magnetic field.

2 Generic Hall effect and Hall coefficient

Named after its discoverer, Edwin Hall, the Hall effect is the phenomenon where a potential difference (called the Hall voltage) develops across an electrical conductor, transverse to an electric current in the conductor and in the presence of a magnetic field perpendicular to the conductor.

Without a magnetic field and only under an applied electric current, charge carriers move in a straight-line along the electric field. When a perpendicular magnetic field is applied, the charge carriers experience a Lorentz force that curves their path causing them to accumulate on one side of the conductor thereby creating a charge absence on the other side and hence a potential difference.

This potential difference, as a result, is perpendicular to both the electric and the magnetic fields, which are themselves mutually perpendicular. Mathematically, the potential difference is developed along the direction of $\mathbf{j} \times \mathbf{B}$ for a current $\mathbf{j}$ and an applied field $\mathbf{B}$.

Suppose we had an applied electric field E_x and magnetic field B_z so that charge carriers would move along the x axis, then any displacement along the y direction can safely be dismissed: $\delta v_y = 0$. The condition that satisfies this is the presence of an electric field

$$E_y = -\omega_c \tau E_x = \frac{-eB_z\tau}{mc}E_x$$

The ratio $E_y/j_x B_z$ is called the **Hall coefficient** or **Hall factor** and is often denoted by R_H. If $j_x = ne^2\tau E_x/m$ as usual, we have

$$R_H = \frac{eE_x\tau B_z m}{ne^2\tau E_x B_z mc} = \frac{1}{-enc} \tag{3}$$

Observe from the above equation that R_H depends on carrier concentration n. The greater n is, the lower R_H becomes. Also, see [B1] for an interesting examination of how R_H can be calculated for p-type doped Silicon.

3 Hall effect in impurity semiconductors

Consider an infinitesimal volume of an impurity semiconductor with n_0 electrons and p_0 holes at temperature T. Say we apply an electric field E_y and a magnetic field B_z as before. We will then observe a j_x which results in a Hall voltage across the conductor. We now have

$$\mathbf{J}_\mathbf{E} = e(n\mu)\mathbf{E}$$
$$\therefore\ \mathbf{J}_\mathbf{E} = \sigma\mathbf{E} \tag{4}$$

which is the current density in terms of the conductivity of our semiconductor. Note that here $n\mu$ is really $n_0\mu_n + p_0\mu_p$ for electrons and holes combined.

When B_z is applied, the Lorentz force on charge carriers *due to the magnetic field* becomes

$$\mathbf{F}_{L_B} = -e\left(\frac{\mathbf{v}_n \times \mathbf{B}}{c}\right) = -e\left(\frac{v_x B_z}{c}\right) \tag{5}$$

The balance condition requires that eq. (4) must be equal to the current density due to B_z, giving us[1]

$$\mathbf{J}_E + \mathbf{J}_L = 0$$
$$[e(n_0\mu_n + p_0\mu_p)\mathbf{E}] + [pev_{y_p} + nev_{y_n}] = 0$$

However we must express v_{y_p} and v_{y_n} in terms of v_{n_x} and v_{p_x} for which we turn to $\mathbf{F} = m\mathbf{a}$ in the form

$$\mathbf{F}_n = m_n^*\left(\frac{v_{y_n}}{\tau}\right)\hat{j}$$

and equate it to eq. (5) to get

$$v_{y_n} = \left(\frac{-e\tau v_{n_x} B_z}{m_n^* c}\right) = \frac{\mu_n^2 E_x B_z}{c}$$

using $\mu = e\tau/m^*$, where m^* is the anisotropic **effective mass** in the $\mathbf{E}_H$ direction.

