Groups and Representations - Lecture Notes

Horia Magureanu

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1 Introduction

These notes are mainly based on Prof. Andre Lukas' lecture notes ('Groups and Representations' course, Oxford University). A range of books were used as inspiration, from which I recommend 'Lie Algebras in Particle Physics' by H. Georgi. The classic group theory reference is of course 'Representation Theory: A first course', by Fulton and Harris, where many of the proofs can actually be found. For the differentiable manifolds section, the short book by M. Spivak ('Calculus on Manifolds') was used. For completeness, I will also list 'Matrix Groups - an introduction to Lie group theory' by A. Baker and 'Lie Groups, Lie Algebras and Representations' by B. C. Hall. Apart from Georgi's book, 'Lie groups, Physics and geometry' by R. Gilmore might also be useful for those with a physics background.

2 Groups and Representations

Definition 2.1 A *group* G is a set with a map $\cdot : G \times G \to G$ ("multiplication") which satisfies:
(G_0) $g_1 \cdot g_2 \in G, \forall g_1, g_2 \in G$ (closure)
(G_1) $g_1 \cdot (g_2 \cdot g_3) = (g_1 \cdot g_3) \cdot g_3, \forall g_1, g_2, g_3 \in G$ (associativity)
(G_2) $\exists e \in G : e \cdot g = g, \forall g \in G$ (neutral element)
(G_3) $\forall g \in G, \exists g' \in G : g' \cdot g = e$ (inverse)
If in addition we have $g_1 \cdot g_2 = g_2 \cdot g_1, \forall g_1, g_2 \in G$, the group is called Abelian.

Remarks: (a) A left inverse is also a right inverse.
(b) The inverse for a given group element is unique.
(c) A left-unit is also a right unit.
(d) The neutral element is unique.

Proof. (a) $g' \cdot g = e$ and $g'' \cdot g' = e \Rightarrow g \cdot g' = e \cdot g \cdot g' = g'' \cdot g' \cdot g \cdot g' = g'' \cdot g' = e$. The proofs for (b), (c) and (d) are similar.

□

The inverse for $g \in G$ is also denoted by g^{-1}. Note the following: $(g^{-1})^{-1} = g$ and $(g_1 \cdot g_2)^{-1} = (g_1)^{-1} \cdot (g_2)^{-1}$.

Definition 2.2 A relation $\sim$ is called an *equivalence relation* (on a set S) if and only if:
$(E1)$ $s \sim s, \forall s \in S$ (reflexivity)
$(E2)$ $s \sim s' \Rightarrow s' \sim s, \forall s, s' \in S$ (symmetry)
$(E3)$ $s \sim s'$ and $s' \sim s'' \Rightarrow s \sim s'', \forall s, s', s'' \in S$ (Transitivity)

Definition 2.3 An *equivalence class* is defined as: $[s] = \{s' \in S | \ s' \sim s\}$

Remark Two equivalence classes are either disjoint or equal.

Proof. Consider $[s]$, $[\tilde{s}]$. If $[s] \bigcap [\tilde{s}] = \{\}$, the statement is trivial. Assume that $s' \in [s] \bigcap [\tilde{s}]$. If $s'' \in [s]$, then by definition $s'' \sim s'$. Thus, $s'' \in [\tilde{s}]$, so $[s] \subset [\tilde{s}]$. But the argument is symmetric so $[s] = [\tilde{s}]$.

□

Definition 2.4 Two group elements $g_1, g_2 \in G$ are called *conjugate* iff $\exists g \in G : g_1 = g g_2 g^{-1}$. This is an equivalence relation, with its equivalence classes being the *conjugacy classes* of G. (This can be easily shown be verifying E1-3).

Definition 2.5 A subset $H \subset G$ of a group G is called a *sub-group* if it forms a group by itself, under the 'multiplication' on G. The two sub-groups $\{e\}$ and $G \subset G$ are the trivial sub-groups.
For a group G and a sub-group $H \subset G$, we define a relation:

$$g_1 \sim g_2 :\Leftrightarrow g_1^{-1} g_2 \in H$$

This is an equivalence relation, with equivalence classes $gH = \{g \cdot h | h \in H\}$ being called (left) *cosets*.
Note that the above relation is identical to: $g_2 h = g_1, h \in H$.

Let G be a finite group. The *order* of the group is denoted by $|G|$ (i.e number of elements). Then, every coset gH has the same order $|H|$, so $|H|$ divides $|G|$. Also, if $|G|$ is prime, then G only has trivial sub-groups.

Definition 2.6 A sub-group $H \subset G$ is *normal* iff $gH = Hg, \forall g \in G$. If $H \subset G$ is normal, then the *quotient* $G/H = \{gH | g \in G\}$ can be made into a group with multiplication $(g_1 H)(g_2 H) := (g_1 \cdot g_2)H$. Again, one can verify the group axioms.

Definition 2.7 A map $f : G \to \tilde{G}$ between two groups $G, \tilde{G}$ is called a (group) *homomorphism* iff $f(g_1 g_2) = f(g_1) f(g_2)$. We define $\mathbf{Ker}(f) = \{g \in G | f(g) = \tilde{e}\} \subset G$ and $\mathbf{Im}(f) = \{f(g) | g \in G\} \subset \tilde{G}$.
Remarks: $f(e) = \tilde{e} \Rightarrow e \in Ker(f), f(g^{-1}) = f(g)^{-1}$.
$Im(f)$ is a sub-group of $\tilde{G}$. $Ker(f)$ is a normal sub-group of G.
The map f is one-to-one (injective) $\Leftrightarrow Ker(f) = \{e\}$. It is onto (surjective) $\Leftrightarrow Im(f) = \tilde{G}$.

Theorem 2.8 For a group homomorphism $f : G \to \tilde{G}$, we have:

$$G/Ker(f) \cong Im(f)$$

If f is bijective this is obvious. Otherwise, the quotient has to be used instead of the group itself.
Let V be a vector space over a field F ($\mathbb{R}$ or $\mathbb{C}$). The set $Gl(V)$ of invertible linear maps $l : V \to V$ forms a group, called the general linear group.

Definition 2.9 A *representation* $\mathcal{R}$ of a group G is a group homomorphism $\mathcal{R} : G \to Gl(V)$. This means: $\mathcal{R}(g_1 g_2) = \mathcal{R}(g_1)\mathcal{R}(g_2)$. The dimension of $\mathcal{R}$ is defined as $dim(\mathcal{R}) := dim(V)$.

Definition 2.10 Two representations $\mathcal{R}_1 : G \to Gl(V)$, $\mathcal{R}_2 : G \to Gl(V)$ are called *equivalent* if there is a $\phi \in Gl(V) : \mathcal{R}_1(g) = \phi \mathcal{R}_2(g) \phi^{-1}, \forall g \in G$.
A representation is called *unitary* iff all $\mathcal{R}(g)$ are unitary.
A representation $\mathcal{R}$ is called *faithful* iff $\mathcal{R}$ is injective.

Definition 2.11 A representation $\mathcal{R} : G \to Gl(V)$ is called *reducible* if there is a subspace $U \subset V$, where $U \neq \{0\}, V$ such that $\mathcal{R}(g)U \subset U, \forall g \in G$. Otherwise, the representation is called *irreducible.*
For some object from $U : \begin{pmatrix} 0 \\ u \end{pmatrix}$, in order to obtain another object from U after acting with $\mathcal{R}(g)$ we must have a matrix of the form: $\mathcal{R}(g) \sim \begin{pmatrix} A(g) & 0 \\ C(g) & B(g) \end{pmatrix}$ acting on some object.

A reducible representation $\mathcal{R} : G \to Gl(V)$ is called *fully reducible* iff: $V = V_1 \oplus ... \oplus V_k, \mathcal{R}(g)V_i \subset V_i$ and $\mathcal{R}|_{V_i} =: \mathcal{R}_i$ is irreducible for $i = 1, ..., k$. In the previous example, this could be obtained for $C(g) = 0$ and

A, B irreducible.

Proposition 2.12 Every reducible, unitary representation if fully reducible.

Proof. Since $\mathcal{R}$ is unitary, we must have $\langle \mathcal{R}(g)v_1, \mathcal{R}(g)v_2 \rangle = \langle v_1, v_2 \rangle, \forall v_1, v_2 \in V$. Assume $U \subset V : \mathcal{R}(g)U \subset U$ and let $W = U^\perp$ so $V = U \oplus W$. Then for $u \in U, w \in W$: $\langle \mathcal{R}(g)w, u \rangle = \langle w, \underbrace{\mathcal{R}(g)^{-1}u}_{\in U} \rangle = 0 \Rightarrow \mathcal{R}(g)W \subset W$, and iterate. $\square$

Corollary 2.13 Any representation of a finite dimensional group is unitary (relative to some scalar product) and hence fully reducible (if it was reducible).

Proof. Let $\mathcal{R} : G \to Gl(V)$, inner product $\langle \cdot, \cdot \rangle$ on V. Define $\langle v, w \rangle := \sum_{g \in G} \langle \mathcal{R}(g)v, \mathcal{R}(g)w \rangle \Rightarrow \langle \mathcal{R}(g)v, \mathcal{R}(g)w \rangle = \langle v, w \rangle \Rightarrow \mathcal{R}$ unitary relative to $\langle \cdot, \cdot \rangle$. Then, the statement follows from the previous proposition. $\square$

Lemma 2.14 (Schur's lemma) Let $\mathcal{R} : G \to Gl(V), \tilde{\mathcal{R}} : G \to Gl(W)$ be two *irreducible* representations, V, W complex vector spaces, $\phi : V \to W$ a linear map with $\phi \circ \mathcal{R}(g) = \tilde{\mathcal{R}}(g) \circ \phi, \forall g \in G$. Then:
(i) Either ϕ is an isomorphism or $\phi = 0$.
(ii) If $\mathcal{R} = \tilde{\mathcal{R}}$, then $\phi = \lambda \mathbb{1}, \lambda \in \mathbb{C}$

Proof. (i) Let $v \in Ker(\phi)$ so $\phi(v) = 0 \Rightarrow \phi(\mathcal{R}(g)v) = \tilde{\mathcal{R}}(g)(\phi(v)) = 0 \Rightarrow \mathcal{R}(g)Ker(\phi) \subset Ker(\phi)$. Similarly $\tilde{\mathcal{R}}Im(\phi) \subset Im(\phi)$. Hence, by definition of irreducible representations: $Ker(\phi) = \{0\}$ or V and $Im(\phi) = \{0\}$ or W. One must now check all these combinations: e.g. $Ker(\phi) = \{0\} \Rightarrow \phi$ injective and $Im(\phi) = W$ as it cannot be $\{0\}$ since their dimensions must add to $dim(V)$; thus, ϕ surjective so it is an isomorphism etc.

(ii) Consider the space $Eig_\phi(\lambda) = \{v \in V | \phi v = \lambda v\}$. At least one eigenvalue $\lambda \neq 0$, since $det(\phi - \lambda \mathbb{1}) = 0$ has solutions in $\mathbb{C}$, so $Eig_\phi(\lambda) \neq \{0\}$. Take a non-zero eigenvector v from this space: $\phi(\mathcal{R}(g)(v)) = \mathcal{R}(g)(\phi(v)) = \mathcal{R}(g)(\lambda v) = \lambda \mathcal{R}(g)(v)$, so $\mathcal{R}(g)(v) \in Eig_\phi(\lambda)$ as well. Then $Eig_\phi(\lambda)$ is invariant unde $\mathcal{R} \Rightarrow Eig_\phi(\lambda) = V$ since $\mathcal{R}$ is irreducible, so $\phi = \lambda \mathbb{1}$. $\square$

Corollary 2.15 All complex irreducible representations of an Abelian group are *one-dimensional*.

Proof. G abelian: $g\tilde{g} = \tilde{g}g \Rightarrow \mathcal{R}(g)\mathcal{R}(\tilde{g}) = \mathcal{R}(\tilde{g})\mathcal{R}(g), \forall g, g \in G$. Fix g and call $\mathcal{R}(g) = \phi$ so this now satisfies Schur's lemma. Thus: $\mathcal{R}(g) = \lambda(g)\mathbb{1}$, consistent with $\mathcal{R}$ being irreducible if and only if $dim\mathcal{R} = 1$ (i.e otherwise, the resulting diagonal matrix is reducible).

$\square$

Examples (a) Consider the Abelian group: $G = \mathbb{Z}_n, \mathbb{Z}_n = \{0, 1, ...n-1\}$, with $k \cdot l = (k+l) mod\ n$ and let $\mathcal{R} : \mathbb{Z}_n \to \mathbb{C}^*$ be an irreducible representation of this group (from the corollary it must be one dimensional):
$\mathcal{R}(1)^n = \mathcal{R}(1^n) = \mathcal{R}(0) = 1 \Rightarrow \mathcal{R}(1) = e^{\frac{2\pi i q}{n}}, q = 0, ..., n-1$ and thus: $\mathcal{R}(k) = \mathcal{R}(1)^k = e^{\frac{2\pi i q k}{n}}$

$\mathbb{Z}_n$ has n complex irreducible representations $\mathcal{R}_q, q = 0, ..., n-1$, given by $\mathcal{R}_q(k) = e^{\frac{2\pi i q k}{n}}$.

(b) The Unitary group $U(1) = \{z \in \mathbb{C} |\ |z| = 1\}$

The irreducible representations of $U(1)$ are given by $\mathcal{R}_q$, where $q \in \mathbb{Z}$ and $\mathcal{R}_q(e^{i\phi}) = e^{iq\phi}$.

2.1 New representations from old ones

In this subsection we will see how we can build new representations starting with some known representations of a group:

• Trivial representation: $\mathcal{R}(g) = \mathbb{1}, \forall g \in G$.

•Direct sum: given $\mathcal{R}_1, \mathcal{R}_2$, define $\mathcal{R} = \mathcal{R}_1 \oplus \mathcal{R}_2$ by: $\mathcal{R}(g) = \begin{pmatrix} \mathcal{R}_1(g) & 0 \\ 0 & \mathcal{R}_2(g) \end{pmatrix}$. This is fully reducible.

• Dual representation: Let us first recall that the dual space of some vector space V is defined as: $V' = \{f | f : V \to \mathbb{R}\}$, with $dimV = dimV'$ and if e^i form a basis of V, then the dual basis elements e'_j satisfy: $e'_i(e_j) = \delta_{ij}$. Then, given a representation $\mathcal{R} : G \to Gl(V)$, we define the dual representation $\mathcal{R}' : G \to Gl(V')$ by: $(\mathcal{R}'(g)\phi)(\mathcal{R}(g)v) = \phi(v), \forall \phi \in V', \forall v \in V$. Thus: $\mathcal{R}'(g) = (\mathcal{R}(g)^{-1})^T$. One can check that: $\mathcal{R}'(g\tilde{g}) = \mathcal{R}'(g)\mathcal{R}'(\tilde{g})$. Note also that if $\mathcal{R}$ is unitary, then the dual representation is the complex conjugate representation: $\mathcal{R}' = \mathcal{R}^*$.

• Tensor product representations: Given two vector spaces V and W, with basis elements $e^i, i = 1, ...n$ and $\epsilon^a, a = 1, ...n$, then the space $V \otimes W$ has basis $e^i \otimes \epsilon^a$. Also, $dim(V \otimes W) = dim(V)dim(W)$. Consider now two representations $\mathcal{R}_V : G \to Gl(V)$, $\mathcal{R}_W : G \to Gl(W)$. Then, define:

$$(\mathcal{R}_V \otimes \mathcal{R}_W(g))(v \otimes w) := (\mathcal{R}_V(g)v) \otimes (\mathcal{R}_W(g)w)$$

This object can be constructed using the Kronecker product for matrices: $(\mathcal{R}_V \otimes \mathcal{R}_W)(g) \longleftrightarrow [\mathcal{R}_V] \times [\mathcal{R}_W]$.

• Induced representation on $Hom(V, W)$: Given two vector spaces V, W over a field F, define the set of all vector space homomorphisms of V onto W as $Hom(V, W)$. ....

• Clebsch-Gordon decomposition: $\mathcal{R}_1 \otimes \mathcal{R}_2 = \oplus \mathcal{R}_s$, so $dim(\mathcal{R}_1 \otimes \mathcal{R}_2) = dim(\mathcal{R}_1)dim(\mathcal{R}_2) = \sum dim(\mathcal{R}_s)$.

• Sub-group and branching rules: Let $H \subset G$ be a sub-group; a $\mathcal{R}^{(G)}$ a representation of G induces a representation $\mathcal{R}^{(H)}$ of H by: $\mathcal{R}^H(h) = \mathcal{R}^G(h), h \in H$. However, there is no guarantee $\mathcal{R}^H$ remains/becomes (ir)reducible. For these one needs some branching rules.

3 Finite groups

Definition 3.1 Let $\mathcal{R} : G \to Gl(V)$ be a representation of the finite group G. Define: $\boxed{\chi_{\mathcal{R}}(g) := tr(\mathcal{R}(g))}$. $\chi_{\mathcal{R}}$ is called the *character* of the representation $\mathcal{R}$. Note that we could have used other basis invariant quantities (such as the determinant), but this is the simplest definition.

Note that $\chi_{\mathcal{R}}$ is constant on conjugacy classes, so it is a class function: $\chi_{\mathcal{R}}(\tilde{g}g\tilde{g}^{-1}) = tr(\mathcal{R}(\tilde{g})\mathcal{R}(g)\mathcal{R}(\tilde{g})^{-1}) = tr(\mathcal{R}(g)) = \chi_{\mathcal{R}}(g)$. Also: $\chi_{\mathcal{R}}(e) = dim(\mathcal{R})$ since $e \longmapsto id$.

Proposition 3.2 The following statements about the character are true:
(i) $\chi_{\mathcal{R}_1 \oplus \mathcal{R}_2}(g) = \chi_{\mathcal{R}_1}(g) + \chi_{\mathcal{R}_2}(g)$
(ii) $\chi_{\mathcal{R}_1 \otimes \mathcal{R}_2}(g) = \chi_{\mathcal{R}_1}(g)\chi_{\mathcal{R}_2}(g)$
(iii) $\chi_{\mathcal{R}'}(g) = \chi_{\mathcal{R}}(g)^*$

Proof. (i) $\mathcal{R}_1 \oplus \mathcal{R}_2(g) \to \begin{pmatrix} \mathcal{R}_1(g) & 0 \\ 0 & \mathcal{R}_2(g) \end{pmatrix}$, so the statement is clear.
(ii) Follows from the Kronecker product: $tr(A \times B) = tr(A)tr(B)$.
(iii) Diagonalise $\mathcal{R}$. Since $g^k = e$, for some k (i.e the group is finite): $\mathcal{R}(g) = diag(\lambda_1, ...\lambda_k)$, with λ_i being the k^{th} roots of unity. Then: $\mathcal{R}'(g) = (\mathcal{R}(g)^{-1})^T \to diag(\lambda_1^*, ..., \lambda_n^*) \Rightarrow tr(\mathcal{R}'(g)) = tr(\mathcal{R}(g))^*$. □

Proposition 3.3 For a representation $\mathcal{R} : G \to Gl(V)$, define the singlets $V^G = \{v \in V | \mathcal{R}(g)v = v, \forall g \in G\}$ and the linear map $p := \frac{1}{|G|}\sum_{g \in G} \mathcal{R}(g) : V \to V$. Then, p is a *projector* onto V^G.

Proof. We need to show that $Im(p) = V^G$ and that $p \cdot p = p$.
(i) Let $v \in Im(p)$, so $v = p(w) = \frac{1}{|G|}\sum_{g \in G} \mathcal{R}(g)w$. Also: $\mathcal{R}(\tilde{g})v = \frac{1}{|G|}\sum_{g \in G} \mathcal{R}(\tilde{g}g)w = \frac{1}{|G|}\sum_{g' \in G} \mathcal{R}(g')w = v \Rightarrow v \in V^G$.
(ii) Let $v \in V^G$ so $\mathcal{R}(g)v = v, \forall g \in G$. Then $p(v) = \frac{1}{|G|}\sum_{g \in G} \mathcal{R}(g)v = v \Rightarrow v \in Im(p)$.
(iii) $p \cdot \underbrace{p(v)}_{w} = p(\underbrace{w}_{\in Im(p)}) = w = p(v)$, so $p \cdot p = p$ □

Note that: $dim(V^G) = tr(p) = \frac{1}{|G|}\sum_{g \in G} \chi_{\mathcal{R}}(g)$.

Let us now return to the $Hom(V, W)$ space (i.e the space of all linear maps). Consider two complex, irreducible representations $\mathcal{R}_V : G \to Gl(V)$, $\mathcal{R}_W : G \to Gl(W)$. Then:

$$Hom(V, W)^G = \{\phi \in Hom(V, W) | \underbrace{\mathcal{R}_{Hom(V,W)}(g)(\phi)}_{\mathcal{R}_W(g)\phi\mathcal{R}_V(g)^{-1}} = \phi, \forall g \in G\} = \{\phi \in Hom(V, W) | \mathcal{R}_W(g) \cdot \phi = \phi \cdot \mathcal{R}_V(g), \forall g \in G\}$$

Note that by Schur's lemma we have: $dim(Hom(V, W)^G) = \begin{cases} 1, & \text{if } \mathcal{R}_V \cong \mathcal{R}_W \\ 0, & \text{otherwise} \end{cases}$. Remember also that $Hom(V, W) = V' \otimes W \Rightarrow \chi_{Hom(V,W)}(g) = \chi_V(g)^*\chi_W(g)$. However, the previous result implies that: $dim(Hom(V, W)^G) = \frac{1}{|G|}\sum_{g \in G} \chi_V(g)^*\chi_W(g)$. This result is important as it gives an indication of how we could define the inner product of these characters.

Definition 3.4 Between two class functions $\chi, \tilde{\chi} : G \to \mathbb{C}$, define an inner product by: $(\chi, \tilde{\chi}) := \frac{1}{|G|}\sum_{g \in G} \chi(g)^*\tilde{\chi}(g)$.

Theorem 3.5 Let $\mathcal{R}_1, ..., \mathcal{R}_k$ be the complex irreducible representations of G and let $\chi_1, ..., \chi_k$ be their corresponding characters. Then: $(\chi_i, \chi_j) = \delta_{ij}$. Hence, the characters of the complex irreducible representations form an ortho-normal system under $(\cdot, \cdot)$.

Corollary 3.6 The number of complex irreducible representations of a finite group G is less or equal to the number of conjugacy classes of G.

Theorem 3.7 Let $\mathcal{R}_1, ..., \mathcal{R}_k$ be all the complex irreducible representations of G and let $\chi_1, ..., \chi_k$ be their corresponding characters. Let $\mathcal{R}$ be any complex representation of G.
(i) The representation $\mathcal{R}_i$ appears in $\mathcal{R}$ exactly $(\chi_i, \chi_{\mathcal{R}})$ many times.
(ii) $\mathcal{R}$ is completely determined by its character $\chi_{\mathcal{R}}$.
(iii) $\mathcal{R}$ is irreducible if and only if $(\chi_{\mathcal{R}}, \chi_{\mathcal{R}}) = 1$.

Proof. (i) Let $\mathcal{R} = \mathcal{R}_1^{\oplus m_1} \oplus ... \oplus \mathcal{R}_k^{\oplus m_k}$. Then: $\chi_{\mathcal{R}} = \sum_j m_j \chi_j \Rightarrow (\chi_i, \chi_{\mathcal{R}}) = \sum_j m_j (\chi_i, \chi_j) = m_i$. It is clear that (ii) follows from (i). For (iii): $(\chi_{\mathcal{R}}, \chi_{\mathcal{R}}) = \sum_i \sum_j m_i m_j \delta_{ij} = \sum_i m_i^2$, and since $m_i \in \mathbb{N}$, the equality is proved. □

Definition 3.8 An algebra is a vector space with multiplication. For a finite group, the *group algebra* A_G consists of all formal linear combinations $v = \sum_{\tilde{g} \in G} v(\tilde{g})\tilde{g}$, for $\tilde{g}$ being linearly independent. Note that $dim(A_G) = |G|$. We also define the *regular representation* $\mathcal{R}_{reg} : G \to Gl(A_G)$ by: $\mathcal{R}_{reg}(g)v := gv$. Thus: $dim(\mathcal{R}_{reg}) = dim(A_G) = |G|$.

