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The derived category of coherent sheaves

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1. Introduction

The concept of mirror symmetry appeared in the 1980's in the context of string theory compactifications, with the first precise example developed in the work of Candelas, de la Ossa, Green and Parkes in 1991. This concept involves the relation between the so-called A and B models, which are topologically twisted $\mathcal{N} = 2$ sigma models with Calabi-Yau target spaces. In 1994, Kontsevich proposed a mathematically rigorous framework for this symmetry, which broadened this picture.

In this conjectured symmetry, a complex algebraic manifold Y is said to be the 'mirror' dual of a given Calabi-Yau manifold X if the *bounded derived category of coherent sheaves on Y* is equivalent to the bounded derived category constructed from the *Fukaya category of X*. The reason why just the Fukaya category cannot be equivalent to the derived category of coherent sheaves is the lack of a triangulated structure in the former. In this review we will briefly describe the first of these notions, without making any attempts in understanding the equivalence between the two inherently different categories. We define derived categories and sheaves, leading up to the relevant description of 'B-branes'. We will also briefly touch on the physics of the B-model, but will leave out the details and proofs of some of the statements.

Our main references will be [1, 2] for the physics reviews, as well as [3] for the more formal aspects.

2. Derived Categories

The aim of this section is to introduce the derived category of an abelian category and study its structure. We do this by first inverting quasi-isomorphisms and then present a more suitable description of the derived category in terms of the homotopy category.

2.1 Quasi-isomorphisms

The first instance of the 'need' of the derived category of an abelian category appears when discussing chain complexes. For the rest of this section let $\mathcal{A}$ be an abelian category, with $Ch(\mathcal{A})$ the category of cochain complexes in $\mathcal{A}$. In this category, the morphisms are morphisms of complexes, which must also commute with the 'differentials' of the chain complexes. Given such a morphism between two cochain complexes $f : a^\bullet \to b^\bullet$, recall that there is an induced homomorphism on cohomology:

$$f_* : H^n(a^\bullet) \to H^n(b^\bullet)\ . \tag{2.1}$$

If this induced homomorphism is an isomorphism, then f is called a *quasi-isomorphism*. The question that arises is whether there exists a category of cochain complexes in which quasi-isomorphic complexes are isomorphic. This is in fact the *derived category* and we have the following lemma.

Lemma 2.1. *There exists a category $D(\mathcal{A})$ and a functor $Q : Ch(\mathcal{A}) \to D(\mathcal{A})$ called the localisation functor, such that:*

1. *Q inverts quasi-isomorphisms, i.e. if $s : a \to b$ is a quasi-isomoprhism, then $Q(s) : Q(a) \to Q(b)$ is an isomorphism.*
2. *Q is universal with this property, i.e. if $Q' : Ch(\mathcal{A}) \to D'$ is another functor inverting quasi-isomorphisms, then there exists a functor $F : D(A) \to D'$, such that $Q' \cong F \circ Q$.*

Proof. It is intructive to prove the first part of the statement. We follow the exposition of [1] and view $Ch(\mathcal{A})$ as an oriented graph Γ, with the cochain complexes lying at the vertices of the graph and the morphisms being the directed edges. Then, we construct the graph Γ_* by adding additional edges $s^{-1} : b \to a$ for each quasi-isomorphism $s : a \to b$ in $Ch(\mathcal{A})$. There then exists a unique equivalence relation $\sim$ on the set of finite paths in Γ_* generated by the following relations.

1. $s \cdot s^{-1} \sim id_b$ and $s^{-1} \cdot s \sim id_a$ for each quasi-isomorphism $s : a \to b$ in $Ch(\mathcal{A})$.
2. $g \cdot f \sim g \circ f$ for composable morphisms $f : a \to b$ and $g : b \to c$ of $Ch(\mathcal{A})$.

