Lecture notes

Fields due to moving charges

V.H. Belvadi

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These notes are based on a series of lectures on classical electrodynamics given to graduate physics students at YCM, University of Mysore, during the academic year 2016–2017. This document represents only a part of the entire course which covers, among other things, classical electrodynamics, plasma physics and optics.

Please e-mail `hello@vhbelvadi.com` with your thoughts or suggestions, or if you spot any errors. These notes are—and will probably always remain—a work in progress. They may be updated and, if they are, the latest version of this document will always be available for download at `http://vhbelvadi.com/notes/` for anybody interested in it.

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Changelog:

v.1.0	Initial.
v.2.0	Added a new review section in the beginning.
v.3.0	Added some proofs inline.
v.4.0	Added new appendices and moved proofs there.
v.5.0	Corrected errors in §2.1 and §4.
v.6.0	Corrected 'Lorenz' gauge, wrongly called 'Lorentz'.

P.S. LaTeX needs much better line breaking capabilities for equations. If packages better than breqn and the split and multiline environments built into amsmath exist, or if you have written one yourself, please send me an e-mail. Thank you.

While these lecture notes are not self-contained, considerable effort has been put into explaining everything in a manner satisfying enough that looking for assistive sources will be driven more by awakened interest than by necessity. Since the scope of this document is only part of the entire course on classical electrodynamics, it is understandable that some knowledge would be required of the preceding topics to understand what is contained here. Lastly, I found that some students in class seemed to have a rather shaky foundation in these preceding topics.

To this end, below is a review of some important ideas that will be useful to know before we begin the main part of our discussion. A new appendix has also been added at the end to provide proofs for certain results used in these notes (particularly §1.2), and to explain foundational and accompanying ideas so that the structure of these notes is not deviated from. Also, note that ∇ is not emboldened to denote $\vec{\nabla}$. I trust readers will be able to decipher this just fine anyway.

0 Review

1. Maxwell's equations tell us how electric charges generate electric fields and electric currents generate magnetic fields. There are four equations:

$$\nabla \cdot \mathbf{E} = \frac{\rho}{\epsilon_0} \qquad \text{(Ref. 1)} \qquad\qquad \nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} \qquad \text{(Ref. 3)}$$

$$\nabla \cdot \mathbf{B} = 0 \qquad \text{(Ref. 2)} \qquad\qquad \nabla \times \mathbf{B} = \mu_0 \mathbf{J} + \mu_0 \epsilon_0 \frac{\partial \mathbf{E}}{\partial t} \qquad \text{(Ref. 4)}$$

2. We can use the fact that

$$\mathbf{B} = \nabla \times \mathbf{A} \qquad \text{(Ref. 5)}$$

in (Ref. 3) to write $\nabla \times \left(\mathbf{E} + \frac{\partial \mathbf{A}}{\partial t}\right) = 0$, which, being the curl of a vector and going to zero, means we can write this as the gradient of some scalar potential, say, V. Therefore,

$$\mathbf{E} = -\nabla V - \frac{\partial \mathbf{A}}{\partial t} \qquad \text{(Ref. 6)}$$

3. Note that (Ref. 5) and (Ref. 6) satisfy (Ref. 2) and (Ref. 3) respectively. We can extend this to (Ref. 1) and (Ref. 4) as well, to cover all four of Maxwell's equations, by substituting (Ref. 6) in (Ref. 1) as,

$$\nabla^2 V + \frac{\partial}{\partial t}(\nabla \cdot \mathbf{A}) = -\frac{\rho}{\epsilon_0} \qquad \text{(Ref. 7)}$$

and substituting (Ref. 5) and (Ref. 6) in (Ref. 4) we get,

$$\left(\nabla^2 \mathbf{A} - \mu_0 \epsilon_0 \frac{\partial^2 \mathbf{A}}{\partial t^2}\right) - \nabla\left(\nabla \cdot \mathbf{A} + \mu_0 \epsilon_0 \frac{\partial V}{\partial t}\right) = -\mu_0 \mathbf{J} \qquad \text{(Ref. 8)}$$

4. Our next topic of interest are gauge transformations, particularly the Coulomb and Lorenz gauges. The term *gauge freedom* refers to the fact that we can add any constant potential to an existing potential (since it is often the potential difference and not the potential itself that matters). And such a transformation wherein several different potentials thus achieved all lead to the same field is called a *gauge transformation*, to wit: the altered or transformed potentials are such that they leave the field itself invariant.

   Coulomb gauge takes

   $$\nabla \cdot \mathbf{A} = 0 \qquad \text{(Ref. 9)}$$

   and *Lorenz gauge* takes

   $$\nabla \cdot \mathbf{A} = -\mu_0 \epsilon_0 \frac{\partial V}{\partial t} \qquad \text{(Ref. 10)}$$

5. The Lorenz gauge applied to (Ref. 8) leads us to the *d'Alembertian operator*, represented by,

   $$\Box^2 = \nabla^2 - \mu_0 \epsilon_0 \frac{\partial^2}{\partial t^2} \qquad \text{(Ref. 11)}$$

   Thereby,

   $$\Box^2 V = -\frac{\rho}{\epsilon} \qquad \text{(Ref. 12)}$$
   $$\Box^2 \mathbf{A} = -\mu_0 \mathbf{J} \qquad \text{(Ref. 12)}$$

   The beauty of the Lorenz gauge lies in its treatment of both V and $\mathbf{A}$ using the same (d'Alembertian) operator. The equations (Ref. 12) are together called the *inhomogeneous wave equations.*

1 Retarded time and retarded potential

Information about a charge, manifested as the potential field arising because of it, travels at a finite speed. In other words, if a charge is at some point $\mathbf{r}'$, called the *source point*, and the observer is at some $\mathbf{r}$, called the *field point*, and the charge moves some distance over time t, the field point receives information about the charge not as it is *now*, but rather as it was when it was at $\mathbf{r}'$, at some time t_r (say), due to the finite velocity of information travel.

The time, t_r, tells us when the information *left the charge.* If the distance from the source point to the field point is $|\mathbf{r}' - \mathbf{r}|$, then the delay in receiving information is $|\mathbf{r}' - \mathbf{r}|/c$, making the total time taken,

$$t = t_r + \frac{|\mathbf{r}' - \mathbf{r}|}{c}$$

or, the *retarded time*,

$$t_r = t - \frac{|\mathbf{r}' - \mathbf{r}|}{c}$$

That is, the information was "sent" at some retarded time, t_r, ago, which is some time $|\mathbf{r}' - \mathbf{r}|/c$ earlier than the time, t, at which it was received. These are generally called *retardation effects* and require, therefore, that we consider, explicity, time dependence in potentials A and $\mathbf{V}$.

1.1 Relativistic effects

Suppose that the charge is currently inside a volume element $d\tau$ within which it gets displaced to $\mathbf{r}'$. Then the potentials $V(\mathbf{r})$ and $\mathbf{A}(\mathbf{r})$ which we know to be

$$V(\mathbf{r}) = \frac{1}{4\pi\epsilon_0}\int_\tau \frac{\rho(\mathbf{r}')\mathrm{d}\tau}{|\mathbf{r}' - \mathbf{r}|}, \text{ and}$$

$$\mathbf{A}(\mathbf{r}) = \frac{\mu_0}{4\pi}\int_\tau \frac{\mathbf{J}(\mathbf{r}')\mathrm{d}\tau}{|\mathbf{r}' - \mathbf{r}|}$$

now become functions of the retarded time, t_r, as well. Therefore,

$$V(\mathbf{r}, t) = \frac{1}{4\pi\epsilon_0}\int_\tau \frac{\rho(\mathbf{r}', t_r)\mathrm{d}\tau}{|\mathbf{r}' - \mathbf{r}|}, \text{ and} \tag{1}$$

$$\mathbf{A}(\mathbf{r}, t) = \frac{\mu_0}{4\pi}\int_\tau \frac{\mathbf{J}(\mathbf{r}', t_r)\mathrm{d}\tau}{|\mathbf{r}' - \mathbf{r}|} \tag{2}$$

These are called *retarded potentials* and give us the values of the potentials, $V(\mathbf{r})$ and $\mathbf{A}(\mathbf{r})$, at the field point, $\mathbf{r}(t)$, due to the (earlier position of the) charge at $\mathbf{r}'(t_r)$.

Since we define the charge density, ρ, as $\rho = \frac{e}{\tau}$ and $\mathbf{J} = ne\mathbf{v}_d$ for the same volume element as mentioned above (and for drift velocity $\mathbf{v}_d$ of the electrons), we can re-write equations (1) and (2) as[1],

$$V(\mathbf{r}, t) = \frac{e}{4\pi\epsilon_0|\mathbf{r}' - \mathbf{r}|}, \text{ and} \tag{3}$$

$$\mathbf{A}(\mathbf{r}, t) = \frac{\mu_0 e\mathbf{v}_d}{4\pi|\mathbf{r}' - \mathbf{r}|} \tag{4}$$

We can also show relativistic effects with a simple mathematical trick: multiply and divide equation (4) by ϵ_0 and use $c = 1/\sqrt{\mu_0\epsilon_0}$ to achieve a $\beta = \mathbf{v}/c$ term leaving us with an equation that tells us how $\mathbf{A}(\mathbf{r}, t)$ varies relativistically.

$$\mathbf{A}(\mathbf{r}, t) = \frac{e}{4\pi\epsilon_0 c}\frac{\vec{\beta}}{|\mathbf{r}' - \mathbf{r}|} \tag{5}$$

[1]We use $\mathbf{J} = \frac{I}{\tau} = \frac{ne\mathbf{v}_d\tau}{\tau}$. So, when integrating, we are left with a $\mathbf{J}\tau = \frac{I\tau}{\tau} = e\mathbf{v}_d$ term, ignoring the number density, n.

1.2 Spatial derivatives

It is time to introduce, for simplicity, the definition $|\mathbf{r}' - \mathbf{r}| = \mathscr{r}$, incorporating the cursive script[2] $\mathscr{r}$ made famous by Griffiths (see appendix C2), to help us define certain terms more elegantly.

We know that $\nabla := \hat{\mathbf{x}}_i \frac{\partial}{\partial x_i}$ and $\mathbf{r} = x_i \hat{\mathbf{x}}_i$. This gives us a few useful results as outlined below. Note that while the results are shown for $\boldsymbol{\mathscr{r}}$ they apply to $\mathbf{r}$ just the same. Please consult *Appendix A* for proofs.

1. $\nabla \mathscr{r} = \hat{\boldsymbol{\mathscr{r}}}$ and $\nabla r = \hat{\mathbf{r}}$
2. $\nabla(1/\mathscr{r}) = -\hat{\boldsymbol{\mathscr{r}}}/\mathscr{r}^2$ and for $1/r$
3. $\nabla \cdot \boldsymbol{\mathscr{r}} = \hat{\nabla} \cdot \mathbf{r} = 3$
4. $\nabla \cdot \left(\frac{\hat{\boldsymbol{\mathscr{r}}}}{\mathscr{r}^2}\right) = 4\pi\delta^3\boldsymbol{\mathscr{r}}$ and for $\left(\frac{\hat{\mathbf{r}}}{r^2}\right)$
5. $\nabla \cdot \left(\frac{\hat{\boldsymbol{\mathscr{r}}}}{\mathscr{r}}\right) = \frac{1}{\mathscr{r}^2}$ and for $\left(\frac{\hat{\mathbf{r}}}{r}\right)$
6. $\nabla^2 \left(\frac{1}{\mathscr{r}}\right) = -4\pi\delta^3\boldsymbol{\mathscr{r}}$ and for $\left(\frac{1}{r}\right)$

1.3 Proof of time dependence

We now have to prove our assumptions[3] in equations (1) and (2) that ρ and $\mathbf{J}$ are functions of t_r in addition to $\mathbf{r}$. We prove this by showing their agreement with the inhomogeneous wave equation (Ref. 12).

