Lecture notes

Radiating electrodynamic systems

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These notes are based on a series of lectures on classical electrodynamics given to graduate physics students at YCM, University of Mysore, during the academic year 2016–2017. This document represents only a part of the entire course which covers, among other things, classical electrodynamics, plasma physics and optics.

Please e-mail `hello@vhbelvadi.com` with your thoughts or suggestions, or if you spot any errors. These notes are—and will probably always remain—a work in progress. They may be updated and, if they are, the latest version of this document will always be available for download at `http://vhbelvadi.com/notes/` for anybody interested in it.

A note on references: Footnotes are marked by superscript numbers ^{x}, equations by parentheses (x) and references to the appendix by square brackets [x]. This is a deviation from my previous lecture notes but I favour this because such a pattern makes it easier for me to prepare these notes. As always, these are typeset with LaTeX on a Mac.

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Changelog:

	v.1.0	Initial.
	v.2.0	Corrected errors in §2.2.
TO-DO:	v.3.0	Discuss arbitrarily moving point charges.
TO-DO:	v.4.0	Derive reduction formulae. (Is this necessary?)

P.S. LaTeX needs much better line breaking capabilities for equations. If packages better than breqn and the split and multiline environments built into amsmath exist, or if you have written one yourself, please send me an e-mail. Thank you.

Accelerating charges emit electromagnetic radiations. Over the course of our discussion, we will look at dipolar oscillation of a charge taking retarded time into account[1], establish formulae governing radiations due to moving charges and so on.

1 The radiation zone of oscillating electric dipoles

A charge oscillating obediently with a displacement, d, is described by a function of time with the usual oscillatory equation

$$q(t) = q_0 \cos(\omega t) \tag{1}$$

with $\omega = 2\pi f$ containing information about the frequency of such an oscillation. The same oscillation in terms of the dipole moment, $\mathbf{p}(t)$, then is

$$\mathbf{p}(t) = p_0 \cos(\omega t)\hat{\mathbf{z}} \tag{2}$$

with $p_0 \equiv q_0 d$. We have assumed oscillation along the z-axis.

Note that this dipole may be thought of as two spherical conductors connected by a thin wire, with a charge, q, flowing alternately between each sphere. As a result, when one sphere acquires a charge $+q$, the other goes to $-q$ relatively. When the condition in (1) holds, this setup is known variously as a dipole antenna, or a *Hertzian dipole* or a *Hertzian antenna*, after the German physicist, Heinrich Hertz.

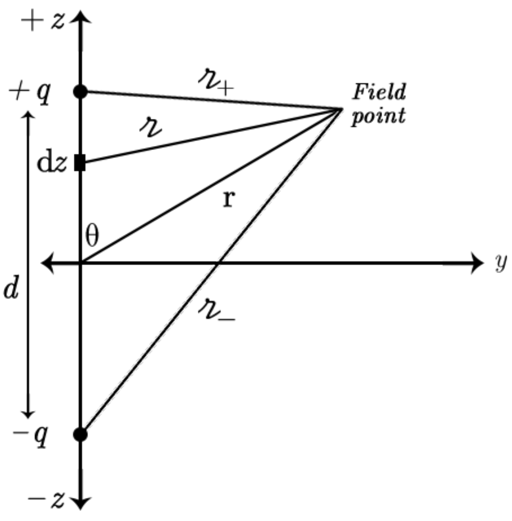


Figure 1: A Hertzian antenna.

It is time to set up our system. As shown in fig. 1, say the line connecting the field point to the origin is some $\mathbf{r}$. Say it forms an angle θ with the positive z-direction of oscillation such that $\theta \neq 90°$ so the field point is *not* on the x- or y-axes.

Further, let $\mathscr{r}_+$ be the distance from the field point to the extreme position of oscillation in the positive z-direction. Likewise, let $\mathscr{r}_-$ be the line connecting the field point to the extreme position of oscillation in the negative z-drection. By the law of cosines,

$$\mathscr{r}_\pm = \sqrt{r^2 \mp rd\cos\theta + \left(\frac{d}{2}\right)^2} \tag{3}$$

[1]It is highly recommended that you read up on prerequisite ideas such as the Liénard-Wiechert potentials and fields, and the concept of retarded time, since we will make use of, or refer to, all of these in our discussion here. There are lecture notes like this one on these prerequisite topics titled "Fields due to moving charges" available at `http://vhbelvadi.com/notes/` to download.

1.1 Potentials in the far field region

The retarded potential, V($\mathbf{r}$,t), generally described by

$$V(\mathbf{r},t) = \frac{1}{4\pi\epsilon_0}\int_\tau \frac{\rho(\mathbf{r}',t_r)}{\mathscr{r}}\mathrm{d}\tau$$

is now retarded by $t_r = t - (\mathscr{r}_\pm/c)$ to become

$$V(\mathbf{r},t) = \frac{1}{4\pi\epsilon_0}\left[\frac{q_0\cos\left[\omega\left(t-\frac{\mathscr{r}_+}{c}\right)\right]}{\mathscr{r}_+} - \frac{q_0\cos\left[\omega\left(t-\frac{\mathscr{r}_-}{c}\right)\right]}{\mathscr{r}_-}\right] \tag{4}$$

We now make a few assumptions[2]

1. d is extremely small. In other words, $d \ll r$.
2. d is smaller than the wavelength of the oscillation, or, in terms of the frequency, $d \ll (c/\omega)$, since $c = f\lambda$ for a wavelength of λ.
3. The distance to the oscillatory extremes from the field point is greater than the wavelength, or, $(c/\omega) \ll r$.

Let us examine what implications these assumptions have. The first assumption requires that we eliminate the $(d/2)^2$ term in equation (3) and use the binomial expansion [A1] to get

$$\begin{aligned}\mathscr{r}_\pm &\approx \left[r^2 \mp rd\cos\theta\right]^{\frac{1}{2}}\\ &\cong r\left(1 \mp \frac{d}{2r}\cos\theta\right)\end{aligned} \tag{5}$$

and, as a result,

$$\frac{1}{\mathscr{r}_\pm} \cong \frac{1}{r}\left(1 \pm \frac{d}{2r}\cos\theta\right) \tag{6}$$

Therefore, using (5) in (4),

$$\begin{aligned}\cos\left[\omega\left(t-\frac{\mathscr{r}_\pm}{c}\right)\right] &\cong \cos\left[\omega\left(t-\frac{r}{c}\right) \pm \omega\left(\frac{rd}{2rc}\cos\theta\right)\right]\\ &= \cos\left[\omega\left(t-\frac{r}{c}\right)\right]\cos\left[\omega\left(\frac{d}{2c}\cos\theta\right)\right]\\ &\qquad \mp \sin\left[\omega\left(t-\frac{r}{c}\right)\right]\sin\left[\omega\left(\frac{d}{2c}\right)\cos\theta\right]\end{aligned}$$

[2]The three assumptions together: $d \ll (c/\omega) \ll r$, so the maximum extreme-to-extreme displacement of the oscillation is smaller than the wavelength and the straight line distance form the origin of oscillation to our field point field point, from where we observe the system/phenomenon, is greater that the wavelength.

but, from the second approximation, since $d \ll (c/\omega) \Rightarrow (\omega d/2c)$ is an extremely small quantity. As a result,

$$\cos\left(\frac{\omega d}{2c}\cos\theta\right) \to \cos(0) \to 1$$

and, by the small angle approximation,

$$\sin\left(\frac{\omega d}{2c}\cos\theta\right) \to \frac{\omega d}{2c}\cos\theta$$

giving us

$$\cos\left[\omega\left(t - \frac{\mathscr{r}_{\pm}}{c}\right)\right] \cong \cos\left[\omega\left(t - \frac{r}{c}\right)\right] \mp \left(\frac{\omega d}{2c}\cos\theta\right)\sin\left[\omega\left(t - \frac{r}{c}\right)\right] \tag{7}$$

Substituting (6) and (7) in (4) we arrive at

$$V(\mathbf{r}, \theta, t) = \frac{p_0 \cos\theta}{4\pi\epsilon_0 r}\left\{-\frac{\omega}{c}\sin\left[\omega\left(t - \frac{r}{c}\right)\right] + \frac{1}{r}\cos\left[\omega\left(t - \frac{r}{c}\right)\right]\right\} \tag{8}$$

which is the potential of an oscillating dipole.