Theorem 3.9 Let $\mathcal{R}_1, ..., \mathcal{R}_k$ be all the complex irreducible representations of G and let $\chi_1, ..., \chi_k$ be their corresponding characters. Then:
(i) $\mathcal{R}_{reg} = \mathcal{R}_1^{\oplus dim(\mathcal{R}_1)} \oplus ... \oplus \mathcal{R}_k^{\oplus dim(\mathcal{R}_k)}$
(ii) $\sum_{i=1}^k (dim(\mathcal{R}_i))^2 = |G|$

Proof. Note that since $dim(\mathcal{R}_{reg}) = |G|$, (ii) follows from (i). For (i) choose a basis of A_G: $\{\tilde{g} \in G\}$. Then $\mathcal{R}_{reg}(g)\tilde{g} = g\tilde{g} = \begin{cases} \tilde{g}, & \text{if } g = e \\ \neq \tilde{g}, & \text{otherwise} \end{cases}$. Consequently, $\chi_{reg}(g) = \begin{cases} |G|, & \text{if } g = e \\ 0, & \text{if } g \neq e \end{cases}$, since in the latter case $g\tilde{g}$ is not of the form $\alpha\tilde{g}$. Hence: $(\chi_i, \chi_{reg}) = \frac{1}{|G|} \sum_{g \in G} \chi_i(g)^* \chi_{reg}(g) = \frac{1}{|G|} \underbrace{\chi_i(e)^*}_{dim(\mathcal{R}_i)} \underbrace{\chi_{reg}(e)}_{|G|} = dim(\mathcal{R}_i)$. □

Examples (a) $G = \mathbb{Z}_3 = \{0, 1, 2\}$. This is an *Abelian* group, so each element is a conjugacy class. Since it is Abelian, all complex irreducible representations are one-dimensional (by corollary 2.15), so using $\alpha = e^{2\pi i/3}$, the 3 irreducible representations are: $\mathcal{R}_0(g) = 1$, $\mathcal{R}_1(g) = \alpha^g$ and $\mathcal{R}_2(g) = \alpha^{2g}$. The character table is shown below (the conjugacy classes are written in the top row):

	0	**1**	**2**
χ_0	1	1	1
χ_1	1	α	α^2
χ_2	1	α^2	α

Since $1 + \alpha + \alpha^2 = 0$ and $|G| = 3$, it follows that: $(\chi_i, \chi_j) = \delta_{ij}$. Define $\mathcal{R} : \mathbb{Z}_3 \to Gl(\mathbb{R}^3)$ by $\mathcal{R}(g) = A^g$, where $A = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$, so $A^3 = \mathbb{I}_3$. This clearly cannot be an irreducible representation, as these must be one-dimensional; the character is given by: $\chi_{\mathcal{R}} = (3, 0, 0)$ (i.e. first entry corresponds to $g = \mathbf{0}$ etc.). Then, $(\chi_i, \chi_{\mathcal{R}}) = 1$, for all i's, so: $\mathcal{R} = \mathcal{R}_0 \oplus \mathcal{R}_1 \oplus \mathcal{R}_2$ is in fact the regular representation. The group algebra for $\mathbb{Z}_3$ is given by: $A_{\mathbb{Z}_3} = \{a\mathbf{0} + b\mathbf{1} + c\mathbf{2} |\ a, b, c \in \mathbb{C}\}$. Then:

$$\mathcal{R}_{reg}(\mathbf{0}) \begin{cases} \mathbf{0} \\ \mathbf{1} \\ \mathbf{2} \end{cases} = \begin{cases} \mathbf{0} \\ \mathbf{1} \\ \mathbf{2} \end{cases}, \mathcal{R}_{reg}(\mathbf{1}) \begin{cases} \mathbf{0} \\ \mathbf{1} \\ \mathbf{2} \end{cases} = \begin{cases} \mathbf{1} \\ \mathbf{2} \\ \mathbf{0} \end{cases} \text{ and } \mathcal{R}_{reg}(\mathbf{2}) \begin{cases} \mathbf{0} \\ \mathbf{1} \\ \mathbf{2} \end{cases} = \begin{cases} \mathbf{2} \\ \mathbf{0} \\ \mathbf{1} \end{cases}.$$

Hence, this allows us to build the matrices:

$$\mathcal{R}_{reg}(\mathbf{0}) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \mathcal{R}_{reg}(\mathbf{1}) = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} = A \text{ and } \mathcal{R}_{reg}(\mathbf{2}) = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix} = A^2$$

(b) Quaternionic group, $H = \{\pm 1, \pm i, \pm j, \pm k\}$, with $i^2 = j^2 = k^2 = -1$ and $ij = k$ - and cyclic permutations (non-Abelian). Note that $|G| = 8$. The conjugacy classes are given by: $C_1 = \{1\}$, $C_{-1} = \{-1\}$, $C_i = \{\pm i\}$, $C_j = \{\pm j\}$ and $C_k = \{\pm k\}$, since $jij^{-1} = -jij = -jk = -i$ etc. Thus, we have at most 5 irreducible representations (by corollary 3.6). Also: $\sum[dim(\mathcal{R}_i)]^2 = 8$ by theorem 3.9. There is always a trivial representation (which is one-dimensional), so we must have one 2-dim and four 1-dim representations. These are:

$\mathcal{R}_1$: $\mathcal{R}_1(g) = 1, \forall g \in H$
$\mathcal{R}_i$: $\mathcal{R}_i(\pm 1) = 1, \mathcal{R}_i(\pm i) = 1, \mathcal{R}_i(\pm j) = -1, \mathcal{R}_i(\pm k) = -1$
$\mathcal{R}_j$: $\mathcal{R}_j(\pm 1) = 1, \mathcal{R}_j(\pm i) = -1, \mathcal{R}_j(\pm j) = 1, \mathcal{R}_j(\pm k) = -1$
$\mathcal{R}_k$: $\mathcal{R}_k(\pm 1) = 1, \mathcal{R}_k(\pm i) = -1, \mathcal{R}_k(\pm j) = -1, \mathcal{R}_k(\pm k) = 1$, and since Pauli matrices multiply in a cyclic way:
$\mathcal{R}_2$: $\mathcal{R}_2(\pm 1) = \pm\mathbb{I}_2, \mathcal{R}_2(\pm i) = \pm i\sigma_3, \mathcal{R}_2(\pm j) = \pm i\sigma_3, \mathcal{R}_2(\pm k) = \pm i\sigma_1$.

The character table can then be computed:

	C_1	C_{-1}	C_i	C_j	C_k
# elements	1	1	2	2	2
χ_1	1	1	1	1	1
χ_i	1	1	1	-1	-1
χ_j	1	1	-1	1	-1
χ_k	1	1	-1	-1	1
χ_2	2	-2	0	0	0

We can build higher dimensional representations using these irreducible representations. For instance, consider the 4-dimensional one: $\mathcal{R}_4(\pm 1) = \pm\mathbb{I}_4$ and:

$$\mathcal{R}_4(\pm i) = \pm \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix}, \mathcal{R}_4(\pm j) = \pm \begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}, \mathcal{R}_4(\pm k) = \pm \begin{pmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix}.$$

Then, its character is $\chi_{\mathcal{R}_4} = (4, -4, 0, 0, 0)$, and $(\chi_{1,i,j,k}, \chi_{\mathcal{R}_4}) = 0$ and $(\chi_2, \chi_{\mathcal{R}_4}) = 2$. Thus: $\mathcal{R}_4 = \mathcal{R}_2 \oplus \mathcal{R}_2$. Also, the tensor product representation $\mathcal{R}_2 \otimes \mathcal{R}_2$ has a character $\chi_{\mathcal{R}_2\otimes\mathcal{R}_2} = (4, 4, 0, 0, 0)$ and $(\chi_{1,i,j,k}, \chi_{\mathcal{R}_2\otimes\mathcal{R}_2}) = 1$, $(\chi_2, \chi_{\mathcal{R}_2\otimes\mathcal{R}_2}) = 0$, so we have: $\mathcal{R}_2 \otimes \mathcal{R}_2 = \mathcal{R}_1 \oplus \mathcal{R}_i \oplus \mathcal{R}_j \oplus \mathcal{R}_k$.

Lemma 3.10 Let $\alpha : G \to G$ be a *class function*, $\mathcal{R} : G \to Gl(V)$ a representation and $\phi_{\alpha,\mathcal{R}} = \sum_{g\in G} \mathcal{R}(g) : V \to V$. Then:
(*i*) $[\phi_{\alpha,\mathcal{R}}, \mathcal{R}(h)] = 0, \forall h \in G$
(*ii*) If $\mathcal{R}$ is irreducible, then $\phi_{\alpha,\mathcal{R}} = \lambda\mathbb{I}$.

Proof. $\phi_{\alpha,\mathcal{R}}(\mathcal{R}(h)v) = \sum_{g\in G} \alpha(g)\mathcal{R}(g)\mathcal{R}(h)v \overset{g\to hgh^{-1}}{=} \sum_{g\in G} \alpha(hgh^{-1})\mathcal{R}(hg)v = \sum_{g\in G} \alpha(g)\mathcal{R}(h)\mathcal{R}(g)v$, where the last equality follows since α is a class function. It is then clear that this is equal to $\mathcal{R}(h)(\phi_{\alpha,\mathcal{R}}v)$, so the first statement is proved. (ii) follows from (i) by using Schur's lemma. □

Theorem 3.11 The *number of complex irreducible representations* of G **equals** the *number of conjugacy classes* of G.

Proof. We show that any class function $\alpha : G \to \mathbb{C}$ with $(\alpha, \chi_i) = 0$ for all characters χ_i of the irreducible representations satisfy: $\alpha = 0$ (i.e. there are no other class functions perpendicular to the χ_i's, so these form a basis of this space). Define $\phi_{\alpha,i} = \sum_{g\in G} \alpha^*(g)\mathcal{R}_i(g)$, for $i = 1, ..., k$. Then, using the previous lemma:

$\phi_{\alpha,i} = \lambda\mathbb{I} \Rightarrow \lambda = \frac{1}{dim(\mathcal{R}_i)} tr(\phi_{\alpha,i}) = \frac{1}{dim(\mathcal{R}_i)} \sum_{g\in G} \alpha^*(g) tr(\mathcal{R}_i(g)) = \frac{|G|}{dim(\mathcal{R}_i)}(\alpha, \chi_i)$. For the last equality we simply used the definitions of the character and of their inner product. Consequently: $\lambda = 0$, so we must have $\phi_{\alpha,i} = 0$ for all $i = 1, ..., k$. Then, recalling the definition of the regular representation (def 3.8): $0 = \phi_{\alpha,reg} = \sum_{g\in G} \alpha^*(g) g \Rightarrow \alpha(g) = 0, \forall g \in G \Rightarrow \alpha = 0$. □

Theorem 3.12 $P_i = \frac{dim(\mathcal{R}_i)}{|G|} \sum_{g\in G} \chi_i(g)^* g : A_G \to A_G$ is a projector onto $V_i^{\oplus dim(V_i)} \subset A_G$, where V_i is the vector space associated with the irreducible representation $\mathcal{R}_i$. Note that this is not a projector onto an irreducible representation, but into $dim(V_i)$ copies of $\mathcal{R}_i$.

Proof. Consider a map $\psi = \frac{1}{|G|} \sum_{g\in G} \chi_i(g)^* \mathcal{R}(g) : V \to V$, with $\mathcal{R}$ an irreducible representation. Then, from lemma 3.10: $\psi = \lambda\mathbb{I}$, so, as before:

$$\lambda = \frac{tr(\psi)}{dim(\mathcal{R})} = \frac{(\chi_i, \chi_\mathcal{R})}{dim(\mathcal{R})} = \begin{cases} \dim(\mathcal{R}), & \text{if } \mathcal{R} = i \\ 0, & \text{otherwise} \end{cases}$$

Now consider $P_i|_{V_j} = \begin{cases} \mathbb{I}, & \text{if } i = j \\ 0, & \text{otherwise} \end{cases}$ □

Example <u>Permutation groups</u> $S_n = \{\sigma : \{1, ..., n\} \to \{1, ..., n\} |\ \sigma \text{ is bijective}\}$

Note that $|S_n| = n!$. Also define the map $sgn : S_n \to \{1, -1\} \cong \mathbb{Z}_2$ by: $sgn(\sigma) := \prod_{i<j} \frac{\sigma(j)-\sigma(i)}{j-i}$. Hence $sgn(\sigma_1\sigma_2) = sgn(\sigma_1)sgn(\sigma_2)$, so this defines a group homomorphism.

The conjugacy classes of this group are in *one-to-one correspondence with partitions* of $n = \lambda_1 + ... + \lambda_k$, for $\lambda_1 \geq \lambda_2 \geq ... \geq \lambda_k \geq 1$. It is often convenient to write the group elements in terms of cycles, for example:

$$\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 2 & 3 & 1 & 5 & 4 \end{pmatrix} = (123)(45)$$

It is clear now that the previously mentioned λ's are the lengths of these cycles. Note that if two permutations have the same λ's, we would only have to 'rename' them such that they are conjugate. To a partition $\lambda = (\lambda_1, ..., \lambda_k)$ of n, we can associate a *Young tableaux* with k rows:

$$\begin{array}{|c|c|c|c|} \hline 1 & 2 & ... & \\ \hline & & \multicolumn{2}{c}{} \\ \cline{1-2} n & \multicolumn{3}{c}{} \\ \cline{1-1} \end{array}$$

This allows us to define: $R_\lambda := \{g \in S_n | g \text{ preserves each } \textbf{row}\}$ and $C_\lambda := \{g \in S_n | g \text{ preserves each } \textbf{column}\}$, and $P_\lambda = c\left[\sum_{g\in R_\lambda} g\right]\left[\sum_{g\in C_\lambda} sgn(g) g\right]$. Then, $\exists c \in \mathbb{C} : P_\lambda^2 = P_\lambda$, so this is a projector. $P_\lambda A_{S_n}$ corresponds to an irreducible representation of S_n. Let us consider $S_2 = \{e = \begin{pmatrix} 1 & 2 \\ 1 & 2 \end{pmatrix}, g = \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}\}$. There are two partitions, with their corresponding Young tableaux shown below: $\lambda = (2, 0)$ $\begin{array}{|c|c|} \hline 1 & 2 \\ \hline \end{array}$ and $\lambda = (1, 1)$ $\begin{array}{|c|} \hline 1 \\ \hline 2 \\ \hline \end{array}$.

Then: $R_{(2,0)} = S_2$, $C_{(2,0)} = \{e\}$ and $R_{(1,1)} = \{e\}$, $C_{(1,1)} = S_2$ and the corresponding projectors are $P_{(2,0)} = c(e + g)$, $P_{(1,1)} = c(e - g)$, with c being fixed by the condition that $P_\lambda^2 = P_\lambda$. Then: $V_{(2,0)} = P_{(2,0)} A_{S_2} = Span(e + g)$ and $V_{(1,1)} = P_{(1,1)} A_{S_2} = Span(e - g)$.

$\mathcal{R}_{reg}(e)(e + g) = (e + g)$, $R_{(2,0)}(e) = 1$
$\mathcal{R}_{reg}(g)(e + g) = (e + g)$, $R_{(2,0)}(g) = 1$

$\mathcal{R}_{reg}(e)(e - g) = (e - g)$, $R_{(1,1)}(e) = 1$
$\mathcal{R}_{reg}(g)(e - g) = -(e - g)$, $R_{(1,1)}(g) = -1$

4 Lie groups

Definition 4.1 A group G is a *Lie group* if it is a differentiable manifold and if the group multiplication and inversion are differential maps.

Let us take a detour into Differentiable manifolds before going further into the analysis of Lie groups.

4.1 Differentiable manifolds

Definition 4.2 If U and V are open sets in $\mathbb{R}^n$, a differentiable function $h : U \to V$ with a differentiable inverse $h^{-1} : V \to U$ will be called a diffeomorphism.

Definition 4.3 A subset M of $\mathbb{R}^n$ is called a k-dimensional manifold if for every point $x \in M$ the following condition is satisfied:
(M) There is an open set U containing x, an open set $V \subset \mathbb{R}^n$, and a diffeomorphism $h : U \to V$ such that: $h(U \bigcap M) = V \bigcap(\mathbb{R}^k \times \{0\}) = \{y \in V : y^{k+1} = ... = y^n = 0\}$. In other words, $U \bigcap M$ is, 'up to diffeomorphism', simply $\mathbb{R}^k \times \{0\}$ (i.e. it locally resembles $\mathbb{R}^n$).

Theorem 4.4 A subset M of $\mathbb{R}^n$ is a k-dimensional manifold if and only if for each point $x \in M$ the following 'coordinate condition' is satisfied - there is an open set U containing x, an open set $W \bigcap \mathbb{R}^k$, and a 1-1 differentiable function $f : W \to \mathbb{R}^n$ such that:
(1) $f(W) = M \bigcap U$
(2) $f'(y)$ has rank k for each $y \in W$
(3) $f^{-1} : f(W) \to W$ is continuous. (such a function is called a coordinate system around x)

The proof of this theorem is beyond the aim of these notes, but it can be found, for example, in M. Spivak - 'Calculus on Manifolds' (p.111). Let us now introduce the concept of 'tangent space', first on $\mathbb{R}^n$ and then on any differentiable manifold.

Definition 4.5 If $p \in \mathbb{R}^n$, the set of all pairs (p, v), for $v \in \mathbb{R}^n$, is denoted $\mathbb{R}^n_p$, and called the *tangent space* of $\mathbb{R}^n$ at p. This set is made into a vector space by defining: $(p, v) + (p, w) = (p, v + w)$, and $a{\cdot}(p, v) = (p, av)$. (The vector $(p, v) \in \mathbb{R}^n_p$ may be pictured as an arrow with same direction and length as $v \in \mathbb{R}^n$, but with initial point p). A *vector field* is a function F such that $F(p) \in \mathbb{R}^n_p$ for each $p \in \mathbb{R}^n_p$. For each p there are numbers $F^1(p), ..., F^n(p)$ such that: $F(p) = F^1(p) \cdot (e_1)_p + ... + F^n(p) \cdot (e_n)_p$. The vector field is continuous, differentiable etc., if the functions F^i are.

Definition 4.6 Let M be a k-dimensional manifold in $\mathbb{R}^n$ and let $f : W \to \mathbb{R}^n$ be a coordinate system around $x = f(a)$. Since $f'(a)$ has rank k, the linear transformation $f_* : \mathbb{R}^k_a \to \mathbb{R}^n_x$ is 1-1, and $f_*(\mathbb{R}^k_a)$ is a k-dimensional subspace of $\mathbb{R}^n_x$. If $g : V \to \mathbb{R}^n$ is another coordinate system, with $x = g(b)$, then: $g_*(\mathbb{R}^k_b) = f_*(f^{-1} \circ g)_*(\mathbb{R}^k_b) = f_*(\mathbb{R}^k_a)$. Thus, the k-dimensional subspace $f_*(\mathbb{R}^k_a)$ does not depend on the coordinate system f, and is usually denoted by T_xM, being called the tangent space of M at x.

Definition 4.7 A vector field $\xi : C^\infty(M) \to C^\infty(M)$ is a linear map which satisfies: $\xi(fg) = f\xi(g) + g\xi(f)$. The map assigns a vector $\xi(f) \in T_xM$ for each $x \in M$: $\xi_x(f) = \xi(f)(x)$. Then, the tangent space is the set: $T_xM = \{\xi_x | \xi \text{ is a (local) vector field on } M\}$.

Definition 4.8 Consider a differentiable function $F : M_1 \to M_2$. Then, the *tangent map* to F at x, $T_xF : T_xM_1 \to T_{F(x)}M_2$ is defined by $T_xF(v)(f) := v(f \circ F)$, where f is a smooth function on M_2, $v \in T_xM_1(f \circ F \in M_1$ so v can act on it).

Note that Lebiniz's rule can be obtained from: $T_x(F \circ G)(v)(f) = v(f \circ F \circ G) = T_xG(v)(f \circ F) = w(f \circ F) = T_{G(x)}F(w)(f) \Rightarrow T_x(F \circ G) = T_{G(x)}(F)T_x(G)$, which is precisely the chain rule.

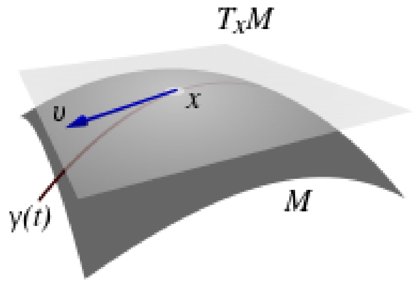


Figure 1: Tangent space

Also: $T_x(id_M)(v)(f) = v(f) \Rightarrow T_x(id_M)(v) = 0$ or $T_x(id_M) = \mathbb{I}$. Additionally: $T_x(F)^{-1} = T_{F(x)}(F^{-1})$. It is useful to introduce coordinates, for two continuous maps $\phi(x) = (x^1, ..., x^n)$, $\psi(y) = (y^1, ..., y^n)$, so then:

$$T_x(\phi)v = v^i(x)\frac{\partial}{\partial x^i}, T_x(\psi)v = w^i(y)\frac{\partial}{\partial y^i}$$

From the above equality it follows that: $dim(T_xM) = n = dim(M)$. Let us also introduce $\mathcal{F} := \psi \circ F \circ \phi^{-1}$ so:

$$T_{(x^1,...,x^n)}\mathcal{F}\left(v^i\frac{\partial}{\partial x^i}\right)(f)(x) = v^i\frac{\partial}{\partial x^i}(f \circ F)(x) = v^i\frac{\partial \mathcal{F}}{\partial x^i}\frac{\partial f}{\partial y^i} = \left(\frac{\partial \mathcal{F}}{\partial x}(x)\right)^j_i v^i\frac{\partial}{\partial y^j}(f)$$

which is indeed the Jacobi matrix.

Definition 4.9 The map $L_g : G \to G$ defined by $L_g x := gx$ is called a *left-translation*. A vector field ξ on G is called *left-invariant* if and only if $T_x L_g(\xi_x) = \xi_{gx}$.

Note that for two vector fields $\xi, \eta$the quantities ξ^2 and η^2 have second order derivatives, so they cannot be vector fields. However, the commutator: $[\xi, \eta] = \xi^i\frac{\partial}{\partial x^i}(\eta^j\frac{\partial}{\partial x^j}) - \eta^j\frac{\partial}{\partial x^j}(\xi^i\frac{\partial}{\partial x^i}) = \left(\xi^i\frac{\partial \eta_j}{\partial x^i} - \eta^i\frac{\partial \xi_j}{\partial x^i}\right)\frac{\partial}{\partial x^j}$ is a vector field since the 2nd order derivatives cancel.

Lemma 4.10 If ξ, η are left-invariant vector fields on a Lie group G, then $[\xi, \eta]$ is left-invariant as well.

Proof. Let $(g^\#\xi)_x := T_{gx}L_{g^{-1}}(\xi_{gx})$ so ξ is left invariant if and only if $g^\#\xi = \xi$. Then: $(g^\#\xi)_x(f) = T_{gx}L_{g^{-1}}(\xi_{gx})(f) = \xi_{gx}(f \circ g^{-1}) = \xi(f \circ g^{-1})(gx)$, so $(g^\#\xi)(f) = \xi(f \circ g^{-1}) \circ g$. Hence, computing the commutator:

$$\begin{aligned}[g^\#\xi, g^\#\eta](f) = g^\#\xi(\eta(f \circ g^{-1}) \circ g) - g^\#\eta(\xi(f \circ g^{-1}) \circ g) = \xi(\eta(f \circ g^{-1}) \circ g) - \eta(\xi(f \circ g^{-1}) \circ g) = \\ = [\xi, \eta](f \circ g^{-1}) \circ g = g^\#([\xi, \eta])(f)\end{aligned}$$ □

Definition 4.11 A *Lie algebra* $\mathcal{L}$ is a vector space with a bracket $[\cdot, \cdot] : \mathcal{L} \times \mathcal{L} \to \mathcal{L}$ which is bi-linear and satisfies:
(i) $[\xi, \eta] = -[\eta, \xi]$
(ii) $[\xi, [\eta, \lambda]] + [\eta, [\lambda, \xi]] + [\lambda, [\xi, \eta]] = 0$ (Jacobi identity)

Definition 4.12 Denote by $\mathcal{L}(G)$ the space of left-invariant vector fields on the Lie group G. Then $\mathcal{L}(G)$ is a Lie algebra and is called the *Lie algebra of* G.

Note that the Lie algebra of a Lie group can be identified with the tangent space at the identity of the group

$\boxed{\mathcal{L}(G) \cong T_eG}$ (as vector spaces), via $\xi \mapsto \xi_e$. Hence: $dim(\mathcal{L}(G)) = dim(T_eG) = dim(G)$.
In physics it is common to introduce a basis $(\xi_i)_{i=1,\dots,n}$ of $\mathcal{L}(G)$ and define the basis dependent quantities f_{ij}^k known as *structure constants* by:

$$[\xi_i, \xi_j] = f_{ij}^k \xi_k$$

Definition 4.13 A *representation of a Lie algebra*, $\mathcal{L}$ is a linear map $\Gamma : \mathcal{L} \to End(V)$ that preserves the commutator: $\Gamma([\xi, \eta]) = [\Gamma(\xi), \Gamma(\eta)], \forall \eta, \xi \in \mathcal{L}$. Using the basis introduced above, this translates as: $[\Gamma(\xi), \Gamma(\eta)] = f_{ij}^k \Gamma(\xi_k)$.

Here $End()$ is the set of all linear maps, and $Aut()$ is the set of invertible maps (matrices).

Definition 4.14 Define a group automorphism $C_g : G \to G$ ('conjugation') by: $C_g(x) := gxg^{-1}$, so $C : G \mapsto Aut(G)$, $g \mapsto C_g$. We will also define the following two 'adjoint' representations:

- **Ad**: $G \to Aut(\mathcal{L}G)$ defined by: $Ad(g) := T_eC_g$
- **ad**: $\mathcal{L}(G) \to End(\mathcal{L}G)$ defined by: $ad = T_eAd$

Remark (i) Ad is indeed a representation since: $C_{g_1 \cdot g_2}(x) = g_1 \cdot g_2 x (g_1 \cdot g_2)^{-1} = C_{g_1} \cdot C_{g_2}(x)$, so $Ad(g_1 \cdot g_2) = T_eC_{g_1 \cdot g_2} = T_e(C_{g_1} \cdot C_{g_2}) = T_e(C_{g_1}) \cdot T_e(C_{g_2}) = Ad(g_1)Ad(g_2)$.

(ii) ad is indeed a representation of Lie algebra, but the proof is slightly longer, but we will use the following theorem, without proving it here.

Theorem 4.15 $ad(\xi_e)(\eta_e) = [\xi, \eta]_e$

Then: $ad([\xi, \eta])(\lambda) = [[\xi, \eta], \lambda] = [\xi, [\eta, \lambda]] - [\eta, [\xi, \lambda]] = ad(\xi) \cdot ad(\eta)(\lambda) - ad(\eta) \cdot ad(\xi)(\lambda) = [ad(\xi), ad(\eta)](\lambda)$.

