Notes on BPS quivers

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Abstract:

October 28, 2024

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1 String theory construction

The spectra of BPS states of 4d $\mathcal{N} = 2$ theories can be calculated from the quantum mechanics of an associated BPS quiver. Originally, these quivers arose in string theory constructions, where BPS objects are D-branes and the quantum mechanical description of the BPS spectrum corresponds to the world-volume theory of the D-branes. This construction is due to Douglas and Moore [1] and we shall comment on it momentarily.

1.1 D-branes on CY3

First, let us put this story into context. In the late 90's, there has been much interest in the understanding of the theory of D-branes on Calabi-Yau manifolds. Much was already known about D-branes in flat space and K3 compactifications, but things get more complicated in the case of Calabi-Yau threefolds. The issue is that the 'conventional geometric prediction' is modified by stringy and quantum corrections, leading also to changes on the D-brane world volume action.

D-branes on Calabi-Yau manifolds divide naturally into two classes: A-branes and B-branes. The notation comes from 2d topological quantum field theories, and we summarize it below following [2]. Consider a 2d $\mathcal{N} = (2, 2)$ non-linear sigma model $\phi : \Sigma \to X$, where Σ is the string worldsheet and X is the Calabi-Yau target manifold. This model defines string theory only up to nonperturbative effects in the string coupling, which are invisible from this point of view. Note that if X is CY, the $\mathcal{N} = (2, 2)$ theory is actually superconformal. Then, closed string states form representations of the superconformal algebra, which are best studied by passing to a TQFT associated to the $\mathcal{N} = (2, 2)$ SCFT.

There are 2 such TQFTs that occur, called the A-model and the B-model, discovered by Witten. These are obtained by twisting the SCFT by modifying the bundles in which fermions take value. In the A-model, correlators involving $\mathcal{Q}$-exact operators vanish and, thus, we restrict attention to $\mathcal{Q}$-cohomology. Moreover, this turns out to be equivalent to de Rham cohomology. The action of the A-model depends on the complexified Kähler form but any change in the complex structure is trivial:

$$S_A = -2\pi i \int_\Sigma \phi^*(B + iJ) + \mathcal{Q}(\ldots) \ . \tag{1.1}$$

However, the A-model receives worldsheet instanton corrections, which come from the $\mathcal{Q}$-exact term in the action and reduce to *holomorphic maps* $\bar{\partial}\phi^i = 0$.

Now, in the B-model, there is typically a non-zero chiral anomaly unless the target is CY. The action now depends only on the complex structure of the target and $\mathcal{Q}$-cohomology becomes Dolbeault cohomology. Note also that instantons are trivial in this case, being just constant maps from Σ to points in the target (and thus no infinite sums in the correlators). We then have the following statement of mirror symmetry:

$$\mathrm{A\,model\,on}\,Y \longleftrightarrow \mathrm{B\,model\,on}\,X \ . \tag{1.2}$$

Having introduced the A and B models, let us discuss the branes in each of these. Naively, one might think of D-branes simply as subspaces on which open strings may end, which is the interpretation in flat spacetime. However, string theory modifies such classical notions and this picture is not good for general backgrounds unless viewed is the context of large distances.

Consider first the A-model, where A-branes are 3-branes wrapped on special Lagrangian submanifolds. As in the usual picture, open strings end on these D-branes, which are just subspaces L_a:

$$\phi(\partial\Sigma) \subset \bigcup_a L_a \ . \tag{1.3}$$

Now, for the B-model, considering again branes as defined by Dirichlet and Neumann boundary conditions in the target threefold, B-branes are 2p-branes wrapped on holomorphic cycles and carrying holomorphic vector bundles. This might seem backwards from the discussion in the previous section, as the 2p-cycles and the masses of B branes are controlled by Kähler moduli and thus calculable in the A-twisted topological field theory. However, in going from the open to closed string channel, the boundary conditions on the $U(1)$ currents in the $\mathcal{N} = (2,2)$ superconformal algebra change sign. Since these determine which supersymmetry is preserved, the net result is the interchanging of the A and B twists.

So, what we have learned so far is the B-brane masses receive instanton corrections, while the large volume results for A brane masses are exact. Now, going away from the large volume limit, this simple description in terms of subspaces no longer holds. In particular, B-branes become objects in the derived category of coherent sheaves on X, where coherent sheaves are a generalisation of vector bundles.

Let us also mention that these branes from the TQFT do not automatically correspond to BPS states in the untwisted theory. That is because the BPS condition is stronger that that imposed on branes in the twisted theories. Thus, for the A and B-branes to correspond to BPS states, one needs to impose a further condition called 'stability'. We will only mention here B-brane stability. Let us first note that B-brane decay is not classical due to the non-perturbative α' corrections. What happens physically when a D-brane is allowed to decay is that the open string becomes massless.

