Comments on the U-plane of 5d SCFTs

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Enumerative invariants, quantum fields and string theory correspondences
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Outline

1. Seiberg-Witten solution and rational elliptic surfaces
2. The E_n 5d SCFTs
3. Modularity and BPS quivers
4. Galois covers, isogenies and orbifolds
5. *5d partition functions: an overview

Seiberg-Witten solution and rational elliptic surfaces

4d $\mathcal{N} = 2$ SQCD

Coulomb branches of 4d $\mathcal{N} = 2$ theories widely studied in the past decades. Consider first SQCD with $G = SU(2)$ gauge group.

- Classical vacua given by vanishing of scalar potential $V = \frac{1}{2}\mathrm{Tr}\,[\bar{\phi}, \phi]^2 \stackrel{!}{=} 0$.
- In such vacua, the scalar field ϕ belongs to the Cartan subalgebra $\mathfrak{h}$ of G:

$$\phi = \begin{pmatrix} a & 0 \\ 0 & -a \end{pmatrix}, \qquad a \in \mathbb{C} \; .$$

- The VEV of ϕ breaks the gauge group to $SU(2) \to U(1)$ by Higgs mechanism, leading to a $U(1)$ LEET. This moduli space is the Coulomb Branch, being parametrized by the gauge invariant operator:

$$u = \langle \mathrm{Tr}\,(\phi^2) \rangle = 2a^2 + \ldots \; .$$

- Note that the weak coupling point is $u \approx \infty$.
- The CB metric is determined by a holomorphic function: $\tau = \frac{\partial^2 \mathcal{F}}{\partial a^2}$.

- Electric-magnetic duality of a U(1) vector multiplet:

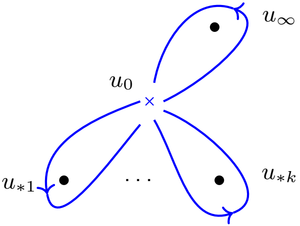
$$\tau \to \mathbb{M}\,\tau \ , \qquad \begin{pmatrix} a_D \\ a \end{pmatrix} \to \mathbb{M} \begin{pmatrix} a_D \\ a \end{pmatrix} \ , \qquad \mathbb{M} \in \mathrm{SL}(2,\mathbb{Z}) \ .$$

- In semi-classical region already, complexified gauge coupling is not single-valued due to one-loop correction. The LEEA is obtained by integrating out the massive degrees of freedom. This description breaks down at the loci where certain BPS states become massless.

These singularities induce monodromies such that:

$$\prod_{\text{sing}} \mathbb{M}_* = 1 \ .$$

Think of u-plane as a $\mathbb{P}^1$ with a distinguished point at $u \approx \infty$.



- Seiberg and Witten proposed that τ should be interpreted as the complex structure modulus of an elliptic curve, such that:

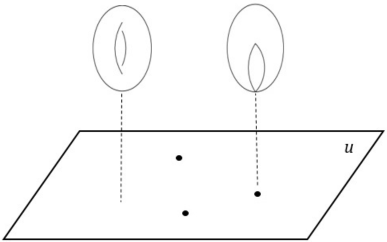
$$a = \oint_A \lambda_{SW} \ , \qquad a_D = \oint_B \lambda_{SW} \ .$$

- One can generally bring the curves to Weierstrass form:

$$y^2 = 4x^3 - g_2(u)x - g_3(u) \ ,$$

with the singularities given be the zeroes of the discriminant locus:

$$\Delta(u) = g_2(u)^3 - 27 g_3(u)^2 \ .$$



The Seiberg-Witten geometry corresponds to the elliptic fibration:

$$E_u \longrightarrow \mathcal{S} \longrightarrow \mathbb{P}^1 \cong \{u\} \ .$$

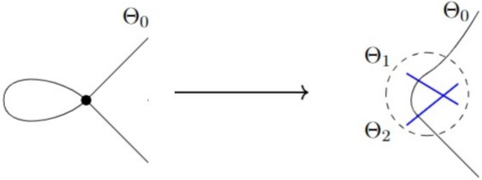
Definition. An elliptic surface is a genus one fibration $f : S \to C$ from a smooth projective surface S to a smooth projective curve C, with a section $\sigma_0 : C \to S$.