Consider equation (1) with the $\rho(\mathbf{r})$ shown explicitly, i.e. ρ depends on $\mathbf{r}$ in the term $\boldsymbol{\mathscr{r}}$ and the term $t_r = t - \frac{\boldsymbol{\mathscr{r}}}{c}$, so we split ρ to show these two terms explicitly and use spatial derivatives 1 and 2 as described in §1.2.

Note that to define ρ as a function of both r and t, we may put $\nabla\rho = \frac{\partial \rho}{\partial r}$ which gives us $\frac{\partial \rho}{\partial t_r}\frac{\partial t_r}{\partial r} = \dot{\rho}\nabla t_r$ which, from the definition of t_r above, is $-\frac{1}{c}\dot{\rho}\nabla\mathscr{r}$.

$$\nabla V = \frac{1}{4\pi\epsilon_0}\int\limits_\tau \left[\frac{\nabla\rho}{\mathscr{r}} + \rho\nabla\frac{1}{\mathscr{r}}\right] d\tau = \frac{1}{4\pi\epsilon_0}\int\limits_\tau \left[\frac{-\dot{\rho}}{c}\frac{\hat{\boldsymbol{\mathscr{r}}}}{\mathscr{r}} - \rho\frac{\hat{\boldsymbol{\mathscr{r}}}}{\mathscr{r}^2}\right] \mathrm{d}\tau$$

We now consider the divergence of the ∇V term:

$$\nabla^2 V = \frac{1}{4\pi\epsilon_0}\int\limits_\tau \left\{\frac{-1}{c}\left[\frac{\hat{\boldsymbol{\mathscr{r}}}}{\mathscr{r}} \cdot (\nabla\dot{\rho}) + \dot{\rho}\nabla \cdot \left(\frac{\hat{\boldsymbol{\mathscr{r}}}}{\mathscr{r}}\right)\right] - \left[\frac{\hat{\boldsymbol{\mathscr{r}}}}{\mathscr{r}^2} \cdot (\nabla\rho) + \rho(\nabla) \cdot \left(\frac{\hat{\boldsymbol{\mathscr{r}}}}{\mathscr{r}^2}\right)\right]\right\} \mathrm{d}\tau$$

and use $\nabla\dot{\rho} = -\frac{1}{c}\ddot{\rho}\nabla\mathscr{r} = -\frac{1}{c}\ddot{\rho}\hat{\boldsymbol{\mathscr{r}}}$ along with spatial derivatives 4 and 5 to arrive at,

$$\nabla^2 V = \frac{1}{4\pi\epsilon_0}\int\limits_\tau \left[\frac{1}{c}\frac{\ddot{\rho}}{\mathscr{r}} - 4\pi\rho\delta^3(\boldsymbol{\mathscr{r}})\right] \mathrm{d}\tau$$

[2]Find it on Griffith's website: `http://www.reed.edu/physics/faculty/griffiths.html`

[3]For a discussion that arrives at these results independently, without first assuming V and $\mathbf{J}$ as functions of t_r, see the notes on *Retarded potentials and fields and radiation by charged particles* by Bo E. Sernelius: `https://people.ifm.liu.se/boser/elma/Lect15.pdf`

and then split the integral and substitute as necessary[4] to arrive at,

$$\begin{aligned}
\nabla^2 V &= \frac{1}{c^2}\frac{\partial^2 V}{\partial t^2} - \frac{1}{\epsilon_0}\rho(\mathbf{r},t) \\
\text{or,}\quad \left(\nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right) V &= -\frac{\rho}{\epsilon_0} \\
\Rightarrow \Box^2 V &= -\frac{\rho}{\epsilon_0} \qquad (6)
\end{aligned}$$

which clearly resembles the inhomogeneous wave equation. This can similarly be proved for the retarded vector potential in equation (2) as well.

2 Liénard-Wiechert potentials

Alfred-Marie Liénard and Emil Wiechert independently developed what is now named after them as the Liénard-Wiechert potentials[5] that describe the electromagnetic field due to an arbitrarily moving point charge. They are relativistically correct and time-varying (agreeing with §1) and of an extremely general form[6].

2.1 Derivation

Recall the retarded potentials due to a moving charge. For convenience, here they are again:

$$V(\mathbf{r},t) = \frac{1}{4\pi\epsilon_0}\int\limits_\tau \frac{\rho(\mathbf{r}', t_r)\mathrm{d}\tau}{\mathscr{r}}, \text{ and} \qquad (1)$$

$$\mathbf{A}(\mathbf{r},t) = \frac{\mu_0}{4\pi}\int\limits_\tau \frac{\mathbf{J}(\mathbf{r}', t_r)\mathrm{d}\tau}{\mathscr{r}} \qquad (2)$$

Let us slightly redefine the source point from $\mathbf{r}'$ to some $\mathbf{r}'(t')$. We will enocunter this soon. For now all we need to realise is that having $\mathbf{r}'(t')$ implies that the position of our source charge is time-dependent, i.e. the charge is in motion, while not being constrained to any particular type of motion, since all we know is that its position changes with time. That is to say, we have an arbitrarily moving charge that forces us to re-define our separation vector as,

$$\mathscr{r} = |\mathbf{r} - \mathbf{r}'(t')|$$

[4] Use $c = 1/\sqrt{\mu_0\epsilon_0}$ and the definition of V from equation (1) and transpose equation (6) as needed to look like (Ref. 12).

[5] This is pronounced like `lee-yen-(h)ard Veek-(h)urt` with a silent (h).

[6] The L-W potentials do not require any assumptions regarding the motion of the point charge, such as that it is either relativistic or non-relativistic; they do not require a specific (minimum or maximum) distance of the observer; and they do not place any constraints on how the charge must move (indeed it can be completely arbitrary). However, they do not account for quantum mechanical effects. This is unsurprising, since the work was done shortly before the 20^{th} century.

For a charge q, we can then also re-define the charge and current densities using the Dirac delta function:

$$\rho(\mathbf{r},t) = q\delta^3(\mathbf{r}-\mathbf{r}'(t')) \tag{7}$$

and, similarly[7],

$$\mathbf{J}(\mathbf{r},t) = q\frac{\partial \mathbf{r}'(t')}{\partial t}\delta^3(\boldsymbol{\mathscr{r}}) = q\mathbf{v}(t')\delta^3(\mathbf{r}-\mathbf{r}'(t'))$$

These are easily verified: $\rho = q$ where q lies and $\rho = 0$ elsewhere; likewise, $\mathbf{J} = q\mathbf{v}$ where q is and when q passes through a point and $\mathbf{J} = 0$ elsewhere and at all other times before q passes through.

However, these are in terms of some t' and not the retarded time that we are more concerned with. We can bring in the retarded time using a second Dirac delta function so that if the equation is now valid at $\mathbf{r}'(t')$, it will further be valid when $t' = t_r$. In other words, we introduce the delta function $\delta(t'-t_r)$, by definition, and make $\rho(\mathbf{r},t') \to \rho(\mathbf{r},t_r)$.

$$\rho(\mathbf{r},t_r) = q\delta^3(\mathbf{r}-\mathbf{r}'(t'))\int \delta(t'-t_r)\mathrm{d}t'$$

This can likewise be done for $\mathbf{J}$ as well. Next, substituting this equation into our retarded scalar potential[8] given by equation (1),

$$V(\mathbf{r},t) = \frac{q}{4\pi\epsilon_0}\int_{\tau} \mathrm{d}\tau \left(\frac{\delta^3(\mathbf{r}-\mathbf{r}'(t'))}{\mathscr{r}}\right)\int \mathrm{d}t' \left(\delta(t'-t_r)\right)$$

The delta function now sets $\mathbf{r} = \mathbf{r}'(t')$ and we can integrate with respect to $\mathrm{d}\tau$ easily[9] to arrive at,

$$V(\mathbf{r},t) = \frac{q}{4\pi\epsilon_0}\int \mathrm{d}t' \left(\frac{\delta(t'-t_r)}{\mathscr{r}}\right)$$

whose convenience is apparent if we expand the terms:

$$V(\mathbf{r},t) = \frac{q}{4\pi\epsilon_0}\int \mathrm{d}t' \left(\frac{1}{|\mathbf{r}-\mathbf{r}'(t')|}\right)\delta\left(t' - \left(t - \frac{|\mathbf{r}-\mathbf{r}'(t')|}{c}\right)\right) \tag{8}$$

which means all that is left now is to solve this equation.

We have a term that looks like $\delta(f(x))$ in the argument. Suppose $f(x) = a$, then $\mathrm{d}a = f'(x)\mathrm{d}x$ or $\mathrm{d}x = \mathrm{d}a/f'(x)$. Therefore,

$$\int \mathrm{d}x\,\delta(f(x)) = \int \mathrm{d}(a)\left(\frac{\delta(a)}{f'(x)}\right) = \frac{1}{f'(x)} \tag{9}$$

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[7] Note a result we have already used a couple of times so far: $\frac{\partial}{\partial t_r} = \frac{\partial}{\partial t}$.

[8] We will, like in §1.3, work everything out for the scalar potential, V, knowing that the same reasoning will lead us to an equivalent answer for the vector potential $\mathbf{A}$ as well.

[9] Since nowhere other than at $\mathbf{r} = \mathbf{r}'(t')$ does the δ^3 function have a non-zero value, there is no point worrying about our integral when $\mathbf{r} \neq \mathbf{r}'(t')$. It will go to zero anyway. At $\mathbf{r} = \mathbf{r}'(t')$, the δ^3 function goes to 1 and the integral over $\mathrm{d}\tau$ conveniently vanishes.

In equation (8) the integral over $\mathrm{d}t'$ of the delta function reduces to the form in equation (9) where $f(x)$ is specifically $f(t')$.

