16MSPAH101 — ADVANCED CLASSICAL MECHANICS

INTRODUCTION TO HAMILTONIAN MECHANICS

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1 THE LEGENDRE TRANSFORMATION

WE BEGIN OUR discussion of Hamiltonian mechanics with the a quick look at a mathematical technique known as the Legendre transformation. The Legendre transformation for some function $f(u_1, u_2, \dots, u_n)$ generates another function $f'(u'_1, u'_2, \dots, u'_n)$ where u' and u are related as

$$u'_i = \frac{\partial f}{\partial u_i}$$

and may be done using

$$f' = u_i u'_i - f \tag{1}$$

which is quite straightforward. Firstly this tells us that f' is only a function of u_i and not, interestingly, of u'_i. To see this consider the differential of the transformed function:

These transformations are the work of the XVIII century french mathematician Adrien-Marie Legendre.

$$\begin{aligned} \mathrm{d}f' &= u_i \mathrm{d}u'_i + u'_i \mathrm{d}u_i - \frac{\partial f}{\partial u_i}\mathrm{d}u_i \\ &= u_i \mathrm{d}u'_i + \left(u'_i - \frac{\partial f}{\partial u_i}\right)\mathrm{d}u_i \end{aligned}$$

However, since $u'_i = \partial f/\partial u_i$ as stated above we have simply

$$\mathrm{d}f' = u_i \mathrm{d}u'_i$$

which means f' is not a function of u', just of u.

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Secondly, we ask what would happen if f' was in fact a function of u'? Its differential would then be given by

$$\mathrm{d}f' = \frac{\partial f'}{\partial u'_i}\,\mathrm{d}u'_i$$

Compare the two equations we have for $\mathrm{d}f'$ now and it is easy to see that

$$u_i = \frac{\partial f'}{\partial u'_i}$$

In other words, this brings us back to the first equation we wrote down above and, consequently, a symmetric result follows eq. (1) giving us

$$f = u_i u'_i - f' \tag{2}$$

a sort of 'reverse' Legendre transformation.

Keep in mind that everywhere you see repeating indices, i.e. to instances of the same subscript, summation is implied. So, for example, eq. (1) is actually written as

$$f' = \sum_i^n u_i u'_i - f$$

but we omit the $\sum$ as agreed upon in Einstein's summation convention.

Lastly, we need to account for the possibility that $f \equiv f(u, v)$ where the us transform but the vs do not. In this case, as we did before, consider the differential of the transformed function:

$$\begin{aligned}
\mathrm{d}f' &= (u_i \mathrm{d}u'_i + u'_i \mathrm{d}u_i) - \mathrm{d}f(u, v) \\
&= (u_i \mathrm{d}u'_i + u'_i \mathrm{d}u_i) - \frac{\partial f}{\partial u_i}\mathrm{d}u_i - \frac{\partial f}{\partial v_i}\mathrm{d}v_i \\
\therefore \mathrm{d}f' &= u_i \mathrm{d}u'_i - \frac{\partial f}{\partial v_i}\mathrm{d}v_i
\end{aligned}$$

where the two terms get cancelled in the last step for the same reasons as they did earlier, telling us that f' is a function of u and v both. Once again like we did before we now consider what would happen in f' was in fact a function of u' too. Its differential would then become

$$\mathrm{d}f' = \frac{\partial f'}{\partial u_i}\mathrm{d}u_i + \frac{\partial f'}{\partial v_i}\mathrm{d}v_i$$

Compare the two equations we now have for $\mathrm{d}f'$ and you will see that

$$\frac{\partial f}{\partial v_i} = -\frac{\partial f'}{\partial v'_i} \tag{3}$$

This is the relation between any non-transforming variable, here v, and its transformed functions, $f \leftrightarrow f'$ undergoing a Legendre transformation.

2 THE HAMILTONIAN

So far the Legendre transformation has largely been mathematical; it is now time for us to give it a more physical form. To start with, consider the Lagrangian of some system: $\mathscr{L}(q_k, \dot{q}_k, t)$. Following our discussion on the Legendre transformation we can transform $\mathscr{L}$ into some $\mathscr{H}$ using a transformation that involves the variable $\dot{q}_k$ alone. So our non-participants are q_k and t and our transformed function then is

$$\mathscr{H} \equiv \mathscr{H}(q_k, \partial\mathscr{L}/\partial\dot{q}_k, t)$$

However we know that $\partial\mathscr{L}/\partial\dot{q}_k$ is nothing but the generalised momentum p_k which means the function is actually given by

$$\mathscr{H} \equiv \mathscr{H}(q_k, p_k, t)$$

So if $\mathscr{L} \to \mathscr{H}$ we have, from eq. (1), the transformation

$$\mathscr{H}(q_k, p_k, t) = p_k\dot{q}_k - \mathscr{L}(q_k, \dot{q}_k, t) \tag{4}$$

where the function $\mathscr{H}$ is known as the **Hamiltonian**.

