Symplectic Quotients

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1 Introduction

In this project we aim to review the symplectic quotient construction. We start by reviewing group actions on symplectic manifolds and compute the moment maps for a few examples. We then state the Marsde-Weinstein-Meyer theorem and illustrate it with a few examples. Finally, we will also mention the Kähler and HyperKähler quotient constructions.

2 Symplectic Quotients

Definition 2.1. *A compact Lie group G is said to act on a manifold M if for all elements $h \in G$ we can associate homeomorphisms $\phi_h : M \to M$ such that:*

i) $\phi_{h_1} \cdot \phi_{h_2} = \phi_{h_1 h_2}$
ii) for the unit element $e \in G$, the corresponding map ϕ_e is the identity
iii) the map $\phi : G \times M \to M$ given by $\phi(h, x) = \phi_h(x)$ is continuous and differentiable.

The correspondence between group elements and the homeomorphisms ϕ_g is called the *action* of G on M and is itselft a group homomorphism $\psi : G \to Diff(M)$. If the map ϕ mentioned in the above definition is a smooth map, then the group action ψ is also smooth.

In the case of symplectic manifolds, G is said to act *symplectically* on M is for all elements $h \in G$ we have $h\omega = \omega$. Let us also recall that Kähler manifolds are symplectic manifolds, with the symplectic form being the Kähler form. Before discussing Kähler quotients, let us first consider quotient manifolds in general.

Definition 2.2. *For a group action $\psi : G \to Diff(M)$, the orbit of G through a point $p \in M$ is defined as the set $\{\phi_g(p) : g \in G\}$. We call the stabilizer of $p \in M$ the set containing the group elements for which p is a fixed point, i.e. $G_p := \{g \in G : \phi_g(p) = p\}$.*

G is said to act *freely* on the manifold if the only element with fixed points is the identity element; equivalently, all stabilizers are trivial e. Given such a group action on the manifold it is natural to consider the orbit space M/G; we can gain some understanding of this by considering the following geometric setup [1]. Note that the elements $\xi \in \mathfrak{g}$ induce vector fields $\rho(\xi) = d(e^{t\xi} \cdot p)/dt|_{t=0}$ on M and if G acts freely on M then at each point $p \in M$, these vector fields form a subspace V_pM of the tangent space T_pM, called the *vertical space*, with $dim(V_pM) = dim(\mathfrak{g})$. This is essentially the tangent space to the orbits of G through the point p. Then, given the (continuous) projection map $P : M \to M/G$, it follows that the tangent space at the point $P(p) \in M/G$ is isomorphic to the quotient T_pM/V_pM.

We can further consider a Riemannian metric g on M, with G acting as isometries. We would like to define a metric on the quotient space M/G. We decompose the tangent space into orthogonal spaces, i.e. the vertical and horizontal spaces (H_pM). We have seen already that the projection map P induces an isomorphism between H_pM and the tangent space of the quotient M/G, so tangent vectors $\tilde{\zeta}$ have unique (horizontal) lifts to $\zeta \in H_pM$. Thus, the metric g' on M/G is defined by:

$$g'(\tilde{\zeta}_1, \tilde{\zeta}_2) = g(\zeta_1, \zeta_2) \ . \tag{2.1}$$

Let us note that since the metric g is preserved by the action of G, the above metric is independent of the base point. Additionally, the manifold M can also be tought of as a principal G-bundle over M/G, but we will come to this later. The manifold M can have additional structure and, in general, the quotient will not inherit the same structure. However, there is a way of preserving the Kähler structure, through the *symplectic quotient* construction. For this we will need to introduce the moment map, but we will take a short detour, introducing some basic notions.

Let us start by recalling [2] that a *symplectic manifold* is a smooth manifold M equipped with a nondegenerate closed 2-form ω. For instance, for $M = \mathbb{R}^{2n}$, with coordinates $(x_1, ..., x_n, y_1, ..., y_n)$, the standard symplectic form is $\omega_0 = \sum dx_k \wedge dy_k$. Similarly, for $M = \mathbb{C}^n$, with coordinates $(z_1, ..., z_n)$, the standard symplectic form is $\omega_0 = \frac{i}{2} \sum dz_k \wedge d\bar{z}_k$. Finally, for $M = S^2 \subset \mathbb{R}^3$, in cylindrical coordinates the standard symplectic form is $\omega = d\theta \wedge dz$, away from the poles.