The third approximation, that $r \gg \lambda$, is what brings us to the *far field region* that we are interested in. Since $r \gg \lambda \Rightarrow r \gg (c/\omega) \Rightarrow 2\pi r \gg ct$, which means we can neglect ct before $2\pi r$. Therefore,

$$\omega\left(t - \frac{r}{c}\right) \Rightarrow 2\pi - \frac{2\pi r}{ct} \Rightarrow 2\pi(1 - r)$$

With r being large, the cosine of large (negative or positive) angles tends to zero. Therefore,

$$cos\left[\omega\left(t - \frac{r}{c}\right)\right] \to 0$$

and plugging this into (8) gives us the far field scalar potential due to our oscillating dipole as

$$\boxed{V(\mathbf{r}, \theta, t) = -\frac{p_0\omega}{4\pi\epsilon_0 c}\left(\frac{\cos\theta}{r}\right)\sin\left[\omega\left(t - \frac{r}{c}\right)\right]} \tag{9}$$

In much the same way we now need to set up the vector potential, $\mathbf{A}(\mathbf{r}, \theta, t)$. Returning to fig. 1, let us define a region, $\mathrm{d}z$, to help us picture it while it creeps into our equation. Now, a current oscillating between the same two spheres that we used before will simply constitute an alternating current.

$$\mathbf{I}(t) = \frac{\mathrm{d}q}{\mathrm{d}t}\hat{\mathbf{z}} = -q_0\omega\sin(\omega t)\hat{\mathbf{z}}, \text{ given that } q = q_0\cos(\omega t)$$

The vector potential then is given by the integral of this current in a region dz over the entire displacement from $-\frac{d}{2}$ to $+\frac{d}{2}$.

$$\mathbf{A} = \frac{\mu_0}{4\pi} \int_{-d/2}^{+d/2} \mathrm{d}z \; \frac{-q_0 \omega \sin\left[\omega\left(t - \frac{\mathscr{r}}{c}\right)\right]\hat{\mathbf{z}}}{\mathscr{r}}$$

The integral goes to $1\Big|_{-d/2}^{+d/2} = d$ allowing us to simply write this as the value of the argument at the centre/origin, making $\mathscr{r} \to r$ (see fig. 1) and thus, with $q_0 d \equiv p_0$, we get the far field vector potential due to our oscillating dipole.

$$\boxed{\mathbf{A}(\mathbf{r}, \theta, t) = -\frac{\mu_0 p_0 \omega}{4\pi r} \sin\left[\omega\left(t - \frac{r}{c}\right)\right]\hat{\mathbf{z}}} \tag{10}$$

Note that both (9) and (10) fall as $1/r$ and are the potentials present in the far field region, or the *radiation zone*, of an oscillating dipole.

1.2 Fields in the far field region

The electric ($\mathbf{E}$) and magnetic ($\mathbf{B}$) fields are determined from the potentials, V and $\mathbf{A}$, as given by (9) and (10) above, using their usual mathematical definitions.

$$\mathbf{E} = -\nabla V - \frac{\partial \mathbf{A}}{\partial t} \qquad\qquad \mathbf{B} = \nabla \times \mathbf{A}$$

From (9) we know that $V \equiv V(\mathbf{r}, \theta, t)$, so we have to evaluate it in spherical polar coördinates with two spatial coördinates, r and θ according to the definiton of a gradient in spherical polar coördinates [A2].

$$\nabla V = -\frac{p_0 \omega}{4\pi\epsilon_0 c}\left\{\cos\theta\left(-\frac{1}{r^2}\sin\left[\omega\left(t - \frac{r}{c}\right)\right] - \frac{\omega}{rc}\cos\left[\omega\left(t - \frac{r}{c}\right)\right]\right)\hat{\mathbf{r}} \right.$$
$$\left. -\frac{\sin\theta}{r^2}\sin\left[\omega\left(t - \frac{r}{c}\right)\right]\hat{\theta}\right\}$$

We have assumed that $r \gg (c/\omega)$, or $(r\omega/c) \gg 1$, so $\omega[t - r/c]$ becomes a small enough term to ignore and, in turn, so do the two sine terms in the above equation.

$$\therefore\ \nabla V = -\frac{p_0 \omega}{4\pi\epsilon_0 c}\left(\frac{\cos\theta}{r}\right)\cos\left[\omega\left(t - \frac{r}{c}\right)\right] \tag{11}$$

Calculating the term $\partial\mathbf{A}/\partial t$ is much more straightforward, but we will have to first convert our equation from its Cartesian form in (10) into spherical polar coördinates according to [A4].

$$\mathbf{A}(\mathbf{r}, \theta, t) = -\frac{\mu_0 p_0 \omega}{4\pi r}\sin\left[\omega\left(t - \frac{r}{c}\right)\right]\hat{\mathbf{z}} \cong -\frac{\mu_0 p_0 \omega}{4\pi r}\sin\left[\omega\left(t - \frac{r}{c}\right)\right]\left(\cos\theta\,\hat{\mathbf{r}} - \sin\theta\,\hat{\theta}\right)$$

Therefore,

$$\frac{\partial \mathbf{A}}{\partial t} = -\frac{\mu_0 p_0 \omega^2}{4\pi r} \cos\left[\omega\left(t - \frac{r}{c}\right)\right]\left(\cos\theta\,\hat{\mathbf{r}} - \sin\theta\,\hat{\theta}\right) \tag{12}$$

Substituting (11) and (12) in the standard equation for electric field we get

$$\mathbf{E} = -\nabla V - \frac{\partial \mathbf{A}}{\partial t} \Rightarrow \boxed{\mathbf{E} = -\frac{\mu_0 p_0 \omega^2}{4\pi}\left(\frac{\sin\theta}{r}\right)\cos\left[\omega\left(t - \frac{r}{c}\right)\right]\hat{\theta}} \tag{13}$$

The magnetic field is calculated using the curl of a vector in spherical polar coördinates [A3]. Knowing that the field curls around the ϕ direction[3], we may ease our problem by focusing on the $\hat{\phi}$ component exclusively. (To visualise, the r component is the z-axis (not to be confused with the r marked in fig. 1), the θ component is the angle between the z-axis and the r line as marked in fig. 1, and the ϕ component is the angle traced perpendicular to z coming out/going into the page, and this is what the magnetic field traces.)

$$\begin{aligned}
\nabla \times \mathbf{A} &= \frac{1}{r}\left(\frac{\partial}{\partial t}(rA_\theta) - \frac{\partial A_r}{\partial \theta}\right)\hat{\phi} \\
&= -\left(\frac{\mu_0 p_0 \omega}{4\pi}\right)\left(\frac{1}{r}\right)\left[\frac{\partial}{\partial r}\left(\sin\left[\omega\left(t - \frac{r}{c}\right)\right]\sin\theta\right)\right. \\
&\qquad\qquad\left. -\frac{\partial}{\partial \theta}\left(\sin\left[\omega\left(t - \frac{r}{c}\right)\right]\left(\frac{\cos\theta}{r}\right)\right)\right]\hat{\phi} \\
&= -\frac{\mu_0 p_0 \omega}{4\pi}\left(\frac{1}{r}\right)\left[\frac{\omega}{c}\sin\theta\cos\left[\omega\left(t - \frac{r}{c}\right)\right] + \frac{\sin\theta}{r}\sin\left[\omega\left(t - \frac{r}{c}\right)\right]\right]\hat{\phi}
\end{aligned}$$

yet again, the sine term vanishes thanks to our approximation that $r\omega/c \gg 1$, leaving us with

$$\mathbf{B} = \nabla \times \mathbf{A} \Rightarrow \boxed{\mathbf{B} = -\left(\frac{\mu_0 p_0 \omega^2}{4\pi c}\right)\left(\frac{\sin\theta}{r}\right)\cos\left[\omega\left(t - \frac{r}{c}\right)\right]\hat{\phi}} \tag{14}$$

the equation for the far zone magnetic field.

Notice how both (13) and (14) are remarkably similar. In fact $\mathbf{B}$ is simply ($\mathbf{E}$/c) in a mutually perpendicular direction (along $\hat{\phi}$ and $\hat{\theta}$ respectively). They are also perpendicular to the direction of propagation. Further, somewhat interestingly, they are both in phase and drop like $1/r$ with increase in distance. Also, they are both spherical waves and not plane waves.

We know what the **B** field physically looks like as it curls around the z-axiz in our example. A useful way of picturing the **E** field physically is to think of ripples in a pond, emanating from where the dipole antenna is located, spreading (or *radiating*) on the y-z plane in fig. 1.