- All but finitely many fibres F_v are smooth genus one curves. The singular fibers can be resolved through blow-ups:

$$F_v = \sum_{i=0}^{m_v - 1} \mu_{v,i} \Theta_{v,i} \ ,$$

  where m_v is the number of (distinct) irreducible components, $\Theta_{v,i}$ the irreducible components and $\mu_{v,i}$ the multiplicity of $\Theta_{v,i}$.

- The component denoted by $\Theta_{v,0}$ is the unique component of F_v which intersects the zero section $[\sigma_0]$.



- If F_v is irreducible, then it must be either a rational curve with a node (type I_1), or a rational curve with a cusp (type II).
- All possible reducible fibers have been classified by Kodaira:

fiber	ord(g_2)	ord(g_3)	ord(Δ)	m_v	$\mathbb{M}_*$	$\mathfrak{g}$
I_k	0	0	k	k	T^k	$\mathfrak{su}(k)$
I_k^*	2	3	$k+6$	$k+5$	$-T^k$	$\mathfrak{so}(2k+8)$
I_0^*	≥ 2	≥ 3	6	5	$-\mathbb{I}$	$\mathfrak{so}(8)$
II	≥ 1	1	2	1	$(ST)^{-1}$	-
II^*	≥ 4	5	10	9	(ST)	E_8
III	1	≥ 2	3	2	S^{-1}	$\mathfrak{su}(2)$
III^*	3	≥ 5	9	8	S	E_7
IV	≥ 2	2	4	3	$(ST)^{-2}$	$\mathfrak{su}(3)$
IV^*	≥ 3	4	8	7	$(ST)^2$	E_6

An elliptic surface S is rational if it is birationally equivalent to $\mathbb{P}^2$. In this case, the base curve C is the projective line $\mathbb{P}^1$:

$$E \hookrightarrow S \longrightarrow \overline{\mathcal{M}}_{CB} \cong \mathbb{P}^1 \ ,$$

where $\overline{\mathcal{M}}_{CB}$ is the u-plane with the point at infinity added.

- The configurations of singular fibres of rational elliptic surfaces have been classified by Persson and Miranda. Important constraint:

$$\sum_v \operatorname{ord}(\Delta) = 12 \ .$$

- Singularity at infinity determined by 1-loop β function for $SU(2)$ SQCD with N_f fundamentals:

$$F_\infty = I^*_{4-N_f} \ , \qquad \mathbb{M}_\infty = -T^{4-N_f} \ .$$

Fixing the fibre at infinity

- More generally, the four configurations with only 2 singular fibers must describe all 4d $\mathcal{N}=2$ SCFTs of rank-one: [Caorsi, Cecotti, 2019]

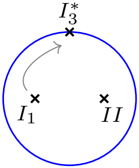
$$(II^*, II)\,, \quad (III^*, III)\,, \quad (IV^*, IV)\,, \quad (I_0^*, I_0^*)\ .$$

with the fiber at infinity chosen accordingly (*e.g.* $F_\infty = II$ for MN E_8).

- Massless dyon of charge (m, q) induces monodromy conjugate to T:

$$\mathbb{M}_{(m,q)} = BTB^{-1} = \begin{pmatrix} 1+mq & q^2 \\ -m^2 & 1-mq \end{pmatrix}\,, \qquad \textit{i.e. an } I_1 \textit{ singularity.}$$

- Persson's list can be used to predict RG flows – *e.g.* for $SU(2)$ $N_f = 1$, we have $F_\infty = I_3^*$ and find (I_3^*, I_1, I_1, I_1) as well as (I_3^*, I_1, II).



'Zooming in' around the II singularity leads to the $(II_\infty^*; II)$ configuration describing the simplest Argyres-Douglas theory. [Argyres, Douglas, 1995]

Q: What about $F_\infty = I_n$??

