An introduction to ABJM

Horia Magureanu

E-mail: magureanu@maths.ox.ac.uk

Contents

1 Overview

It is by now well established that in $d = 4$ any Lagrangian $\mathcal{N} = 3$ theory is so strongly contstrained that it automatically has $\mathcal{N} = 4$ supersymmetry. An indication of this fact comes from building CPT invariant $\mathcal{N} = 3$ multiplets, which have the field content of an $\mathcal{N} = 4$ multiplet. That is not to say that $\mathcal{N} = 3$ theories do not exist – in fact, they do, but they do not admit Lagrangian descriptions. This is similar to the 6d $\mathcal{N} = (2, 0)$ theory, which also does not have a Lagrangian description but can be constructed as the worldvolume theory of M5-branes. In a similar fashion, string theory constructions were proposed for 4d $\mathcal{N} = 3$ theories [1].

In 3d, we have a more intricate story. 3d $\mathcal{N} = 8$ theories come in two classes: either $\mathcal{N} = 8$ SYM theories or the Bagger-Lambert-Gustavsson (BLG) theories, which are superconformal Chern-Simons-matter systems whose supersymmetry enhances to $\mathcal{N} = 8$. $\mathcal{N} = 7$ theories turn out to be automatically $\mathcal{N} = 8$, even if they do not have a Lagrangian description (*i.e.* a stronger statement compared to the 4d $\mathcal{N} = 3$ to $\mathcal{N} = 4$ theories). This leaves us with the 3d $\mathcal{N} = 6$ models, which have the same amount of supersymmetry as 4d $\mathcal{N} = 3$ theories, that is, 12 supercharges. Here we have the Aharony-Bergman-Jafferis-Maldacena (ABJM) theories [2] and the Aharony-Bergman-Jafferis (ABJ) models [3], which are Chern-Simons-matter theories.[1] Note that some ABJM and ABJ models enhance quantum mechanically to $\mathcal{N} = 8$ SUSY. For a more detailed overview of these models we refer to [4].

[1] There are also some other variants, where the gauge groups are special unitary or orthosymplectic.

The aim of these notes is to discuss the $\mathcal{N}=6$ case in more detail. In particular, the ABJM theory is believed to describe the low energy limit of the worldvolume theory of a stack of M2 branes. We will also look at the relation between 3d $\mathcal{N}=6$ ABJM theories and 4d $\mathcal{N}=3$ S-folds.

2 Chern-Simons theories with matter

Let us start by recalling some basic aspects of the 3d $\mathcal{N}=2$ supersymmetry algebra. First, the vector multiplet consists of a gauge field A_μ, a real scalar σ, two-component Dirac spinors $\lambda, \bar{\lambda}$ (the gaugini), and an auxiliary scalar field D:

$$V: \quad A_\mu,\, \sigma,\, \lambda,\, \bar{\lambda},\, D\ . \tag{2.1}$$

For those more accustomed to 4d $\mathcal{N}=1$, the scalar σ is simply the A_4 component upon dimensional reduction. We also have a chiral multiplet consisting of a scalar field ϕ, a fermion ψ and an auxiliary field F, similarly to the 4d $\mathcal{N}=1$ chiral multiplet:

$$\Phi: \quad \phi,\, \psi,\, F\ . \tag{2.2}$$

In 3d, an important and somewhat peculiar symmetry occurs in the presence of abelian gauge fields. For every $U(1)$ gauge field A_μ, one can define a current:

$$j_T^\mu = \frac{1}{4\pi}\epsilon^{\mu\nu\rho}F_{\mu\nu}\ , \tag{2.3}$$

which is conserved due to Bianchi identity. The corresponding global symmetry $U(1)_T$ is often called a topological symmetry and the states charged under $U(1)_T$ carry non-trivial $U(1)$ magnetic charge.