The balance condition now gives us

$$[e(n_0\mu_n + p_0\mu_p)\mathbf{E}_H] = -\left[\frac{(p_0\mu_p^2 - n_0\mu_n^2)(eE_x B_z)}{c}\right]$$

and, using eq. (4) and substituting for σ we end up with

$$\mathbf{E}_H = \frac{J_x B_z}{c}\left[\frac{p_0\mu_p^2 - n_0\mu_n^2}{e(p_0\mu_p + n_0\mu_n)^2}\right]$$

where we can replace $\mu_n/\mu_p = b$ to get the **Hall field** as

$$\mathbf{E}_H = \frac{J_x B_z}{ec}\left[\frac{p_0 - n_0 b^2}{(p_0 + n_0 b)^2}\right] \tag{6}$$

[1]Here terms of the form v_{n_x} are the same as v_x for n, i.e. electrons. The change in notation is only to show that we the v_y in the two terms of $\mathbf{J}_L$ are separately for holes and electrons.

4 Hall mobility

One of the main laboratory uses of the Hall effect is to measure the mobility of charge carriers in a semiconductor. Mobility measured using the Hall effect is known as **Hall mobility**.

We know that the Hall field and the applied field must balance each other:

$$-e\mathbf{E}_H = -ev_x B_z/c$$

or

$$\mathbf{E}_H = v_x B_z/c$$

Electron current is usually given by $I = -env_x tw$ for a semiconductor of width w and thickness t. Therefore, substituting for v_x,

$$\mathbf{E}_H = \frac{-IB}{enctw}$$

Using eq. (3) we can write

$$\mathbf{E}_H = \frac{-IBR_H}{tw}$$

and, using $w\mathbf{E}_H = V_H$, for the Hall voltage we get

$$R_H = \frac{V_H t}{IB}$$

Given R_H we can write σR_H as $-en\mu/enc = \mu/c$ (or just μ in SI units for reasons explained before). The Hall mobility is thus is given by

$$\mu_H = \frac{\sigma V_H t}{IB} \tag{7}$$

where V_H is the Hall voltage, t the thickness of the sample, I the current applied and B the magnetic field. Note that subscripts have been left out of this equation, e.g. μ_H instead of μ_{H_n} and B instead of B_z etc. since this is the general case that is to be adapted as required.

5 Temperature and mobility

The phonon[2] concentration in a semiconductor increases with temperature, so does scattering. With an increase in scattering, the time between successive collisions, τ, drops, thereby decreasing carrier mobility as well. The actual variation of mobility with temperature depends on the semiconductor itself: $\mu \propto T^{-2.4}, T^{-2.2}, T^{-1.7}, T^{-2.3}$ for Si electrons and holes and Ge electrons and holes respectively.

[2]Like a photon but associated with lattice vibrations. All vibrations have an energy associated with them, and a quantum of energy associated with a lattice vibration is called a phonon.

6 Magnetoresistance — *qualitative only*

When a semiconductor is placed in a magnetic field, it exhibits the tendency to change its resistance due to the applied magnetic field. This phenomenon is known as **magnetoresistance** and, based on the nature of variation of resistance, i.e. increase with or against the applied field, it is categorised as positive or negative.

There are also various extents to which a magnetic field can affect a multi-layer semiconductor, based on which we classify them as Giant, Extraordinary or Tunnel Magnetoresistance. If the resistance depends on the angle between the existing electric current and the applied magnetic field, the effect is called anisotropic magnetoresistance.

7 Cyclotron resonance to measure effective mass

When we discussed cyclotron resonance in §1 we saw that, under a constant and uniform magnetic field, B, charge carriers of a semiconductor gyrate about the B direction with an average frequency given by

$$\omega_c = \frac{eB}{m^* c}$$

where m^* is the effective mass as discussed in §3.

When a transverse oscillating field of the same frequency ω_c is applied, resonance is observed in the carriers and they absorb energy during each half-cycle of orbit about B.