Assuming mirror symmetry to be true, the concept of stability that appears for B-branes is called Π-stability, and depends on the central charge of a given B-brane. Aasymptotically, this is given by:

$$Z(\mathcal{E}) = \int_X e^{-(B+iJ)} \mathrm{ch}(\mathcal{E}) \sqrt{\mathrm{Td}(X)} \ , \tag{1.4}$$

with $\mathcal{E}$ being an object in the derived category of coherent sheaves on X. Then, Π-stability offers conditions for B-brane decay depending on whether the central charges align or not. In the large volume limit, where the classical description of B-branes in terms vector bundles supported over subspaces of the CY holds, the relevant notion of stability is μ-stability. In this limit, the world-volume approach to D-branes is assumed to be accurate.

1.2 Quivers at last

Having discussed D-branes on CY threefolds, let us now see what the connection to quiver gauge theories is. This connection started with the study of orbifolds $\mathbb{C}^d/G$, for G some finite subgroup of $SU(d)$, and the need to understand B-branes on such spaces. We have, first, the following statement.

B-branes on the orbifolds $\mathbb{C}^d/G$ and open strings between them are described by the derived category of McKay quiver representations, with relations.

McKay quivers are the extended Dynkin diagram associated to G and represents exactly the configuration of $\mathbb{P}^1$'s in the exceptional divisor upon blowing-up the orbifold singularity. The distinguished set of D-branes on the orbifold associated to the irreducible representations of G are called *fractional branes*. Meanwhile, 0-branes on X correspond to quivers for the regular representation of G.

Alternatively, we can get these quivers from the Douglas-Moore construction by considering II compactification on $\mathbb{R}^{1,3} \times \mathbb{C}^3/G$ with a D3-brane filling the space-like directions of $\mathbb{R}^{1,3}$, such that it appears as a 0 brane in the $\mathbb{C}^3/G$ directions. The world-volume theory of this D3 brane is a 4d $\mathcal{N} = 1$ theory in $\mathbb{R}^{1,3}$. This theory can be analysed by considering a collection of D-branes in $\mathbb{C}^3$ corresponding to the preimage of the quotient by G and then imposing G-invariance. Thus, the quiver is encoded in the decomposition of the representation V of G under which the D-branes transform:

$$V = \bigoplus_k m_k V_k \ , \tag{1.5}$$

with V_k the irreducible representations of G. The associated quiver representation is then the quiver gauge theory having gauge group $\prod_k U(m_k)$.

Hence, we find the picture that BPS quivers are 4d $\mathcal{N}=1$ gauge theories living on stacks of D3 branes probing $\mathbb{C}^3/G$ orbifolds. Alternatively, one can perform 3 T-dualities along the world-volume of the D3-branes (which, recall, are along the $\mathbb{R}^{1,3}$ directions), to get D0 branes in IIA geometry. As a result, we have the alternative view that BPS quivers are 1d $\mathcal{N}=4$ quiver quantum mechanics living on the worldline of D0 branes in IIA. These are, of course, the BPS states of the theory.

1.3 Brane tilings and all that

The understanding of these 4d $\mathcal{N}=1$ quiver theories living on the world-volume of the branes has been of great interest in the last 20 years. In particular, we have seen that these theories leave on D3 branes probing the tip of the CY cone, with the D3 brane filling the spatial directions of $\mathbb{R}^{1,3}$. This setup is dual to IIB compactifications on $AdS_5 \times X_5$, where X_5 is a Sasaki-Einstein 5-manifolds whose metric determines the CY metric.

Much interest has been shown, in particular, in determining these quiver gauge theories directly from the CY, for more general cases then orbifolds. In the case of toric threefolds, the first such method is called the *inverse algorithm*. This starts from the toric diagram of the CY and uses an embedding into an orbifold $\mathbb{C}^3/G$; then, a partial resolution of the orbifold, which is equivalent to the threefold, is used in order to determine the quiver. This algorithm is rather cumbersome and was improved by the *fast inverse algorithm*, which passes from the toric diagram to the *brane tiling*, before obtaining the quiver gauge theory. The forward algorithm, developed in [3, 4], starts from the quiver gauge theory and obtains the toric data of the CY3. Alternatively, the fast forward algorithm goes again through the associated brane tiling, from which the toric data is encoded in the determinant of the Kasteleyn matrix.

Brane tilings [5] are bipartite graphs that encode the brane configurations that engineer the quiver gauge theory. In particular, recall that a D3 brane probing a $\mathbb{C}^3/\mathbb{Z}_m$ orbifold is T-dual to a collection of m NS5 branes, with the D3 brane being mapped to a D5 brane. Similarly, for the $\mathbb{C}^3/\mathbb{Z}_m\times\mathbb{Z}_n$ orbifolds, the dual configuration has m NS5 and n NS5' branes, with the D5 suspended between these NS5 branes. Such systems are believed to exist for more general threefolds. Note that the NS5 and D5 branes share 4 directions; the NS5 branes wrap a homolorphic curve Σ with the D5 stretching inside holes of the NS5 skeleton. That is, the D5 branes are bounded by NS5 branes in 2 directions. The intersection of the plane formed by these directions with the compact curve Σ leads to a bipartite graph, due to the orientation of the fundamental strings stretching between neighbouring D5 branes.