[3] See the previous lecture on "Fields due to moving charges" (`http://vhbelvadi.com/notes/`)

1.3 The Poynting vector and the power of radiation

The energy transferred per unit area, or the *energy flux density*, of a radiation is given by the *Poynting vector*[4]. This is given by

$$\mathbf{S} = \frac{1}{\mu_0}\,\mathbf{E}\times\mathbf{B}$$

which, using (13) and (14), gives us the energy radiated by an oscillating dipole in the $\hat{\mathbf{r}}$ direction, since $\mathbf{E}$ and $\mathbf{B}$ are along $\hat{\theta}$ and $\hat{\phi}$ respectively and we are essentially left with $\hat{\theta}\times\hat{\phi} = \hat{\mathbf{r}}$.

$$\mathbf{S} = \left(\frac{\mu_0 p_0^2\omega^4}{16\pi^2 c}\right)\left(\frac{\sin\theta}{r}\right)^2 \cos^2\left[\omega\left(t-\frac{r}{c}\right)\right]\hat{\mathbf{r}} \tag{15}$$

The intensity is given by the time-average of the Poynting vector[5]. The time factor exists only inside the cosine term so we use the fact that average value of the cosine function is (1/2) (since $\cos^2\varphi$ goes from 0 to +1) and get

$$\langle\mathbf{S}\rangle = \left(\frac{\mu_0 p_0^2\omega^4}{32\pi^2 c}\right)\left(\frac{\sin\theta}{r}\right)^2\hat{\mathbf{r}} \tag{16}$$

Due to the sine term, there is intensity goes to zero along the $\hat{\mathbf{r}}$ direction, i.e. along the z-axis direction of propagation. It achieves a maximum value perpendicular to this direction. In other words, the intensity is mapped like a torus.

We now have, in (15), the Poynting vector, $\mathbf{S}$, giving us the rate of energy radiated per unit area. Its time average, $\langle\mathbf{S}\rangle$, in (16), gives us the intensity, which is the energy radiated per unit area per time.

If we go on to integrate this over the entire area, $\mathbf{A}$, of, say, a spherical charge (or, strictly speaking, our spheres in the setup shown in fig. (1)), we end up with the energy expended per time, or the *power of radiation*, often denoted by $\langle P\rangle$.

$$\langle P\rangle = \int\langle\mathbf{S}\rangle\cdot\mathrm{d}\mathbf{A} = \left(\frac{\mu_0 p_0^2\omega^4}{32\pi^2 c}\right)\int\left(\frac{\sin\theta}{r}\right)^2 r^2\sin\theta\,\mathrm{d}\theta\,\mathrm{d}\phi$$

$$\Rightarrow \langle P\rangle = \frac{\mu_0 p_0^2\omega^4}{12\pi c} \tag{17}$$

To integrate over the area of the spherical surface we have used the result in [A5] to write the area down in spherical polar form. To solve the integral itself, we use the result in [A6], which shows that the surface integral reduces to $8\pi/3$.

The power, therefore, has nothing to do with the size of the charge but rather with the charge itself (via $p_0 = q_0 d$) and, most usefully, through the frequency to which it is most sensitive (thanks to the ω^4 term).

[4] Since we will not be deriving the Poynting vector mathematically, the curious reader will find this helpful: `http://webpages.ursinus.edu/lriley/courses/p212/lectures/node26.html`

[5] Intensity is, by definition, the energy radiated per unit area *per unit time*.

2 Radiation from point charges

First derived by the Irish physicist, Sir Joseph Larmor [B6], the Larmor formula describes the power radiated by a point charge in non-relativistic acceleration[6].

2.1 Power radiated by a point charge (the Larmor formula)

We know that the electric and magnetic fields generated by arbitrarily moving point charges[7] are given by

$$\mathbf{E}(\mathbf{r}, t) = \left(\frac{q\mathscr{r}}{4\pi\epsilon_0 \, (\boldsymbol{\mathscr{r}} \cdot \mathbf{w})^3} \right) \big[(c^2 - v^2)\mathbf{w} + \boldsymbol{\mathscr{r}} \times (\mathbf{w} \times \mathbf{a}) \big] \tag{18}$$

$$\text{and} \quad \mathbf{B}(\mathbf{r}, t) = \frac{1}{c} \, \hat{\boldsymbol{\mathscr{r}}} \times \mathbf{E}(\mathbf{r}, t) \tag{19}$$

where $\mathbf{w} = c\hat{\boldsymbol{\mathscr{r}}} - \mathbf{v}$.

The Poynting vector then is

$$\begin{aligned} \mathbf{S} &= \frac{1}{\mu_0}(\mathbf{E} \times \mathbf{B}) \\ &= \frac{1}{\mu_0 c}[\mathbf{E} \times (\hat{\boldsymbol{\mathscr{r}}} \times \mathbf{E})] \\ \therefore \mathbf{S} &= \frac{1}{\mu_0 c}[E^2 \hat{\boldsymbol{\mathscr{r}}} - (\hat{\boldsymbol{\mathscr{r}}} \cdot \mathbf{E})\mathbf{E}] \end{aligned} \tag{20}$$

Suppose we want to calculate the field at some point $\mathscr{r}$ away from our moving point charge, we need to compute the integral of the Poynting vector over some infinitesimal area that is under observation, at the time $t - t_r = \mathscr{r}/c$.

This is no different from constructing a huge[8] sphere of radius $\mathscr{r}$ about the position of our moving charge at retarded time, t_r, and then calculating the power through some infinitesimal area on this sphere. Given that the power is dependent on the area, which, in turn, is dependent on $\mathscr{r}^2$, we need only concern ourselves with terms varying as $\mathscr{r}^2$ or $1/\mathscr{r}^2$ in (18) and (19), i.e. the *radiation fields.*

Consider the radiation field term in (18) to begin with.

$$\mathbf{E}_r = \left(\frac{q\mathscr{r}}{4\pi\epsilon_0 \, (\boldsymbol{\mathscr{r}} \cdot \mathbf{w})^3} \right) [\boldsymbol{\mathscr{r}} \times (\mathbf{w} \times \mathbf{a})] \tag{21}$$

[6]And deceleration, naturally.

[7]Yet again, this is from our previous lecture on "Fields due to moving charges". Notes for this lecture are available at `http://vhbelvadi.com/notes/` to download.

[8]More specifically, a sphere whose radius is much larger than the wavelength, in agreement with our assumption that r (and hence $\mathscr{r}$) $\gg c/\omega$ in §1.1.

With the mathematcial definition for $\mathbf{E}_r$ above involving a cross product with the vector, $\boldsymbol{\mathscr{r}}$, it is clear that $\mathbf{E}_r$ is perpendicular to $\hat{\boldsymbol{\mathscr{r}}}$. As a result of this, $\hat{\boldsymbol{\mathscr{r}}} \cdot \mathbf{E} = 0$, and (20) becomes

$$\mathbf{S}_r = \frac{1}{\mu_0 c} E_r^2 \, \hat{\boldsymbol{\mathscr{r}}} \tag{22}$$

The instantaneous velocity, $\mathbf{w}$, of the charge at time t_r with (say) $\mathbf{v} \ll c$, leaves us with $\mathbf{w} = c\hat{\boldsymbol{\mathscr{r}}} - \mathbf{v} \Rightarrow \mathbf{w}/c = \hat{\boldsymbol{\mathscr{r}}} - \mathbf{v}/c \Rightarrow \mathbf{w}/c = \hat{\boldsymbol{\mathscr{r}}}$ or $\mathbf{w} = c\hat{\boldsymbol{\mathscr{r}}}$. Therefore,

$$\mathbf{E}_r = \left(\frac{q\mathscr{r}}{4\pi\epsilon_0 c^3 \mathscr{r}^3}\right)[\mathscr{r}\hat{\boldsymbol{\mathscr{r}}} \times (c\hat{\boldsymbol{\mathscr{r}}} \times \mathbf{a})] = \left(\frac{q}{4\pi\epsilon_0 c^2 \mathscr{r}}\right)[\hat{\boldsymbol{\mathscr{r}}} \times (\hat{\boldsymbol{\mathscr{r}}} \times \mathbf{a})]$$

$$\Rightarrow \mathbf{E}_r = \left(\frac{\mu_0 q}{4\pi\mathscr{r}}\right)[(\hat{\boldsymbol{\mathscr{r}}} \cdot \mathbf{a})\hat{\boldsymbol{\mathscr{r}}} - \mathbf{a}] \tag{23}$$

Returning to the Poynting vector, once again,

$$\begin{aligned}\mathbf{S}_r &= \frac{1}{\mu_0 c}\left(\frac{\mu_0 q}{4\pi\mathscr{r}}\right)^2 [(\hat{\boldsymbol{\mathscr{r}}} \cdot \mathbf{a})\hat{\boldsymbol{\mathscr{r}}} - \mathbf{a}]^2 \; \hat{\boldsymbol{\mathscr{r}}} \\ &= \left(\frac{\mu_0 q^2}{16\pi^2 \mathscr{r}^2 c}\right)\left[(\hat{\boldsymbol{\mathscr{r}}} \cdot \mathbf{a})^2 + a^2 - 2\mathbf{a}(\hat{\boldsymbol{\mathscr{r}}} \cdot \mathbf{a})\right] \; \hat{\boldsymbol{\mathscr{r}}} \\ &= \left(\frac{\mu_0 q^2}{16\pi^2 \mathscr{r}^2 c}\right)\left[a^2 - (\hat{\boldsymbol{\mathscr{r}}} \cdot \mathbf{a})^2\right] \; \hat{\boldsymbol{\mathscr{r}}}\end{aligned}$$