(iii) Given a basis (ξ_i) of $\mathcal{L}(G)$: $\boxed{[ad(\xi_i)]_j^k = f_{ij}^k}$ (i.e. the structure constants are representation matrices for the ad representation)

Before looking at some examples, there are still some 'bigger' questions to be answered. The first one is whether or not the tangent map at the identity of a group homomorphism always gives a Lie algebra homomorphism. To answer this, consider the following diagrams:

$$\begin{array}{ccc}
G & \xrightarrow{F} & \tilde{G} \\
\downarrow C_g & & \downarrow C_{F(g)} \\
G & \xrightarrow{F} & \tilde{G}
\end{array}
\qquad
\begin{array}{ccc}
T_eG & \xrightarrow{T_eF} & T_{\tilde{e}}\tilde{G} \\
\downarrow Ad(g) & & \downarrow Ad(F(g)) \\
T_eG & \xrightarrow{T_eF} & T_{\tilde{e}}\tilde{G}
\end{array}
\qquad
\begin{array}{ccc}
T_eG & \xrightarrow{T_eF} & T_{\tilde{e}}\tilde{G} \\
\downarrow ad(\xi) & & \downarrow ad(T_eF(\xi)) \\
\underbrace{T_eG}_{\cong \mathcal{L}(G)} & \xrightarrow{T_eF} & \underbrace{T_{\tilde{e}}\tilde{G}}_{\cong \mathcal{L}(\tilde{G})}
\end{array}$$

We want to see if the first diagram commutes: $F \cdot C_g(x) = F(gxg^{-1}) = F(g)F(x)F(g)^{-1} = C_{F(g)} \cdot F(x)$. Thus, we find that: $F \cdot C_g = C_{F(g)} \cdot F$. Hence, taking the tangent map of this last equation:

$$T_eF \circ \underbrace{T_eC_g}_{Ad(g)} = \underbrace{T_eC_{F(g)}}_{Ad(F(g))} \circ T_eF \xrightarrow[map]{tangent} T_eF \circ ad(\xi)(\eta) = ad \circ T_eF(\xi) \circ T_eF(\eta) \Rightarrow T_eF([\xi, \eta]) = [T_eF(\xi), T_eF(\eta)]$$

where in the last line we have used the previously mentioned theorem (4.15). Hence, it is clear that we can indeed obtain a Lie algebra homomorphism (representation) from the tangent maps at identity.

Our next concern is whether all Lie algebra homomorphisms (representations) are tangent maps of group homomorphisms (representations) or not. To answer this, we will state the following theorem, without proving it (this can be found in, for example, Fulton and Harris, p.119).

Theorem 4.16 A linear map $f : \mathcal{L}(G) \to \mathcal{L}(G)$ is the tangent map of a group homomorphism F (so $f = T_eF$) if and only if f is a Lie algebra homomorphism.

The last important question we need to answer is the following: 'can the group G be recovered from its Lie Algebra $\mathcal{L}(G)$?'. In order to be able to answer it, we will introduce a few concepts.

Definition 4.17 For a left invariant vector field ξ, with $v = \xi_e$, define the *integral curve* $\alpha_v : [a, b] \to G$ passing through e at time $t = 0$ (i.e $\alpha_v(0) = e$) by: $\frac{\partial}{\partial t}|_t \alpha_v(t) := \xi_{\alpha_v(t)} = \xi(\alpha_v(t)) := T_e\alpha_v\left(\frac{\partial}{\partial t}\right)$.

Definition 4.18 The *exponential map* $Exp : T_eG \to G$ is defined by $Exp(v) := \alpha_v(1)$.

Theorem 4.19 1) Exp is differentiable at the origin and $\boxed{T_eExp = id_{\mathcal{L}(G)}}$.
2) Exp maps $\mathcal{L}(G) \cong T_eG$ diffeomorphic into a neighbourhood of $e \in G$.
3) For $F : G \to \tilde{G}$ hom., we have $\boxed{F \circ Exp = \tilde{Exp} \circ T_eF}$.

Proof. 1) Differentiability of Exp follows from the general properties of solutions to differential equations: $s \mapsto \alpha_{tv}(s)$ and $s \mapsto \alpha_v(ts)$ belong to the same vector tv, so they are equal. Thus: $Exp(tv) = \alpha_{tv}(1) = \alpha_v(t)$. Define $m_v(t) := tv \Rightarrow \dot{m}_v(0) = v$, so: $Exp \circ m_v(t) = \alpha_v(t) \Rightarrow T_0Exp \circ \dot{m}_v(0) = v$ (in the last line we take the partial derivative with respect to t).

2) $T_0Exp = id_{\mathcal{L}(G)}$ so Exp local diffeomorphism around origin by inverse function theorem $((f^{-1})'(b) = 1/f'(a))$.

3) Let α_v be an integral curve on G, $\beta_w = F \circ \alpha_v$ an integral curve on $\tilde{G}$: $w = \dot{\beta}_w(0) = T_eF\dot{\alpha}_v(0) = T_eFv$. Hence: $\tilde{Exp} \circ T_eFv = \tilde{Exp}(T_eFv) = \tilde{Exp}(w) = \beta_w(1) = F \circ \alpha v(1) = F \circ Exp(v)$. Consequently, the following diagram commutes:

$$\begin{tikzcd} G \arrow[r, "F"] & \tilde{G} \\ \mathcal{L}(G) \arrow[u, "Exp"] \arrow[r, "T_eF"] & \mathcal{L}(\tilde{G}) \arrow[u, "\tilde{Exp}"'] \end{tikzcd}$$

□

The main point of the above results is that we can classify Lie groups by classifying their Lie algebras and that we can understand the representations of these groups by understanding Lie algebras representations (thanks to the commutativity of the above diagram). Furthermore, Lie group homomorphisms are in one to one correspondence with Lie algebras homomorphisms.

4.2 Matrix Lie Groups

Consider a Lie group $G \subset Gl(V)$ (where $V = \mathbb{R}^n$ or $\mathbb{C}^n$) and parametrisation $g = g(t)$, where $t = (t^i)_{i=1,...,n}$ around $\mathbb{I} \in G$ and $g(0) = 1$, $g = (g_\mu^\nu)$, $\mu, \nu = 1, ..., dim(V)$. Expanding now around $t = 0$: $g(t) = \mathbb{I} + \frac{\partial g}{\partial t^i}(0)t^i + \mathcal{O}(t^2)$, so:

$$\boxed{\mathrm{T}_i := \tfrac{\partial g}{\partial t^i}(0)}$$

are called the generators of the Lie algebra.

- vector fields: $\xi_t = \xi^i \frac{\partial}{\partial t^i}|_t = \xi^i(t) \frac{\partial g_\mu^\nu}{\partial t^i}(t) \frac{\partial}{\partial g_\mu^\nu} = \xi^i(t) tr \left(\frac{\partial g}{\partial t^i}(t) \frac{\partial}{\partial g^T} \right)$

- Tangent space at $\mathbb{I}$ is spanned by the generators T_i: $\xi_{t=0} = \xi^i(0) tr \left(T_i \frac{\partial}{\partial g^T} \right)$ and:

$$T_{\mathbb{I}} G = \{ v^i tr \left(T_i \frac{\partial}{\partial g^T} \right) | v \in \mathbb{R}^n \} \cong \{ v^i T_i | v \in \mathbb{R}^n \} \cong \mathcal{L}(G)$$

- Left invariant vector fields: for ξ left invariant we have: $\xi_{gx} = T_x(g)\xi_x$. Note also that $(gx)_\mu{}^\nu = g_\mu{}^\sigma x_\sigma{}^\nu$ implies that: $T_x(g)^{\nu\rho}_{\mu\sigma} = \frac{\partial (gx)_\mu{}^\nu}{\partial x_\rho{}^\sigma} = g_\mu{}^\rho \delta_\sigma{}^\nu$. Thus:

$$\xi^i(gt) \frac{\partial g_\mu{}^\nu}{\partial t^i}(gx) \frac{\partial}{\partial g_\mu{}^\nu} = T_x(g)^{\nu\rho}_{\mu\sigma} \underbrace{\xi^i(t) \frac{\partial g_\rho{}^\sigma}{\partial t^i}(t) \frac{\partial g}{\partial g_\mu{}^\nu}}_{\xi_x} \Rightarrow \xi^i(gt) \frac{\partial g}{\partial t^i}(gx) = \xi^i(t) g \frac{\partial g}{\partial t^i}(x)$$

Consider now $x = \mathbb{I}, t = 0$, so: $\xi^i \frac{\partial g}{\partial t^i} = \xi^i(0) g T_i$. We can further introduce a basis $\xi_j{}^i$, with $\xi_j{}^i(0) = \delta_j{}^i$ so: $\xi_j{}^i{}_{\frac{\partial T}{\partial t^i} = gT_j}$ so we can define:

$$L_i := \xi_i{}^j(t) \frac{\partial}{\partial t^j} = \xi_i{}^j(t) tr \left(\frac{\partial g}{\partial t^j} \frac{\partial}{\partial g^T} \right) = tr \left(g T_i \frac{\partial}{\partial g^T} \right)$$

We then have an explicit map: $\underbrace{\mathcal{L}(G)}_{v^i L_i} \to \underbrace{T_{\mathbb{I}} G}_{v^i T_i} : L_i \to L_{i,\mathbb{I}} = tr \left(T_i \frac{\partial}{\partial g^T} \right) \sim T_i$ (i.e. $L_i \leftrightarrow T_i$).

- Commutator:

$$\begin{aligned} [L_i, L_j] = tr \left(g T_i \frac{\partial}{\partial g^T} \right) tr \left(g T_j \frac{\partial}{\partial g^T} \right) - tr \left(g T_j \frac{\partial}{\partial g^T} \right) tr \left(g T_i \frac{\partial}{\partial g^T} \right) = \\ = tr \left(g \left[T_i, T_j \right] \frac{\partial}{\partial g^T} \right) \in \mathcal{L}(G) \end{aligned}$$

It must be true that $[T_i, T_j] = f_{ij}^k T_k \Rightarrow [L_i, L_j] = f_{ij}^k L_k$.

- Exponential map: Consider the curve $t^i = t^i(s)$, with $t^i(0) = 0$ and $\frac{dt^i}{ds} = v^j \xi_j{}^i$ and define $T := v^i T_i$. Then $t^i(s)$ is an integral curve for the left-invariant vector field $v^j \xi_j{}^i$.

$$\alpha_v(s) = g(t(s)) \Rightarrow \frac{d\alpha_s}{dt} \frac{\partial g}{\partial t^i} \frac{\partial t^i}{\partial s} \frac{\partial g}{\partial t^i} v^j \xi_j{}^i = v^j g T_j = \alpha_v(s) T$$

Also since: $\alpha_v(0) = g(t(0)) = g(0) = \mathbb{I}$, it follows that: $\alpha_v(s) = e^{sT}$ so:

$$Exp(T) = \alpha_v(1) = e^T$$

4.2.1 Adjoint Representation

Recall that we made use of a group automorphism (*conjugation*), defined as: $C_g(M) = gMg^{-1}$. Using this and the fact that: $F \cdot Exp = \tilde{Exp} \cdot T_eF$, it follows that:

$$Ad(g)(T) = T_eC_g(T) = exp^{-1} \cdot C_g \cdot exp(T) = exp^{-1} \cdot (ge^Tg^{-1}) = gexp^{-1}(e^T)g^{-1}$$

$$Ad(g)(T) = gTg^{-1}$$

The other important result to keep in mind is: $ad(T)(S) = [T, S]$. With these, we can know consider a few examples.

4.2.2 $SU(2)$ and $SO(3)$

Def: $SU(2) = \{U \in Aut(\mathbb{C}^2) | U^\dagger U = \mathbb{I}, det(U) = 1\} = \{\begin{pmatrix} \alpha & \beta \\ -\beta^* & \alpha^* \end{pmatrix}, \alpha, \beta \in \mathbb{C}$ and $|\alpha|^2 + |\beta|^2 = 1\}$

It is then Clear that $SU(2) \cong S^3$ as a manifold, so it is compact and simply-connected. The fundamental representation is: $\mathcal{R}(U) = U$. Let us now look at the Lie algebra of this group, so consider: $U = \mathbb{I} + T + \mathcal{O}(T^2)$. Then, imposing: $U^\dagger U = \mathbb{I}_2$ we need: $T = -T^\dagger$. Furthermore expanding $det(\mathbb{I} + T) = 1 + tr(T) + ...$, we also impose: $tr(T) = 0$. Summarising these results:

$$su(2) = \mathcal{L}(SU(2)) = \{T \in End(\mathbb{C}^2) | T = -T^\dagger, tr(T) = 0\} = Span(T_i)_{i=1,2,3} \text{ where } T_i = -i\frac{\sigma_i}{2}$$

We introduced the Pauli matrices, which are a convenient basis to work in. Recall:

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

Some of the important properties we will need are listed below:

$$\sigma_i\sigma_j = \delta_{ij}\mathbb{I}_2 + i\epsilon_{ij}^{\ \ k}\sigma_k \Rightarrow \begin{cases} [\sigma_i, \sigma_j] = 2\epsilon_{ij}^{\ \ k}\sigma_k \\ \{\sigma_i, \sigma_j\} = 2\delta_{ij}\mathbb{I}_2 \end{cases} \quad \text{and } tr(\sigma_i\sigma_j) = 2\delta_{ij}$$

$$\sigma_2\sigma_i^*\sigma_2 = -\sigma_i$$

It then follows that: $[T_i, T_j] = \epsilon_{ij}^{\ \ k}T_k$, so the structure constants: $f_{ij}^{\ \ k} = \epsilon_{ij}^{\ \ k}$ are the representation matrices of the *ad* representation. Using the properties of the Pauli matrices, one can also compute the exact form of the exponential map; note also that the *exp* gives back the entire $SU(2)$ groups, since the manifold is connected.

$$exp(v^iT_i) = cos\left(\frac{v}{2}\right)\mathbb{I} + isin\left(\frac{v}{2}\right)\frac{\boldsymbol{v} \cdot \boldsymbol{\sigma}}{v}$$

Another representation of interest is the complex conjugate of the fundamental representation. Note that since:

$$\mathcal{R}^*(U) = U^* = e^{T^*} = \sigma_2e^T\sigma_2^{-1} = \sigma_2\mathcal{R}(U)\sigma_2^{-1}$$

the two representations are in fact equivalent. Let us introduce some terminology.

Definition 4.20 If $\mathcal{R}^*(g) = \mathcal{R}(g)$, for all g, then we call $\mathcal{R}$ a *real* representation. If $\mathcal{R}^*$ and $\mathcal{R}$ are inequivalent, we cal $\mathcal{R}$ a *complex* representation. Furthermore, if $\mathcal{R}^*(g) \neq \mathcal{R}(g)$ but they are still equivalent, $\mathcal{R}$ is called *pseudoreal*.

Lastly, let us consider the adjoint representation of this group. Define a 'coordinate map' $\tau : \mathbb{R}^3 \to su(2)$, by:

$\tau(\boldsymbol{X}) := X^i T_i$, so that $su(2) \overset{\tau}{\cong} \mathbb{R}^3$. The inner product on this vector space is defined as:

$$\boldsymbol{X} \cdot \boldsymbol{Y} = -2tr\left(\tau(\boldsymbol{X})\tau(\boldsymbol{Y})\right)$$

One can check that this gives the expected result (i.e. $X^i Y_i$) by using the properties of the Pauli matrices listed above. Let us now consider the adjoint representation is these coordinates: $\mathcal{R} : SU(2) \to Aut(\mathbb{R}^3)$, defined by: $\mathcal{R}(U) = \tau^{-1} \cdot Ad(U) \cdot \tau$. What happens with the inner product under this representation?

$$\begin{aligned}
(\mathcal{R}(\mathcal{U})\boldsymbol{X}) \cdot (\mathcal{R}(\mathcal{U})\boldsymbol{Y}) &= -2tr\left(\tau\left(\mathcal{R}(\mathcal{U})\boldsymbol{X}\right)\tau\left(\mathcal{R}(\mathcal{U})\boldsymbol{Y}\right)\right) \\
&= -2tr\left(Ad(U)(\tau(\boldsymbol{X}))Ad(U)(\tau(\boldsymbol{Y}))\right) \\
&= -2tr(U\tau(\boldsymbol{X})U^\dagger U\tau(\boldsymbol{Y})U^\dagger) \\
&= -2tr(\tau(\boldsymbol{X})\tau(\boldsymbol{Y})) = \boldsymbol{X} \cdot \boldsymbol{Y}
\end{aligned}$$

Hence, the inner product is left invariant. Consequently, $\mathcal{R}(U)$ must be a rotation, i.e. $\mathcal{R}(U) \in O_3 = \{O \in Aut(\mathbb{R}^3) | O^T O = \mathbb{I}_3\}$. However, note that O_3 is not connected, as $det(O) = \pm 1$; but $SU(2) \cong S^3$ is connected and since the identities map into each other, it must be the case that $SU(2)$ is mapped into the connected part of $O(3)$ with determinant equal to 1 (as $det(\mathbb{I}_3) = 1$), which is precisely $SO(3)$). Hence: $\mathcal{R}(U) \in SO(3)$.

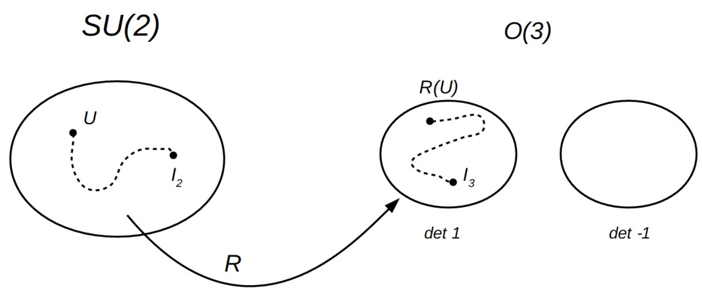


Figure 2: $SU(2) \mapsto SO(3)$

What is the $Ker(\mathcal{R})$? Let $U \in Ker(\mathcal{R})$ so: $\mathcal{R}(U) = \mathbb{I}_3 \Leftrightarrow \boldsymbol{X} = \tau^{-1} \cdot Ad(U) \cdot \tau(\boldsymbol{X}), \forall \boldsymbol{X}$ or: $\tau(X) = Ad(U)(\tau(X))$ and thus we have: $T = Ad(U)(T) = UTU^\dagger, \forall T \in SU(2)$. It follows that T and U commute and applying Schur's lemma we must have: $U = \lambda \mathbb{I}_2$. However, since $U \in SU(2)$, we can only have $\lambda = \pm 1$. One can also show that the image of $\mathcal{R}$ is the entire $SO(3)$, but we will only state this fact:

$$Ker(\mathcal{R}) = \{\pm \mathbb{I}_2\} \text{ and } Im(\mathcal{R}) = SO(3) \Rightarrow SO(3) \cong SU(2)/\mathbb{Z}_2$$

Consequently, $SU(2)$ is a two fold cover of $SO(3)$, or, $SO(3)$ is the adjoint representation of $SU(2)$. We can also look at the Lie algebra of SO(3):

$$so(3) = \mathcal{L}(SO(3)) = \{T \in End(\mathbb{R}^3) | T = -T^T\} = Span(\epsilon_{ijk})_{i=1,2,3}$$

We will return to this example in a different section (See 4.3.2).

4.3 Lie Algebras

Let $\mathcal{L}$ be a Lie algebra with commutator $[\cdot,\cdot]$, elements $T, S, ... \in \mathcal{L}$, basis $(T_i)_{i=1,...,n}$, with: $[T_i, T_j] = f_{ij}^k T_k$.

Definition 4.21 (i) A subset $\mathcal{A} \in \mathcal{L}$ is called a sub (Lie) algebra if and only if $\mathcal{A}$ is a linear sub-space and $[\mathcal{A}, \mathcal{A}] \subset \mathcal{A}$ (i.e. closed under the bracket).

(ii) A sub-algebra is called Abelian if and only if $[\mathcal{A}, \mathcal{A}] = 0$.

(iii) A sub-algebra is called ideal if and only if $[\mathcal{L}, \mathcal{A}] \subset \mathcal{A}$. An ideal is called non-trivial if and only if $\mathcal{A} \neq \{0\}, \mathcal{L}$.

(iv) The derived series $(\mathcal{D}^k\mathcal{L})_{k=1,2,...}$ of $\mathcal{L}$ is defined recursively by: $\mathcal{D}^1\mathcal{L} = [\mathcal{L}, \mathcal{L}]$ and $\mathcal{D}^k\mathcal{L} = [\mathcal{D}^{k-1}\mathcal{L}, \mathcal{D}^{k-1}\mathcal{L}]$. Then, $\mathcal{L}$ is called solvable if and only if $\mathcal{D}^k\mathcal{L} = \{0\}$, for some non-zero number k.

Definition 4.22 (i) A Lie algebra is called simple iff it contains no non-trivial ideal (so only trivial ones). (ii) A Lie algebra is called semi-simple iff it has no non-zero solvable ideals. This definition might not be useful in practice, so we will use the following lemma.

Lemma 4.23 A semi-simple Lie algebra $\mathcal{L}$ has no non-zero Abelian ideals.

Proof. "$\Rightarrow$" Assume $\mathcal{L}$ has a non-zero Abelian ideal $\mathcal{A}$, so: $\mathcal{D}^1\mathcal{A} = [\mathcal{A}, \mathcal{A}] = 0 \Rightarrow \mathcal{A}$ is solvable so $\mathcal{L}$ is not semi-simple.
"$\Leftarrow$" $\mathcal{A} \subset \mathcal{L}$ so $[\mathcal{L}, \mathcal{A}] \subset \mathcal{A} \Rightarrow [\mathcal{D}^1\mathcal{A}, \mathcal{A}] = [[\mathcal{A}, \mathcal{A}], \mathcal{L}] \xrightarrow{Jacobi} = [\underbrace{[\mathcal{A}, \mathcal{L}]}_{\subset \mathcal{A}}, \mathcal{A}] \in \mathcal{D}^1\mathcal{A}$, so $\mathcal{D}^1\mathcal{A}$ is ideal. Iterating, $\mathcal{D}^k\mathcal{A}$ is an ideal. Assume now $\mathcal{L}$ is not semi-simple so $\mathcal{A}$ is a solvable ideal with $\mathcal{D}^{k-1}\mathcal{A} \neq 0$ and $\mathcal{D}^k\mathcal{A} = 0$ for some k. Thus: $[\mathcal{D}^{k-1}\mathcal{A}, \mathcal{D}^{k-1}\mathcal{A}] = 0$, so $\mathcal{D}^{k-1}\mathcal{A}$ is an Abelian ideal. $\square$

Lemma 4.24 Let $\mathcal{B}$, $\mathcal{C} \in \mathcal{L}$ be solvable ideals. Then, $\mathcal{B} + \mathcal{C}$ is also a solvable ideal.

Definition 4.25 The sum of all solvable ideals (non-zero) in $\mathcal{L}$ is called the *radical* of $\mathcal{L}$ and is denoted by $rad(\mathcal{L})$. Then, $\mathcal{L}/rad(\mathcal{L})$ is semi-simple.

Theorem 4.26 (Levi) Consider the sequence: $0 \to rad(\mathcal{L}) \to \mathcal{L} \to \mathcal{L}/rad(\mathcal{L}) \to 0$. The sequence splits, that is to say that $\mathcal{L} = rad(\mathcal{L}) \oplus_s \mathcal{A}$, where $\mathcal{A}$ is semi-simple, and the s stands for semi-direct sum. Let $\mathcal{P} = rad(\mathcal{L})$ so we have: $[\mathcal{P}, \mathcal{P}] \subset \mathcal{P}$, $[\mathcal{A}, \mathcal{A}] \subset \mathcal{A}$, $[\mathcal{P}, \mathcal{A}] \subset \mathcal{P}$.

Theorem 4.27 A semi-simple Lie-algebra is a direct sum of simple Lie Algebras.

The proof of the above theorems can be found in Fulton, Harris, p.499, and 480, respectively.

$$\mathcal{L} = rad(\mathcal{L}) \oplus_s (\mathcal{L}_1 \oplus ... \oplus \mathcal{L}_n)$$

where $\mathcal{L}_i$ are simple, $[\mathcal{L}_i, \mathcal{L}_j] = 0$ for $i \neq j$. At the group level, this allows us to compute the Lie algebra of Cartesian products of groups: $G = G_1 \times G_2 = \{(g_1, g_2 | g_1 \in G_1, g_2 \in G_2\}$ with $[\mathcal{L}(G_1), \mathcal{L}(G_2)] = 0$ as

$$\mathcal{L}(G) \cong T_eG = T_eG_1 \oplus T_eG_2 \cong \mathcal{L}(G_1) \oplus \mathcal{L}(G_2).$$

Lemma 4.28 (i) $\mathcal{L}$ is semi-simple is equivalent to the *ad* representation being faithful.
(ii) $\mathcal{L}$ is simple is equivalent to the *ad* representation being irreducible.

Proof. Add proof - Notes page 50. □

4.3.1 The Killing Form

Definition 4.29 The symmetric bi-linear form $\Gamma : \mathcal{L} \times \mathcal{L} \to \mathbb{R}$ defined by:

$$\Gamma(T, S) = tr\,(ad(T)ad(S))$$

is called the *Killing form* of $\mathcal{L}$.

We can introduce coordinates as before: $\gamma_{ij} = tr\,(ad(T_i)ad(T_j)) = (ad(T_i))^l_{\ k}\,(ad(T_j))^k_{\ l}$. Thus $\boxed{\gamma_{ij} = f^l_{ik} f^k_{jl}}$. We will see later that this quantity will behave as a 'metric'.

Lemma 4.30 The Killing form has the following property: $\Gamma(T, ad(U)S) = -\Gamma(ad(U)S, T)$.

Proof. Add proof from notes - page 51. □

Theorem 4.31 Let $\mathcal{L}$ be semi-simple. Then, this is equivalent to saying that Γ is *non-degenerate*. In other words, if $\Gamma(T, S) = 0, \forall T \in \mathcal{L}$, we must have $S = 0$ (i.e. no 'vector' perpendicular to the whole algebra. These statements are also equivalent to saying that (γ_{ij}) is invertible.