How did we choose u_i and u_i' in eq. (4)? Recall that u_i and u_i' are related by $u_i' = \partial f/\partial u_i$ which, here, refers to $p_k = \partial\mathscr{L}/\partial\dot{q}_k$ making $u_i \equiv \dot{q}_k$ and $u_i' \equiv p_k$. As before, summation is implicit.

3 THE CANONICAL EQUATIONS

The reversible nature of the Legendre transformation described earlier allows us to use the relation

$$p_k = \frac{\partial\mathscr{L}}{\partial\dot{q}_k} \tag{5}$$

to similarly transform from $\dot{q}_k$ to p_k, but this time via $\mathscr{H}$, as

$$\dot{q}_k = \frac{\partial\mathscr{H}}{\partial p_k} \tag{5}$$

At this point let us not forget that the Lagrangian is made up of two other non-participating variables q_k and t. Their relations to the transformed function is given by eq. (3) as

$$\frac{\partial\mathscr{H}}{\partial q_k} = -\frac{\partial\mathscr{L}}{\partial q_k} \qquad \text{and} \qquad \frac{\partial\mathscr{H}}{\partial t} = -\frac{\partial\mathscr{L}}{\partial t} \tag{6}$$

Now recall the Euler-Lagrange equation

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial\mathscr{L}}{\partial\dot{q}_k}\right) = \frac{\partial\mathscr{L}}{\partial q_k}$$

We can combine eq. (5) and (6) with the Euler-Lagrange equation to arrive at two key equations. The first of these arises from the Euler-Lagrange equation given above where the parenthetical term is replaced by the generalised momentum and written using eq. (3) as follows:

$$\dot{p}_k = \frac{\partial \mathscr{L}}{\partial q_k}$$

$$\Rightarrow \dot{p}_k = -\frac{\partial \mathscr{H}}{\partial q_k}$$

As an exercise try deriving these canonical equations from Hamilton's principle.

and the second equation, for $\dot{q}_k$, is as given by eq. (5) above. Together these simple but powerful transformation equations

$$\dot{q}_k = \frac{\partial \mathscr{H}}{\partial p_k} \quad \text{and} \quad \dot{p}_k = -\frac{\partial \mathscr{H}}{\partial q_k} \tag{7}$$

are known as **Hamilton's canonical equations**.

All of this ties up nicely if we make one final observation: use the fact that so long as the Lagrangian is time-independent i.e. $\partial \mathscr{L}/\partial t = 0$ in a conservative system we can write

$$2\mathrm{T} = \left(\frac{\partial \mathscr{L}}{\partial \dot{q}_k}\right)\dot{q}_k$$

The intermediate result used here is easy to see if we consider a Cartesian system where $q_k \equiv x$ as follows:

$$\mathscr{L}(q_k, \dot{q}_k) \equiv \mathscr{L}(x, \dot{x}) = \frac{m\dot{x}^2}{2}$$

$$\Rightarrow \frac{\partial \mathscr{L}}{\partial \dot{q}_k} = m\dot{x}$$

$$\therefore \dot{q}_k\left(\frac{\partial \mathscr{L}}{\partial \dot{q}_k}\right) = m\dot{x}^2 = 2\mathrm{T}$$

so that eq. (4) reduces to

$$\begin{aligned}
\mathscr{H} &= p_k\dot{q}_k - \mathscr{L} \\
&= \left(\frac{\partial \mathscr{L}}{\partial \dot{q}_k}\right)\dot{q}_k - \mathscr{L} \\
&= 2\mathrm{T} - (\mathrm{T} - \mathrm{V}) \\
\therefore \mathscr{H} &= \mathrm{T} + \mathrm{V} = \mathrm{E}
\end{aligned} \tag{8}$$

In other words, for conservative systems the Hamiltonian gives us the total energy of a system. This fact becomes particularly useful in the domain of quantum mechanics where, often, we simply call the total energy of a system the 'Hamiltonian' of that system.

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