Definition 2.3. *A vector field X on a symplectic manifold (M, ω) is symplectic if the contraction $\iota_x \omega$ is closed. It is said to be Hamiltonian if the contraction is exact.*

A consequence of the above definition is that the first de Rham cohomoogy group of M measures the obstruction for symplectic vector fields to be Hamiltonian. Note also that for a symplectic vector field X the Lie derivative of the symplectic form vanishes:

$$\mathcal{L}_X \omega = d\iota_X \omega + \iota_X d\omega = 0 \ . \tag{2.2}$$

This is also the case for Hamiltonian vector fields X, for which we denote by H the *Hamiltonian function* satisfying $\iota_X \omega = dH$. For instance, for the standard symplectic structure $(S^2, d\theta \wedge dz)$, the vector field $X = \partial/\partial\theta$ is Hamiltonian, with $\iota_X(d\theta \wedge dz) = dz$, so the Hamiltonian function associated to X is $H(\theta, z) = z + const.$

Definition 2.4. *Let $\mathfrak{g}$ be the Lie algebra of G, with dual $\mathfrak{g}^*$ and inner product $\langle , \rangle : \mathfrak{g}^* \times \mathfrak{g} \to \mathbb{R}$. The moment map $\mu : M \to \mathfrak{g}^*$ corresponding to the (symplectic) action of G on the (symplectic) manifold M is defined such that:*

$$d \langle \mu, \xi \rangle = \iota_{\rho(\xi)} \omega \ ,$$

where $\xi \in \mathfrak{g}$ and $\rho(\xi)$ are the induced vector fields on M, and $\langle \mu, \xi \rangle (p) = \langle \mu(p), \xi \rangle$ for $p \in M$.

Note that if such a moment map exists, then the group action ϕ is called a *Hamiltonian action*. The above definition[1] tells us that the contraction of the vector field $\rho(\xi)$ with ω is closed. In fact, this can be deduced from the fact that the Lie derivative $\mathcal{L}_{\rho(\xi)} \omega = 0$, which is another

[1]In general we also require μ to be G-equivariant, but we will 'neglect' this in the definition of μ as we will not prove the Marsden-Meyer theorem.

way to say that the Lie group action preserves the symplectic structure. As a first example, we will consider different actions on the cotangent bundle $T^*\mathbb{R}^3 \cong \mathbb{R}^3 \times \mathbb{R}^3$, with its canonical symplectic structure: $\omega = dx^i \wedge dp^i$, where we view x as a position vector and q as the momentum. Consider first the action of the translation group on this manifold; then, for the induced vector field $a^i \frac{\partial}{\partial x^i}$, the contraction $\iota_{a^i \frac{\partial}{\partial x^i}} \omega = a^i dp^i$. Hnece, in this case, the moment map $\mu : T^*\mathbb{R}^3 \mapsto \mathbb{R}^3$ is given by $\mu(x, p) = p$, i.e. the linear momentum. If we instead look at the action of $SO(3)$, with Lie algebra $\mathfrak{so}(3) \cong \mathbb{R}^3$, on the same space, the moment map is instead given by the cross product: $\mu(x, p) = x \times p$ which can be interpreted as giving the 'angular momentum'.

For future examples, it will be instructive to consider the $U(1) \cong S^1$ action on $M = \mathbb{C}$, with standard symplectic form $\omega_0 = \frac{i}{2} dz \wedge d\bar{z}$, given by $z \mapsto t^k z$, for $t \in U(1)$ and $k \in \mathbb{Z}$ fixed. The moment map is then given by:

$$\mu(z) = -\frac{1}{2} k |z|^2 \ , \tag{2.3}$$

which we can check by writing the symplectic form in polar coordinates: $\omega_0 = r dr \wedge d\theta$ and $\mu(re^{i\theta}) = -\frac{1}{2} k r^2$. Then, for the single generator of $\mathfrak{u}(1) \cong \mathbb{R}$, the vector field induced on $M = \mathbb{C}$ is $k \frac{\partial}{\partial \theta}$ and the definition of the moment map leads to the above result.