Energy in k-space is given by $\epsilon(k) = \hbar^2 k^2/2m^*$, or, specifically for a spherical surface,

$$m^* = \frac{\hbar^2}{\frac{\mathrm{d}^2\epsilon}{\mathrm{d}k^2}}$$

Since Fermi surfaces tend to be ellipsoidal rather than perfect spheres we have

$$m_i^* = \frac{\hbar^2 \mathrm{d}k_i^2}{\mathrm{d}^2\epsilon}$$

which is a tensor for $i = x, y, z$. For now, though, within tolerable error, we will consider them to be spheres. This tensorial expression is known as the **effective mass tensor**.

Using the three equations of $\mathbf{v}$ from §1 and assuming the nature of E_x, x and y to be

$$E_x = E_0 e^{i\omega t} \qquad x = x_0 e^{i\omega t} \qquad y = y_0 e^{i\omega t}$$

we can write the components of the Lorentz force as

$$F_{L_x} = m_p^* \frac{\mathrm{d}^2 x}{\mathrm{d}t^2} = \frac{e}{c} v_y B_z + eE_0 e^{i\omega t}$$

$$F_{L_y} = m_p^* \frac{\mathrm{d}^2 y}{\mathrm{d}t^2} = -\frac{e}{c} v_x B_z$$

$$F_{L_z} = m_p^* \frac{\mathrm{d}^2 z}{\mathrm{d}t^2} = 0$$

We assume that any magnetic effect of the applied oscillating field is negligible. The fact that F_{L_z} is zero simply means that the particle velocity in the z direction is zero.

We can write only the x and y components which are of interest to us as

$$\frac{\mathrm{d}^2 x}{\mathrm{d}t^2} = \omega_c \frac{\mathrm{d}y}{\mathrm{d}t} + \frac{eE_0 e^{i\omega t}}{m_p^*}$$

$$\frac{\mathrm{d}^2 y}{\mathrm{d}t^2} = -\omega_c \frac{\mathrm{d}x}{\mathrm{d}t}$$

and, assuming solutions of the form $x = x_0 e^{i\omega t}$ and $y = y_0 e^{i\omega t}$ we can substitute these into the above equations and solve for the amplitudes

$$x_0 = \left(\frac{eE_0}{m_p^*}\right) \Bigg/ \left(\omega_c^2 - \omega^2\right) \tag{8}$$

and

$$y_0 = \frac{i\omega_c x_0}{\omega} = \left(\frac{i\omega_c}{\omega m_p^*}\right)\left(\frac{e\omega_c}{\omega_c^2 - \omega^2}\right) \tag{9}$$

Knowing B_z and ω_c then we can measure x_0 and y_0 with an AC field and determine m^*.