The supersymmetric Yang-Mills-Chern-Simons-matter Lagrangian that we can build from these fields is given by:[2]

$$\mathcal{L} = \mathcal{L}_{SYM} + \mathcal{L}_{CS} + \mathcal{L}_{\bar{\Phi}\Phi} + \mathcal{L}_W + \mathcal{L}_{\bar{W}}\ . \tag{2.4}$$

The SYM term, for instance, reads:

$$\mathcal{L}_{SYM} = \frac{1}{g^2}\left(\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \frac{1}{2}D_\mu\sigma D^\mu\sigma - i\bar{\lambda}\gamma^\mu D_\mu\lambda - i\bar{\lambda}[\sigma,\lambda] - \frac{1}{2}D^2\right)\ , \tag{2.5}$$

again very similar to the 4d $\mathcal{N}=1$ SYM Lagrangian (just the dimensional reduction of the latter). In terms of the $\mathcal{N}=2$ vector superfield V, we can write the Lagrangian as an integral over superspace coordinates (the 3d $\mathcal{N}=2$ superspace is very similar to that of 4d $\mathcal{N}=1$, with coordinates $x^\mu, \theta, \bar{\theta}$):

$$\mathcal{L}_{SYM} = \frac{1}{g^2}\int d^2\theta d^2\bar{\theta}\left(\frac{i}{4}\bar{D}^\alpha e^{-V}D_\alpha e^V\right)^2 = \frac{1}{g^2}\int d^2\theta d^2\bar{\theta}\,\Sigma^2\ , \tag{2.6}$$

where the expression in brackets (Σ) is the field strength superfield.

[2]See [5] for a nice review.

2.1 CS terms

The pure Chern-Simons term takes the form:

$$S_{CS} = \frac{k}{4\pi}\int_{\mathcal{M}} \mathrm{Tr}\left(A\wedge dA + \frac{2}{3}A\wedge A\wedge A\right) \ , \tag{2.7}$$

with the field strength:

$$F = dA + [A, A] \ . \tag{2.8}$$

This action is defined on any 3-manifold $\mathcal{M}$, and does not depend on the metric. In fact, under a finite gauge transformation:

$$A \to U^{-1}AU + U^{-1}dU \ , \tag{2.9}$$

the CS action changes by:

$$\delta S_{CS} = \frac{k}{4\pi}\int_{\mathcal{M}} d\left(A\wedge dUU^{-1}\right) - \frac{k}{12\pi}\int_{\mathcal{M}} U^{-1}dU\wedge U^{-1}dU\wedge U^{-1}dU \ . \tag{2.10}$$

One notices that the first term is a total derivative, while the second term is a topological property of the manifold, namely the *winding number* of the gauge transformation around the 3-manifold $\mathcal{M}$:

$$w = \frac{1}{24\pi^2}\int_{\mathcal{M}} (U^{-1}dU)^3 \ \in \mathbb{Z} \ , \tag{2.11}$$

which is always an integer! Note also that w is non-zero for *large gauge transformations*, *i.e.* the ones not connected to the identity. The possible winding numbers of a manifold $\mathcal{M}$ turn out to be classified by the third homotopy group of the gauge group $\pi_3(G)$. Now, invariance of the path integral under gauge transformations implies the condition

$$k \in \mathbb{Z} \ , \tag{2.12}$$

for non-abelian simple gauge groups, for which $\pi_3(G) = \mathbb{Z}$. The classical Chern-Simons theory is characterized by its equations of motion, which read:

$$0 = \frac{\delta S}{\delta A} = \frac{k}{2\pi}F \ . \tag{2.13}$$

These equations are satisfied if and only if the curvature vanishes everywhere $F = 0$, in which case we are talking about flat connections of principal G-bundles P on 3-manifolds $\mathcal{M}$:

$$G \to P \to \mathcal{M} \ . \tag{2.14}$$

The flat connections are in one-to-one correspondence with equivalence classes of homomorphisms from the fundamental group of $\mathcal{M}$ to the gauge group G up to conjugation. We will, however, couple the CS theory to dynamical matter fields, and consider instead 3d $\mathcal{N} = 2$ CS theories. For these, the Lagrangian contains a supersymmetric completion:

$$\mathcal{L}_{CS} = \frac{k}{4\pi}\left(i\epsilon^{\mu\nu\rho}\Big(A_\mu\partial_\nu A_\rho - \frac{2i}{3}A_\mu A_\nu A_\rho\Big) - 2D\sigma + 2i\bar{\lambda}\lambda\right) \ , \tag{2.15}$$

and, as for the SYM term, we can rewrite it as an integral over superspace coordinates:

$$\mathcal{L}_{CS} = \frac{k}{4\pi}\int d^2\theta d^2\bar{\theta}\, V\Sigma \ , \tag{2.16}$$

for the abelian case and:

$$\mathcal{L}_{CS} = \frac{k}{4\pi}\int d^2\theta d^2\bar{\theta}\int_0^1 dt \frac{i}{2} V\bar{D}^\alpha e^{-tV} D_\alpha e^{tV} \ , \tag{2.17}$$

in the non-abelian scenario.