The E_n 5d SCFTs

Consider Type IIA string theory on $\mathbb{R}^4 \times \tilde{\boldsymbol{X}}$, with $\boldsymbol{X}$ a Calabi-Yau manifold. If $\boldsymbol{X}$ is non-compact, we have a 4d $\mathcal{N}=2$ QFT in the infrared. [Katz, Klemm, Vafa, 1996]

Uplift to M-theory leads to a five-dimensional QFT. Taking a crepant resolution $\tilde{\boldsymbol{X}}$ of a canonical singularity $\boldsymbol{X}$, we are on the Coulomb branch of a 5d SCFT. Thus, we have a 4d $\mathcal{N}=2$ KK theory: [Witten, 1995][Nekrasov, 1996]

$$D_{S^1}\mathcal{T}_{\boldsymbol{X}}^{5d} \text{ on } \mathbb{R}^4 \cong \mathcal{T}_{\boldsymbol{X}}^{5d} \text{ on } \mathbb{R}^4 \times S^1 \ .$$

- Similar to usual 4d $\mathcal{N}=2$ $SU(2)$ gauge theories, we have a low-energy $U(1)$ scalar:

  $$a = i(\varphi + iA_5) \ , \qquad e^{2\pi i A_5} = e^{\int_{S^1} A} \ ,$$

  and define the gauge invariant order parameter: [Nekrasov, 1996][Lawrence, Nekrasov, 1997]

  $$U = \frac{1}{2}\left(e^{2\pi i a} + e^{-2\pi i a}\right) + \dots \ ,$$

  as well as complexified mass parameters: $M = e^{2\pi i(\beta m + iA_5)}$.

The scaling dimension of the gauge coupling in 5d is negative, so action is non-renormalizable. However, as stated before, five-dimensional SCFTs engineered in string theory. For rank-one: [Seiberg, 1996][Morrison, Seiberg, 1996]

$$\tilde{\boldsymbol{X}} = \mathrm{Tot}(\mathcal{K} \to S) \ , \qquad \text{for } S = dP_n \text{ or } \mathbb{F}_0 \ .$$

- The E_n SCFTs are the UV completion of $5d$ $\mathcal{N}=1$ $SU(2)$ gauge theories with $N_f = n-1$ fundamentals. Their flavour symmetry algebra reads:

$$\begin{aligned} &E_0 = \varnothing \ , && E_2 = \mathfrak{su}(2) \oplus \mathfrak{u}(1) \ , && E_5 = \mathfrak{so}(10) \ , \\ &\widetilde{E}_1 = \mathfrak{u}(1) \ , && E_3 = \mathfrak{su}(3) \oplus \mathfrak{su}(2) \ , && E_n = \mathfrak{e}_n \quad (n = 6, 7, 8) \ . \\ &E_1 = \mathfrak{su}(2) \ , && E_4 = \mathfrak{su}(5) \ , && \end{aligned}$$

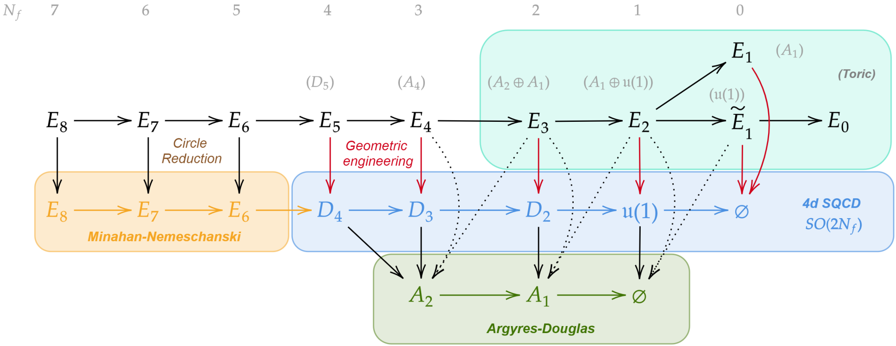
- SW curves for these theories derived in [Ganor, Morrison, Seiberg, 1996][Eguchi, Sakai, 2002] and also discussed in [Huang, Klemm, Poretschkin, 2013].
- From one-loop correction of 4d $\mathcal{N}=2$ KK theories, we have:

$$\mathcal{F} = \mu_0 a^2 + \frac{2}{(2\pi i)^3} Li_3\left(e^{4\pi i a}\right) - \frac{1}{(2\pi i)^3} \sum_{j=1}^{n-1} \sum_{\pm} Li_3\left(M_i e^{\pm 2\pi i a}\right) \Longrightarrow F_\infty = I_{9-n} \ .$$

RG flows

- Using Persson's list we can find some *unexpected* RG flows from the E_n SCFTs to AD theories, by turning on non-trivial holonomies along S^1.

$$5d\ E_n\ SCFT \longrightarrow 5d\ SU(2) + N_f = n-1\ fund.$$



Mordell-Weil Group

An important object used in the classification of rational elliptic surfaces is the *Mordell-Weil group of sections* of the RES.