In equation (8) we have our $f(x)$ from the delta function as,

$$\begin{aligned} f(t') &= t' - \left(t - \frac{|\mathbf{r} - \mathbf{r}'(t')|}{c}\right) \\ \frac{\mathrm{d}f(t')}{\mathrm{d}t'} &= 1 + \left(\frac{1}{c}\frac{\mathrm{d}}{\mathrm{d}t'}|\mathbf{r} - \mathbf{r}'(t')|\right) \end{aligned} \tag{10}$$

Therefore for the final term on the right hand side,

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t'}|\mathbf{r} - \mathbf{r}'(t')|^2 &= 2\left(\mathbf{r} - \mathbf{r}'(t')\right)\frac{\mathrm{d}}{\mathrm{d}t'}\left(\mathbf{r} - \mathbf{r}'(t')\right) \\ &= 2\left(\mathbf{r} - \mathbf{r}'(t')\right)\left(\frac{\mathrm{d}}{\mathrm{d}t'}\mathbf{r} - \frac{\mathrm{d}}{\mathrm{d}t'}\mathbf{r}'(t')\right) \\ &= -2\left(\mathbf{r} - \mathbf{r}'(t')\right)\frac{\mathrm{d}\mathbf{r}'(t')}{\mathrm{d}t'} \end{aligned}$$

Now if $\mathbf{r}' = -r \cdot \hat{x}$, meaning if we set our problem up such that the charge is moving in the negative x-direction; and, further, if $\frac{\mathrm{d}\mathbf{r}'}{\mathrm{d}t'} = \beta c \cdot \hat{x} = \mathbf{v}$, meaning that our charge is traveling with a velocity[10] of βc, also in the negative x-direction, then,

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t'}|\mathbf{r} - \mathbf{r}'(t')|^2 &= -2\beta c\left(\mathbf{r} - \mathbf{r}'(t')\right) \cdot \hat{x} \\ \Rightarrow |\mathbf{r} - \mathbf{r}'(t')|\frac{\mathrm{d}}{\mathrm{d}t'}\left(\mathbf{r} - \mathbf{r}'(t')\right) &= -\beta c\left(\mathbf{r} - \mathbf{r}'(t')\right) \cdot \hat{x} \\ \Rightarrow \frac{\mathrm{d}}{\mathrm{d}t'}\left(\mathbf{r} - \mathbf{r}'(t')\right) &= -\beta c\frac{\left(\mathbf{r} - \mathbf{r}'(t')\right)}{|\mathbf{r} - \mathbf{r}'(t')|} \cdot \hat{x} \\ \frac{\mathrm{d}}{\mathrm{d}t'}\left(\mathbf{r} - \mathbf{r}'(t')\right) &= -\frac{\left(\mathbf{r} - \mathbf{r}'(t')\right)}{|\mathbf{r} - \mathbf{r}'(t')|} \cdot \mathbf{v}(t_r) \end{aligned}$$

We now simply have to plug this result back into equation (10).

$$\Rightarrow \frac{\mathrm{d}f(t')}{\mathrm{d}t'} = 1 - \frac{1}{c}\left[\frac{\mathbf{r} - \mathbf{r}'(t')}{|\mathbf{r} - \mathbf{r}'(t')|}\right] \cdot \mathbf{v}$$

And then into equation (9), realising that $x = t'$.

$$\Rightarrow \int \mathrm{d}t'\,\delta(f(t')) = \frac{1}{\left(\frac{\mathrm{d}f(t')}{\mathrm{d}t'}\right)} = \frac{1}{1 - \frac{1}{c}\left(\frac{\mathbf{r} - \mathbf{r}'(t')}{|\mathbf{r} - \mathbf{r}'(t')|}\right) \cdot \mathbf{v}}$$

And finally into equation (8), bringing us back to our original problem.

[10] Observe that such a definition of the velocity of our charge puts $0 < \beta < 1$.

$$\Rightarrow V(\mathbf{r},t) = \frac{q}{4\pi\epsilon_0} \int\limits_{\text{at } t_r} \mathrm{d}t' \left(\frac{1}{|\mathbf{r}-\mathbf{r}'(t')|}\right) \delta\left(t' - \left(t - \frac{|\mathbf{r}-\mathbf{r}'(t')|}{c}\right)\right)$$

$$= \left(\frac{q}{4\pi\epsilon_0}\right)\left(\frac{1}{|\mathbf{r}-\mathbf{r}'(t_r)|\left[1-\frac{1}{c}\left(\frac{\mathbf{r}-\mathbf{r}'(t_r)}{|\mathbf{r}-\mathbf{r}'(t_r)|}\right)\right]}\cdot\mathbf{v}\right)$$

$$\therefore \quad \boxed{V(\mathbf{r},t) = \left(\frac{qc}{4\pi\epsilon_0}\right)\left(\frac{1}{c\,|\mathbf{r}-\mathbf{r}'(t_r)| - (\mathbf{r}-\mathbf{r}'(t_r))\cdot\mathbf{v}(t_r)}\right)} \tag{11}$$

Similarly, using $\mathbf{J} = \rho V$, we can arrive at an equation for the vector potential as well, which we will find to be

$$\boxed{\mathbf{A}(\mathbf{r},t) = \left(\frac{\mu_0 qc}{4\pi}\right)\left(\frac{\mathbf{v}(t_r)}{c\,|\mathbf{r}-\mathbf{r}'(t_r)| - (\mathbf{r}-\mathbf{r}'(t_r))\cdot\mathbf{v}(t_r)}\right)} \tag{12}$$

Equations (11) and (12) are known[11] as the *Liénard-Wiechert potentials.* We can also write them more consicely in terms of our separation vector, $\boldsymbol{\mathscr{r}}$, as

$$V(\mathbf{r},t) = \left(\frac{qc}{4\pi\epsilon_0}\right)\left(\frac{1}{c\mathscr{r} - \boldsymbol{\mathscr{r}}\cdot\mathbf{v}}\right) \quad \text{and}$$

$$\mathbf{A}(\mathbf{r},t) = \left(\frac{\mu_0 qc}{4\pi}\right)\left(\frac{\mathbf{v}}{c\mathscr{r} - \boldsymbol{\mathscr{r}}\cdot\mathbf{v}}\right)$$

2.2 Understanding the $\{-|\mathbf{r}-\mathbf{r}'(t_r)|\cdot\mathbf{v}(t_r)\}$ term

Generally, for a point charge, the charge and current density terms understandably approach proportionality with q (with ρ in fact becoming equal to q) leaving us with an equation that should look like

$$V(\mathbf{r},t) = \frac{1}{4\pi\epsilon_0}\frac{q}{\boldsymbol{\mathscr{r}}}$$

for $\boldsymbol{\mathscr{r}} = |\mathbf{r}-\mathbf{r}'(t_r)|$, and likewise for $\mathbf{A}(\mathbf{r},t)$.

However, in the Liénard-Wiechert potentials, note that there is an extra term, $-|\mathbf{r}-\mathbf{r}'(t_r)|\cdot\mathbf{v}(t_r)$ or $-\boldsymbol{\mathscr{r}}\cdot\mathbf{v}(t_r)$, in the denominator besides $\boldsymbol{\mathscr{r}}$. There is an verbal, non-mathematical explanation for this which makes sense for particles that are not

[11] Note that equation (12) reduces to $\mathbf{A}(\mathbf{r},t) = (\mathbf{v}/c^2)V(\mathbf{r},t)$ if we take the term $(\mathbf{v}/c^2)$ out from the right hand side and use $c^2 = (\mu_0\epsilon_0)^{-1}$.

point-sized[12], but is worth going over solely for the physical picture it provides, departing for a brief moment from the abstractness of the Dirac delta function.

Let us suppose that the source charge is one-dimensional along our axis of view, i.e. along the x-axis and moving in the x-direction if the observer is looking in the negative x-direction with the source charge coming at him. In this case, the charge tends to appear longer than it actually is since light from the near end travels a shorter distance than the light from the far end. Alternatively, the light from the far end left *earlier in time* ($\sim t_r$) when compared to that from the near end. This is not necessarily a relativistic effect and especially has nothing to do with the rest mass of the charge.

Say the source charge is of length l_1, traveling with a velocity of v. Say the far end is some distance l_2 away from the observer when the light/information leaves the far end. It implies then that the far end of the charge covers a distance of $l_2 - l_1$ by the time the light from the far end reaches the observer. Since the time taken for the light from the far end to reach the observer and train to cover this distance of $l_2 - l_1$ is the same, we can equate their times:

$$\frac{l_2}{c} = \frac{l_2 - l_1}{\mathbf{v}} \Rightarrow l_2 = \frac{l_1}{1 - \frac{\mathbf{v}}{c}}$$

This scaling factor can be extended to motion *not* parallel to the observer's view as well:

$$l_2 = \frac{l_1}{1 - \hat{\boldsymbol{\mathscr{r}}} \cdot \frac{\mathbf{v}}{c}}$$

where $\hat{\boldsymbol{\mathscr{r}}}$ is a unit vector from the charge to the observer. Note, further, that this applies to volumetric scaling too (although only the dimension along the line of sight of the observer will be scaled, as evinced by the $\hat{\boldsymbol{\mathscr{r}}}$ term.)[13]

$$\tau_2 = \frac{\tau_1}{1 - \hat{\boldsymbol{\mathscr{r}}} \cdot \frac{\mathbf{v}}{c}}$$

Applying this to our equation for, say, the scalar potential, we get

$$V(\mathbf{r}, t) = \frac{1}{4\pi\epsilon_0} \frac{q}{\boldsymbol{\mathscr{r}}\left(1 - \hat{\boldsymbol{\mathscr{r}}} \cdot \frac{\mathbf{v}}{c}\right)} = \frac{1}{4\pi\epsilon_0} \frac{qc}{c\boldsymbol{\mathscr{r}} - \boldsymbol{\mathscr{r}} \cdot \mathbf{v}}$$

which is nothing but the Liénard-Wiechert scalar potential. Naturally, the same reasoning works for the vector potential, $\mathbf{A}(\mathbf{r}, t)$ as well.

[12] Rather unsurprisingly, the mathematical derivation in §2.1 is valid for all charges, point-sized or not, which is why we ought to prefer that from a physicist's perspective. For a *fuller* explanation of this verbal reasoning, see appendix C2, §10.3.1.

[13] While this verbal reasoning does not appear to make sense for point-sized particles as we said in footnote 12, because there is no "near end" and "far end", Griffiths (ibid.) rightly points out that since the scaling factor itself does not contain any length term, it should apply just as well to point particles. Yet this is not as satisfying as our previous, rigorous mathematical treatment.

3 Fields due to an arbitrarily moving point charge

3.1 Deriving the Liénard-Wiechert fields

The Liénard-Wiechert potentials can tell us more about the fields generated by moving point charges[14]. We will, in other words, go on to calculate **E** and **B** using equations (11) and (12). Also recall equations (Ref. 5) and (Ref. 6) from our initial review. These are the equations we are interested in:

$$V(\mathbf{r},t) = \left(\frac{qc}{4\pi\epsilon_0}\right)\left(\frac{1}{c\boldsymbol{\mathscr{r}} - \boldsymbol{\mathscr{r}}\cdot\mathbf{v}}\right) \tag{11}$$

$$\mathbf{A}(\mathbf{r},t) = \left(\frac{\mu_0 qc}{4\pi}\right)\left(\frac{\mathbf{v}}{c\boldsymbol{\mathscr{r}} - \boldsymbol{\mathscr{r}}\cdot\mathbf{v}}\right) \tag{12}$$

$$\mathbf{E} = -\nabla V - \frac{\partial \mathbf{A}}{\partial t} \tag{Ref. 6}$$

$$\mathbf{B} = \nabla \times \mathbf{A} \tag{Ref. 5}$$

For equation (Ref. 6) we need to first calculate ∇V. Consider, therefore, the gradient of $V(\mathbf{r},t)$ from equation (11) above.

$$\nabla V = \frac{qc}{4\pi\epsilon_0}\,\frac{-1}{\left(\mathscr{r}c - \boldsymbol{\mathscr{r}}\cdot\mathbf{v}\right)^2}\nabla\left(\mathscr{r}c - \boldsymbol{\mathscr{r}}\cdot\mathbf{v}\right) \tag{13}$$

We have two terms to compute here: $\nabla\mathscr{r}c$ and $\nabla(\boldsymbol{\mathscr{r}}\cdot\mathbf{v})$. The first one is pretty straightforward. Since $\mathscr{r} = c\,(t - t_r)$,

$$\nabla\mathscr{r} = -c\nabla t_r \Rightarrow \nabla\mathscr{r}c = -c^2\nabla\ t_r \tag{13.1}$$

As for the second term, we turn to the product rule (see appendix B5a) to help us in expanding it.

$$\nabla(\boldsymbol{\mathscr{r}}\cdot\mathbf{v}) = \boldsymbol{\mathscr{r}}\times(\nabla\times\mathbf{v}) + (\boldsymbol{\mathscr{r}}\cdot\nabla)\mathbf{v} + \mathbf{v}\times(\nabla\times\boldsymbol{\mathscr{r}}) + (\mathbf{v}\cdot\nabla)\boldsymbol{\mathscr{r}} \tag{13.2}$$

And now we solve each of these four terms[15] one by one, remembering that all functions of time, such as **a**, **v** etc. are particularly functions of t_r.

$$\begin{aligned}
\nabla\times\mathbf{v} &= \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \partial_x & \partial_y & \partial_z \\ v_x & v_y & v_z \end{vmatrix} \\
&= \left(\partial_j v_k - \partial_k v_j\right)\hat{i} \\
&= \left(\partial_{t_r} v_k \partial_j t_r - \partial_{t_r} v_j \partial_k t_r\right)\hat{i} \\
&= -\mathbf{a}\times\nabla t_r \\
\therefore\ \boldsymbol{\mathscr{r}}\times(\nabla\times\mathbf{v}) &= -\boldsymbol{\mathscr{r}}\times(\mathbf{a}\times\nabla t_r)
\end{aligned} \tag{13.2.1}$$

[14] These are sometimes also called the Liénard-Wiechert fields.