Note that these equivalence classes of finite paths in Γ are morphisms in $Ch(\mathcal{A})$; we define the category $D(\mathcal{A})$ as the category whose objects are the vertices of Γ_* and whose morphisms are the equivalence classes of finite paths in Γ_*. The functor $Q : Ch(\mathcal{A}) \to D(\mathcal{A})$ can be thus defined by sending morphisms f of $Ch(\mathcal{A})$ to length one paths in Γ_*. $\square$

While this construction proves the existence of the derived category, it is not of much use in practice. For this, we define the *homotopy category* $H(\mathcal{A})$, obtained from $Ch(\mathcal{A})$ by identifying cochain homotopic morphisms. One can in fact show that if $f, g : a^\bullet \to b^\bullet$ are homotopic, then the induced maps on cohomology f^*, g^* are equal [4]. As a result, the localisation functor Q induces a functor:

$$Q_H : H(\mathcal{A}) \to D(\mathcal{A}) \ , \tag{2.2}$$

whose properties we shall explore momentarily.

Lemma 2.2. *Let $H(\mathcal{A})$ be the homotopy category of an abelian category $\mathcal{A}$.*

1. *Given the morphisms $M \xrightarrow{f} N \xleftarrow{s} N'$ in $H(\mathcal{A})$, with s being a quasi-isomorphism, there exists a complex M' such that the diagram:*

$$\begin{tikzcd} M' \arrow[r, "g"] \arrow[d, "t"'] & N' \arrow[d, "s"] \\ M \arrow[r, "f"] & N \end{tikzcd} \tag{2.3}$$

commutes, with t being a quasi-isomorphism.

2. *If $f : M \to N$ is a morphism in $H(\mathcal{A})$, then there is a quasi-isomorphism $s : M' \to M$ with $f \circ s = 0$ in $H(\mathcal{A})$ if and only if there is a quasi-isomorphism $t : N \to N'$ with $t \circ f = 0$ in $H(\mathcal{A})$.*

We will not prove this lemma here, but remark that these statements only hold due to homotopy equivalence and would not be true in $Ch(\mathcal{A})$ instead.

Definition 2.3. *The derived category $D(\mathcal{A})$ is the category with the same objects as $Ch(\mathcal{A})$ and $H(\mathcal{A})$, but with the morphisms $M \to N$ given by pairs, sometimes called roofs, of a morphism and a quasi-isomorphism: $M \xleftarrow{s} M' \xrightarrow{f} N$. We will denote such morphisms by (f, s). Two such diagrams define a morphism in $D(\mathcal{A})$ if there exists a commutative diagram in $H(\mathcal{A})$ of the form:*

$$\begin{tikzcd} & M' \arrow[dl, "s_1"'] \arrow[dr, "f_1"] & \\ M & P \arrow[u, "t_1"] \arrow[d, "t_2"] & N \\ & M'' \arrow[ul, "s_2"] \arrow[ur, "f_2"'] & \end{tikzcd} \tag{2.4}$$

with t_1, t_2 quasi-isomorphisms.

In the above definition, one also needs to clarify what we mean by compositions of morphisms in $D(\mathcal{A})$ in this picture. Given the roofs $M \overset{s}{\leftarrow} M' \overset{f}{\rightarrow} N$ and $N \overset{t}{\leftarrow} N' \overset{g}{\rightarrow} P$, the composition $(f,s) \circ (g,t)$ is defined by the commutative diagram in $H(\mathcal{A})$:

$$
\begin{tikzcd}
 & & N'' \arrow[dl, "t'"'] \arrow[dr, "f'"] & & \\
 & M' \arrow[dl, "s"'] \arrow[dr, "f"] & & N' \arrow[dl, "t"'] \arrow[dr, "g"] & \\
M & & N & & P
\end{tikzcd}
\tag{2.5}
$$

where t' is a quasi-isomorphism. Let us also note that any two morphisms in $D(\mathcal{A})$ can be put over a 'common denominator', giving $D(\mathcal{A})$ an additive structure. However, we stress that the derived category is not abelian.

2.2 Triangulated categories

As mentioned at the end of the previous subsection, the derived category is not an abelian category and thus there is no notion of a short exact sequence in $D(\mathcal{A})$. However, there is a similar notion called a *distinguished triangle* in this category.