- (Rational) elliptic surfaces can be thought of as elliptic curves over function fields E/K, with $K = k(U)$. [Schutt, Shioda] On elliptic curve, one can define an additive group law by requiring that three collinear points add to O, the marked point of the elliptic curve.
- The K-rational points of E/K form a group (Mordell-Weil group), which, by the celebrated MW theorem, is finitely generated.

$$\Phi = \mathbb{Z}^r \oplus \Phi_{tor} \ .$$

- Importantly, the K-rational points of E/K are isomorphically mapped to rational sections of the elliptic surface. Under this map, O becomes the zero-section.

- *Claim 1:* Abelian flavour symmetry is $U(1)^r$, with r the rank of the MW group. Proof uses F-theory inspired arguments and local mirror symmetry. [Aspinwall, Morrison, Park, Mayrhofer, Till, Weigand, Cvetic, Lin, Moore, Monnier ...]
- Torsion part of Φ restricts global form of the flavour symmetry. Define:

$$\mathcal{Z}^{[1]} = \{P \in \Phi_{tor} : (P) \text{ intersects } \Theta_{v,0} \text{ for all } F_{v\neq\infty}\} \ ,$$

and denote by $\mathcal{F}$ the cokernel of the inclusion map $\mathcal{Z}^{[1]} \to \Phi_{tor}$. Thus, we have the short exact sequence:

$$0 \to \mathcal{Z}^{[1]} \to \Phi_{tor} \to \mathcal{F} \to 0 \ .$$

Claim: The flavour symmetry group of the theory $\mathcal{T}_{F_\infty}$ is given by:

$$G_F = \widetilde{G}_F/\mathcal{F} \ ,$$

with $\widetilde{G}_F$ the simply connected group with algebra $\mathfrak{g}_F$ from Kodaira fibers.

We find the following results for the 4d $\mathcal{N}=2$ SCFTs:

$\{F_v\}$	$\Phi_{\rm tor}$	*4d theory*	$\mathfrak{g}_F$	G_F
II^*, II	-	AD H_0	-	-
		E_8 MN	E_8	E_8
III^*, III	$\mathbb{Z}_2$	AD H_1	A_1	$SU(2)/\mathbb{Z}_2$
		E_7 MN	E_7	$E_7/\mathbb{Z}_2$
IV^*, IV	$\mathbb{Z}_3$	AD H_2	A_2	$SU(3)/\mathbb{Z}_3$
		E_8 MN	E_6	$E_6/\mathbb{Z}_3$
I_0^*, I_0^*	$\mathbb{Z}_2\times\mathbb{Z}_2$	$SU(2)\,, N_f=4$	D_4	$\mathrm{Spin}(8)/\mathbb{Z}_2\times\mathbb{Z}_2$

New observation for AD H_2, others also discussed recently elsewhere. For the E_n theories, we have: $G_F = E_n/Z(E_n)$, with E_n being the simply connected group with the corresponding Lie algebra. [Closset, Schafer-Nameki, Wang, Del Zotto, García Etxebarria, Hosseini, Bhardwaj, Hubner, Apruzzi, Cordova, Shao, Buican, Nishinaka]

- Given the previous short exact sequence, it is natural to conjecture:

$$\mathcal{Z}^{[1]} \cong \text{ 1-form symmetry of } \mathcal{T}_{F_\infty} \ ,$$

while when Φ_{tor} is a non-trivial extension, we have:

$$\Phi_{tor} \cong \text{ 2-group symmetry of } \mathcal{T}_{F_\infty} \ .$$

- There are not many examples of such theories. In fact, we only have:

