16MSPAH101 — ADVANCED CLASSICAL MECHANICS

CANONICAL TRANSFORMATIONS

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Generating function	*Derivatives*	*Trivial case*
$\chi'(q, \mathrm{Q}, t)$	$p_i = \partial_{q_i}\chi'$ and $\mathrm{P}_i = -\partial_{\mathrm{Q}_i}\chi'$	$\chi' = q_i\mathrm{Q}_i$
$\chi''(q, \mathrm{P}, t) - \mathrm{Q}_i\mathrm{P}_i$	$p_i = \partial_{q_i}\chi''$ and $\mathrm{Q}_i = \partial_{\mathrm{P}_i}\chi''$	$\chi' = q_i\mathrm{P}_i$
$\chi'''(p, \mathrm{Q}, t) + q_i p_i$	$q_i = -\partial_{p_i}\chi'''$ and $\mathrm{P}_i = -\partial_{\mathrm{Q}_i}\chi'''$	$\chi' = p_i\mathrm{Q}_i$
$\chi''''(p, \mathrm{P}, t) + q_i p_i - \mathrm{Q}_i\mathrm{P}_i$	$q_i = -\partial_{p_i}\chi''''$ and $\mathrm{Q}_i = \partial_{\mathrm{P}_i}\chi''''$	$\chi' = p_i\mathrm{P}_i$

1 EXAMPLES OF CANONICAL TRANSFORMATIONS

1.1 *Some general results on q and p*

Consider the generating function $\chi'' = q_i p_i$ with the transformation equations $p_i = \partial_{q_i}\chi'' = \mathrm{P}_i$ and $\mathrm{Q}_i = \partial_{\mathrm{P}_i}\chi''$ with $\mathrm{K} = \mathrm{H}$ since $\partial_t\chi'' = 0$. This is the trivial case of the χ'' listed above. We already know that $\mathrm{P}_i = \partial_{q_i}\chi''$ and $q_i = \partial_{\mathrm{P}_i}\chi''$ which means $p_i = \mathrm{P}_i$ and $q_i = \mathrm{Q}_i$, leaving the new coördinates untouched.

The general form of this arises from a generating function containing $f(q,t)$ as expected but along with some $g(q,t)$, a differentiable function of q and possibly t. That is $\chi'' = f_i(q_1, \ldots, q_n; t)\mathrm{P}_i + g(q_1, \ldots, q_n; t)$ which, when $g = 0$, satisfies $\mathrm{Q}_i = \partial_{\mathrm{P}_i}\chi'' = f_i(q_1, \ldots, q_n; t)$ and tells us that point transformations are always canonical.

When $g \neq 0$ we end up with writing $p_i = \partial_{q_i}\chi''$ as $p_j = \mathrm{P}_i\partial_{q_j}f_i + \partial_{q_j}g$. These are essentially matrices with i rows and j columns, which means we could rearrange this equation to get $\mathrm{P} = \left[\partial_q f\right]^{-1}\left(p - \partial_q g\right)$.

As a second exercise consider $\chi' = q_i\mathrm{Q}_i$ which, as evinced by the single prime, is what we have defined as a generating function of the first kind. This trivially transforms surprisingly $p_i = \partial_{q_i}\chi' = \mathrm{Q}_i$ and $\mathrm{P}_i = -\partial_{\mathrm{Q}_i}\chi' = -q_i$ which tells us that p and q are interconvertible. This result is ample proof that q and p differ only in name and that, celebrating the generality of analytical mechanics, they can transform into each other with considerable ease.

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In a system with, say, two degrees of freedom with a generating function that is a combination like $\chi' + \chi'' = q_1 Q_1 + q_2 Q_2$ we end up with the relations $Q_1 = q_1$, $P_1 = p_1$, $Q_2 = p_2$ and $P_2 = -q_2$. Generating functions and their derivative coördinates and momenta are, therefore, superimposable.

