$5d$ Seiberg-Witten Geometry

Horia Magureanu

University of Oxford

October 28, 2024

Outline

$4d$ Review

[Tachikawa '13, Martone '20, Closset '20]

- Vector Multiplet: $(\phi, \lambda^I, A_\mu, D^{IJ})$
- Hypermultiplet: (q_I, η, χ, F_I)

$$\mathcal{L}_{SYM} = Im\left(-\frac{\tau}{8\pi}\int d^2\theta \mathcal{W}^\alpha \mathcal{W}_\alpha\right) + \frac{1}{g^2}\int d^2\theta d^2\bar{\theta} Tr\left(\bar{\Phi} e^{-2V}\Phi\right),$$

$$\mathcal{L}_{mat} = \int d^2\theta d^2\bar{\theta}\left(Q^\dagger e^{-2V} Q\right) + \int d^2\theta \mathcal{W}(Q, \widetilde{Q}, \Phi) + cc.$$

- Classical vaccum moduli space: $\left[\phi, \bar{\phi}\right] = 0$ for pure SYM.
- ϕ belongs to Cartan subalgebra, e.g. for $G = SU(2)$:

$$\phi = \begin{pmatrix} a & 0 \\ 0 & -a \end{pmatrix}.$$

- The *Coulomb Branch*: $\mathcal{M}_C = \mathfrak{h}_{\mathbb{C}}/W_G$ is parametrized by the gauge invariant $u = Tr\phi = 2a^2$ for $G = SU(2)$, where $W_{SU(2)} = \mathbb{Z}_2$.

Coulomb Branch

- The VEV of ϕ breaks the gauge group by the Higgs mechanism:

$$G \longrightarrow U(1)^r.$$

- Perturbative massive vector field, the W-boson: $M_W = 2|a|$.
- Monopoles exist in classical theory, with energies $M \propto \tau a$ for $\theta \neq 0$. Classical particles obey: $M \geq |na + (2\tau a)m + \sum f_i \mu_i|$.
- In quantum theory, the 'equivalent' inequality is: $M \geq |Z|$ so define:

$$Z = na(u) + ma_D(u) + \sum f_i \mu_i.$$

- $U(1)^r$ EFT fully described by holomorphic function - *prepotential*:

$$\tau = \frac{\partial^2 \mathcal{F}}{\partial a^2}, \quad a_D = \frac{\partial \mathcal{F}}{\partial a}.$$

Seiberg-Witten Geometry

- The prepotential receives quantum corrections:

$$\mathcal{F}=\frac{1}{2}\tau_0 a^2+\frac{i}{\pi}a^2 Log\left(\frac{a^2}{\Lambda^2}\right)+\frac{a^2}{2\pi i}\sum_{k=1} c_k\left(\frac{\Lambda}{a}\right)^{4k}.$$

- Electro-magnetic duality: $\tau \longrightarrow \frac{a\tau+b}{c\tau+d}$ is reminiscent of the $SL(2,\mathbb{Z})$ action on T^2 modular parameter. The Seiberg-Witten curve is an elliptic curve: $y^2=x^3-g_2(u)x-g_3(u)$ which encodes information about the physical theory:

$$a=\oint_A \lambda_{SW}, \quad a_D=\oint_B \lambda_{SW},$$

- The SW differential: $\partial\lambda_{SW}/\partial u=dx/y$ plays a central role, giving the periods, as well as the flavour charges $(Res(\lambda_{sw}))$. The monodromies are relevant for determining the BPS spectrum.

$4d$ $N=2$ pure $SU(2)$ theory

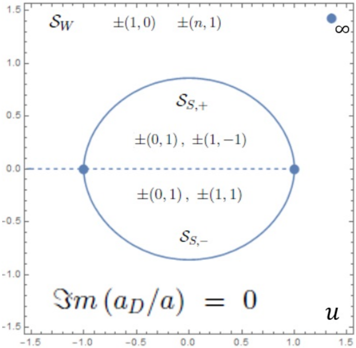
- The SW curve is given by: $y^2=(x^2-u)^2-\Lambda^4$. [Seiberg, Witten '94]. Periods of elliptic curves satisfy an 'universal' Picard-Fuchs equation [Klemm, Lian, Roan, Yau '94]. The pure $SU(2)$ periods are [Bilal, Ferrari '96] :

$$a_D(u)=\frac{i(u-1)}{2}\,{}_2F_1\left(\tfrac{1}{2},\tfrac{1}{2};2;\tfrac{1-u}{2}\right),$$

$$a(u)=\sqrt{2(u+1)}\,{}_2F_1\left(-\tfrac{1}{2},\tfrac{1}{2};1;\tfrac{2}{u+1}\right).$$

