Resolution of Singularities

Horia Magureanu

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1 Introduction

The aim of this project is to introduce the tools required to 'resolve' singularities. We will start by stating some basic results of algebraic geometry in this section, which will be useful in the latter discussions. Divisors are defined in section 2, while section 3 contains blow-ups and a few examples of resolutions of singularities. Finally, in section 4, we will briefly discuss Du Val singularities and their link to Dynkin diagrams.

Definition 1.1. *The* ***local dimension*** $dim_p X$ *of a variety* X *at a point* $p \in X$ *is the maximum over all lengths of chains starting with* $X_0 = \{p\}$. *The* ***dimension*** *of* X *is the maximum of the lengths of all chains. The* ***codimension*** *of an irreducible subvariety* $Y \subset X$ *is:*

$$codimY = max_m \left(\exists \text{ chain } Y \subsetneq X_1 ... \subsetneq X_m \right)$$

Definition 1.2. *Given an affine variety* X *with an open set* U, *we introduce the set of regular functions on* U *as:*

$$\mathcal{O}_X(U) = \{ f : U \to k : f \text{ regular at each } p \in U \}$$

A function f is regular at a point p iff on some open set $W \subset U$, such that $p \in W$, $f = g/h$, with $g, h \in k[X]$ and $h(w) \neq 0$ on W. Let us also introduce the notion of **germ of a function**. For a topological space X, the germ of a function near $p \in X$ is an equivalence class $[(U, f)]$, with $f : U \to k$ a function defined on a neighbourhood $U \subset X$ of p, such that functions defined on two neighbourhoods are identified if they agree on a smaller neighbourhood of p. We denoted by $\mathcal{O}_X(U)$ the k-algebra of functions regular at all points in the open U, and now by $\mathcal{O}_{X,p}$ the k-algebra of germs of regular functions at p. In fact, one can show that:

$$k[X] = \cap \mathcal{O}_{X,p} \tag{1.1}$$

where the union is over all $p \in X$. This is another way of saying that the the set of rational functions is equal to the union of functions with no poles at a point p. Note also that if X is an affine variety then $\mathcal{O}_X(X) \cong k[X]$.

Definition 1.3. *For an irreducible quasi-projective variety* X, *a* ***rational map*** f *is a collection of regular maps defined on the same target* Y *which are identified if they agree on some non-empty open subset. It is essentially an equivalence class* $[(U, F)]$ *with* $U \subset X$ *open and* $F : U \to Y$ *a morphism of quasi-projective varieties.*

The rational map $f[(U, F)]$ is **dominant** if the image $F(U) \subset Y$ is *dense*. Consequently, a **birational equivalence** is a *dominant rational map* between irreducible quasi-projective varieties which has a rational inverse.

Definition 1.4. *We define the linear polynomial* $d_p F = \sum \frac{\partial F}{\partial x_j}(p)(x_j - p_j)$. *Then, the* ***Zariski tangent space*** *to an affine variety* X, *with* $\mathbb{I}(X) = \langle F_1, ..., F_N \rangle$ *is:* $T_p X = \mathbb{V}(d_p F_1, ..., d_p F_N)$, *or, equivalently, the intersection of hyperplanes* $\mathbb{V}(d_p F_i) = ker d_p F_i$.

A point $p \in X$ is **smooth** if $dim T_p X = dim_p X$, i.e. the dimension of the Zariski tangent space 'jumps' up at singular points. Thus, a way of finding **singular** points of a d dimensional affine variety X is computing the Jacobian $J\left(\partial F_i / \partial x_j\right)$ and looking for points where all the $(n-d) \times (n-d)$ minors vanish. We will see some examples in a latter section.

2 Divisors

2.1 Divisors on Curves

In this section we will briefly introduce the notion of divisors (on curves), following [1] and [2]. These will give us an intuitive way of thinking of divisors, which will be treated more generally in the next subsection. Let us consider a meromorphic function f on an algebraic curve C, with zeros $p_1, ..., p_k$ with multiplicities $m_1, ..., m_k$ and poles $q_1, ..., q_l$ with multiplicities $n_1, ...n_l$. This information determines a function up to a constant. Define then:

$$(f) = \sum_i^k m_i p_i - \sum_i^l n_i q_i \tag{2.1}$$

This is the *divisor* of the function f. In general, we define divisors on curves as follows.

Definition 2.1. *A divisor D on a curve C is a sum $D = \sum n_p p$ over all points $p \in C$, with $n_p \in \mathbb{Z}$ and $n_p \neq 0$ for finitely many points. The degree of the divisor is defined as the sum of all these n_p's.*

Following the above definition one notes that for a function $deg(f) = 0$. A divisor is principal if $D = div(f)$ for some function f, and two divisors are said to be equivalent if they 'differ' by a principal divisor.