$$\begin{aligned}
\text{4d pure } SU(2): &\quad (I_4^*; I_1, I_1), && \Phi = \mathcal{Z}^{[1]} = \mathbb{Z}_2 \ , \\
D_{S^1}E_0: &\quad (I_9; I_1, I_1, I_1), && \Phi = \mathcal{Z}^{[1]} = \mathbb{Z}_3 \ , \\
D_{S^1}E_1: &\quad (I_8; I_2, I_1, I_1), && \Phi = \mathbb{Z}_4 \ , \mathcal{Z}^{[1]} = \mathbb{Z}_2 \ , \\
&\quad (I_8; I_1, I_1, I_1, I_1), && \Phi_{tor} = \mathcal{Z}^{[1]} = \mathbb{Z}_2 \ ,
\end{aligned}$$

which agree with known results. [Gaiotto, Kapustin, Seiberg, Willet 2014; Morrison, Schafer-Nameki, Willet 2020; Albertini, Del Zotto, García Etxebarria, Hosseini 2020]

Modularity and BPS quivers

Modularity

- In certain cases, the configuration of interest is modular. That is, the rational elliptic surface is constructed from the quotient $\mathbb{H}/\Gamma$ with $\mathbb{H}$ the upper half-plane and $\Gamma \in \mathrm{PSL}(2,\mathbb{Z})$, together with a finite number of points. [Aspman, Furer, Manschot]
- Subgroups of $\mathrm{PSL}(2,\mathbb{Z})$ have a set of special points: elliptic points - points with non-trivial stabilizer, and cusps - points in the $\mathrm{PSL}(2,\mathbb{Z})$ orbit of $\tau = i\infty$. These points are mapped to the U-plane singularities. [Shioda, 1972]
- When RES is modular, $U = U(\tau)$ is a modular function for Γ and periods are modular forms. Can also map U-plane to upper half-plane isomorphically: [Aspman, Furrer, Manschot 2020-2021; HM, Closset 2021; HM 2022][Alim, Scheidegger, Yau, Zhou, 2013]

$$\mathcal{F}_\Gamma = \bigsqcup \alpha_i \mathcal{F}_0 \ ,$$

with $\{\alpha_i\}$ a set of coset representatives and $\mathcal{F}_0$ the fundamental domain of $\mathrm{PSL}(2,\mathbb{Z})$.

Advantage is that upper-half plane knows about *monodromies*!

Recall that monodromy induced by dyon of charge (m, q) becoming massless is:

$$\mathbb{M}_{(m,q)} = BTB^{-1} = \begin{pmatrix} 1+mq & q^2 \\ -m^2 & 1-mq \end{pmatrix} .$$

- Given a basis of BPS states with magnetic-electric charges $\gamma_i = (m_i, q_i)$, the BPS quiver is obtained by assigning a quiver node (i) to each light dyon, and a (effective) number of n_{ij} arrows from node (i) to (j) given by the Dirac pairing:

$$n_{ij} = \langle \gamma_i, \gamma_j \rangle = m_i q_j - q_i m_j \ .$$

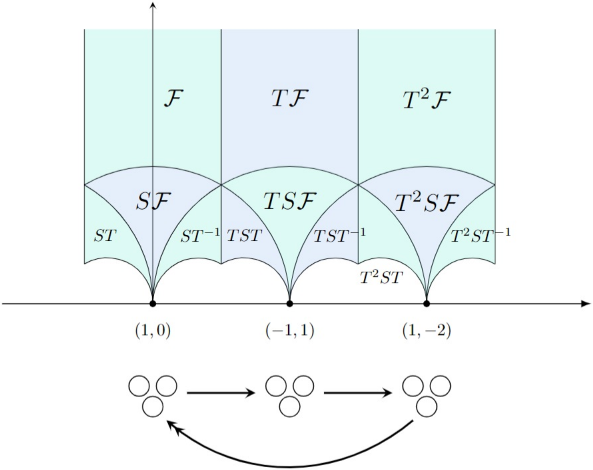
- We then have the following correspondence:

$$I_1 \text{ cusp at } \tau = \frac{q}{m} \in \mathbb{Q} \quad \longleftrightarrow \quad \text{BPS particle of charge } \pm(m, -q) \ .$$

- In practice, one first needs to check existence of quiver points! Note: modular configurations are 'rare' but classification is possible. [HM, 2022]

$D_{S^1}\mathcal{T}^{5d}$	E_7	E_6	E_5	E_4	E_3	E_1	E_0
Γ	$\Gamma^0(2)$	$\Gamma^0(3)$	$\Gamma^0(4)$	$\Gamma^1(5)$	$\Gamma^0(6)$	$\Gamma^0(8)$	$\Gamma^0(9)$

Example: a certain configuration of $D_{S^1}E_6$ with $(I_3^\infty; 3I_3)$ with monodromy group $\Gamma(3)$.



