Mathematical and Theoretical Physics

String Theory I

University of Oxford

Hilary Term 2019

Bosonic String Theory of Orbifolds

1 Introduction

We will start by introducing the concept of orbifolds and looking at some simple examples of such objects ([1], [2], [3]). The following section will present the classical theory of strings and a review of the old covariant approach to quantizing the theory. We will also include a description of the physical spectrum of the closed string as this will be of great use in the following sections. The $\mathbb{R}^1/\mathbb{Z}_2$ and $S^1/\mathbb{Z}_2$ orbifold theories are then developed and the *twisted* and *untwisted* sectors of the spectra are described at lower levels. Finally, we will briefly discuss torus amplitudes and see how modular invariance forces the introduction of the *twisted sector* in the orbifold theory.

1.1 Orbifolds

Given a smooth manifold $\mathcal{M}$ with a discrete isometry group G, consider the quotient space $\mathcal{M}/G$; if there exist points in $\mathcal{M}$ that are not affected by the action of G (i.e. *fixed points*), then the resulting space contains singularities at these points and it is called an orbifold[1]. Note that a point in the newly formed space corresponds to a point $x \in \mathcal{M}$ and all its images gx under the action of G ($g \in G$); thus, each point is identified with its orbit and hence the name orbifold. The simplest example of such a space can be obtained by considering the action of $\mathbb{Z}_2$ on the circle S^1. The circle is obtained by the real line identification $x \sim x + 2\pi R$, for $R \in \mathbb{R}$; then, the identification $x \sim -x$ (corresponding to the $\mathbb{Z}_2$ symmetries since the operation squares to one) transforms the circle into an interval, as shown in Figure 1. In this particular case, there are 'fixed points' at both ends of the interval. Similarly, the $\mathbb{R}/\mathbb{Z}_2$ orbifold is obtained from the identification $x \sim -x$ on the real axis; it is thus clear that the space obtained in this way is a (semi-infinite) line with a singularity at $x = 0$.

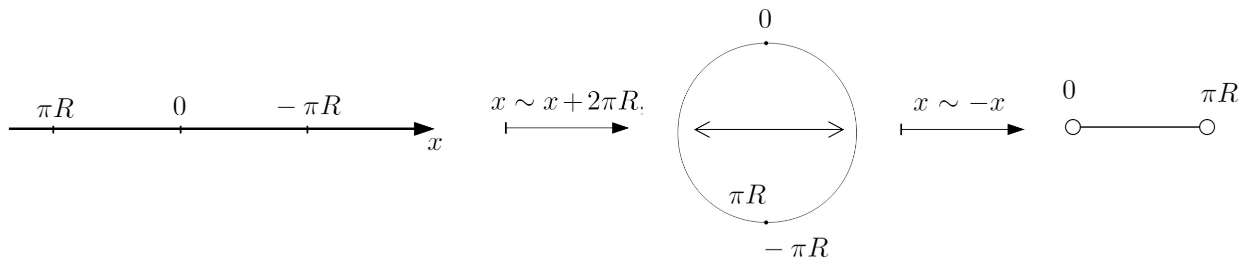

Figure 1: The $S^1/\mathbb{Z}_2$ orbifold obtained from the real line using two consecutive identifications

Slightly more complicated orbifolds are obtained from $\mathbb{R}^2$, or equivalently $\mathbb{C}$, by forming the quotient space $\mathbb{R}^2/\mathbb{Z}_2$ (or $\mathbb{C}/\mathbb{Z}_2$). The $\mathbb{Z}_2$ action is generated by the symmetry $\theta : x_i \mapsto -x_i$; here, we only need to consider the points in the upper half plane since the points in the lower half plane of $\mathbb{R}^2$ are just their θ images. Thus, the remaining θ identification needs to be done for the points on the x_1 axis. 'Glueing' the half plane on this axis we thus form a cone; note that this orbifold has a conical singularity at $(x_1, x_2) = (0, 0)$ (since this point is not affected by the action of $\mathbb{Z}_2$). In general, not all such quotients are singular spaces; $\mathbb{C}/\mathbb{Z}$ or $\mathbb{C}/(\mathbb{Z} \times \mathbb{Z})$, are in fact smooth manifolds (cylinder and torus, respectively).

It is interesting to see what string states we could expect on an orbifold background geometry. It is very likely that we will find states that already exist on $\mathcal{M}$ and are invariant under the action of G. These states are called *untwisted states* on the orbifold $\mathcal{M}/G$. Moreover, there is another class of (closed) string states that appear because of the action of G; on the original

[1] Strictly speaking this is not a definition of an orbifold but rather a particular class of spaces that have an orbifold structure.

manifold, a string connecting a point and one of its images (under the action of G) would not be an allowed closed string state, but after the identification is performed, such states are allowed in the quotient space. These states are called *twisted states* and are described by: $X^\mu(\sigma+2\pi) = gX^\mu(\sigma)$, for $g \in G$, with σ being the string world-sheet spatial coordinate. For $g = \mathbb{I}$, one obtains the untwisted states. Before analysing into more detail the spectrum on orbifolds, let us review the closed string spectrum in $\mathbb{R}^{\mathcal{D}-1,1}$ spacetime.

2 String Theory in $\mathbb{R}^{\mathcal{D}-1,1}$

2.1 Classical string

Let us start with a classical string by considering the Polyakov action, with a world-sheet metric γ_{ab} and coordinates $\xi = (\tau, \sigma)$:

$$S_P[X^\mu(\tau,\sigma), \gamma_{ab}] = -\frac{T}{2}\int d\tau d\sigma \sqrt{-\gamma}\gamma^{ab}\frac{\partial X^\mu}{\partial \xi^a}\frac{\partial X^\nu}{\partial \xi^b}\eta_{\mu\nu} \ . \tag{2.1}$$

Apart from Poincaré invariance, the action is also invariant under world-sheet reparametrisations so any two out of the three independent components of γ_{ab} can be eliminated. In the conformal gauge: $\gamma_{ab} = e^{2\omega(\tau,\sigma)}\eta_{ab}$, the action can be written as:

$$S_P^{conf.g.}[X^\mu] = -\frac{T}{2}\int d\tau d\sigma(-\partial_\tau X\cdot\partial_\tau X + \partial_\sigma X\cdot\partial_\sigma X) \ , \tag{2.2}$$

with the additional constraint $\delta S/\delta\gamma^{ab} = 0$.[2] It is useful to define the stress-energy tensor as: $T_{ab} = -\frac{2}{T}\frac{1}{\sqrt{-\gamma}}\delta S/\delta\gamma^{ab}$. In this particular gauge, it is straightforward to compute the equations of motion from the Polyakov action: $\partial_\tau^2 X^\mu - \partial_\sigma^2 X^\mu = 0$. These are just standard wave equations. Furthermore, the components of the stress energy-momentum tensor are: $T_{ab} = \partial_a X\cdot\partial_b X - \frac{1}{2}\gamma_{ab}\gamma^{cd}\partial_c X\cdot\partial_d X$, i.e.:

$$\begin{aligned} T_{\tau\tau} = T_{\sigma\sigma} = \tfrac{1}{2}(\partial_\tau X\cdot\partial_\tau X + \partial_\sigma X\cdot\partial_\sigma X) \overset{!}{=} 0 \ , \\ T_{\tau\sigma} = T_{\sigma\tau} = \partial_\tau X\cdot\partial_\sigma X \overset{!}{=} 0 \ . \end{aligned} \tag{2.3}$$

Note that closed strings must satisfy the (periodic) boundary conditions: $X^\mu(\tau,\pi) = X^\mu(\tau,0)$ and $\partial_\sigma X^\mu(\tau,\pi) = \partial_\sigma X^\mu(\tau,0)$. In an orbifold background, these will hold for the non-compactified dimensions, while for the compactified ones we might get different boundary conditions. For instance, for $\mathcal{D} = 26$, we could have, for example the following identifications:

$$\begin{aligned} &\mathbb{R}^1/\mathbb{Z}_2: \qquad X^{25} \cong -X^{25} \ , \\ &S^1/\mathbb{Z}_2: \qquad X^{25} \cong \pm X^{25} + 2\pi R m, \quad m\in\mathbb{Z} \ , \end{aligned} \tag{2.4}$$

which lead to different boundary conditions for the twisted sectors. The general solution to the above wave equation (for a closed string) satisfying periodic boundary conditions is of the form: $X^\mu(\tau,\sigma) = X_R^\mu(\tau,\sigma) + X_L^\mu(\tau,\sigma)$, where:

[2] The starting point in describing the classical string is the Nambu-Goto action. This can be obtained from the Polyakov action by integrating out the world-sheet metric γ_{ab}. Hence, when working with the Polyakov action, we need to impose an additional constraint on this metric.

$$
\begin{aligned}
X^\mu_R(\tau,\sigma) &= \frac{x^\mu}{2} + \frac{l^2 p^\mu}{2}(\tau-\sigma) + \frac{il}{2}\sum_{n\neq 0}\frac{1}{n}\alpha^\mu_n e^{-2in(\tau-\sigma)} \ , \\
X^\mu_L(\tau,\sigma) &= \frac{x^\mu}{2} + \frac{l^2 p^\mu}{2}(\tau+\sigma) + \frac{il}{2}\sum_{n\neq 0}\frac{1}{n}\tilde{\alpha}^\mu_n e^{-2in(\tau+\sigma)} \ .
\end{aligned} \tag{2.5}
$$

As reasoned before, this will differ for X^{25} for the particular cases listed above, but we will discuss this in a later section. However, these results will still be useful for the untwisted sector of the spectrum. The quantities x^μ and p^μ are the centre of mass position and momentum of the string. The reality condition on X^μ implies that: $x^\mu, p^\mu \in \mathbb{R}$ and $(\alpha^\mu_n)^* = \alpha^\mu_{-n}$, $(\tilde{\alpha}^\mu_n)^* = \tilde{\alpha}^\mu_{-n}$. To understand their physical significance, we can compute the Noether currents of the action corresponding to spacetime translations: $x^\mu \to x^\mu + \epsilon^\mu$ as follows: $\Pi^\mu_a = \frac{\delta S}{\delta(\partial^a X_\mu)} = T\partial_a X^\mu$ so then the total momentum of the string at $\tau = 0$ is:

$$
\Pi^\mu = T\int_0^\pi d\sigma \frac{X^\mu}{d\tau} = T\pi l^2 p^\mu = p^\mu, \ \text{for } l^2 = \frac{1}{\pi T} \ . \tag{2.6}
$$