- Spectrum determined by:

$$\mathcal{S}_W=M_\infty\mathcal{S}_W,\quad [M_\infty,G_{\mathbb{Z}_2}]=0.$$



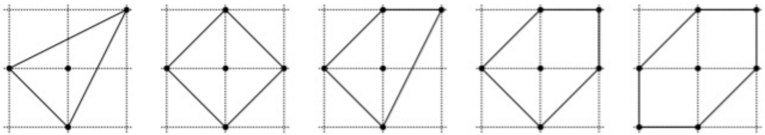
$5d$ $\mathcal{N}=1$ theory on S^1

- $5d$ theory on a circle can be viewed as a $4d$ theory with an infinite number of fields (KK modes). One loop and instanton contributions aquire thus an infinite sum which can be regulated by considering the fourth derivative [Nekrasov '97].The result for pure $SU(2)$ is:

$$\mathcal{F}^{S^1}_{1-loop} = \frac{1}{(2\pi i)^3}\left(Li_3\left(e^{-4\pi i\widetilde{u}}\right) + Li_3\left(e^{4\pi i\widetilde{u}}\right)\right),$$

- The $4d$ limit is obtained for $\widetilde{u} = \beta a$, $\beta \to 0$.
- This argument is not entirely applicable to the instanton contributions, but somewhat believed to be true.
- We will see that these trilogs do appear in the instanton contribution to the prepotential of the KK theory.

$5d$ SCFTs



(a) E_0. (b) E_1. (c) $\widetilde{E}_1$. (d) E_2. (e) E_3.

- Calabi-Yau condition implies that the the toric vectors ar co-planar. The Kähler moduli space of the E_0 geometry (collapse of $dP_0 = \mathbb{P}^2$ inside a non-compact CY_3) can be thought of as the moduli space of deformations of the $\mathbb{C}^3/\mathbb{Z}_3$ orbifold. [Aspinwall, Greene, Morrison, Diaconescu, Gomis].This naturally leads to *fractional branes*.
- $\mathcal{N}=1$ LEEFT on the Coulomb branch is determined by a prepotential which can be obtained from the geometry [Intrilligator, Morrison, Seiberg '97]:

$$\mathcal{F}_{E_0} = \frac{3}{2}\phi^3, \quad \mathcal{F}_{E_1} = h_0\phi^2 + \frac{4}{3}\phi^3.$$

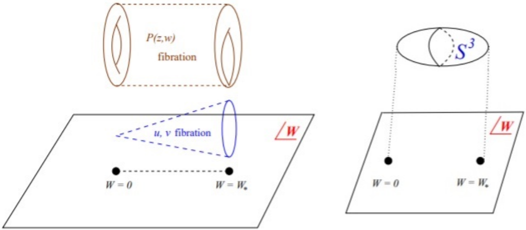
Local Mirror Symmetry

- $4d$ pure $SU(2)$ theory can be obtained from IIA compactification on non-compact CY_3 with A_1 singularity, i.e. a vanishing $\mathbb{P}^1_f$ fibered over another $\mathbb{P}^1_b$. This *Local A model* depends on Kähler moduli t_b, t_f and thus receives worldsheet instanton corrections.
- Local mirror symmetry can be used to avoid these corrections in the *local B model*. The data of this local geometry gives rise to a Riemann surface embedded inside the 'mirror' CY_3, together with a meromorphic $1-$form. The quantum corrected Kähler moduli are given by the periods of this meromorphic form. [Batyrev, Borisov].
- Consider the SCFTs on a finite-size circle, leading to a KK theory. It is possible to describe these using a smooth brane configuration in M-theory. [Kol '97, Brandhuber et al '97].The mirror geometry of **X** is given by a local threefold specified by: [Hori, Vafa '00, Hori, Vafa, Iqbal '00].

$$W = P(z, w) := \sum_{(p,q)\in\Delta} c_{(p,q)} z^p w^q, \quad W = uv.$$

Setup

- These can be seen as a T^3 fibration (locally), reminiscent of the SYZ conjecture for compact mirror CYs. A particularly important limit in these geometries is the 'large radius limit' of the Calabi-Yau, where the α'-corrections are supressed and the classical analysis of D-branes holds.



The Seiberg-Witten curve is given by:

$$P(z,w)=0.$$

- The D-brane charges can be determined using anomaly inflow arguments [Green, Harvey, Moore '96]. These enter the central charge expression and thus give an asymptotic behaviour for the periods.