Proof. "$\Leftarrow$" Let Γ be non-degenerate, $\mathcal{A} \subset \mathcal{L}$ an Abelian ideal and consider a basis $\{T_a, T_\alpha\}$ of $\mathcal{L}$, where the indices a are for the basis of $\mathcal{A}$. Also let $t \in \mathcal{L}, S \in \mathcal{A}$. Then: $ad(T) \circ ad(S)(T_a) = [T, [S, T_a]] = 0$ since $\mathcal{A}$ is Abelian. Similarly: $ad(T) \circ ad(S)(T_\alpha) = [T, [S, T_\alpha]] \in \mathcal{A}$. Consequently:

$$\Gamma(T, S) = tr\,(ad(T)ad(S)) = tr\left(\begin{array}{c|c} 0 & * \\ \hline 0 & 0 \end{array}\right) = 0$$

so that $\Gamma(T, S) = 0$, for all T and thus $S = 0$.
"$\Rightarrow$" This direction is somewhat similar - see Fulton, Harris, p.480. □

Theorem 4.32 Given a compact group G, the Killing form on its Lie algebra is negative semi-definite, i.e. $\Gamma(T, T) \leq 0, \forall T \in \mathcal{L}$.

Proof. See Brocker, tom Dieck - 'Representations of Compact Lie Groups', p.214. □

4.3.2 Structure constants and Conventions

The Jacobi identity for these generators is: $[[T_i, T_j], T_k] + [[T_k, T_i], T_j] + [[T_j, T_k], T_i] = 0$, from which we obtain:

$$f_{ij}^{\ l} f_{kl}^{\ n} + f_{jk}^{\ l} f_{il}^{\ n} + f_{ki}^{\ l} f_{jl}^{\ n} = 0$$

Clearly $f_{ij}^{\ k} = -f_{ji}^{\ k}$. Let us define: $f_{ijk} = \gamma_{kl} f_{ij}^{\ l}$, so γ acts as a metric. Then:

$$\begin{aligned} f_{ijk} = f_{km}^{\ n} f_{ij}^{\ l} f_{ln}^{\ m} &\overset{\text{Jacobi}}{=} f_{km}^{\ n}(f_{jn}^{\ l} f_{il}^{\ m} + f_{mi}^{\ l} f_{jl}^{\ m}) \\ &= f_{km}^{\ n} f_{jn}^{\ l} f_{il}^{\ m} + f_{km}^{\ n} f_{mi}^{\ l} f_{jl}^{\ m} \\ &= tr(f_i f_j f_k) + tr(\tilde{f}_i \tilde{f}_j \tilde{f}_k) \end{aligned}$$

The point of the above result is not to determine the objects f_i; note that since the trace is invariant under cyclic permutations of i, j, k and $f_{ij}^{\ k}$ antisymmetric in the first two indices, it follows that f_{ijk} is actually *totally antisymmetric*.

Theorem 4.33 For $\mathcal{L}$ a semi-simple (matrix) Lie algebra, $C := \gamma^{ij}T_iT_j$ is called the (quadratic) *Casimir operator*. Then:

$$[C,T] = 0, \forall T \in \mathcal{L}$$

Proof.

$$\begin{aligned}[C,T_l] &= \gamma^{ij}[T_iT_j,T_l] = \gamma^{ij}T_i[T_j,T_l] + \gamma^{ij}[T_i,T_l]T_j = \gamma^{ij}f_{jl}{}^{m}T_iT_m + \gamma^{ij}f_{il}{}^{m}T_mT_j = \\ &= \gamma^{ij}f_{jk}^{m}(T_iT_m + T_mT_i) = \underbrace{\gamma^{ij}\gamma^{mn}(T_iT_m + T_mT_i)}_{\text{symmetric}} \underbrace{f_{jln}}_{antisymmetric} = 0\end{aligned}$$

□

Corollary 4.34 In any irreducible (complex) representation r of $\mathcal{L}$ we have: $C = C(r)\mathbb{I}$. $C(r)$ is a characteristic number of the representation r.

Proof. The proof follows from Schur's lemma.

□

We will now introduce some physics conventions. Assume in the following that $\mathcal{L}(G)$ is simple, G is compact, so γ is negative definite; we can then choose a basis $(T_i)_{i=1,\dots,n}$ such that $\gamma_{ij} = -\delta_{ij}$. Thus: $f_{ij}^{k} = -f_{ijk}$ so both of these tensors are totally ani-symmetric. The Casimir is: $C = \sum_i \left(T_i^{(r)}\right)^2 = C(r)\mathbb{I}$, where $T_i^{(r)} := r(T_i)$. We now claim that for some number $c(r)$, we have:

$$tr\left(T_i^{(r)}T_j^{(r)}\right) = -c(r)\delta_{ij}$$

To show that this is indeed true, define: $M_{jk} = tr\left(T_j^{(r)}T_k^{(r)}\right)$. Also, remember that $T_i^{(ad)}$ are just the structure constants so:

$$\begin{aligned}\left([T_i^{(ad)},M]\right)_{jk} &= \left(T_i^{(ad)}\right)_{jk} tr\left(T_l^{(r)}T_k^{(r)}\right) - tr\left(T_j^{(r)}T_l^{(r)}\right)\left(T_i^{(ad)}\right)_{lk} = tr\left(f_{ijl}T_l^{(r)}T_k^{(r)}\right) - tr\left(f_{ilk}T_j^{(r)}T_l^{(r)}\right) = \\ &= tr\left(\left[T_i^{(r)},T_j^{(r)}\right]T_k^{(r)} + T_j^{(r)}\left[T_i^{(r)},T_k^{(r)}\right]\right) = tr\left(\left[T_i^{(r)},T_j^{(r)}T_k^{(r)}\right]\right) = 0 \text{ since trace is symmetric}\end{aligned}$$

Then, by Schur's lemma: $M = \lambda\mathbb{I}$, which is precisely our claim. How are $C(r)$ and $c(r)$ related?

$$dim(r)C(r) = tr_r(C) = -\sum_i tr\left(\left(T_i^{(r)}\right)^2\right) = \sum_i c(r) = dim(ad)c(r)$$

$$c(r) = \frac{dim(r)}{dim(ad)}C(r)$$

An application of the above convetions is the 1-loop β function for gauge theories. For a gauge group G, gauge coupling g, the vector fields transform in the ad representation of $\mathcal{L}(G)$ (so $r_v = ad$), with the (Weyl) fermions and scalars transforming in r_W and r_s, respectively. Then, the variation of g is governed by:

$$\mu\frac{dg}{d\mu} = \beta(g),$$

where μ is related to the energy. β is given by:

$$\beta(g) = -\frac{1}{32\pi^2}\left[\frac{11}{3}c(r_v) - \frac{2}{3}c(r_w) - \frac{1}{6}c(r_s)\right]g^3$$

Let us now return to the $SU(2)$ example we discussed in Section 4.2.2. In the basis discussed above, the Killing form becomes: $\gamma_{ij} = f_{ik}^{\ l} f_{jl}^{\ k} = \epsilon_{ik}^{\ l} \epsilon_{jl}^{\ k} = -2\delta_{ij}$. It is often useful to use a different basis for this group, given by:

$$E_{\pm} = \frac{1}{2}(iT_1 \pm T_2), H = -iT_3 \text{ such that: } [H, H] = 0, [H, E_{\pm}] = \pm E_{\pm}, [E_+, E_-] = \frac{1}{2}H$$

This is in fact a *complexification* of the $SU(2)$ Lie algebra, usually denoted by: $su(2)_{\mathbb{C}}$. In this basis, the Casimir is: $C = \gamma^{ij} T_i T_j = -\frac{1}{2}\sum T_i^2 = \frac{1}{2}J^2$ - 'total angular momentum'. Also, the Killing form becomes:

$$\gamma = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$$

Consider a representation of $su(2)_{\mathbb{C}}$: r_j, with $j \in \mathbb{Z}/2$, characterised by the Casimir $J^2 = j(j+1)$. Then, the representation vector space V_j is spanned by $|j, m\rangle$, with $m = -j, ..., j$ (this can be seen by considering the effect of $J_3 J_{\pm}$ on the eigenstates), so $dim(r_j) = 2j + 1$. The states form an orthonormal basis, i.e. $\langle j, m'| \, |j, m\rangle = \delta_{mm'}$. Now, for $J_3 = H$ and $J_{\pm} = 2E_{\pm}$, we have:

$$\begin{cases} J_3 \, |j, m\rangle = m \, |j, m\rangle \\ J_{\pm} \, |j, m\rangle = \sqrt{j(j+1) - m(m \pm 1)} \, |j, m \pm 1\rangle \end{cases}$$

4.4 Cartan Weyl Basis

The Cartan-Weyl basis is a generalization to arbitrary Lie algebras of the $su(2)_{\mathbb{C}}$ basis we have seen in a previous example: $(H, E_\pm)$ or $(J_3, J_\pm)$. In the following definitions, let $\mathcal{L}$ be a complexified semi-simple Lie algebra.

Definition 4.35 A *maximal* (i.e. there is no other sub-algebra that includes it), diagonalisable, Abelian sub-algebra $\mathcal{H}$ of $\mathcal{L}$ is called a *Cartan* sub-algebra. The dimension of $\mathcal{H}$ is called the rank of $\mathcal{L}$:

$$\boxed{rk(\mathcal{L}) := dim(\mathcal{H})}$$

Up to this point we have not discussed the existence, construction or well-definedness of this sub-algebra. However, we will skip these details for now and only leave a reference for the interested reader: Fulton, Harris - AppendixD.

Let us consider an eigenvalue problem in the *ad* representation. For $H \in \mathcal{H}$: $ad(H)(T) = \alpha(H)T$, where $\alpha \in \mathcal{H}'$ (the dual space of $\mathcal{H}$) is called a *root*. Then, denote by $\mathcal{L}_\alpha$ the eigenspace of this root (i.e. the span of all T that satisfy the above equation). The *Cartan decomposition* of $\mathcal{L}$ is:

$$\mathcal{L} = \mathcal{H} \oplus \bigoplus_\alpha \mathcal{L}_\alpha := \mathcal{L}_0 \oplus \bigoplus_\alpha \mathcal{L}_\alpha$$

The space $\Delta = \{\alpha \in \mathcal{H}' | \alpha \text{ root}\}$ is called the *root space*. Also, the lattice Δ_R generated by Δ (i.e. integer linear combinations of elements of Δ) is called the root lattice.

Let $T \in \mathcal{L}_\alpha, S \in \mathcal{L}_\beta, U \in \mathcal{L}_\gamma$ so $[H,T] = \alpha(H)T$ and $[H,S] = \beta(H)S$. Then: $ad(H)(ad(T)S) = [H,[T,S]] = [T,[H,S]] - [S,[H,T]] = (\alpha(H) + \beta(H))[T,S]$, so that:

$$[\mathcal{L}_\alpha, \mathcal{L}_\beta] \subset \mathcal{L}_{\alpha+\beta}$$

Hence: $ad(T)ad(S)(U) = [T,[S,U]] \in \mathcal{L}_{\alpha+\beta+\gamma}$ by the same argument and since $\Gamma(T,S) = tr(ad(T)ad(S))$ we have the following important statement:

$$\boxed{T \in \mathcal{L}_\alpha, S \in \mathcal{L}_\beta \text{ and } \alpha + \beta \neq 0 \Rightarrow \Gamma(T,S) = 0 \text{ or } \mathcal{L}_\alpha \perp \mathcal{L}_\beta \text{ for } \alpha + \beta \neq 0.}$$

Theorem 4.36 (Structure of Cartan decomposition)

(*i*) $\Gamma|_{\mathcal{H}\times\mathcal{H}}$ is non-degenerate. This means that for all roots $\alpha \in \Delta$ there exists a unique $H_\alpha \in \mathcal{H}$ with $\Gamma(H, H_\alpha) = \alpha(H)$. We define:

$$(\alpha, \beta) := \Gamma(H_\alpha, H_\beta)$$

(*ii*) For $\alpha \in \Delta$, $-\alpha$ is also a root.
(*iii*) For $T \in \mathcal{L}_\alpha, S \in \mathcal{L}_{-\alpha}$ we have: $[T,S] = \Gamma(T,S)H_\alpha$. One can choose T, S such that $\Gamma(T,S) = 1$.
(*iv*) $dim(\mathcal{L}_\alpha) = 1$ for all $\alpha \in \Delta$.
(*v*) Let $\alpha \in \Delta$, The from $\{k\alpha | k \in \mathbb{Z}\}$, only α and $-\alpha$ are roots.
(*vi*) For $H, \tilde{H} \in \mathcal{H}$ we have: $\Gamma(H, \tilde{H}) = \sum_{\alpha\in\Delta} \alpha(H)\alpha(\tilde{H})$.
(*vii*) Δ contains a basis of $\mathcal{H}'$.

Proof. (*i*) Let $\mathcal{L}$ be semi-simple, and Γ non-degenerate on $\mathcal{L}\times\mathcal{L}$. For $H \in \mathcal{H}$ with $\Gamma(H, \tilde{H}) = 0, \forall \tilde{H} \in \mathcal{L}$, write $S \in \mathcal{L}$ as: $S = \tilde{H} + \sum_{\alpha\in\Delta} S_\alpha$, where $S_\alpha \in \mathcal{L}$. Hence: $\Gamma(H,S) = \Gamma(H,\tilde{H}) + \sum_{\alpha\in\Delta} \Gamma(H, S_\alpha) = 0 \Rightarrow H = 0$. Here we have used the root property is the above box (with $\mathcal{H} = \mathcal{L}_0$). Note that any non-degenerate bilinear form induces an isomorphism between $\mathcal{H}$ and $\mathcal{H}'$.

(*ii*) Assume $\mathcal{L}_{alpha} = \{0\}$ (i.e. not a root) $\Rightarrow \mathcal{L}_\alpha \perp \mathcal{L}_\beta, \forall \beta \in \Delta$ be the same property; furthermore $\mathcal{L}_\alpha \perp \mathcal{H}$ so $\mathcal{L}_\alpha \perp \mathcal{L}$. But since Γ is non-degenerate on $\mathcal{L}$, this cannot be possible.

(*iii*) $[\mathcal{L}_\alpha, \mathcal{L}_\beta] \subset \mathcal{L}_{\alpha+\beta}$ so $[\mathcal{L}_\alpha, \mathcal{L}_{-\alpha}] \subset \mathcal{L}_0 = \mathcal{H}$. Let $H \in \mathcal{H}, T \in \mathcal{L}_\alpha, S \in \mathcal{L}_{-\alpha}$ so: $\Gamma([H,[T,S]]) = \Gamma([[H,T],S]) = \alpha(H)\Gamma(T,S) = \Gamma(H,H_\alpha)\Gamma(T,S) = \Gamma(H,\Gamma(T,S)H_\alpha)$, where we have used (i) and lemma (*** - Killing Form). Thus: $[T,S] = \Gamma(T,S)H_\alpha$.
For the second statement, as in (i) we can show that Γ is non-degenerate on $\mathcal{L}_\alpha \oplus \mathcal{L}_{-\alpha}, S \in \mathcal{L}_{-\alpha}, T \in \mathcal{L}_\alpha$ such that: $\Gamma(T,S) \neq 0$. We can then normalise (T,S) so $\Gamma(T,S) = 1$.

(*iv*)(*v*) Let $[T,S] = H_\alpha$ for $T \in \mathcal{L}_\alpha, S \in \mathcal{L}_{-\alpha}, H_\alpha \in \mathcal{H}$. The vector space: $V := \mathbb{C}S + \mathbb{C}H_\alpha + \sum_{k\geq 1}\mathcal{L}_{k\alpha}$ is invariant under $ad(H_\alpha)$ so: $tr(ad(H_\alpha)|_V) = tr(ad([T,S])_V) = tr([ad(T),ad(S)]_V) = 0$. Note that the ad is a representation so it can be pulled in/out of the commutator. On the other hand, we can compute the trace directly by building the matrix. Let $F \in \mathcal{L}_{k\alpha}$:

$$\left.\begin{array}{l} ad(H_\alpha)(S) = [H_\alpha, S] = -\alpha(H_\alpha)S = -(\alpha,\alpha)S \\ ad(H_\alpha)(H_\alpha) = 0 \\ ad(H_\alpha)(F) = [H_\alpha, F] = k\alpha(H_\alpha)F = k(\alpha,\alpha)S \end{array}\right\} \Rightarrow tr(ad(H_\alpha)|_V) = (\alpha,\alpha)\left(-1 + \textstyle\sum_{k\geq 1} k dim(\mathcal{L}_{k\alpha})\right)$$

Since this must be 0, it follows that $dim(\mathcal{L}_\alpha) = 1$ and $dim(\mathcal{L}_{k\alpha}) = 0$ for $k > 1$.

(*vi*) Let $\tilde{H} \in \mathcal{H}, T \in \mathcal{L}_\alpha$: $ad(H) \circ ad(\tilde{H})(T) = [H,[\tilde{H},T]] = \alpha(H)\alpha(\tilde{H})T$. It is then clear that:

$$\Gamma(H,\tilde{H}) = \sum_{\alpha\in\Delta} \alpha(H)\alpha(\tilde{H})$$

(*vii*) Assume $Span(\Delta) \neq \mathcal{H}'$. Let $(\alpha_1, ..., \alpha_n)$ be a basis of $Span(\Delta)$ and complete basis $(\alpha_1, ..., \alpha_n, ..., \alpha_N)$ for $\mathcal{H}'$. The dual basis $(H_1, ..., H_N)$ of $\mathcal{H}$ satisfies: $\alpha_i(H_j) = \delta_{ij} \Rightarrow \alpha(H_N) = 0, \forall \alpha \in \Delta$. From (vi) we have that $\Gamma(H_N, H), \forall H \in \mathcal{H}$, but since Γ is non-degenerate on $\mathcal{H} \times \mathcal{H}$, this is a contradiction. $\square$

From the theorem above, we have a basis (H_i, E_α), where $i = 1, ..., rk(\mathcal{L}), \alpha \in \Delta$ of $\mathcal{L}$, with $\mathcal{H} = Span(H_i)$, $\mathcal{L}_\alpha = Span(E_\alpha), \Gamma(E_\alpha, E_{-\alpha}) = 1$. This basis is called the **Cartan Weyl basis**.
In this basis, the Killing form is: $\gamma_{ij} = \Gamma(H_i, H_j) = \sum_{\alpha\in\Delta} \alpha_i\alpha_j$, where $\alpha_i := \alpha(H_i)$.

$$\left.\begin{array}{l} \gamma_{i\alpha} = \Gamma(H_i, E_\alpha) = 0 \\ \gamma_{\alpha\beta} = \Gamma(E_\alpha, E_\beta) = 0 \text{ if } \alpha+\beta \neq 0 \\ \gamma_{\alpha,-\alpha} = \Gamma(E_\alpha, E_{-\alpha}) = 1 \end{array}\right\} \Rightarrow \Gamma \sim \left[\begin{array}{c|ccccc} \gamma_{ij} & & & 0 & & \\ \hline & 0 & 1 & & & \\ & 1 & 0 & & & \\ 0 & & & 0 & 1 & \\ & & & 1 & 0 & \\ & & & & & \ddots \end{array}\right] \begin{array}{l} \mathcal{H} \\ \mathcal{L}_\alpha \\ \mathcal{L}_{-\alpha} \\ \mathcal{L}_\beta \\ \mathcal{L}_{-\beta} \\ \vdots \end{array}$$

Let us also summarize some other results here:

$$\mathcal{L} = \mathcal{H} \oplus \bigoplus_\alpha \mathcal{L}_\alpha := \mathcal{L}_0 \oplus \bigoplus_\alpha \mathcal{L}_\alpha$$

Given a root α and $H_\alpha \in \mathcal{H}$ such that: $\Gamma(H, H_\alpha) = \alpha(H), \forall H \in \mathcal{H}$, we must have:

$$H_\alpha = \alpha^i H_i = \gamma^{ij}\alpha_j H_i$$

Furthermore, the inner product of the roots becomes:

$$(\alpha,\beta) = \gamma^{ij}\alpha_i\beta_j$$

To see where the last results from the above box come from, recall that for each root α, $H_\alpha \in \mathcal{H}$ defined by: $\Gamma(H, H_\alpha) = \alpha(H), \forall H \in \mathcal{H}$. We must have $H_\alpha = \lambda^i H_i$ for some λ^i: $\alpha_j = \alpha(H_j) = \Gamma(H_\alpha, H_j) = \lambda^i\Gamma(H_i, H_j) = \lambda^i\gamma_{ij} \Rightarrow \lambda^i = \gamma^{ij}\alpha_j =: \alpha^i$, which is why we stated before that γ will be defined as a 'metric'. Hence: $H_\alpha = \alpha^i H_i$. Note also that: $(\alpha,\beta) = \Gamma(H_\alpha, H_\beta) = \beta(H_\alpha) = \alpha^i\beta(H_i) = \alpha^i\beta_i = \gamma^{ij}\alpha_i\beta_j$.

Let us now investigate the Commutation relations in the Cartan-Weyl basis:

$$\begin{cases} [H_i, H_j] = 0 & \text{as } \mathcal{H} \text{ is Abelian} \\ [H_i, E_\alpha] = \alpha_i E_\alpha & \text{from the definition of } \mathcal{H} \\ [E_\alpha, E_\beta] = \begin{cases} N_{\alpha\beta} E_{\alpha+\beta} & \text{if } \alpha+\beta \neq 0 \text{ and } \alpha+\beta \text{ root} \\ 0 & \text{if } \alpha+\beta \neq 0 \text{ and } \alpha+\beta \text{ not a root (i.e. } \mathcal{L}_{\alpha+\beta} = \emptyset) \end{cases} \\ [E_\alpha, E_{-\alpha}] = H_\alpha = \alpha^i H_i & \in \mathcal{H} = \mathcal{L}_0 \end{cases}$$

For $\alpha \in \Delta$ consider $\{H_\alpha, E_\alpha, E_{-\alpha}\}$ so: $\boxed{\begin{cases} [H_\alpha, E_{\pm\alpha}] = \pm(\alpha,\alpha) E_{\pm\alpha} \\ [E_\alpha, E_{-\alpha}] = H_\alpha \end{cases} \to su(2)_{\mathbb{C}}}$

Representations and weights

Consider a map $r : \mathcal{L} \to End(V)$. We find a basis of commong eigenvectors of $r(H), H \in \mathcal{H}$:

$$r(H)(v) = w(H)v, w \in \mathcal{H}', v \in V$$

Then, w is called a **weight**, and: $V = \oplus_w V_w$. Let $v \in V_w$ and consider $r(E_\alpha)v$, so try: $r(H)(r(E_\alpha)v) = r(E_\alpha)(r(H)v) + [r(H), r(E_\alpha)]v$. But r is a representation, so it can be 'pulled out' of the commutator and since $r([H, E_\alpha]) = \alpha(H) r(E_\alpha)$, we obtain:

$$r(H)(r(E_\alpha)v) = (w(H) + \alpha(H)) r(E_\alpha)v$$

In other words, $r(E_\alpha)v \in V_{w+\alpha}$, i.e. it is a vector with weight $w + \alpha$. Suppose now that we have two weights w_1, w_2 of a representation, such that $w_1 - w_2 \notin \Delta_R$. Then: $\bigoplus_{\alpha \in \Delta_R} V_{w_1+\alpha}$ would correspond to a sub-representation of V. Hence, **if r is irreducible, then any two weights must differ by roots**. The idea is to construct the representation by starting with one weight and then add roots until the process terminates. Note that roots are the weight of the *ad* representation.

Let us consider two representations $\mathcal{R}_V : G \to Aut(V)$ and $\mathcal{R}_{\tilde{V}} : G \to Aut(\tilde{V})$. Assume that: $\mathcal{R}_V = \mathbb{I} + r_V(T) + ...$ and similarly for $\mathcal{R}_{\tilde{V}}$. Recall from section 2.1 that the tensor product representation can be obtained by taking the Kronecker product: $\mathcal{R}_V(g) \times \mathcal{R}_{\tilde{V}}(g) = \mathbb{I} + \underbrace{r_V(T) \times \mathbb{I} + \mathbb{I} \times r_{\tilde{V}}(T)}_{r_{V\otimes\tilde{V}}(T)} + ...$

Going to the Cartan-Weyl basis: $\left.\begin{matrix} r_V(H)v = w(H)v \\ r_{\tilde{V}}(H)\tilde{v} = \tilde{w}(H)\tilde{v} \end{matrix}\right\} r_{V\otimes\tilde{V}}(H) v \otimes \tilde{v} = (w(H) + \tilde{w}(H)) v \otimes \tilde{v}$

If v has weight w, and $\tilde{v}$ has weight $\tilde{w}$, then $v \otimes \tilde{v}$ has weight $w + \tilde{w}$. We can also show that the weights of the dual representation are given by: $w'(H) = -w(H)$ (see notes - p.74).

Values of weights

Recall that $Span(H_\alpha, E_\alpha, E_{-\alpha}) \cong su(2)_{\mathbb{C}}, \forall \alpha \in \Delta$. Also: $[H_\alpha, E_{\pm\alpha}] = \pm(\alpha,\alpha) E_{\pm\alpha}$, which implies that $(w, \alpha) := w(H_\alpha) \in \frac{(\alpha,\alpha)}{2}\mathbb{Z}$. To see why this is the case, we can use the analogy to $su(2)_{\mathbb{C}}$, where in the the $(J_3, J_\pm)$ basis the analogue of (w, α) is the 'magnetic quantum number' $(m) \in \mathbb{Z}/2$. Of course, in our case, the commutator is not normalized, which is why we have a slightly different set. Then, the set $\Delta_w = \{w \in \mathcal{H}' | \frac{2(w,\alpha)}{(\alpha,\alpha)} \in \mathbb{Z}, \forall \alpha \in \Delta\}$ is called the **weight lattice**. It follows that **all weights of all representations of $\mathcal{L}$ are elements of** Δ_R.