Definition 2.2. *The set of meromorphic functions f for which $(f) + D$ is an effective divisor is denoted by $\mathcal{L}(D)$. Here, by effective we mean that all the n_p in the above definition are non-negative.*

It can be shown that $\mathcal{L}(D)$ is a finite-dimensional vector space, and for equivalent divisors D, D' we have $l(D) := dim\mathcal{L}(D) = dim\mathcal{L}(D') =: l(D')$. Let us consider the following example: $f(z) = (z-1)/(z+1)^3$. This has zeros at $z = 1, \infty$ and a triple pole at $z = -1$, and thus: $div(f) = 1[1] + 1[\infty] - 3[-1]$. Consider alternatively $D = 2[1]$; then, $\mathcal{L}(D)$ is the set of functions with at most a pole of order 2 at $z = 1$, i.e. functions of the type $g(z)/(z-1)^2$ with g a polynomial of degree at most 2 (otherwise the overall function would have poles at infinity).

2.2 Divisors

We will now consider a more general definition of divisors. The analysis is based on [3, 4].

Definition 2.3. *A (Weil)* ***divisor*** *D of a variety X is a linear combination of codimension one irreducible subvarieties $D = \sum n_i V_i$, with $n_i \in \mathbb{Z}$ and $V_i \subset X$. If all $n_i \geq 0$, then the divisor is called effective.*

For instance, in the case $X = \mathbb{A}^n$, irreducible codimension one subvarieties C are defined by a single equation f. Thus, $C = div(f)$, so every divisor on $\mathbb{A}^n$ is in fact principal. For $X = \mathbb{P}^n$, irreducible codimension one subvarieties are defined by a single homogeneous equation. Then, for a function f, write $f = F/G$, where F, G are of the same degree with the factorisation into irreducible polynomials $F = \prod H_i^{k_i}$, $G = \prod L_j^{m_j}$. Thus, $div(f) = \sum k_i C_i - \sum m_j D_j$, where C_i, D_j are the hypersurfaces obtained from $H_i = 0$ and $L_j = 0$, respectively.

As we have seen in the case of meromorphic functions on curves, divisors collect information on poles and zeros. We can introduce $\mathcal{L}(D)$ as for divisors on curves, but the divisor D is now defined as in 2.3. Also, the degree is defined again as the sum of all n_i. Following the above definition, for all $n_i \neq 0$, the variety $V_1 \cup ... \cup V_r$ is called the *support* of the divisor and is denoted by $SuppD$. Let us also denote by $|D|$ the set of all divisors D' that are effective and equivalent to D. Lastly, we would like to introduce canonical divisors as well, but to do this we first need to define differential forms on varieties. To do so, one needs to use regular functions in a neighbourhood of a point $x \in X$. The definition is thus slightly more involved, but it essentially reduces to the following remark.

Remark. *A differential r-form*[1] $\omega \in \Omega^r[X]$ *on* X *can be written in a neighbourhood of any point* $x \in X$ *as:*

$$\omega = \sum g_{i_1 ... i_r} df_{i_1} \wedge ... \wedge df_{i_r}$$

where $g_{i_1...i_r}$ *and* f_i *are regular functions in a neighbourhood of* x.

We will be interested in the module $\Omega^n[U]$, which is a free $k[U]$-module of rank 1, so an element of this module is of the form: $\omega = g du_1 \wedge ... \wedge du_n$, with n the dimension of the variety X. Let us now consider the pairs (ω, U) with U an open set on X and the equivalence relation: $(\omega, U) \sim (\omega', U')$ if $\omega = \omega'$ on $U \cap U'$. This equivalence class is called a *rational differential r-form*, and the set of all such objects on X is denoted by $\Omega^r(X)$. It can be shown that this is a vector space of dimension $\binom{n}{r}$.

Definition 2.4. *A* ***canonical divisor*** *is the divisor associated* [2] *with a rational differential n-form on an n-dimensional variety X (i.e. 'zeros' - 'poles' of the n-form).*

The canonical divisors satisfy the following properties:

- $div(f\omega) = divf + div\omega$
- $div\omega \geq 0$ iff $\omega \in \Omega^n[X]$

Note that any $n-$form ω can be written as $\omega = f\omega_1$, so the divisors of ω and ω_1 are equivalent, thus forming a divisor class called **canonical class** on X; this is usually denoted by K_X. Note also that the second property implies that $divf + div\omega_1 \geq 0$, so: $\Omega^n[X] = \mathcal{L}(div(\omega_1))$. For $n = 1$, the canonical class invariant $l(K_X) = dim(\mathcal{L}(div(\omega)))$ is just the genus of the curve X.

Let us compute the canonical divisors on $X = \mathbb{P}^1$, with coordinates $[x : y]$. The canonical form is thus a 1-form. We have two coordinate charts, with $u = x/y$ in the first one and $v = y/x$ in the second one. Thus, for $u, v \neq 0$, we find: $du = -dv/v^2$ and $dv = -du/u^2$. Consequently, $div(du) = -2[1 : 0]$ and $div(dv) = -2[0 : 1]$. Note that the function $f = \frac{x}{y}$ has $div(f) = [0 : 1] - [1 : 0]$, which suffices to show that the divisors of du and dv are equivalent and thus part of the canonical class. Consequently, $K_{\mathbb{P}^1} = -2y$, for any point $y \in \mathbb{P}^1$.

