

## Chapter 9: Introduction to Control

August 20, 2015

# Outline

## 1 Basics

- Control Theory
- Demo: Inverted Pendulum

## 2 Control Goals

- Examples
- Exercise

## 3 Closed-loop systems

- Sensitivity - Robustness
- Types of systems and Steady State Error
- Noise and disturbance rejection

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- Control Theory
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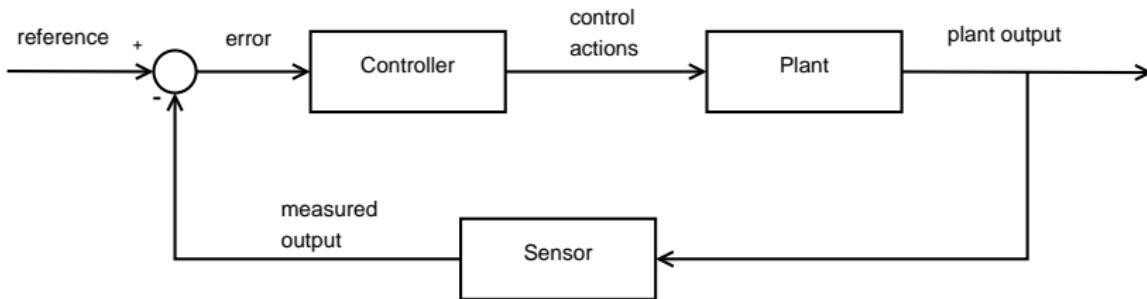
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# What is control theory?

**Control theory** is an interdisciplinary branch of engineering and mathematics that deals with the behavior of dynamical systems with inputs, and how their behavior is modified by feedback. The usual objective of control theory is to control a system, often called plant, so its output follows a desired reference signal which may be a fixed or changing value.



# Controllers

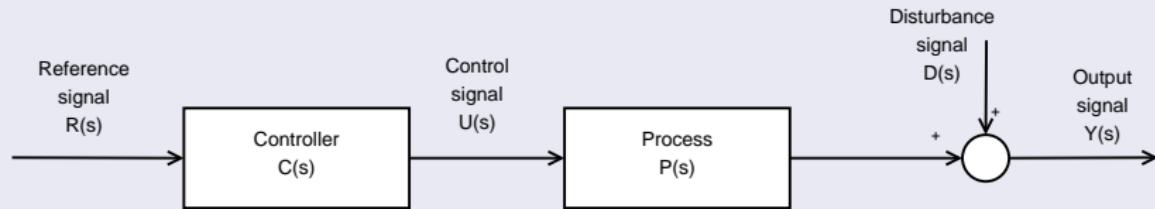
## Types of controllers

- On-off controllers
  - e.g., Thermostat at home
- **PID controllers, Lead and lag compensators (this course)**
  - Cruise-control in your car
  - Temperature, level, flow, pressure, and pH control in chemical plants
- More advanced controllers
  - State-space feedback controllers (e.g., LQR)
  - Model Predictive Controllers (MPCs)
  - Fuzzy Controllers
  - Neuro-fuzzy Controllers
  - ...

# Open-loop System

## Definition

In an open loop control system the actual output signal  $Y(s)$  has no effect on the control action  $U(s)$ .



$$Y(s) = P(s)U(s) = P(s)C(s)R(s)$$

# Open-loop System

## Example

- You are pouring a glass of water, but you **cannot look at the glass**.
- The desired output is a full glass of water within a reasonable time.
- The input can have two values: on or off (assume a quite primitive tap).
- It will not be easy to do this successfully.



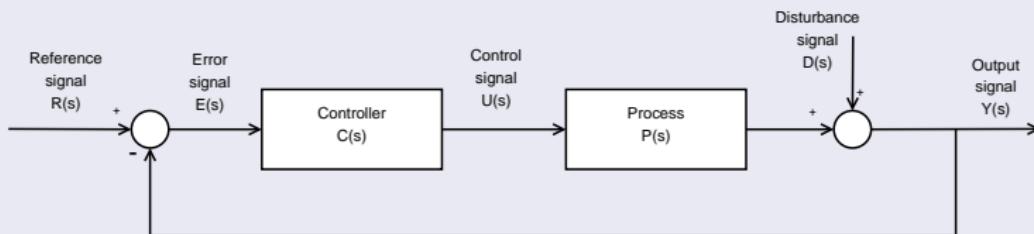
The solution is evident: look at the glass while pouring!