Thus, it is now clear that p^μ is in fact the total momentum of the classical string. In the following we will use the notation: $\alpha^\mu_0 = \frac{1}{2} l p^\mu = \tilde{\alpha}^\mu_0$. Before quantising the string, let us introduce light cone coordinates on the world sheet:

$$
\begin{aligned}
\sigma^+ &= \tau + \sigma \ , \qquad \partial_+ = \frac{1}{2}(\partial_\tau + \partial\sigma) \ , \qquad \partial_+\sigma_+ = \partial_-\sigma_- = 1 \ , \\
\sigma^- &= \tau - \sigma \ , \qquad \partial_+ = \frac{1}{2}(\partial_\tau - \partial\sigma) \ , \qquad \partial_+\sigma_- = \partial_-\sigma_+ = 0 \ .
\end{aligned} \tag{2.7}
$$

Then, the world-sheet metric becomes:

$$
(\eta_{a'b'}) = \begin{pmatrix} 0 & -1/2 \\ -1/2 & 0 \end{pmatrix} \text{ and } (\eta^{a'b'}) = \begin{pmatrix} 0 & -2 \\ -2 & 0 \end{pmatrix} . \tag{2.8}
$$

In these coordinates, the traceless property of the stress energy momentum tensor: $\eta^{ab}T_{ab} = 0$ implies that T_{+-}=0 (since T can always be brought to a symmetric form). Furthermore, the conservation law: $\eta^{ab}\partial_a T_{bc} = 0$ implies that $\partial_- T_{++} = 0$ so $T_{++} = T_{++}(\sigma_+)$ and similarly $\partial_+ T_{--} = 0$ so $T_{--} = T_{--}(\sigma_-)$. We can then construct an infinite set of conserved charges:

$$
Q_f = \int d\sigma f(\sigma_+) T_{++}(\sigma_+) \Rightarrow \frac{\partial}{\partial\tau} Q_f = \int d\sigma (2\partial_- + \partial_\sigma) f(\sigma_+) T_{++}(\sigma_+) = 0 \ . \tag{2.9}
$$

The last equality vanishes because of the boundary conditions of the closed string. Note that we have not yet imposed the constraints (2.3) on the stress tensor yet. We can introduce a basis for these arbitrary functions f as follows:

$$
\begin{aligned}
L_n &:= \frac{T}{2}\int_0^\pi d\sigma e^{2in\sigma_-} T_{--}(\sigma_-) \ , \\
\tilde{L}_n &:= \frac{T}{2}\int_0^\pi d\sigma e^{2in\sigma_+} T_{++}(\sigma_+) \ .
\end{aligned} \tag{2.10}
$$

As we have already noticed, $\partial L_n/\partial\tau = 0$, so we can pick any value of τ when analysing the spectrum; it is convenient to express L and $\tilde{L}$ at $\tau = 0$. Using the fact that: $T_{--} = \partial_- X \cdot \partial_- X = \partial_- X_R \cdot \partial_- X_R$, we find:

$$
\begin{aligned}
L_m &= \frac{T}{2}\int_0^\pi d\sigma e^{-2im\sigma}\partial_- X_R\cdot\partial_- X_R \\
&= \frac{Tl^2}{2}\int_0^\pi d\sigma e^{-2im\sigma}\Big(\alpha_0+\sum_{n\neq 0}\alpha_n e^{2in\sigma}\Big)\cdot\Big(\alpha_0+\sum_{n'\neq 0}\alpha_{n'}e^{2in'\sigma}\Big) \\
&= \frac{1}{2}\int_0^\pi d\sigma\sum_{n,n'} e^{-2i\sigma(m-n-n')}\alpha_n\cdot\alpha_{n'} = \frac{1}{2}\sum_{n=-\infty}^{\infty}\alpha_{m-n}\cdot\alpha_n\ , \\
\tilde{L}_m &= \frac{1}{2}\sum_{n=-\infty}^{\infty}\tilde{\alpha}_{m-n}\cdot\tilde{\alpha}_n\ .
\end{aligned}
\tag{2.11}
$$

Let us finally impose the additional constraints (i.e. vanishing of the stress tensor). This corresponds to setting all L's and $\tilde{L}$'s to 0. Using the fact that $\alpha_0^2 = l^2p^2/4 = \tilde{\alpha}_0^2$, adding and subtracting the $L_0 = \tilde{L}_0 = 0$ constraints we obtain:

$$
\begin{aligned}
&\text{`Mass shell condition':} && M^2 = -p^2 = \frac{2}{\alpha'}\sum_{n=1}^{\infty}(\alpha_{-n}\cdot\alpha_n + \tilde{\alpha}_{-n}\cdot\tilde{\alpha}_n)\ , \\
&\text{`Level matching condition':} && \sum_{n=1}^{\infty}\alpha_{-n}\cdot\alpha_n = \sum_{n=1}^{\infty}\tilde{\alpha}_{-n}\cdot\tilde{\alpha}_n\ .
\end{aligned}
\tag{2.12}
$$

It is instructive to derive an expression for the (classical) Hamiltonian starting from the action: $H = \int_0^\pi d\sigma\big(\Pi_\tau\cdot\dot{X} - \mathcal{L}\big)$, which then leads to:

$$
H = \frac{1}{2}\sum_{n=-\infty}^{\infty}\alpha_{-n}\cdot\alpha_n\ . \tag{2.13}
$$

2.2 Canonical quantization

The natural way to 'quantize' this theory is to promote the Poisson brackets, for the oscillator coordinates, in phase space, to commutators of operators acting on the string Fock space. In order to understand how this may be achieved, we have to compute the Poisson bracket structure in the classical system. For canonical coordinates at equal times τ:

$$
\big[\Pi_\tau^\mu(\sigma),\Pi_\tau^\nu(\sigma')\big]_{\rm PB} = 0,\quad \big[X^\mu(\sigma),X^\nu(\sigma')\big]_{\rm PB} = 0 \quad\text{and}\quad \big[\Pi_\tau^\mu(\sigma),X^\nu(\sigma')\big]_{\rm PB} = \eta^{\mu\nu}\delta(\sigma-\sigma')\ . \tag{2.14}
$$

In the last equality we replaced the Kronecker delta by a Dirac delta function since σ is a continuous variable. Recalling that $\Pi_\tau^\mu = T\partial_\tau X$, we find:

$$
\begin{aligned}
\big[\dot{X}^\mu(\sigma),X^\nu(\sigma')\big]_{\rm PB} &= \Big[l^2p^\mu + l\sum_{n\neq 0}\big(\alpha_n^\mu e^{2in\sigma}+\tilde{\alpha}_n^\mu e^{-2in\sigma}\big), x^\nu + \frac{il}{2}\sum_{m\neq 0}\frac{1}{m}\big(\alpha_m^\nu e^{2im\sigma}+\tilde{\alpha}_m^\nu e^{-2im\sigma}\big)\Big]_{\rm PB} \\
&= l^2\big[p^\mu,x^\nu\big]_{\rm PB} + \frac{il^2}{2}\sum_{m,n\neq 0} e^{2i\sigma(m+n)}\big[\alpha_n^\mu,\alpha_m^\nu\big]_{\rm PB} + e^{-2i\sigma(m+n)}\big[\tilde{\alpha}_n^\mu,\tilde{\alpha}_m^\nu\big]_{\rm PB} + \dots
\end{aligned}
\tag{2.15}
$$

We expect that the Poisson brackets of x^μ and p^μ with all the α_n and $\tilde{\alpha}_m$ are 0 (except possibly $n = 0$); additionally, the α_n's and $\tilde{\alpha}_n$'s are completely independent variables so we set their Poisson brackets to 0 as well. Note also that $[p^\mu, x^\nu]_{\rm PB} = \eta^{\mu\nu}$ so the (2.14) conditions are satisfied for:

$$
[\alpha_m^\mu,\alpha_n^\nu]_{\rm PB} = [\tilde{\alpha}_m^\mu,\tilde{\alpha}_n^\nu]_{\rm PB} = im\eta^{\mu\nu}\delta_{m+n,0}\ . \tag{2.16}
$$

We will prove these in a more formal way for the twisted sector of an orbifold theory. The conserved charges L_m and $\tilde{L}_m$ will satisfy:

$$
\begin{aligned}
[L_m, L_n]_{\mathrm{PB}} &= \frac{1}{4}\sum_{p,q}[\alpha_{m-p}\cdot\alpha_p, \alpha_{n-q}\cdot\alpha_q]_{\mathrm{PB}} \\
&= \frac{1}{4}\sum_{p,q}\eta_{\mu\nu}\eta_{\rho\sigma}(\alpha^\mu_{m-p}\alpha^\rho_{n-q}[\alpha^\nu_p,\alpha^\sigma_q] + \alpha^\mu_{m-p}[\alpha^\nu_p,\alpha^\rho_{n-q}]\alpha^\sigma_q + \alpha^\rho_{n-q}[\alpha^\mu_{m-p},\alpha^\sigma_q]\alpha^\nu_p + [\alpha^\mu_{m-p},\alpha^\rho_{n-q}]\alpha^\sigma_q\alpha^\nu_p) \\
&= \frac{i}{4}\sum_p(\eta_{\mu\rho}\alpha^\mu_{m-p}\alpha^\rho_{n+p}p + \eta_{\mu\sigma}\alpha^\mu_{m-p}\alpha^\sigma_{n+p}p + \eta_{\nu\rho}\alpha^\nu_p\alpha^\rho_{n-p+m}(m-p) + \eta_{\nu\sigma}\alpha^\nu_p\alpha^\sigma_{m-p+n}(m-p)) \\
&= \frac{i}{2}\sum_p(p\alpha_{m-p}\cdot\alpha_{n+p} + (m-p)\alpha_p\cdot\alpha_{m_n-p}) = i(m-n)L_{m+n}\ .
\end{aligned}
\tag{2.17}
$$

In the last line we simply changed the summation variable $p\mapsto p-n$ in the first term to obtain the final result. A similar expression holds for the $\tilde{L}$ charges. This algebra is called the *Witt algebra*. We are now ready to promote the Poisson brackets to commutators. We thus postulate the following commutation relations at equal time in the Schrodinger picture:

$$
[\alpha^\mu_m,\alpha^\nu_n] = [\tilde{\alpha}^\mu_m,\tilde{\alpha}^\nu_n] = m\eta^{\mu\nu}\delta_{m+n,0}\ , \quad \text{and} \quad [p^\mu, x^\nu] = -i\eta^{\mu\nu}\ . \tag{2.18}
$$