To define the latter, let us recall from [4] the definition of a translation of chain complexes. In particular, given a complex M in $Ch(\mathcal{A})$, the translated complex $M[p]$ is defined by $M[p]^i = M^{i+p}$, with the differential $d^i_{M[p]} = (-1)^p d^{i+p}_M$. It is, of course, straightforward to define morphisms between shifted complexes $f[p] : M[p] \to N[p]$ starting from the original complexes, i.e. $p = 0$, and imposing $f[p]^i = f^{i+p}$. Recall also the definition of the mapping cone of $f = f[0]$ as the complex $Cone(f)$, with:

$$Cone(f)^i = M^{i+1} \oplus N^i \ , \tag{2.6}$$

and differential:

$$d^i(m,n) = \left(-d^{i+1}_M(m) \ , d^i_N(n) + f^{i+1}(m)\right) \ . \tag{2.7}$$

This immediately leads to the sequence of complexes:

$$M \longrightarrow N \longrightarrow Cone(f) \longrightarrow M[1] \ . \tag{2.8}$$

Additionally, there is a short exact sequence in $Ch(\mathcal{A})$:

$$0 \longrightarrow N \longrightarrow Cone(f) \longrightarrow M[1] \longrightarrow 0 \ , \tag{2.9}$$

which induces a long exact sequence of cohomology, by the Snake lemma. Given this construction we define the following.

Definition 2.4. *A distinguished triangle in the derived category $D(\mathcal{A})$ is a triple of objects and morphisms:*

$$A \longrightarrow B \longrightarrow C \longrightarrow A[1] \ , \tag{2.10}$$

such that there exists a morphism $f : M \to N$ and isomorphisms s, t, u in $D(\mathcal{A})$ such that the diagram:

$$\begin{tikzcd}
A \arrow[r] \arrow[d, "s"] & B \arrow[r] \arrow[d, "t"] & C \arrow[r] \arrow[d, "u"] & A[1] \arrow[d, "s[1]"] \\
M \arrow[r, "f"] & N \arrow[r] & Cone(f) \arrow[r] & M[1]
\end{tikzcd} \tag{2.11}$$

commutes.

Schematically, such a distinguished triangle is usually depicted as one of the following:

$$\begin{tikzcd}
A \arrow[rr] & & B \arrow[dl] \\
& C \arrow[ul, dashed] &
\end{tikzcd}
\qquad
\begin{tikzcd}
A \arrow[rr] & & B \arrow[dl] \\
& C \arrow[ul, "{[1]}"] &
\end{tikzcd} \tag{2.12}$$

with the dashed arrow indicating that the morphism from F is to $D[1]$ rather than to D. Let us remark that the translation operation introduced before defined a functor $[p] : Ch(\mathcal{A}) \to Ch(\mathcal{A})$, which descends to a functor $[p] : D(\mathcal{A}) \to D(\mathcal{A})$. The category $D(\mathcal{A})$ is in fact an example of a triangulated category, which is more generally defined as follows.

Definition 2.5. *A triangulated category $\mathcal{C}$ is an additive category, together with:*

1. *a translation functor $[p] : \mathcal{C} \to \mathcal{C}$ which is an isomorphism.*

2. *a set of distinguised triangles:*

$$A \xrightarrow{a} B \xrightarrow{b} C \xrightarrow{c} A[1] \ , \tag{2.13}$$

where morphisms between triangles are commutative diagrams of the type:

$$\begin{tikzcd}
A \arrow[r, "a"] \arrow[d, "f"] & B \arrow[r, "b"] \arrow[d, "g"] & C \arrow[r, "c"] \arrow[d, "h"] & A[1] \arrow[d, "f[1]"] \\
A' \arrow[r, "a"] & B' \arrow[r, "b"] & C' \arrow[r, "c"] & A'[1]
\end{tikzcd} \tag{2.14}$$

Additionally, triangulated categories satisfy a set of axioms.

In the following we will describe the axioms satisfied by triangulated categories. Note that these are not fully independent.

TR 1. *a) For any object A, the triangle $A \xrightarrow{id} A \xrightarrow{0} 0 \xrightarrow{0} A[1]$ is distinguished.*

b) If a triangle is isomorphic to a distinguished triangle, then it, too, is distinguished.

c) Any morphism $a : A \to B$ can be completed to a distinguished triangle of the form (2.13).