Focus for now on the preimage $N = \mu^{-1}(\eta)$, for $\eta \in \mathfrak{g}^*$ a coadjoint fixed point. In the case where $\eta = 0 \in \mathfrak{g}^*$, N is a submanifold of M of dimension $dim(N) = dim(M) - dim(G)$ (since N contains the points in the original manifold that map into the same element of $\mathfrak{g}^*$). In such a case, assuming that the action of G on $N = \mu^{-1}(0)$ is free, the topological quotient $\mu^{-1}(\eta)/K$ is a manifold of dimension $dim(M) - 2dim(G)$. Marsden, Weinstein and Meyer proved that if these conditions are satisfied than this quotient also has a natural symplectic form. We summarize the above ideas in the following theorem.

Theorem 2.1. *Let (M, ω) be a symplectic manifold with a Lie group action G and moment map $\mu : M \to \mathfrak{g}^*$. If the G action on $\mu^{-1}(0)$ is free, then:*

*i) The **symplectic reduction** $M//G = \mu^{-1}(0)/G$ is a smooth manifold of dimension $dim(M) - 2dim(G)$ and $\mu^{-1}(0)$ can be considered as a principal G bundle over the reduced space.*

ii) The unique symplectic form ω' on this manifold can be obtained from the symplectic form on M using the quotient map $\pi : \mu^{-1}(\eta) \to \mu^{-1}(\eta)/G$ as:
$$\pi^* \omega' = i^* \omega$$
where $i : \mu^{-1}(\eta) \to M$ is the inclusion map.

We will not prove this theorem, but will try to illustrate its content through some examples. Before doing so, we first consider a 'generalization' of the theorem. To obtain a symplectic manifold using such a construction for an element $\eta \in \mathfrak{g}^*$ that is not the identity, we need $N = \mu^{-1}(\eta)$ to be preserved by the group action. We claim that this happens if $\eta \in Z(\mathfrak{g}^*)$. As already mentioned before, the 'level' 0 is always preserved but this is not usually the case for the other elements of $\mathfrak{g}^*$, unless we limit to a torus $\mathbb{T}^n$ action on the manifold. The symplectic quotient $\mu^{-1}(\eta)/G$ can also lead to *orbifolds*. More generally, if $\eta \in \mathfrak{g}^*$ is not fixed under the G action and has stabilizer/isotropy subgroup H, then the quotient $\mu^{-1}(\eta)/H$ also has a symplectic structure [2].

For the action of $U(1) \cong S^1$, with Lie algebra $\mathfrak{u}(1) \cong \mathbb{R}$, on $\mathbb{C}^n$, given by $\alpha \cdot (z_1, ..., z_n) = (\alpha z_1, ..., \alpha z_n)$, the moment map $\mu : \mathbb{C}^n \to \mathbb{R}$ is given by:

$$\mu(z_1, ..., z_n) = \sum_{i=1}^{n} |z_i|^2 \ . \tag{2.4}$$

as can be checked explicitly using the definition 2.4 and the previous examples. It is then clear that $\mu^{-1}(0) = \{0\}$, so $\mu^{-1}(0)/U(1)$ is just a single point. For $\eta = 1$, we find $\mu^{-1}(1) = S^{2n-1}$, in which case the symplectic reduction becomes $\mathbb{C}^n//S^1 = S^{2n-1}/S^1 = \mathbb{P}^{n-1}$. What is the induced symplectic form on this manifold? Given the inclusion map $i : S^{2n-1} \to \mathbb{C}^n$, we first need to find the restriction of ω_0 to $\mu^{-1}(1) \cong S^{2n-1}$. Assuming $(x_1, ..., x_n, y_1, ..., y_n)$ are the $\mathbb{C}^n$ coordinates:

$$\sum_{k=1}^{n} |z_k|^2 = 1 \Rightarrow \sum_{k=1}^{n} x_k dx_k + y_k dy_k = 1 \ . \tag{2.5}$$

The restriction to the sphere is then realized using the pullback of the inclusion map (we assume $y_n \neq 0$ and use $(x_1, ..., x_n, y_1, ..., y_{n-1})$ as the sphere coordinates):

$$\begin{aligned} i^*\omega_0 &= \sum_{k=1}^{n-1} (x_k dy_k - y_k dx_k) + x_n dy_n - y_n dx_n = \\ &= \sum_{k=1}^{n-1} \left(x_k - \frac{x_n y_k}{y_n} \right) dy_k - \sum_{k=1}^{n} \left(y_k - \frac{x_n x_n}{y_n} \right) dx_k \ . \end{aligned} \tag{2.6}$$