Furthermore, it must be the case that: $(\alpha^\mu_n)^\dagger = \alpha^\mu_{-n}$, $(\tilde{\alpha}^\mu_n)^\dagger = \tilde{\alpha}^\mu_{-n}$ and also $x^\dagger = x$, $p^\dagger = p$. Let us build the Fock space of this model. Defining the oscillator ground state $|0,0\rangle_{\mathrm{osc}}$ as the state that is annihilated by:

$$
\begin{aligned}
&\alpha^\mu_n\,|0,0\rangle_{\mathrm{osc}} = 0\ , \\
&\tilde{\alpha}^\mu_n\,|0,0\rangle_{\mathrm{osc}} = 0\ , \qquad n = 1,2,\dots\ .
\end{aligned}
\tag{2.19}
$$

we then have: $\mathcal{H}_{Fock} = Span\{\prod_{i=1}^k \alpha^{\mu_i}_{-n_i}\,|0,0\rangle_{\mathrm{osc}}\} \otimes Span\{\prod_{i=1}^k \tilde{\alpha}^{\mu_i}_{-n_i}\,|0,0\rangle_{\mathrm{osc}}\}$. We can also define the *number operators* $N = \sum_{k=1}\alpha_{-k}\cdot\alpha_k$ and similarly for $\tilde{N}$, such that the eigenvalues of these operators corresponds to the 'level' of the state. We also quantize the $(0,0)$-modes by: $\hat{p}^\mu\,|0,0;k\rangle = k^\mu\,|0,0;k\rangle$. Let us now consider the expressions for L_n's. For $n\neq 0$, we can write an analogous expression to the classical case for these charges as the α operators involved will commute with each other. However, there is an issue with the classical expression for $L_0 \propto \sum\alpha_{-k}\alpha_k$. As we already know, these operators **do not commute**, so if we were to use this expression we would quickly run into trouble: $\sum\alpha_{-k}\cdot\alpha_k\,|0,0;p\rangle = \alpha' p^2/4 + \sum_{k=1}k(D-2)$, which diverges! Thus, normal ordering the expression we have:

$$
\begin{aligned}
L_0 =& \frac{1}{2}\alpha_0\cdot\alpha_0 + \sum_{k=1}^\infty \alpha_{-k}\cdot\alpha_k := \frac{\alpha' p^2}{4} + N\ , \\
L_m =& \frac{1}{2}\sum_{k=-\infty}^\infty \alpha_{m-k}\cdot\alpha_k\ .
\end{aligned}
\tag{2.20}
$$

As reasoned above, for $m+n\neq 0$ we expect the Witt algebra to hold for these operators. However, due to the ambiguities coming from normal ordering of the operators, these is an additional term for $m=-n$:

$$
[L_m, L_n] = (m-n)L_{m+n} + \frac{D}{12}(m^3-m)\delta_{m+n,0} \tag{2.21}
$$

This is called the *Virasoro* algebra, with central charge $D/12$.

2.3 Physical Spectrum

We now want to cut down the Fock space to some sub-space that we will call the 'physical' Hilbert space, by imposing the analogous constraints to the classical ones (2.3). Again, because of the normal-ordering ambiguities we cannot set the action of all L_m's on *physical states* to be zero. It is enough to impose only:

$$\begin{aligned} L_m \left|\phi\right\rangle = 0 \ , \qquad & \tilde{L}_m \left|\phi\right\rangle = 0 \ , \qquad && \text{for } m \geq 1 \ , \\ (L_0 - a)\left|\phi\right\rangle = 0 \ , \qquad & (\tilde{L}_0 - a)\left|\phi\right\rangle = 0 \ . && \end{aligned} \tag{2.22}$$

Here a is just a constant arising from the fact the our choice of L_0 was not unique, but only a normal ordered version of the classical expression. The Hamiltonian of the system becomes: $H = L_0 + \tilde{L}_0 - 2a$, in direct analogy to the classical expression. The above conditions are the definition of a physical state $|\phi\rangle$. The L_0 constraint on such states reduces to two conditions similar to (2.12):

$$\begin{cases} (L_0 - a)\left|\phi\right\rangle = (\frac{\alpha' p^2}{4} + N - a)\left|\phi\right\rangle = 0 \ , \\ (\tilde{L}_0 - a)\left|\phi\right\rangle = (\frac{\alpha' p^2}{4} + \tilde{N} - a)\left|\phi\right\rangle = 0 \ , \end{cases} \Rightarrow \begin{cases} \alpha' M^2 = -4a + 2(N + \tilde{N}) \ , \\ N = \tilde{N} \ . \end{cases} \tag{2.23}$$

We also define *spurious* states to be the states obeying the L_0 and $\tilde{L}_0$ constraints for which: $\langle \phi | \phi_{phys} \rangle = 0$ for any physical state $|\phi_{phys}\rangle$. Note also that physical spurious states are *null*, so we define the reduced Hilbert space as: $\mathcal{H}_{reduced} = \mathcal{H}_{phys}/(\mathcal{H}_{phys} \cap \mathcal{H}_{spurious})$. For a sensible causal theory we require the physical Hilbert space to be free from negative-norm states, states called *ghosts*. As discussed in [4], for instance, the spectrum has this property[3] for $a = 1$ and $D = 26$ or $a \leq 1$ and $D \leq 25$. We will mainly consider the former case (i.e. critical bosonic string) which contains additional null states.

Let us now analyze the physical state spectrum. It can be shown the the operators $N, \tilde{N}$ defined above satisfy: $[N, \alpha_n^\mu] = -n\alpha_n^\mu$ and $[\tilde{N}, \tilde{\alpha}_n^\mu] = -n\tilde{\alpha}_n^\mu$ so they do indeed behave as number operators. For *level* 0 *states* ($N = \tilde{N} = 0$) we only need to impose the L_0 and $\tilde{L}_0$ constraints, such that:

$$|0, 0; p\rangle \rightarrow \alpha' p^2 = 4a = 4 \text{ `closed string tachyon' } . \tag{2.24}$$

At *level* 1 ($N = \tilde{N} = 1$), the most general state has the form: $|\Omega; p\rangle := \Omega_{\mu\nu}\alpha_{-1}^\mu \tilde{\alpha}_{-1}^\nu |0, 0; p\rangle$ and the L_0 constraint reads: $\alpha' p^2 = 4a - 4 = 0$ in the critical case. We can decompose the polarization tensor:

$$\begin{cases} \gamma_{\mu\nu}\alpha_{-1}^\mu \tilde{\alpha}_{-1}^\nu |0, 0; p\rangle \ , & \text{symmetric} \\ B_{\mu\nu}\alpha_{-1}^\mu \tilde{\alpha}_{-1}^\nu |0, 0; p\rangle \ , & \text{antysimmetric} \\ \phi\alpha_{-1} \cdot \tilde{\alpha}_{-1} |0, 0; p\rangle \ . & \end{cases} \tag{2.25}$$

The symmetric part of the polarization tensor corresponds to what is referred to as a *graviton*. For these states we also need to impose L_1 and $\tilde{L}_1$ constraints, which is how one can deduce the right form of the dilaton state: $\left[(\zeta \cdot \alpha_{-1})(\alpha_0 \cdot \tilde{\alpha}_{-1}) + (\alpha_0 \cdot \alpha_{-1})(\zeta \cdot \tilde{\alpha}_{-1}) + \alpha_{-1} \cdot \tilde{\alpha}_{-1}\right] |0, 0; p\rangle$.

[3] These emerge from the so called *no ghost theorem*.

3 $\mathbb{R}^1/\mathbb{Z}_2$ orbifold theory

We will consider $D = 26$ again. For closed strings on the $\mathbb{R}^1/\mathbb{Z}_2$ orbifold, only the x^{25} coordinate of the space-time is affected. As we have already seen in section 1, this coordinate is restricted to $x^{25} \geq 0$ by the identification $x^{25} \sim -x^{25}$. The string coordinates still obey the wave equation, but the boundary conditions are slightly different. As detailed in [5], we can introduce a formal operator U that implements the $\mathbb{Z}_2$ identification on the string coordinates:

$$\begin{aligned} UX^j(\tau,\sigma)U^{-1} &= X^j(\tau,\sigma)\ , \qquad \text{for } j = 0,1,...,24 \\ UX^{25}(\tau,\sigma)U^{-1} &= -X^{25}(\tau,\sigma)\ . \end{aligned} \tag{3.1}$$

3.1 Untwisted sector

As discussed in section 1, there are two types of states on the orbifold. The first type are the U invariant states from the initial $\mathbb{R}^{25,1}$ theory. It is clear that not all states in this space have this symmetry, but in general one can build such states as a superposition of simpler states. The string coordinates X^j will have the same form as in (2.5) for $j = 0,1,...,24$; however, for a U-invariant state, the last component will have slightly different properties: $X^{25}(\tau,\sigma) = x_0^{25} + l^2p^{25}\tau + \frac{il}{2}\sum_{n\neq 0}\frac{1}{n}\left(\alpha_n^{25}e^{-2in(\tau-\sigma)} + \tilde{\alpha}_n^{25}e^{-2in(\tau+\sigma)}\right)$. From the action of U on these coordinates we can deduce:

$$\begin{aligned} U\alpha_n^jU^{-1} &= \alpha_n^j, \quad U\alpha_n^{25}U^{-1} = -\alpha_n^{25}, \quad Ux_0^{25}U^{-1} = -x_0^{25} \\ U\tilde{\alpha}_n^jU^{-1} &= \tilde{\alpha}_n^j, \quad U\tilde{\alpha}_n^{25}U^{-1} = -\tilde{\alpha}_n^{25}, \quad Up^{25}U^{-1} = -p^{25} \end{aligned} \tag{3.2}$$

Recall that the Hamiltonian is $H = L_0 + \tilde{L}_0 - 2a$, where $L_0 = \frac{1}{2}\alpha_0^j\alpha_{0,j} + \frac{1}{2}\alpha_0^{25}\alpha_0^{25} + \sum\alpha_{-k}^j\alpha_{k,j} + \alpha_{-k}^{25}\alpha_k^{25}$ and similarly for $\tilde{L}_0$. Then, using the above actions of U, it follows that $UHU^{-1} = H$, so H is U invariant. This essentially means that U is a symmetry of the Hamiltonian so states that are invariant under U at some time will remain U invariant at later times. Thus, in the orbifold theory we only need to keep states that had this property in the initial theory. Note that (in this untwisted sector), the mass shell and level-matching conditions (2.23) will still hold.