1.2 *Harmonic oscillators*

This example will finally clarify just how and where we find use for the four generating functions discussed earlier. Consider a one-dimensional simple harmonic oscillator, i.e. an object simple harmonically moving to and fro along a single direction. The Hamiltonian, given by sum of the kinetic and potential energies, of such a system is known (from previous lectures) to be

$$\mathscr{H} = \frac{p^2}{2m} + k\,\frac{q^2}{2}$$

If $\omega^2 = k/m$ we have

$$\mathscr{H} = \frac{1}{2m}\left(p^2 + m^2\omega^2 q^2\right)$$

The term inside the parentheses looks suspiciously like it is waiting to be written as $\cos^2\theta + \sin^2\theta$. To achieve this an appropriate substitution would be

$$p = f(\mathrm{P})\cos \mathrm{Q} \qquad \text{and} \qquad qm\omega = f(\mathrm{P})\sin \mathrm{Q}$$

naturally because we have no justification to claim that $p = \cos \mathrm{Q}$ straight away. The correction function $f(\mathrm{P})$ is of course unknown. Substituting these we have

$$\mathscr{H} = \frac{f^2(\mathrm{P})}{2m} = \mathscr{K}$$

and the only thing now left to do is to find $f(\mathrm{P})$.

This is where our generating functions come in. The first type, χ', is of the form $\chi'(q, \mathrm{Q}, t)$ with the trivial case $\chi' = q_i \mathrm{Q}_i$ so that $\mathrm{Q}_i = p_i$ and $\mathrm{P}_i = -q_i$. These give rise to the following generating function:

$$\begin{aligned}
\chi' &= q_i \mathrm{Q}_i \\
&= q_i p_i \\
&= \frac{f^2(\mathrm{P})\sin \mathrm{Q}\cos \mathrm{Q}}{m\omega} \\
&= \frac{q^2 m^2\omega^2 \sin \mathrm{Q}\cos \mathrm{Q}}{\sin^2 \mathrm{Q} m\omega} \\
&= q^2 m\omega \cot \mathrm{Q}
\end{aligned}$$

Recall that $\partial_\theta \cot\theta = -\csc\theta$.

We now use this in $p = \partial_q \chi'$ and $\mathrm{P} = -\partial_{\mathrm{Q}} \chi'$ to arrive at

$$p = 2m\omega \cot \mathrm{Q} \qquad \text{and} \qquad \mathrm{P} = \frac{m\omega q^2}{\sin^2 \mathrm{Q}}$$

which gives us

$$q = \sin \mathrm{Q}\sqrt{\frac{\mathrm{P}}{m\omega}} \qquad \text{and} \qquad p = 2\cos \mathrm{Q}\sqrt{pm\omega}$$

Comparing this with $p = f(\mathrm{P})\cos\mathrm{Q}$ we get $f(\mathrm{P}) = 2\sqrt{\mathrm{P}m\omega}$ and the new Hamiltonian becomes

$$\mathscr{K} = \frac{f^2(\mathrm{P})}{2m} = 2\mathrm{P}\omega$$

which, on further comparison to $\mathrm{P} = \mathrm{E}/m$ tells us the factor of 2 is extra and that our generating function needs to be corrected accordingly. Suppose we take the generating function

$$\chi' = \frac{q^2 m\omega \cot\mathrm{Q}}{2}$$

we will end up with $\mathscr{K}(\mathrm{P},\mathrm{Q}) = \omega\mathrm{P}$ as expected, signalling that this is in fact the correct generating function. The new Hamiltonian is **cyclic in Q**. Consequently, Hamilton's equation for Q is

$$\dot{\mathrm{Q}} = \partial_{\mathrm{P}}\mathrm{K} = \omega \implies \mathrm{Q} = \omega t + \beta \text{ (say)}$$

where β is a constant of integration, and, on substituting this back into the equation for q above we get

$$q = \sin(\omega t + \beta)\sqrt{\frac{2\mathrm{E}}{m\omega^2}}$$

The additional factor of 2 comes once we rework the previous steps with the corrected generating function (which, if you recall, itself differs from our originally expected function by a factor of half).

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