Galois covers, isogenies and orbifolds

Torsion sections of the Mordell-Weil generate automorphisms of the RES $\langle t_P \rangle$. In fact, the automorphism group of a rational elliptic surface $\mathcal{S}$ is isomorphic to:

$$\mathrm{Aut}(\mathcal{S}) = \mathrm{MW}(\mathcal{S}) \rtimes \mathrm{Aut}_\sigma(\mathcal{S}) \ ,$$

where $\mathrm{Aut}_\sigma(\mathcal{S})$ is the subgroup of automorphisms preserving the zero section.

- One can then consider quotients $\mathcal{S}/\langle t_P \rangle$ by such automorphisms. For 4d $\mathcal{N} = 2$ theories, quotients by subgroups of $\mathrm{Aut}_\sigma(\mathcal{S})$ are equivalent to discrete gaugings. [Caorsi, Cecotti, 2019] [Argyres, Martone]
- Meanwhile, quotients by automorphisms generated by a torsion section of the Mordell-Weil group lead to isogenous elliptic surfaces. An m-isogeny between two elliptic curves is a non-constant morphism $\varphi_m : E \to \widetilde{E}$ accompanied by a unique dual isogeny $\hat{\varphi}_m : \widetilde{E} \to E$ such that:

$$\varphi_m \circ \hat{\varphi}_m = m\widetilde{E} \ , \qquad\qquad \hat{\varphi}_m \circ \varphi_m = mE \ .$$

- These turn out to be related to the concept of Galois covers. [Cecotti, del Zotto, 2015]

Galois Covers

- BPS states form a category that is isomorphic to the (bounded) derived category of $\mathcal{J}$-modules, with $\mathcal{J}$ is the Jacobian algebra: *e.g.* [Closset, del Zotto, 2019]

$$\mathcal{J} = \mathbb{C}Q/(\partial W) \ .$$

- For $(Q, \mathcal{W})$ and $\mathbb{G}$ a group of $\mathbb{C}$-linear automorphisms of $\mathcal{A} = \mathbb{C}Q/(\partial W)$ acting freely on the objects of $\mathcal{A}$, then there exists a quiver $(\widetilde{Q}, \widetilde{\mathcal{W}})$ whose category of quiver representations is the *orbit category*: [Cecotti, del Zotto, 2015]

$$\mathcal{A}/\mathbb{G} = \mathbb{C}\widetilde{Q}/(\partial\widetilde{\mathcal{W}}) = \Big(\mathbb{C}Q/(\partial\mathcal{W})\Big)/\mathbb{G} \ .$$

- The *Galois covering* functor:

$$F : \mathcal{A} \longrightarrow \mathcal{A}/\mathbb{G} \ ,$$

take the objects of $i \in \mathcal{A}$, to $i \mapsto \mathbb{G}i \in \mathcal{A}/\mathbb{G}$. Moreover, denoting by $\mathcal{A}(i,j)$ the morphism spaces between nodes i and j of the quiver Q, the morphisms of the orbit category $\mathcal{B} = \mathcal{A}/\mathbb{G}$ are given by:

$$\mathcal{B}(\mathbb{G}i, \mathbb{G}j) = \bigoplus_{g\in\mathbb{G}} \mathcal{A}(i, gj) \ .$$

Simplest example involves 4d SQCD theories – pure $SU(2)$ theory $(\Gamma^0(4))$ and the massless $SU(2)$ $N_f = 2$ theory $(\Gamma(2))$, with $\mathbb{G} = \mathbb{Z}_2$:

$$\mathcal{E}_{\tilde{\gamma}_1} \twoheadrightarrow \mathcal{E}_{\tilde{\gamma}_2} \qquad\qquad \begin{tikzcd} \mathcal{E}_{\gamma_1} \arrow[r] \arrow[dr] \arrow[d, leftrightarrow, dashed, bend right] & \mathcal{E}_{\gamma_2} \arrow[d, leftrightarrow, dashed, bend left] \\ \mathcal{E}_{\gamma_3} \arrow[r] \arrow[ur] & \mathcal{E}_{\gamma_4} \end{tikzcd}$$