We denote the closed string states in $\mathbb{R}^{25,1}$ by $|N,\tilde{N};\boldsymbol{p},p\rangle$, where $\boldsymbol{p}$ is a 25-dimensional vector and p is the momentum in the 25^{th} direction. We observe that the states $|N,\tilde{N};\boldsymbol{p},0\rangle$ are U-invariant, so it is then clear that $|0,0;\boldsymbol{p},0\rangle$ is a ground state of the orbifold theory. This state is tachyonic, with $\alpha' M_{(25)}^2 = -4$. The action of U on p^{25} can be interpreted in terms of string states as follows[4]:

$$U\,|N,\tilde{N};\boldsymbol{p},p\rangle \propto |N,\tilde{N};\boldsymbol{p},-p\rangle \tag{3.3}$$

As a result, superpositions of states, such as: $|N,\tilde{N};\boldsymbol{p},p\rangle + |N,\tilde{N};\boldsymbol{p},-p\rangle$ are also U-invariant, so we can form *level* 0 states: $|0,0;\boldsymbol{p},p\rangle + |0,0;\boldsymbol{p},-p\rangle$. These obey: $\alpha'(M_{(25)}^2 - p^2) = -4$. The *level* 1 states in the original theory have the following form: $\Omega_{\mu\nu}\alpha_{-1}^{\mu}\alpha_{-1}^{\tilde{\nu}}\,|0,0;\boldsymbol{p},p\rangle$. As we have already seen before, Ω can be decomposed into a symmetric tensor $\gamma_{\mu\nu}$, an antisymmetric tensor $B_{\mu\nu}$ and a scalar ϕ. Then, these are further divided into:

[4] This is in fact an equality up to a minus sign, coming from the number of α_n^{25} and $\tilde{\alpha}_n^{25}$ operators involved.

$$
\begin{aligned}
\gamma_{\mu\nu}(x^\rho) &= \big(\gamma_{ij}(x^k), \gamma_{25,j}(x^k), \gamma_{25,25}(x^k)\big)\\
B_{\mu\nu}(x^\rho) &= \big(B_{ij}(x^k), B_{25,j}(x^k)\big)\\
\phi(x^\rho) &= \big(\phi(x^k)\big)
\end{aligned}
\tag{3.4}
$$

where the greek indices run from 0 to 25 and the latin ones only up to 24. Let us first consider *level* 1 states for which $p^{25} = 0$. These are in direct analogy to the $\mathbb{R}^{25,1}$ states (graviton, dilaton and the B state in $\mathbb{R}^{24,1}$), apart from the states that 'include' α_{-1}^{25}:

$$
\begin{aligned}
&\big(\zeta \cdot \alpha_{-1}\tilde{\alpha}_{-1}^{25} + \zeta \cdot \tilde{\alpha}_{-1}\alpha_{-1}^{25}\big)\,|0,0;\boldsymbol{p},0\rangle\\
&\big(\zeta \cdot \alpha_{-1}\tilde{\alpha}_{-1}^{25} - \zeta \cdot \tilde{\alpha}_{-1}\alpha_{-1}^{25}\big)\,|0,0;\boldsymbol{p},0\rangle\\
&\alpha_{-1}^{25}\tilde{\alpha}_{-1}^{25}\,|0,0;\boldsymbol{p},0\rangle
\end{aligned}
\tag{3.5}
$$

The first two states correspond to $\gamma_{25,j}$ and $B_{25,j}$, respectively, while the last one corresponds to $\gamma_{25,25}$. The scalar product indicated in these states is over $\mathbb{R}^{24,1}$ only. However, keep in mind that these states must be U-invariant, so given the action of U on α_{-1}^{25}, it is clear that only the third state listed above satisfies this condition. Note that all these state are massless, $\alpha' M_{(25)}^2 = 0$. We can also consider states for which $p \neq 0$, and build superpositions of states as we have done for *level* 0. Imposing the U symmetry on such states we can form a basis:

$$
\begin{aligned}
&\alpha_{-1}^{i}\tilde{\alpha}_{-1}^{j}\big(\,|0,0;\boldsymbol{p},p\rangle + |0,0;\boldsymbol{p},-p\rangle\,\big)\\
&\alpha_{-1}^{25}\tilde{\alpha}_{-1}^{j}\big(\,|0,0;\boldsymbol{p},p\rangle - |0,0;\boldsymbol{p},-p\rangle\,\big)\\
&\alpha_{-1}^{i}\tilde{\alpha}_{-1}^{25}\big(\,|0,0;\boldsymbol{p},p\rangle - |0,0;\boldsymbol{p},-p\rangle\,\big)\\
&\alpha_{-1}^{25}\tilde{\alpha}_{-1}^{25}\big(\,|0,0;\boldsymbol{p},p\rangle + |0,0;\boldsymbol{p},-p\rangle\,\big)
\end{aligned}
\tag{3.6}
$$

3.2 Twisted sector

Beyond the initial states that are invariant under U, we can have new states on the orbifold arising from the new boundary condition: $X^{25}(\tau, \sigma + \pi) = -X^{25}(\tau, \sigma)$. We can think of these as 'open strings' in the initial space-time which become 'closed string' states because of the additional constraint. Let us now try to find an expression for X^{25}. A general solution of the wave equation $(\partial_\tau^2 - \partial_\sigma^2)X^{25} = 0$ has the form $X^{25}(\tau, \sigma) = X_R^{25}(\sigma_-) + X_L^{25}(\sigma_+)$. Taking derivatives with respect to σ_+ and σ_- of the boundary condition we find:

$$
\big(X_L^{25}\big)'(\sigma_+ + \pi) = -\big(X_L^{25}\big)'(\sigma_+) \quad \text{and} \quad \big(X_R^{25}\big)'(\sigma_- - \pi) = -\big(X_R^{25}\big)'(\sigma_-) \tag{3.7}
$$

As opposed to the other coordinates, X^j, which are π periodic in σ, we now need to restrict the sum in (2.5) to odd multiples of integers $(n/2)$ to account for the new periodicity, which gets rid of the additional factors of 2:

$$
\begin{aligned}
X_R^{25}(\tau, \sigma) &= x_R + il \sum_{n \text{ odd}} \frac{1}{n}\alpha_{\frac{n}{2}}^{25} e^{-in(\tau-\sigma)}\\
X_L^{25}(\tau, \sigma) &= x_L + il \sum_{n \text{ odd}} \frac{1}{n}\tilde{\alpha}_{\frac{n}{2}}^{25} e^{-in(\tau+\sigma)}
\end{aligned}
\tag{3.8}
$$

Note that the boundary condition on X^{25} implies that: $x_R + x_L = 0$. Analogous to the previous case, we can introduce the string momentum $\Pi_\tau^{25} = T\partial_\tau X^{25}$, such that the (2.14) Poisson brackets are satisfied. These are then promoted to quantum mechanical commutators in the usual way. Then, denoting by prime the σ derivatives and by dot the τ derivatives:

$$\begin{aligned}
\Big[X^{25}(\tau,\sigma),\dot{X}^{25}(\tau,\sigma')\Big] = -iT^{-1}\delta(\sigma-\sigma') &\xRightarrow{d/d\sigma} \Big[X'^{25}(\tau,\sigma),\dot{X}^{25}(\tau,\sigma')\Big] = -iT^{-1}\frac{d}{d\sigma}\delta(\sigma-\sigma') \\
&\xRightarrow{d/d\tau} \Big[\dot{X}^{25}(\tau,\sigma),\dot{X}^{25}(\tau,\sigma')\Big] = 0 \\
\Big[X^{25}(\tau,\sigma),X^{25}(\tau,\sigma')\Big] = 0 &\xRightarrow[d/d\sigma']{d/d\sigma} \Big[X'^{25}(\tau,\sigma),X'^{25}(\tau,\sigma')\Big] = 0 \\
\Big[\dot{X}^{25}(\tau,\sigma),\dot{X}^{25}(\tau,\sigma')\Big] = 0 &
\end{aligned} \tag{3.9}$$

These allow us to compute the following quantities[5]:

$$\begin{aligned}
&\Big[(\dot{X}^{25}\pm X'^{25})(\tau,\sigma),(\dot{X}^{25}\pm X'^{25})(\tau,\sigma')\Big] = \pm 2iT^{-1}\frac{d}{d\sigma}\delta(\sigma-\sigma') \\
&\Big[(\dot{X}^{25}\pm X'^{25})(\tau,\sigma),(\dot{X}^{25}\mp X'^{25})(\tau,\sigma')\Big] = 0
\end{aligned} \tag{3.10}$$

Note that $X'^{25}(\tau,\sigma)+\dot{X}^{25}(\tau,\sigma) = 2\big(X_L^{25}\big)'(\sigma_+)$ and $X'^{25}(\tau,\sigma)-\dot{X}^{25}(\tau,\sigma) = 2\big(X_R^{25}\big)'(\sigma_-)$, where the primes on $X_{R,L}^{25}$ are now total derivatives. This allows us to write:

$$\begin{aligned}
\Big[\big(X_R^{25}\big)'(\tau-\sigma),\big(X_R^{25}\big)'(\tau+\sigma')\Big] &= 4l^2\sum_{q,h\text{ odd}} e^{-iq(\tau-\sigma)}e^{-ih(\tau-\sigma')}\Big[\alpha^{25}_{\frac{q}{2}},\alpha^{25}_{\frac{h}{2}}\Big] \\
&= -2iT^{-1}\frac{d}{d\sigma}\delta(\sigma-\sigma')
\end{aligned} \tag{3.11}$$

Multiplying both sides of the above equation by $e^{-im\sigma}e^{-in\sigma'}$, for $m,n\in\mathbb{Z}$ odd and integrating over both σ and σ' we find:

$$\begin{aligned}
&\text{LHS: } 4l^2\sum_{q,h\text{ odd}} e^{-i\tau(q+h)}\Big[\alpha^{25}_{\frac{q}{2}},\alpha^{25}_{\frac{h}{2}}\Big]\int_0^\pi\int_0^\pi d\sigma d\sigma' e^{i\sigma(q-m)+i\sigma'(h-n)} = 4l^2\pi^2 e^{-i\tau(m+n)}\Big[\alpha^{25}_{\frac{m}{2}},\alpha^{25}_{\frac{n}{2}}\Big] \\
&\text{RHS: } -2iT^{-1}\int_0^\pi\int_0^\pi d\sigma d\sigma' e^{-im\sigma}e^{-in\sigma'}\frac{d}{d\sigma}\delta(\sigma-\sigma') \\
&\quad = -2iT^{-1}\int_0^\pi d\sigma' e^{-in\sigma'}\left(\Big[e^{-im\sigma}\delta(\sigma-\sigma')\Big]_{\sigma=0}^{\sigma=\pi} + im\int_0^\pi e^{-im\sigma}\delta(\sigma-\sigma')\right) \\
&\quad = -2iT^{-1}\Big(-e^{-in\pi}-1+im\int_0^\pi d\sigma' e^{-i\sigma'(n+m)}\Big) = 2T^{-1}m\pi\delta_{m+n,0}
\end{aligned} \tag{3.12}$$

Thus, we finally obtain:

$$\Big[\alpha^{25}_{\frac{m}{2}},\alpha^{25}_{\frac{n}{2}}\Big] = \frac{m}{2}\delta_{m+n,0} \quad \Big[\tilde{\alpha}^{25}_{\frac{m}{2}},\tilde{\alpha}^{25}_{\frac{n}{2}}\Big] = \frac{m}{2}\delta_{m+n,0} \quad \Big[\alpha^{25}_{\frac{m}{2}},\tilde{\alpha}^{25}_{\frac{n}{2}}\Big] = 0 \tag{3.13}$$