In the brane tiling, the D5 branes correspond to faces, while NS5 branes are nodes. The strings stretched between D5 branes through NS5 branes are edges between two polygons. Moreover, regions where k strings interact locally are k-valent vertices in the tiling.

2 Field theory interpretation

2.1 4d $\mathcal{N} = 2$ theories

BPS quivers play an important role in 4d $\mathcal{N} = 2$ theories, with the first field theory interpretation for the BPS quivers appearing in [6, 7]. There, it was shown how the spectra of BPS states for 4d $\mathcal{N} = 2$ theories can be calculated from the quantum mechanics of an associated quiver. The starting point was to split the BPS spectrum into particles and antiparticles by essentially splitting the central charge plane into two half-planes. Then, a minimal basis of particles will consist of $2r + f$ BPS hypermultiplets, equal to the rank of the lattice of electric, magnetic and flavour charges Γ. Thus, every charge BPS particle of charge γ can be expressed in this basis as

$$\gamma = \sum_{i}^{2r+f} n_i \gamma_i \ , \qquad n_i \in \mathbb{N} \ . \tag{2.1}$$

Given such a basis, a quiver is obtained by drawing a node for each element γ_i in the basis and a number of arrows between nodes γ_i and γ_j equal to their Dirac pairing. Thinking back to the fractional brane quivers, recall that near the orbifold point the representations having only one $m_i \neq 0$ were precisely the fractional branes, which formed a basis of BPS states. The idea here is a generalization of this fractional brane picture.

The superpotential of the quantum mechanics appear whenever there are non-trivial cycles in the quiver and is fixed by alternative means. However, assuming this to be known, the moduli space of supersymmetric ground states with charge γ (that is, when the quiver has gauge group $\prod_i^{2r+f} U(n_i)$) is given by:

$$\mathcal{M}_\gamma = \left\{ B^a_{ij} | \ F, \ D\text{-terms} = 0 \right\} / \prod_i U(n_i) \ , \tag{2.2}$$

where B^a_{ij} are matter fields between nodes i and j. If $\mathcal{M}_\gamma$ is non-empty, then there exists a particle in the spectrum with charge γ. Moreover, its spin and degeneracy are completely determined by $\mathcal{M}_\gamma$.

There is an alternative description of $\mathcal{M}_\gamma$, which makes use of Π-stability. That is, $\mathcal{M}_\gamma$ is given as the solution to only the F-term equations of motion, modulo the complexified gauge group $\prod_i \mathrm{GL}(n_i, \mathbb{C})$, augmented by a stability condition. Of course, this second prescription makes use of quiver representation theory, so $\mathcal{M}_\gamma$ consists of the stable quiver representations, modulo the action of the complexified gauge group.

2.2 Geometric engineering and 5d BPS quivers

It is by now known that geometric engineering of 4d $\mathcal{N} = 2$ theories secretly involves some additional KK modes, due to the IIA/M-theory duality. It is the latter description that is more natural from the CY perspective, as the strict 4d limit involves some 'deformation' of the geometry. However, BPS spectra of such 4d KK theories are much more involved than their 4d counterparts.

BPS states form a category $\mathcal{J}^{\text{BPS}}$, which is equivalent to the derived category of coherent sheaves on the Calabi-Yau threefold X, $D^b(X)$. Moreover, these are also equivalent to the derived category of quiver representations, such that we have the following equivalences:

$$\mathcal{J}^{\text{BPS}} \cong D^b(X) \cong D^b(\text{Rep}(Q, \mathcal{W})) \ . \tag{2.3}$$

We have already discussed that these categories need to be supplemented by certain stability conditions in order to describe 'physical' BPS states. An important object in this context is the heart of a triangulated category, which is an abelian category such that $\mathcal{J} = D^b\mathcal{A}$. Note that a choice of $\mathcal{A}$ corresponds to a choice of splitting the charge lattice between particles and antiparticles and, thus, to a choice of a quiver in the mutation class.

An important point that we have avoided so far is the existence of a quiver description for the BPS spectrum. Generally, the existence of quiver points is far from being trivial, but such points do exist for orbifolds; there, the fractional branes become mutually local around the orbifold point and their central charges align, such that the BPS quiver correctly describes the BPS spectrum. In non-orbifold theories a similar picture is expected to hold: there is a locus in moduli space where the Calabi-Yau develops a conical singularity, and furthermore the fractional branes supported on this singularity become mutually BPS. This is usually also referred to as an "orbifold point" even though the space may not admit an orbifold symmetry [8].

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