Positive roots

Let us choose an $l \in \mathcal{H}'$ (i.e. a direction in root space), such that: $\Delta = \Delta_+ + \Delta_-$, where $\Delta_\pm = \left\{ \alpha \in \Delta \middle| \begin{matrix} l(\alpha) > 0 \\ l(\alpha) < 0 \end{matrix} \right\}$

are called the positive/negative roots. Note that $\Delta_+ = -\Delta_-$.

Definition 4.37 Let $r : \mathcal{L} \to End(V)$ be a representation of $\mathcal{L}$. A non-zero vector $v \in V$ is called the **highest weight vector** if and only if: $E_\alpha v = 0, \forall \alpha \in \Delta_+$.

Lemma 4.38 For a semi-simple (complex) Lie-algebra $\mathcal{L}$ we have:
(i) every finite dimensional representation r of $\mathcal{L}$ has a highest weight vector v.
(ii) Succesive applications of $E_\alpha, \alpha \in \Delta_-$ on v gives an irreducible sub-representation of r. If r is irreducible, then it is obtained in this way.
(iii) For r irreducible, the highest weight vector is unique up to a re-scaling.

Proof. See notes - p.76 □

Definition 4.39 The weight λ of the highest weight vector is called the highest weight of the representation.

Definition 4.40 A positive (negative) root is called simple if it cannot be written as a sum of two positive (negative) roots.

Note: (a) The simple positive roots form a basis of $\mathcal{H}'$, since Δ spans $\mathcal{H}'$.
(b) For a basis $(\alpha_i)_{i=1,...,rk(\mathcal{L})}$ of $\mathcal{H}'$, a weight w is characterised by the integers $a_i = \frac{2(w,\alpha_i)}{(\alpha_i,\alpha_i)} \in \mathbb{Z}$. The vector $(a_1, ..., a_{rk(\mathcal{L})}) \in \mathbb{Z}^{rk(\mathcal{L})}$ is called the **Dynkin label** of the weight w.
(c) The matrix $A_{ij} = \frac{2(\alpha_i,\alpha_j)}{(\alpha_j,\alpha_j)}$ (since α_i form a basis, they are simple roots) is called the **Cartan matrix** of $\mathcal{L}$ and its rows are the Dynkin labels of the positive roots.

Theorem 4.41 The highest weight of all finite dimensional irreducible representations of $\mathcal{L}$ are given by the Dynkin labels $(a_1, ..., a_{rk(\mathcal{L})})$, where all $a_i > 0$.

Proof. See Fulton, Harris - Appendix D. □

5 Examples

5.1 Lorentz Group

5.2 SU(n), tensor methods

5.2.1 SU(n)

Let us start with some basic definitions: $\eta = diag(\underbrace{-1,...,-1}_{p},\underbrace{1,...,1}_{q})$, with $p+q=n$.

$$U(p,q) = \{U \in Gl(\mathbb{C}^n) | U^\dagger \eta U = \eta\}$$
$$SU(p,q) = \{U \in U(p,q) | det(U) = 1\}$$
$$U(n) = U(n,0) = \{U \in Gl(\mathbb{C}^n) | U^\dagger U = \mathbb{I}\}$$
$$SU(n) = SU(n,0) = \{U \in U(n) | det(U) = 1\}$$

There are a few global properties we will only state, without proving. All $(S)U(p,q)$ are connected. Furthermore, $(S)U(n)$ are compact. From now on we will focus on the $SU(n)$ group.

Lie algebra: $U = \mathbb{I} + T + ... \Rightarrow T = -T^\dagger$ and $tr(T) = 0 \Rightarrow su(n) = \{T \in End(\mathbb{C}^n) | T = -T^\dagger, tr(T) = 0\}$. It is then easy to see that: $dim_{\mathbb{R}} su(n) = n^2 - 1$. The Cartan sub-algebra is then the set of traceless anti-hermitian diagonal matrices: $\mathcal{H} = \{diag(ia_1, ..., ia_n) | a_i \in \mathbb{R}, \sum_i a_i = 0\}$, so $rk(su(n)) = n-1$. We can also introduce the complexified Lie algebra, defined as:

$$su(n)_{\mathbb{C}} = \{T \in End(\mathbb{C}^n) | tr(T) = 0\} = su(p,q)_{\mathbb{C}} = sl(n,\mathbb{C}) =: A_{n-1}$$

Cartan-Weyl basis: Define the matrices: $H_i = diag(0,...,0,1,0,...,0)$, with the 1 in the i^{th} position and E_{ij} as the matrix with zero entries everywhere, except the i^{th} row and the j^{th} column, where its value is 1. Then: $\mathcal{H} = \{b_i H_i | b_i \in \mathbb{C}, \sum_i b_i = 0\}$. We now claim that the Cartan-Weyl decomposition is given by:

$$\mathcal{L} = \mathcal{H} \oplus \bigoplus \mathbb{C} E_{ij}$$

Define a dual basis $\{L_i\}$ such that: $L_i(H_j) = \delta_{ij}$. Then: $\mathcal{H}' = \frac{\mathbb{C}(L_1,...,L_n)}{\mathbb{C}(L_1+...+L_n)}$. Why is the denominator needed? Note that there are n matrices in the basis of $\mathcal{H}'$, but its rank is equal to $n-1$, so we must also take into account the 'traceless' condition in $\mathcal{H}$. Let $H = \sum_k b_k H_k \in \mathcal{H}$ so:

$$ad(H)(E_{ij}) = [H, E_{ij}] = ... = (b_i - b_j)E_{ij} = (L_i - L_j)(H)E_{ij} := L_{ij}(H)E_{ij}$$

Hence, E_{ij} has root L_{ij}, so $\Delta = \{L_{ij} := L_i - L_j | i \neq j\}$ are the roots, which agrees with the above decomposition.

Killing form: Let $H = \sum b_i H_i$, $\tilde{H} = \sum \tilde{b}_j H_j$, with $\sum bi = \sum \tilde{b}_j = 0$. We can then compute:

$$\Gamma(H,\tilde{H}) = \sum_{i\neq j} L_{ij}(H) L_{ij}(\tilde{H}) = \sum_{i\neq j} (b_i - b_j)(\tilde{b}_i - \tilde{b}_j) = \sum_{i,j} (b_i - b_j)(\tilde{b}_i - \tilde{b}_j) = 2n \sum_i b_i \tilde{b}_i$$

We can also compute the Killing form on roots; let us first define $L =_i L_i \in \mathcal{H}'$ and still using $H = \sum b_i H_i$ from before, there must exist a unique H_L (see 'Structure of Cartan decomposition' theorem), $H_L = \sum c_i H_i$ with $\sum c_i = 0$, such that: $\Gamma(H_L, H) = L(H), \forall H \in \mathcal{H}$. Then: $2n \sum c_i b_i = \sum l_i b_i \Rightarrow \sum (2nc_i - l_i) b_i = 0 \Rightarrow c_i = \frac{1}{2n}(l_i - k)$, where $k = \frac{1}{n}\sum l_j$ (k was chosen such that the constraint on the c_i's is still satisfied). Then:

$$(L,\tilde{L}) = \Gamma(H_L, H_{\tilde{L}}) = 2n \sum (l_i - k)(\tilde{l}_i - \tilde{k}) = \frac{1}{2n}\left[\sum l_i \tilde{l}_i - \frac{1}{n}\sum l_i \sum \tilde{l}_j\right]$$

Positive and simple roots: Define ordering $\hat{l}(_i L_i) := \sum k_i l_i$, where $k_1 > k_2 > ... > k_n$. Then we have: $\hat{l}(L_{ij}) = k_i - k_j$ and it is clear that: $\Delta_+ = \{L_{ij} | i < j\}$ and $\Delta_- = \{L_{ij} | i > j\}$. We also have the set of simple positive roots: $\{\alpha_i = L_{i,i+1} | i = 1, ..., n-1\}$, from which all the other L_{ij}'s can be obtained (eg. $L_{13} = \alpha_1 + \alpha_2$ etc.). Similarly, the set of simple negative roots is: $\{L_{i+1,i} | i = 1, ..., n-1\}$.

Cartan Matrix: Let $\lambda = \sum \lambda_i L_i$ and using the previous results it is straightforward to show that: $(L_{ij}, L_{ij}) = 1/n$ and $(\lambda, L_{ij}) = \frac{1}{2n}(\lambda_i - \lambda_j)$ so: $\frac{2(\lambda, L_{ij})}{(L_{ij}, L_{ij})} = \lambda_i - \lambda_j$. We can also build the Cartan matrix:

$$A_{ij} = \frac{2(\alpha_i, \alpha_j)}{(\alpha_j, \alpha_j)} = \frac{2(L_{i,i+1}, L_{j,j+1})}{(L_{j,j+1}, L_{j,j+1})} = \begin{cases} 2, & i = j \\ -1, & |i-j| = 1 \\ 0, & \text{otherwise} \end{cases} \Rightarrow A = \begin{pmatrix} 2 & -1 & 0 & 0 & 0 & & \\ -1 & 2 & -1 & 0 & 0 & & \\ 0 & -1 & 2 & -1 & 0 & & \\ 0 & 0 & -1 & 2 & -1 & & \\ & & & & \ddots & & \\ & & & & & 2 & -1 \\ & & & & & -1 & 2 \end{pmatrix} \to (n-1)\times(n-1) \text{ matrix}$$

Graphical representation of Cartan Matrix:
- simple positive roots $\alpha_i \leftrightarrow \circ^i$
- connect $\circ^i$ and $\circ^j$ by a link with $-A_{ij}$ lines, for $i < j$

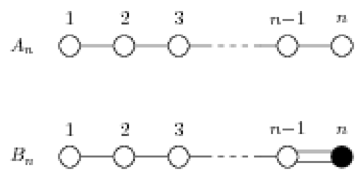


Figure 3: Dynkin Diagrams

Weight lattice: Using the previous results, with $\lambda = \sum \lambda_i L_i : \frac{2(\lambda, L_{ij})}{(L_{ij}, L_{ij})} = \lambda_i - \lambda_j \in \mathbb{Z}, \forall L_{ij}$. Then the weight lattice is: $\Delta_W = \frac{\mathbb{Z}(L_1, \ldots, L_n)}{\mathbb{Z}(L_1 + \ldots + L_n)}$. Here, the denominator is needed as there is an additional degree of freedom associated with the weights - i.e. we can still shift by all integers (will use this soon).

Dynkin Label: $a = (a_1, \ldots, a_{n-1})$ with $a_i = \frac{2(\lambda, \alpha_i)}{(\lambda_i, \lambda_i)} = \lambda_i - \lambda_{i+1}$

We can arbitrarily set $\lambda_n = 0$ (by the quotient in Δ_W), so: $\begin{cases} \lambda_{n-1} = a_{n-1} \\ \lambda_{n-2} = a_{n-2} + a_{n-1} \\ \vdots \\ \lambda_1 = a_1 + \ldots + a_{n-1} \end{cases}$

Also remember that the irreducible representations are in $1-1$ correspondence with the highest weight Dynkin labels $(a_1, \ldots, a_{n-1})$, where $a_i \geq 0$.

Fundamental Representation: on $V \cong \mathbb{C}^n$. Define the ascending operators E_{ij}, for $i < j$, as the matrix with **one** non-zero entry on the i^{th} row and j^{th} column. Then, it is clear that the vector $e_1 = (1, 0, \ldots, 0)$ is the highest weight vector, since $E_{ij} e_1 = 0$, for any i, j. Also, for some $H = \sum b_i H_i \in \mathcal{H}$: $He_i = \sum_j b_j H_j e_i = b_i e_i = L_i(H) e_i$, so e_i has weight L_i. It is then clear that L_1 is the highest weight of the fundamental representation and that the Dynking label is: $a = (1, 0, \ldots, 0)$.

Tensors: Recall that the tensor product representation is given by: $\mathcal{R}_{V \otimes V}(g)(v_1 \otimes v_2) = \mathcal{R}(g)(v_1) \otimes \mathcal{R}(g)(v_2)$. We introduce the 2-index symmetric and antisymmetric tensors, defined by:

$$S^2V \text{ generated by: } v_1 \otimes_S v_2 = \tfrac{1}{2}(v_2 \otimes v_2 - v_2 \otimes v_1) \text{ and } \Lambda^2 V \text{ generated by: } v_1 \wedge v_2 = \tfrac{1}{2}(v_2 \otimes v_2 - v_2 \otimes v_1)$$

These do form representations and, in fact, $\mathcal{R}_{\Lambda^k V}$ and $\mathcal{R}_{S^p V}$ are also representations. Note that because of the ordering of k_i's from before, the highest weight vector of $\Lambda^k V$ is $e_1 \wedge \ldots \wedge e_k$ (this is the non-zero state with highest l), with its corresponding weight: $L_1 + \ldots + L_k$. Consequently, its Dynkin label is: $(0, 0, \ldots, 0, 1, 0, \ldots, 0)$, where the 1 is at the $k^t h$ position.

Let us also consider S^kU. Say $u \in U$ is the highest weight vector of U, with the highest weight λ. Then, it follows that $u^k \in S^kU$ is the highest weight vector of S^kU, with highest weight $k\lambda$. Consequently, the irreducible representation with highest weight Dynkin label given by $(a_1, ..., a_{n-1})$ is contained in: $S^{a_1}V \otimes S^{a_2}(\Lambda^2 V) \otimes ... \otimes S^{a_{n-1}}(\Lambda^{n-1}V)$, with highest weight: $e_1^{a_1} \otimes (e_1 \wedge e_2)^{a_2} \otimes ... \otimes (e_1 \wedge ... \wedge e_{n-1})^{a_{n-1}}$. However, there is still an issue with this - while it is true that it contains irreducible representations, it is not always itself irreducible. We will come back to this issue in the next section.

	d	h	D	$SU(2)$	$SU(3)$	$SU(4)$
□	1	1	N	2	3	4
□ □ □ $\}\ N-1$	1	$(N-1)!$	N	2	$\overline{3}$	$\overline{4}$
□ □ □ □ $\}\ N$	1	$N!$	1	1	1	1
□□ □ □ $\}\ N-1$	$N-1$	$\frac{N!}{N-1}$	N^2-1	3	8	15
□□	1	2	$\frac{N(N+1)}{2}$	3	6	10
□ □	1	2	$\frac{N(N-1)}{2}$	1	$\overline{3}$	6
□□□	1	6	$\frac{N(N+1)(N+2)}{6}$	4	10	20
□□ □	2	3	$\frac{N(N+1)(N-1)}{3}$	2	8	20
□ □ □	1	6	$\frac{N(N-1)(N-2)}{6}$	0	1	$\overline{4}$
□□□□	1	24	$\frac{N(N+1)(N+2)(N+3)}{24}$	5	15	35
□□□ □	3	8	$\frac{N(N+1)(N+2)(N-1)}{8}$	3	15	45
□□ □□	2	12	$\frac{N^2(N+1)(N-1)}{12}$	1	$\overline{6}$	20
□□ □ □	3	8	$\frac{N(N+1)(N-1)(N-2)}{8}$	0	3	15
□ □ □ □	1	24	$\frac{N(N-1)(N-2)(N-3)}{24}$	0	0	1

Figure 4: Young Tableaux

5.2.2 Tensor Methods

Before doing a more systematical description of tensor methods, let us first have a look at how tensors might be used in this context. To each representation, we associate a tensor which transforms in the following way:

· fundamental representation $\boldsymbol{n}$: $\phi_\mu \mapsto U_\mu{}^\nu \phi_\nu$
· complex conj. repr. $\bar{\boldsymbol{n}}$: $\phi^\mu \mapsto (U^*)^\mu{}_\nu \phi^\nu$
· tensor in $\boldsymbol{n}^{\otimes p} \otimes \boldsymbol{n}^{\otimes q}$: $\phi^{\nu_1 ... \nu_q}_{\mu_1 ... \mu_p} \mapsto U_{\mu_1}^{\ \rho_1} ... U_{\mu_p}^{\ \rho_p} (U^*)^{\sigma_1}_{\ \nu_1} ... (U^*)^{\sigma_q}_{\ \nu_q} \phi^{\sigma_1 ... \sigma_q}_{\rho_1 ... \rho_p}$

Remarks: δ^ν_μ and $\epsilon^{\mu_1 \cdots \mu_n}$ are $SU(n)$ invariant tensors, so they can be used to construct new tensors. Note also that (anti)symmetrisation commutes with $SU(n)$ actions, so (anti)symmetrised tensors form representations. To see why this is the case, assume: $\phi_{\mu\nu} = \pm\phi_{\nu\mu}$. Then, this transforms as:

$$\phi_{\mu\nu} \mapsto \phi'_{\mu\nu} = U_\mu{}^\rho U_\nu{}^\sigma \phi_{\rho\sigma} = \pm U_\mu{}^\rho U_\nu{}^\sigma \phi_{\sigma\rho} = \pm \phi'_{\sigma\rho}$$

We can build $SU(n)$ invariants such as: $\phi^\mu_\mu = \delta^\mu_\nu \phi_\mu{}^\nu$ and $\epsilon^{\mu_1 \dots \mu_n} \phi_{\mu_1 \dots \mu_n}$. In addition to this, $\phi^\mu = \epsilon^{\mu\mu_1 \dots \mu_n} \phi_{\mu_1 \dots \mu_n}$ transforms in $\bar{\boldsymbol{n}}$, so we can conclude that: $\bar{\boldsymbol{n}} \sim \Lambda^{n-1} V \sim (0, ..., 0, 1)$. It can be also shown that the Dynkin label of the adjoint representation is $(1, 0, ..., 0, 1)$.

Let us now approach this topic in a slightly more formal way. Consider some d ranked tensors $V^{\otimes d}$, with $V \cong \mathbb{C}^n$ and the permutation group S_d. This acts as: $\sigma(v_1 \otimes ... \otimes v_d) = v_{\sigma(1)} \otimes ... \otimes v_{\sigma(d)}$. Recall that the partitions $d = \lambda_1 + ... + \lambda_k$, with $\lambda_1 \geq ... \geq \lambda_k$ can be associated with a Young tableaux with $\lambda = (\lambda_1, ..., \lambda_k)$. We have shown in a previous example that the projectors are given by:

$$P_\lambda = c \underbrace{\left[\sum_{\sigma \in R_\lambda} \sigma\right]}_{\text{row-preserving}} \underbrace{\left[\sum_{\sigma \in C_\lambda} sgn(\sigma)\sigma\right]}_{\text{col-preserving}}$$

The action of $\sigma \in S_d$ on $V^{\otimes d}$ commutes with $SU(n)$, so we can define $P_\lambda(V) = P_\lambda V^{\otimes d}$, which form $SU(n)$ representations.

Theorem 5.1 (i) $P_\lambda(V)$ contains the same irreducible representation with highest weight $\lambda = \sum_i \lambda_i L_i$, where $a_i = \lambda_i - \lambda_{i+1}$, $(\lambda_i) = (a_1 + ... + a_{n-1}, a_2 + ... + a_{n-1}, ..., a_{n-1}, 0)$ and the associated Dynkin label $(a_1, ..., a_{n-1})$, as $S^{a_1} V \otimes S^{a_2}(\Lambda^2 V) \otimes ... \otimes S^{a_{n-1}}(\Lambda^{n-1} V)$.

(ii) $P_\lambda(V)$ is irreducible.

Proof. TBC. $\square$

Corollary 5.2 The Young tableaux with less than n rows are in $1-1$ correspondence with the irreducible complex representations of $su(n)_\mathbb{C}$.

$$\begin{array}{|c|c|c|c|} \hline \mu_1 & \mu_2 & \cdots & \mu_{\lambda_1} \\ \hline \nu_1 & \nu_2 & \cdots & \\ \cline{1-3} \vdots & & & \\ \cline{1-1} & & & \\ \cline{1-1} \end{array} \leftrightarrow \phi_{\mu_1 \dots \mu_{\lambda_1}; \nu_1 \dots \nu_{\lambda_2}; \dots}$$

The tensor shown above is symmmetric in row indices, anti-symmetric in column indices.

Definition 5.3 A standard tableaux is a Young tableaux with the numbers $1, ..., n$ filled into the boxes, such that:
(i) numbers do not decrease from left to right
(ii) numbers increase from top to bottom

Consequently, we can deduce that: $dim(P_\lambda(V))$ = number of standard tableaux. The following table shows some examples of Young tableaux. The S and A stand for the symmetric and anti-symmetric parts of $\boldsymbol{n} \otimes \boldsymbol{n}$. The standard tableaux are obtained using the above definition and the dimension of the representation can then be easily calculated. Note that $dim\left((\boldsymbol{n} \otimes \boldsymbol{n})_S\right) + dim\left((\boldsymbol{n} \otimes \boldsymbol{n})_A\right) = n$, as it should be.

Representation	Young tableaux	Tensor	Standard tableaux	Dimension
$\boldsymbol{n}$	$\ydiagram{1}$	ψ_μ	$\ytableaushort{1}, \dots, \ytableaushort{n}$	n
$(\boldsymbol{n} \otimes \boldsymbol{n})_S$	$\ydiagram{2}$	$\psi_{(\mu,\nu)}$	$\ytableaushort{11}, \ytableaushort{12}, \dots$	$\frac{1}{2}n(n+1)$
$(\boldsymbol{n} \otimes \boldsymbol{n})_A$	$\ydiagram{1,1}$	$\psi_{[\mu,\nu]}$	$\ytableaushort{1,2}, \ytableaushort{1,3}, \dots$	$\frac{1}{2}n(n-1)$

The above results are hold for any $SU(n)$. Let us consider $SU(3)$ for now. Recall also that the Dynkin label can be easily obtained from the Young tableaux, as discussed in section 5.2.1.

Representation	Young tableaux	Dynkin label	Tensor
$\mathbf{1}$	$\ydiagram{1,1,1}$	$(0,0)$	$\psi_{[\mu\nu\rho]} = \epsilon_{\mu\nu\rho}\psi$
$\mathbf{3}$	$\ydiagram{1}$	$(1,0)$	ψ_μ
$\bar{\mathbf{3}}$	$\ydiagram{1,1}$	$(0,1)$	$\psi_{[\mu\nu]}$
$\mathbf{6}$	$\ydiagram{2}$	$(2,0)$	$\psi_{(\mu\nu)}$
$\mathbf{8}$	$\ydiagram{2,1}$	$(1,1)$	

The tensor of the first few representations listed are easy to compute. However, for $\mathbf{8}$, it is not that obvious. We first need to find the permutations that preserve to columns and the rows and then compute the projector:

$$P_\lambda = \underbrace{\left(e + \begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{pmatrix}\right)}_{\text{row-preserving}} \underbrace{\left(e - \begin{pmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \end{pmatrix}\right)}_{\text{col-preserving}} = e + \begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{pmatrix} - \begin{pmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \end{pmatrix} - \begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{pmatrix}$$

Then, the tensor of interest is: $P_\lambda \psi_{\lambda_1\lambda_2\lambda_3} = \psi_{\lambda_1\lambda_2\lambda_3} + \psi_{\lambda_2\lambda_1\lambda_3} - \psi_{\lambda_3\lambda_2\lambda_1} - \psi_{\lambda_3\lambda_1\lambda_2}$. We can also consider tensor products of representations, such that:

$$P_\lambda(V) \otimes P_\mu(V) = \bigoplus_\nu N_{\lambda\mu} P_\nu(V)$$

One can determine the coefficients $N_{\lambda\mu}$ using the so-called *Littlewood-Richardson* rule (See Fulton, Harris, Appendix A). However, there is a practical rule which is much simpler:

i) Write one of the Young Tableaux as: $\ytableaushort{aa{\dots}a,bb{\dots},{\vdots}}$

ii) Attach boxes of the first tableaux to the second one in all possible ways, starting with a's, then b's etc., such that:

   i) no two same letters appear in the same column

   ii) the result is always a Young tableaux

iii) Read all letters from left to right in the first row, then the second row etc. This sequence must form a *lattice permutation* (i.e. to the left of any symbol there are no fewer a's than b's etc.)

Let us now apply this rule to some representations of $SU(3)$:

$$\mathbf{3}\otimes\mathbf{3} = \ytableaushort{a} \otimes \ytableaushort{{}} = \ytableaushort{{},a} \oplus \ytableaushort{{}a} = \bar{\mathbf{3}}\oplus\mathbf{6}$$

$$\mathbf{3}\otimes\bar{\mathbf{3}} = \ytableaushort{a} \otimes \ytableaushort{{},{}} = \ytableaushort{{},{},a} \oplus \ytableaushort{{}a,{}} = \mathbf{1}\oplus\mathbf{8}$$

5.2.3 $SU(3)$ explicitly

A common basis for the generators of the $SU(3)$ Lie algebra are the *Gell Mann* matrices, given by:

$$\lambda_i = \begin{pmatrix} \sigma_i & 0 \\ 0 & 0 \end{pmatrix}, i = 1, 2, 3 : su_I(2)$$

$$\lambda_4 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \lambda_5 = \begin{pmatrix} 0 & 0 & -i \\ 0 & 0 & 0 \\ i & 0 & 0 \end{pmatrix} : su_v(2)$$

$$\lambda_6 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \lambda_7 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{pmatrix} : su_v(2)$$

$$\lambda_8 = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix}$$

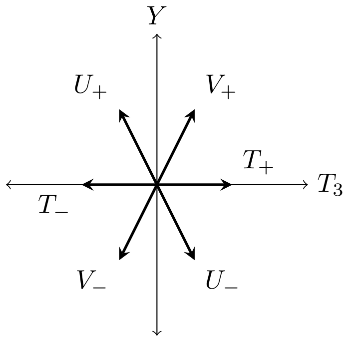
Note that: $tr(\lambda_I\lambda_J) = 2\delta_{IJ}$. We will define $T_I = \frac{1}{2}\lambda_I$, and using the intuition from the complexification of $su(2)$, introduce:

$$T_\pm = T_1 \pm iT_2, \; U_\pm = T_6 \pm iT_7 \text{ and } V_\pm = T_4 \pm iT_5$$

We now want to determine the Cartan-Weyl basis. The Cartan is composed of the matrices commuting with the space spanned by the Gell Mann matrices, so: $\mathcal{H} = Span(T_3, Y := \frac{2}{\sqrt{3}}T_8)$. Note that $Q = T_3 + \frac{1}{2}Y \in \mathcal{H}$ as well and that $rk(su(3)) = 2$. The commutators of interest are listed below:

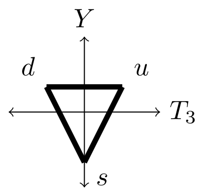
$$\left.\begin{array}{ll} [T_3, T_\pm] = \pm T_\pm, & [Y, T_\pm] = 0 \\ [T_3, U_\pm] = \mp\frac{1}{2}U_\pm, & [Y, U_\pm] = \pm U_\pm \\ [T_3, V_\pm] = \pm\frac{1}{2}V_\pm, & [Y, V_\pm] = \pm V_\pm \end{array}\right\} \Rightarrow \begin{array}{r} \alpha_{T_\pm} = (\pm 1, 0) \\ \alpha_{U_\pm} = (\mp\frac{1}{2}, \pm 1) \\ \alpha_{V_\pm} = (\pm\frac{1}{2}, \pm 1) \end{array} \text{ are the roots in the } T_3, Y \text{ basis}$$

The corresponding root diagram is shown below.