Consider also $C = \mathbb{V}(x^2 - yz)$ in $\mathbb{P}^2$ and denote the points on C by $(x : y : z)$. On the $z \neq 0$ chart, we introduce the coordinates $r = x/z$, $s = y/z$. It is straightforward to show that in this chart the curve becomes $r^2 - s = 0$ and $ds = 2rdr$, so ds has a zero at $[0 : 0 : 1]$. For $z = 0$, a point on the curve must have coordinates $[0 : 1 : 0]$ and we can show that in this chart with $t = x/y$, we have $ds = -\frac{2}{t^3}dt$. Thus, the divisor of ds is $div(ds) = [0 : 0 : 1] - 3[0 : 1 : 0]$. In fact, this is equivalent to $div(dr)$ as well.

[1] Strictly speaking, the regular differential r-forms on X, i.e. functions sending a point of X to a skewsymmetric multilinear form, form a $k[X]$-module $\Omega^r[X]$, with ω being an element of this module.

[2] It is not entirely obvious why this would be a divisor. Given $\omega = g^{(i)} du_1^{(i)} \wedge ... \wedge du_n^{(i)}$ on U_i, g can be found in any other patch by a coordinate transformation and since the Jacobian is non-singular, $g^{(i)}/g^{(j)}$ is regular and non-zero on $U_i \cap U_j$, forming a compatible system of functions and thus defining a divisor on X.

3 Resolutions

3.1 Blow-ups

Definition 3.1. *The* ***blow-up*** *of* $\mathbb{A}^n$ *at the origin is the set of lines in* $\mathbb{A}^n$*, with a given choice of point:*

$$B_0\mathbb{A}^n = \{(\boldsymbol{x}, \boldsymbol{y}) : \mathbb{A}^n \times \mathbb{P}^{n-1} : \boldsymbol{x} \in \boldsymbol{y}\} = \mathbb{V}(x_i y_j - x_j y_i)$$

where the x *coordinates are along* $\mathbb{A}^n$ *and the* $y's$ *along the* $\mathbb{P}^{n-1}$ *with* $\boldsymbol{x}$ *proportional to* $\boldsymbol{y}$*. Note that the morphism* $\pi : B_0\mathbb{A}^n \to \mathbb{A}^n$*, with* $\pi(x, [y]) = x$ *is birational, with inverse* $x \mapsto (x, [x])$*, for* $x \neq 0$*.*

Consider $n = 2$ for example; here, the blow-up of $\mathbb{A}^2$ at the origin is equivalent to associating to each point of $\mathbb{A}^2$ the slope of the line through the origin. We thus have two **affine charts** of the blowup. However, since the origin is still missing, the blow-up is defined as the Zariski closure of the above, sometimes denoted by $\tilde{\mathbb{A}}^2$. We denote by $\mathbb{E}$ the **exceptional divisor** $\mathbb{E}_0 = \pi^{-1}(0) = \{0\} \times \mathbb{P}^{n-1}$, such that $\pi : B_0\mathbb{A}^n_0 \to \mathbb{A}^n$ is an isomorphism and π collapses $\mathbb{E}_0$ to the origin. For an affine variety $X \subset \mathbb{A}^n$ with $0 \in X$, the **blow-up** of X at the origin (also called the *strict transform*) is:

$$B_0X = closure\left(\pi^{-1}\left(X \setminus \{0\}\right)\right) \tag{3.1}$$

Note that π is a birational map from B_0X to X, and the exceptional divisor is $E = \pi^{-1}(0) \cap B_0X$. We will see some examples in the following sections.

3.2 Resolutions

Definition 3.2. *The* $X_1, ..., X_k \subset \mathbb{A}^n$ *algebraic varieties have* ***normal crossings*** *at a point* $p \in \mathbb{A}^n$ *if there exists a system of coordinates in a neighbourhood of* p*, such that each* X_i *is either empty or the vanishing locus of some of the coordinates (this is actually incomplete - also require that each* X_i *should be given by the vanishing of a different set of coordinates). In other words, we can choose coordinates locally such that each* X_i *is a coordinate subspace or a collection of coordinate hyperplanes. Then, a variety* Y *is* ***transversal*** *to the exceptional locus* $\mathbb{E}$ *of a blow-up of* X*, if* Y *and the components of* $\mathbb{E}$ *have normal crossings.*

Blow-ups can be used to desingularise a variety X, which is essentially the content of Hironaka's theorem. Before doing this for some particular singular varieties, let us first introduce the concept of resolution. We will denote by $Sing(X)$ the set of points on X that are singular.

