
Invariants of
4-Manifolds

Classification Schemes

- $2d$: Oriented closed 2-manifolds classified by genus.
- $3d$: *Thurston's Geometrization Program*: Every closed 3-manifold can be decomposed into pieces that each have one of eight types of *geometric structure*. This program can be applied in $2d$ as well - *Uniformization theorem*.
- In $5d$ and higher dimensions the classification is based on h-cobordism theorem and surgery theory.
- From this point of view $4d$ manifolds are the most complicated ones: no analogue of geometrization (used in $2d$, $3d$), the so-called Whitney trick used for $d > 5$ fails.

Classification Schemes

Theorem. (Markov) There is no algorithm that can tell whether two arbitrary closed 4-manifolds are diffeomorphic.

- A general classification of 4-manifolds is impossible, but, one can still try to classify manifolds with fixed fundamental group, e.g. $\pi_1(X) = 1$. In fact, simply connected, smooth 4-manifolds are classified up to *homeomorphism* by their intersection forms.
- Dimension 4 is the lowest dimension where the distinction between smooth and topological manifolds appears. In particular, there exist 4-manifolds that are *homeomorphic*, but not *diffeomorphic*!

Diffeomorphism $\longrightarrow$ *Homeomorphism* $\longrightarrow$ *Homotopy equivalence*

(Smooth structure) | *(Topological spaces)*

Gauge Theory

- Intersection form is $Q : H^2(M, \mathbb{Z}) \times H^2(M, \mathbb{Z}) \to \mathbb{Z}$, with:

$$Q(\alpha, \beta) = \int_M \alpha \wedge \beta \ .$$

- Donaldson turned his attention to gauge theory, which lead to the so called 'Donaldson invariants' of four-manifolds. These are polynomials on $H_0(X) \oplus H_2(X)$ which are invariants of the smooths structure of X.
- Witten gave a QFT description of Donaldson invariants, in terms of topologically twisted $4d\ \mathcal{N} = 2$ SYM. Since this theory is topological, one can give an entirely different description using the Low Energy Effective Theory.
- This led to the reformulation of the Donaldson invariants in terms of 'more tractable' invariants, the so-called Seiberg-Witten invariants.

Outline

1. Donaldson Polynomials
2. Witten's Conjecture
3. IR Description of the Invariants

References

- Labastida, Marino - TQFT and Four Manifolds (see also Marino - Phd Thesis)
- G. Moore - Lectures on the Physical Approach to Donaldson and SW Invariants
- Moore, Witten - 9709193, Manschot, Moore - 2104.06492

Donaldson Polynomials

Moduli Space of ASD Connections

- Let X be a smooth, oriented, compact 4-manifold with P and principal-$G = SU(2)$ bundle:

$$\pi : P \longrightarrow X \ .$$

- The connection A on P is a one-form with values in the Lie algebra of G, i.e. a section of $T^*P \otimes \mathfrak{g}$. Denote by $\mathcal{A}$ the space of all connections on P.
- Then, the ASD condition defines a subspace of $\mathcal{A}$, and define the moduli space of ASD connections as:

$$\mathcal{M}_{ASD} = \{[A] \in \mathcal{A}/\mathcal{G} \mid F_A^+ = 0\} \ .$$

- Its (*virtual*) dimension can be computed using AS index theorem.

Invariants of 4-manifolds

- The usual way of obtaining invariants of a certain geometrical structure is to start with $(X, \mathcal{S}, g)$ a manifold X, a geometrical structure $\mathcal{S}$ and some additional data g and write down a nonlinear elliptic PDE.
- One can then define the moduli spaces of solutions to this PDE $\mathcal{M}(M, \mathcal{S}, g)$, and then, define *invariants* which 'count' these $\mathcal{M}$ and give an answer *independent* of the extra data g.
- For instance, if $dim\ \mathcal{M} = 0$, then $\mathcal{M}$ is just a set of finitely many points, and the invariants count these points with certain signs.
- *Problems*: $\mathcal{M}$ is usually NOT compact, nor smooth!

Donaldson Polynomials

- Donaldson invariants are defined in terms of integrals of differential forms on the moduli space of (irreducible) ASD connections.
- The construction is quite technical and revolves around the so-called *Donaldson map*:

$$\mu : H_i(X) \longrightarrow H^{4-i}(\mathcal{M}_{ASD}) \ .$$

- Donaldson polynomials are defined as:

$$\mathcal{D}^k_X(x^l S_{i_1} \dots S_{i_p}) = \int_{\mathcal{M}^k_{ASD}} \mathcal{O}^l \wedge I_2(S_{i_1}) \wedge \dots I_1(S_{i_p}) \ ,$$

where $x \in H_0(X)$, $S \in H_2(X)$ and $\mathcal{O} \in H^4(\mathcal{M})$, $I_2 \in H^2(\mathcal{M})$.