As for the untwisted sector, the action of U on α_n^{25} and $\tilde{\alpha}_n^{25}$ produces a minus sign. Keep in mind that the expression we derived for $X^{25}(\tau,\sigma)$ does not have an α_0^{25} (nor $\tilde{\alpha}_0^{25}$) dependence, so we do not expect the twisted states to have a p^{25} momentum component. Hence, we will label the twisted states by $|N,\tilde{N};\boldsymbol{p}\rangle$, where, as before, $\boldsymbol{p}$ is the momentum along X^i, $i=0,...,24$. We want to see what the L_m constraints become for these twisted states. Thus, we introduce:

[5] One needs to use the fact that $\frac{d}{d\sigma'}\delta(\sigma-\sigma') = -\frac{d}{d\sigma}\delta(\sigma-\sigma')$

$$
\begin{aligned}
L_0 =& \frac{1}{2}\alpha_0 \cdot \alpha_0 + \sum_{k=1}^{\infty} \alpha_{-k} \cdot \alpha_k + \sum_{\substack{k \text{ odd} \\ k>0}} \alpha^{25}_{-\frac{k}{2}} \alpha^{25}_{\frac{k}{2}} \quad =: \frac{1}{2}\alpha_0 \cdot \alpha_0 + N \\
L_m =& \frac{1}{2} \sum_{k=\infty}^{\infty} \alpha_{m-k} \cdot \alpha_k + \frac{1}{2} \sum_{k \text{ odd}} \alpha^{25}_{m-\frac{k}{2}} \alpha^{25}_{\frac{k}{2}}
\end{aligned}
\tag{3.14}
$$

The scalar product is again over $\mathbb{R}^{24,1}$. It is clear that the level matching condition will still hold; however, the mass-shell condition will be different because of the ordering of the 25^{th} component of the α operators. This arises because the new sum is only over the odd positive integers, as opposed to previous cases. In fact, one can show that the new reordering constant is $a = 15/16$ which then leads to:

$$
\alpha' M^2_{(25)} = 2(N + \tilde{N}) - \frac{15}{4} \tag{3.15}
$$

Let us now see how the N (and $\tilde{N}$) operators act on physical states. To do this, we compute (for $n > 0$):

$$
\left[N, \alpha^{25}_{\frac{n}{2}}\right] = \sum_{\substack{k \text{ odd} \\ k>0}} \left[\alpha^{25}_{-\frac{k}{2}} \alpha^{25}_{\frac{k}{2}}\ ,\ \alpha^{25}_{\frac{n}{2}}\right] = -\sum_{\substack{k \text{ odd} \\ k>0}} \frac{n}{2} \delta_{k,n}\ \alpha^{25}_{\frac{k}{2}} = -\frac{n}{2}\ \alpha^{25}_{\frac{n}{2}} \tag{3.16}
$$

Consequently, the α^{25} operators contribute with half-integers to N (and similarly for $\tilde{N}$). We can now consider the *level* 0 states of the twisted sector: $|0, 0; \boldsymbol{p}\rangle$. These are tachyonic states with $\alpha' M^2_{(25)} = -15/4$. The '*level* 1/2' state has $N = \tilde{N} = 1/2$, i.e. $|1, 1; \boldsymbol{p}\rangle$ and is obtained by:

$$
\alpha^{25}_{-\frac{1}{2}} \tilde{\alpha}^{25}_{-\frac{1}{2}}\ |0, 0; \boldsymbol{p}\rangle \tag{3.17}
$$

This has mass $\alpha' M^2_{(25)} = -7/8$, so it is also tachyonic. We can achieve $N = \tilde{N} = 1$ by either using one α^j_{-1} operator or two $\alpha^{25}_{-\frac{1}{2}}$ operators, so a basis for such states could be:

$$
\begin{gathered}
\Omega_{ij} \alpha^i_{-1} \tilde{\alpha}^j_{-1}\ |0, 0; \boldsymbol{p}\rangle \\
\alpha^{25}_{-\frac{1}{2}} \alpha^{25}_{-\frac{1}{2}} \tilde{\alpha}^{25}_{-\frac{1}{2}} \tilde{\alpha}^{25}_{-\frac{1}{2}}\ |0, 0; \boldsymbol{p}\rangle
\end{gathered}
\tag{3.18}
$$

As opposed to the untwisted sector, we do not have any massless states in the twisted sector.

4 $S^1/\mathbb{Z}_2$ orbifold theory

We will now consider a different orbifold, $S^1/\mathbb{Z}_2$, which is obtained by the identification:

$$
X^{25} \cong \pm X^{25} + 2\pi R m, \quad m \in \mathbb{Z} \tag{4.1}
$$

This theory will also have two types of states. Let us first analyze the untwisted sector. For simplicity, we will also introduce string units:

$$
l = 1, \quad \alpha' = 1/2, \quad T = 1/\pi \tag{4.2}
$$

4.1 Untwisted sector

This sector contains states that obey: $X^{25}(\tau, \sigma + \pi) = X^{25}(\tau, \sigma) + 2\pi R m$, for integer values of m. We can try to obtain a general solution to the wave equation satisfied by X^{25} and apply this new boundary condition:

$$\big(X_L^{25}\big)'(\sigma_+ + \pi) = \big(X_L^{25}\big)'(\sigma_+) \text{ and } \quad \big(X_R^{25}\big)'(\sigma_- - \pi) = \big(X_R^{25}\big)'(\sigma_-) \tag{4.3}$$

This leads to the usual solution for the oscillator coordinates:

$$\begin{aligned} X_R^{25}(\tau-\sigma) &= x_R^{25} + \big(\frac{p^{25}}{2} - L\big)(\tau-\sigma) + \frac{i}{2}\sum_{n\neq 0}\frac{1}{n}\alpha_n^{25}e^{-2in(\tau-\sigma)} \\ X_L^{25}(\tau+\sigma) &= x_L^{25} + \big(\frac{p^{25}}{2} + L\big)(\tau+\sigma) + \frac{i}{2}\sum_{n\neq 0}\frac{1}{n}\tilde{\alpha}_n^{25}e^{-2in(\tau+\sigma)} \end{aligned} \tag{4.4}$$

To satisfy the boundary condition we require: $L = Rm$, $m \in \mathbb{Z}$. This integer is called the *winding* number. Note also that the p^{25} momentum is also quantized to: $p^{25} = n/R$, $n \in \mathbb{Z}$. This restriction assures that the quantum wave function $e^{ip^{25}x^{25}}$ is invariant under $x \to x + 2\pi R$, or, equivalently, that the string states are invariant under this identification (as stated in [4] or [6] for example). We can introduce:

$$\begin{aligned} \alpha_0^{25} &= \frac{p^{25}}{2} - Rm \\ \tilde{\alpha}_0^{25} &= \frac{p^{25}}{2} + Rm \end{aligned} \tag{4.5}$$

When quantizing the theory, we need to consider the L charges as before. Using the $\mathbb{R}^{24,1}$ scalar product again:

$$\begin{aligned} L_0 &= \frac{1}{2}\alpha_0\cdot\alpha_0 + \sum_{k=1}^{\infty}\alpha_{-k}\cdot\alpha_k + \frac{1}{2}\alpha_0^{25}\alpha_0^{25} + \sum_{k=1}^{\infty}\alpha_{-k}^{25}\alpha_k^{25} \\ L_p &= \frac{1}{2}\sum_{k=-\infty}^{\infty}\alpha_{p-k}\cdot\alpha_k + \sum_{k=-\infty}^{\infty}\alpha_{m-k}^{25}\alpha_k^{25} \end{aligned} \tag{4.6}$$

It is then clear that the normal ordering constant is the same as for a closed string in $\mathbb{R}^{25,1}$, i.e. $a = 1$, but the L_0 constraint is slightly different because of the extra term appearing in the expression for α_0^{25}. Hence, comparing to (2.23):

$$\begin{cases} L_0: & \frac{1}{8}\boldsymbol{p}^2 + \frac{1}{2}\alpha_0^{25}\alpha_0^{25} + N - 1 = 0 \\ \tilde{L}_0: & \frac{1}{8}\boldsymbol{p}^2 + \frac{1}{2}\tilde{\alpha}_0^{25}\tilde{\alpha}_0^{25} + \tilde{N} - 1 = 0 \end{cases} \Rightarrow \begin{cases} -\boldsymbol{p}^2 = M^2 = 4(N + \tilde{N} - 2) + 4m^2R^2 + \frac{n^2}{R^2} \\ 0 = nm + N - \tilde{N} \end{cases} \tag{4.7}$$

Let us denote a general orbifold state by $|N, \tilde{N}; \boldsymbol{p}, n, m\rangle$. Introducing again a formal operator U that is a symmetry of the Hamiltonian, the untwisted sector contains states that are U-invariant. In analogy to the $\mathbb{R}^1/\mathbb{Z}_2$ orbifold theory (equations (3.2) and (3.3)), the action of U on α_n^{25} and $\tilde{\alpha}_n^{25}$ brings a minus sign, while acting on the ground states:

$$U\,|0,0;\boldsymbol{p},n,m\rangle = |0,0;\boldsymbol{p},-n,-m\rangle \tag{4.8}$$

Consequently, the *level 0* states of the untwisted sector are $|0,0;\boldsymbol{p},0,0\rangle$ and $\big(|0,0;\boldsymbol{p},n,m\rangle + |0,0;\boldsymbol{p},-n,-m\rangle\big)$, for $n, m \in \mathbb{Z}$. Let us now consider massless states at generic values of R. Using the (4.7) constraints, it is clear that this can only be achieved for $n = m = 0$, so $N = \tilde{N}$ and $N + \tilde{N} = 2$, which suggests that $N = \tilde{N} = 1$. Using the (3.4) decomposition of the general *level* 1 state of the closed string, we observe that the graviton, the dilaton and the antisymmetric B states will also be found in the untwisted sector as they are U invariant. Furthermore, the $\alpha_{-1}^{25}\tilde{\alpha}_{-1}^{25}\,|0,0;\boldsymbol{p},0,0\rangle$ is also U invariant, as for the untwisted sector of $\mathbb{R}^1/\mathbb{Z}_2$. We can also build new states as superpositions of states that are not U invariant, as we have done in (3.6). However, there are more basis elements in this case: four coming from changing $p \mapsto m$ (with $n = 0$) in (3.6) and another four from $p \mapsto n$ (with $m = 0$).

$$\begin{aligned}
&\alpha^i_{-1}\tilde{\alpha}^j_{-1}\big(|0,0;\boldsymbol{p},n,m\rangle + |0,0;\boldsymbol{p},-n,-m\rangle \big) \\
&\alpha^{25}_{-1}\tilde{\alpha}^j_{-1}\big(|0,0;\boldsymbol{p},n,m\rangle - |0,0;\boldsymbol{p},-n,-m\rangle \big) \\
&\alpha^i_{-1}\tilde{\alpha}^{25}_{-1}\big(|0,0;\boldsymbol{p},n,m\rangle - |0,0;\boldsymbol{p},-n,-m\rangle \big) \\
&\alpha^{25}_{-1}\tilde{\alpha}^{25}_{-1}\big(|0,0;\boldsymbol{p},n,m\rangle + |0,0;\boldsymbol{p},-n,-m\rangle \big)
\end{aligned} \tag{4.9}$$

Note that, for non-trivial values of m, n we can find a specific value of R for which there exist additional massless states[6]. This is $R = \sqrt{\alpha'}$. In fact, it turns out that the orbifold at this value of R is equivalent to the S^1 compactification at $R = 2\sqrt{\alpha'}$, but we will not discuss this relation here.