2.2 Matter Lagrangian

We will be interested in coupling the $\mathcal{N} = 2$ CS theory to matter fields Φ_i. This can be realized by adding matter terms to the Lagrangian, which are of the type:

$$\mathcal{L}_{CS} + \int d^2\theta d^2\bar{\theta}\, \Phi^\dagger e^V \Phi + \int d^2\theta\, W(\Phi) + cc \ , \tag{2.18}$$

exactly the same as in 4d $\mathcal{N} = 1$. At this point, let us pause and make some important remarks. First, when $W = 0$, the only coupling in the above lagrangian is the CS level k. It was proved in the 90s that for Chern-Simons-matter theories the beta function of this coupling (the coupling is $1/k$, to be more precise) vanishes! In fact, the above Lagrangian turns out to be *clasically marginal* and, it was argued later on by Gaiotto and Yin that it is *exactly marginal* (once quantum corrections are taken into account). Thus, the above Lagrangian should in fact describe an $\mathcal{N} = 2$ SCFT.

There is, however, a very important distinction to the CFTs having a continuous coupling, such as 4d $\mathcal{N} = 4$ SYM. There, we can deform the theory by adding the corresponding marginal operator to the Lagrangian: *e.g.* in $\mathcal{N} = 4$ can just add the Lagrangian itself $\delta\mathcal{L} = \alpha\mathcal{L}$, for some infinitesimal α, which effectively leads to a small shift in the gauge coupling! Here, however, k is *quantized*, so continuous shifts of $\mathcal{L}$ are not allowed and, thus, *each CS level k corresponds to an isolated CFT.*

We can also turn on superpotential terms, which, despite being clasically marginal, turn out to be relevant for couplings $\alpha \ll 1/k$ and irrelevant for $\alpha \gg 1/k$. This implies that there exists an RG fixed point at finite α, and we still expect such points to be isolated.

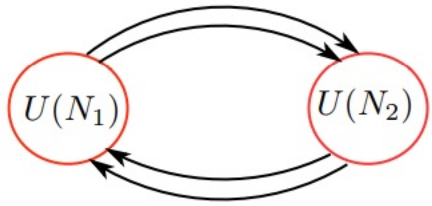

Figure 1: The ABJM quiver.

2.3 The ABJM theory

The famous ABJM theory is a Chern-Simons matter theory with gauge group $U(N)_k \times U(N)_{-k}$, where the subscripts are the CS levels of each $U(N)$ factor. The matter fields are four chiral multiplets, whose superfields we will denote by (a_1, a_2, b_1, b_2), with a_1, a_2 transforming in the $(N, \bar{N})$ representation of the gauge group, while b_1, b_2 transform in $(\bar{N}, N)$.

Denoting by V_1, V_2 the $\mathcal{N} = 2$ vector superfields, we have:

$$\mathcal{L}_{\text{ABJM}} = \mathcal{L}_{CS}(V_1, k) + \mathcal{L}_{CS}(V_2, -k) + \int d^4\theta \sum \Big(- a_i^\dagger e^{V_1} a_i e^{-V_2} - b_i^\dagger e^{-V_1} b_i e^{V_2} \Big) \\ + \int d^2\theta \, \frac{4\pi}{k} \mathrm{Tr} \left(\epsilon^{ij} \epsilon^{kl} a_i b_k a_j b_l \right) + cc \; . \tag{2.19}$$

With generic superpotential coupling, the Lagrangian has $\mathcal{N} = 2$ supersymmetry, but here we fix it by actually requiring the superpotential to have $\mathcal{N} = 3$ SUSY! That is, $\mathcal{N} = 3$ Lagrangians can be built by starting with the matter content of an $\mathcal{N} = 4$ theory (the same as 4d $\mathcal{N} = 2$); the $\mathcal{N} = 4$ vector multiplet contains an additional $\mathcal{N} = 2$ chiral multiplet φ in the adjoint representation, which leads to an $\mathcal{N} = 4$ superpotential of the form:

$$W^{\mathcal{N}=4} = \Phi_i^\dagger \varphi \Phi_i \; . \tag{2.20}$$

The CS term now includes a term of the type φ^2, which breaks the $\mathcal{N} = 4$ SUSY explicitly to $\mathcal{N} = 3$. Since there is no kinetic term for φ, as we view it as an auxiliary field in this construction, we can integrate it out. This procedure fixes the coupling of the superpotential in terms of the CS level, as above, leading to enhanced $\mathcal{N} = 3$ SUSY at the conformal point. Note that the $\mathcal{N} = 4$ symmetry is broken to $\mathcal{N} = 3$ by mass terms given to the additional scalar in the $\mathcal{N} = 4$ vector multiplet.