# Closed-loop system

## Definition

In a closed-loop system the error signal  $E(s)$ , which is the difference between the process output  $Y(s)$  and the reference  $R(s)$ , is fed to the controller which generates the proper control actions  $U(s)$  to reduce the error and bring the process output to its desired value.

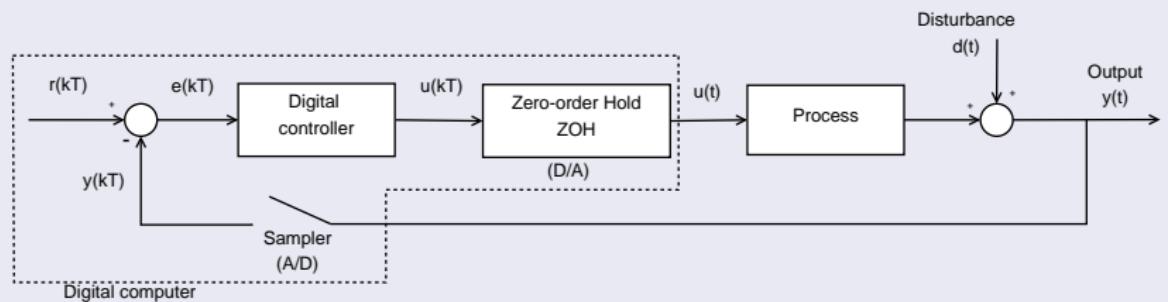
## Classical control-loop



$$Y(s) = \frac{P(s)C(s)}{1 + P(s)C(s)}R(s) \quad \text{when } D(s) = 0$$



## Digital Control Loop



# Outline

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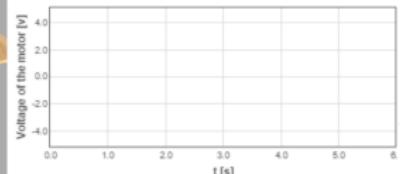
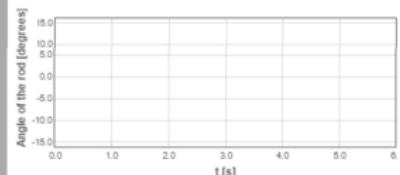
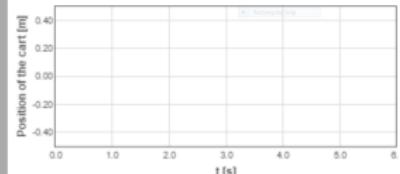
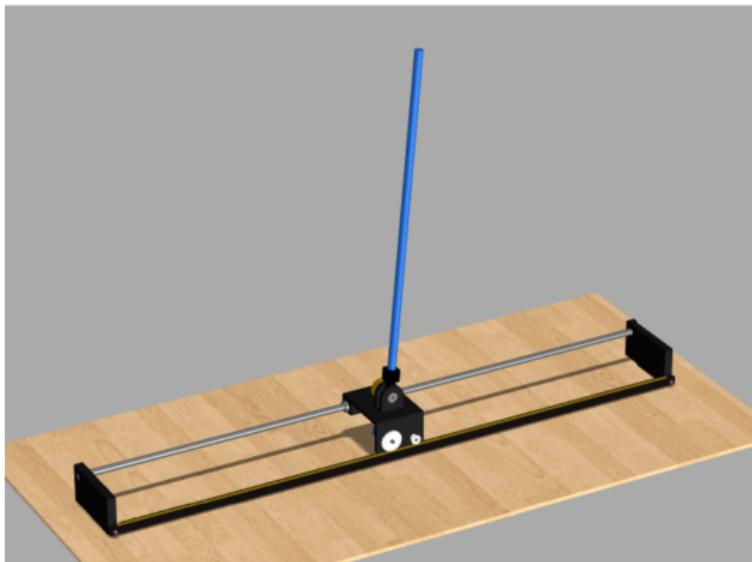
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## Virtual Lab: Inverted Pendulum



Simulation	Setpoint of the cart [m]	Disturbance	Initial conditions	Camera view	Controller
<button>Start</button> <button>Stop</button> <button>Reset</button>	<input checked="" type="radio"/> Random Steps <input type="radio"/> Manual : 0	<input type="button" value="Push ⏪"/> <input type="button" value="Push ⏩"/>	Cart position [m]: 0 Rod angle [deg]: 5	<input type="button" value="Default"/> <input type="button" value="Front"/> <input type="button" value="Top"/>	$u(t) = -K(x(t) - xd(t))$ $k_1: -9.1287 \quad k_2: -52.4975$ $k_3: -14.9912 \quad k_4: -6.3533$ <input type="button" value="Default"/>

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# What is good control?