Now if the angle between the direction of acceleration, $\hat{\mathbf{a}}$, and the vector $\hat{\boldsymbol{\mathscr{r}}}$ to our point of interest[9] is ϕ, then $[a^2 - (\hat{\boldsymbol{\mathscr{r}}} \cdot \mathbf{a})^2] = [a^2 - (\hat{\boldsymbol{\mathscr{r}}} \cdot \hat{\mathbf{a}})^2 a^2]$ and, taking a^2 out commonly from both terms we get, $a^2[1 - (\hat{\boldsymbol{\mathscr{r}}} \cdot \hat{\mathbf{a}})^2] = a^2[1 - \cos^2\phi] = a^2 \sin^2\phi$ making our Poynting vector

$$\mathbf{S}_r = \frac{\mu_0 q^2 a^2 \sin^2\phi}{16\pi^2 \mathscr{r}^2 c} \, \hat{\boldsymbol{\mathscr{r}}} \tag{24}$$

A good self-check at this point is to see if our radiation energy is still torus-shaped as we found it to be in (16). It is quickly apparent from the $\sin^2$ term that it is, so we are indeed on the right track. We can now safely proceed onto the power itself, given, this time, as the surface integral over our entire imaginary sphere[10].

$$\begin{aligned}P &= \oint \mathbf{S}_r \cdot \mathrm{d}\mathbf{A} \\ &= \frac{\mu_0 q^2 a^2}{16\pi^2 c}\int \frac{\sin^2\phi}{\mathscr{r}^2}\mathscr{r}^2 \sin\theta \, \mathrm{d}\theta \, \mathrm{d}\phi\end{aligned}$$

$$\Rightarrow \boxed{P = \frac{\mu_0 \, q^2 a^2}{6\,\pi c}} \tag{25}$$

[9] This is the vector form the position of our charge at time t_r to the point where we hope to measure the power of its radiation and is not to be confused with the position vector between $\pm q$ and the field point as in fig. 1. We are no longer interested in the antenna setup from that figure.

[10] We can drop the angle bracket notation, $\langle P\rangle$, hereon since we are no longer looking at the power in terms of the time average of the Poynting vector, or any average for that matter.

Note that we have, once again, used the result in [A6] to show that the surface integral reduces to $8\pi/3$. Also, equation (25) is popularly known as the *Larmor formula*.

2.2 Liénard's relativistic generalisation of Larmor's formula

In §2.1 we made the assumption that $\mathbf{v} \ll c$, which need not always be the case. The french physicist, Alfred-Marie Liénard, managed to generalise (25) and showed that Larmor's formula was but a specific case of his relativistic generalisation.

If we factor a mass term, m, into (25) we get

$$P = \frac{\mu_0\, q^2 a^2 m^2}{6\,\pi c\, m^2}$$

and we can now re-write this in terms of the momentum as

$$P = \frac{\mu_0\, q^2}{6\,\pi c\, m^2}\left|\frac{\mathrm{d}(m\mathbf{v})}{\mathrm{d}t}\right|^2$$

We can now use $\mathrm{d}t = \gamma\,\mathrm{d}\tau$ and $\mathbf{p} = m\mathbf{v}$ in the above equation, with τ being the proper time that can equated to the coördinate time, t, in our equation above, through γ, known as the Lorentz factor[11].

$$\begin{aligned}
P &= \frac{\mu_0\, q^2}{6\,\pi c\, m^2}\left|\frac{\mathrm{d}\mathbf{p}}{\gamma\,\mathrm{d}\tau}\right|^2 \\
&= \frac{\mu_0\, q^2}{6\,\pi c\, m^2\,\gamma^2}\left|\frac{\mathrm{d}\mathbf{p}}{\mathrm{d}\tau}\right|^2 \\
&= \frac{\mu_0\, q^2}{6\,\pi c\, m^2}(1-\beta^2)\left|\frac{\mathrm{d}\mathbf{p}}{\mathrm{d}\tau}\right|^2 \qquad \text{using } \gamma^2 = (1-\beta^2)^{-1} \\
&= \frac{\mu_0\, q^2}{6\,\pi c\, m^2}\left[\left(\frac{\mathrm{d}\mathbf{p}}{\mathrm{d}\tau}\right)^2 - \beta^2\left(\frac{\mathrm{d}\mathbf{p}}{\mathrm{d}\tau}\right)^2\right] \\
&= \frac{\mu_0\, q^2}{6\,\pi c\, m^2}\left[\left(\frac{\mathrm{d}\mathbf{p}}{\mathrm{d}\tau}\right)^2 - \frac{1}{c^2}\left(\frac{\mathrm{d}\mathcal{E}}{\mathrm{d}\tau}\right)^2\right] \qquad \text{for } v^2p^2 = \mathcal{E}^2 \text{ as } \mathbf{v} \to c \qquad (26)
\end{aligned}$$

$$\Rightarrow \boxed{P = \frac{\mu_0\, q^2}{6\,\pi c\, m^2}\left[-\frac{\mathrm{d}p_\mu}{\mathrm{d}\tau}\frac{\mathrm{d}p^\mu}{\mathrm{d}\tau}\right]} \qquad \text{(in covariant form, according to [A9])} \qquad (27)$$

[11] Discussing the Lorentz factor is outside the scope of this lecture. For now suffice it to say that the Lorentz factor is defined as $\gamma = \left(1 - v^2/c^2\right)^{-1/2}$ and is prominent in relativistic discussions. Along with this are two other useful quantities: the proper time, τ, which is related to the coördinate time, t, as $t = \gamma\,\tau$ and the commonly used symbol, β, which denotes the ratio v/c.

This is called *Liénard's generalisation* (or simply the *relativistic generalisation*) of Larmor's formula and reduces to Larmor's formula (25) when $\beta \ll 1$. Alternatively, by substituting $\mathcal{E} = \gamma\, mc^2$ and $\mathbf{p} = \gamma\, m\mathbf{v}$ into (26) we get[12] the non-covariant form,

$$P = \frac{\mu_0\, q^2 \gamma^6}{6\,\pi c} [\, \dot{\beta}^2 - (\beta \times \dot{\beta})^2 \,] \tag{28}$$

or, by further reduction,

$$P = \frac{\mu_0\, q^2 \gamma^6}{6\,\pi c} \left[a^2 - \left| \frac{\mathbf{v} \times \mathbf{a}}{c} \right|^2 \right] \tag{28}$$

3 Implications of the relativistic Larmor formula

3.1 Radiation energy loss in linear accelerators

From (27) it is clear that the power of radiation is inversely related to mass, so the lighter the particle, the more intense is its radiation effect or, specifically, its power loss. It is largest, therefore, for electrons, and this is one of the implications of the relativistic generalisation of the Larmor formula.

The second implication relies on understanding that we involved the four-momentum while computing the power of radiation by point charges in §2.2. In case of a linear accelerator, the motion is strictly one-dimensional, so our momentum is no longer a four-vector but instead simply $(\mathrm{d}p/\mathrm{d}t)$ in one dimension. And the power becomes

$$P = \frac{\mu_0\, q^2}{6\,\pi c\, m^2} \left[\frac{\mathrm{d}p}{\mathrm{d}t} \right]^2$$

Under the assumption that $\beta \to 1$ (or $\mathbf{v} \to c$) the equivlence between the derivatives of energy and momentum can be shown as

$$\frac{\mathrm{d}\mathcal{E}}{\mathrm{d}x} = \frac{\mathrm{d}(p\,\mathbf{v})}{\mathrm{d}x} = \frac{\mathrm{d}p}{\mathrm{d}x}\left(\frac{\mathrm{d}\mathbf{x}}{\mathrm{d}t}\right) = \frac{\mathrm{d}p}{\mathrm{d}t}\left(\frac{\mathrm{d}\mathbf{x}}{\mathrm{d}x}\right) = \frac{\mathrm{d}p}{\mathrm{d}t}$$

Now, given that the rate of change of momentum is equal to the change in energy per unit distance, the power becomes

$$P = \frac{\mu_0\, q^2}{6\,\pi c\, m^2} \left[\frac{\mathrm{d}\mathcal{E}}{\mathrm{d}x} \right]^2 \tag{29}$$

Suppose we have a linear accelerator set up to supply a power of $\left(\frac{\mathrm{d}\mathcal{E}}{\mathrm{d}t}\right)$ the ratio of