TR 2. *The triangle in TR1 is distinguished if and only if the triangle:*

$$
\begin{tikzcd}
 & C \arrow[dl] & \\
A[1] \arrow[rr, "{[1]}" description] & & B \arrow[ul]
\end{tikzcd}
\tag{2.15}
$$

is also distinguished.

In this case, one should compare (2.15) with (2.12). Thus, the above axiom allows one to move the [1] edge around the triangle, at the cost of translating the objects and morphisms accordingly. In particular, if in (2.12) we have the morphism $a : A \to B$, here we have $a[1] : A[1] \to B[1]$.

TR 3. *Given two triangles and the vertical maps f, g as in* (2.14)*, there exists a morphism h that completes the commutative diagram in* (2.14).

TR 4. *The Octahedral Axiom:*

$$
\begin{tikzcd}
 & B \arrow[dr] \arrow[ddl] & \\
D \arrow[ur, "{[1]}" description] \arrow[d, dashed, "{[1]}" description] & & E \arrow[ll] \arrow[ddl, dashed] \\
C \arrow[rr, "{[1]}" description] \arrow[dr, dashed] & & A \arrow[u, dashed] \\
 & F \arrow[uul, dashed] \arrow[ur, dashed, "{[1]}" description] &
\end{tikzcd}
\tag{2.16}
$$

The last axiom states that four faces of the octahedron are distinguished triangles, while the other four faces commute. At this stage we have defined the derived category, but would like to understand what are the properties of this category. For now, we will aim to understand the following lemma [5, 6].

Lemma 2.6. *Given an abelian category $\mathcal{A}$ and objects $X, Y \in \mathcal{A}$, $k \in \mathbb{Z}$, there is a canonical isomorphism:*

$$
Ext^k_{\mathcal{A}}(X, Y) \cong Hom_{D(\mathcal{A})}(X, Y[k]) \; . \tag{2.17}
$$

The following statement is equivalent to saying that the derived category is a natural framework to define and study derived functors.

2.3 Derived functors

Let us recall [4] that left/right derived functors are defined as functors between abelian categories $F : \mathcal{A} \to \mathcal{B}$, where $\mathcal{A}$ is assumed to have enough projectives/injectives. In particular, for the category of R-modules, for the R-modules A and B, one has:

$$Ext^i_R(A, B) = R^i\left(Hom_R(A, -)\right)(B) = R^i\left(Hom_R(-, B)\right)(A) \ , \tag{2.18}$$

where on the right hand side we have the right derived functor of $Hom_R(A, -)$ and $Hom_R(-, B)$, respectively. That is, the Ext modules are defined as the cohomology of $Hom(A, I^\bullet)$ or $Hom(P_\bullet, A)$, where $I^\bullet$ and $P_\bullet$ are injective and projective resolutions of B and I, respectively [4]:

$$Ext^n_R(A, B) = H^n\left(Hom_R(A, I^\bullet)\right) \ . \tag{2.19}$$

Let us now return to the lemma 2.6 and assume that the abelian category $\mathcal{A}$ has enough injectives. We will follow the argument of [5]. Under this assumption, there exists an injective resolution of B:

$$0 \longrightarrow B \longrightarrow I^0 \longrightarrow I^1 \longrightarrow \dots \ , \tag{2.20}$$

which, by definition, means that there is a quasi-isomorphism $s : B \to I^\bullet$, with B viewed as a 'trivial' complex, of the form:

$$\begin{array}{ccccccccc} \cdots & \longrightarrow & 0 & \longrightarrow & B & \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow & \cdots \\ & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\ \cdots & \longrightarrow & 0 & \longrightarrow & I^0 & \longrightarrow & I^1 & \longrightarrow & I^2 & \longrightarrow & \cdots \end{array} \tag{2.21}$$