Definition 3.3. *A* ***nonembedded resolution of singularities*** *of an algebraic variety* X *is a birational map* $\phi : \tilde{X} \to X$*, where* $\tilde{X}$ *is smooth, such that* ϕ *is an isomorphism over* $X \backslash Sing(X)$*. If* X *is embedded in a smooth ambient variety* W*, an* ***embedded resolution*** *of singularities of* X *is a birational morphism from another smooth variety* W' *to* W*, such that the total transform of* X *in* W' *has only normal crossings.*

Given a resolution π of X with exceptional divisor $\mathbb{E} = \mathbb{E}_1 \cup ... \cup \mathbb{E}_s$, the resolution is called *good* if $\mathbb{E}$ is a normal crossing divisor, i.e. the $\mathbb{E}_i's$ intersect each other transversally. Let us now consider some examples.

3.2.1 Examples

The Cusp is defined as $C = \mathbb{V}(y^2 - x^3) \subset \mathbb{A}^2$. Note that the Jacobian is $J = (-3x^2, 2y)$, so this is singular at the origin, as can be shown in Figure (2). We blow-up the ambient space at this singular point, such that:

$$B_0\mathbb{A}^2 = \{((x, y), [u : v]) \in \mathbb{A}^2 \times \mathbb{P}^1 : xv = yu\} \tag{3.2}$$

We now need to look at both of the charts. In the first one, $u = 1$, $y = xv$ so the cusp equation becomes $x^2(v^2 - x) = 0$ and we easily identify the strict transform $\tilde{C} = \mathbb{V}(v^2 - x)$ and the exceptional divisor $E : x = 0, v^2 - x = 0$, i.e. $((0, 0), [1 : 0])$, a line in the $x - y$ plane. Note that the new curve is now smooth. For the second patch: $v = 1$ so $\tilde{C} = \mathbb{V}(1 - yu^3)$, which is again smooth.

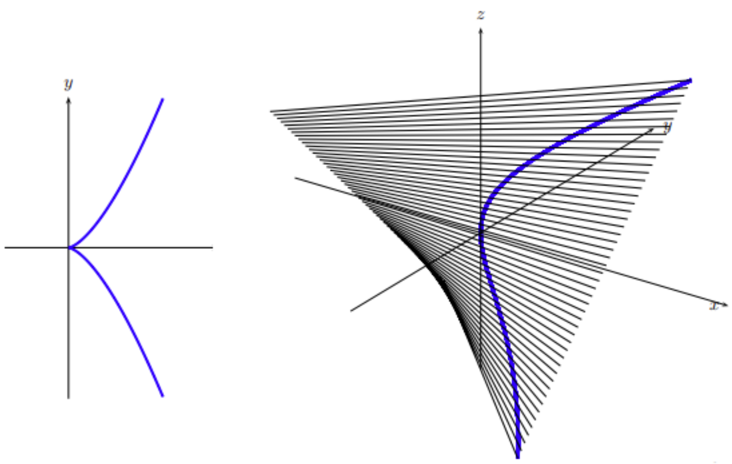

Figure 1: The Cusp and its blow-up at the origin ([5])

The Nodal Curve is defined as $C = \mathbb{V}(y^2 - x^3 - x^2)$. The Jacobian is $J = (-x(3x + 2), 2y)$, so all minors vanish for $y = 0$ and $x = 0$ or $x = -2/3$. However, only the first point is on the curve, so it is a singularity. Blowing-up $\mathbb{A}^2$ as before, we obtain in the two charts:

$$\begin{aligned} u = 1 : \quad & v^2 - x - 1 = 0 \\ v = 1 : \quad & 1 - yu^3 - u^2 = 0 \end{aligned} \tag{3.3}$$

These are both smooth, so the blow-up is indeed a resolution of the singularity. Let us also note that $\pi^{-1}(0)$ gives the entire $\mathbb{P}^1$, so its intersection with B_0C leads to the union of two lines in $\mathbb{P}^1$: in the $u = 1$ chart, $E = ((0, 0), [1 : \pm 1])$, while in the $v = 1$ chart it is $E = ((0, 0), [\pm 1 : 1])$ (these are essentially the same two projective lines).

For the **quadratic cone** discussed in [6], $x^2 - y^2 - z^2 = 0$ in $\mathbb{A}^3$, the Jaccobian is given by: $J = (2x, -2y, -2z)$, so all the 1×1 minors vanish only at the origin. Then, let us consider the ambient space transformation π^{-1}: $x = u, y = uv, z = uw$, which leads to the equation of X: $u^2(1 - v^2 - w^2) = 0$. Thus, $\pi^{-1}(X)$ has two components, the plane $u = 0$ and the cylinder X': $v^2 + w^2 = 1$. The latter has $J = (0, 2v, 2w)$ and there is no point on the cylinder where all the minors vanish. Note that the u plane is in fact the inverse image of the origin of X, i.e. the singular point of X and is called the exceptional hypersurface $\mathbb{E}$. The *strict transform* X' is obtained as $\pi^{-1}(X) \setminus \mathbb{E}$. Note that X' and $\mathbb{E}$ have only normal crossings; this is because at each point along where the two intersect, we can find a coordinate neighbourhood such that both X'

and $\mathbb{E}$ are coordinate subspaces. Thus, this resolution is an embedded resolution. In fact, the map used here is the blow-up of X at the origin.