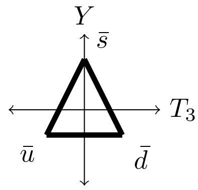


Let us now consider some representations. In the fundamental representation, **3**, we have: $u = e_1, d = e_2, s = e_3$. Then, in the $T_3 = diag(\frac{1}{2}, -\frac{1}{2}, 0), Y = diag(\frac{1}{3}, \frac{1}{3}, -\frac{2}{3})$ basis, the weights will be:

$$\left.\begin{array}{ll} T_3 u = \frac{1}{2}u, & Yu = \frac{1}{3}u \\ T_3 d = -\frac{1}{2}d, & Yd = \frac{1}{3}d \\ T_3 s = 0, & Ys = -\frac{2}{3}s \end{array}\right\} \Rightarrow \begin{array}{c} \lambda_u = \left(\frac{1}{2}, \frac{1}{3}\right) \\ \lambda_d = \left(-\frac{1}{2}, \frac{1}{3}\right) \\ \lambda_s = \left(0, -\frac{2}{3}\right) \end{array}$$

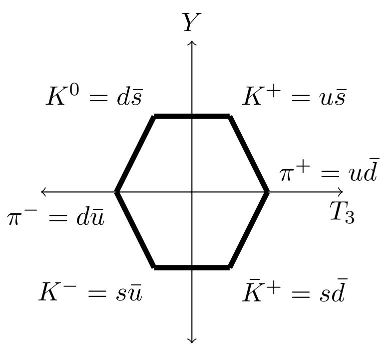


Then, for the $\bar{\mathbf{3}}$ representation: $\bar{u} = e_1, \bar{d} = e_2, \bar{s} = e_3$ and $T_3 = diag(-\frac{1}{2}, \frac{1}{2}, 0), Y = (-\frac{1}{3}, -\frac{1}{3}, \frac{2}{3})$, and the weight diagram is shown below, with: $\lambda_{\bar{u}} = (-\frac{1}{2}, -\frac{1}{3}), \lambda_{\bar{d}} = (\frac{1}{2}, -\frac{1}{3}), \lambda_{\bar{s}} = (0, \frac{2}{3})$



With these results, we can now compute $\mathbf{3} \otimes \bar{\mathbf{3}}$ explicitly. Note that weights add up when we take the tensor product of the representations. Note that there are 3 states are the origin of the weight diagram, states that we will label: $\pi^0, \eta^0 \in \mathbf{8}$ and $\eta' \in \mathbf{1}$.

Tensor States	Weight
$u \otimes \bar{u}$	$(0,0)$
$u \otimes \bar{d}$	$(1,0)$
$u \otimes \bar{s}$	$(1/2,1)$
$d \otimes \bar{u}$	$(-1,0)$
$d \otimes \bar{d}$	$(0,0)$
$d \otimes \bar{s}$	$(-1/2,1)$
$s \otimes \bar{u}$	$(-1/2,-1)$
$s \otimes \bar{d}$	$(1/2,-1)$
$s \otimes \bar{s}$	$(0,0)$



To find out what combinations of u, s, d are in π^0 and η^0, we start with π^+ for instance, and apply T_-; we can also start at K^0 and apply U_-. However, we firstly need to find an expression for T_- and U_- in this representation.

$$\left.\begin{array}{ll} T_-^{\mathbf{3}} = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} & T_-^{\bar{\mathbf{3}}} = \begin{pmatrix} 0 & -1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \\ U_-^{\mathbf{3}} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} & U_-^{\bar{\mathbf{3}}} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 0 & 0 \end{pmatrix} \end{array}\right\} \Rightarrow \begin{array}{l} T_-^{\mathbf{3}\otimes\bar{3}} = \mathbb{I} \times T_-^{\bar{3}} + T_-^{\mathbf{3}} \times \mathbb{I} \\ U_-^{\mathbf{3}\otimes\bar{3}} = \mathbb{I} \times U_-^{\bar{3}} + U_-^{\mathbf{3}} \times \mathbb{I} \end{array}$$

Then: $T_-(u\bar{d}) = u(T_-^3\bar{d}) + (T_-^3 u)d = -uu + \bar{d}\bar{d}$ and similarly: $U_-(d\bar{s}) = -d\bar{d} + s\bar{s}$. Consequently, by choosing pi^0, the other state that is part of **8** is found by imposing orthogonality; to obtain the last $(0,0)$ state we must form a state that is orthogonal to both of these two. Hence:

$$\pi^0 = \frac{1}{\sqrt{2}}(u\bar{u} - d\bar{d}),\ \eta^0 = \frac{1}{\sqrt{6}}(u\bar{u} + d\bar{d} - 2s\bar{s}),\ \eta' = \frac{1}{\sqrt{3}}(u\bar{u} + d\bar{d} + s\bar{s})$$

The last point we want to discuss is the *embedding* of $SU(2)$ into $SU(3)$. Given a matrix $U_(2) \in SU(2)$, this embedding is done by:

$$\begin{pmatrix} U_2 & 0 \\ 0 & 1 \end{pmatrix}$$

We now want to see the branching of the representations of $SU(3)$ under this embedding. It is clear that: $\mathbf{3}_{SU(3)} \to (\mathbf{2} \oplus \mathbf{1})_{SU(2)}$, from which: $\bar{\mathbf{3}}_{SU(3)} \to \bar{\mathbf{2}} \oplus \mathbf{1} = (\mathbf{2} \oplus \mathbf{1})_{SU(2)}$. Then, we can consider other representations, such as:

$$(\mathbf{8} \oplus \mathbf{1})_{SU(3)} = (\mathbf{3} \otimes \bar{\mathbf{3}})_{SU(3)} = (\mathbf{2} \oplus \mathbf{1}) \otimes (\mathbf{2} \oplus \mathbf{1})_{SU(2)} = (\mathbf{3} \oplus 2 \oplus \mathbf{2} \oplus \mathbf{1} \oplus \mathbf{1})_{SU(2)}$$

Therefore: $\mathbf{8}_{SU(3)} \to (\mathbf{3} \oplus 2 \oplus \mathbf{2} \oplus \mathbf{1})_{SU(2)}$

5.3 SO(n), spinor representations

5.3.1 SO(n)

As for $SU(n)$, let us start with some basic definitions: $\eta = diag(\underbrace{1,...,1}_{p},\underbrace{-1,...,-1}_{q})$, with $p+q=n$.

$O(p,q) = \{O \in Aut(\mathbb{R}^n) | O^T \eta O = \eta\}$, so $det(O) = \pm 1$
$SO(p,q) = \{O \in O(p,q) | det(O) = 1\}$
$O(n) = O(n,0) = \{O \in Aut(\mathbb{R}^n) | O^T O = \mathbb{I}_n\}$
$SO(n) = SO(n,0) = \{O \in O(n) | det(O) = 1\}$

Global properties: $O(n)$ has two disconnected components. $SO(n)$ is connected, but not simply connected. Furthermore, we have the following isomorphism: $SO(n) \cong Spin(n)/\mathbb{Z}_2$. (There is another isomorphism involving one of these $Spin$ groups: $Spin(6) \cong SU(4)$.)

Lie algebra: $O = \mathbb{I} + T + ... \Rightarrow T = -\eta T^T \eta$, so: $so(p,q) = \{T \in End(\mathbb{R}^n) | T = -\eta T^T \eta\} = Span(\sigma_{\mu\nu})$. The σ matrices mentioned above, are just a convenient basis of this lie algebra: $(\sigma_{\mu\nu})^\rho{}_\sigma = \eta^\rho{}_\mu \eta_{\nu\sigma} - \eta_{\mu\sigma}\eta^\rho{}_\nu$. It is clear that the dimension of this Lie algebra is: $dim(so(p,q)) = n(n-1)/2$. The σ matrices satisfy the so-called Lorentz algebra:

$$[\sigma_{\alpha\beta}, \sigma_{\gamma\delta}] = \eta_{\alpha\delta}\sigma_{\beta\gamma} + \eta_{\alpha\gamma}\sigma_{\delta\beta} + \eta_{\beta\delta}\sigma_{\gamma\alpha} + \eta_{\beta\gamma}\sigma_{\alpha\delta}$$

The Lie algebra can be complexified into:

$$so(p,q)_{\mathbb{C}} \cong so(n)_{\mathbb{C}} = \begin{cases} \mathcal{D}_m & \text{if } n = 2m \\ \mathcal{B}_m & \text{if } n = 2m+1 \end{cases}$$

The Cartan sub-algebra is given by: $\mathcal{H} = Span(\sigma_{12}, \sigma_{34}, ..., \sigma_{2m-1,2m})$, such that $rk(so(2m)) = rk(so(2m+1)) = m$, as we cannot build another σ matrix with only one extra row.

Cartan-Weyl basis

(i) $\mathcal{D}_m, n = 2m$. It is convenient to use the signature: $\eta = \begin{pmatrix} \mathbb{I}_m & 0 \\ 0 & -\mathbb{I}_m \end{pmatrix}$. Also, let $P = \frac{1}{2}\begin{pmatrix} \mathbb{I} & \mathbb{I} \\ \mathbb{I} & \mathbb{I} \end{pmatrix}$, so: $P^T \eta P = \begin{pmatrix} 0 & \mathbb{I} \\ \mathbb{I} & 0 \end{pmatrix}$. The Lie algebra generators can be written as: $T = \begin{pmatrix} A & B \\ B^T & C \end{pmatrix}$, with $A = -A^T, C = -C^T$. It turns out that a good basis is:

$$P^T T P = \begin{pmatrix} \tilde{A} & \tilde{B} \\ \tilde{C} & -\tilde{A}^T \end{pmatrix}, \tilde{B} = -\tilde{B}^T, \tilde{C} = -\tilde{C}^T$$

(ii) $\mathcal{B}_m, n = 2m+1$. In a similar fashion, we can introduce $\eta = \left(\begin{array}{cc|c} \mathbb{I}_m & 0 & 0 \\ 0 & -\mathbb{I}_m & 0 \\ \hline 0 & 0 & 1 \end{array}\right)$, $P = \frac{1}{2}\left(\begin{array}{cc|c} \mathbb{I} & \mathbb{I} & 0 \\ \mathbb{I} & \mathbb{I} & 0 \\ \hline 0 & 0 & 1 \end{array}\right)$, so $P^T \eta P = \left(\begin{array}{cc|c} 0 & \mathbb{I} & 0 \\ \mathbb{I} & 0 & 0 \\ \hline 0 & 0 & 1 \end{array}\right)$. Hence, as before:

$$P^T T P = \left(\begin{array}{cc|c} \tilde{A} & \tilde{B} & \tilde{E} \\ \tilde{C} & -\tilde{A}^T & \tilde{F} \\ \hline -\tilde{F}^T & -\tilde{E}^T & 0 \end{array}\right), \tilde{B} = -\tilde{B}^T, \tilde{C} = -\tilde{C}^T$$

Define $H_i = E_{ii} - E_{m+1,m+1}$, for $i = 1, ..., m$, where E_{ij} is the matrix with zero entries everywhere except the i^{th} row and j^{th} column, where the entry is 1. Then the Cartan sub-algebra is given by: $\mathcal{H} = \{\sum_{i=1}^m b_i H_i\}$. We can further define a dual basis (L_i), with $L_i(H_j) = \delta_{ij}$, so: $\mathcal{H}' = \{\sum_i l_i L_i\}$. Then, the $\mathcal{D}_m$ basis is given by:

$$\begin{array}{llcc} & & \text{root} & \\ A \text{ block} & X_{ij} = E_{ij} - E_{m+j,m+i} & L_i - L_j & i \neq j \\ B \text{ block} & Y_{ij} = E_{i,m+j} - E_{j,m+i} & L_i + L_j & i < j \\ C \text{ block} & Z_{ij} = E_{m+i,j} - E_{m+j,i} & -L_i - L_j & i < j \end{array}$$

Apart from these basis elements, $\mathcal{B}_m$ has some additional ones:

$$\begin{array}{lc} & \text{root} \\ U_i = E_{i,2m+1} - E_{2m+1,m+i} & L_i \\ V_i = E_{m+i,2m+1} - E_{2m+1,i} & -L_i \end{array}$$

The Cartan-Weyl decomposition can then be written as:

$$\mathcal{D}_m = \mathcal{H} \oplus \bigoplus_{i \neq j} \mathbb{C}X_{ij} \oplus \bigoplus_{i<j} \mathbb{C}Y_{ij} \oplus \bigoplus_{i<j} \mathbb{C}Z_{ij} \text{ and } \mathcal{B}_m = \mathcal{D}_m \oplus \bigoplus_i \mathbb{C}U_i \oplus \bigoplus_i \mathbb{C}V_i$$

The root systems are: $\begin{cases} \Delta(\mathcal{D}_m) = \{\pm L_i \pm L_j | i \neq j\} \\ \Delta(\mathcal{B}_m) = \mathcal{D}_m \cup \{\pm L_i\} \end{cases}$

Killing form: Using the above definition of H_i, and the fact that $H_i H_j = 0$, for $i \neq j$, we can compute the Killing form:

$$\Gamma(\sum_i b_i H_i, \sum_j \tilde{b}_j H_j) = N(n) \sum b_i \tilde{b}_i \text{ where } N(n) = \begin{cases} 2n-2, & \text{if } n = 2m+1 \\ 2n-4, & \text{if } n = 2m \end{cases}$$

Consequently, for $L = \sum_i l_i L_i \Rightarrow H_L = \frac{1}{N(n)} \sum l_i H_i$. We can also compute: $(\sum l_i L_i, \sum \tilde{l}_j L_j) = \frac{1}{N(n)} \sum l_i \tilde{l}_i$.

Positive and simple roots, Cartan matrix: $\hat{l}(\sum l_i L_i) = \sum k_i l_i$, with $k_1 > k_2 > ... > k_n > 0$. Let us start with $\mathcal{D}_m$, for which: $\Delta_+ = \{L_i + L_j | i < j\} \cup \{L_i - L_j | i < j\}$. Then, as for $SU(n)$, introduce: $\alpha_i = L_i - L_{i+1}$, for $i = 1, ..., m-1$ and $\alpha_m = L_{m-1} + L_m$, in order to make sure that Δ_+ can be reconstructed. Using the above result for the Killing form, it follows that: $(\alpha_i, \alpha_i) = 2/N(n)$, for $i = 1, ...m$ (i.e. all simple positive roots have the same length). Then, for some $\lambda = \sum \lambda_i L_i$, we have:

$$(\lambda, \alpha_i) = \frac{1}{N(n)} \begin{cases} \lambda_i - \lambda_{i+1}, & i = 1, ..., m-1 \\ \lambda_{m-1} + \lambda_m, & i = m \end{cases} \quad \text{so: } \frac{2(\lambda, \alpha_i)}{(\alpha_i, \alpha_i)} = \begin{cases} \lambda_i - \lambda_{i+1}, & i = 1, ..., m-1 \\ \lambda_{m-1} + \lambda_m, & i = m \end{cases}$$

Consequently, the Cartan matrix is: $A(\mathcal{D}_m) = \begin{pmatrix} 2 & -1 & & & & & & \\ -1 & 2 & -1 & & & & & \\ & -1 & 2 & -1 & & & & \\ & & -1 & 2 & -1 & & & \\ & & & & \ddots & & & \\ & & & & & 2 & -1 & -1 \\ & & & & & -1 & 2 & 0 \\ & & & & & -1 & 0 & 2 \end{pmatrix} \to m \times m$ matrix

The corresponding Dynkin diagram is shown in Figure 3.
For $\mathcal{B}_m$: $\Delta_+ = \{L_i + L_j | i < j\} \cup \{L_i - L_j | i < j\} \cup \{L_i\}$, and, in this case, a good choice would be: $\alpha_i = L_i - L_{i+1}$ for $i = 1, ..., m-1$ and $\alpha_m = L_m$. Hence: $(\alpha_i, \alpha_i) = \begin{cases} \frac{2}{N(n)}, & i = 1, ..., m-1 \\ \frac{1}{N(n)}, & i = m \end{cases}$. The first $m-1$ roots are called 'longer' roots, with α_m being the 'short' root. This is drawn as a filled circle in the Dynkin diagram.

$$(\lambda, \alpha_i) = \frac{1}{N(n)} \begin{cases} \lambda_i - \lambda_{i+1}, & i = 1, ..., m-1 \\ 2\lambda_m, & i = m \end{cases} \quad \text{so: } \frac{2(\lambda, \alpha_i)}{(\alpha_i, \alpha_i)} = \begin{cases} \lambda_i - \lambda_{i+1}, & i = 1, ..., m-1 \\ 2\lambda_m, & i = m \end{cases}$$

Hence, the Cartan matrix is: $A(\mathcal{B}_m) = \begin{pmatrix} 2 & -1 & & & & & \\ -1 & 2 & -1 & & & & \\ & -1 & 2 & -1 & & & \\ & & -1 & 2 & -1 & & \\ & & & & \ddots & & \\ & & & & & 2 & -2 \\ & & & & & -1 & 2 \end{pmatrix} \to m \times m$ matrix

The corresponding Dynkin diagram can be also seen in Figure 3.

Weight lattice: Using the previous results we can compute, for both $\mathcal{D}_m$ and $\mathcal{B}_m$, the following quantity: $\frac{2(\lambda_i, \pm L_i \pm L_j)}{(\pm L_i \pm L_j, \pm L_i \pm L_j)} = \pm\lambda_i \pm \lambda_j \overset{!}{\in} \mathbb{Z}$. Furthermore, for $\mathcal{B}_m$ we also have: $\frac{2(\lambda, \pm L_i)}{(\pm L_i, \pm L_i)} = \pm 2\lambda_i \overset{!}{\in} \mathbb{Z}$. Hence, the weight lattice is given by: $\Delta_w = \mathbb{Z}(L_1, ..., L_m, \frac{1}{2}(L_1 + ... + L_m))$.

Irreducible representations: Let us start with $\mathcal{D}_m$ again. We have already computed the Dynkin labels (when calculating the Cartan matrix):

$$a_i = \frac{2(\lambda, \alpha_i)}{(\alpha_i, \alpha_i)} = \begin{cases} \lambda_i - \lambda_{i+1}, & i = 1, ..., m-1 \\ \lambda_{m-1} + \lambda_m, & i = m \end{cases}$$

As for $SU(n)$, the weight of the representation with the Dynkin label $(a_1, ..., a_m)$ is given by:

$$\lambda = (a_1 + ... + a_{m-2})L_1 + (a_2 + ... + a_{m-2})L_2 + ... + a_{m-2}L_{m-2} + a_{m-1}\alpha + a_m\beta$$

where $\alpha = \frac{1}{2}(L_1 + ... + L_m)$ and $\beta = \frac{1}{2}(L_1 + ... + L_{m-1} - L_m)$. Note that the first $m-2$ terms are identical to what we have seen for $SU(n)$. However, the fact that the a_m's do not coincide with those from $SU(n)$ brings two additional terms. This representation can be found in:

$$S^{a_1}V \otimes S^{a_2}(\Lambda^2 V) \otimes ... \otimes S^{a_{m-2}}(\Lambda^{m-2}V) \otimes S^{a_{m-1}}\Gamma_\alpha \otimes S^{a_m}\Gamma_\beta$$

We are already familiar with the first part of this expression; however, we have not shown that the last two representations (called **spinor representations**) actually exist! We will come back to this issue later.

Let us go back to $\mathcal{B}_m$ for now. Starting from the Dynkin label we can compute the weight of the representation as in the previous case:

$$\lambda = (a_1 + ... + a_{m-2})L_1 + (a_2 + ... + a_{m-2})L_2 + ... + a_{m-2}L_{m-2} + a_{m-1}L_{m-1} + a_m\alpha$$

Then, the representation can be obtained by the same 'trick'.

Remark The Dynkin label of the adjoint is always $(0, 1, 0, ..., 0)$, for both $\mathcal{D}_m$ and $\mathcal{B}_m$.

5.3.2 Spinor representations of $SO(1, d-1)$

We will restrict our discussion to the $SO(1, d-1)$ groups, with $\eta = diag(-1, 1, ..., 1)$. Let us introduce:

$$k := \left[\frac{d}{2}\right] - 1 \begin{cases} \frac{d-2}{2}, & d \text{ even} \\ \frac{d-3}{2}, & d \text{ odd} \end{cases}$$

The **Clifford algebra** is generated by the $2^{k+1} \times 2^{k+1}$ dimensional $\gamma_\mu, \mu = 0, ..., d-1$, which satisfy:

$$\{\gamma_\mu, \gamma_\nu\} = 2\eta_{\mu\nu}$$

so $\gamma_\mu^2 = \eta_{\mu\mu}$. At this point, it is not clear that such objects exist, so let us start with some low dimensions. For :

$$d = 2 \text{ can use: } \gamma_0 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} = -i\sigma_0 \text{ and } \gamma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \sigma_1$$

From here, it is easier to construct the γ matrices for higher dimensions, using the following iterative process:

$$d \mapsto d+2 : \begin{cases} \gamma_\mu^{(d+2)} = \gamma_\mu^{(d)} \otimes \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \mu = 0, ..., d-1 \\ \gamma_d^{(d+2)} = \mathbb{I} \otimes \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \\ \gamma_{d+1}^{(d+2)} = \mathbb{I} \otimes \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix} \end{cases}$$

Note that since the two matrices that appear in the construction of $\gamma_d^{(d+2)}$ and $\gamma_{d+1}^{(d+2)}$ commute with the matrix appearing in the construction of $\gamma_\mu^{(d+2)}$, the anti-commutation relations should be preserved. The above procedure works for d even; if this is not the case, we will need to introduce the matrix: $\gamma = i^{-k}\gamma_0\gamma_1...\gamma_{d-1}$, which is a generalization of the γ^5 used in the context of relativistic quantum mechanics. Note that: $\gamma^2 = \mathbb{I}$. Then, for even d:

$$d \mapsto d+1 : \begin{cases} \gamma_\mu^{(d+1)} = \gamma_\mu^{(d)}, \quad \mu = 0, ..., d-1 \\ \gamma_d^{(d+1)} = \gamma \end{cases}$$

The size of the matrices remains the same, but we add one more matrix to the basis.

Let us investigate further the properties of these matrices. Recall that the the $\sigma_{\mu\nu}$ matrices we introduced in the previous section as the generators of the Lie-algebra of $so(p,q)$ satisfy the Lorentz algebra. We introduce some completely different objects:

$$\sigma_{\mu\nu} = \frac{i}{4}[\gamma_\mu, \gamma_\nu]$$

It then follows from the anti-commutation relations that these σ's actually generate the same algebra! In fact, we will see that they are linked to the spinor representations.

Claim For $d = 2m+1$, the $\sigma_{\mu\nu}$ matrices defined above correspond to the spinor representation Γ_α.

Proof. Recall that an element of the Cartan can be written as: $\tilde{H} = \left(\begin{array}{cc|c} B & 0 & 0 \\ 0 & -B & 0 \\ \hline 0 & 0 & 0 \end{array}\right), B = diag(b_1, ..., b_m)$. Also, $\tilde{U} = \left(\begin{array}{cc|c} 0 & 0 & 0 \\ 0 & 0 & u \\ \hline -u^T & 0 & 0 \end{array}\right)$ corresponds to the positive roots L_i. Using the previous definition of the matrix P, we then have: $H = P\tilde{H}P = \left(\begin{array}{cc|c} 0 & B & 0 \\ -B & 0 & 0 \\ \hline 0 & 0 & 0 \end{array}\right) = \sum_{i=1}^m b_i\sigma_{i,m+1}$, where these σ's are the 'original ones' (i.e. not the ones defined in terms of the γ matrices. Also: $U = P\tilde{U}P = \left(\begin{array}{cc|c} 0 & 0 & u \\ 0 & 0 & u \\ \hline -u^T & -u^T & 0 \end{array}\right), U_i = \sigma_{i,2m+1} - \sigma_{m+i,2m+1}$. The highest weight state is defined by: $U_i v = 0 \Rightarrow \gamma_i\gamma_{m+i}v = v$. We want to find the highest weight of this state:

$$Hv = \sum_{i=1}^m \frac{1}{2} b_i \gamma_i \gamma_{m+1} v = \sum_{i=1}^m \frac{1}{2} b_i v = \frac{1}{2}\sum_{i=1}^m L_i(H)v = \alpha(H)v$$

with $\alpha = \frac{1}{2}(L_1 + ... + L_m)$. We would still have to show that this spinor representation is irreducible, but we will not do it here.