In the derived category $D(\mathcal{A})$, this quasi-isomorphism then induces an isomorphism:

$$Hom_{D(\mathcal{A})}(A, B[n]) \cong Hom_{D(\mathcal{A})}(A, I[n]) \ , \tag{2.22}$$

which can be further shown to be isomorphic to $Hom_{H(\mathcal{A})}(A, I[n])$. For the proof of the latter statement we refer the reader to [3][1]. At this stage, recall the definition of the homotopy category:

$$Hom_{H(\mathcal{A})}(A, I[n]) = Hom_{Ch(\mathcal{A})}(A, I[n])/\sim \ , \tag{2.23}$$

where by $\sim$ we represent the chain homotopy equivalence relation. Recall also that morphisms of chain complexes (i.e. chain maps) are defined such that they commute with the boundary operators. In our case, we have for a morphisms $f : A \to I[n]$ the following diagram:

$$\begin{array}{ccccccccccc} \cdots & \longrightarrow & 0 & \longrightarrow & A & \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow & \cdots \\ & & \downarrow & & \downarrow{\scriptstyle f_0} & & \downarrow & & \downarrow & & \\ \cdots & \longrightarrow & I^{n-1} & \xrightarrow{\epsilon_{n-1}} & I^n & \xrightarrow{\epsilon_n} & I^{n+1} & \xrightarrow{\epsilon_{n+1}} & I^{n+2} & \longrightarrow & \cdots \end{array} \tag{2.24}$$

[1]See Corollary 10.4.7 of [3].

and thus we must have $\epsilon_n \circ f_0 = 0$. Thus, $Hom_{Ch(\mathcal{A})}(A, I[n])$ consists of all maps in $Hom_{\mathcal{A}}(A, I^n)$ that satisfy this relation, which are precisely the cocycles of $Hom_{\mathcal{A}}(A, I^n)$. Finally, the equivalence relation in (2.23) imposes another constraint on the chain maps. That is, chain morphisms f, g that differ by a chain homotopy:

$$
\begin{tikzcd}
\cdots \arrow[r] & 0 \arrow[r] \arrow[d] & A \arrow[r] \arrow[d, shift right, "g_0"'] \arrow[d, shift left, "f_0"] \arrow[dl, "t_0"'] & 0 \arrow[r] \arrow[d] & 0 \arrow[r] \arrow[d] & \cdots \\
\cdots \arrow[r] & I^{n-1} \arrow[r, "\epsilon_{n-1}"] & I^n \arrow[r, "\epsilon_n"] & I^{n+1} \arrow[r, "\epsilon_{n+1}"] & I^{n+2} \arrow[r] & \cdots
\end{tikzcd}
\tag{2.25}
$$

namely: $f_0 - g_0 = t_0 \epsilon_{n-1}$ are in the same equivalence class. This is, however, equivalent to modding out by coboundaries of $Hom_{\mathcal{A}}(A, I^n)$, and thus we have:

$$Hom_{H(\mathcal{A})}(A, B[n]) \cong Hom_{H(\mathcal{A})}(A, I[n]) \cong H^n(Hom_{\mathcal{A}}(A, I^\bullet)) \ . \tag{2.26}$$

Thus, this proves lemma 2.6 for this particular scenario. It is in fact more common to use (2.17) as the definition of Ext groups in the derived category, which by the above argument reduce to the usual Ext groups when the abelian category has enough injectives.

3. Coherent Sheaves

The next step in understanding the category of B-branes is to introduce sheaves, which we will do in this section.

3.1 Sheaves

For the rest of the section let X be a topological space. We then define the following.

Definition 3.1. *A presheaf $\mathcal{F}$ on X consists of the following data.*

1. *For every open set $U \subset X$ we associate an abelian group $\mathcal{F}(U)$.*
2. *If $V \subset U$ are open sets, there is a restriction homomorphism $\rho_{UV} : \mathcal{F}(U) \to \mathcal{F}(V)$.*

We further require that: $\mathcal{F}(\emptyset) = 0$, while ρ_{UU} is the identity map. Additionally, for the open sets $W \subset V \subset U$, we impose $\rho_{UW} = \rho_{VW}\rho_{UV}$.

For an element $\sigma \in \mathcal{F}(U)$, we will denote by σ_V the restriction $\rho_{UV}(\sigma)$.