The **Whitney umbrella** is an example of a surface that cannot be resolved by blowing-up points ([7, 8]). To resolve this surface, we will need to consider blowing-up subvarieties. In order to understand how to do this, we will consider this particular example: $S = \mathbb{V}(x^2 - y^2 z)$. The Jacobian is given by $J = (2x, -2yz, -2y)$, so $SingS = \{x = 0, y = 0\}$ contains the z axis. Let us first see what happens when we blow-up the ambient space at the origin:

$$B_0\mathbb{A}^3 = \{((x, y, z), [u : v : w]) \in \mathbb{A}^3 \times \mathbb{P}^2 : xv = yu, xw = zu, yw = zv\} \tag{3.4}$$

In the $w = 1$ chart, $y = zv$, $x = zu$ such that the blow-up of S becomes $B_0S = \mathbb{V}(u^2 - v^2 z)$. This is essentially the same as S! The blow-up of the ambient space along the z axis can be computed using the following procedure:

$$B_Z\mathbb{A}^3 = \{((x, y, z), [u : v]) \in \mathbb{A}^3 \times \mathbb{P}^1 : xv = yu\} \tag{3.5}$$

Thus, on the $u = 1$ chart we find the strict transform $1 - zv^2 = 0$, while on the $v = 1$ chart this is: $u^2 - z = 0$. These are both smooth, so the blow-up along the z axis is a resolution of the Whitney umbrella.

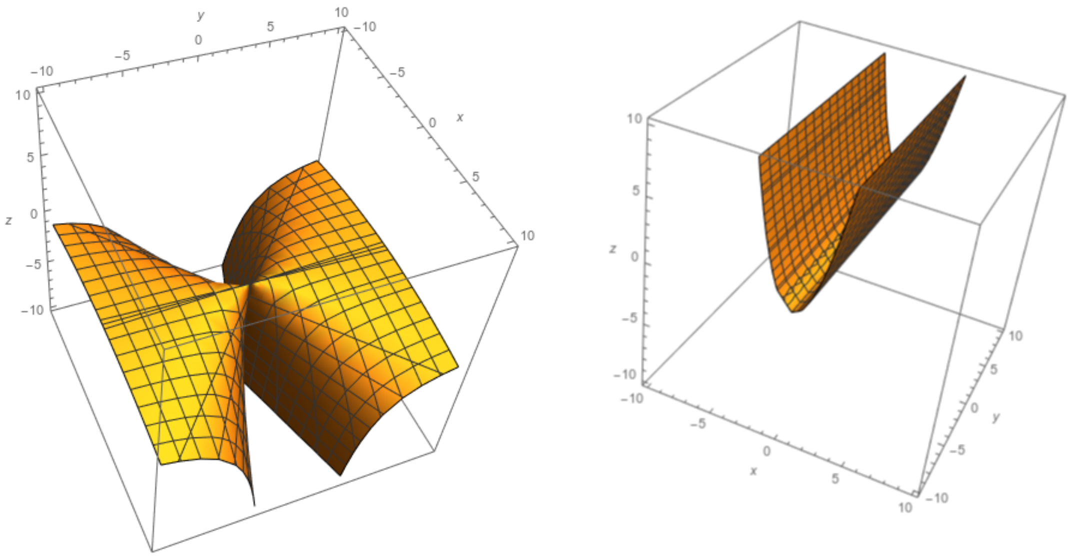

Figure 2: The Whitney umbrella and its strict transform in the $v = 1$ chart.

3.3 Intersection Numbers

Before discussing further blow-ups, let us first introduce intersection numbers (Chapter 4 in [4], but also [3, 9]). We are interested in the intersection of codimension 1 subvarieties on a

n-dimensional non-singular variety X and, in particular, in the case that there is a finite number of intersection points. Naturally, this also leads to the intersection of divisors. Let us start by considering the intersection of n such divisors.

Definition 3.4. *The effective divisors $D_1, ..., D_n$ on the $n-$dimensional non-singular variety X are in* ***general position*** *at a point $x \in X$ if $dim \bigcap SuppD_i = 0$ at x.*

This means that in some neighbourhood of a point x of interest, $\bigcap SuppD_i$ consists of x only. Assume for now that the divisors D_i are in general position at a point $x \in X$ and that their local equations in some neighbourhood of this point are given by f_i.