[12]The steps involved in reduction here are quite tedious and have been left out.

power radiated (29) to the power supplied becomes

$$\begin{aligned}\frac{P}{(\mathrm{d}\mathcal{E}/\mathrm{d}t)} &= \frac{\mu_0\, q^2}{6\,\pi c\, m^2}\left[\frac{\mathrm{d}\mathcal{E}}{\mathrm{d}x}\right]^2\left[\frac{\mathrm{d}t}{\mathrm{d}\mathcal{E}}\right]\\ &= \frac{\mu_0\, q^2}{6\,\pi c\, m^2}\left[\frac{\mathrm{d}\mathcal{E}}{\mathrm{d}x}\right]\left[\frac{\mathrm{d}t}{\mathrm{d}x}\right]\\ \Rightarrow \frac{P}{(\mathrm{d}\mathcal{E}/\mathrm{d}t)} &= \frac{\mu_0\, q^2}{6\,\pi c\, m^2\,\mathbf{v}}\left[\frac{\mathrm{d}\mathcal{E}}{\mathrm{d}x}\right] \qquad (30)\end{aligned}$$

It is clear therefore that the energy loss (30) depends on the rate of change of energy with distance as suggested by the $\mathrm{d}\mathcal{E}/\mathrm{d}x$ term on the right hand side. In other words, for there to be enough energy loss to affect our particle in a linear accelerator, the power radiated, P, must be comparable to the power fed in, $\mathrm{d}\mathcal{E}/\mathrm{d}t$, or, $\frac{P}{(\mathrm{d}\mathcal{E}/\mathrm{d}t)} \approx 1$ must be satisfied.

Suppose we now take it to the relativistic case where $\mathbf{v} \to c$, and multiply and divide by c^2 to introduce a second mc^2 term, we end up with

$$\Rightarrow \frac{P}{(\mathrm{d}\mathcal{E}/\mathrm{d}t)} = \frac{\mu_0\, c^2}{6\,\pi}\left[\frac{1}{m\, c^2}\right]\left[\frac{q^2}{mc^2}\right]\left[\frac{\mathrm{d}\mathcal{E}}{\mathrm{d}x}\right]$$

In (32), we are interested in two terms: the $1/mc^2$ term and the q^2/mc^2 term. The first gives us the reciprocal of energy and the second gives us the distance[13]. Together they tell us the change in energy with distance, specifically, the energy gain. To the factor $\mu_0\, c^2/6\pi$ we can multiply and divide 4 to get

$$\frac{2}{3}\,\frac{\mu_0\, c^2}{4\,\pi} = \frac{2}{3} \qquad (31)$$

This happens because, in CGS units, the Coulomb constant, $1/4\pi\epsilon_0$ or $\mu_0 c^2/4\pi$, goes to 1. Consequently, we will soon see that the resulting factor of $2/3$ is small enough to ignore. So we now have

$$\frac{P}{(\mathrm{d}\mathcal{E}/\mathrm{d}t)} \approx \left[\frac{1}{m\, c^2}\right]\left[\frac{q^2}{mc^2}\right]\left[\frac{\mathrm{d}\mathcal{E}}{\mathrm{d}x}\right] \qquad (32)$$

Now let us suppose that our particle under consideration is the electron. We know that the rest energy of electrons, given by $\mathcal{E} = \gamma\, m_0 c^2$, is about 0.511 MeV. This is given by the first term[14]. If the electron in our linear accelerator covers a

[13]The reason e^2/mc^2 has dimensions of length becomes clear if we work in CGS units. In this system, the elementary charge is measured in statC, which is $cm^{3/2} g^{1/2} s^{-1}$ and, more familiarly, mass is measured in gram and the speed of light in $cm\, s^{-1}$. Therefore e^2/mc^2 is going to have dimensions of $cm^3 g\, s^{-2} g^{-1} cm^{-2} s^2 = cm$, to wit, dimensions of length. We employ CGS units in electrostatics and electrodynamics because the manner in which they are defined makes a constants easier to handle: for example, see (31). Also refer to [A7] for an interesting primer on moving between SI and CGS systems.

[14]Note my misuse of the word "term" here. A term is one that is separated by either a plus or a minus sign; here I simply refer to each set of square-bracketed symbols.

distance of e^2/mc^2, as suggested by the second term, it is covering about 3×10^{13} cm. It should be apparent by now just how convenient our choice to use CGS units has made this problem.

We said above that the energy loss was dependent on $\frac{P}{(\mathrm{d}\mathcal{E}/\mathrm{d}t)}$ being roughly 1 or, in other words, $P \approx \mathrm{d}\mathcal{E}/\mathrm{d}t$. This means the entire right hand side of (32) must go to 1. That is to say, $(q^2/mc^2)/(mc^2)$ must be equal to $\mathrm{d}\mathcal{E}/\mathrm{d}x$ so that they go to 1, allowing $P \approx \mathrm{d}\mathcal{E}/\mathrm{d}t$. From the above quantities we find that the energy gain, given by $(q^2/mc^2)/(mc^2)$ in (32) is of the order of $\approx 10^{14}\,\mathrm{MeV\,m^{-1}}$.

In other words, for there to be sufficient energy loss to affect our electrons in the linear accelerator, there must be an energy gain (i.e. supplied energy) over distance (given by $\mathrm{d}\mathcal{E}/\mathrm{d}x$) of the order of $10^{14}\,\mathrm{MeV\,m^{-1}}$. This is *huge*. In today's linear accelerators, energy gains are around $10^2\,\mathrm{MeV\,m^{-1}}$. Energy loss through radiation in linear accelerators, therefore, is negligible[15].

3.2 Bremsstrahlung

The umbrella term, *Bremsstrahlung*, refers to any radiation produced during (and as a result of) the deceleration of any particle. The term itself means "breaking radiation" or "deceleration radiation".

Let us turn our attention to that form of (28) which is given in terms of β.

$$P = \frac{\mu_0\, q^2\gamma^6}{6\,\pi c}[\, \dot{\beta}^2 - (\beta \times \dot{\beta})^2 \,] \tag{28}$$

We can find two cases in this depending on whether the acceleration and velocity of the charge are parallel or not[16].

1. If $\mathbf{a} \parallel \mathbf{v}$, then $\beta \times \dot{\beta} = 0$ which quickly gives us

$$P_{\hat{a}\|\hat{v}} = \frac{\mu_0\, q^2\gamma^6}{6\,\pi c}[\, \dot{\beta}^2 \,] \tag{33}$$

2. If $\mathbf{a} \perp \mathbf{v}$, then $\beta \cdot \dot{\beta} = 0$ but we do not have a $\beta \cdot \dot{\beta}$ term so we will have to try to bring it in.

   In equation (28) above, add and subtract a $\beta^2\dot{\beta}^2$ term. We now have the bracketed term as

$$\dot{\beta}^2 - \beta^2\dot{\beta}^2 - (\beta \times \dot{\beta})^2 + \beta^2\dot{\beta}^2 = \dot{\beta}^2(1-\beta^2) - (\beta \times \dot{\beta})^2 + \beta^2\dot{\beta}^2$$

[15] Why then do we prefer circular accelerators? The trouble with linear accelerators is that we can only accelerate particles as long as there is some length of the linear accelerator left; in case of a circular accelerator we can just let them keep circulating and accelerating as long as we please

[16] Needless to say, the case that $\mathbf{a} \parallel \mathbf{v}$ is what common sense often dictates, but it need not always be so. In any uniform circular motion, for example, the velocity is tangential and the acceleration is radial, making $\mathbf{a} \perp \mathbf{v}$, and in a projectile launched at some angle $\neq 90°$, the acceleration and the velocity form an angle $0 < \pm\varphi \leq 90°$ at any given time, with $\mathbf{a} \perp \mathbf{v}$ right at the highest point in the trajectory and they vectors are never parallel at any point in time.