Definition 3.2. *A sheaf $\mathcal{F}$ on X is a presheaf satisfying the following additional conditions.*

1. *If $U, V \subset X$ and $\sigma \in \mathcal{F}(U)$, $\tau \in \mathcal{F}(V)$ with $\sigma_{U\cap V} = \tau_{U\cap V}$, then there exists $\nu \in \mathcal{F}(U \cup V)$ such that $\nu_U = \sigma$ and $\nu_V = \tau$.*

2. If $\sigma \in \mathcal{F}(U \cup V)$ and $\sigma_U = \sigma_V = 0$ then $\sigma = 0$.

One of the most important examples of a sheaf on an algebraic variety X is the so called *structure sheaf* or sheaf of regular functions $\mathcal{O}_X$. In this case, every abelian group $\mathcal{F}(U) = \mathcal{O}_X(U)$ is in fact a commutative ring, namely the ring of regular functions on U, while the restrictions are the obvious resctriction maps on open subsets. The sheaf property is equivalent to saying that these local functions 'glue' in a unique way to a global function on U.

More generally, the collections $\{\mathcal{F}(U)\}$ are called *sections* of $\mathcal{F}$ over U, being locally defined. The sheaves on an algebraic variety X form, in fact, a category, with the morphisms of sheaves $f : \mathcal{F} \to \mathcal{G}$ defined on the abelian groups associated to the open subsets $U \subset X$. Additionally, we require that for $V \subset U$ these morphisms commute with the restriction homomorphisms, i.e. the following diagram:

$$\begin{array}{ccc} \mathcal{F}(U) & \xrightarrow{f(U)} & \mathcal{G}(U) \\ \big\downarrow{\scriptstyle \rho_{UV}} & & \big\downarrow{\scriptstyle \rho'_{UV}} \\ \mathcal{F}(V) & \xrightarrow{f(V)} & \mathcal{G}(V) \end{array} \tag{3.1}$$

commutes. The sections of the structure sheaf, i.e. the sets of regular functions over open sets U, $\mathcal{O}_X(U)$, are abelian groups under addition, but also have a ring structure under multiplication. As a result, we can also consider the sheaf of $\mathcal{O}_X$-modules, namely for every open set we now have an $\mathcal{O}_X$-module instead. Since $\mathcal{O}_X$ is itself an $\mathcal{O}_X$-module, we may also consider:

$$\mathcal{O}_X^{\oplus n} = \mathcal{O}_X \oplus \cdots \oplus \mathcal{O}_X \ , \tag{3.2}$$

which is also an $\mathcal{O}_X$-module, usually called the free $\mathcal{O}_X$-module of rank n. By an abuse of notation, we will denote by $\mathcal{O}_X^{\oplus n}$ the corresponding sheaf of $\mathcal{O}_X$-modules as well.

Definition 3.3. *A sheaf $\mathcal{E}$ is called a locally free sheaf of rank n if there is an open covering $\{U_\alpha\}$ of X such that $\mathcal{E}(U_\alpha) \cong \mathcal{O}_X(U_\alpha)^{\oplus n}$ for all α.*

We end this section discussing the relation between locally free sheaves and holomorphic vector bundles. Consider a rank n vector bundle:

$$E \xrightarrow{\pi} X \ , \tag{3.3}$$

with local trivialization $\{U_\alpha\}$. If $n = 1$ and E is a trivial complex line bundle, then it is straightforward to shee that $\mathcal{O}_X(U_\alpha)$ is the group of holomorphic sections of the bundle over the upen set U_α. More generally, the group of holomorphic sections of E over U_α is given by the locally free sheaf $\mathcal{O}_X(U_\alpha)^{\oplus n}$. For a locally free sheaf $\mathcal{E}$ that is not neccesarily a sheaf of free $\mathcal{O}_X$-modules, we have by definition an isomorphism:

$$\phi_\alpha : \mathcal{E}(U_\alpha) \to \mathcal{O}_X(U_\alpha)^{\oplus n} \ . \tag{3.4}$$

Thus, on the nontrivial intersections $U_\alpha \cap U_\beta$ we can define:

$$\phi_\beta \phi_\alpha^{-1} : \mathcal{O}_X(U_\alpha \cap U_\beta)^{\oplus n} \longrightarrow \mathcal{O}_X(U_\alpha \cap U_\beta)^{\oplus n} \ , \tag{3.5}$$

which become the transition functions of the holomorphic bundle $E \to X$. Thus, there is a one-to-one correspondence between holomorphic vector bundles of rank n on X and locally free sheaves of rank n on X. The importance of holomorphic vector bundles in this context comes by considering branes in the B-model. It turns out that B-branes are described by holomorphic vector bundles, with the base X being a Calabi-Yau threefold (or holomorphically embedded submanifolds of X).