[15] The notation $\partial_x := \partial/\partial x$ is used for simplicity. We will also be using Einstein's summation convention henceforth, also for simplicity. And everywhere i, j, k will refer to the x, y and z components respectively, e.g. $v_i = v_x$, $\partial_j = \partial/\partial y$ and so on.

$$
\begin{aligned}
(\boldsymbol{\mathscr{r}} \cdot \nabla)\mathbf{v} &= (\mathscr{r}_i \cdot \partial_i)\,\mathbf{v} \\
&= (\mathscr{r}_i \cdot \partial_{t_r} \cdot \partial_i t_r)\,\mathbf{v} \\
&= \mathbf{a}(\boldsymbol{\mathscr{r}} \cdot \nabla t_r)
\end{aligned} \tag{13.2.2}
$$

$$
\begin{aligned}
\nabla \times \boldsymbol{\mathscr{r}} &= \nabla \times \mathbf{r} - \nabla \times \mathbf{r}' \\
\text{(like (13.2.1))} \qquad &= 0 - (\partial_j r_k - \partial_k r_j)\hat{i} \\
&= -(\partial_{t_r} r_k \partial_j t_r - \partial_{t_r} r_j \partial_k t_r)\hat{i} \\
&= \mathbf{v} \times \nabla t_r \\
\therefore\ \mathbf{v} \times (\nabla \times \boldsymbol{\mathscr{r}}) &= \mathbf{v} \times (\mathbf{v} \times \nabla t_r)
\end{aligned} \tag{13.2.3}
$$

$$
\begin{aligned}
(\mathbf{v} \cdot \nabla)\boldsymbol{\mathscr{r}} &= (\mathbf{v} \cdot \nabla)(\mathbf{r} - \mathbf{r}'(t_r)) \\
&= (v_i \partial_i)\,\mathbf{r} - (v_i \partial_i)\mathbf{r}' \\
&= (v_i \partial_i i) - (v_i \partial_{t_r} \cdot \partial_i t_r)\mathbf{r}' \\
&= \mathbf{v} - \mathbf{v}(\mathbf{v}' \cdot \nabla t_r) \\
\text{since } \mathbf{r}' \equiv (\Sigma r_i) \qquad &= \mathbf{v} - \mathbf{v}(\mathbf{v} \cdot \nabla t_r)
\end{aligned} \tag{13.2.4}
$$

Equation (13.2.4) works because we have simply defined $\mathbf{r}'$ as $\mathbf{r}$ when it is a function of t_r in particular. Besides that, $\mathbf{r}$ and $\mathbf{r}'$ are basically the same thing, hence we assert their equivalence.

We now put equations (13.2.1–13.2.4) into equation (13.2) and solve further using the BAC-CAB rule (see appendix B5b).

$$
\begin{aligned}
\nabla(\boldsymbol{\mathscr{r}} \cdot \mathbf{v}) &= [-\boldsymbol{\mathscr{r}} \times (\mathbf{a} \times \nabla t_r)] + [\mathbf{a}(\boldsymbol{\mathscr{r}} \cdot \nabla t_r)] + [\mathbf{v} \times (\mathbf{v} \times \nabla t_r)] + [\mathbf{v} - \mathbf{v}(\mathbf{v} \cdot \nabla t_r)] \\
&= [-\mathbf{a}(\boldsymbol{\mathscr{r}} \cdot \nabla t_r) + \nabla t_r(\boldsymbol{\mathscr{r}} \cdot \mathbf{a})] + [\mathbf{a}(\boldsymbol{\mathscr{r}} \cdot \nabla t_r)] \\
&\qquad + [\mathbf{v}(\mathbf{v} \cdot \nabla t_r) - \nabla t_r(\mathbf{v} \cdot \mathbf{v})] + [\mathbf{v} - \mathbf{v}(\mathbf{v} \cdot \nabla t_r)] \\
&= (\boldsymbol{\mathscr{r}} \cdot \mathbf{a} - v^2)\,\nabla t_r + \mathbf{v}
\end{aligned}
$$

We have now simplified both terms in equation (13) so we put the above equation and equation (13.1) into equation (13) to arrive at

$$
\nabla V = \frac{qc}{4\pi\epsilon_0} \frac{1}{(\mathscr{r}c - \boldsymbol{\mathscr{r}} \cdot \mathbf{v})^2} \left[(c^2 + \boldsymbol{\mathscr{r}} \cdot \mathbf{a} - v^2)\,\nabla t_r + \mathbf{v}\right] \tag{14}
$$

Our next task then is to find ∇t_r. We can use equation (13.1) for this along with the product rule (see appendix B5a).

$$
\begin{aligned}
\nabla \mathscr{r} &= -c\nabla t_r \\
\Rightarrow -c\nabla t_r &= \nabla\sqrt{\boldsymbol{\mathscr{r}} \cdot \boldsymbol{\mathscr{r}}} \\
&= \frac{1}{2\sqrt{\boldsymbol{\mathscr{r}} \cdot \boldsymbol{\mathscr{r}}}} \nabla(\boldsymbol{\mathscr{r}} \cdot \boldsymbol{\mathscr{r}}) \\
&= \frac{1}{2\mathscr{r}} \left\{[\boldsymbol{\mathscr{r}} \times (\nabla \times \boldsymbol{\mathscr{r}}) + (\boldsymbol{\mathscr{r}} \cdot \nabla)\boldsymbol{\mathscr{r}}] + [\boldsymbol{\mathscr{r}} \times (\nabla \times \boldsymbol{\mathscr{r}}) + (\boldsymbol{\mathscr{r}} \cdot \nabla)\boldsymbol{\mathscr{r}}]\right\} \\
&= \frac{1}{\mathscr{r}} [\boldsymbol{\mathscr{r}} \times (\nabla \times \boldsymbol{\mathscr{r}}) + (\boldsymbol{\mathscr{r}} \cdot \nabla)\boldsymbol{\mathscr{r}}]
\end{aligned} \tag{15}
$$

We already know that $\nabla \times \boldsymbol{\mathscr{r}} = \mathbf{v} \times \nabla t_r$ from our solution for equation (13.2.3), which means we can now use the BAC-CAB rule (see appendix B5b) to show that $\boldsymbol{\mathscr{r}} \times (\nabla \times \boldsymbol{\mathscr{r}}) = \boldsymbol{\mathscr{r}} \times (\mathbf{v} \times \nabla t_r) = \mathbf{v}(\boldsymbol{\mathscr{r}} \cdot \nabla t_r) - \nabla t_r(\boldsymbol{\mathscr{r}} \cdot \mathbf{v})$ is one of the two terms in the braces in equation (15).

For the other term in the braces, we see that

$$\begin{aligned}(\boldsymbol{\mathscr{r}} \cdot \nabla)\boldsymbol{\mathscr{r}} &= (\mathscr{r}_i \cdot \partial_i)|\mathbf{r} - \mathbf{r}'(t_r)| \\ &= (\mathscr{r}_i \cdot \partial_i)\mathbf{r} - (\mathscr{r}_i \cdot \partial_{t_r} \cdot \partial_i t_r)\mathbf{r}'(t_r) \\ &= \boldsymbol{\mathscr{r}} - \mathbf{v}(\boldsymbol{\mathscr{r}} \cdot \nabla t_r)\end{aligned}$$

Putting these together into equation (15)

$$\begin{aligned}-c\nabla t_r &= \frac{1}{2\mathscr{r}}\{[\boldsymbol{\mathscr{r}} \times (\nabla \times \boldsymbol{\mathscr{r}}) + (\boldsymbol{\mathscr{r}} \cdot \nabla)\boldsymbol{\mathscr{r}}] + [\boldsymbol{\mathscr{r}} \times (\nabla \times \boldsymbol{\mathscr{r}}) + (\boldsymbol{\mathscr{r}} \cdot \nabla)\boldsymbol{\mathscr{r}}]\} \\ &= \frac{1}{\mathscr{r}}[\boldsymbol{\mathscr{r}} \times (\nabla \times \boldsymbol{\mathscr{r}}) + (\boldsymbol{\mathscr{r}} \cdot \nabla)\boldsymbol{\mathscr{r}}] \\ &= \frac{1}{\mathscr{r}}[\mathbf{v}(\boldsymbol{\mathscr{r}} \cdot \nabla t_r) - \nabla t_r(\boldsymbol{\mathscr{r}} \cdot \mathbf{v}) + \boldsymbol{\mathscr{r}} - \mathbf{v}(\boldsymbol{\mathscr{r}} \cdot \nabla t_r)] \\ \Rightarrow \nabla t_r\left(-c + \frac{\boldsymbol{\mathscr{r}} \cdot \mathbf{v}}{\mathscr{r}}\right) &= \frac{\boldsymbol{\mathscr{r}}}{\mathscr{r}} \\ \therefore\ \nabla t_r &= \frac{-\boldsymbol{\mathscr{r}}}{\mathscr{r}c - \boldsymbol{\mathscr{r}} \cdot \mathbf{v}} \qquad (16)\end{aligned}$$

Finally we can combine equations (14) and (16) to arrive at

$$\begin{aligned}\nabla V &= \frac{qc}{4\pi\epsilon_0}\frac{1}{(\mathscr{r}c - \boldsymbol{\mathscr{r}} \cdot \mathbf{v})^2}\left[(c^2 + \boldsymbol{\mathscr{r}} \cdot \mathbf{a} - v^2)\nabla t_r + \mathbf{v}\right] \\ &= \frac{qc}{4\pi\epsilon_0}\frac{1}{(\mathscr{r}c - \boldsymbol{\mathscr{r}} \cdot \mathbf{v})^2}\left[(c^2 + \boldsymbol{\mathscr{r}} \cdot \mathbf{a} - v^2)\left(\frac{-\boldsymbol{\mathscr{r}}}{\mathscr{r}c - \boldsymbol{\mathscr{r}} \cdot \mathbf{v}}\right) + \mathbf{v}\right] \\ &= \frac{qc}{4\pi\epsilon_0}\frac{1}{(\mathscr{r}c - \boldsymbol{\mathscr{r}} \cdot \mathbf{v})^3}\left[(\mathscr{r}c - \boldsymbol{\mathscr{r}} \cdot \mathbf{v})\mathbf{v} - \boldsymbol{\mathscr{r}}(c^2 + \boldsymbol{\mathscr{r}} \cdot \mathbf{a} - v^2)\right] \qquad (17)\end{aligned}$$

So now we have ∇V. But recall from equation (Ref. 6) that the electric field is given by

$$\mathbf{E} = -\nabla V - \frac{\partial \mathbf{A}}{\partial t}$$

which means we need to find $\frac{\partial \mathbf{A}}{\partial t}$ next.