Good control depends on the application

## Control Goals

- Stability
- Disturbance rejection
- Reference tracking
- Robustness
- ...

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# Examples: Stability



Space shuttles are like inverted pendulums. Control systems make sure they do not flip over.

# Examples: Disturbance rejection

- Your body will try to keep your internal temperature as constant as possible, no matter how hot/cold it is outside.



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## Examples: Reference tracking



Audi has a system for automatic driving in traffic jams. The vehicle will follow the car in front at an appropriate distance.

[https://www.youtube.com/watch?v=Qa\\_ZSRj0WM0](https://www.youtube.com/watch?v=Qa_ZSRj0WM0)

# Examples: Robustness



It is not possible to drive a formula 1 car using your knowledge of regular cars. However, you can drive a wide variety of road cars.

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# Exercise: Could you name the correct property?



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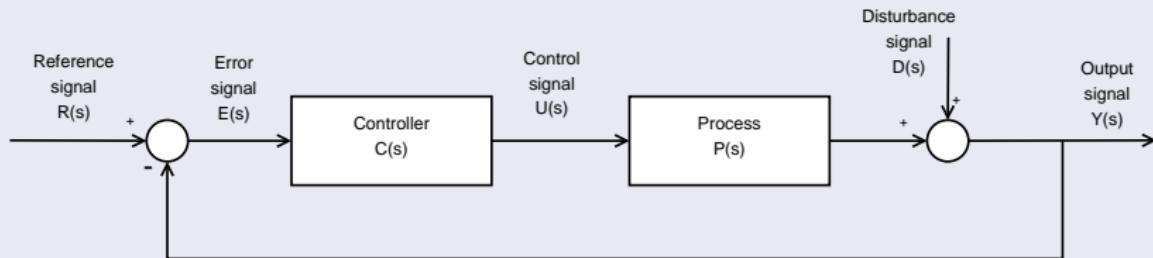
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# Transfer function of a closed-loop system

## Classical control loop



$$Y(s) - D(s) = P(s)U(s) \quad \text{with } U(s) = C(s)E(s)$$

$$Y(s) - D(s) = P(s)C(s)E(s) \quad \text{with } E(s) = R(s) - Y(s)$$

$$Y(s) - D(s) = P(s)C(s)(R(s) - Y(s))$$

$$Y(s) - D(s) = P(s)C(s)R(s) - P(s)C(s)Y(s)$$

$$Y(s) = \frac{P(s)C(s)}{1 + P(s)C(s)}R(s) + \frac{1}{1 + P(s)C(s)}D(s)$$

# Transfer function from $R(s)$ to $Y(s)$

## Definition

We define  $H(s)$  as the transfer function from  $R(s)$  to  $Y(s)$

$$H(s) \triangleq \frac{Y(s)}{R(s)} = \frac{P(s)C(s)}{1 + P(s)C(s)} \quad \text{when } D(s) = 0.$$

- This transfer function  $H(s)$  will help us to evaluate tracking
- Almost perfect tracking: the output  $Y(s)$  will follow  $R(s)$  very closely  $\Rightarrow H(s) \approx 1$

# Transfer function from $D(s)$ to $Y(s)$

## Definition

We define  $M(s)$  as the transfer function from  $D(s)$  to  $Y(s)$

$$M(s) \triangleq \frac{Y(s)}{D(s)} = \frac{1}{1 + P(s)C(s)} \quad \text{when } R(s) = 0.$$

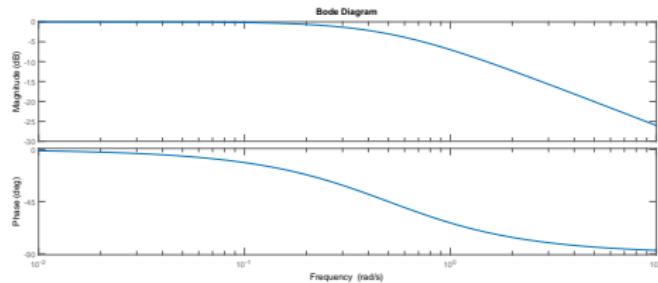
If the disturbance rejection of the control system is very good, the disturbances will have almost no effect on the output  $\Rightarrow M(s) \approx 0$

if  $\begin{cases} |H(j\omega)| \cong 1 & \text{then } |P(j\omega)C(j\omega)| \text{ (**open loop gain**) is very} \\ |M(j\omega)| \cong 0 & \text{large.} \end{cases}$