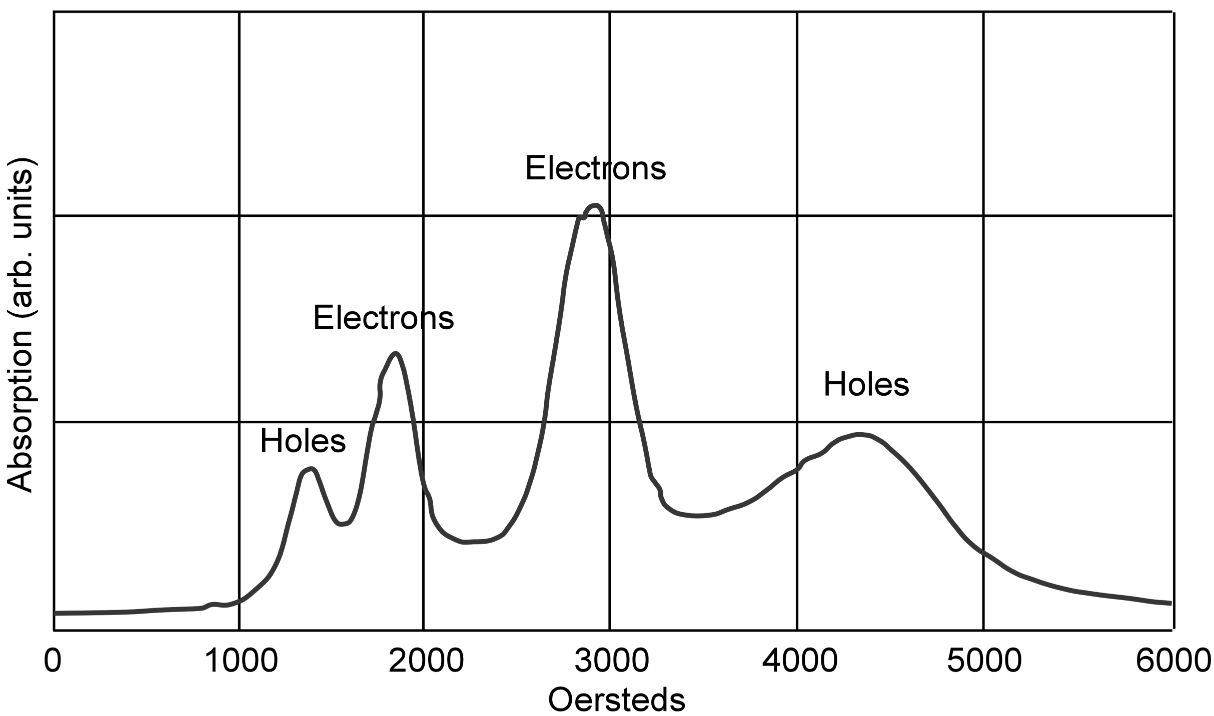

Figure 1: Laughlin/Stanford/AP273/Spring 2007

8 Spherical and ellipsoidal energy surfaces

For eq. (8) and (9), as $\omega \to \omega_c$ we have

$$y_0 = ix_0 = x_0 e^{i\pi/2}$$

and we know that

$$x(t) = x_0 e^{i\omega_c t}$$

which means

$$y(t) = y_0 e^{i\omega t} = x_0 e^{i(\omega_c t + \pi/2)}$$

Energies fall at resonance (two different frequencies for electrons and holes) because a large majority of carriers absorb EM field energy. However—

1. The presence of multiple resonance peaks was unexpected but was observed
2. The orientation of the crystal caused changes in the observed peaks
3. Multiple valence bands were observed

To explain these, an ellipsoidal surface has to be considered instead of a spherical one.

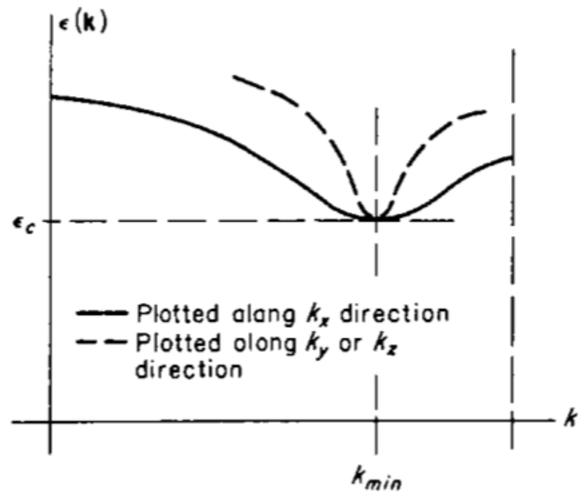

Figure 2: ϵ–k curve, from [B3]

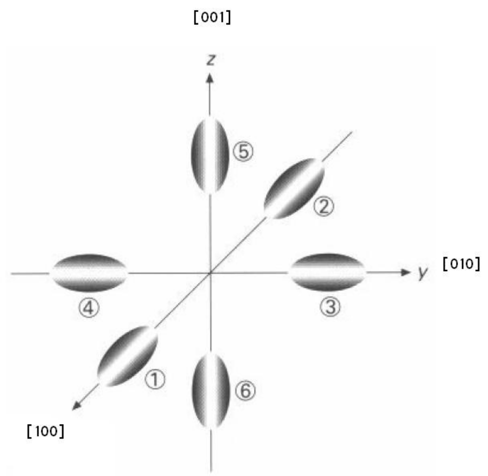

Figure 3: Laughlin/Stanford/AP273/Spring 2007

The energy curves, as we have discussed already, depend on the effective mass m^* of our semiconductor. However, the presence of multiple resonance peaks and valence bands means a different effective mass exists along different axes.