M-theory Setup

- Our setup is essentially a five-dimensional uplift of the setup studied in [Witten '97]. The $5d$ SCFTs can be described in terms of (p,q) webs in IIB, which have a T-dual description in IIA, in terms of $NS5$ and $D4$ branes. These become a single smooth $M5$-brane in the shape of a curve, that we identify with the Seiberg-Witten curve.

	x^0	x^1	x^2	x^3	$\boldsymbol{x}^4$	x^5	x^6	x^7	x^8	x^9	$\boldsymbol{x}^{10}$
D4	$\times$	$\times$	$\times$	$\times$	$\cdot$	$\cdot$	Δx	$\cdot$	$\cdot$	$\cdot$	
NS5	$\times$	$\times$	$\times$	$\times$	$\times$	$\times$	$\cdot$	$\cdot$	$\cdot$	$\cdot$	
M5	$\times$	$\times$	$\times$	$\times$	$\longleftarrow$	$\longleftrightarrow$	$\longrightarrow$	$\cdot$	$\cdot$	$\cdot$	$\leftrightarrow$

- The x^4, x^{10} directions are periodic so introduce single valued coordinates (in $4d$ limit $w \to v$ [Kim, Yagi '14]):

$$w = e^{iRv}, \quad z = e^{-\frac{x^6+ix^{10}}{R_M}}, \quad \text{with } v = \frac{x^4+ix^5}{\alpha'}$$

E_0 Periods

- The mirror curve for the E_0 theory is:

$$P(t,w) = c_1 w + t(c_0 w + c_3) + c_2 t^2 w^2 = 0.$$

- One complex structure parameter: $z = c_1 c_2 c_3 / c_0^3$. Choose a parametrization: $c_1 = c_2 = c_3 = 1$, $c_0 = -3U$. We can then put the above curve in Weierstrass form, to find the discriminant:

$$\Delta = g_2(U)^3 - 27 g_3(U)^2 = 27(U^3 - 1).$$

- The discriminant gives us the U-plane singularities: ω^n, $n = 0, 1, 2$, $\omega = e^{2\pi i/3}$. The (toric) geometry also allows us to obtain the Picard-Fuchs equation satisfied by the periods:

$$\left[\theta_z^3 - z\left(\theta_z + \frac{1}{3}\right)\left(\theta_z + \frac{2}{3}\right)\theta_z\right]\Pi(z) = 0$$

E_0 Periods

- This equation has a trivial solution: $\Pi(z) = const$. To find the other solutions, let $\omega(z) = \theta_z \Pi(z)$, in which case the differential equation becomes a standard hypergeometric equation. The solutions are *zero-balanced* hypergeometric functions:

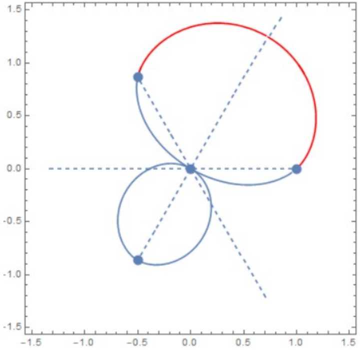
$$\omega_1 = \frac{1}{2\pi i}\, {}_2F_1\left(\frac{1}{3}, \frac{2}{3}; 1; z\right), \quad \omega_2 = \frac{1}{2\pi\sqrt{3}} F_1\left(\frac{1}{3}, \frac{2}{3}; 1; 1-z\right).$$

- Analytic continuation can be done using the Gauss-Ramanujan identity for balanced hypergeometric functions.
- Keeping track of integration constants, we can find a convenient basis for the actual periods $(1, \Pi_{D2}(z), \Pi_{D4}(z))$ [Aspinwall]. What about the U-plane periods? This is a non-trivial map: $z = -\frac{1}{27U^3} \longrightarrow U$.

U-plane Singularities

- The E_0 theory has three 'conifold' singularities and one large-volume singularity in the U-plane. In the z-plane, the $z \to \infty$ is the orbifold point (recall that the local CY_3 is $\mathbb{C}^3/\mathbb{Z}_3$), and thus the states becoming massless at the 'conifold' singularities are the three *fractional branes*.

- We can try to determine the spectrum using the monodromies and this $\mathbb{Z}_3$ symmetry, as in the case of $4d$ theories. This is more challenging as $[M_\infty, G_{\mathbb{Z}_3}] \neq 0$. Π-stability conditions already studied in the literature. [Douglas, Fiol, Romelsberger '00].