Using the result in [A7] that $(\beta \cdot \dot{\beta})^2 = \beta^2\dot{\beta}^2 - (\beta \times \dot{\beta})^2$ the last two terms may be consolidated.

$$\dot{\beta}^2(1-\beta^2) - (\beta \times \dot{\beta})^2 + \beta^2\dot{\beta}^2 = \dot{\beta}^2(1-\beta^2) + (\beta \cdot \dot{\beta})^2$$

Substituting this back into (28) we get

$$\begin{aligned} P &= \frac{\mu_0\, q^2\gamma^6}{6\,\pi c}\left[\,\dot{\beta}^2(1-\beta^2) + (\beta \cdot \dot{\beta})^2\,\right] \\ &= \frac{\mu_0\, q^2\gamma^4}{6\,\pi c}\left[\,\frac{\dot{\beta}^2(1-\beta^2) + (\beta \cdot \dot{\beta})^2}{1-\beta^2}\,\right] \\ \Rightarrow P &= \frac{\mu_0\, q^2\gamma^4}{6\,\pi c}\left[\,\dot{\beta}^2 + \frac{(\beta \cdot \dot{\beta})^2}{1-\beta^2}\,\right] \end{aligned}$$

We can now apply the condition that $\beta \cdot \dot{\beta} = 0$ to get

$$P_{\hat{a}\perp\hat{v}} = \frac{\mu_0\, q^2\gamma^4\dot{\beta}^2}{6\,\pi c} \tag{34}$$

It is clear from (33) and (34) that during bremsstrahlung, for the $\hat{a} \parallel \hat{v}$ case, such as in a linear accelerator, due to the γ^6 term, the energy drops as its sixth power. In the $\hat{a} \perp \hat{v}$ case, such as in a cyclotron, due to the γ^4 term, the energy loss is as its fourth power, since $\mathcal{E} = \gamma mc^2$.

This same set of relations also tell us that as the mass, m, drops, so does the radiated energy. For example, given that $\gamma = \mathcal{E}/mc^2$, we know that γ increases as the mass decreases. So, in the $\hat{a} \parallel \hat{v}$ case, the radiated power varies as m^{-6} while in the $\hat{a} \perp \hat{v}$ case it varies as m^{-4}, meaning that, as we saw in §3.1, low mass particles radiate more. Additionally, the same particle, under comparable conditions, loses more energy through bremsstrahlung in the $\hat{a} \perp \hat{v}$ (cyclotron) case than in the $\hat{a} \parallel \hat{v}$ (linear accelerator) case. This is also in agreement with our discussion in §3.1.

4 Reactive forces

4.1 The Abraham-Lorentz formula

A summary of all our ideas so far would look something like the description that follows. Any accelerated charge radiates (with reference to the acceleration term or radiation terms in **E** and **B**) and the fields generated as a result depend almost entirely on the position of the particle at some time in the past (retarded time) the farther the observer gets from the charge.

Electric dipoles in oscillation radiate power through electric and magnetic fields that are mutually perpendicular to each other and to the direction of motion. The energy radiated looks like a torus centred at the present position of any moving point charge (due to the sine dependence of $\mathbf{S}$) and its magnetic field curls around its direction of propogation. The power radiated is minimum in the direction of propogation and maximum perpendicular to it and depends on the four-momentum in relativistic scenarios. This, in turn, leads to an energy loss since the energy that is "radiated away" comes from the kinetic energy of the particle; in linear accelerators this is not much, but in circular accelerators it increases substantially. Any deceleration also causes bremsstrahlung which sees lesser radiation and hence lesser energy loss the heavier the particle is, and vice versa; additionally, the effects are more pronounced if the acceleration and velocity are perpendicular.

So far so good. However, Newton's third law beckons us from the shadows: if some radiative force exist, if some fields exist that are generated by some particle, should its own fields not exert a reactive force back on the particle?

Such a force does exist, as it turns out, and we can refer to it as $\mathbf{F}_{AL}$ for now[17]. It is known as the *radiation reaction* force.

Let us return to the Larmor formula (25) which tells us that the power radiated by a moving point charge is

$$P = \frac{\mu_0 q^2 a^2}{6\pi c} \tag{25}$$

From our discussions in §3.1 we are in a position to identify the $\mu_0/6\pi c$ term as equivalent to $2/3c^3$ in CGS units. Therefore,

$$P = \frac{2}{3}\frac{q^2 a^2}{c^3}$$

which, like everything in CGS units, is simpler to handle[18].

For convenience, let us assume that our particle is in periodic motion. (We have, somwhat symbolically, returned to a setup reminiscent of the one we used in §1.) Suppose this periodic motion completes an integral number of cycles within the time period $|t - t'|$. Then, by the definition of any reactive force, the work done by the radiation reaction (or the *Abraham-Lorentz force*) on the charged particle will be the negative of the Larmor power radiated during the same time period. After all it is this radiation of energy that the Abraham-Lorentz force is *reacting* to.

Generally, we know that power, P, and force, $\mathbf{F}$, are related as $P = \mathbf{F} \cdot \mathbf{v}$ for velocity $\mathbf{v}$ of the object in question. And the work done on an object and its power are related as $\int \mathrm{d}t\, P$, where the integration is over time, t. Staying in agreement with the law of conservation of energy, we must then be able to write, for our

[17]I am hesitant to use the subscript $_r$ since we have already used it to refer to the radiation field terms in §2.1. Instead we shall prefer $_{AL}$ in reference to Abraham and Lorentz.

[18]It is important to remember, at this point, that by invoking the convenience of CGS units we have also implicitly re-defined certain other terms, for example our charge, q, which is now in esu.

previously defined radiation reaction force,

$$\mathbf{F}_{AL} \cdot \mathbf{v} = -P$$

with the minus indicating the fact that the above term is in reaction to the Larmor power. Consequently, if we carry forward the assumption that the power and force may be equated this way[19], then the work done is

$$\begin{aligned}
\int_t^{t'} \mathrm{d}t\, \mathbf{F}_{AL} \cdot \mathbf{v} &= -\int_t^{t'} \mathrm{d}t\, P \\
&= -\int_t^{t'} \mathrm{d}t\, \frac{2}{3} \frac{q^2 a^2}{c^3} \\
&= -\frac{2}{3} \frac{q^2}{c^3} \int_t^{t'} \mathrm{d}t\, \left(\frac{\mathrm{d}\mathbf{v}}{\mathrm{d}t} \cdot \frac{\mathrm{d}\mathbf{v}}{\mathrm{d}t} \right) \\
\text{(Integrating by parts)} \qquad &= -\frac{2}{3} \frac{q^2}{c^3} \left\{ \left(\mathbf{v} \cdot \frac{\mathrm{d}\mathbf{v}}{\mathrm{d}t} \right) \Bigg|_t^{t'} - \int_t^{t'} \mathrm{d}t\, \frac{\mathrm{d}^2\mathbf{v}}{\mathrm{d}t^2} \cdot \mathbf{v} \right\}
\end{aligned}$$

Since the velocity (and by extension the acceleration) is alike on the extreme ends of the periodic motion, the first term in the braces goes to zero. We are then left with just the second term involving the second time derivative of velocity, the *jerk*[20].

$$\int_t^{t'} \mathrm{d}t\, \mathbf{F}_{AL} \cdot \mathbf{v} = \frac{2}{3} \frac{q^2}{c^3} \int_t^{t'} \mathrm{d}t\, (\dot{\mathbf{a}} \cdot \mathbf{v})$$

By simple comparison we see that

$$\boxed{\mathbf{F}_{AL} = \frac{2}{3} \frac{q^2}{c^3} \cdot \dot{\mathbf{a}}} \tag{35}$$

This is known as the *Abraham-Lorentz formula* for the reactive force acting on an accelerating charged particle. It is named after the German physicist, Max Abraham, and his Dutch counterpart, Hendrik Antoon Lorentz, who worked independently at the start of the 1900s.

4.2 Observations on the Abraham-Lorentz force

Admittedly, §4.1 gives a less-than-satsifactory way of arriving at the Abraham-Lorentz formula (35) mostly due to the many assumptions we make to arrive at it. While it is not incorrect, this formula is only fairly accurate in non-relativistic conditions and does not hold at all as $\mathbf{v} \to c$. Some observations on (35) follow.

[19] As we will see in §4.2, this is incorrect but is within acceptable margins of error.

[20] Successive time derivatives of position are called velocity, acceleration and jerk.

1. In finding P we used only the acceleration fields. However, given that $\mathbf{F}_{AL}$ reacts to all energy radiated away due to the generated fields, it is important to also account for the energy loss due to the velocity terms of (18) and (19), especially at near field zones.

2. Since (35) relies on $\dot{\mathbf{a}}$ rather than just $\mathbf{a}$, it means that $\mathbf{F}_{AL}$ can wrongly go to zero even though a charge is radiating. Think of a particle in constant acceleration: due it its acceleration, it continues to radiate and we expect there to exist a radiation reaction force, but (35) tells us that, due to $\dot{\mathbf{a}} = 0$ for a constant $\mathbf{a}$, there is no $\mathbf{F}_{AL}$ at all, which is clearly untrue.

3. We have considered periodic motion; we should not have to. By extension, our assumption of periodicity negates the effect of velocities exactly at t and t', which need not be true for arbitrary motion.

4. Under specific circumstances (such as when $\mathbf{F}_{AL}$ is assumed to be the only force acting on a moving charged particle) the Abraham-Lorentz formula insists on breaking causality.