The Marsden-Weinstein-Meyer theorem tells us that the pullback $\pi^*\hat{\omega}$ of the symplectic form on the reduced manifold, $\mathbb{P}^{n-1}$ in this case, should be equal to the above form on the sphere. In fact, we can check that the standard Fubini-Study form $\omega_{FS} = \partial\bar{\partial} Log\left(\sum_{k=1}^{n} z_k \bar{z}_k\right)$ on $\mathbb{P}^n$, with $z_k = x_k + i y_k$, gives precisely the above result. We note that there exists an alternative definition of complex projective spaces, as (algebraic) quotients $\mathbb{P}^{n-1} = (\mathbb{C}^n \setminus \{0\})/\mathbb{C}^*$, where $\mathbb{C}^*$ is the complexification of $U(1)$. This $\mathbb{C}^*$ acts on $\mathbb{C}^n$ as $(z_1, ..., z_n) \mapsto (\lambda z_1, ..., \lambda z_s, \lambda^{-1} z_{s+1}, ..., \lambda^{-1} z_n)$ This is not a coincidence, but rather a consequence of the identification of the symplectic (Kähler) quotient with the algebraic quotient[2]: $X/G_{\mathbb{C}}$.

Let us consider the following example: the action of $U(1)$ on $\mathbb{C}^4$, given by $u \cdot (z_1, z_2, z_3, z_4) = (u z_1, u z_2, u^{-1} z_3, u^{-1} z_4)$. Similar to the previous example, the moment map is given by:

$$\mu(z_1, z_2, z_3, z_4) = |z_1|^2 + |z_2|^2 - |z_3|^2 - |z_4|^2 = \eta \ . \tag{2.7}$$

Since the center of the coalgebra $Z(\mathfrak{u}(1)^*)$ is just $\mathfrak{u}(1)^*$, we can consider $\mu^{-1}(\eta)$ for any $\eta \in \mathfrak{u}(1)^*$. However, using the above moment map we can see that a rescaling of the $\mathbb{C}^4$ coordinates limits

[2]One should in fact consider only the semi-stable points [3] in X in this quotient, i.e. the points $z \in X$ such that invariant polynomials are bounded away from zero on the orbit of $G_{\mathbb{C}} z$

this to three points of interest: $\eta = 0, \pm 1$. Note that for $\eta = 0$, the resulting quotient is singular since $\mu^{-1}(0)$ contains the origin of the $\mathbb{C}^4$, which is a fixed point of the $U(1)$ action. For $\eta > 0$, the $\mu^{-1}(\eta)$ hypersurface does not contain the subspace $z_1 = z_2 = 0$, while for $\eta < 0$ it is clear that $z_3 = z_4 = 0$ is not included in $\mu^{-1}(\eta)$.

This example has been studied in the context of non-linear sigma models by Witten [4]. Following his conventions, denote by Z_+, Z_0 and Z_- the three possible quotients, with the subscript indicating the sign of η. Let $\tilde{V} = \{\boldsymbol{z} \in \mathbb{C}^4 : (|z_1|^2 + |z_2|^2)(|z_3|^2 + |z_4|^2) \neq 0\}$ and denote by $V_{12} = \{(z_1, z_2, 0, 0) \in \mathbb{C}^4\}$ and $V_{34} = \{(0, 0, z_3, z_4) \in \mathbb{C}^4\}$. It turns out that the symplectic quotient $\tilde{V}//U(1)$ can be identified with the algebraic quotient $\tilde{V}/\mathbb{C}^*$. To find the three symplectic quotients Z, one then has to add one of the two sets V_{12}, V_{34}, or the origin. For instance, for $\eta > 0$, we have already seen that $\mu^{-1}(\eta)$ does not contain the subspace $z_1 = z_2 = 0$, i.e. V_{34}. Thus, we can argue that:

$$Z_+ = (\tilde{V} \cup V_{12})/\mathbb{C}^* \ . \tag{2.8}$$

In this case, $z_{1,2} \neq 0$ are points in a copy of $\mathbb{P}^1$, which we denote by $\mathbb{P}^1_{(12)}$, following [4]. In this sense, Z_+ can be thought of as a $\mathbb{C}^2$ fibration over this $\mathbb{P}^1_{(12)}$, with the fiber coordinates being $z_{3,4}$. Similarly, for $\eta < 0$, we find $Z_- = (\tilde{V} \cup V_{34})/\mathbb{C}^*$, which can also be thought as a $\mathbb{C}^2$ fibration over $\mathbb{P}^1_{(34)}$. For $\eta = 0$, we only need to add the origin to $Z_0 = (\tilde{V} \cup \{\mathbf{0}\})/\mathbb{C}^*$, but we see that this is a singularity. In fact, the three quotients Z_+, Z_0, Z_- are all Calabi-Yau threefolds; starting with the $(4, 0)$ holomorphic form on $\mathbb{C}^4$: $dz_1 \wedge dz_2 \wedge dz_3 \wedge dz_4$ which is invariant under the $U(1)$ action, we can contract it with the vector field generating the circle action. Thus, the resulting holomorphic three-form is non-vanishing at all points of the quotients, being in fact the volume form.

It was also argued in [4] that the isolated singularity at the origin of the Calabi-Yau threefold Z_0 can be resolved while preserving the Calabi-Yau condition, by replacing the origin with a copy of $\mathbb{P}^1$ - i.e. either $\mathbb{P}^1_{(12)}$ or $\mathbb{P}^1_{(34)}$. These resolutions are precisely $Z_\pm$, so the change of sign in η corresponds to a transition between these two resolutions. Consequently, this is a *flop* transition in the Calabi-Yau, with the $\mathbb{P}^1$ base shrinking to zero size at $\eta = 0$.

2.1 Kähler Quotients

Let us also review quotients that preserve additional structure. Given a Kähler manifold with a group action, we would like to obtain a new Kähler structure on the quotient. Recall that for a $2n$-dimensional Kähler manifolds with complex structure J and metric g, we can define a symplectic structure ω by:

$$\omega(X, Y) = g(JX, Y) \ . \tag{2.9}$$

Given a group action on M preserving the metric and the complex structure, we can realize a symplectic quotient as before $\hat{M} = \mu^{-1}(\eta)/G$. This has a naturally induced metric as we have already seen in the previous section. In fact, we claim that this induced metric is Kählerian. The idea of the proof is to find the Levi-Civita connection on $\hat{M}$ and show that it is torsion-free and that it preserves the metric.

2.2 HyperKähler Quotients

Finally, we would like to mention the generalization to hyperKähler quotients [5]. The hyperKähler structure can be thought of as an extension of the Kähler structure to quaternions.

Definition 2.5. *A Riemannian manifold (M, g) is HyperKähler if it is equipped with three covariantly constant complex structures $\boldsymbol{I}, \boldsymbol{J}, \boldsymbol{K}$, satisfying the quaternionic algebraic relations: $\boldsymbol{I}^2 = -1$, $\boldsymbol{J}^2 = -1$, $\boldsymbol{K}^2 = -1$, $\boldsymbol{IJ} = \boldsymbol{K}$ etc.*

The consequence of the above definition is that, at each point of the manifold, the tangent space is a quaternionic vector space and the structure group is reduced to $Sp(n)$. Let us also note that each complex structure defines a symplectic form: $\omega_1(X, Y) = g(\boldsymbol{I}X, Y)$ etc., so each triple $(g, \boldsymbol{I}, \omega_1)$ gives the manifold a Kähler structure. We would like to use the previous symplectic quotient construction to form a Hyperhahler manifold as a quotient. We consider such a manifold with a compact Lie group G action that preserves the metric and the three symplectic structures. Consequently, we can define three moment maps μ_i for each symplectic structure and apply the symplectic quotient construction for each one of them. In fact, each of the three quotients will inherit a Kähler structure from the original manifold, but in order to obtain a HyperKähler structure we need a slightly different construction.

Theorem 2.2. *Given a group G-action on a hyperKähler manifold $(M, g, \boldsymbol{I}, \omega)$, we define the moment map $\mu : M \to \mathfrak{g}^* \otimes \mathbb{R}^3$ by $\mu(x) = (\mu_1(x), \mu_2(x), \mu_3(x))$. Then the quotient $\mu^{-1}(0)/G$ with inherited metric, complex structures and symplectic forms is a HyperKähler manifold.*

This construction also arrises in supersymmetric quantum field theories, such as in the Higgs branches of $4d$ $\mathcal{N} = 2$ theories or theories with the same amount of supersymmetry.