4.2 Twisted sector

The twisted sector contains states that obey the boundary condition: $X^{25}(\tau, \sigma+\pi) = -X^{25}(\tau,\sigma) + 2\pi R m$, for $m \in \mathbb{Z}$. Consider first the case where $m = 0$. We can solve again the wave equation satisfied by X^{25} subject to this boundary condition, to find a mode expansion in terms of half-integers:

$$X_0^{25}(\tau,\sigma) = i \sum_{n \text{ odd}} \frac{1}{n} \alpha^{25}_{\frac{n}{2}} e^{-in(\tau-\sigma)} + i \sum_{n \text{ odd}} \frac{1}{n} \tilde{\alpha}^{25}_{\frac{n}{2}} e^{-in(\tau+\sigma)} \tag{4.10}$$

It is clear that we cannot have neither momenta nor windings in the expansion because of the $X \sim -X$ identification. Consequently, the string cannot move away from the $X_0^{25} = 0$ fixed point, meaning that the string states are localized near this fixed point. Recall that this orbifold (see Fig. 1) has two fixed points so there will also be states localized at $X_\pi^{25} = \pi R$. These are in fact the states that obey the $m = 1$ boundary condition and the expansion will contain an additional constant term: X_π^{25} '=' $X_0^{25} + \pi R$. Here, the equality only indicates a similar mode expansion. Then: $X_\pi^{25}(\tau, \sigma + \pi) = -X_0^{25}(\tau,\sigma) + \pi R = 2\pi R - X_\pi^{25}(\tau,\sigma)$, so the boundary condition is indeed satisfied. These are the only unique fixed points since the remaining ones (i.e. different values of m) get identified with one of these two, as discussed in section 1. Similarly to the $\mathbb{R}^1/\mathbb{Z}_2$ twisted sector, the normal-ordering constant is $a = 15/16$ and the mass shell condition is (3.15). There are two *level* 0 states, one corresponding to each fixed point: $|0,0;\boldsymbol{p}\rangle_0$ and $|0,0;\boldsymbol{p}\rangle_\pi$; they are again tachyonic, with $\alpha' m^2 = -15/4$. The first excited states have $N = \tilde{N} = 1/2$, i.e. $\alpha^{25}_{-1}\tilde{\alpha}^{25}_{-1}|0,0;\boldsymbol{p}\rangle$ and are also tachyonic. Note that there are no twisted massless states.

5 Torus amplitudes

5.1 Modular Group

The torus is one of the four Riemann surfaces with Euler number equal to 0. Given two complex numbers λ_1, λ_2, it is defined (see [7], [6]) by the identification $z \sim z + n\lambda_1 + m\lambda_2$, for $n, m \in \mathbb{Z}$. The rescaling of $z = az$ leads to a rescaling of both $\lambda_{1,2}$, but their ratio $\tau = \lambda_2/\lambda_1$ is conformally invariant. Thus, the above identification might be written only in terms of this new parameter: $z \sim z + n + m\tau$. Note, however, that there are global diffeomorphisms (not smoothly connected to the identity) that in fact change τ but leave the torus unchanged. A more detailed discussion

[6]These states exist in the physical spectrum of a 26 dimensional theory compactified on a spacetime circle of radius R (S^1) and are listed in Appendix C.

of such transformations can be found in Appendix B. For now, we claim that a convenient basis for such transformations is:

$$T : \tau \to \tau + 1, \quad S : \quad \tau \to \frac{-1}{\tau} \tag{5.1}$$

The group generated in this way is $SL(2,\mathbb{Z})$, with the general transformation: $\tau \to \frac{a\tau+b}{c\tau+d}$ and $ac - bd = 1$. Note that changing the sign of all the parameters corresponds to the same transformation, so the group of global diffeomorphisms is in fact $PSL(2,\mathbb{Z}) = SL(2,\mathbb{Z})/\mathbb{Z}_2$, which is called the *modular group of the torus*. We will see in the next section that *1-loop* scattering amplitudes of closed strings correspond to integrations over a torus (or topologically equivalent spaces) and thus associated quantities must be modular invariant.

5.2 A review of Scattering Amplitudes

The discussion in this section follows from [4]. We start by considering a 'string diagram' with strings ('tubes') that extend into the past/future. Since under a Weyl rescaling of the world-sheet metric $\gamma_{ab} \to e^{\phi}\gamma_{ab}$ the string (Polyakov) action remains invariant, we can compactify the world-sheet by projecting these external string states to points. However, we need to introduce some local operators at such points; these are the vertex operators: $V_\Lambda(k)$ for emission/absorption of string states of type Λ of momentum k. Such operators can then be used to calculate the amplitude for scattering of multiple particles of different momenta:

$$\mathcal{A} \sim g^{M-2} \int \frac{\mathcal{D}X\mathcal{D}\gamma_{\alpha\beta}}{V_{\text{diff}\times\text{Weyl}}} e^{-S_P} \cdot \prod_{i=1}^{M} V_{\Lambda_i}(k_i) \tag{5.2}$$

where S_P is the Polyakov action, g is the coupling constant and the integrals are path integrals on the compact world sheet. The denominator in the above expression accounts for the overcounting of configurations that are related by diffeomorphisms and Weyl rescalings. Then, tree amplitudes are obtained for surfaces that are topologically equivalent to spheres while for 1-loop amplitudes one needs to perform the integration over a torus (for n-loop diagrams need genus n surfaces). It turns out that in the case of tree amplitudes, the reparametrization $\gamma_{\alpha\beta} = e^{\phi}\gamma'_{\alpha\beta}$ holds globally (clearly holds locally as it is a symmetry of the Polyakov action) and thus a conformally flat metric could be used. Hence, the computation is greatly simplified as the theory reduces to a free field theory in flat $2D$. However, we will abandon this path integral formalism for now and perform the integration over the Fock space. We will prefer to use τ dependent vertex operators $V_\Lambda(k,\tau)$, which for open strings are: $V_\Lambda(k,\tau) = e^{i\tau L_0}V_\Lambda(k,0)e^{-i\tau L_0}$. This easily follows from the expression of the Hamiltonian in terms of L_0. For convenience, we define $x = e^{i\tau} \in \mathbb{R}$, corresponding to a Wick rotation in the time coordinate, so: $V_\Lambda(k,x) = x^{L_0}V_\Lambda(k,1)x^{-L_0}$. For the closed string, the existence of right and left moving modes indicates a factorization:

$$V(k,\tau,\sigma) = V_R(k/2,\sigma_-)V_L(k/2,\sigma_+) \tag{5.3}$$

Then, evolving V_R and V_L as before, but with the generator of σ translations (i.e. $P \propto L_0 - \tilde{L}_0$), rather than that of τ (i.e. H) and integrating over σ we can find an expression for $V_\Lambda^{\text{cl}}(k,\tau)$:

$$V_\Lambda^{\text{cl}}(k,\tau) = \frac{1}{\pi}\int_0^\pi d\sigma e^{-2i\sigma(L_0-\tilde{L}_0)}V_R(k/2,\tau)V_L(k/2,\tau)e^{2i\sigma(L_0-\tilde{L}_0)} \tag{5.4}$$

Then, using the previous notation we immediately find: $V(k,x) \sim x^{\tilde{L}_0}V_R x^{-\tilde{L}_0}x^{L_0}V_L x^{-L_0}$ and since V_R commutes with L_0 and V_L with $\tilde{L}_0$: $V(k,x) = x^{L_0+\tilde{L}_0}V(k,1)x^{-L_0-\tilde{L}_0}$. We will also need an expression for propagators. In analogy to a scalar field satisfying the Klein Gordon equation: $(p^2+m^2)\phi = 0$, for which the propagator is $(p^2+m^2)^{-1}$, for a closed string that satisfies $(L_0 - a)\left|\phi\right\rangle = (\tilde{L}_0 - a)\left|\phi\right\rangle = 0$ we can guess the form:

$$\Delta = \frac{1}{2}\frac{1}{L_0+\tilde{L}_0-2a} = \frac{1}{2}\int_0^1 z^{L_0+\tilde{L}_0-2a-1}dz \tag{5.5}$$

However, closed string states also obey $(L_0-\tilde{L}_0)\left|\phi\right\rangle = 0$; thus, in order to make sure that only states that obey this additional constraint will propagate:

$$\Delta = \int_0^1 d\rho \int_0^{2\pi} \frac{d\phi}{4\pi}\rho^{L_0+\tilde{L}_0-2a-1}e^{i\phi(L_0-\tilde{L}_0)} = \frac{1}{4\pi}\int_{|z|\leq 1}\frac{d^2z}{|z|^2}z^{L_0-a}\bar{z}^{\tilde{L}_0-a} \tag{5.6}$$

In the second equality we introduced $z = \rho e^{i\phi}$, so $d^2z = \rho d\rho d\phi$. We are now ready to write 1-loop scattering amplitudes for closed strings. Hence, neglecting the denominator in (5.2) and a possible sum over permutations of the vertex operators (see [8]), the torus amplitude for coupling M closed string states is:

$$\begin{aligned}
\mathcal{A} &\sim \int d^Dp\; Tr\big(\Delta V(k_1,1)\Delta V(k_2,1)...V(k_M,1)\big) \\
&\sim \int d^Dp\; Tr\Big[\int dz_1 z_1^{L_0}\bar{z}_1^{\tilde{L}_0}V(k_1,1)\int dz_2 z_2^{L_0}\bar{z}_2^{\tilde{L}_0}V(k_2,1)\;...\;\frac{1}{|w|^2}w^{-a}\bar{w}^{-a}\Big] \\
&\sim \int d^Dp \int\prod_{i=1}^{M}dz_i\; Tr\Big[V(k_1,z1)V(k_2,z_1z_2)...V(k_2,z_1...z_M)\frac{1}{|w|^2}w^{L_0-a}\bar{w}^{\tilde{L}_0-a}\Big] \\
&\sim \int d^Dp\; Tr\Big(V(k_1,\rho_1)...V(k_M,\rho_M)w^{L_0}\bar{w}^{\tilde{L}_0}\Big)
\end{aligned} \tag{5.7}$$

In the second line we used the form of the propagator while in the third line the 'τ' evolution of the vertex operators was introduced. Lastly, the changes of variables: $\rho_r = z_1...z_r$, with $w = \rho_M$ were performed in the last line but we omitted the additional integrals as they are of no interest in the following arguments. The trace runs over all oscillator modes. Using the fact that $H = L_0 + \tilde{L}_0$ and $P = L_0 - \tilde{L}_0$, it easy to notice that $w^{L_0}\bar{w}^{\tilde{L}_0}$ can be brought to the form $e^{-yH}e^{ixP}$, for some x, y. This can be seen from a simpler reasoning than the one above. As argued in [6], we can start by considering a field theory on a circle and then evolve for Euclidian time $2\pi\tau_2$. Then, translating by $2\pi\tau_1$ in the σ^1 direction and identifying the ends we indeed obtain a torus of modulus $\tau = \tau_1 + i\tau_2$. Thus, the torus partition function (which is the equivalent of setting $M = 0$ in the above expression) would be:

$$Z(\tau) = Tr\Big(e^{2\pi i\tau_1 P}e^{-2\pi\tau_2 H}\Big) \sim Tr\Big(q^{L_0}\bar{q}^{\tilde{L}_0}\Big) \tag{5.8}$$

We implicitly assumed that the trace contains an integral over momenta and used $q = e^{2\pi i\tau}$. Rather than trying to show that the above amplitude is modular invariant in general, we will focus on a particular example.