Note that from the above construction, the trivial line bundle corresponds to the structure sheaf $\mathcal{O}_X$. Let us also recall the the holomorphic vector bundles over the complex projective plane $\mathbb{P}^1$ are classified by Grothendieck's Lemma, being isomorphic to:

$$E \cong \mathcal{O}(a_1) \oplus \mathcal{O}(a_2) \oplus \cdots \oplus \mathcal{O}(a_n) \ , \tag{3.6}$$

for some unique integers $a_1 \geq a_2 \geq \cdots \geq a_k$. It is thus customary to introduce the locally free sheaves $\mathcal{O}_{\mathbb{P}^1}(n)$ as the sheaves associated to the holomorphic line bundles $\mathcal{O}(n)$ over $\mathbb{P}^1$.

3.2 Sheaf Cohomology

In the previous subsection we introduced the category of sheaves, which, however, turns out to be a non-abelian category. Moreover, locally free sheaves also form a category, which is also not abelian. To remedy this issue we define the following.

Definition 3.4. *A sheaf $\mathcal{F}$ of $\mathcal{O}_X$-modules is called coherent*[2] *if, for every sufficiently small open set $U \subset X$ with ring of regular functions $\mathcal{O}_X(U)$, there is an exact sequence of $\mathcal{O}_X(U)$-modules:*

$$\mathcal{O}_X(U)^{\oplus m} \longrightarrow \mathcal{O}_X(U)^{\oplus n} \longrightarrow \mathcal{F}(U) \longrightarrow 0 \ , \tag{3.7}$$

that is compatible with restrictions, for some n, m.

Here we will be slightly loose with the terminology, but let us try to explain the additional information carried by coherent sheaves. As claimed before, the category of locally free sheaves is not an abelian category. As a result, there are morphisms that do not have kernels or cokernels, or morhpisms that are not strict. Heuristically, the category of coherent sheaves is obtained by 'adding' all (co)kernels and the morphisms between these new objects and the locally free sheaves, leading to an abelian category which contains the locally free sheaves.

[2] To be more precise, this is the definition of a quasi-coherent sheaf.

We will illustrate this construction with an example, on an open subset of the algebraic variety X, that is locally isomorphic to $\mathbb{C}^3$. Consider thus the following morphism:

$$\mathcal{O}_X^{\oplus 3}(U) \xrightarrow{\varphi} \mathcal{O}_X(U) \ , \tag{3.8}$$

described by:

$$\varphi : (f_1, f_2, f_3) \mapsto x f_1 + y f_2 + z f_3 \ , \tag{3.9}$$

with the functions f_i being regular functions on X and (x, y, z) the affine coordinates on $\mathbb{C}^3$. The image of this map involves all regular functions on X^*, and thus the cokernel of φ involves regular functions at the origin of X. Thus, let us introduce the *skyscraper sheaf* $\mathcal{O}_p$, defined as:

$$\mathcal{O}_p(U) = \begin{cases} \mathbb{C} \ , & \textit{if } \{\mathbf{0}\} \in U \ , \\ 0 \ , & \textit{otherwise} \ . \end{cases} \tag{3.10}$$

The skyscraper sheaf is a sheaf of $\mathcal{O}_X$-modules, but is not a locally free sheaf, as its rank 'jumps'. There is also a natural map: $\mathcal{O}(U) \to \mathcal{O}_p(U)$, taking a function to its value at the origin. Thus, this is indeed the cokernel of the morphism φ, leading to the exact sequence:

$$\mathcal{O}^{\oplus 3}(U) \longrightarrow \mathcal{O}(U) \longrightarrow \mathcal{O}_p(U) \longrightarrow 0 \ . \tag{3.11}$$