What is the symmetry of this theory? In general, 3d theories with $\mathcal{N}$ supercharges will have an $SO(\mathcal{N})$ symmetry, which, for the ABJM theory, appears to be $SO(3)$, or, more precisely, $SU(2)_R$. We have the following representations under $SU(2)_R$ of the vector multiplet fields:

Field	A_μ	$\lambda, \bar{\lambda}$	$\sigma \oplus \varphi, \bar{\varphi}$	D's
$SU(2)_R$	$\mathbf{1}$	$\mathbf{3} \oplus \mathbf{1}$	$\mathbf{3}$	$\mathbf{3}$

(2.21)

For the chiral multiplets, all fields are doublets under $SU(2)_R$: e.g. the bottom components (a_i, b_i^*) form such a doublet.

From the way we wrote the superpotential, we also have an $SU(2) \times SU(2)$ symmetry acting separately on the a's and on the b's, which is a symmetry of the whole Lagrangian. So now we have three $SU(2)$ groups, but it is obvious that the $SU(2)_R$ symmetry does not commute with the other two factors. In fact, together, the symmetries generate an $SU(4)$, under which the fields:

$$(a_1, a_2, b_1^*, b_2^*) \tag{2.22}$$

transform as a **4**. Since $SU(4)$ and $SO(6)$ have the same algebra, this is a first hint that the ABJM action preserves in fact $\mathcal{N} = 6$ SUSY! Writing out explicitly the action in component fields, one can check that this is indeed the case [2].

3 ABJM and the AdS/CFT correspondence

There are two main perspectives we can take when discussing extended objects (M-branes) of M-theory:

- Look at solutions of 11-dimensional supergravity.
- Look at field theories on the worldvolumes of the membranes.

This is at the heart of the AdS/CFT correspondence. The ABJM theory is supposedly the worldvolume theory of M2-branes; the proposal of ABJM is that the above Chern-Simons SCFT is dual to M-theory on $AdS_4 \times S^7/\mathbb{Z}_k$, for a particular $\mathbb{Z}_k$ quotient of the 7-sphere.

3.1 Brane construction

Consider a IIB setup with two NS5 branes stretched along 012345 directions, but separated along the 6 compact direction - place them at $x^6 = 0$ and $x^6 = \pi R$, where $x^6 \sim x^6 + 2\pi R$. Then, let us add a stack of N D3 branes, stretched along the 012 directions, which wrap the 6th compact direction:

	0	1	2	3	4	5	6	7	8	9
NS5	$\times$	$\times$	$\times$	$\times$	$\times$	$\times$	$x^6 = 0$			
NS5'	$\times$	$\times$	$\times$	$\times$	$\times$	$\times$	$x^6 = \pi R$			
N D3	$\times$	$\times$	$\times$				$\times$			

(3.1)

The presence of all branes breaks the Lorentz group to $SO(1,2) \times SO(3) \times SO(3)$, where the two $SO(3)$ factors come from the 345 and 789 rotations, respectively. Note also that these two $SO(3)$ factors generate an $SO(4)_R$ symmetry [6], which already suggests that we are dealing with a 3d $\mathcal{N} = 4$ theory. In this construction, gauge groups arise at the intersection of D3-branes with NS5-branes. In this particular example, we have two $U(N)$ gauge groups from the intersection of the D3 stack with the NS5-branes, along the compact direction. Moreover, there are bifundamental hypers coming from open strings between the D3-branes across the NS5 brane.

To get Chern-Simons levels, we have to replace one of the NS5-branes by a $(1,k)$-fivebrane; this can be achieved by first adding k D5-branes to the setup:

	0	1	2	3	4	5	6	7	8	9
k D5	$\times$	$\times$	$\times$	$\times$	$\times$		$x^6=0$			$\times$

(3.2)

which breaks SUSY to $\mathcal{N}=2$ only (R-symmetry from 78 directions). Then, we 'merge' the k D5-branes with the NS5-brane along 01234 into an intermmediate $(1,k)$ five-brane. This web deformation is achieved by turning on a mass term for the chiral multiplets coming from the addition of the D5-brane. This turns out to produce a CS level k for one of the $U(N)$ factors, and a $-k$ for the other one.