Definition 3.5. *The* ***intersection multiplicity*** *of $D_1, ..., D_n$ at $x \in X$ is* [3]*:*

$$(D_1 \dots D_n)_x = dim\,(\mathcal{O}_{X,x}/(f_1, \dots, f_n))$$

Let us now try to understand the meaning of these multiplicities. For this it is useful to consider two curves C_1, C_2 in $\mathbb{C}$ given by $x^2 - y = 0$ and $y - a^2 = 0$, respectively (i.e. a parabola and a straight line). When $a \neq 0$, the curves intersect at two points $(\pm a, a^2)$, both with 'multiplicity' 1, but when $a = 0$, there is a single point, with 'multiplicity' 2. Using the above definition, the coordinate ring of the intersection of the two curves is:

$$\frac{\mathbb{C}[x,y]}{(x^2 - y, y - a^2)} = \frac{\mathbb{C}[x]}{(x^2 - a^2)} = \begin{cases} \frac{\mathbb{C}[x]}{(x-a)} \oplus \frac{\mathbb{C}[x]}{(x+a)} & \text{if } a \neq 0 \\ \frac{\mathbb{C}[x]}{(x^2)} & \text{if } a = 0 \end{cases}$$

Hence, for $a \neq 0$, $dim(\mathbb{C}[x]/(x \pm a)) = 1$, but it is 2 at $a = 0$, which is the result we intuitively expected.

If the divisors are not effective, we can let $D_i = D_i' - D_i''$, with the new divisors being effective and thus extend the above definition by multilinearity.

Definition 3.6. *The* ***intersection number*** *of n divisors in general position is:*

$$D_1 \dots D_n = \sum_{x \in \bigcap SuppD_i} (D_1 \dots D_n)_x$$

One can prove that (see [4]) this additive and, given two equivalent divisors $D_n = D_n'$, we have $D_1 \dots D_n = D_1 \dots D_n'$. We can generalise these results for the case $k \neq n$, by first changing to definition of 'general position' to $dim \bigcap SuppD_i = n - k$. This then means that there exist irreducible $(n-k)$-dimensional varieties C_j such that $\bigcap SuppD_i = \bigcup C_j$. Thus, we assign to each C a number: $dim\,(\mathcal{O}_C/(f_1, ..., f_k))$, which is the intersection multiplicity of $D_1, ..., D_k$ along C. Note that in the case $n = k$, C_j is simply a point and the multiplicity is precisely the one previously defined. Let us further note that given n divisors D_i' on X, there exist equivalent divisors $D_i \sim D_i'$, such that the new set of divisors are in general position. Thus, we can now compute the intersection number of any n divisors on a n-dimensional variety X.

One important feature of these last remarks is that we can also consider **self-intersection numbers** of divisors. Consider for instance $X = \mathbb{P}^2$ and let C be a line on X. Then, there exists $C' \sim C'' \sim C$, with C', C'' in general position, such that: $C^2 = C'C''$. To see why

[3]Note that the intersection multiplicity does not depend on the local choice of f_i equations, but only on the divisors D_i.

this is the case, consider $\mathbb{P}^1$ for simplicity. We want to show that all the divisors containing precisely one point are equivalent. In this respect, consider the points $p = [a : b]$ and $q = [c : d]$ the function of the coordinate ring: $f = \frac{bx-ay}{dx-cy}$. Then, $div(f) = [p] - [q]$, so the two divisors are indeed equivalent. In this way, we can produce a similar argument for lines in $\mathbb{P}^2$. In this particular example, C' and C'' are just (distinct) lines that intersect at a single point, so $C^2 = 1$.

3.4 Exceptional Divisors of blow-ups

Let us now return to the discussion of blow-ups. Let us state without proof the following:

Theorem 3.1. *Given the blow-up $\pi : B_\xi(X) \to X$ of a surface X at a non-singular point $\xi \in X$ with exceptional divisor $E = \pi^{-1}(\xi)$, then: $E.E = -1$ and $E \cong \mathbb{P}^1$. Such a curve is called a (-1) curve.*

Proof. Let us try to sketch the proof of the 'theorem'. Let (x, y) be the local parameters at ξ, which for simplicity we take to be the origin, and $[t_0 : t_1]$ the $\mathbb{P}^1$ coordinates. Then: $t_0 y = t_1 x$, so in the $t_0 \neq 0$ chart, we can write $x = u, y = uv$. Note that E has local equation $u = 0$. Consider further a curve (divisor) $C \subset X$ going through the origin with (local) equation $y = 0$. The inverse image of C, denoted by $\pi^*(C)$, contains two components: E and C', the closure of $\pi^{-1}(C \setminus \xi)$, such that[4] $\pi^*(C) = C' + kE$. We now note that C' has local equation $v = 0$, so it is transverse to E and thus $E.C' = 1$. Using the fact[5] that $\pi^*(C).L = 0$ we find:

$$E.\pi^*(C) = E.\big(C' + E\big) = 1 + E.E = 0$$

□

Theorem 3.2 (Caltelnuovo)**.** *Given a surface X with a (-1) curve $E \subset X$, then there is a morphism $\pi : X \to X'$ with X' non-singular, such that π is the blow-up of X' with exceptional divisor E.*

These theorems tell us that (-1) curves and blow-ups of non-singular points are related. Consequently, blowing-up singular points we would expect exceptional divisors with self intersection numbers different than -1. Another theorem shows that in fact these numbers are strictly negative, so we would expect that $E^2 = -2$.