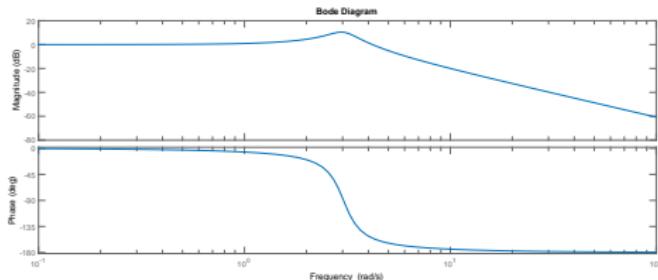
A large open loop amplification might lead to instabilities!

# Exercise: Which controller do you prefer?

The closed-loop transfer functions for two different controllers for high precision surgery are:



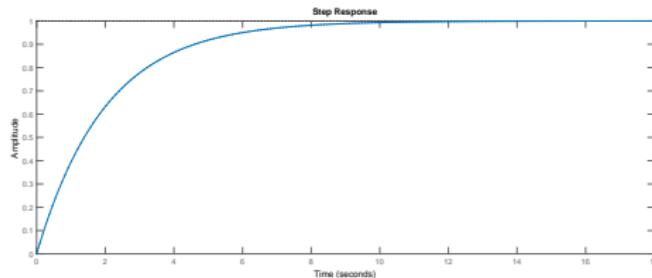
$$\frac{Y(s)}{R(s)} = \frac{0.5}{s + 0.5}$$



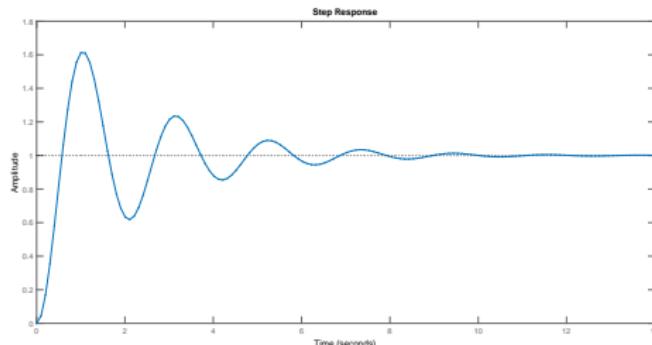
$$\frac{Y(s)}{R(s)} = \frac{10}{1.09s^2 + s + 10}$$

# Exercise: Which controller do you prefer?

The output  $Y(s)$  of the closed-loop system when  $R(s)$  is a step function is as follows:



$$\frac{Y(s)}{R(s)} = \frac{0.5}{s + 0.5}$$

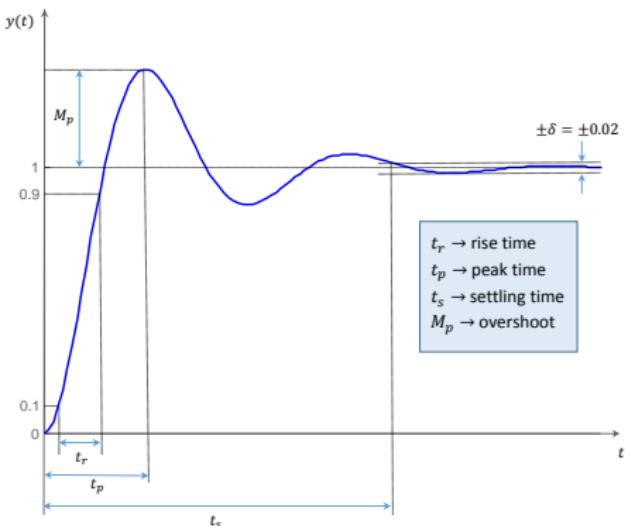


$$\frac{Y(s)}{R(s)} = \frac{10}{1.09s^2 + s + 10}$$

# Quality of reference tracking

Look at the step response of the transfer function from  $R(s)$  to  $Y(s)$ . The quality can be determined using these criteria:

- Rise-time
- Settling time
- Steady-state error
- Overshoot
- ...