Note that, so far we only have $\mathcal{N}=1$ SUSY, from the addition of the D5-branes. To get $\mathcal{N}=3$ SUSY, one needs to rotate the $(1,k)$ five-brane relative to the other NS5-brane in the 37 and 48 planes, such that the $(1,k)$ five-brane stretches along 012 directions and $[3,7]_\theta$, $[4,8]_\theta$, $[5,9]_\theta$, with θ being the initial angle of the $(1,k)$ brane in the $[5,9]$ plane.

M-theory uplift. Next up, we want to uplift this setup to M-theory. We do this in 2 steps. First, we T-dualize to IIA along the compact direction (x^6). Recall that T-duality in a direction tangent to a Dp-brane reduces it to a D$(p-1)$-brane, while a T-duality in a direction orthogonal turns it into a D$(p+1)$-brane. Hence, here T-duality transforms the D3-branes into D2-branes. Upon lifting to M-theory, these D2-branes become M2-branes.

Meanwhile, T-duality maps NS5-branes into background geometries. More precisely, the IIB NS5-brane is mapped to the IIA KK monopole, which is a D6-brane. The corresponding supergravity solution is an asymptotically locally flat (ALF) space (Taub-NUT), whose metric reads:

$$ds^2_{ALF} = h(r)d\boldsymbol{r}\cdot d\boldsymbol{r} + \frac{1}{4}h(r)^{-1}(d\psi + \boldsymbol{\omega}\cdot d\boldsymbol{r})^2 \ , \tag{3.3}$$

where $\boldsymbol{r}\in\mathbb{R}^3$, $\psi\in[0,4\pi)$ and $\nabla\times\boldsymbol{\omega} = \nabla(1/r)$. Meanwhile, the harmonic function $H(r)$ is given by:

$$H(r) = \frac{1}{g^2} + \frac{1}{2r} \ . \tag{3.4}$$

Note that as $r\to 0$, the metric becomes flat $\mathbb{R}^4$, while in the asymptotic regime $r\to\infty$, the metric is locally $\mathbb{R}^3\times S^1$, with the S^1 parametrized by ψ. The asymptotic radius of this circle is g. This is thus a non-trivial fibration of S^1 over $\mathbb{R}^3$ – the Taub-NUT space. Uplifting this to M-theory, we are left with a KK-monopole in 11 dimensions, *i.e.* pure geometry.

Returning to our setup, we need to T-dualize $(1, k)$ five-branes. This leads to a KK monopole associated with the 6 circle direction, together with 'k $D6$-branes'. Uplifting to M-theory, the 'D6' branes become KK monopole associated to the 10 direction [7]. Thus, the $(1, k)$ five-brane leads to a geometry of the type $\mathbb{R}^{1,2} \times X_8$, where X_8 is the space we get by superposing the two monopole solutions. It turns out that locally, x_8 is of the type $\mathbb{C}^4/\mathbb{Z}_k$, where the $\mathbb{Z}_k$ acts as $z_I \to e^{\frac{2\pi i}{k}} z_I$ on the complex coordinates.

3.2 Gravity side

To understand the gravity side of the duality, let us quickly review the duality involving the theory leaving on the worldvolume of M2-branes in flat space, and the $AdS_4 \times S^7$ solution [8]. 11-dimensional supergravity can be thought of as the low-energy limit of the more misterious M-theory. The bosonic action for 11d supergravity is given by:

$$S = \frac{1}{(2\pi)^8 l_p^9} \left(\int d^{11}x \sqrt{-G_{11}} \big(R - \frac{1}{2}|G_4|^2\big) - \frac{1}{2} \int A_3 \wedge G_4 \wedge G_4 \right) \;, \tag{3.5}$$

with l_p the 11d Planck length. M2 branes couple electrically to A_3, while G_4 is the field strength of the A_3 form. Meanwhile, G_{11} is just the 11-dimensional metric and R is the Ricci scalar.