From equation (12) we have

$$\mathbf{A}(\mathbf{r}, t) = \left(\frac{\mu_0 qc}{4\pi}\right)\left(\frac{\mathbf{v}}{c\mathscr{r} - \boldsymbol{\mathscr{r}} \cdot \mathbf{v}}\right)$$

which means

$$\begin{aligned}\left(\frac{4\pi}{\mu_0 qc}\right)\frac{\partial \mathbf{A}(\mathbf{r}, t)}{\partial t} &= \frac{\partial \mathbf{v}}{\partial t}\left(\frac{1}{c\mathscr{r} - \boldsymbol{\mathscr{r}} \cdot \mathbf{v}}\right) + \mathbf{v}\frac{\partial}{\partial t}\left(\frac{1}{c\mathscr{r} - \boldsymbol{\mathscr{r}} \cdot \mathbf{v}}\right) \\ &= \frac{\partial \mathbf{v}}{\partial t_r}\frac{\partial t_r}{\partial t}\left(\frac{1}{c\mathscr{r} - \boldsymbol{\mathscr{r}} \cdot \mathbf{v}}\right) - \left(\frac{\mathbf{v}}{(c\mathscr{r} - \boldsymbol{\mathscr{r}} \cdot \mathbf{v})^2}\right)\frac{\partial}{\partial t}(c\mathscr{r} - \boldsymbol{\mathscr{r}} \cdot \mathbf{v}) \qquad (18)\end{aligned}$$

Like with equation (13) we find the various terms in equation (18) separately. Specifically $\frac{\partial \mathscr{r}}{\partial t}$ and $\frac{\partial \boldsymbol{\mathscr{r}}}{\partial t}$. We start with the definition of $\boldsymbol{\mathscr{r}}$.

$$
\begin{aligned}
\boldsymbol{\mathscr{r}} &= \mathbf{r} - \mathbf{r}'(t_r) \\
\Rightarrow \frac{\partial \boldsymbol{\mathscr{r}}}{\partial t} &= -\mathbf{v}\,\frac{\partial t_r}{\partial t}
\end{aligned}
\tag{19}
$$

and from the definition of t_r,

$$
\begin{aligned}
t_r &= t - \frac{\mathscr{r}}{c} \\
\Rightarrow \frac{\partial \mathscr{r}}{\partial t} &= c\,\left(1 - \frac{\partial t_r}{\partial t}\right)
\end{aligned}
\tag{20}
$$

Substituting equations (19) and (20) into equation (18) we have

$$
\begin{aligned}
\left(\frac{4\pi}{\mu_0 q c}\right)\frac{\partial \mathbf{A}(\mathbf{r},t)}{\partial t} &= \frac{\partial \mathbf{v}}{\partial t_r}\,\frac{\partial t_r}{\partial t}\left(\frac{1}{c\mathscr{r} - \boldsymbol{\mathscr{r}}\cdot\mathbf{v}}\right) \\
&\qquad - \left(\frac{\mathbf{v}}{(c\mathscr{r} - \boldsymbol{\mathscr{r}}\cdot\mathbf{v})^2}\right)\,\left(c\frac{\partial \mathscr{r}}{\partial t} - \frac{\partial \boldsymbol{\mathscr{r}}}{\partial t}\cdot\mathbf{v} - \boldsymbol{\mathscr{r}}\frac{\partial \mathbf{v}}{\partial t_r}\frac{\partial t_r}{\partial t}\right) \\
&= \frac{\partial t_r}{\partial t}\left[\frac{\mathbf{a}}{c\mathscr{r} - \boldsymbol{\mathscr{r}}\cdot\mathbf{v}} - \frac{\mathbf{v}}{(c\mathscr{r} - \boldsymbol{\mathscr{r}}\cdot\mathbf{v})^2}(-c^2 + v^2 - \boldsymbol{\mathscr{r}}\cdot\mathbf{a})\right] \\
&\qquad - \left(\frac{c^2\mathbf{v}}{(c\mathscr{r} - \boldsymbol{\mathscr{r}}\cdot\mathbf{v})^2}\right)
\end{aligned}
\tag{21}
$$

We now have to compute $\frac{\partial t_r}{\partial t}$ so we turn to its definition again.

$$
\begin{aligned}
\mathscr{r} &= c\,(t - t_r) \\
\frac{\partial \mathscr{r}}{\partial t} &= c\,(1 - \frac{\partial t_r}{\partial t}) \\
\Rightarrow c - c\,\frac{\partial t_r}{\partial t} &= \frac{\partial}{\partial t}(\sqrt{\boldsymbol{\mathscr{r}}\cdot\boldsymbol{\mathscr{r}}}) \\
-c\,\frac{\partial t_r}{\partial t} &= \frac{1}{2\mathscr{r}}\frac{\partial}{\partial t}(\boldsymbol{\mathscr{r}}\cdot\boldsymbol{\mathscr{r}}) - c \\
\frac{\partial t_r}{\partial t} &= \frac{1}{2\mathscr{r}c}\frac{\partial}{\partial t}(r^2 + r'^2 - 2\mathbf{r}\cdot\mathbf{r}') - 1 \\
&= \frac{1}{2\mathscr{r}c}\,(0 + 2r'\partial_{t_r}r'\,\partial_t t_r - 2\mathbf{r}\cdot\partial_{t_r}\mathbf{r}'\partial_t t_r - 0) - 1 \\
&= \frac{(r'\partial_{t_r}r' - \mathbf{r}\cdot\partial_{t_r}\mathbf{r}')\,\partial_t t_r - \mathscr{r}c}{\mathscr{r}c}
\end{aligned}
$$

$$
\Rightarrow \frac{\partial t_r}{\partial t}\ \text{(or simply } \partial_t t_r\text{)} = \frac{-\mathscr{r}c}{\mathscr{r}c - r'\partial_{t_r}r' - \mathbf{r}\cdot\partial_{t_r}\mathbf{r}'} = \frac{\mathscr{r}c}{\mathscr{r}c - (r'\partial_{t_r}r' - r\hat{r}\cdot\partial_{t_r}r'\hat{r}')}
$$

$$
\therefore\ \partial_t t_r = \frac{\mathscr{r}c}{\mathscr{r}c - (r' - r)\,\partial_{t_r}r'} = \frac{\mathscr{r}c}{\mathscr{r}c - \boldsymbol{\mathscr{r}}\cdot\mathbf{v}}
\tag{22}
$$

Plugging equation (22) into equation (21) we get

$$
\begin{aligned}
\left(\frac{4\pi}{\mu_0 qc}\right)\frac{\partial \mathbf{A}(\mathbf{r},t)}{\partial t} &= \left(\frac{\mathscr{r}c}{\mathscr{r}c-\boldsymbol{\mathscr{r}}\cdot\mathbf{v}}\right)\left[\frac{\mathbf{a}}{c\mathscr{r}-\boldsymbol{\mathscr{r}}\cdot\mathbf{v}}-\frac{\mathbf{v}}{(c\mathscr{r}-\boldsymbol{\mathscr{r}}\cdot\mathbf{v})^2}(-c^2+v^2-\boldsymbol{\mathscr{r}}\cdot\mathbf{a})\right] \\
&\qquad -\left(\frac{c^2\mathbf{v}}{(c\mathscr{r}-\boldsymbol{\mathscr{r}}\cdot\mathbf{v})^2}\right) \\
\frac{\partial \mathbf{A}(\mathbf{r},t)}{\partial t} &= \left(\frac{\mu_0 qc}{4\pi(c\mathscr{r}-\boldsymbol{\mathscr{r}}\cdot\mathbf{v})^3}\right)\big[\mathscr{r}c\mathbf{a}(c\mathscr{r}-\boldsymbol{\mathscr{r}}\cdot\mathbf{v})-\mathscr{r}c\mathbf{v}(-c^2+v^2-\boldsymbol{\mathscr{r}}\cdot\mathbf{a}) \\
&\qquad -c^2\mathbf{v}(c\mathscr{r}-\boldsymbol{\mathscr{r}}\cdot\mathbf{v})\big]
\end{aligned}
$$

In order to get the $qc/4\pi\epsilon_0$ factor that is in equation (17) we use the fact that $c^2=(\mu_0\epsilon_0)^{-1} \Rightarrow c\mu_0=(c\epsilon_0)^{-1}$ and then take c^2 out as a common factor from the square-bracketed terms.

$$
\begin{aligned}
\partial_t\mathbf{A} &= \left(\frac{q}{4\pi\epsilon_0 c(c\mathscr{r}-\boldsymbol{\mathscr{r}}\cdot\mathbf{v})^3}\right)\left[(\mathscr{r}c\,\mathbf{a}-c^2\mathbf{v})(c\mathscr{r}-\boldsymbol{\mathscr{r}}\cdot\mathbf{v})-\mathscr{r}c\,\mathbf{v}(-c^2+v^2-\boldsymbol{\mathscr{r}}\cdot\mathbf{a})\right] \\
&= \left(\frac{qc}{4\pi\epsilon_0(c\mathscr{r}-\boldsymbol{\mathscr{r}}\cdot\mathbf{v})^3}\right)\left[\left(\frac{\mathscr{r}\mathbf{a}}{c}-\mathbf{v}\right)(c\mathscr{r}-\boldsymbol{\mathscr{r}}\cdot\mathbf{v})-\left(\frac{\mathscr{r}\mathbf{v}}{c}\right)(-c^2+v^2-\boldsymbol{\mathscr{r}}\cdot\mathbf{a})\right]
\end{aligned}
\tag{23}
$$

At long last we have, in equations (17) and (23), both the terms we need to find the electric field, $\mathbf{E}$, as given by equation (Ref. 6).

$$
\begin{aligned}
\mathbf{E}(\mathbf{r},t) &= -\nabla V-\partial_t\mathbf{A} \\
&= \left(\frac{qc}{4\pi\epsilon_0(c\mathscr{r}-\boldsymbol{\mathscr{r}}\cdot\mathbf{v})^3}\right)\Big\{\left[(c\mathscr{r}-\boldsymbol{\mathscr{r}}\cdot\mathbf{v})(-\mathbf{v})+\boldsymbol{\mathscr{r}}(c^2-v^2+\boldsymbol{\mathscr{r}}\cdot\mathbf{a})\right] \\
&\qquad -\left[\left(\frac{\mathscr{r}\mathbf{a}}{c}-\mathbf{v}\right)(c\mathscr{r}-\boldsymbol{\mathscr{r}}\cdot\mathbf{v})+\left(\frac{\mathscr{r}\mathbf{v}}{c}\right)(c^2-v^2+\boldsymbol{\mathscr{r}}\cdot\mathbf{a})\right]\Big\}
\end{aligned}
$$

Note that the term $(\mathscr{r}c-\boldsymbol{\mathscr{r}}\cdot\mathbf{v})$ recurs often. We can define[16] a vector $\mathbf{w}\equiv c\hat{\boldsymbol{\mathscr{r}}}-\mathbf{v}$ so that $\boldsymbol{\mathscr{r}}\cdot\mathbf{w}=c\boldsymbol{\mathscr{r}}\cdot\hat{\boldsymbol{\mathscr{r}}}-\boldsymbol{\mathscr{r}}\cdot\mathbf{v}=c\mathscr{r}-\boldsymbol{\mathscr{r}}\cdot\mathbf{v}$. This makes our expression simpler.