4 Rational Double Points

Definition 4.1. *A point x of a surface X is a **Du Val singularity** if there is a (minimal) resolution $f : \tilde{X} \to X$ such that $K_{\tilde{X}}.E_i = 0$, for all exceptional curves E_i.*

These singularities do not affect the canonical class, so the surface invariants remain the same after this mapping. In fact, one can show that the above definition implies that $C_i^2 = -2$ (for minimal resolutions) and $C_i \cong \mathbb{P}^1$. Another important result is that the Du Val singularities are given by the following equations:

$$\begin{aligned} A_n : &\quad x^2 + y^2 + z^{n+1} = 0 \quad \text{for } n \geq 1 \\ D_n : &\quad x^2 + y^2 z + z^{n-1} = 0 \quad \text{for } n \geq 4 \\ E_6 : &\quad x^2 + y^3 + z^4 = 0 \\ E_7 : &\quad x^2 + y^3 + yz^3 = 0 \\ E_8 : &\quad x^2 + y^3 + z^5 = 0 \end{aligned} \tag{4.1}$$

[4]Note that C' appears with coefficient 1 since the blow-up is an isomorphism outside ξ. Furthermore, one can show that k is actually the multiplicity of C at ξ, which is 1 here.

[5]For a divisor E on $B_\xi(X)$ whose components are exceptional curves of π, then for any divisor D on X we have: $E.\pi^*(D) = 0$. This result is proved in Chapter 4, Section 3.2 of [4].

One immediately observes that the 'names' of these singularities are identical to those of the finite Lie groups. In fact, the Dynkin diagrams corresponding to these groups are a way of visualising the singularities.

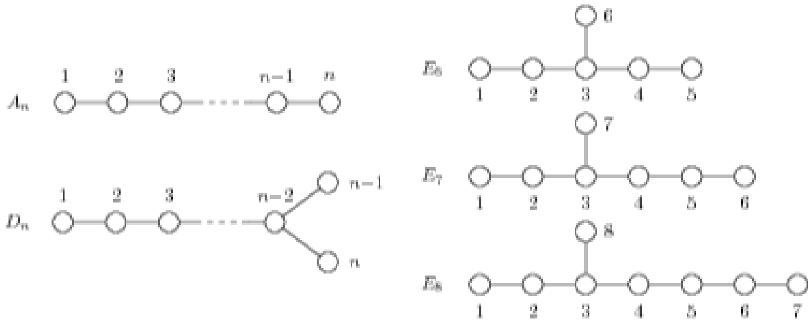

Figure 3: Dynkin Diagrams

4.1 The A_1 singularity

In this subsection we will consider the singularity of $S = \mathbb{V}(x^2+y^2+z^2) \subset \mathbb{A}^3$. This is also called the ordinary double point singularity ([10, 11]). The Jacobian is given by: $J = (2x, 2y, 2z)$, so all the minors vanish only at the origin, which is the A_1 singularity. Let us start by blowing-up the origin of the ambient space, i.e.:

$$B_0\mathbb{A}^3 = \{((x, y, z), [u : v : w]) \in \mathbb{A}^3 \times \mathbb{P}^2 : xv = yu, xw = zu, yw = zu\} \tag{4.2}$$

$\mathbb{P}^2$ can be covered by 3 coordinate charts, so let us consider the $u \neq 0$ chart (we have the freedom to take $u = 1$ in this chart). We now want to find the blow-up of S, defined as the closure of $\pi^{-1}(X \setminus \{0\})$. Then, under the (inverse) map, the equation defining S becomes: $x^2(1 + v^2 + w^2) = 0$. To obtain B_0S, we should allow x to be arbitrary, so in this chart $B_0S = \mathbb{V}(1 + v^2 + w^2)$ is a cylinder. Note that this is clearly non-singular, so we resolved the singularity at the origin. Then, the exceptional divisor is the intersection of the plane $\pi^{-1}(0)$, i.e. $((0, 0, 0), [u : v : w])$, with the cylinder B_0S, as discussed in the previous section, so: $1 + v^2 + w^2 = 0$ and $x = 0$. This is in fact birationally equivalent to a line, so E is rational. Note that the three charts of B_0S are symmetric, so E is smooth.

One can further show that $E^2 = -2$. To do this, we proceed as in the proof of Theorem 3.1. Thus, we take again a divisor $C \subset X$; it follows that $E.\pi^*(C) = 0$ (see footnote 5). If we take $z = 0$ as the local equation of C, then C' (the closure of $\pi^{-1}(C \setminus \{0\})$) will be defined by $w = 0$. Thus, the intersections of C and E are given by the solutions of these local equations, which reduce to: $1 + v^2 = 0$, so $v = \pm i$. Both points have multiplicity one, so $E.E + 2 = 0$, as expected.