E_1 Periods

- The E_1 mirror geometry involves two complex structure parameters $z_b = e^{2\pi i T_b}$, $z_f = e^{2\pi i T_f}$, with an obvious $\mathbb{Z}_2$ fiber-base duality, as the collapsing $4-$cycle is $\mathbb{P}^1 \times \mathbb{P}^1$. There are now two parameters in the theory: $\lambda = z_b/z_f$ and $U^2 = z_f/4$. The vanishing locus of the discriminant is:

$$U = \pm 1 \pm \lambda.$$

- The theory has two second order Picard-Fuchs equations which can be again reduced to a single second order PDE for fixed λ. Of particular importance are the cases $\lambda^2 = 1$: where two singularities collide at the origin and $\lambda^2 = -1$: which shows a $\mathbb{Z}_4$ quantum symmetry.
- In these two cases, the geometric periods can be written in terms of zero-balanced hypergeometric functions with argument: $Z = \frac{4}{U^2}$ and $Z = -\frac{4}{U^4}$ respectively.

E_1 Prepotential

- The periods are given by:

$$\Pi_{f,b} = \frac{1}{2\pi i} Log(z_{f,b}) + \mathcal{O}(z),$$
$$\Pi_D = \Pi_b(U)\Pi_f(U) + \frac{1}{6} + \mathcal{O}(z)$$

where $\mathcal{O}(z)$ represents the quantum corrections. These are supressed in the $5d$ limit, obtained by identifying: $U = e^{-2\beta\phi}, \lambda = e^{\beta h_0}$. The classical contribution to the prepotential is:

$$\mathcal{F} = \frac{(2\pi i)^3}{2} \int T_D(U) dT_f \longrightarrow \mathcal{F}_{5d} = \lim_{\beta\to\infty} \frac{1}{\beta^3}\mathcal{F} = \frac{4}{3}\phi^3 + h_0\phi^2.$$

- Perhaps more interesting are the quantum corrections:

$$\mathcal{F} = 2Li_3(Q_f) + 2Li_3(Q_b) + 4Li_3(Q_fQ_b) + 6Li_3(Q_f^2Q_b) + 6Li_3(Q_fQ_b^2)\ldots$$

E_1 Prepotential

- These agree with a similar computation done in [Huang, Klemm '13]. We can then try to take a $4d$ limit to reproduce the pure $SU(2)$ theory. We argue that the correct limit is:

$$Q_f \sim e^{\beta \widetilde{a}}, \quad Q_b \sim \beta^4 \Lambda^4$$

- This double scaling limit similar to what was suggested in [Katz, Klemm, Vafa '96], includes the *quantum corrected periods since* $Q = Exp(2\pi i \Pi)$. The non-trivial argument needed for this computation is an assumption on the growth of the instanton contributions:

$$d_{n,m} \sim \gamma_n m^{4n-3}.$$

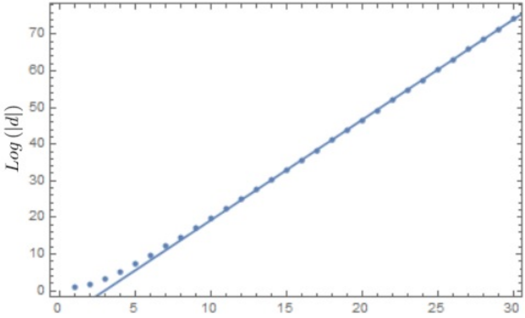
- In this way, we exactly reproduce the instanton corrections to the four dimensional prepotential of the pure $SU(2)$ theory!

E_0 Prepotential

- Performing a similar analysis for E_0, we find:

$$\mathcal{F} = \mathcal{F}_{class} + 3Li_3(Q) - 6Li_3(Q^2) + 27Li_3(Q^3) - 192Li_3(Q^4)\ldots$$

- We computed these corrections up to 30^{th} instanton. The growth of these numbers seems to be exponential.



Summary and Outlook

- $5d$ Seiberg-Witten geometry can be determined using local mirror symmetry or in analogy to $4d$ theories from M-theory.
- The SW curves can be used to determine the prepotential and the BPS spectrum. Effects of the 1-form symmetries of the $5d$ theories are reflected in the U-plane [Morrison, Schafer-Nameki, Willet '20, Albertini, del Zotto, Etxebarria, Hosseini '20].
- There are some interesting limits one can consider. Can we say more about the circle reduction of the $5d$ E_n SCFTs?
- Argyres-Douglas limits: [Closset, del Zotto '20, Bonelli, del Monte, Tanzini '20]

Thank you for your attention!