$$
\begin{aligned}
\mathbf{E}(\mathbf{r},t) &= \left(\frac{qc}{4\pi\epsilon_0\,(\boldsymbol{\mathscr{r}}\cdot\mathbf{w})^3}\right)\left[(\boldsymbol{\mathscr{r}}\cdot\mathbf{w})(-\frac{\mathscr{r}\mathbf{a}}{c})+(c^2-v^2+\boldsymbol{\mathscr{r}}\cdot\mathbf{a})(\boldsymbol{\mathscr{r}}-\frac{\mathscr{r}\mathbf{v}}{c})\right] \\
&= \left(\frac{qc}{4\pi\epsilon_0\,(\boldsymbol{\mathscr{r}}\cdot\mathbf{w})^3}\right)\left[\left(\frac{\mathscr{r}}{c}\right)[-\mathbf{a}(\boldsymbol{\mathscr{r}}\cdot\mathbf{w})]+(c^2-v^2+\boldsymbol{\mathscr{r}}\cdot\mathbf{a})\left(\frac{\mathscr{r}}{c}\right)\mathbf{w}\right] \\
&= \left(\frac{q\mathscr{r}}{4\pi\epsilon_0\,(\boldsymbol{\mathscr{r}}\cdot\mathbf{w})^3}\right)\left\{(c^2-v^2)\mathbf{w}+[\mathbf{w}(\boldsymbol{\mathscr{r}}\cdot\mathbf{a})-\mathbf{a}(\boldsymbol{\mathscr{r}}\cdot\mathbf{w})]\right\}
\end{aligned}
$$

$$
\therefore\quad \boxed{\mathbf{E}(\mathbf{r},t)=\left(\frac{q\mathscr{r}}{4\pi\epsilon_0\,(\boldsymbol{\mathscr{r}}\cdot\mathbf{w})^3}\right)\left[(c^2-v^2)\mathbf{w}+\boldsymbol{\mathscr{r}}\times(\mathbf{w}\times\mathbf{a})\right]}\quad \text{(see appendix B5b)} \tag{24}
$$

[16]We arrive at this by removing $\boldsymbol{\mathscr{r}}$ as a common factor: $c\mathscr{r}-\boldsymbol{\mathscr{r}}\cdot\mathbf{v}\to\boldsymbol{\mathscr{r}}\cdot(c\hat{\boldsymbol{\mathscr{r}}}-\mathbf{v})\to\boldsymbol{\mathscr{r}}\cdot\mathbf{w}$.

Now we know from equations (11) and (12) that

$$\mathbf{A} = \mu_0\epsilon_0 V\mathbf{v} = \frac{1}{c^2}V\mathbf{v}$$

Therefore,

$$\begin{aligned}\nabla\times\mathbf{A} &= \frac{1}{c^2}\nabla\times(V\mathbf{v})\\ &= \frac{1}{c^2}\left[V(\nabla\times\mathbf{v}) - \mathbf{v}\times(\nabla V)\right]\end{aligned}$$

From equation (13.2.1) we know $\nabla\times\mathbf{v}$, and from equation (17) we know ∇V, both of which we can use here. Further, we will also use equation (16) for ∇t_r.

$$\begin{aligned}\nabla\times\mathbf{A} = \frac{1}{c^2}\Bigg[&V(-\mathbf{a}\times\nabla t_r)\\ &-\mathbf{v}\times\left(\frac{qc}{4\pi\epsilon_0}\frac{1}{(\boldsymbol{\mathscr{r}}\cdot\mathbf{w})^3}\left[(\boldsymbol{\mathscr{r}}\cdot\mathbf{w})\mathbf{v} - \boldsymbol{\mathscr{r}}(c^2 - v^2 + \boldsymbol{\mathscr{r}}\cdot\mathbf{a})\right]\right)\Bigg]\end{aligned}$$

$$\begin{aligned}= \frac{1}{c^2}\Bigg\{&\left[\frac{qc}{4\pi\epsilon_0}\left(\frac{1}{\boldsymbol{\mathscr{r}}\cdot\mathbf{w}}\right)\left(-\mathbf{a}\times\frac{-\boldsymbol{\mathscr{r}}}{\boldsymbol{\mathscr{r}}\cdot\mathbf{w}}\right)\right]\\ &-\left[\frac{qc}{4\pi\epsilon_0}\left(\frac{1}{(\boldsymbol{\mathscr{r}}\cdot\mathbf{w})^3}\right)\big(\mathbf{v}\times(\boldsymbol{\mathscr{r}}\cdot\mathbf{w})\mathbf{v} - (\mathbf{v}\times\boldsymbol{\mathscr{r}})(c^2 - v^2 + \mathbf{r}\cdot\mathbf{a})\big)\right]\Bigg\}\end{aligned}$$

We now take certain common terms out,

$$= \frac{q}{4\pi\epsilon_0 c}\left(\frac{1}{(\boldsymbol{\mathscr{r}}\cdot\mathbf{w})^3}\right)\left\{\left[(\boldsymbol{\mathscr{r}}\cdot\mathbf{w})\,(-\mathbf{a}\times-\boldsymbol{\mathscr{r}})\right] - \left[\big(0 - (\mathbf{v}\times\boldsymbol{\mathscr{r}})(c^2 - v^2 + \mathbf{r}\cdot\mathbf{a})\big)\right]\right\}$$

and use the property of scalar distribution over a cross product (see appendix B5e),

$$= \frac{q}{4\pi\epsilon_0 c}\frac{1}{(\boldsymbol{\mathscr{r}}\cdot\mathbf{w})^3}\left\{\left[-(\boldsymbol{\mathscr{r}}\cdot\mathbf{w})\mathbf{a}\times-\boldsymbol{\mathscr{r}}\right] + \left[\mathbf{v}(c^2 - v^2 + \mathbf{r}\cdot\mathbf{a})\times\boldsymbol{\mathscr{r}}\right]\right\}$$

since $(-\mathbf{a}\times\mathbf{b}) = -(\mathbf{a}\times\mathbf{b})$ is like multiplying by -1 using the rule in appendix B5e,

$$= \frac{q}{4\pi\epsilon_0 c}\frac{1}{(\boldsymbol{\mathscr{r}}\cdot\mathbf{w})^3}\left\{-\left[-(\boldsymbol{\mathscr{r}}\cdot\mathbf{w})\mathbf{a}\right] + \left[(c^2 - v^2)\mathbf{v} + (\mathbf{r}\cdot\mathbf{a})\mathbf{v}\right]\right\}\times\boldsymbol{\mathscr{r}}$$

$$= \frac{-q}{4\pi\epsilon_0 c}\frac{1}{(\boldsymbol{\mathscr{r}}\cdot\mathbf{w})^3}\boldsymbol{\mathscr{r}}\times\left\{(c^2 - v^2)\mathbf{v} + (\mathbf{r}\cdot\mathbf{a})\mathbf{v} + (\boldsymbol{\mathscr{r}}\cdot\mathbf{w})\mathbf{a}\right\}$$

Lastly, notice that in the domain of $\boldsymbol{\mathscr{r}}$ alone[17], the velocities $\mathbf{w}\equiv-\mathbf{v}$.

[17] We know that $\mathbf{w}$ is a velocity with two components, the scalar c and the vector $-\mathbf{v}$, by definition. In this particular equation, since $\mathbf{v}$ is going to be in a cross product, we need only concern ourselves with the vector component in $\mathbf{w}$, which allows us to effectively equate it to just its vector component, knowing what we are really exploiting here is the proportionality between $\mathbf{v}$ and $\mathbf{w}$ velocities. As for the $\boldsymbol{\mathscr{r}}\times$ term, we split it as $\mathscr{r}\hat{\boldsymbol{\mathscr{r}}}$ since the $\mathscr{r}$ lets us compare this equation to that of the electric field in equation (24).

Therefore, using the BAC-CAB rule (see appendix B5b),

$$\nabla \times \mathbf{A} = \frac{-q}{4\pi\epsilon_0 c} \frac{1}{(\boldsymbol{\mathscr{r}} \cdot \mathbf{w})^3} \boldsymbol{\mathscr{r}} \times \left\{ -(c^2 - v^2)\mathbf{w} + [-(\mathbf{r} \cdot \mathbf{a})\mathbf{w} + (\boldsymbol{\mathscr{r}} \cdot \mathbf{w})\mathbf{a}] \right\}$$

$$= \frac{q\mathscr{r}}{4\pi\epsilon_0 c} \frac{1}{(\boldsymbol{\mathscr{r}} \cdot \mathbf{w})^3} \boldsymbol{\mathscr{r}} \times \left\{ (c^2 - v^2)\mathbf{w} + [\hat{\boldsymbol{\mathscr{r}}} \times (\mathbf{w} \times \mathbf{a})] \right\}$$

$$\therefore\ \nabla \times \mathbf{A}(\mathbf{r}, t) = \frac{1}{c}\, \hat{\boldsymbol{\mathscr{r}}} \times \mathbf{E}(\mathbf{r}, t)$$

By extension, the magnetic field is

$$\nabla \times \mathbf{A} = \boxed{\mathbf{B}(\mathbf{r}, t) = \frac{1}{c}\, \hat{\boldsymbol{\mathscr{r}}} \times \mathbf{E}(\mathbf{r}, t)} \tag{25}$$

As you may have realised by now, all mathematical operations/calculations are to be done *at retarded time*. Equations (11) and (12) give us the Liénard-Wiechert potentials and, in turn, through equations (24) and (25) they describe the electric and magnetic fields generated by a point charge in arbitrary motion with an arbitrary velocity—perhaps the most general case possible.

3.2 Some observations on the Liénard-Wiechert fields

Our discussion of the Liénard-Wiechert potentials and fields (§2 − §3.1) have been mathematically rigorous, but we ought to be just as interested as what they mean physically, not least because they do a remarkably good job at describing electrodynamics everywhere but in the quantum realm.

1. The most immediately apparent fact is that, as equation (25) tells us, the magnetic field, **B**, the electric field, **E**, and the propagation direction, $\hat{\boldsymbol{\mathscr{r}}}$, are *mutually perpendicular*.

2. From the first term in equation (24) we see that the electric field reduces as the inverse square of the distance from the particle, **w**. This is called the *velocity field*. Due to its similarity in terms of an inverse square relation, this term is also called the *generalised Coulomb field*.

3. The second term in equation (24) varies as the first power of $\mathscr{r}$ and, being proportional to the acceleration of the particle, is called the *acceleration field*. It is also known as the *radiation field*. Likewise in equation (25) as well.

4. In equation (25), the first term[18] goes to zero at $\mathbf{v} = 0$, telling us that a stationary charge does not generate a magnetic field.

5. Applying the Liénard-Wiechert fields to the Lorentz force law[19] makes for a somewhat general, all-encompassing equation of electrodynamics, e.g. see the results of §4.

[18] For clarity see the full equation in the last step for the derivation of $\nabla \times \mathbf{A}$.

[19] $\mathbf{F} = Q(\mathbf{v} \times \mathbf{B})$

4 The case of a charge moving with a constant velocity

4.1 Liénard-Wiechert potentials at constant velocity

If we have a charge moving with velocity, $\mathbf{v}$, and acceleration, $\mathbf{a} = 0$, passing through the origin at $t = 0$, then its position vector at retarded time $\mathbf{r}'(t_r) \to \mathbf{v}t_r$.