# Model errors

- The plant characteristics may be variable or time-varying. System modeling techniques identify the plant within a certain model class and with a certain amount of inaccuracy. So there always exists a plant uncertainty, which cannot be described exactly by the mathematical models.
- Control systems need to be made robust against this plant variability and uncertainty.

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# Sensitivity

Sensitivity is a measure of the relative change in the system due to a relative change in a chosen parameter. Here we look at  $\Delta Y(s)$  due to  $\Delta P(s)$ .

$$Y(s) + \Delta Y(s) = \frac{(P(s) + \Delta P(s))C(s)}{1 + (P(s) + \Delta P(s))C(s)} R(s) + \frac{1}{1 + (P(s) + \Delta P(s))C(s)} D(s)$$

Look at the effect on the system without disturbances ( $D(s)=0$ )

$$\Delta Y(s) = \frac{(P(s) + \Delta P(s))C(s)}{1 + (P(s) + \Delta P(s))C(s)} R(s) - \frac{P(s)C(s)}{1 + P(s)C(s)} R(s)$$

# Sensitivity

$$\begin{aligned} &= \frac{(P(s) + \Delta P(s))C(s)(1 + P(s)C(s)) - P(s)C(s) - P(s)C(s)(P(s) + \Delta P(s))C(s)}{(1 + (P(s) + \Delta P(s))C(s))(1 + P(s)C(s))} R(s) \\ &= \frac{\Delta P(s)C(s)}{(1 + (P(s) + \Delta P(s))C(s))(1 + P(s)C(s))} R(s) \\ &= Y(s) \frac{\Delta P(s)}{P(s)} \frac{1}{1 + (P(s) + \Delta P(s))C(s)} \end{aligned}$$

# Sensitivity

Now take the relative change due to this model uncertainty and take the limit for  $\Delta x \rightarrow 0$ ; this gives the following (measure of the) sensitivity:

$$S_P^Y(s) = \frac{\frac{\partial Y}{\partial P}(s)}{\frac{\partial P}{P}(s)} = \frac{\partial Y}{\partial P}(s) \cdot \frac{P(s)}{Y(s)} = \frac{1}{1 + P(s)C(s)}$$

- Again, a very large  $|P(s)C(s)|$  looks like a good choice, but again there is a risk for instability!
- Note that the sensitivity can be determined for any parameter

# Example 1

## Problem statement and solution

Suppose we want to stabilize the system  $P(s) = \frac{1}{(s-a)}$ . We only know that  $a \in [0.20; 0.80]$ . We propose to use a proportional controller  $C(s) = K$ . The closed-loop transfer function becomes

$$H(s) = \frac{P(s)C(s)}{1 + P(s)C(s)} = \frac{\frac{1}{s-a}K}{1 + \frac{1}{s-a}K} = \frac{K}{s - a + K}$$

The controller stabilizes the system if  $-a + K > 0$ . If we choose  $K$  very large the system will be robust against changes in  $a$ .

## Example 2

### Problem statement and solution

Suppose we want to stabilize the system  $P(s) = \frac{s}{(s-a)}$  with  $a \in [0.20; 0.80]$ . If we choose the control law  $C(s) = \frac{K}{s}$  this results in the same transfer function.

$$H(s) = \frac{P(s)C(s)}{1 + P(s)C(s)} = \frac{\frac{s}{s-a} \frac{K}{s}}{1 + \frac{s}{s-a} \frac{K}{s}} = \frac{K}{s - a + K}$$

Again we choose  $K > a$  to ensure stability. But how does the controller perform on the slightly perturbed system  $\tilde{P}(s) = \frac{(s+\epsilon)}{(s-a)}$ ? The transfer function from  $R(s)$  to  $Y(s)$  becomes

$$\tilde{H}(s) = \frac{\frac{s+\epsilon}{s-a} \frac{K}{s}}{1 + \frac{s+\epsilon}{s-a} \frac{K}{s}} = \frac{(s+\epsilon)K}{s^2 + (K-a)s + \epsilon K}$$

## Example 2

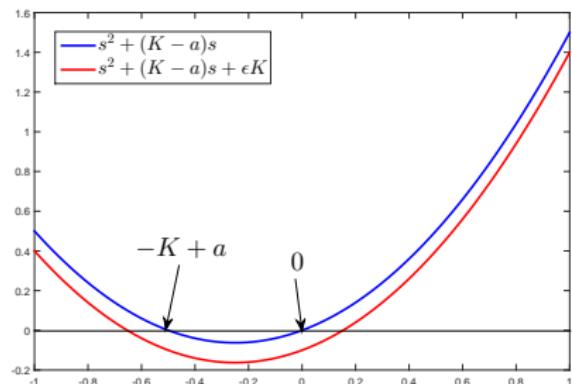
$$\tilde{S}(s) = \frac{(s + \epsilon)K}{s^2 + (K - a)s + \epsilon K}$$