But from the definition 3.4, the skyscraper sheaf is in fact a coherent sheaf, which thus shows how coherent sheaves can complete the category of locally free sheaves to an abelian category. Note that this is still a subgategory of the category of sheaves of $\mathcal{O}_X$-modules. In this category we can also discuss about (co)homology, as usual, which is usually called *sheaf cohomology*, being defined as the right derived functor of the $Hom(\mathcal{O}_X, -)$ functor:

$$H^n(X, \mathcal{F}) = R^n Hom(\mathcal{O}_X, -)(\mathcal{F}) \ . \tag{3.12}$$

This is, of course, the usual definition of the Ext groups in (2.18). Let us also point out that sheaf cohomology is equivalent to Ĉech cohomology on X. Finally, given this technology, we can talk about the derived category of coherent sheaves. To be pedantic, the category of B-branes is the *bounded* derived category of coherent sheaves on X, usually denoted by $D^b(X)$. That is, the elements of $D^b(X)$ are bounded chain complexes. This category is also a triangulated category, with a translation functor, for which we define as before:

$$Ext_X^i(\mathcal{E}, \mathcal{F}) = Hom_{D^b(X)}(\mathcal{E}, \mathcal{F}[i]) \ . \tag{3.13}$$

3.3 Triangulated structure

While the derived category of coherent sheaves is the correct category of B-branes, many of the objects in this category might not actually correspond to D-branes. In fact, one has to impose additional conditions on the sheaves to ensure that they correspond to B-model branes. In the physical theory, these conditions are equivalent to the BPS condition and reduce to the Hermitian-Yang-Mills condition on the curvature of the holomorphic vector bundles over X.

The reason behind these conditions comes from the topological twisting[3] procedure of the $2d$ $\mathcal{N} = (2,2)$ supersymmetric non-linear σ-model. In the 'untwisted' theory, D-branes satisfy the BPS condition, which turns out to be stronger than the condition imposed on the branes in the topological theory. Thus, there is a need for an additional condition, but we will not discuss this issue here. The mirror symmetry conjecture relates the topologically twisted theories and, as a result, the equivalence between the A and B-model brane categories does not require this additional data.

Let us now see how the derived category encodes information about bound states of D-branes. For this, we will use the fact that the derived category is a triangulated category, in which case the axioms of triangulated categories translate to rules for D-brane decays. Consider first the Ext^1 group of two sheafs $\mathcal{E}$ and $\mathcal{F}$ in the abelian category of coherent sheaves, which, as is well known (see [4] for instance), classifies all extensions of $\mathcal{E}$ by $\mathcal{F}$, that is all short exact sequences:

$$0 \longrightarrow \mathcal{F} \longrightarrow \mathcal{G} \longrightarrow \mathcal{E} \longrightarrow 0 \ , \tag{3.14}$$

that are non-split. Each such short exact sequence in the category of coherent sheaves induces a distinguished triangle in the derived category:

$$\begin{tikzcd} \mathcal{F} \arrow[rr] & & \mathcal{G} \arrow[dl] \\ & \mathcal{E} \arrow[ul, "{[1]}"] & \end{tikzcd} \tag{3.15}$$

This is easily seen by noticing that the above construction is isomorphic to a triple coming from the mapping cone construction in the derived category, as in (2.8). Thus, $\mathcal{E}$ in the above distinguished triangle is quasi-isomorphic to the mapping cone of the morphism $f : \mathcal{F} \to \mathcal{G}$. As a result, there is a morphism $\mathcal{E} \to \mathcal{F}[1]$, and the above distinguished triangle does exist.

Assuming for now that these sheaves correspond to B-model branes, the sheaf $\mathcal{G}$ is interpreted as a bound state of the sheaves $\mathcal{F}$ and $\mathcal{E}$, binded via an open string. Under this interpretation, the axioms of the triangulated category obtain a physical interpretation. For instance, TR2 states that if the decay $B \to A \oplus C$ is allowed,

[3] This is equivalent to 'modifying' the bundles in which the fermions are valued.

then we must also allow the decay $C \to A[1] \oplus B$. This might appear peculiar at the beginning, but one can show that $A[1]$ is the 'anti-brane' of A [2]. As a result, D-brane/anti-D-brane annihilation is built into the derived category description.

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