From $t_r = t - |\mathbf{r} - \mathbf{v}t|/c$ therefore, the retarded time can be written as

$$
\begin{aligned}
|\mathbf{r} - \mathbf{v}t_r| &= c(t - t_r) \\
\text{squaring,} \qquad r^2 + v^2 t_r^2 - 2\mathbf{r}\cdot\mathbf{v}t_r &= c^2(t^2 + t_r^2 - 2tt_r)
\end{aligned}
$$

And re-writing in terms of t_r using the formula $x = (-b \pm \sqrt{b^2 - 4ac})/2a$ we get[20]

$$
\Rightarrow t_r = \frac{(c^2 t - \mathbf{r}\cdot\mathbf{v}) - \sqrt{(c^2 t - \mathbf{r}\cdot\mathbf{v})^2 + (c^2 - v^2)(r^2 - c^2 t^2)}}{c^2 - v^2} \tag{26}
$$

We now focus on the recurring $c\mathscr{r} - \boldsymbol{\mathscr{r}}\cdot\mathbf{v}$ term using the definitions

$$
\mathscr{r} = c(t - t_r)
$$

and

$$
\begin{aligned}
\boldsymbol{\mathscr{r}} &= \mathbf{r} - \mathbf{v}t_r \\
\Rightarrow \mathscr{r}\hat{\boldsymbol{\mathscr{r}}} &= \mathbf{r} - \mathbf{v}t_r \\
\therefore \hat{\boldsymbol{\mathscr{r}}} &= \frac{\mathbf{r} - \mathbf{v}t_r}{c(t - t_r)}
\end{aligned}
$$

So we get

$$
\begin{aligned}
\frac{c\mathscr{r} - \boldsymbol{\mathscr{r}}\cdot\mathbf{v}}{c} = \mathscr{r}\left(1 - \frac{\hat{\boldsymbol{\mathscr{r}}}\cdot\mathbf{v}}{c}\right) &= [c(t - t_r)]\left[1 - \frac{(\mathbf{r} - \mathbf{v}t_r)}{c(t - t_r)}\cdot\frac{\mathbf{v}}{c}\right] \\
&= c(t - t_r) - \frac{\mathbf{v}\cdot\mathbf{r}}{c} + \frac{v^2 t_r}{c} \\
\text{take } \frac{1}{c} \text{ out to get a familiar term} \qquad &= \frac{1}{c}[(c^2 t - \mathbf{r}\cdot\mathbf{v}) - (c^2 - v^2)t_r] \\
\text{substituting } t_r \text{from equation (26)} \qquad &= \frac{1}{c}\sqrt{(c^2 t - \mathbf{r}\cdot\mathbf{v})^2 + (c^2 - v^2)(r^2 - c^2 t^2)}
\end{aligned}
$$

$$
\therefore c\mathscr{r} - \boldsymbol{\mathscr{r}}\cdot\mathbf{v} = \sqrt{(c^2 t - \mathbf{r}\cdot\mathbf{v})^2 + (c^2 - v^2)(r^2 - c^2 t^2)} \tag{27}
$$

Finally, we substitute this term into our standard Liénard-Wiechert potentials:

$$
\boxed{V(\mathbf{v}, t) = \left(\frac{qc}{4\pi\epsilon_0}\right)\left(\frac{1}{\sqrt{(c^2 t - \mathbf{r}\cdot\mathbf{v})^2 + (c^2 - v^2)(r^2 - c^2 t^2)}}\right)} \tag{28}
$$

[20] Note that we definitively use $-$ here instead of $\pm$ since, at $v = 0$, $t_r = t - r/c$ and not $t + r/c$.

and

$$\boxed{\mathbf{A}(\mathbf{r},t) = \left(\frac{\mu_0 qc}{4\pi}\right)\left(\frac{\mathbf{v}}{\sqrt{(c^2t-\mathbf{r}\cdot\mathbf{v})^2+(c^2-v^2)(r^2-c^2t^2)}}\right)} \tag{29}$$

4.2 Liénard-Wiechert fields at constant velocity

The case of the Liénard-Wiechert **E** and **B** fields generated by a charge moving with a constant velocity is particularly interesting. Although this is named after them and although we will be working from the equations they came up with, to wit, equations (28) and (29), it is worth noting that this particular case was worked out by Oliver Heaviside (see appendix C5) just shy of a decade earlier.

Recall equation (24) which said

$$\mathbf{E}(\mathbf{r},t) = \left(\frac{q\mathscr{r}}{4\pi\epsilon_0\,(\boldsymbol{\mathscr{r}}\cdot\mathbf{w})^3}\right)\left[(c^2-v^2)\mathbf{w}+\boldsymbol{\mathscr{r}}\times(\mathbf{w}\times\mathbf{a})\right]$$

Setting the acceleration to zero ($\mathbf{a}=0$) and assuing that the charge is moving with some velocity $\mathbf{v}$ so that the position vector $\mathbf{r}\to\mathbf{v}t$ for the same reason as in §4.1,

$$\mathbf{E}(\mathbf{r},t) = \frac{q(c^2-v^2)\mathscr{r}\mathbf{w}}{4\pi\epsilon_0\,(\boldsymbol{\mathscr{r}}\cdot\mathbf{w})^3} = \frac{q(c^2-v^2)\mathscr{r}(c\hat{\boldsymbol{\mathscr{r}}}-\mathbf{v})}{4\pi\epsilon_0\,(\boldsymbol{\mathscr{r}}\cdot\mathbf{w})^3} = \frac{q(c^2-v^2)\mathscr{r}\mathbf{w}}{4\pi\epsilon_0\,(\boldsymbol{\mathscr{r}}\cdot\mathbf{w})^3}$$

Notice that we have the two terms $\mathscr{r}\mathbf{w}$ and $\boldsymbol{\mathscr{r}}\cdot\mathbf{w}$ that we need to simplify. Therefore, $\mathscr{r}\mathbf{w} = c\boldsymbol{\mathscr{r}} - \mathscr{r}\mathbf{v} = c(\mathbf{r}-\mathbf{v}t) - c(t-t_r)\mathbf{v} = c(\mathbf{r}-\mathbf{v}t) = c\mathbf{R}$. (See the term $\mathbf{R}$ defined after equation (27) below.)

$$\Rightarrow\ \mathbf{E} = \frac{q(c^2-v^2)c\mathbf{R}}{4\pi\epsilon_0\,(\boldsymbol{\mathscr{r}}\cdot\mathbf{w})^3} \tag{30}$$

And for the second one, we can use the result in equation (27).

$$c\mathscr{r} - \boldsymbol{\mathscr{r}}\cdot\mathbf{v} = \sqrt{(c^2t-\mathbf{r}\cdot\mathbf{v})^2+(c^2-v^2)(r^2-c^2t^2)} \tag{27}$$

However, we note that the right hand term can be shown to equal something other than the left hand term as well. Since we previously defined $\boldsymbol{\mathscr{r}} = \mathbf{r}-\mathbf{v}t_r$ particularly at the retarded time, we can also define some $\mathbf{R} = \mathbf{r}-\mathbf{v}t$ at any time, t.

$$\mathbf{R} = \mathbf{r}-\mathbf{v}t \Rightarrow \mathbf{R}\cdot\mathbf{v} = \mathbf{r}\cdot\mathbf{v} - v^2t \Rightarrow \mathbf{r}\cdot\mathbf{v} = \mathbf{R}\cdot\mathbf{v} + v^2t \tag{31}$$

$$\text{also,}\qquad R^2 = r^2 + v^2t^2 - 2\mathbf{r}\cdot\mathbf{v} \Rightarrow r^2 = R^2 - v^2t^2 + 2\mathbf{r}\cdot\mathbf{v} \tag{32}$$

$$\text{Substituting equation (31) in (32)}\qquad r^2 = R^2 + v^2t^2 + 2\mathbf{R}\cdot\mathbf{v}t \tag{33}$$

Using equations (31) and (33) in the argument (i.e. the term inside the square root) in equation (27) above,

$$\begin{aligned}(c^2t-\mathbf{r}\cdot\mathbf{v})^2+(c^2-v^2)(r^2-c^2t^2) &= (c^2t-\mathbf{R}\cdot\mathbf{v}-v^2t)^2 - [(c^2-v^2)\\ &\qquad (R^2+(v^2-c^2)t^2+2\mathbf{R}\cdot\mathbf{v}t)]\\ &= (\mathbf{R}\cdot\mathbf{v})^2+(c^2-v^2)R^2\end{aligned}$$

If we have an angle of ϕ between $\mathbf{R}$ and $\mathbf{v}$, then

$$
\begin{aligned}
(c^2t - \mathbf{r}\cdot\mathbf{v})^2 + (c^2 - v^2)(r^2 - c^2t^2) &= (Rv\cos\phi)^2 + (c^2 - v^2)R^2 \\
&= R^2\left[c^2 - v^2\left(1 - cos^2\phi\right)\right] \\
&= R^2c^2\left[1 - \frac{v^2}{c^2}(\sin^2\phi)\right]
\end{aligned} \tag{34}
$$

Using equation (34) in (30),

$$
\begin{aligned}
\mathbf{E} &= \frac{q\,(c^2 - v^2)c\mathbf{R}}{4\pi\epsilon_0\left(R^2c^2\left[1 - \frac{v^2}{c^2}(\sin^2\phi)\right]\right)^{\frac{3}{2}}} \\
&= \frac{q\,(1 - \frac{v^2}{c^2})c\hat{\mathbf{R}}R}{4\pi\epsilon_0\left[1 - \frac{v^2}{c^2}\sin^2\phi\right]^{\frac{3}{2}}R^3}
\end{aligned}
$$

$$
\therefore \boxed{\mathbf{E}(\mathbf{r},t) = \left(\frac{q}{4\pi\epsilon_0}\right)\left[\frac{\left(1 - \frac{v^2}{c^2}\right)}{\left(1 - \frac{v^2}{c^2}\sin^2\phi\right)^{\frac{3}{2}}}\right]\left(\frac{\hat{\mathbf{R}}}{R^2}\right)} \tag{35}
$$

which gives us the Liénard-Wiechert elctric field of a point charge that is moving with a constant velocity.

From equation (25) we know that

$$
\mathbf{B}(\mathbf{r},t) = \frac{1}{c}\,\hat{\boldsymbol{\mathscr{r}}}\times\mathbf{E} \tag{25}
$$

We try to bring in $\mathbf{R}$ by adding and subtracting $\mathbf{v}t$ into the definition of $\hat{\boldsymbol{\mathscr{r}}}$.

$$
\begin{aligned}
\hat{\boldsymbol{\mathscr{r}}} = \frac{\mathbf{r} - \mathbf{v}t_r}{\mathscr{r}} &= \frac{\mathbf{r} - \mathbf{v}t_r + (\mathbf{v}t - \mathbf{v}t)}{\mathscr{r}} \\
&= \frac{(\mathbf{r} - \mathbf{v}t) + (t - t_r)\mathbf{v}}{\mathscr{r}}
\end{aligned}
$$

And, since $\mathbf{R} = (\mathbf{r} - \mathbf{v}t)/\mathscr{r}$ and $\mathscr{r} = c\,(t - t_r)$ we get

$$
\hat{\boldsymbol{\mathscr{r}}} = \frac{\mathbf{R}}{\mathscr{r}} + \frac{\mathbf{v}}{c}
$$

Therefore, equation (25) becomes[21]

$$
\boxed{\mathbf{B}(\mathbf{r},t) = \frac{1}{c^2}\,\mathbf{v}\times\mathbf{E}} \tag{36}
$$

giving us the Liénard-Wiechert magnetic field of a point charge that is moving with a constant velocity.

[21] In substituting $\hat{\boldsymbol{\mathscr{r}}}$ into equation (25) we use the fact that a cross product is distributive over addition to show that $\mathbf{R}\times\mathbf{R}$ sends the first term to zero and we are only left with the $\mathbf{v}/c$ term in our definition of $\hat{\boldsymbol{\mathscr{r}}}$ and, therefore, $\mathbf{B}\propto\mathbf{v}\times\mathbf{E}$.

4.3 Some observations on the constant velocity case

The constant velocity case is critical in helping us understand more about the Liénard-Wiechert potentials and fields in addition to what we discussed in §3.2.