When the ϵ–k curve is plotted plotted parallel to the x axis the curve corresponds to an effective mass, say, $m^*_{\parallel}$. When it is plotted along the y axis (here, since we have B_z, but any axis perpendicular to the previous one) we have a corresponding 'perpendicular' effective mass $m^*_\perp$.

From this point forward, we change the magnetic orientation from B_z to B_y just to show that the equations work regardless of the direction of $\mathbf{B}$. It may just as well be done with B_x.

The relation between energy ϵ and $\mathbf{k}$ for an ellipsoidal surface can be shown to be

$$\epsilon - \epsilon_c = \hbar^2 \left[\frac{k_x^2 + k_y^2}{2m^*_\perp} + \frac{(k_z - k_{z_0})^2}{2m^*_\parallel} \right]$$

where the tensor components $(1/m^*)_{\alpha\beta}$ are to be determined. Since $\partial^2\epsilon/\partial k_\alpha \partial k_\beta = 0$ for $\alpha \neq \beta$ all the off-diagonal elements are zero. The remaining diagonal elements turn out to be as follows [B3]:

$$\left(\frac{1}{m^*}\right) = \begin{bmatrix} \frac{1}{m^*_\perp} & 0 & 0 \\ 0 & \frac{1}{m^*_\perp} & 0 \\ 0 & 0 & \frac{1}{m^*_\parallel} \end{bmatrix}$$

Using $\mathbf{F} = \left(e\mathbf{E} + \frac{e}{c}\mathbf{v} \times \mathbf{B}\right)$ along with the new effective mass tensor above, we get three equations for force:

$$m^*_\perp \frac{\mathrm{d}v_x}{\mathrm{d}t} = \frac{e}{c}(v_y B_z - v_z B_y) + eE_x$$

$$m^*_\perp \frac{\mathrm{d}v_y}{\mathrm{d}t} = \frac{e}{c}(v_z B_x - v_x B_z) + eE_y$$

$$m^*_\parallel \frac{\mathrm{d}v_z}{\mathrm{d}t} = \frac{e}{c}(v_x B_y - v_y B_x) + eE_z$$

as the general equations. If any B_0 is applied so that $B_x = B_0 \sin\theta$, $B_z = B_0 \cos\theta$ and $B_y = 0$, and $E_y = E_0 e^{i\omega t}$, $E_x = E_z = 0$. we end up with

$$\frac{\mathrm{d}v_x}{\mathrm{d}t} = \omega_\perp v_y \cos\theta$$

$$\frac{\mathrm{d}v_y}{\mathrm{d}t} = \omega_\perp v_z \sin\theta - \omega_\perp v_x \cos\theta + \frac{eE_0}{m^*_\perp} e^{i\omega t}$$

$$\frac{\mathrm{d}v_z}{\mathrm{d}t} = -\omega_\parallel v_y \sin\theta$$

where

$$\omega_\perp = \frac{eB_z}{m^*_\perp c} \qquad \omega_\parallel = \frac{eB_z}{m^*_\parallel c}$$

The two masses $m^*_\perp$ and $m^*_\parallel$ result in corresponding $\omega_\parallel$ and $\omega_\perp$ which are related by Schottky's relation (which we will prove in the next section):

$$\omega_c^2 = \omega_\perp^2 \cos^2\theta + \omega_\parallel \omega_\perp \sin^2\theta \tag{10}$$

where ω_c is the cyclotron resonance frequency, θ is the angle between the applied field and the major ellipsoidal axis.