3 SUSY

In this section we review the Kähler quotient construction arising in the context of non-linear σ models. Consider a general $4d$ $\mathcal{N} = 1$ supersymmetric theory with n chiral multiplets $\Phi = (\phi, \psi, F)$, whose Lagrangian is fully determined by a real superfield $K(\Phi, \bar{\Phi})$ called the *Kähler potential* which encodes the kinetic terms and by a chiral superfield $W(\Phi)$ which is a holomorphic function of Φ:

$$\mathcal{L} = \int d^2\theta d^2\bar{\theta} K(\Phi, \bar{\Phi}) + \int d^2\theta W(\Phi) + \int d^2\bar{\theta} \bar{W}(\bar{\Phi}) \ . \tag{3.1}$$

Expanding the Lagrangian in components, the kinetic terms for the scalar fields of the chiral multiplets are $g_{i\bar{j}} \partial_\mu \bar{\phi}^{\bar{j}} \partial^\mu \phi^i$, where $g_{i\bar{j}} = \partial_i \bar{\partial}_{\bar{j}} K$, ∂_i indicating derivatives with respect to the scalar fields. The Lagrangian defines a *non-linear sigma model*, where the fields $\phi(x)$ are viewed as maps from the four-dimensional space time to a *target space* $\phi : \mathbb{R}^{1,3} \to \mathcal{M}$, defined by $x^\mu \mapsto \phi(x)$; hence, the scalar fields are local coordinates on $\mathcal{F}$. Consequently, $g_{i\bar{j}}$ is a Riemannian metric on this space. In fact, the existence of the Kähler potential implies that the target space of our $4d$ $\mathcal{N} = 1$ theory is a Kähler manifold.

More generally, fields ϕ are sections of a bundle $\mathcal{F} \to E \to M$, with the fiber F being the target space, while the base M in this construction is the above space-time manifold. In the case of supersymmetric gauge theories with gauge group G, E iss an affine bundle whose sections are connections on a principal G-bundle over M. Let us note that the symmetries of such a model are the Poincare group for the flat space $M = \mathbb{R}^{1,3}$ and the internal symmetries of the target space, i.e. those generated by the Killing vectors of the fiber:

$$\delta\phi^i = \alpha^A k_A \phi^i = \alpha^A k_A^i \ , \tag{3.2}$$

where α^A are constant parameters and $k_A = k_A^i \partial_i$ are Killing vectors generating the Lie algebra of the isometry group.

with the Lagrangian density:

$$\mathcal{L} = \int d^\theta d^2\bar{\theta} \Phi^\dagger e^V \Phi + \frac{1}{4g^2} \int d^2\theta Tr\left(\mathcal{W}^\alpha \mathcal{W}_\alpha\right) + \int d^2\theta W(\Phi) + c.c. \ , \tag{3.3}$$

where Φ is a chiral multiplet transforming is a representation of the gauge group G, V is a Lie algebra valued vector multiplet, while $\mathcal{W}$ is the associated field strength and $W(\Phi)$ is the superpotential. Given Ω a chiral multiplet in the adjoint representation of the Lie algebra of G and the group element $g = e^{i\Omega}$, one can check that the above Lagrangian is invariant under:

$$\Phi \mapsto g \cdot \Phi \quad e^V \mapsto (g^{-1})^\dagger e^V g^{-1} \ . \tag{3.4}$$

Let us note that these induce the transformation $\mathcal{W}_\alpha \mapsto e^{i\Omega} \mathcal{W}_\alpha e^{-i\Omega}$ on the field strength. Let us note, that in this general form where the scalar component of V is kept finite, the above also includes the transformations of the complexified gauge group[3]. Let us note that this vector multiplet also contains an auxiliary field (in the adjoint of $\mathfrak{g}$) which we will denote by $D = D^a T_a$, for T_a the generators of $\mathfrak{g}$. We are interested in the (classical) vacuum moduli space of the gauge theory, which is parametrized by the values of ϕ and is defined as the vanishing locus of $\partial_\phi W$ and of the *moment map operators.*

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[3] If we were to use the Wess-Zumino gauge where, among others, to scalar of the vector superfield is set to 0, the $G_{\mathbb{C}}$ invariance would be broken down to G, but we do not impose this here.