Donaldson Polynomials

- One usually defines the formal sum $S = \sum_{i=1}^{b2} v_i S_i$, and the Donaldson generating functional as:

$$Z_{DW}(p, v_i) = \sum_{k=0}^{\infty} \mathcal{D}_X^k \left(e^{px+S}\right) \ .$$

- The amazing fact is that Donaldson invariants distinguish different smooth structures on 4-manifolds: there exist smooth 4-manifolds M_1, M_2 which are homeomorphic, but whose Donaldson invariants are different, so they are not diffeomorphic. No known algebro-topological invariants do this!
- They are difficult to calculate as they require detailed knowledge of the instanton moduli space, but, nevertheless, they are known for certain 4-manifolds.

Witten's Conjecture

Twisted $\mathcal{N} = 2$ SYM

- Witten famously conjectured that Donaldson invariants are closely related to the topologically twisted $\mathcal{N} = 2$ SYM theory. Consider the pure $SU(2)$ SYM theory in Euclidean signature, with the symmetry group:

$$SU(2)_+ \times SU(2)_- \times SU(2)_R \times U(1)_R \ .$$

- Before twisting, the supercharges and fields transform as follows:

$$\begin{array}{lcllcl}
\overline{Q}^I_{\dot\alpha} & : & (1,2,2)_1 \ , & \overline{\psi}^I_{\alpha} & : & (2,1,2)_{-1} \ , \\
Q^I_{\alpha} & : & (2,1,2)_{-1} \ , & \psi^I_{\dot\alpha} & : & (1,2,2)_1 \ , \\
A_\mu & : & (2,2,1)_0 \ , & \phi & : & (1,1,1)_2 \ , \\
D_{IJ} & : & (1,1,3)_0 \ , & \bar{\phi} & : & (1,1,1)_{-2} \ .
\end{array}$$

Twisted $\mathcal{N} = 2$ SYM

- The topological twist is performed by coupling the fields to the $SU(2)_+$ spin connection according to the way they transform under the $SU(2)_R$ group:

$$SU(2)_+ \times SU(2)_R \longrightarrow SU(2)_{diag} \ .$$

- This leads to a well defined theory on *any* four-manifold, with (at least) one supercharge preserved:

$$Q = \delta_I^{\dot{\alpha}} \overline{Q}_{\dot{\alpha}}^I \ , \ \text{with } Q^2 = 0 \ .$$

- Under twisting, A_μ remains a vector, while from the fermions we get a scalar η, a 1-form ψ_μ and a SD 2-form $\chi_{\mu\nu}$. From D_{IJ} we also get a SD 2-form. Additionally:

$$Q\phi = 0 \ .$$

Twisted $\mathcal{N} = 2$ SYM

- The next step is to construct observables. ϕ transforms in the adjoint representation, so a gauge invariant Q-closed operator is:

$$\mathcal{O}^{(0)} = Tr\left(\phi^2\right) \ .$$

- Starting from $\mathcal{O}^{(0)}$, we can produce new operators $\mathcal{O}^{(i)}$ using the *topological descent* procedure:

$$d\mathcal{O}^{(k)} = Q\mathcal{O}^{(k+1)} \ .$$

- Using SUSY variations, one finds:

$$\begin{aligned} \mathcal{O}^{(1)} &= Tr\left(\frac{1}{\sqrt{2}}\phi\psi_\mu\right)dx^\mu \ , \\ \mathcal{O}^{(2)} &= -\frac{1}{2}Tr\left(\frac{1}{\sqrt{2}}\phi\left(F_{\mu\nu} + D_{\mu\nu}\right) - \frac{1}{4}\psi_\mu\psi_\nu\right)dx^\mu \wedge dx^\nu \ . \end{aligned} \tag{1}$$

- The relevant observables are found by integrating over corresponding k-cycles:

$$I_2(S) = \int_S \mathcal{O}^{(2)} \ .$$

- Standard result in TQFT's is that correlation functions of Q-exact operators don't depend on the gauge coupling constant, and one can thus evaluate the functional integral in the saddle-point approximation. This boils down to analysing zero-modes and quadratic fluctuations of fermionic/bosonic modes.
- Similar to localization computations, bosonic zero-modes can be found by considering supersymmetric configurations, i.e. $Q(fermions) = 0$:

$$F^+ = 0 \ , \qquad D^+ = 0 \ , \qquad \nabla\phi = 0 \ .$$

Witten's Conjecture

- Hence, the zero modes of the gauge field are the ASD equations, and the integral over the collective coordinates reduces to an integral over $\mathcal{M}_{ASD}$.
- Keeping track of the fermionic modes, one find that the zero-mode measure in the path integral becomes:

$$\prod_{i=1}^{D} da_i d\psi_i \ ,$$

  with $D = dim(\mathcal{M}_{ASD})$, a_i and ψ_i the bosonic and fermionic zero modes.
- This is the natural measure for integration of differential forms on $\mathcal{M}_{ASD}$, with the Grassmannian variables ψ_i interpreted as a basis of one-forms on $\mathcal{M}_{ASD}$. We are thus led to *Witten's Conjecture*!