The figure on the right plots

$$s^2 + (K - a)s$$

$$s^2 + (K - a)s + \epsilon K$$

For  $\epsilon$  negative the system becomes unstable. The control system is not robust and useless in practical situations. You should not use control laws which rely on a pole-zero cancellation.



# Sensitivity - Robustness

## Steady-state error

The steady state error is defined as follows:

$$\begin{aligned}\lim_{t \rightarrow \infty} e(t) &= \lim_{t \rightarrow \infty} (r(t) - y(t)) \\ &= \lim_{s \rightarrow 0} s(R(s) - Y(s)) \quad (\text{final value theorem})\end{aligned}$$

A very small steady state error (preferably zero) indicates that the controller tracks the reference very well.

# Sensitivity - Robustness

## Open-loop system

For an open loop system with a step reference, the steady-state error is given by

$$e_{ol}(\infty) = \lim_{s \rightarrow 0} s(1 - C(s)P(s))R(s) = 1 - C(0)P(0)$$

- So, the open loop system has no steady state error (for a step reference) if the controller is calibrated such that  $C(0)P(0) = 1$
- $\Rightarrow$  a precise calibration of the DC gain (not robust)

# Sensitivity - Robustness

## Closed-loop system

The steady-state error for a step reference is given by

$$\begin{aligned} e_{cl}(\infty) &= \lim_{s \rightarrow 0} s \left( 1 - \frac{C(s)P(s)}{1 + C(s)P(s)} \right) R(s) \\ &= \lim_{s \rightarrow 0} \frac{s}{1 + C(s)P(s)} R(s) = \frac{1}{1 + C(0)P(0)} \end{aligned}$$

- The steady state error is small if  $C(0)P(0)$  is very large
- Again, calibrating the DC gain
- The difference is that we only need a large gain, which is far less demanding than having to make it equal to 1

# An open loop controller is not robust

We can now show how this results in a great advantage of the closed-loop strategy over the open loop strategy:

- If  $P$  changes slightly (for instance due to a factor that has not been taken up into the model) to  $P + \Delta P$
- Then making  $e_{ol}(\infty)$  small would require to calibrate anew
- Whereas  $e_{cl}(\infty)$  would remain small, as long as  $(P(0) + \Delta P(0))C(0)$  remains large

Hence an open loop configuration can not control the output **robustly** against changes in  $P$

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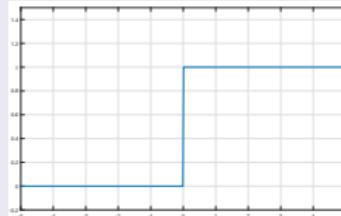
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# Steady state error: step

## Definition

$$\epsilon(t) = \begin{cases} 0, & t \leq 0 \\ \frac{1}{2}, & t = 0 \\ 1, & t \geq 0 \end{cases}$$



For a step reference  $r(t) = a\epsilon(t) \Rightarrow R(s) = \frac{a}{s}$

$$e_{cl}(\infty) = \frac{s}{1 + C(0)P(0)} \frac{a}{s}$$

So  $e_{cl}(\infty) = 0$  if  $C(s)P(s)$  has a least one pole in zero  
( $C(0)P(0) \rightarrow \infty$ )

# Steady state error: ramp

For a ramp reference function  $r(t) = at\epsilon(t) \Rightarrow R(s) = \frac{a}{s^2}$ , we have

$$e_{cl}(\infty) = \lim_{s \rightarrow 0} \frac{s}{1 + C(s)P(s)} \frac{a}{s^2}$$

So,  $e_{cl}(\infty) = 0$  if  $C(s)P(s)$  has at least two poles in zero.