□

The Hermitian conjugate of the γ matrices are easy to compute: $\gamma_0^\dagger = -\gamma_0, \gamma_i^\dagger = \gamma_i$ so: $\gamma_\mu = \gamma_0 \gamma_\mu \gamma_0$. Consequently: $\sigma_{\mu\nu}^\dagger = -\gamma_0 \sigma_{\mu\nu} \gamma_0$.

Dirac spinors: The spinor $\psi \in \mathbb{C}^N, N = 2^{k+1}$ forms a complex representation of $so(1, d-1)$, called the Dirac spinor. The infinitesimal transformation is: $\delta\psi = i\epsilon^{\mu\nu}\sigma_{\mu\nu}\psi$. This is a 2^{k+1} dimensional complex representation and it is irreducible for odd d.

Weyl Spinors: For d even: $[\sigma_{\mu\nu}, \gamma] = 0$, define the projector $P_{L,R} = \frac{1}{2}(\mathbb{I} \pm \gamma)$, so $P_{L,R}^2 = P_{L,R}$ since $\gamma^2 = \mathbb{I}$. Also:

$$tr(P_{L,R}) = \frac{1}{2}(tr(\mathbb{I}) \pm tr(\gamma)) = \frac{1}{2}(d \pm (-i)^k \underbrace{tr(\gamma_0\gamma_1 ... \gamma_{d-1})}_{0}) = d/2$$

Hence: $\psi_{L,R} = P_{L,R}\psi$ (where ψ is a Dirac spinor) are 2^k dimensional complex representations, called left and right-handed Weyl spinors. They do corresponds to Γ_α and Γ_β. The proof is similar to what we have already seen for d odd.

Complex conjugation: Note that γ_μ and γ_μ^* satisfy the same anti-commutation relations. This suggests that they are related by some basis transformation. Let us define: $B_1 = \gamma_3\gamma_5 ...$ ($B_1 = \mathbb{I}$ for $d = 2$) and $B_2 = \gamma B_1$. Then:

$$\left.\begin{array}{ll} B_1\gamma_\mu B_1^{-1} = (-1)^k\gamma_\mu^*, & B_2\gamma_\mu B_2^{-1} = (-1)^{k+1}\gamma_\mu^* \\ B_1\gamma B_1^{-1} = (-1)^k\gamma^*, & B_2\gamma B_2^{-1} = (-1)^k\gamma^* \end{array}\right\} \Rightarrow B\sigma_{\mu\nu}B^{-1} = -\sigma_{\mu\nu}^*, \text{ where } B = \begin{cases} B_1 \text{ or } B_2, & d \text{ even} \\ B_1, & d \text{ odd} \end{cases}$$

Charge Conjugation and Majorana spinors: Define $\psi^C = C\psi := B^{-1}\psi$. Then:

$$[i\sigma_{\mu\nu}, C]\psi = i\sigma_{\mu\nu}B^{-1}\psi^* - B^{-1}(i\sigma_{\mu\nu}\psi)^* = -iB^{-1}\sigma_{\mu\nu}^*\psi^* + iB^{-1}\sigma_{\mu\nu}^*\psi^* = 0$$

Also: $(\psi^c)^c = (B^{-1}\psi^*)^c = B^{-1}(B^{-1})^*\psi = (B^*B)^{-1}\psi$. The next question we need to ask is what is this quantity B^*B; before imposing the Majorana condition, we need to see the dimensions where this quantity is just the identity. In fact, using the previous definitions of B_1,B_2 and commuting the γ matrices, one can show that:

$$B_1^*B_1 = (-1)^{k(k+1)/2}\mathbb{I} = I, \text{ for } k = 0, 3 \text{ } mod \text{ } 4 \text{ and also: } B_2^*B_2 = (-1)^{k(k21)/2}\mathbb{I} = I, \text{ for } k = 0, 1 \text{ } mod \text{ } 4$$

Hence, for $d = 0, 1, 2, 3, 4$ mod 8 there exists a matrix B with $B^*B = \mathbb{I}$. In such cases, we can compose a **Majorana condition** for a 2^{k+1} real-dimensional spinor:

$$\psi^c = \psi$$

Majorana-Weyl Spinor: Since Weyl spinors only exists for d even, a Majorana-Weyl spinor could only exist in a dimension d which satisfies: $d = 0, 2, 4$ mod 8. Recall that the γ matrix was introduced in the definition of the Weyl spinor, while the C matrix - for the Majorana spinor. Hence, for consistency, we need to make sure that these do commute:

$$[\gamma, C]\psi = \gamma B^{-1}\psi^* - B^{-1}\gamma^*\psi^* \stackrel{!}{=} 0 \Rightarrow B\gamma B^{-1} = \gamma^* \Leftrightarrow k \text{ even}$$

Hence, Majorana-Weyl spinors could only exist for $d = 2$ mod 8. The following table shows the dimension of the various spinors, for some values of d.

d	k	Dirac	Weyl	Majorana	Maj-Weyl
4	1	8	4	4	–
5	1	8	–	–	–
10	4	64	32	32	16
11	4	64	–	32	–

It turns out that the $d = 10$ Majorana-Weyl spinors are the representations used in SUSY.

Before wrapping up this section, let us also look at some invariants. Define $\bar{\psi} = \psi^\dagger \gamma_0$. Then, the infinitesimal transformation of the quantity $\bar{\psi}\psi$ is given by:

$$\delta(\bar{\psi}\psi) = (\delta\bar{\psi})\psi + \bar{\psi}\delta\psi = (i\epsilon^{\mu\nu}\sigma_{\mu\nu}\psi)^\dagger\gamma_0\psi + \psi^\dagger\gamma_0(i\epsilon^{\mu\nu}\sigma_{\mu\nu}\psi) = i\epsilon^{\mu\nu}(-\psi^\dagger\sigma^\dagger_{\mu\nu}\gamma_0\psi + \psi^\dagger\gamma_0\sigma_{\mu\nu}\psi) = 0$$

where in the last part we used the fact that $\sigma^\dagger_{\mu\nu} = -\gamma_0\sigma_{\mu\nu}\gamma_0$. Thus, we see that $\bar{\psi}\psi$ is indeed an invariant.

6 Classification of Simple Lie algebras

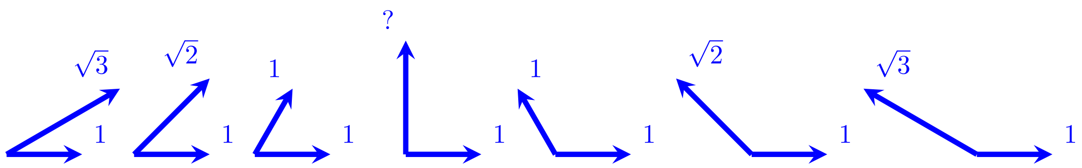
Let us define the following quantities:

$$\begin{cases} ||\alpha|| = (\alpha,\alpha)^{1/2} & \text{'length of roots'} \\ cos\theta = \frac{(\alpha,\beta)}{||\alpha||\cdot||\beta||} & \text{'angle between roots'} \\ n_{\beta\alpha} = \frac{2(\beta,\alpha)}{(\alpha,\alpha)} \in \mathbb{Z} & \end{cases}$$

Note that: $n_{\beta\alpha} = 2cos\theta\frac{||\alpha||}{||\beta||} = n_{\alpha\beta}\frac{||\alpha||^2}{||\beta||^2}$, so: $n_{\alpha\beta}n_{\beta\alpha} = 4cos^2\theta \in [0,4]$. Since $n_{\alpha\beta} \in \mathbb{Z}$, it must be the case that $n_{\alpha\beta}n_{\beta\alpha} = \{0,1,2,3,4\}$. Without loss of generality, let us assume that $||\beta|| \geq ||\alpha||$, so that $|n_{\beta\alpha}| \geq |n_{\alpha\beta}|$. Let us first consider the case where $n_{\alpha\beta}n_{\beta\alpha} = 4$, i.e. $cos\theta = \pm 1$, meaning that β must allign or anti-allign with α: $\beta = k\alpha$. However, using the properties of roots, it must be the case that $k = \pm 1$. The non-trivial solutions are shown in the table below:

$n_{\beta\alpha}$	3	2	1	0	-1	-2	-3
$n_{\alpha\beta}$	1	1	1	0	-1	-1	-1
$\frac{\lVert\beta\rVert}{\lVert\alpha\rVert}$	$\sqrt{3}$	$\sqrt{2}$	1	?	1	$\sqrt{2}$	$\sqrt{3}$
$cos\theta$	$\sqrt{3}/2$	$\sqrt{2}/2$	$1/2$	0	$-1/2$	$-\sqrt{2}/2$	$-\sqrt{3}/2$
θ	$\pi/6$	$\pi/4$	$\pi/3$	$\pi/2$	$2\pi/3$	$3\pi/4$	$5\pi/6$

Using the data from the table, we can form the 'building blocks' of the root diagrams:



Theorem 6.1 Let α, β be roots. The **α-string through** β consists of all roots of the form: $\beta - p\alpha, \beta - (p-1)\alpha, ..., \beta + q\alpha$, such that all of these are roots. Then:
(i) The α string through α consists of $-\alpha, 0, \alpha$.
(ii) For $\beta \neq \pm\alpha$, we have: $p - q = n_{\beta\alpha}$.
(iii) The α string through β has at most four roots, so: $p + q \leq 3$, for $\beta \neq \pm\alpha$.
(iv) For $\beta \neq \pm\alpha$, if: $\begin{cases} (\beta,\alpha) > 0 & \Rightarrow \alpha - \beta \text{ is a root} \\ (\beta,\alpha) < 0 & \Rightarrow \alpha + \beta \text{ is a root} \end{cases}$

Proof. (i) This is clear from the properties of the roots.

(ii) Recall that $(H_\alpha, E_\alpha, E_{-\alpha})$ form an $su(2)_\mathbb{C}$ algebra:

$$\left.\begin{array}{l} ad(H_\alpha)(E_{\beta-p\alpha}) = (\beta - p\alpha, \alpha)E_{\beta-p\alpha} \\ ad(H_\alpha)(E_{\beta+q\alpha}) = (\beta + q\alpha, \alpha)E_{\beta+q\alpha} \end{array}\right\} \Rightarrow \left.\begin{array}{l} \beta - p\alpha, ..., \beta, ..., \beta + q\alpha \\ -j, ..., 0, ..., j \end{array}\right\} su(2)_\mathbb{C} \text{ repr. with } j \in \mathbb{Z}/2$$

Consequently, we must have: $\left.\begin{array}{l} (\beta - p\alpha, \alpha) = -(\alpha,\alpha)j \\ (\beta + q\alpha, \alpha) = (\alpha,\alpha)j \end{array}\right\} \Rightarrow p - q = n_{\beta\alpha}.$

(iii) Set $\beta' = \beta - p\alpha$ and focus on the α string through β', given by: $\beta', \beta' + \alpha, ..., \beta' + q'\alpha$. From (ii) we have that: $q' = |n_{\beta\alpha|} \leq 3$ (from the table).

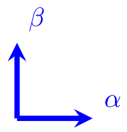
(iv) $p - q = n_{\beta\alpha} = \frac{2(\beta,\alpha)}{(\alpha,\alpha)}$. By considering the sign of $n_{\beta,\alpha} \propto (\beta, \alpha)$, one can determine whether the string can be extended to the right or to the left, from where the statement follows.

□

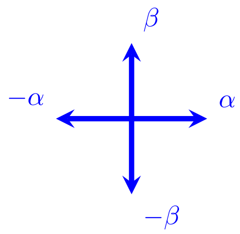
6.1 Classification by 'Drawing' - Root diagrams

1) For $n = rk(\mathcal{L}) = 1$ we have $A_1 \cong su(2)_{\mathbb{C}}$ and the diagram: 

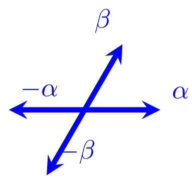
2 For $n = rk(\mathcal{L}) = 2$, we have more cases:
(a) $\theta = \pi/2$: we start by drawing the 'building block' for this specific value of θ:



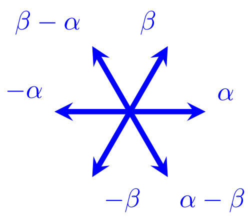
The negatives of the two roots shown must also exists so: $A_1 \oplus A_1 \cong su(2)_{\mathbb{C}} \oplus su(2)_{\mathbb{C}} \cong so(4)_{\mathbb{C}}$



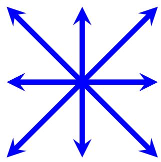
(b) $\theta = \pi/3$. We again start with the 'building block' for this angle, and compute the negatives, obtaining four roots:



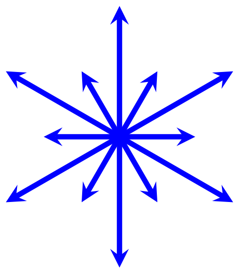
Then, using statement (iv) of the above theorem, we can combine the α and β roots to obtain one more root (and its negative) so: $A_2 \cong su(3)_{\mathbb{C}}$



(c) $\theta = \pi/4$. Proceeding in the same way as in the previous case, we find: $B_2 = so(5)_{\mathbb{C}}$



(d) $\theta = \pi/6$. Again, starting with the 'building block' and using the theorem we find: G_2



This procedure works well for $n = 1, 2$, but as n increases, it is extremely hard (impractical) to compute such diagrams. Hence, we need a more sophisticated method.

6.2 Dynkin Diagrams

Lemma 6.2 (*i*) If $\alpha \neq \beta$ are simple roots, then $\alpha - \beta$ (and $\beta - \alpha$) are not roots.
(*ii*) For two simple roots α, β, we have: $(\alpha, \beta) \leq 0$.

Proof. (*i*) Note that $\alpha = \beta + (\alpha - \beta)$, so if $(\alpha - \beta)$ would be a root, then α cannot be a simple root, which is a contradiction. Similarly write $\beta = \alpha + (\beta - \alpha)$, so $(\beta - \alpha)$ cannot be a root either.

(*ii*) From the fourth statement in Theorem 6.1, we know that if $(\beta, \alpha) > 0$, then $\alpha - \beta$ must be a root. However, for α, β simple roots, $\alpha - \beta$ cannot be a root, so it must be the case that: $(\beta, \alpha) \leq 0$. □

We can now return to the root table from the beginning of this section and identify the **simple roots**, using the above lemma. Thus, since $n_{\beta\alpha} \propto (\beta, \alpha) \leq 0$, these are given by: $\theta = \pi/2,\ 2\pi/3,\ 3\pi/4,\ 5\pi/6$. The Dynkin diagrams are built using these simple roots. The underlying rules for these diagrams are shown in Figure 5.

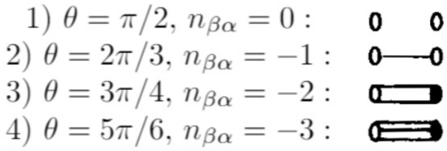


Figure 5: Dynkin Diagrams Rules

In the third rule, since $n_{\beta\alpha} \neq \pm 1$, the two roots has different lengths. We used an empty circle for 'longer roots' and a filled one for 'short roots'. Note that the first two rules also apply for the short roots.

Let us now try to find the relation to the Cartan matrix. For this, let (α_i) be a set of simple roots. Then:

$$A_{ij} = \frac{2(\alpha_i, \alpha_j)}{(\alpha_j, \alpha_j)} = \begin{cases} 2 & ,\ i = j \\ -l_{\alpha_i\alpha_j} & ,\ ||\alpha_i|| \geq ||\alpha_j||,\ i \neq j \\ -1 \text{ or } 0 & ,\ ||\alpha_i|| \leq ||\alpha_j||,\ i \neq j \end{cases},\ \text{where: } l_{\alpha_i\alpha_j} = \begin{cases} -n_{\alpha_i\alpha_j} & \text{for } ||\alpha_i|| \geq ||\alpha_j|| \\ -n_{\alpha_j\alpha_i} & \text{for } ||\alpha_i|| < ||\alpha_j|| \end{cases}$$

In other words, $l_{\alpha_i\alpha_j}$ is the number of lines between the roots α_i and α_j.

6.2.1 Classification of Dynkin Diagrams

From the previous table we can deduce that: $cos\theta = -\frac{\sqrt{l_{\beta\alpha}}}{2}$.

Definition 6.3 A Dynkin diagram with n nodes is called **admissible** if there exist n linear independent vectors (α_i) such that $cos\theta_{\alpha_i\alpha_j} = -\sqrt{l_{\alpha_i\alpha_j}}/2, \forall i, j = 1, ..., n$.
The idea is to classify algebras by classifying Dynkin diagrams. The above requirement is for the simple Lie algebras but, it turns out that it is strong enough to make this classification.

Lemma 6.4 All sub-diagrams of an admissible diagram are admissible.

Proof. The proof is clear from the definition of admissible Dynkin diagrams. □

Lemma 6.5 A diagram with n nodes has at most $n-1$ links (combining double/triple lines as only one link).

Proof. Define a vector $\alpha = \sum_i \frac{\alpha_i}{||\alpha_i||} \Rightarrow \alpha \neq 0$ since the α_i's are linearly independent. Then:

$$0 < ||\alpha||^2 = \sum_{i,j} \frac{(\alpha_i, \alpha_j)}{||\alpha_i|| \cdot ||\alpha_j||} = n + 2 \sum_{\alpha_i, \alpha_j \text{linked}} \frac{(\alpha_i, \alpha_j)}{||\alpha_i|| \cdot ||\alpha_j||} = n + 2 \sum_{\alpha_i, \alpha_j \text{linked}} cos\theta_{\alpha_i \alpha_j} = n + 2 \sum_{\alpha_i, \alpha_j \text{linked}} \frac{-\sqrt{l_{\alpha_i \alpha_j}}}{2}$$

Consequently: $||\alpha||^2 \leq n-$ number of links.

$\square$

The previous lemma excludes all diagrams that have n links or more. Combining the previous two lemmas, we notice that all diagrams that include a loop are excluded, as shown in Figure 6 (**a**).

Lemma 6.6 A node is connected to at most 3 lines.

Proof. Let us start by assuming that a node α is connected to the nodes $\beta_1, ..., \beta_p$, with $(\beta_i, \beta_j) = 0$, for $i \neq j$ (i.e. the β nodes are linked; if they would be, then they would form loops, which are not allowed by the above two lemmas.) Let α_0 be the projection of α onto $Span(\beta_1, ..., \beta_p)$ and let $\gamma = \alpha - \alpha_0$. It is then clear that $(\gamma, \beta_i) = 0$ (since γ is perpendicular to the space spanned by the β vectors). Then, writing:

$$\alpha = \lambda_0 \gamma + \sum_{i=1}^{p} \lambda_i \beta_i \Rightarrow \begin{cases} (\gamma, \alpha) = \lambda_0 (\gamma, \gamma) \\ (\beta_i, \alpha) = \lambda_i (\beta_i, \beta_i) \end{cases} \Rightarrow \alpha = \frac{(\gamma, \alpha)}{||\gamma||^2} \gamma + \sum_i \frac{(\beta_i, \alpha)}{||\beta_i||^2} \beta_i$$

From here we can compute $||\alpha||^2$ and then:

$$||\alpha||^2 = \frac{(\gamma, \alpha)^2}{||\gamma||^2} \Rightarrow \sum_i \frac{(\beta_i, \alpha)^2}{||\alpha||^2 ||\beta_i||^2} = \sum_i cos^2\theta_{\alpha\beta_i} = 1 - \frac{(\gamma, \alpha)^2}{||\alpha||^2 ||\gamma||^2} < 1$$

To see why the above quantity is strictly less then 1, consider the case where it is 1, meaning that $(\gamma, \alpha) = 0$. This implies that $\alpha \in Span(\beta_1, ..., \beta_p)$, which leads to a diagram that is not admissible. Hence: $\sum_i l_{\alpha\beta_i} < 4$.

$\square$

This lemma excludes a few more diagrams, shown in Figure 6(**b**).

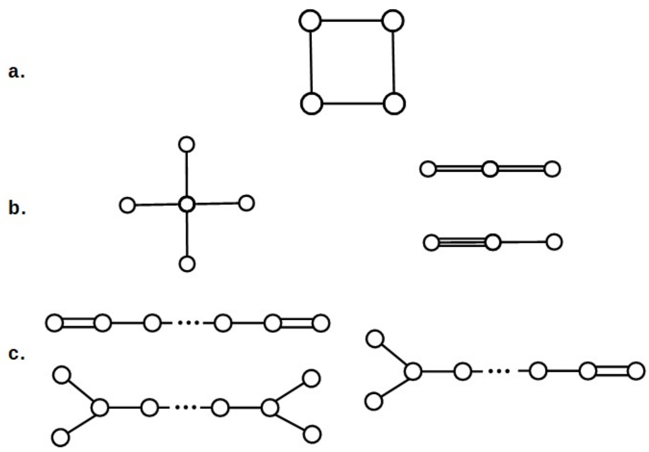


Figure 6: Dynkin Diagrams excluded by: **a**. Lemma 6.5, **b**. Lemma 6.6, **c**. Lemma 6.7

Lemma 6.7 A simple chain (no double/triple links) in an admissible diagram can be replaced by a single node (keeping lines to other nodes unchanged), to lead to an admissible diagram.

Proof. Suppose we have a simple chain $\alpha_1, ..., \alpha_p$, as part of a diagram. Define: $\alpha = \sum_i \frac{\alpha_i}{||\alpha_i||}$, so by Lemma 6.5:

$$||\alpha||^2 = p - \sum_{\alpha_i, \alpha_j \text{linked}} \sqrt{l_{\alpha_i \alpha_j}} = p - (p-1) = 1$$

Consider a root $\beta \neq \alpha$ that is not part of this chain. Then:

$$cos_{\beta\alpha} = \frac{(\beta, \alpha)}{||\beta|| \cdot ||\alpha||} = \sum_{i=1}^{p} \frac{(\beta, \alpha_i)}{||\beta_i|| \cdot ||\alpha_i||} = \begin{cases} cos\theta_{\beta\alpha_1} \\ cos\theta_{\beta\alpha_p} \end{cases}$$

as β can only be linked to α_1 or α_p. Thus, we effectively replaced the chain (α_i) by the single node α.

□

The above lemma excludes a few more diagrams, shown in Figure 6(**c**).

Lemma 6.8 The diagram shown in Figure 7(**a**) is excluded.

Proof. Let $\alpha_1, ..., \alpha_5$ be the nodes of this diagram and let:

$$v = c_1 \frac{\alpha_1}{||\alpha_1||} + c_2 \frac{\alpha_2}{||\alpha_2||}, w = c_3 \frac{\alpha_3}{||\alpha_3||} + c_4 \frac{\alpha_4}{||\alpha_4||} + c_5 \frac{\alpha_5}{||\alpha_5||}$$

Then: $||v||^2 = c_1^2 + c_2^2 - c_1 c_2, ||w||^2 = c_3^2 + c_4^2 + c_5^2 - c_3 c_4 - c_4 c_5$, and $(v, w) = -c_2 c_3 / \sqrt{2}$, since v and w share only one link. Without loss of generality, choose $c_1 = 1, c_2 = 2, c_3 = 3, c_4 = 2, c_5 = 1$, so: $(v, w)^2 = 18, ||v||^2 = 3, ||w||^2$. However, this contradicts the Cauchy-Schwarz inequality since: $(v, w)^2 < ||v||^2 \cdot ||w||^2$.

□

Toghether with the previous lemma, the second diagram in Figure 7(**a**) is also excluded.

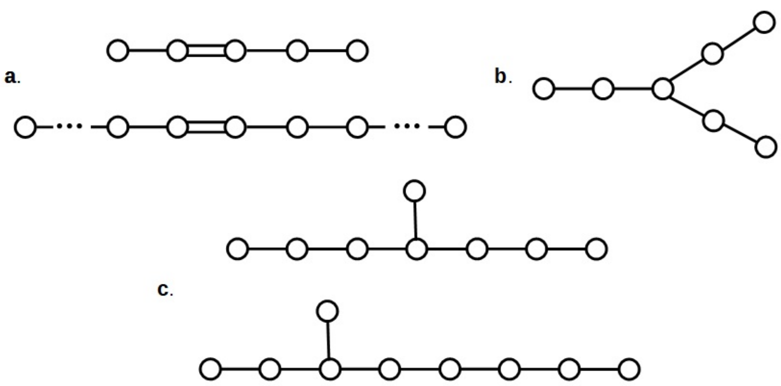


Figure 7: Dynkin Diagrams excluded by: **a**. Lemma 6.8, **b**....., **c**. ....

Lemma 6.9 The diagram shown in Figure 7(**b**) is excluded.

Proof. Label the nodes of the diagram by $\alpha_1, ..., \alpha_7$, with α_1 being the central node. Consider:

$$u = \frac{1}{\sqrt{3}} \left(2\frac{\alpha_2}{||\alpha_2||} + \frac{\alpha_3}{||\alpha_3||} \right), v = \frac{1}{\sqrt{3}} \left(2\frac{\alpha_4}{||\alpha_4||} + \frac{\alpha_5}{||\alpha_5||} \right), w = \frac{1}{\sqrt{3}} \left(2\frac{\alpha_6}{||\alpha_6||} + \frac{\alpha_7}{||\alpha_7||} \right)$$

where $\alpha_{2,4,6}$ are all connected to the central node. As in the proof for lemma 6.6, it follows from the fact that $\alpha_1 \notin Span(u,v,w)$ that:

$$1 > \frac{(\alpha_1,u)^2}{||\alpha_1||\cdot||u||^2} + \frac{(\alpha_1,v)^2}{||\alpha_1||\cdot||v||^2} + \frac{(\alpha_1,w)^2}{||\alpha_1||\cdot||w||^2} = \frac{1}{3}+\frac{1}{3}+\frac{1}{3} = 1$$

which is a contradiction. Hence, the diagram is indeed excluded.