IR Description of the Invariants

Mapping the UV to the IR

- For a TQFT, the $Z_{DW}(p,S)$ partition function defined by the UV path integral should be computable in terms of the low energy effective action:

$$Z_{DW}(p,S) = \langle e^{p\mathcal{O}+\mathcal{O}(S)} \rangle_{UV} = \langle e^{p\mathcal{O}_{IR}+\mathcal{O}_{ir}(S)+\ldots} \rangle_{IR} \ ,$$

  as one can rescale the metric $g_{\mu\nu} \to t g_{\mu\nu}$ and take $t \to \infty$.

- One can thus describe the DW invariants in terms of a twisted version of the low-energy effective theory. Recall that the pure SYM theory has a classical vaccum moduli space on $\mathbb{R}^4$ given by:

$$[\phi, \bar{\phi}] = 0 \implies \phi = \begin{pmatrix} a & 0 \\ 0 & -a \end{pmatrix} \ .$$

- This VEV of ϕ generically breaks $SU(2) \to U(1)$. The CB is parametrised by $u = \mathcal{O}^{(0)} = a^2 + \ldots$.

LEEFT

- The strict IR limit contains only massless abelian gauge fields and their superpartners, which correspond to the moduli problem of abelian instantons on the four-manifold.
- The LEEFT description needs to be modified at the singular points of the Coulomb branch: $u = \pm\Lambda^2$, where certain BPS multiplets become massless. The IR evaluation of the DW invariants is schematically given by:

$$Z_{DW}(p,S) = Z_u + Z_{SW}(p,s) \ ,$$

where Z_{SW} refers to the contributions from the singularities.

- The Z_u contribution can be computed explicitly using the Seiberg-Witten solution:

$$Z_u(p,S) = \int_u da d\bar{a} A(u)^{\chi} B(u)^{\sigma} e^{pu+s^2 T} \Psi \ .$$

- Here Ψ is called the 'photon' integral and is essentially fully determined by the action of the twisted theory: $\Psi \sim e^{-S_{LEEFT}}$. T is some modular form for $\Gamma^0(4)$.
- $A(u)$ and $B(u)$ are gravitational corrections which also appear in the Nekrasov partition function ($\mathcal{A}(u) = log(A(u))$ etc.):

$$\begin{aligned} &- Log \mathcal{Z}_{\mathbb{C}^2 \times S^1}(u, \epsilon_1, \epsilon_2) \approx \\ &\approx \frac{2\pi i}{\epsilon_1 \epsilon_2} \left(\mathcal{F}(u) + (\epsilon_1 + \epsilon_2) H(u) + \epsilon_1 \epsilon_2 \mathcal{A}(u) + \frac{\epsilon_1^2 + \epsilon_2^2}{3} \mathcal{B}(u) \right) \ , \end{aligned}$$

- The SW contributions can be determined from a similar analysis to what we've done in the UV. The LEEA is modified by adding a hypermultiplet field L_{HM}. The supersymmetric configurations are then given by:

$$F^{+}_{\dot{\alpha}\dot{\beta}} + 4i\overline{M}_{(\dot{\alpha}}M_{\dot{\beta})} = 0 \ , \qquad D_L^{\alpha\dot{\alpha}} M_{\dot{\alpha}} = 0 \ .$$

- These are the so-called *Monopole equations*, with $M, \overline{M}$ the spinors obtained after twisting the scalars of the $\mathcal{N} = 2$ hypermultiplet. Here we assume X is a spin manifold, with a $U(1)$-bundle $L \to X$, such that:

$$M \in \Gamma\left(S^{+} \otimes L\right) \ , \qquad \overline{M} \in \Gamma\left(S^{+} \otimes L^{-1}\right) \ .$$

while D_L is the Dirac operator for the bundle $S^{+} \otimes L$.

SW Invariants

- The SW invariants can be defined in a similar way to the Donaldson invariants. However, a massive simplification is given by the fact that the moduli space of the monopole equations is *compact*, in contrast to the ASD moduli space!
- These invariants turn out to be much easier to compute, compared to the Donaldson invariants.

Generalizations

- One can include matter - $SU(2)$ theories with $N_f = 1, 2, 3$ hypers have already been considered in the literature.
- On the physics side, we need a well defined QFT, which gives certain constraints on G and matter content.
- On the maths side, there are many constraints. $\mathcal{M}_{ASD}$ is in general singular and non-compact. The *Generics metric theorem* states that for $k > 0$, $G = SU(2)$ or $SO(3)$ and $b_2^+(X) > 0$, $\mathcal{M}_{ASD}$ is smooth for generic metrics. There is no such theorem for other groups, but there exists some work on $SU(N)$ gauge groups.
- Regarding theory with matter, it turns out that even for pure $SU(2)$, the UV computation is incredibly difficult.

Thank you!