$$\left( \lim_{s \rightarrow 0} sC(s)P(s) = \infty \right)$$

# Type of a system

- The type of a system is determined by the number of poles  $P(s)C(s)$  has in zero, and hence it is linked to what type of references it can track perfectly
- Write  $P(s)C(s)$  as  $\frac{K \prod_{k=1}^m (s-z_k)}{s^i \prod_{k=1}^{n-i} (s-p_k)}$  with  $i$  the amount of poles in zero
- We say this system is of type  $i$ 
  - It will be able to track references of shape  $at^0\epsilon(t)$  up to  $at^{i-1}\epsilon(t)$  perfectly for  $t \rightarrow \infty$  ( $\lim_{t \rightarrow \infty} e_{cl}(t) = 0$ )
  - A reference of shape  $at^i\epsilon(t)$  will be missed by a finite factor for  $t \rightarrow \infty$  ( $\lim_{t \rightarrow \infty} e_{cl}(t) = C$ )
  - A reference of shape  $at^{i+1}\epsilon(t)$ ,  $at^{i+2}\epsilon(t)$ , ... will be missed by infinity for  $t \rightarrow \infty$  ( $\lim_{t \rightarrow \infty} e_{cl}(t) = \infty$ )

# Example of a type 0 system

## Example

Step reference  $r(t) = a\epsilon(t) \Rightarrow R(s) = \frac{a}{s}$

$$\begin{aligned} e_{cl}(\infty) &= \lim_{s \rightarrow 0} \frac{s}{1 + C(s)P(s)} \frac{a}{s} \\ &= \lim_{s \rightarrow 0} \frac{a}{1 + \frac{K \prod_{k=1}^m (-z_k)}{\prod_{k=1}^{n-i} (-p_k)}} = \frac{a}{1 + K_p} \end{aligned}$$

where  $K_p$  is the static position error constant

# Example of a type 0 system

## Example

Ramp reference  $r(t) = at\epsilon(t) \Rightarrow R(s) = \frac{a}{s^2}$

$$\begin{aligned} e_{cl}(\infty) &= \lim_{s \rightarrow 0} \frac{s}{1 + C(s)P(s)} \frac{a}{s^2} \\ &= \lim_{s \rightarrow 0} \frac{1}{1 + \frac{K \prod_{k=1}^m (-z_k)}{\prod_{k=1}^{n-i} (-p_k)}} \frac{a}{s} \\ &= \lim_{s \rightarrow 0} \frac{a}{s + s \frac{K \prod_{k=1}^m (-z_k)}{\prod_{k=1}^{n-i} (-p_k)}} = \frac{a}{K_v} = \infty \end{aligned}$$

where  $K_v$  is the static velocity error constant

# Steady state errors - type of a system

$$K_p = \lim_{s \rightarrow 0} P(s)C(s)$$

$K_p$  = Static position error constant

$$K_v = \lim_{s \rightarrow 0} sP(s)C(s)$$

$K_v$  = Static velocity error constant

$$K_a = \lim_{s \rightarrow 0} s^2 P(s)C(s)$$

$K_a$  = Static acceleration error constant

And the respective steady state errors for different system types are:

Type	Step $a\epsilon(t)$	Ramp $at\epsilon(t)$	Parabola $\frac{at^2\epsilon(t)}{2}$
0	$\frac{a}{1+K_p}$	$\infty$	$\infty$
1	0	$\frac{a}{K_v}$	$\infty$
2	0	0	$\frac{a}{K_a}$

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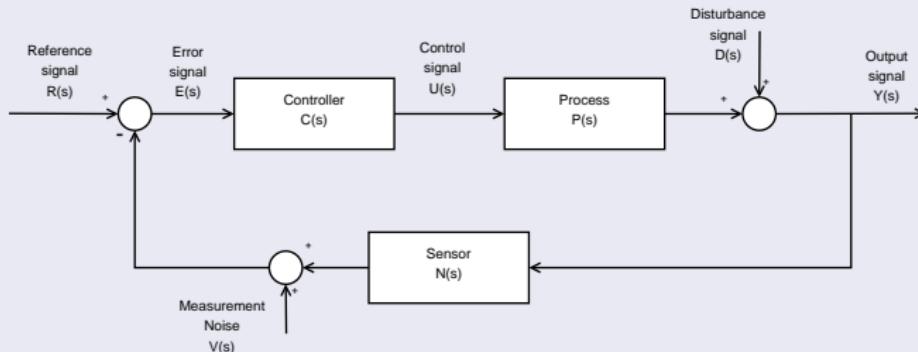
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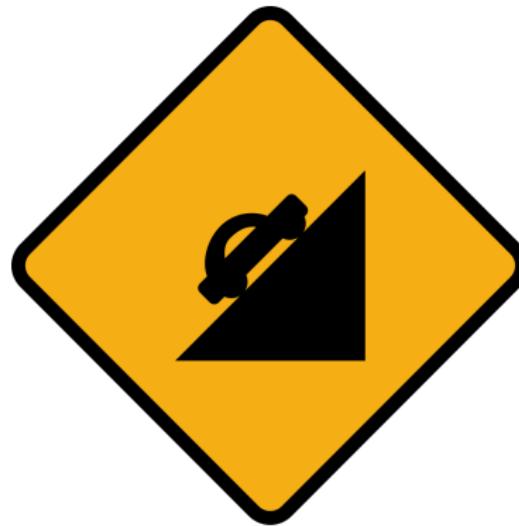
# Noise and disturbance rejection