As many troubles as there may be with (35), there is no denying that it does represent radiation field reaction consicely and under considerable, but not crippling, restrictions. Besides it agrees fully with the law of conservation of energy.

Dirac generalised this formula about thirty-five years later, with the help of Liénard's generalisation of Larmor's formula (28), and gave a more realistic equation to explain radiation reaction taking both arbitrary motion and relativistic motion into account. This is known as the Abraham-Lorentz-Dirac formula but still does not escape causality breaking (called pre-acceleration) where can be shown to react by accelerating even *before* an external force is applied. In fact, such a behaviour of pre-acceleration has not been explained to this day.

. . .

Appendices

A Some useful results

1. **The binomial expansion**

$$(a+b)^n = a^n + na^{n-1}b + \frac{n(n-1)}{2}a^{n-2}b^2 + \cdots$$

which gives us two particularly interesting approximations if we are able to manipulate our argument into the the form $(1+x)$.

For $n = -1$,

$$(1+x)^{-1} = 1 - x$$

and for $n = 1/2$,

$$(1+x)^{\frac{1}{2}} = 1 + \frac{x}{2}$$

and the higher order terms can be ignored for all practical purposes.

2. **Gradient in spherical polar coördinates**

$$\nabla X = \frac{\partial X}{\partial r}\,\hat{\mathbf{r}} + \frac{1}{r}\frac{\partial X}{\partial \theta}\,\hat{\theta} + \frac{1}{r\sin\theta}\frac{\partial X}{\partial \phi}\,\hat{\phi}$$

3. **Curl in spherical polar coördinates**

$$\nabla \times \mathbf{X} = \frac{1}{r\sin\theta}\left(\frac{\partial}{\partial\theta}(\sin\theta X_\phi) - \frac{\partial X_\phi}{\partial\phi}\right)\hat{\mathbf{r}} + \frac{1}{r}\left(\frac{1}{\sin\theta}\frac{\partial X_r}{\partial\phi} - \frac{\partial}{\partial r}(rX_\phi)\right)\hat{\theta}$$
$$+ \frac{1}{r}\left(\frac{\partial}{\partial r}(rX_\theta) - \frac{\partial X_r}{\partial\theta}\right)\hat{\phi}$$

4. **Converting Cartesian unit vectors to spherical polar unit vectors**

$$\hat{\mathbf{x}} = \sin\theta\cos\phi\,\hat{\mathbf{r}} + \cos\theta\cos\phi\,\hat{\phi} - \sin\phi\,\hat{\phi}$$
$$\hat{\mathbf{y}} = \sin\theta\sin\phi\,\hat{\mathbf{r}} + \cos\theta\sin\phi\,\hat{\phi} + \cos\phi\,\hat{\phi}$$
$$\hat{\mathbf{z}} = \cos\theta\,\hat{\mathbf{r}} - \sin\theta\,\hat{\theta}$$

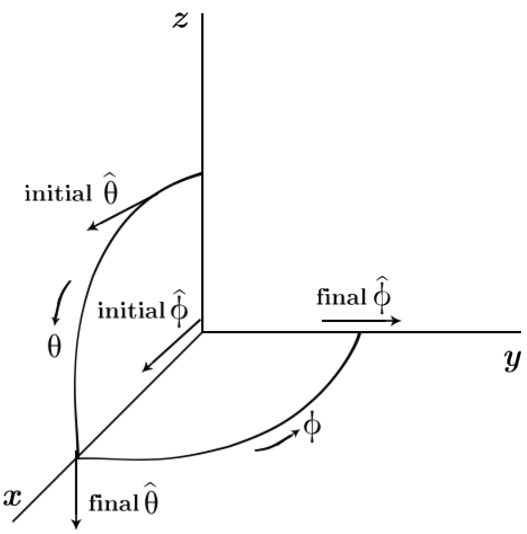
It is actually fairly easy to see how we can arrive at these results. We know the standard relationships between Spherical Polar and Cartesian coördinates:

$x = r\sin\theta\cos\phi$ $\qquad$ $y = r\sin\theta\sin\phi$ $\qquad$ $z = r\cos\theta$

As a result, the unit vector $\hat{\mathbf{r}}$ becomes

$$\frac{\mathbf{r}}{r} = \frac{r\sin\theta\cos\phi\hat{\mathbf{x}} + r\sin\theta\sin\phi\hat{\mathbf{y}} + r\cos\theta\hat{\mathbf{z}}}{r} = \sin\theta\cos\phi\hat{\mathbf{x}} + \sin\theta\sin\phi\hat{\mathbf{y}} + \cos\theta\hat{\mathbf{z}}$$

For $\hat{\theta}$ we can take a more verbal approach. We knoe that the unit vector, $\hat{\theta}$, is the tangent to the arc formed by the angle, θ, rising from the z-axis. Therefore, when θ is zero degrees (with respect to the z-axis), $\hat{\theta}$ can point in the positive $\hat{\mathbf{x}}$ direction, and when θ is a full 90°, the same $\hat{\theta}$ points along the negative z-axis, and has no component on the x-axis. This variation of having a maximum x-component at 0° and zero x-component at 90° is the nature of a cosine function, which means $\hat{\theta}$ varies as $\cos\theta\,\hat{\mathbf{x}}$. Similarly, for ϕ lying on the x-y plane, the unit vector $\hat{\theta}$ initially points along the x-axis for $\phi = 0$ and has no component along the x-axis as $\phi = 90°$. Once again, this is the nature of the $\cos\theta\,\hat{\mathbf{x}}$ function. Therefore, the x-component of $\hat{\theta}$ must be $\cos\theta\cos\phi\,\hat{\mathbf{x}}$.



The above figure probably explains it better than words alone. It shows, diagramatically, our reasoning for the $\hat{\mathbf{x}}$ component of $\hat{\theta}$. Likewise, for the $\hat{\mathbf{y}}$ and $\hat{\mathbf{z}}$ components of $\hat{\theta}$ and then for $\hat{\phi}$, we get two more equations:

$$\hat{\theta} = \cos\theta\cos\phi\,\hat{\mathbf{x}} + \cos\theta\sin\phi\,\hat{\mathbf{y}} - \sin\theta\,\hat{\mathbf{z}} \qquad \hat{\phi} = -\sin\phi\,\hat{\mathbf{x}} + \cos\phi\,\hat{\mathbf{y}}$$

Note that the next bit of this explanation is the answer to exercise 2.5.5 in [B5], so you may want to try it yourself first before going through the solution below.

These three equations may be written in matrix form as

$$\begin{pmatrix} \hat{\mathbf{x}} & \hat{\mathbf{y}} & \hat{\mathbf{z}} \end{pmatrix} \begin{pmatrix} \sin\theta\cos\phi & \cos\theta\cos\phi & -\sin\phi \\ \sin\theta\sin\phi & \cos\theta\sin\phi & \cos\phi \\ \cos\theta & -\sin\theta & 0 \end{pmatrix} = \begin{pmatrix} \hat{r} & \hat{\theta} & \hat{\phi} \end{pmatrix}$$

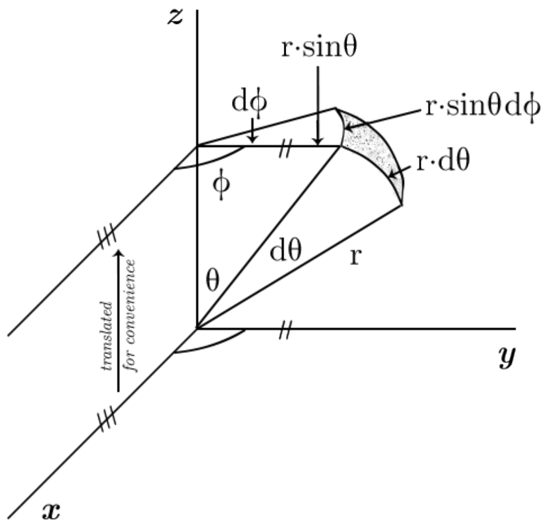
which is in the form $\hat{\mathbf{X}}\mathbf{M} = \hat{\mathbf{P}}$. In terms of the Cartesian unit vectors then, we need $\hat{\mathbf{X}} = \mathbf{M}^{-1}\hat{\mathbf{P}}$, for which we need to find the inverse of the 3×3 matrix in the above equation. This turns out to have a determinant of 1 and its inverse is simply its adjoint, which is

$$\begin{pmatrix} \sin\theta\cos\phi & -\sin\theta\sin\phi & \cos\theta \\ \cos\theta\cos\phi & \cos\theta\sin\phi & -\sin\theta \\ -\sin\phi & \cos\phi & 0 \end{pmatrix}$$

This, when multiplied from the left by $(\hat{r} \quad \hat{\theta} \quad \hat{\phi})$ can give us Polar translations for all three Cartesian unit vectors, but I will only note down the $\hat{\mathbf{z}}$ component below in the interest of this lecture (since we use this result in §1.2).

$$\hat{\mathbf{z}} = \cos\theta\,\hat{r} - \sin\theta\,\hat{\theta}$$

5. **Area of a spherical surface in spherical polar coördinates**
This reasoning only works for infinitesimal surfaces thanks to our use of the forumla $S = r\theta$ to measure the lenght of an arc, S, of a circle of radius r formed by a sweeping angle, θ. However, we work so often with infinitesimal measures while integrating that having results like this is quite helpful.