□

Similarly, one can also rule out the two diagrams from Figure 7(**c**). Using all the above results, we classify the admissible Dynkin diagrams in Figure 8.

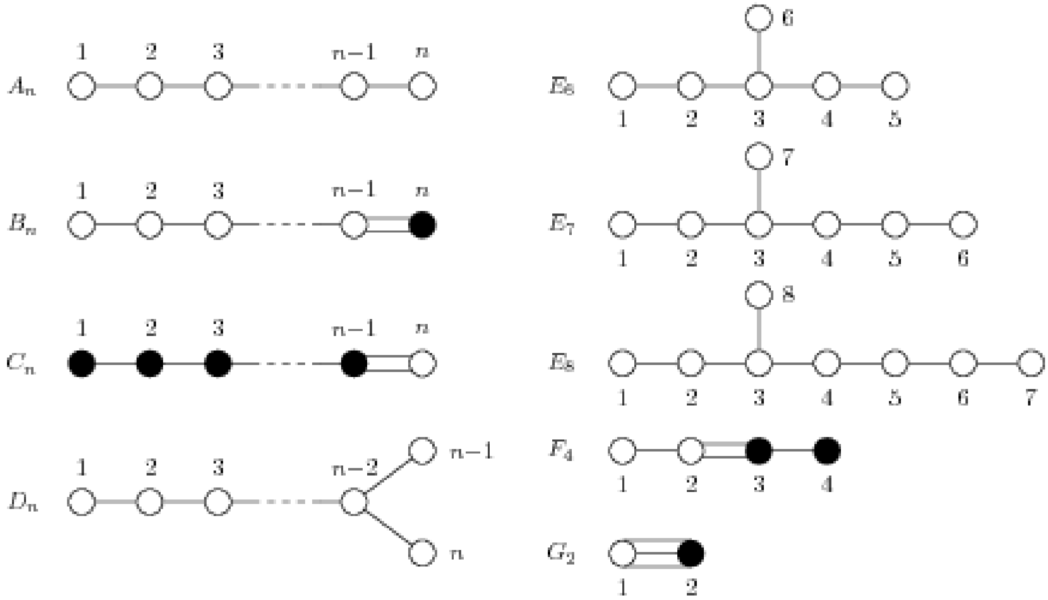


Figure 8: Dynkin Diagrams

The last five diagrams corresponds to the so called EXCEPTIONAL algebras. Note that low-rank exceptional groups are equivalent to the standard groups; also, high rank exceptional algebras: $E_n, n > 8, F_n, n > 4$ or $G_n, n > 2$ do not exist as finite-dimensional algebras (but they might exist as infinite-dimensional ones).

Using the diagrams, it is then straightforward to see some isomorphisms:

$$SO(3) \sim B_1 \sim \bullet \sim \circ \sim A_1 \sim SO(2)$$
$$SO(4) \sim D_2 \sim \circ\ \circ \sim A_1 \times A_1 \sim SU(2) \times SU(2)$$

Similarly, B_2 and C_2 have the same diagram, so $SO(5) \cong Sp(4)$ and from the diagrams of D_3 and A_3 it follows that $SO(6) \cong SU(4)$.

One can also consider the low rank exceptional groups: $E_5 \cong SO(10)$, $E_4 \cong SU(5)$ and $E_3 \cong SU(3) \times SU(2)$.

7 Representations and Dynkin Formalism

Lemma 7.1 Let w be a weight of a representation r, so that $w+\alpha$ is not a weight of r, for a root α. Then, the α-string through w, given by: $w, w-\alpha, ..., w-p\alpha$, has length:

$$p = \frac{2(w,\alpha)}{(\alpha,\alpha)}$$

Proof. (similar to root case) Let us consider the $su(2)_{\mathbb{C}}$ sub-algebra of the representation and let v_w be a vector with weight w so: $r(H_\alpha)v_w = (w,\alpha)v_w$ and $r(H_\alpha)v_{w-p\alpha} = (w-p\alpha,\alpha)v_{w-p\alpha}$. But this must be an $su(2)_{\mathbb{C}}$ representation, i.e. $w \Leftrightarrow j$ and $w - p\alpha \leftrightarrow -(\alpha,\alpha)j$. Thus, we obtain the two equations:

$$(w,\alpha) = -(\alpha,\alpha)j \text{ and } (w-p\alpha,\alpha) = (\alpha,\alpha)j$$

Adding and solving for p concludes the proof.

□

Definition 7.2 Let λ be the highest weight of a representation r and $w = \lambda - \sum_{i=1}^{n} m_i\alpha_i$, with $m_i \in \mathbb{Z}_+$. Then, $\sum m_i$ is called the level of w.

Construction of weight system of r

i) Choose highest weight λ (level 0). (Recall that the Dynkin label has positive entries.)
ii) Assume all weights up to level s are constructed. Then:
   i) find all weights w at level s such that $w + \alpha_i$ is not a weight of r (i.e. the conditions of the previous lemma are satisfied)
   ii) The root α_i can be subtracted $a_i = 2(w,\alpha_i)/(\alpha_i,\alpha_i)$ times from w to obtain new weights of r.
iii) Iterate until the process terminates.

Let us consider a few simple examples:
1) $A_1 = su(2)_{\mathbb{C}}$, with the Cartan matrix $A(A_1) = (2)$. Pick the representation with highest weight $(a_1), a_1 > 0$. Then, the weight system of this representation is simply given by:

$$(a_1), (a_1-2), ..., (-a_1)$$

2) $A_2 = su(3)_{\mathbb{C}}$, with $\begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}$. One can read the Dynkin labels of the positive simple roots from the Cartan matrix: $\alpha_1 = (2,-1)$, $\alpha_2 = (-1,2)$. Let us pick the representations with highest weights $(1,0)$ and $(0,1)$ to start with. Then:

$$(1,0) \xRightarrow[\text{once only}]{-\alpha_1} (-1,1) \xRightarrow{-\alpha_2} (0,-1) \text{ and similarly } (0,1) \xRightarrow{-\alpha_2} (1,-1) \xRightarrow{-\alpha_1} (-1,0)$$

These weights correspond to the **3** and $\mathbf{\bar{3}}$ representations. Let us now consider the representations with highest weights $(2,0)$ and $(1,1)$, for which the weight diagrams are shown below:

$$\begin{array}{c} (2,0) \\ (0,1) \\ (-2,2) \quad (1,-1) \\ (-1,0) \\ (0,-2) \end{array} \qquad\qquad \begin{array}{c} (1,1) \\ (-1,2) \quad (2,-1) \\ (0,0) \quad (0,0) \\ (1,-1) \quad (-1,1) \\ (-1,-1) \end{array}$$

We can work out from the Young tableaux that the first representation is **6**, while the other one is the adjoint representation: **8**. Note that in the second weight system, we wrote the same weight twice, in order to obtain the correct dimension. We do not have any information about the degeneracy so far, but, however, for the adjoint every weight is non-degenerate, except $(0,0)$ which has degeneracy equal to the rank of the Lie algebra (2 here). We will come back to this issue later. Let us now consider tensor products of the above representations:

$$\mathbf{3}\otimes\bar{\mathbf{3}} = \mathbf{8}\oplus\mathbf{1} \rightarrow \begin{array}{c}(1,0)\\(-1,1)\\(0,-1)\end{array} \otimes \begin{array}{c}(0,1)\\(1,-1)\\(-1,0)\end{array} = (0,0)\oplus \begin{array}{c}(1,1)\\ \vdots\end{array}$$

The weights of the resulting representations are obtained by taking sums of all possible combinations. For simplicity, we omitted most of these weights in the above example, as it is clear that $(1,1)$ will be the highest weight of the for the rest of the weights, generating thus **8**.

A more involved calculation is required for higher dimensional representations. Figure 9 shows the weight system for the **16** of $SO(10)$.

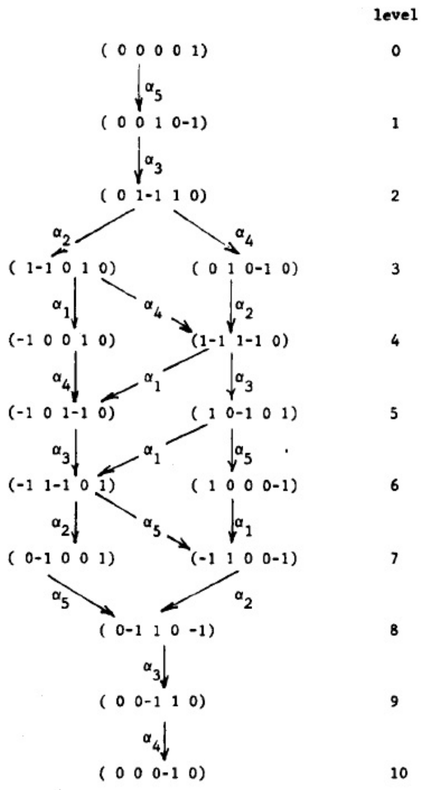


Figure 9: Weight system for the **16** of $SO(10)$

7.1 Dynkin basis and dual basis

Consider a weight w, a let (α_i), $i = 1, ..., n$ be the simple positive roots. Then, recall that the Dynkin label of the weight is given by: $(a_1, ..., a_n)$, where $a_i = 2(w, \alpha_i)/(\alpha_i, \alpha_i)$. We can write w in the so-called *dual basis* as:

$$w = \sum_{i=1}^{n} \bar{w}_i \frac{2}{(\alpha_i, \alpha_i)} \alpha_i$$

where, by convention: $(\alpha_i, \alpha_i) = 2$ for longer roots. We are effectively replacing the Dynkin label by: $(\bar{w}_1, ..., \bar{w}_n)$. Note that:

$$a_i = \sum_j \bar{w}_j \frac{2}{(\alpha_j, \alpha_j)} A_{ji} \Rightarrow \bar{w}_j = \sum_i a_i G_{ij}, \text{ where } G_{ij} = (A^{-1})_{ij} \frac{(\alpha_j, \alpha_j)}{2}$$

In the above A is the Cartan matrix and G is usually referred to as the *metric tensor*.

Killing form:

$$(w, w') = \sum_{i,j} \bar{w}_i \frac{2}{(\alpha_i, \alpha_i)} (\alpha_i, \alpha_j) \frac{2}{(\alpha_j, \alpha_j)} \bar{w}'_j = \sum_{i,j} \bar{w}_i \frac{2}{(\alpha_i, \alpha_i)} A_{ij} \bar{w}'_j = \sum_i a_i \bar{w}'_j$$

We can further write: $(w, w') = \sum_{i,j} a_i G_{ij} a'_j$.

Charges are eigenvalues relative to some chosen elements of the Cartan $\mathcal{H}$. Thus, the charge associated to $H \in \mathcal{H}$, written as: $H = \sum_i \frac{2\bar{q}_i}{(\alpha_i,\alpha_i)} H_{\alpha_i}$, is given by: $(\bar{q}_1, \bar{q}_2, ...)$. To make this statement more concrete, consider a weight: $\lambda = \sum_i \bar{\lambda}_i \frac{2}{(\alpha_i,\alpha_i)} \alpha_i = \sum_{i,j} a_j G_{ij} \frac{2}{(\alpha_i,\alpha_i)} \alpha_i$. Then, the charge associated to this weight is:

$$H(\lambda) = \sum_k \frac{2\bar{q}_k}{(\alpha_k, \alpha_k)} H_{\alpha_k}(\lambda) = \sum_{i,j,k} a_j G_{ij} \frac{2}{(\alpha_i, \alpha_i)} \frac{2\bar{q}_k}{(\alpha_k, \alpha_k)} H_{\alpha_k}(\alpha_i) = \sum_{i,j,k} a_j (A^{-1})_{ji} \frac{2\bar{q}_k}{(\alpha_k, \alpha_k)} (\alpha_i, \alpha_k) \Rightarrow$$

$$\Rightarrow H(\lambda) = \sum_i \bar{q}_i a_i$$

Let us consider a few examples now. For $A_2 = su(3)_{\mathbb{C}}$ we want to specify two such charges, the isospin (T_3) and the physical charge (Q). To make this choice, consider the weights $(1, 0)$ and $(0, 1)$, corresponding to the u and $\bar{s}$ quarks. Thus, we impose:

$$\begin{array}{ll} u: \ T_3(1,0) = \frac{1}{2} & \qquad Q(1,0) = \frac{2}{3} \\ \bar{s}: \ T_3(0,1) = 0 & \qquad Q(0,1) = \frac{1}{3} \end{array}$$

These conditions are then enough to specify the two charges for any other weight. It simply follows that: $\bar{T}_3 = \frac{1}{2}(1, 0)$ and $\bar{Q} = \frac{1}{3}(2, 1)$. We can that consider the K^0 particle for instance, whose weight is $(-1, 1)$: $Q(-1, 1) = 0$, $T_3(-1, 1) = -1/2$.

Let us now return to the degeneracy issues encountered before. It turns out that there is an algorithm for computing the degeneracies, but it is rather complicated. The following theorem describes this algorithm, but we will not prove it here (see Fulton and Harris, section 25.1).

Theorem 7.3 *(Freudenthal)* For a representation r with highest weight λ, the degeneracy g_w of a weight w satisfies:

$$[(\lambda + \delta, \lambda + \delta) - (w + \delta, w + \delta)] g_w = 2 \sum_{\alpha \in \Delta_+, k \geq 0} g_{w+k\alpha} (w + k\alpha, \alpha)$$

where $\delta = \frac{1}{2} \sum_{\alpha \in \Delta_+} \alpha$. In the Dynkin basis: $\delta = (1, ..., 1)$.

Theorem 7.4 *(Weyl formula)* The dimension $dim(\lambda)$ of a representation with highest weight λ is given by:

$$dim(\lambda) = \prod_{\alpha \in \Delta_+} \frac{(\lambda + \delta, \alpha)}{(\delta, \alpha)}$$

Proof. See Fulton, Harris, section 24.1.

□

Freudenthal's recursive relation usually involves long and tedious calculations. However, if one can find $dim(\lambda)$ distinct roots for the representation with highest weight λ, then all the roots are non-degenerate.

Theorem 7.5 The value $C(\lambda)$ of the Casimir in the representation with highest weight λ is:

$$C(\lambda) = (\lambda, \lambda + 2\delta)$$

Proof. Recall from chapter 4 that: $C = \gamma^{IJ} T_I T_J = \sum_{i,j} \gamma^{ij} H_{\alpha_i} H_{\alpha_j} + \sum_{\alpha \in \Delta} E_\alpha E_{-\alpha}$, where $\gamma_{ij} = \Gamma(H_{\alpha_i}, H_{\alpha_j}) = (\alpha_i, \alpha_j) = A_{ij}(\alpha_j, \alpha_j)/2$. Then: $(\gamma^{-1})_{ij} = \frac{2}{(\alpha_i, \alpha_i)}(A^{-1})_{ij}$ so:

$$C = \sum_{i,j} G_{ij} \frac{2H_{\alpha_i}}{(\alpha_i, \alpha_i)} \frac{2H_{\alpha_j}}{(\alpha_j, \alpha_j)} + \sum_{\alpha \in \Delta} E_\alpha E_{-\alpha}$$

For the highest weight λ, with $a_i = \frac{2(\lambda, \alpha_i)}{(\alpha_i, \alpha_i)}$ and highest weight vector v, we then have:

$$C(v) = \sum_{i,j} G_{ij} a_i a_j v + \sum_{\alpha \in \Delta_+} [E_\alpha, E_{-\alpha}]\, v = \left[\Gamma(\lambda, \lambda) + \sum_{\alpha \in \Delta_+} (\lambda, \alpha) \right] v$$

Note that we restricted the sum over Δ to Δ_+ since $E_{-\alpha} v = 0$ for $\alpha \in \Delta_-$, as $E_{\pm\alpha}$ are raising/lowering operators and they act on the highest weight state in this situation. Then, using the definition of δ, it follows that: $Cv = (\lambda, \lambda + 2\delta)v$.

□

Let us return again to the $A_2 = su(3)_{\mathbb{C}}$ example, where the simple positive roots are: $\alpha_1 = (2, -1), \alpha_2 = (-1, 2)$ - clear from the Cartan matrix. Recall that for the **8** representation (adjoint), we had two $(0,0)$ weights, so from the remaining 6 we must have 3 positive and 3 negative. Hence, the other positive root is: $\beta = \alpha_1 + \alpha_2 = (1, 1)$. One can compute the inverse of the Cartan matrix and find the metric tensor:

$$G(A_2) = \frac{1}{3} \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$$

Consider a general representation with highest weight given by $\lambda = (a_1, a_2)$. We will use the fact that: $G\alpha_1 = (1, 0), G\alpha_2 = (0, -1)$ and $G\beta = (1, 1)$. Hence, the Weyl formula reads:

$$dim(a_1, a_2) = \left[(a_1 + 1, a_2 + 1) \frac{G\alpha_1}{\delta G \alpha_1} \right] \left[(a_1 + 1, a_2 + 1) \frac{G\alpha_2}{\delta G \alpha_2} \right] \left[(a_1 + 1, a_2 + 1) \frac{G\beta}{\delta G \beta} \right]$$

$$dim(a_1, a_2) = \frac{1}{2}(a_1 + a_2 + 2)(a_1 + 1)(a_2 + 1)$$

One can see that $dim(1,1) = 8$, as expected. We can also compute the Casimir:

$$C(a_1, a_2) = (a_1, a_2)G(a_1 + 2, a_2 + 2) = \frac{2}{3}(a_1^2 + a_2^2 + a_1a_2 + 3a_1 + 3a_2)$$

Recall also from chapter 4 that:

$$c(a_1, a_2) = \frac{dim(a_1, a_2)}{dim(ad)}C(a_1, a_2)$$

We find that $c(0,1) = c(1,0) = 1$ etc.

7.2 Extended Dynkin Diagrams

We have already classified the Lie algebras using Dynkin diagrams, such that there are no other Dynkin diagrams left. However, there is another type of diagrams, called *extended* which will turn out to be very useful. Such a diagram can be obtaining by adding the negative of the highest root (which is essentially the 'lowest root') θ to the original Dynkin diagram. As already stated, the result is not a Dynkin diagram, as the vectors are not linearly independent now. However, if one root is removed from the extended diagram, it becomes again a Dynkin digram (since $\theta - \alpha$ cannot be a root, for any α).

Let us return to the usual example: A_2, for which the Cartan matrix and the metric tensor are:

$$A = \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}, \; G = \frac{1}{3}\begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$$

The positive simple roots (appearing in the Dynkin diagram) are $\alpha_1 = (2, -1)$ and $\alpha_2 = (-1, 2)$. Then, the negative of the highest weight is: $\theta = (-1, -1)$. Note that $(\theta, \theta) = (\alpha_1, \alpha_1) = (\alpha_2, \alpha_2) = 2$ and $(\theta, \alpha_1 = (\theta, \alpha_2) = -1$. To see how will this weight be connected to the rest of the diagram, we need:

$$\frac{2(\alpha_1, \theta)}{(\theta, \theta)} = -1 \text{ and } \frac{2(\alpha_2, \theta)}{(\theta, \theta)} = -1$$

Thus, θ will be linked to both α_1 and α_2 by one line. One can do this calculation for all Dynkin diagrams. A list of the Extended Dynkin diagrams is shown in Figure 10.

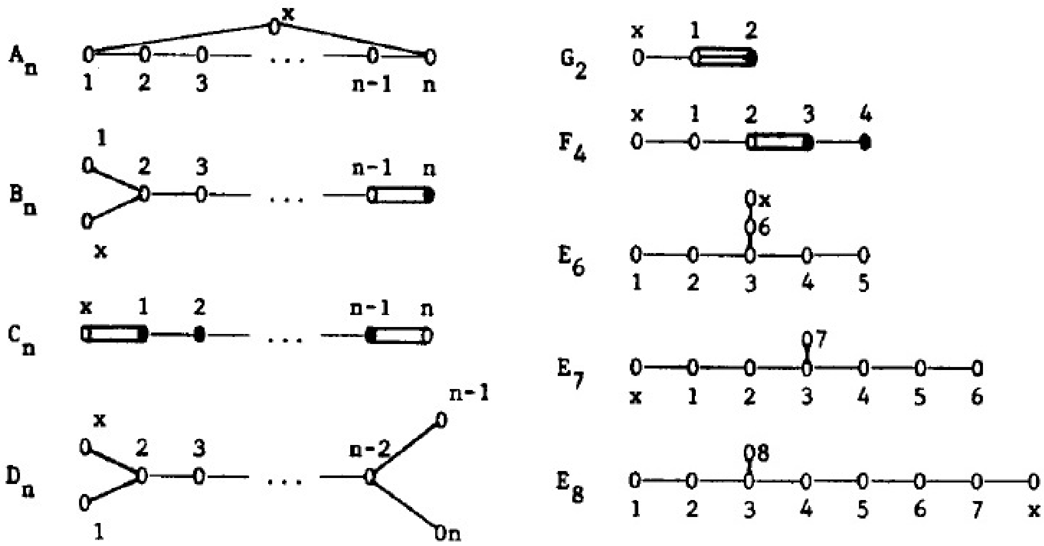


Figure 10: Extended Dynkin Diagrams for Simple Lie Algebras - The extended root is marked by x; black dots correspond to shorter roots.

7.3 Sub-groups and Branching Rules

Definition 7.6 A proper sub-group $G' \subset G$ is called *maximal* if there is no group $\tilde{G}$ such that $G'...\tilde{G}...G$. The definition can be extended to algebras.

Definition 7.7 A sub-algebra $\mathcal{L}' \subset \mathcal{L}$ is called *regular* if $Span(H_i') \subset Span(H_i)$ and $Span(E_\alpha') \subset Span(E_\alpha)$.

Theorem 7.8 The maximal, regular sub-algebra $\mathcal{L}' \subset \mathcal{L}$ can be obtained from the Dynkin diagram as follows:

i) *non semi-simple* $\mathcal{L}'$: given by a $U(1)$ part and a semi-simple part obtained by removing one dot from the Dynkin diagram of $\mathcal{L}$

ii) *semi-simple* $\mathcal{L}'$: obtained by removing one dot from the Extended Dynkin diagram of $\mathcal{L}$

Note that there are irregular sub-algebras for which this does not work.

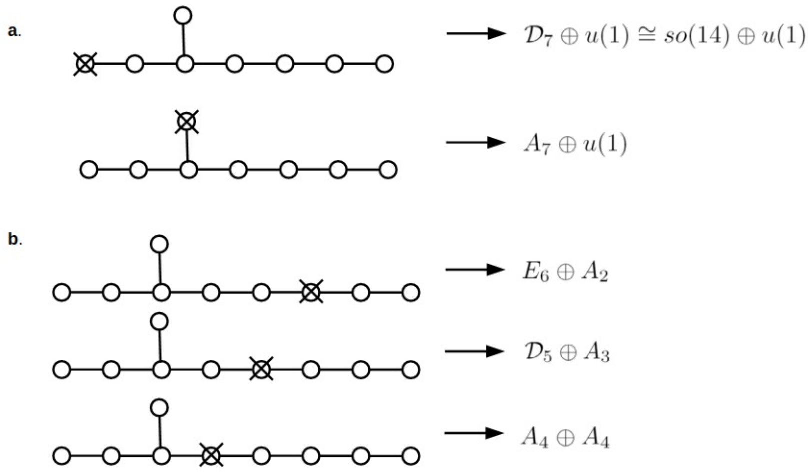


Figure 11: Sub-algebras of E_8 - a. non semi-simple, b. semi-simple

Consider a regular sub-algebra $\mathcal{L}' \subset \mathcal{L}$ and let r be an irreducible representation of $\mathcal{L}$. Under this branching, r might become reducible: $r \mapsto \bigoplus_i r_i$, where r_i are irreducible representations of $\mathcal{L}'$. There is a linear map $\mathcal{P}(\mathcal{L}' \subset \mathcal{L})$ which only depdends on $\mathcal{L}, \mathcal{L}'$ and the embedding $\mathcal{L}' \subset \mathcal{L}$, such that the weight $(a_1, ..., a_n)$ of r and the weight $(a_1', ..., a_m')$ of r' relate as:

$$\begin{pmatrix} a_1' \\ \vdots \\ a_m' \end{pmatrix} = \mathcal{P}(\mathcal{L}' \subset \mathcal{L}) \begin{pmatrix} a_1 \\ \vdots \\ a_n \end{pmatrix}$$

An example is the embedding of $SU(2)$ in $SU(3)$, which we introduced in section 5.2.3. Since the ranks of these two are 1 and 2, respectively, we expect the map to be of the form: $\mathcal{P} = (p_1, p_2)$. We can now look at a couple of representations (3 and $\bar{\mathbf{3}}$) and work out $\mathbb{P}$; the result will then hold for any other representations. Remember that the highest weights of 3 and $\bar{\mathbf{3}}$ are $(1,0)$ and $(0,1)$; we know that both of them are mapped to $\mathbf{2} + \mathbf{1}$ of $SU(2)$ (i.e. Dynkin label (1)), so we must have:

$$\mathcal{P} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = 1 \text{ and also } \mathcal{P} \begin{pmatrix} 0 \\ 1 \end{pmatrix} = 1 \Rightarrow \mathcal{P} = (1,1)$$

This should now work for all representations of $SU(3)$, so let us consider $\mathbf{6}$, whose weight system we have already seen:

$$\begin{array}{ccccccccccc}
 & (2,0) & & & (2) & & & (2) & & & \\
 & (0,1) & & & (1) & & & & & (1) & \\
(-2,2) & & (-1,1) & \to & (0) \quad (0) & & = & (0) & \oplus & & \oplus\,(0) \\
 & (-1,0) & & & (-1) & & & & & (-1) & \\
 & (0,-2) & & & (-2) & & & (-2) & & &
\end{array}$$

Thus, we showed that: $\mathbf{6}_{SU(3)} \mapsto (\mathbf{3} \oplus \mathbf{2} \oplus \mathbf{1})_{SU(2)}$.