5.3 Partition function on orbifolds

Let us now apply the results we obtained in the previous section. We will focus on the $S^1/\mathbb{Z}_2$ orbifold, but the results can be generalised to other orbifold theories. As discussed in [7] and [6], a 'guess' for the partition function of the untwisted sector of the orbifold theory would be:

$$Tr\Big(\mathcal{P}q^{L_0-a}\bar{q}^{\tilde{L}_0-a}\Big) \tag{5.9}$$

Here $\mathcal{P}$ projects all states of the initial theory onto states that are invariant under U. Such a projection operator usually has the form ([9]): $\mathcal{P} = \frac{1}{|G|}\sum_{g\in G} g$, so in our case: $\mathcal{P} = \frac{1}{2}(\mathbb{I}+U)$. We can check that: $\mathcal{P}^2 = \frac{1}{4}(\mathbb{I}+2U+U^2) = \mathcal{P}$ (since $U^2 = \mathbb{I}$), so this is indeed a projection operator. Then:

$$Z = Tr\Big(\frac{\mathbb{I}+U}{2}q^{L_0-a}\bar{q}^{\tilde{L}_0-a}\Big) = \frac{1}{2}Tr\Big(q^{L_0-a}\bar{q}^{\tilde{L}_0-a}\Big) + \frac{1}{2}Tr\Big(Uq^{L_0-a}\bar{q}^{\tilde{L}_0-a}\Big) \tag{5.10}$$

The first term corresponds to the partition function for toroidal compactification: $X^{25} \sim X^{25} + 2\pi R$. In this theory, X^{25} is essentially given by (4.4), so the formalism developed for the untwisted sector of the $S^1/\mathbb{Z}_2$ orbifold would prove to be useful. In fact, using the (5.8) expression for the partition function and the fact that the (4.7) constraints are equivalent to $L_0 \pm \tilde{L}_0$:

$$Z_0 \sim \sum_{m,n=\infty}^{\infty} exp\Bigg[2\pi i\tau_1 nm - \pi\tau_2\Big(\frac{\alpha' n^2}{R^2} + \frac{m^2R^2}{\alpha'}\Big)\Bigg] \tag{5.11}$$

It is rather interesting to note that since the sum is over all n, m integers, there is a strong hint that the transformation $R \mapsto \alpha'/R$ leaves the theory invariant. We will quote from [6] the Poisson ressumation formula:

$$\sum_{k=-\infty}^{\infty} exp\Big[-\pi a k^2 + 2\pi i b k\Big] = a^{-\frac{1}{2}}\sum_{k=-\infty}^{\infty} exp\Big[-\frac{\pi(k-b)^2}{a}\Big] \tag{5.12}$$

Applying this to n in our previous expression:

$$Z_0 \sim \frac{R}{\sqrt{\alpha'\tau_2}}\sum_{m,n=\infty}^{\infty} exp\Bigg[-\frac{\pi R^2}{\tau_2\alpha'}\Big((n-\tau_1 m)^2 + \tau_2^2 m^2\Big)\Bigg] = \frac{R}{\sqrt{\alpha'\tau_2}}\sum_{m,n=\infty}^{\infty} exp\Bigg[-\frac{\pi R^2}{\tau_2\alpha'}\big|n - m\tau\big|^2\Bigg] \tag{5.13}$$

Here $\tau = \tau_1 + i\tau_2$. Now, under the transformation $\tau \to \tau + 1$, i.e. $\tau_1 \to \tau_1 + 1$, we can simply change one of the variables being summed over, $n \to n + m$, such that the partition function remains invariant. For the transformation $\tau \to -1/\tau$, let us try changing $n \to -m$ and $m \to n$, so the exponent becomes:

$$\begin{aligned} -\frac{1}{\tau} = -\frac{\tau_1}{|\tau|^2} + i\frac{\tau_2}{|\tau|^2} \Rightarrow \frac{1}{\tau_2}\Big(\big(n + \frac{m\tau_1}{|\tau|^2}\big)^2 + \frac{\tau_2^2}{|\tau|^2}m\Big) \\ \xrightarrow[n\to -m]{m\to n} \frac{1}{\tau_2}\Big(m^2|\tau|^2 - 2mn\tau_1 + \frac{n^2\tau_1^2}{|\tau|^2} + \frac{n^2\tau_2^2}{|\tau|^2}\Big) \end{aligned} \tag{5.14}$$

This is in fact the same as the previous expression. However, there is an additional factor of $\sqrt{\tau_2}$ in the denominator of the partition function, which leads to a factor of $|\tau|$ in the overall expression of Z_0 under the transformation $\tau \to -1/\tau$. This is because we have not included in the above trace the sum over all the oscillator modes; the expressions for L_0 and $\tilde{L}_0$ contain additional terms, so the whole partition function is:

$$Z_0 = \int \frac{d^{25}k}{(2\pi)^{25}} e^{-\pi\tau_2\alpha' k^2} (q\bar{q})^{-a} \prod_{\mu,h} \sum_{N_h^\mu, \tilde{N}_h^\mu = 0}^{\infty} q^{hN_h^\mu} \bar{q}^{h\tilde{N}_h^\mu} \frac{R}{\sqrt{\alpha'\tau_2}} \sum_{m,n=\infty}^{\infty} exp\left[-\frac{\pi R^2}{\tau_2\alpha'} |n - m\tau|^2 \right] \tag{5.15}$$

The momentum integral is only over 25 dimensions since the last direction has already been taken into account by the integer n that quantizes p^{25}. The index μ runs over all $25 + 1$ spacetime directions, while h runs from 1 to ∞. The sums over the occupation numbers are easy to evaluate: $\sum_{N=0} q^{hN} = (1 - q^h)^{-1}$. Furthermore, there is a factor of $(\tau_2)^{-25/2}$ coming from the momentum integral. Then, we introduce the Dedekind eta function and since $(q\bar{q})$ is independent of τ, we can rewrite the partition function as:

$$\eta(\tau) = q^{1/24} \prod_{n=1}^{\infty} (1 - q^n) \Longrightarrow Z_0 \sim \tau_2^{-26/2} |\eta(\tau)|^{-26} \sum_{m,n=\infty}^{\infty} exp\left[-\frac{\pi R^2}{\tau_2\alpha'} |n - m\tau|^2 \right] \tag{5.16}$$

Thus, using the fact that $\eta(-1/\tau) = (-i\tau)^{1/2}\eta(\tau)$, the extra factors of $|\tau|$ are cancelled by those coming from τ_2 and the partition function is modular invariant. From now on, we will only consider the contributions to the partition function coming from X^{25}, as the rest is identical to the above calculation. Let us now consider the second term coming from the projection operator. Recall that U changes the sign of the momentum and winding numbers of a state $|\boldsymbol{p}, n, m\rangle \to |\boldsymbol{p}, -n, -m\rangle$ so the only states contributing to this term in the partition function are those with $m = n = 0$. Moreover, U changes the signs of α_n^{25} and $\tilde{\alpha}_n^{25}$ so the sums over N^{25} of $q^{hN^{25}}$ have now alternating signs: $1 - q^h + q^{2h} - ... = (1 + q^h)^{-1}$. As a result, the contribution to the (untwisted sector) partition function we are interested in becomes:

$$\frac{1}{2}(q\bar{q})^{-1/24} \prod_{h=1}^{\infty} \sum_{N_h^{25}, \tilde{N}_h^{25}} U q^{hN_h^{25}} \bar{q}^{h\tilde{N}_h^{25}} = \frac{1}{2}(q\bar{q})^{-1/24} \prod_{h=1}^{\infty} |1 + q^h|^{-2} = \left| \frac{\eta(\tau)}{\vartheta_2(\tau)} \right| \tag{5.17}$$

Here we introduced the Jacobi theta function, discussed into more detail in Appendix A. We have already seen that the first term of the partition function is modular invariant. However, using the properties of the theta functions listed in the appendix, it is straightforward to see that this second term is not modular invariant; it stays invariant under T $(\tau \to \tau + 1)$ transformations, but under S $(\tau \to -1/\tau)$ it transforms as:

$$\left| \frac{\eta(\tau)}{\vartheta_2(\tau)} \right| \xleftrightarrow{S} \left| \frac{\eta(\tau)}{\vartheta_4(\tau)} \right| \xleftrightarrow{T} \left| \frac{\eta(\tau)}{\vartheta_3(\tau)} \right| \tag{5.18}$$

This requirement leads us to consider a partition function that includes:

$$Z \sim \left| \frac{\eta(\tau)}{\vartheta_4(\tau)} \right| + \left| \frac{\eta(\tau)}{\vartheta_3(\tau)} \right| \tag{5.19}$$

In fact, we see that modular invariance forces us to include a twisted sector in the orbifold theory. Let us check if the orbifold theory developed in the previous sections does agree with this result. Recall from section 4.2 that the $S^1/\mathbb{Z}_2$ orbifold has two twisted sectors corresponding to the two fixed points. Also, the expansion of X^{25} containts half-integer modes and $p^{25} = 0$. Using the same projector as before and the reasoning for the action of U on physical states:

$$
\begin{aligned}
Tr\Big(\frac{\mathbb{I}+U}{2}q^{L_0-a}\bar{q}^{\tilde{L}_0-a}\Big) &\sim 2\frac{1}{2}(q\bar{q})^{-\frac{1}{16}-\frac{1}{24}}\prod_{h=0}^{\infty}\sum_{N^{25},\tilde{N}^{25}}(\mathbb{I}+U)q^{(h+\frac{1}{2})N^{25}_{h+1/2}}\bar{q}^{(h+\frac{1}{2})\tilde{N}^{25}_{h+1/2}} \\
&= (q\bar{q})^{-\frac{1}{16}-\frac{1}{24}}\prod_{h=0}^{\infty}\Big[\big|1-q^{h+1/2}\big|^{-2}+\big|1-q^{h+1/2}\big|^{-2}\Big]
\end{aligned} \tag{5.20}
$$

Note that the exponent of $(q\bar{q})$ is different than that for the untwisted sector since the normal-ordering constant is $a = 15/16 = 24/24 - 1/16$. Finally, these can be written in terms of the theta functions (see Appendix B), so the full partition function becomes modular invariant:

$$
Z = \frac{1}{2}Z_0 + \left|\frac{\eta(\tau)}{\vartheta_2(\tau)}\right| + \left|\frac{\eta(\tau)}{\vartheta_4(\tau)}\right| + \left|\frac{\eta(\tau)}{\vartheta_3(\tau)}\right| \tag{5.21}
$$

6 Conclusions

We developed a bosonic string theory of orbifolds starting from the closed string in $\mathbb{R}^{25,1}$ and imposing certain constraints on the 25^{th} component of the string coordinate. Beyond the states we would normally expect to have in the new theory, we have seen that the modular invariance requirement imposes the existence of a twisted sector. These states are obtained from half-integer mode expansions, but still satisfy the level-matching condition. However, the mass shell condition differs from the untwisted sector because of the new reordering constant, resulting in the absence of massless states in the twisted sectors that we analysed. The discussion could be generalised to more complicated orbifolds, but the basic ideas introduced in this report would still hold.