4.2 The A_n singularity

The general A_n singularity is the origin of $\mathbb{V}(x^2 + y^2 + z^{n+1}) \subset \mathbb{A}^3$. Blowing-up this point, we have again three charts, two of which will be symmetric. Consider first the $u = 1$ chart, where the strict transform becomes $\mathbb{V}(1 + v^2 + x^{n-1}w^{n+1})$, which is smooth. For the last chart, $w = 1$, the strict transform will be $\mathbb{V}(u^2+v^2+z^{n-1})$, which is precisely an A_{n-2} singularity. Thus, we see that the A_2 singularity should be resolved after blowing it up once. What is then different compared to A_1? The answer is that the blow-up of A_2 leads to an exceptional divisors that contains two irreducible components. For instance, looking at the $u=1$ chart of the mapping, the exceptional divisor is described by: $x=0$ and $1+v^2=0$, with solutions $v=\pm i$. These are two projective lines with intersection number 1.

4.3 The D_4 singularity

We now want see how the Dynkin diagram arises in the description of the D_4 singularity. Following [11], the equation of the D_4 singularity can be brought to $x^2+y^3+z^3=0$ by a coordinate transformation (this becomes clear by simply looking at the two surfaces). The blow-up at the origin will have again three charts. Let us use $[x_1 : y_1 : z_1]$ as $\mathbb{P}^2$ coordinates. However, it turns out that in the $x_1=1$ and $y_1=1$ charts, the strict transform is smooth. Thus, consider only the $z_1=1$ chart, where the strict transform becomes $S_1=\mathbb{V}(x_1^2+zy_1^3+z)$. Here, the Jacobian is $J=(2x_1, 3zy_1^2, 1+y_1^3)$, so there are three new singularities (each being ordinary double points) for $1+y_1^3=0$. We will thus need to blow-up once more these three points. Before doing so, let us note that the exceptional divisor E_0 of the first blow-up has local equations $z=0, x_1=0$.

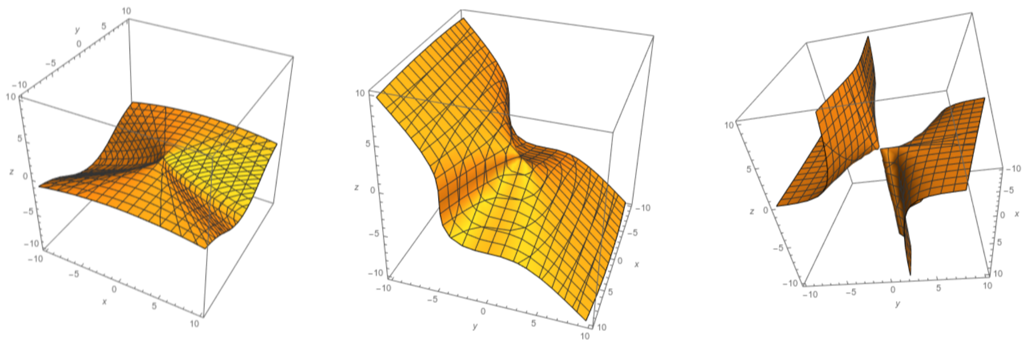

Figure 4: The D_4 singularity for the two equations mentioned in the text and the blow-up of the origin.

We now shift the y_1 coordinate to simplify the problem, such that one of the points of interest is at the origin. The curve we need to resolve is $x_1^2+z(y_1+v_i)^3+z=0$, with v_i one of the three singularities. Since the singularities are ordinary double points, we only need to look at one patch, say $z_2=1$. Here, $y_1=zy_2$, $x_1=zx_2$ and the new exceptional divisor E_i is given by the local equation $x_2^2+3y_2v_i^2=0$. Note that under the mapping (blow-up), the local equation of E_0 is $x_2=0$. Thus, the two curves intersect transversally at a single point. Consequently, we have three (-2) curves, with E_0 intersecting the other three transversally at a single point. This is depicted in Figure 5. The dual of this diagram is precisely the D_4 Dynkin diagram.

Figure 5: The exceptional divisor configuration of D_4 and its dual diagram.

References

[1] Nigel Hitchin. B3.3 Algebraic Curves - Lecture Notes. *University of Oxford*, 2019.

[2] James McKernan. 18.735 - Resolution of Singularities - Lecture Notes. *MIT*, 2007.

[3] Jungkai Alfred Chen. Algebraic Geometry Notes - Chap. 2. *National Taiwan University*, 2014.

[4] Igor R. Shafarevich. In *Basic Algebraic Geomtery 1. Varieties in Projective Space.* Springer, 2013.

[5] Thomas A. Garrity et al. In *Algebraic Geometry - A Problem Solving Approach.* American Mathematical Society, 2013.

[6] Edward Bierstone and Pierre D. Milman. Resolution of Singularities. *MSRI Publications, Volume 37*, 1999.

[7] E. Faber and H. Hauser. Today's Menu: Geometry and Resolution of singular algebraic surfaces. *Bulletin of the American Mathematical Society*, July 2010.

[8] Toni Annala. Resolution of Singularities. *The University of British Colombia.*

[9] Margaret E. Nichols. Intersection Number of Plane Curves. *BA, Oberlin College*, 2013.

[10] Igor Burban. Du Val Singularities - Lecture Notes.

[11] Miles Reid. The Du Val Singularities A_n, D_n, E_6, E_7, E_8.