- BPS spectrum consists of the subset of the BPS category that satisfy certain stability conditions. Thus, if central charges are mapped by the Galois covering functor, BPS spectra are as well! Here:

$$\tilde{Z}_{\gamma_1}, \tilde{Z}_{\gamma_2} \qquad \longleftrightarrow \qquad Z_{\gamma_1} = Z_{\gamma_3},\ Z_{\gamma_2} = Z_{\gamma_4}\ .$$

$\boldsymbol{N_f = 0}$	$\boldsymbol{N_f = 2}$
$\tilde{\gamma}_i + n(\tilde{\gamma}_1 + \tilde{\gamma}_2)$ $-\tilde{\gamma}_i + (n+1)(\tilde{\gamma}_1 + \tilde{\gamma}_2)$	$\gamma_r + n(\gamma_1 + \gamma_2 + \gamma_3 + \gamma_4)$ $-\gamma_r + (n+1)(\gamma_1 + \gamma_2 + \gamma_3 + \gamma_4)$
$i = 1, 2$	$r = 1, 2, 3, 4$

The SW solution is essentially local mirror symmetry: [Katz, Mayr, Vafa, 1996] That is, the CB of $D_{S^1}\mathcal{T}^{5d}_{\boldsymbol{X}}$ is described by:

$$\text{IIA on } \mathbb{R}^4 \times \tilde{\boldsymbol{X}} \longleftrightarrow \text{ IIB on } \mathbb{R}^4 \times \hat{\boldsymbol{Y}} \ .$$

- For the toric geometries, the mirror is given by the Hori-Vafa construction as a hypersurface in $\mathbb{C}^2 \times (\mathbb{C}^*)^2$, or, alternatively, as a double fibration over a complex plane:

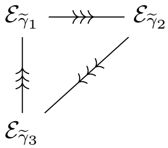
  $$E \times \mathbb{C}^* \to \hat{\boldsymbol{Y}} \to \mathbb{C} \cong \{W\} \ ,$$

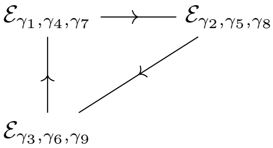
  with elliptic fiber $F(w, t; W) = 0$ and $v_1 v_2 = U - W$, for $(v_1, v_2) \in \mathbb{C}^2$, $(w, t) \in (\mathbb{C}^*)^2$. The SW curve is essentially the Newton polygon of the toric diagram with:

  $$F(w, t; U) = 0 \ .$$

- Q: What is the interpretation of these isogenies on the IIA side?

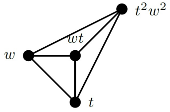
For the KK theories, whose BPS spectra are much more complicated, we have the following example – $D_{S^1}E_0$ $(\Gamma^0(9))$ and $D_{S^1}E_6$ $(\Gamma(3))$ quivers:

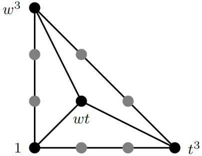




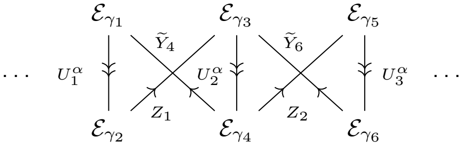
- In IIA, the geometries engineering these quivers are the orbifolds $\mathbb{C}^3/\mathbb{Z}_3$ with action $(1,1,1)$, and $\mathbb{C}^3/(\mathbb{Z}_3\times\mathbb{Z}_3)$ with action $(1,0,2)(0,1,2)$. The latter is a toric deformation of the local dP_6. Can in fact check that the following curves are 3-isogenous:

$$\tilde{t}+\tilde{w}-U\tilde{t}\tilde{w}+t^2w^2=0\ ,\qquad t^3+w^3-Utw-1=0\ .$$





- Previous results suggest that orbifolds in IIA lead to Galois covers, which we can also check for higher-rank theories. For $\mathbb{C}^3$ orbifolds, this can be checked explicitly. [Lawrence, Nekrasov, Vafa, 1998]
- Can generate other Galois covers using threefolds obtained from cones over the $Y^{p,q}$ Sasaki-Einstein manifolds. [Gauntlett, Martelli, Sparks, Waldram, 2004] For instance, the $Y^{2k,0}$ models generate $\mathbb{Z}_k$ covers for local $\mathbb{F}_0$:



with the identification $l \sim l + 2k$ for the nodes, together with $U_1^\alpha \sim U_k^\alpha$, $Z_l \sim Z_{k+1}$ and $\tilde{Y}_{2l} \sim \tilde{Y}_{2l+2k}$. Superpotential given by: [Closset, del Zotto, 2019]

$$\mathcal{W}_k = \sum_{l=1}^{2k} \epsilon_{\alpha\beta} U_l^\alpha Z_l U_{l+1}^\beta \tilde{Y}_{2l+2} \ .$$

We also have the 5d uplift of the 4d $SU(2)$ $N_f = 0$ - $N_f = 2$ Galois cover:

$$
\begin{tikzcd}
\mathcal{E}_{\widetilde{\gamma}_1} \arrow[r, two heads] & \mathcal{E}_{\widetilde{\gamma}_2} \arrow[d, two heads] \\
\mathcal{E}_{\widetilde{\gamma}_4} \arrow[u, two heads] & \mathcal{E}_{\widetilde{\gamma}_3} \arrow[l, two heads]
\end{tikzcd}
\qquad\qquad
\begin{tikzcd}
\mathcal{E}_{\gamma_1,\gamma_5} \arrow[r] & \mathcal{E}_{\gamma_2,\gamma_6} \arrow[d] \\
\mathcal{E}_{\gamma_4,\gamma_8} \arrow[u] & \mathcal{E}_{\gamma_3,\gamma_7} \arrow[l]
\end{tikzcd}
$$

In some particular chambers of $D_{S^1}E_1$ and $D_{S^1}E_5$ quivers, we have BPS spectra: [del Monte, Longhi, 2021]

$D_{S^1}E_1$	$D_{S^1}E_5$
$\widetilde{\gamma}_1 + k(\widetilde{\gamma}_1 + \widetilde{\gamma}_2)$ $-\widetilde{\gamma}_1 + (k+1)(\widetilde{\gamma}_1 + \widetilde{\gamma}_2)$	$\gamma_r + k\sum\gamma_r$ $-\gamma_r + (k+1)\sum\gamma_r$
$\widetilde{\gamma}_3 + k(\widetilde{\gamma}_3 + \widetilde{\gamma}_4)$ $-\widetilde{\gamma}_3 + (k+1)(\widetilde{\gamma}_3 + \widetilde{\gamma}_4)$	$\gamma_s + k\sum\gamma_s$ $-\gamma_s + (k+1)\sum\gamma_s$
$\widetilde{\gamma}_1 + \widetilde{\gamma}_2 + k\widetilde{\gamma}_{D0}$ $-\widetilde{\gamma}_1 - \widetilde{\gamma}_2 + (k+1)\widetilde{\gamma}_{D0}$	$\gamma_a + \gamma_b + k\gamma_{D0}$ $-\gamma_a - \gamma_b + (k+1)\gamma_{D0}$ $\sum\gamma_r + k\gamma_{D0}$ $-\sum\gamma_r + (k+1)\gamma_{D0}$
$k\widetilde{\gamma}_{D0}$	$k\gamma_{D0}$

Conclusions and Outlook

- (Rank-one) Seiberg-Witten geometries have a natural interpretation as rational elliptic surfaces, where RG flows are 'trivial'. Non-abelian part of flavour symmetry algebra encoded in singular fibers, while abelian part and global aspects encoded in MW group.
- Automorphisms of the elliptic surfaces can lead to interesting physical results: discrete gaugings, Galois covers, but not only these!
- Classification of 4d $\mathcal{N} = 2$ SCFTs involves theories with undeformable singularities. RG flows, flavour symmetry algebra still doable from SW geometry, but what about global aspects?
- Modularity allows one to determine light BPS states and BPS quivers, but also shed some light on BPS spectra. Can we find quiver super-potentials from the U-plane? Do fundamental domains retain information about monodromies even when configurations are not modular?

Thank you!

*5d partition functions: an overview