The geometry involved is pretty self-explanatory from the figure above and needs no further verbal description. The shaded region, although part of a sphere, may, since it is infinitesimal, be treated as a flat rectangle, thereby allowing us to use the its length times its breadth to compute its area.

$$\mathrm{d}\mathbf{A} = r \cdot \mathrm{d}\theta \cdot r \cdot \sin\theta\,\mathrm{d}\phi \Rightarrow \mathrm{d}\mathbf{A} = r^2 \sin\theta\,\mathrm{d}\theta\,\mathrm{d}\phi$$

6. **Solving the surface integral** $\oint \sin^3\theta\,\mathrm{d}\theta\,\mathrm{d}\phi$
Consider that θ varies from 0 to π and ϕ varies from 0 to 2π helping us trace half a sphere. (The same can be mirrored to trace the other half, i.e. with θ going from $-\pi$ to 0.) In the interests of §1.3 and §2.1, say we have some integral of the form

$$\oint \frac{\sin^2\theta}{\mathscr{r}^2} \cdot \mathrm{d}\mathbf{A}$$

which may be written using the results of [A5] as

$$\int \frac{\sin^2\theta}{\mathscr{r}^2}\, \mathscr{r}^2 \sin\theta \,\mathrm{d}\theta\,\mathrm{d}\phi$$

or even more explicitly as the double integral that it is.

$$\int_0^{2\pi} \mathrm{d}\phi \int_0^{\pi} \mathrm{d}\theta\ \sin^3\theta$$

Now we can simply apply the reduction formula

$$\int \sin^m x\,\mathrm{d}x = \left(-\frac{\cos x \sin^{m-1} x}{m}\right) + \frac{m-1}{m}\int \sin^{m-2} x\,\mathrm{d}x$$

with $m = 3$ to get

$$\int_0^{2\pi} \mathrm{d}\phi \left[\left(-\frac{1}{3}\sin^2\theta\cos\theta\right) + \frac{2}{3}\int_0^{\pi} \mathrm{d}\theta\ \sin\theta\right]$$

This is now much easier to solve.

$$\begin{aligned}
\oint \frac{\sin^2\theta}{\mathscr{r}^2}\cdot \mathrm{d}\mathbf{A} &= \int_0^{2\pi} \mathrm{d}\phi \left[\left(-\frac{1}{3}\sin^2\theta\cos\theta\right) - \frac{2}{3}\cos\theta\right]_0^{\pi} \\
&= -\frac{1}{3}\int_0^{2\pi} \mathrm{d}\phi\ \left[\sin^2\pi\cos\pi + 2\cos\pi - \sin^2 0\cos 0 - 2\cos 0\right] \\
&= -\frac{1}{3}\int_0^{2\pi} \mathrm{d}\phi\ (-4) \\
&= \frac{4}{3}\,\phi\,\Big|_0^{2\pi} \\
\Rightarrow \oint \frac{\sin^2\theta}{\mathscr{r}^2}\cdot \mathrm{d}\mathbf{A} &= \frac{8\pi}{3}
\end{aligned}$$

7. **A result on the dot product**
For two vectors, $\mathbf{X}$ and $\mathbf{Y}$, with an angle θ between them, their dot and cross products, respectively, are,

$$\mathbf{X}\cdot\mathbf{Y} = XY\cos\theta \qquad \text{and} \qquad \mathbf{X}\times\mathbf{Y} = XY\sin\theta$$

As a result,

$$(\mathbf{X}\cdot\mathbf{Y})^2 = X^2Y^2\cos^2\theta = X^2Y^2(1-\sin^2\theta) = X^2Y^2 - (\mathbf{X}\times\mathbf{Y})^2$$

8. **Finding $\partial t/\partial t_r$**

This is something we have already discussed in the previous lecture[21] nonetheless I will reproduce it below for convenience. We begin the the definition of retarded time, t_r, as $t_r = t - \mathscr{r}/c$ and go from there.

$$\begin{aligned}
\mathscr{r} &= c\,(t - t_r) \\
\frac{\partial \mathscr{r}}{\partial t} &= c\,(1 - \frac{\partial t_r}{\partial t}) \\
\Rightarrow c - c\,\frac{\partial t_r}{\partial t} &= \frac{\partial}{\partial t}(\sqrt{\boldsymbol{\mathscr{r}} \cdot \boldsymbol{\mathscr{r}}}) \\
-c\,\frac{\partial t_r}{\partial t} &= \frac{1}{2\mathscr{r}}\frac{\partial}{\partial t}(\boldsymbol{\mathscr{r}} \cdot \boldsymbol{\mathscr{r}}) - c \\
\frac{\partial t_r}{\partial t} &= \frac{1}{2\mathscr{r}c}\frac{\partial}{\partial t}(r^2 + r'^2 - 2\mathbf{r}\cdot\mathbf{r}') - 1 \\
&= \frac{1}{2\mathscr{r}c}\,(0 + 2r'\partial_{t_r}r'\,\partial_t t_r - 2\mathbf{r}\cdot\partial_{t_r}\mathbf{r}'\partial_t t_r - 0) - 1 \\
&= \frac{(r'\partial_{t_r}r' - \mathbf{r}\cdot\partial_{t_r}\mathbf{r}')\;\partial_t t_r - \mathscr{r}c}{\mathscr{r}c}
\end{aligned}$$

$$\Rightarrow \frac{\partial t_r}{\partial t}\ \text{(or simply } \partial_t t_r) = \frac{-\mathscr{r}c}{\mathscr{r}c - r'\partial_{t_r}r' - \mathbf{r}\cdot\partial_{t_r}\mathbf{r}'} = \frac{\mathscr{r}c}{\mathscr{r}c - (r'\partial_{t_r}r' - r\hat{r}\cdot\partial_{t_r}r'\hat{r}')}$$

$$\therefore \partial_t t_r = \frac{\mathscr{r}c}{\mathscr{r}c - (r' - r)\,\partial_{t_r}r'} = \frac{\mathscr{r}c}{\mathscr{r}c - \boldsymbol{\mathscr{r}}\cdot\mathbf{v}} \tag{36}$$

We have used $\mathbf{w}$ to denote the denominator as $\mathscr{r}c - \boldsymbol{\mathscr{r}}\cdot\mathbf{v} = \hat{\boldsymbol{\mathscr{r}}}\cdot\mathbf{w}$ and hence

$$\frac{\partial t_r}{\partial t} = \frac{\mathscr{r}c}{\hat{\boldsymbol{\mathscr{r}}}\cdot\mathbf{w}}$$

or,

$$\frac{\partial t}{\partial t_r} = \frac{\hat{\boldsymbol{\mathscr{r}}}\cdot\mathbf{w}}{\mathscr{r}c}$$

9. **The Minkowski norm**

Here is an interesting result that finds use in §2.2. For the four-momentum, given by $p \equiv (\frac{\mathcal{E}}{c}, p_x, p_y, p_z)$, we write its Minkowski norm as

$$p \cdot p = p^\mu p_\mu$$

We can use the metric tensor, $g^{\mu\nu}$, defined as

$$g^{\mu\nu}\begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

[21] See "Fields due to moving charges", available at `http://vhbelvadi.com/notes/` to download.

to make $p^\mu \to p_\mu$ according to the rule $A^\mu = g^{\mu\nu} A_\mu$ giving us

$$p^\mu p_\mu = g^{\mu\nu} p_\mu p_\mu = -\frac{\mathcal{E}^2}{c^2} + |\mathbf{p}|^2$$

By extension, therefore, and using $\beta = v/c$

$$-\frac{\mathrm{d}p_\mu}{\mathrm{d}\tau}\frac{\mathrm{d}p^\mu}{\mathrm{d}\tau} = -\frac{1}{c^2}\left(\frac{\mathrm{d}\mathcal{E}}{\mathrm{d}\tau}\right)^2 + \left(\frac{\mathrm{d}\mathbf{p}}{\mathrm{d}\tau}\right)^2 = (1-\beta^2)\left(\frac{\mathrm{d}\mathbf{p}}{\mathrm{d}\tau}\right)^2$$

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