References

[1] K. Becker, B. Becker, and J. H. Schwarz. In *String Theory and M-Theory*. Cambridge University Press, 2007.

[2] Angel M. Uranga. Graduate course in string theory. *Universidad Autonoma de Madrid*, 2005.

[3] Michio Kaku. In *Introduction to Superstrings and M-Theory*. Springer, 1999.

[4] Michael B. Green, John B. Schwarz, and Edward Witten. In *Superstring Theory - Volume I, Introduction*. Cambridge University Press, 2002.

[5] Barton Zwiebach. In *A first course in String Theory*. Cambridge University Press, 2009.

[6] Joseph Polchinski. In *Super String Theory and Beyond, Volume I*. Cambridge University Press, 2005.

[7] Ralph Blumenhagen, Dieter Lust, and Stefan Theisen. In *Basic Concepts of String Theory*. Springer, 2013.

[8] Michael B. Green, John B. Schwarz, and Edward Witten. In *Superstring Theory - Volume 2, Loop Amplitudes, Anomalies and Phenomenology*. Cambridge University Press, 2002.

[9] Anamaria Font and Stefan Theisen. Introduction to string compactification. *Geometric and Topological Methods for Quantum Field Theory, Columbia*, 2003.

Appendix A Jacobi Theta Functions

As introduced in [7], the Jacobi theta functions are defined as:

$$\vartheta\left[\begin{array}{c}\alpha\\ \beta\end{array}\right](z,\tau)=\sum_{n\in\mathbb{Z}} exp\big[i\pi(n+\alpha)^2\tau+2\pi i(n+\alpha)(z+\beta)\big] \tag{A.1}$$

For $z=0$ and $-1/2\leq\alpha,\beta\leq 1/2$, one can relate them to the Dedekind eta function by:

$$\frac{\vartheta\left[\begin{array}{c}\alpha\\ \beta\end{array}\right](\tau)}{\eta(\tau)}=e^{2\pi i\alpha\beta}q^{\frac{\alpha^2}{2}-\frac{1}{24}}\prod_{n=1}^{\infty}\big(1+q^{n+\alpha-\frac{1}{2}}e^{2\pi i\beta}\big)\big(1+q^{n-\alpha-\frac{1}{2}}e^{-2\pi i\beta}\big) \tag{A.2}$$

One usually defines:

$$\vartheta_1=\vartheta\left[\begin{array}{c}1/2\\ 1/2\end{array}\right],\quad \vartheta_2=\vartheta\left[\begin{array}{c}1/2\\ 0\end{array}\right],\quad \vartheta_3=\vartheta\left[\begin{array}{c}0\\ 0\end{array}\right],\quad \vartheta_4=\vartheta\left[\begin{array}{c}0\\ 1/2\end{array}\right] \tag{A.3}$$

Using this notation, we quote some results that are relevant to our analysis:

$$\begin{aligned}\left|\frac{\eta(\tau)}{\vartheta_2(\tau)}\right| &=\frac{1}{2}(q\bar{q})^{-1/24}\prod_{n=1}^{\infty}|1+q^n|^{-2}\\ \left|\frac{\eta(\tau)}{\vartheta_4(\tau)}\right| &=(q\bar{q})^{-\frac{1}{16}-\frac{1}{24}}\prod_{n=0}^{\infty}|1-q^{n+1/2}|^{-2}\\ \left|\frac{\eta(\tau)}{\vartheta_3(\tau)}\right| &=(q\bar{q})^{-\frac{1}{16}-\frac{1}{24}}\prod_{n=0}^{\infty}|1+q^{n+1/2}|^{-2}\end{aligned} \tag{A.4}$$

Lastly, we list the modular transformations of these functions:

$$\begin{aligned}\vartheta\left[\begin{array}{c}\alpha\\ \beta\end{array}\right](\tau+1)&=e^{-\pi i(\alpha^2-\alpha)}\vartheta\left[\begin{array}{c}\alpha\\ \alpha+\beta-1/2\end{array}\right](\tau)\\ \vartheta\left[\begin{array}{c}\alpha\\ \beta\end{array}\right](-1/\tau)&=\sqrt{-i\tau}e^{2\pi i\alpha\beta}\vartheta\left[\begin{array}{c}-\beta\\ \alpha\end{array}\right](\tau)\end{aligned} \tag{A.5}$$

Note that the last transformation holds for $|arg\sqrt{-i\tau}|<\pi/2$. For completeness, we also list the modular transformations of the Dedekind eta function:

$$\begin{aligned}\eta(\tau+1)&=e^{i\pi/12}\eta(\tau)\\ \eta(-1/\tau)&=\sqrt{-i\tau}\eta(\tau)\end{aligned} \tag{A.6}$$

Appendix B Torus - Dehn twists

The torus can be defined using the complex plane identification: $z \sim z+\lambda_1 n+\lambda_2 m$, for $n, m \in \mathbb{Z}$ and $\lambda_1, \lambda_2 \in \mathbb{C}$. Defining $\tau = \lambda_1/\lambda_2$, an alternative definition of the torus is $z \sim z + n + m\tau$, as discussed in Section 5.1. The two global diffeomorphisms that leave the torus invariant but change τ are called the Dehn twists. These are obtained by cutting along a cycle (a or b as shown in Figure 2), twisting one end by 2π and glueing the ends back together.

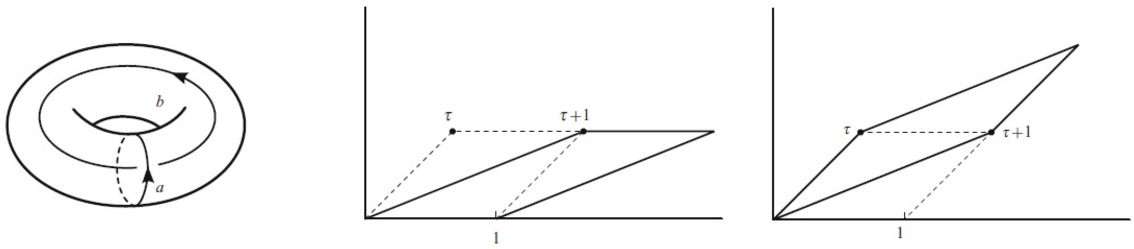

Figure 2: The dotted line in corresponds to the torus definition in terms of τ while the full lines are the actions on τ of the Dehn twists along the cycles a and b respectively.[7]

The Dehn twist around cycle a is equivalent to: $\lambda_1 \to \lambda_1$ and $\lambda_2 \to \lambda_1 + \lambda_2$, or $\tau \to \tau + 1$, the $+1$ factor coming from the additional 2π rotation. For cycle b we instead have $\lambda_1 \to \lambda_1 + \lambda_2$ and $\lambda_2 \to \lambda_2$, which translates to:

$$\tau \to \frac{\lambda_2}{\lambda_1 + \lambda_2} = \frac{\tau}{1+\tau} \tag{B.1}$$

The two transformations can be used as a basis for a more general transformation:

$$\tau \to \frac{a\tau + b}{c\tau + d} \tag{B.2}$$

with $a, b, c, d \in \mathbb{Z}$, and $ad - bc = 1$. In this way we generate the $SL(2, \mathbb{Z})$ group.

Appendix C Additional Massless states in Toroidal Compactification

A closed string theory with the 25^{th} spatial dimension compactified on a circle of radius R is obtained from the identification: $X^{25} \sim X^{25} + 2\pi Rm$, for integer m and $R \in \mathbb{R}$. This is essentially the same theory as the untwisted sector of the $S^1/\mathbb{Z}_2$ orbifold theory discussed in section 4.1, so we just quote the L_0 and $\tilde{L}_0$ constraints:

$$\begin{aligned} M^2_{(25)} =& 4(N + \tilde{N} - 2) + 2m^2R^2 + \frac{n^2}{R^2} \\ 0 =& nm + N - \tilde{N} \end{aligned} \tag{C.1}$$

Here n comes from the discrete values of the p^{25} momentum. We have already seen what the massless states at generic values of R are. The mass-shell condition implies that $N + \tilde{N} - 2 < 0$, so $N, \tilde{N} \in \{0, 1\}$. Combining the two constraints:

$$8\tilde{N} - 8 + \left(\frac{n}{R} - 2mR\right)^2 = 0 \tag{C.2}$$

Setting $R = 1/\sqrt{2} = \sqrt{\alpha'}$, it follows that:

$$4\tilde{N} - 4 + (n-m)^2 = 0 \tag{C.3}$$

We can then see that there are additional massless states for this particular values of R and we list them in the table below:

$$\begin{array}{c|c|c|c}
n & m & N & \tilde{N} \\
\hline
1 & 1 & 0 & 1 \\
-1 & -1 & 0 & 1 \\
1 & -1 & 1 & 0 \\
-1 & 1 & 1 & 0 \\
\hline
2 & 0 & 0 & 0 \\
-2 & 0 & 0 & 0 \\
0 & 2 & 0 & 0 \\
0 & -2 & 0 & 0
\end{array} \tag{C.4}$$

The first four of these states are massless vector states, corresponding to gauge bosons, while the last four are massless scalars. The gauge bosons combine into a non-Abelian theory with gauge group $SU(2) \times SU(2)$, the bosons transforming under the adjoint representations of $SU(2)_{R,L}$.