1. At $v^2 \ll c^2$, equations (35) and (36) reduce to

$$\mathbf{E} = \frac{1}{4\pi\epsilon_0}\frac{q}{R^2}\hat{\mathbf{R}} \qquad \text{(Coulomb law)}$$

$$\mathbf{B} = \frac{\mu_0}{4\pi}\frac{q}{R^2}(\mathbf{v}\times\hat{\mathbf{R}}) \qquad \text{(Biot-Savart law)}$$

2. As is evident from equations (35) and (36), the fields under the constant velocity case are dependent only on the *present time* of the particle as defined with $\mathbf{R}$ around equation (31).

3. The $\hat{\mathbf{R}}$ term in equation (35) tells us that the electric field goes radially outwards starting *from the point charge.*

4. For some given angle, ϕ, between $\hat{\mathbf{R}}$ and $\mathbf{v}$, the ϕ dependence in equations (35) and (36) tells us that, as $v^2/c^2 \to 1$, $\mathbf{E}$ and $\mathbf{B}$ increase as $\sin\phi$ increases (since $[1-(v^2/c^2)\sin\phi]$ drops).

   This means, as the velocity of a charge approaches c, its electric and magnetic fields flatten out in a direction perpendicular to that of motion, with the field lines (and hence the field) increasing in the direction perpendicular to the line of propagation and decreasing in the direction along the line of propagation, giving us the picture of an electromagnetic wave we are quite familiar with.

5. The fact that $\mathbf{B}$ *curls* means the magnetic field lines encircle the charge. Specifically, they curl around the direction of propogation of the charge as suggested by Ampère's right hand screw rule.

Appendices

A Proofs for spatial derivatives

The following proofs assume three dimensions and define $\nabla := \hat{\mathbf{x}}_i \frac{\partial}{\partial x_i}$ and $\mathbf{r} = x_i \hat{\mathbf{x}}_i$, which are indeed their conventional definitions.

The first three are straightforward and can be solved in cartesian coördinates.

1. $\nabla \mathscr{r} = \hat{x}_i \frac{\partial x_i}{\partial x_i} = \hat{x}_i = \hat{\boldsymbol{\mathscr{r}}}$
2. $\nabla \frac{1}{\mathscr{r}} = \hat{x}_i \frac{\partial}{\partial x_i} \left(\frac{1}{x_i} \right) = \frac{-\hat{x}_i}{x_i^2} = -\frac{\hat{\boldsymbol{\mathscr{r}}}}{\mathscr{r}^2}$
3. $\nabla \cdot \boldsymbol{\mathscr{r}} = \hat{x}_i \hat{x}_i \frac{\partial x_i}{\partial x_i} = 3$

The fourth needs its vector, $\hat{\boldsymbol{\mathscr{r}}}$, to be solved in spherical polar coördinates.

4. $\nabla \cdot \frac{\hat{\boldsymbol{\mathscr{r}}}}{\mathscr{r}^2} = \frac{1}{\mathscr{r}^2} \frac{\partial}{\partial \mathscr{r}} \hat{\boldsymbol{\mathscr{r}}} = \frac{1}{\mathscr{r}^2} \frac{\partial}{\partial \mathscr{r}} \left(\mathscr{r}^2 \frac{1}{\mathscr{r}^2} \right) = \frac{1}{\mathscr{r}^2} \frac{\partial}{\partial \mathscr{r}} (1) = 0$

We have used, for $\mathbf{v} \equiv (v_r, v_\theta, v_\phi)$, the spherical polar definition of divergence (see appendix B1) for $\mathbf{v} = \frac{1}{r^2}$ and ignored the θ and ϕ terms since we do not have them anyway.

Observe that the presence of r^2 in the denominator means this is invalid for $r = 0$, the centre of the potential field and the location of the charge itself. To correct for this we use Gauss' theorem over the entire "volume" of the charge:

$$\int\limits_{\text{vol}} (\nabla \cdot \mathbf{v}) \, \mathrm{d}\tau = \oint\limits_{\text{surface}} \mathbf{v} \cdot \mathrm{d}A$$

Therefore, as is usual in spherical polar coördinates, with the surface integral on the right hand side referring to the surface of a sphere, we have,

$$\begin{aligned} \int\limits_{\text{vol}} (\nabla \cdot \frac{\hat{\boldsymbol{\mathscr{r}}}}{\mathscr{r}^2}) \, \mathrm{d}\tau &= \oint\limits_{\text{radius } \mathscr{r}} \left(\frac{\hat{\boldsymbol{\mathscr{r}}}}{\mathscr{r}^2} \right) \cdot \left(\mathscr{r}^2 \sin\theta \, \mathrm{d}\theta \, \mathrm{d}\phi \right) \\ &= \int\limits_{\theta=0}^{\pi} \sin\theta \, \mathrm{d}\theta \cdot \int\limits_{\phi=0}^{2\pi} \mathrm{d}\phi \\ &= -\cos\theta \Big|_0^{\pi} \cdot 2\pi \\ &= 4\pi \end{aligned}$$

This leaves us with a Dirac delta function making $\nabla \cdot \frac{\hat{\boldsymbol{\mathscr{r}}}}{\mathscr{r}^2} = 4\pi\delta^3(\boldsymbol{\mathscr{r}})$ so that the integral is satisfied and, further, the left hand side tends to zero as we go far away from the charge (we already know it reduces as $1/\text{radius}^2$), or, $\delta^3(\boldsymbol{\mathscr{r}}) = 0$ when $\boldsymbol{\mathscr{r}} \neq 0$ and vice versa, where $\int_{-\infty}^{+\infty} \delta^3(\boldsymbol{\mathscr{r}}) \, \mathrm{d}\tau = 1$ when we are pointing to the origin, or, strictly speaking, when $\boldsymbol{\mathscr{r}} = 0$. This, of course, is the normal definition of the Dirac delta function.

5. As above, we solve this first in spherical polar coördinates (see appendix B2) only to obtain a trivial result: $\nabla^2 \frac{1}{\mathscr{r}} = \frac{1}{\mathscr{r}^2}\frac{\partial}{\partial \mathscr{r}}\left(\mathscr{r}^2 \frac{\partial}{\partial \mathscr{r}}\left(\frac{1}{\mathscr{r}}\right)\right) = \frac{1}{\mathscr{r}^2}\frac{\partial}{\partial \mathscr{r}}(-1) = 0$.

   We turn to the Poisson equation to solve this (see the final result in appendix B4). We have

   $$\nabla^2 \varphi = q\nabla^2 \frac{1}{\mathscr{r}} = -4\pi\rho = -4\pi q \delta^3(\mathscr{r})$$

   In other words,

   $$\nabla^2 \frac{1}{\mathscr{r}} = -4\pi\delta^3(\mathscr{r})$$

B Some useful rules

There are many rules of derivatives, vector multiplication and so on that we use in physics, and this neither is nor needs to be a full list of all such rules. Below are identified a few that are useful for our current discussion. Proofs of these results have been deemed to be out of the scope of this document, but they may be added later if I change my mind.

1. **Divergence of a vector in spherical polar coördinates**

   $$\nabla \cdot \mathbf{v} = \frac{1}{r^2}\frac{\partial}{\partial r}(r^2 v_r) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(v_\theta \sin\theta) + \frac{1}{r\sin\theta}\frac{\partial v_\phi}{\partial\phi}$$

2. **Laplacian in spherical polar coördinates**

   $$\nabla^2 a = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial a}{\partial r}\right) + \frac{1}{r^2 \sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta \frac{\partial a}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2 a}{\partial\phi^2}$$

3. **Gauss theorem** (or the *divergence theorem* or *Ostrogradsky's theorem*) tells us that the flux of a vector field through a closed surface ($\mathrm{d}A$) is the net divergence throughout the volume ($\mathrm{d}\tau$) enclosed by that surface.

   $$\int\limits_{\text{vol}} (\nabla \cdot \mathbf{v})\,\mathrm{d}\tau = \oint\limits_{\text{surface}} \mathbf{v} \cdot \mathrm{d}A$$

   Often the surface is spherical. Essentially, this is a standard conservation law that says the net of some *quantity* within a region (add up the sources, subtract the sinks) tells us how much of that *quantity* can flow outside the region. These are equal because no "new" *quantity* can be created.

4. **Poisson's equation** Working in CGS units, in electrostatics, $\nabla \cdot \mathbf{E} = 4\pi\rho$ for a charge density of ρ and electrostatic field $\mathbf{E}$. Also, $\nabla \times \mathbf{E} = 0$, which means $\exists\, \varphi \,\big|\, \mathbf{E} = -\nabla\varphi$. Therefore,

   $$\nabla \cdot \mathbf{E} = \nabla \cdot (-\nabla\varphi) = -\nabla^2\varphi = 4\pi\rho$$

where ρ for a point charge can be defined as $\rho = q\delta^3(\mathscr{r})$. We use the same login in equation (7) in §2.1 too. This simply means the charge density is q where a charge q lies (i.e. at $\mathscr{r} = 0$) and is zero everywhere else. (Integrating over the volume will quickly prove this.)

Coulomb's law tells us that $\mathbf{E} = \frac{q}{\mathscr{r}^2}$. From spatial derivative 3, we realise that $\mathbf{E} = q\left(\frac{1}{\mathscr{r}^2}\right) = -q\nabla\frac{1}{\mathscr{r}}$. Therefore, $\mathbf{E} = -\nabla\varphi \Rightarrow \varphi = \frac{q}{\mathscr{r}}$ giving us

$$\nabla^2 \frac{q}{\mathscr{r}} = -4\pi\rho$$

5. **Product laws**

   (a) $\nabla(\mathbf{X} \cdot \mathbf{Y}) = [\mathbf{X} \times (\nabla \times \mathbf{Y}) + (\mathbf{X} \cdot \nabla)\mathbf{Y}] + [\mathbf{Y} \times (\nabla \times \mathbf{X}) + (\mathbf{Y} \cdot \nabla)\mathbf{X}]$

   (b) $\mathbf{X} \times (\mathbf{Y} \times \mathbf{Z}) = \mathbf{Y}(\mathbf{X} \cdot \mathbf{Z}) - \mathbf{Z}(\mathbf{X} \times \mathbf{Y})$

   (c) $\nabla \times (\mathbf{X} \times \mathbf{Y}) = [(\mathbf{Y} \cdot \nabla)\mathbf{X} - (\mathbf{X} \cdot \nabla)\mathbf{Y}] + [\mathbf{X}(\nabla \cdot \mathbf{Y}) - \mathbf{Y}(\nabla \cdot \mathbf{X})]$

   (d) $\mathbf{X} \times (\nabla \times \mathbf{Y}) = \nabla(\tilde{\mathbf{X}} \cdot \mathbf{Y}) - (\nabla \cdot \tilde{\mathbf{X}})\mathbf{Y}$

   (e) $X(\mathbf{Y} \times \mathbf{Z}) = (X\mathbf{Y} \times \mathbf{Z}) = \mathbf{Y} \times (X\mathbf{X}) = (\mathbf{Y} \times \mathbf{Z})X$

   (f) $\nabla \times (X\mathbf{Y}) = X(\nabla \times \mathbf{Y}) - \mathbf{Y} \times (\nabla X)$

   Notice the patterns here. The $\tilde{\mathbf{X}}$ terms are *not* to be differentiated (this is apparently something Feynman came up with to keep the $\mathbf{X} \times (\nabla \times \mathbf{Y})$ formula in tune with the famous "BAC-CAB" rule, although I only vaguely remember reading this somewhere and could be wrong.)

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