Using $x(t) = x_0 e^{i\omega t}$, $y(t) = y_0 e^{i\omega t}$ and $z(t) = z_0 e^{i\omega t}$ in the above three equations of motion (for acceleration on the left-hand side) we get corresponding equations for x_0, y_0 and z_0 with ω_c in the denominator (left as an exercise).

Equation (10) affects the resonance of charge carriers since, for the orientation $\theta = 0$ two peaks or ellipsoids, depending on whether we refer to a graph or a model, are seen (corresponding to $\omega_c = \omega_\perp$ from the first term), and, for the orientation $\theta = \pi/2$ the other four ellipsoids are seen (corresponding to $\omega_c = \sqrt{\omega_\parallel \omega_\perp}$).

The following are experimentally obtained values: ${}_{Ge}m^*_\parallel = 1.64m_0$, ${}_{Si}m^*_\parallel = 0.98m_0$, ${}_{Ge}m^*_\perp = 0.0819m_0$ and ${}_{Si}m^*_\perp = 0.19m_0$ for intrinsic Ge and Si. Note the large variation in $m^*_\parallel$ to $m^*_\perp$ ratio for Ge and Si.

9 Schottky's relation

The momenta of carrier electrons[3] is given by the three equations

$$p_x = p_{x_0} e^{i\omega t} \qquad p_y = p_{y_0} e^{i\omega t} \qquad p_z = p_{z_0} e^{i\omega t} \tag{11}$$

And the force experienced by an electron is related to its momentum (classically) as

$$\mathrm{d}\mathbf{p}/\mathrm{d}t = -(e/c)(\mathbf{v} \times \mathbf{B}) \tag{12}$$

which gives us

$$\begin{aligned}
\left(\frac{\mathrm{d}p_x}{\mathrm{d}t}\right)_{\text{total}} &= \frac{-e}{c}\left(\frac{p_y B_0 \cos\theta}{m^*_\perp}\right) - \frac{\mathrm{d}p_x}{\mathrm{d}t} = -\omega_\perp p_y \cos\theta - i\omega p_x \\
\left(\frac{\mathrm{d}p_y}{\mathrm{d}t}\right)_{\text{total}} &= \frac{-e}{c}\left(\frac{p_z B_0 \sin\theta}{m^*_\parallel} - \frac{p_x B_0 \cos\theta}{m^*_\perp}\right) - \frac{\mathrm{d}p_y}{\mathrm{d}t} = -\omega_\perp p_x \cos\theta - \omega_\parallel p_z \sin\theta - i\omega p_y \\
\left(\frac{\mathrm{d}p_z}{\mathrm{d}t}\right)_{\text{total}} &= \frac{-e}{c}\left(\frac{-p_y B_0 \sin\theta}{m^*_\perp}\right) - \frac{\mathrm{d}p_z}{\mathrm{d}t} = \omega_\perp p_y \sin\theta - i\omega p_z
\end{aligned} \tag{13}$$

where B_0 is any applied magnetic field, $B_x = B_0 \sin\theta$, $B_z = B_0 \cos\theta$ and $B_y = 0$. The right-hand side of eq. (13) is obtained by differentiating eq. (11) and solving eq. (12) and computing the net momenta.

Solving the determinant of the coefficients of p from eq. (13) we get

$$\begin{vmatrix} i\omega & \omega_\perp \cos\theta & 0 \\ -\omega_\perp \cos\theta & i\omega & \omega_\parallel \sin\theta \\ 0 & -\omega_\perp \sin\theta & i\omega \end{vmatrix} = 0$$

which yields Schottky's equation (10) as its result.

. . .

[3]This idea can be extended to holes too, of course.

Appendices

A Some useful results

1. There's nothing here.

B Bibliography

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