- Measurement noise is often modelled by zero-mean white noise.
- Disturbances are actual changes to the state of the system



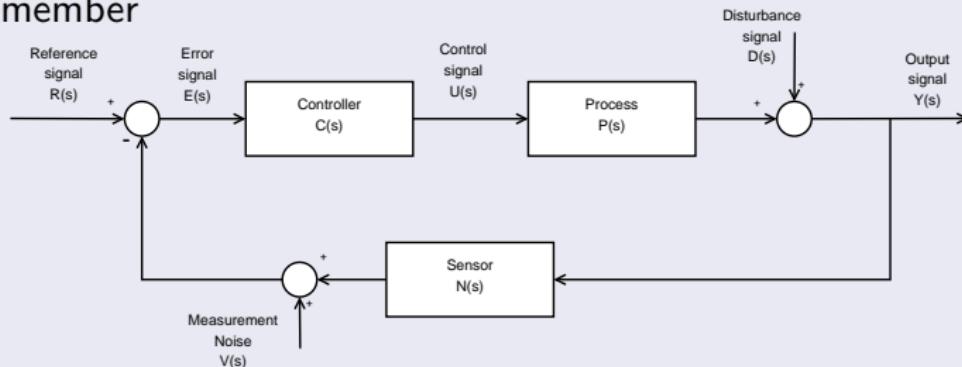
# Noise and disturbance rejection

- Cruise control:
  - Measurement errors on the speed are noise
  - A change in slope is a disturbance



# Noise and disturbance rejection

- Remember

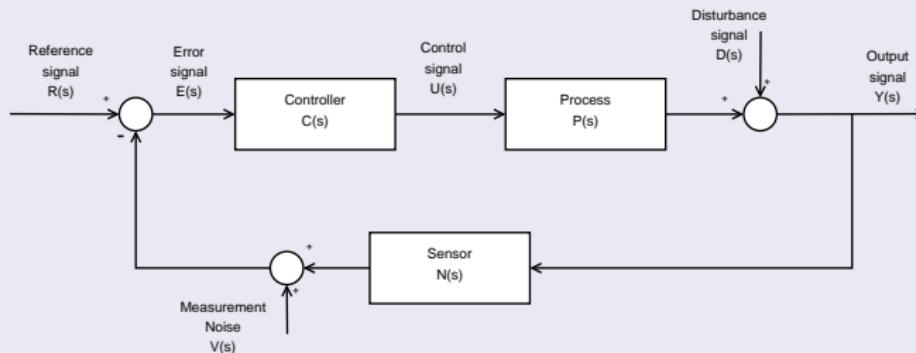


$$Y(s) = \frac{P(s)C(s)}{1 + P(s)C(s)}R(s) + \frac{1}{1 + P(s)C(s)}D(s)$$

- Setting  $M(s) = \frac{1}{1+P(s)C(s)}$  sufficiently small results in the rejection of  $D$ . This can be achieved by choosing  $C(s)$  sufficiently high.

# Noise and disturbance rejection

- What happens to the measurement noise?
  - It will be amplified and applied to the input of the plant which in turn leads to a nervous controller.



# Noise and disturbance rejection

- Good disturbance rejection requires fast control actions to bring the system back to the desired state
- Good noise rejection requires slow control actions
- Note that a controller can not see the difference between measurement noise and disturbances. Slow controllers will be less sensitive to measurement noise but fast controllers will have better disturbance rejection

### **Slow Controllers**

Not sensitive to noise  
Small control actions



### **Fast Controllers**

Good disturbance rejection  
Fast tracking

### **Robust Controllers**

Model errors will not affect the behaviour of the system strongly



### **Aggressive Controllers:**

Exchanges robustness for better performance