A natural question that one could ask is: *What sort of geometries would satisfy the equations of motion?* (*i.e.* the usual Euler-Lagrange equations.) The known solutions have been found by starting with an ansatz of the form:

$$\begin{aligned} ds_{11}^2 &= f^{2a} \eta_{\mu\nu} dx^\mu dx^\nu + f^{2b} g^Y_{\alpha\beta} dy^\alpha dy^\beta \;, \\ A_{\mu\nu\rho} &= \epsilon_{\mu\nu\rho} e^C \;, \end{aligned} \tag{3.6}$$

where the 11-dimensional coordinates x^M are split in (x^μ, y^α). All other components of A_{MNP} and the gravitino field Ψ_M are set to zero. For the *M2-brane solution*, we have $\mu = 0, 1, 2$, with the metric given explicitly by:

$$\begin{aligned} ds_{11}^2 &= f^{-2/3} \eta_{\mu\nu} dx^\mu dx^\nu + f^{1/3} \left(dr^2 + r^2 d\Omega_7^2\right) \;, \qquad f = 1 + \frac{2^5 \pi^2 N l_p^6}{r^6} \;, \\ A_{\mu\nu\rho} &= \epsilon_{\mu\nu\rho} f^{-1} \;, \end{aligned} \tag{3.7}$$

with $d\Omega_7^2$ the metric on the S^7 sphere. The solution also comes with a flux for the dual of the four-form field strength on the seven-sphere:

$$\frac{1}{(2\pi l_p)^6} \int_{S^7} \star G_4 = N \;. \tag{3.8}$$

It is convenient to consider the coordinate reparametrisation:

$$r^6 = k_2 \left(\frac{1}{1 - \tilde{r}^3} - 1 \right) \;, \qquad k_2 = 2^5 \pi^2 N l_p^6 \;. \tag{3.9}$$

such that the solution becomes [9]:

$$ds_{11}^2 = \tilde{r}^2\,\eta_{\mu\nu}dx^\mu dx^\nu + \frac{4k_2^{1/3}}{\tilde{r}^2}d\tilde{r}^2 + k_2^{1/3}d\Omega_7^2 + k_2^{1/3}\left(\frac{1}{1-\tilde{r}^3}-1\right)\left(\frac{4}{\tilde{r}^2}d\tilde{r}^2 + d\Omega_7^2\right) \;. \tag{3.10}$$

We then notice that as $\tilde{r} \to 1$, the solution becomes asymptotically flat (approaches Minkowski space-time). Meanwhile, as $\tilde{r} \to 0$, one approaches the *event horizon*. The near horizon geometry is $AdS_4 \times S^7$, which only involves the first 3 terms in the above metric. Note that the AdS_4 metric here is written in Poincaré coordinates.[3]

Energy - radius relation and the Large N limit. The more rigurous statement of what we have done so far is that the *low-energy* field theory living on the world-volume of a stack of M2-branes is dual to some particular *near-horizon* geometry. But how are the two limits – *i.e.* IR limit on the field theory side and near-horizon limit on the gravity side – related?

Consider again the $AdS_4 \times S^7$ metric obtained before:

$$ds^2 = \tilde{r}^2 dx^\mu dx_\mu + \left(\frac{2R}{\tilde{r}}\right)^2 d\tilde{r}^2 + R^2\, ds_{S^7}^2 \;, \tag{3.11}$$

for some conveniently chosen R. To discuss the energy/radius relation in the AdS_4/CFT_3 correspondence, note that the dilatation symmetry acts as:

$$x^\mu \to \lambda x^\mu \;, \tag{3.12}$$

on the (x^0, x^1, x^2) coordinates, requiring the r coordinate to transform as [10]:

$$\tilde{r} \to \lambda^{-1}\tilde{r} \;. \tag{3.13}$$

But, we also have $\sqrt{g_{00}} = \tilde{r}$, so energies are redshifted as $\tilde{r}$, giving thus (schematically) $E_{CFT} \sim \tilde{r}$.

Finally, we should also comment on the importance of the 'large N limit'. The supergravity solution is to be trusted whenever the eleven dimensional Planck length is taken to 0 size $l_p \to 0$, while keeping ratio $r/l_p^{\frac{3}{2}}$ fixed, with r the separation of the M2-branes. This combination has to remain fixed because the scalar field describing the motion of the M2-brane has scaling dimension 1/2 [8]. Then, one finds the 'radii' of the sphere and of the AdS_4 to be:

$$R_{S^7} \sim R_{AdS_4} \sim l_p N^{1/6} \;. \tag{3.14}$$

But the radii are fixed in Planck units as we take $l_p \to 0$, so the supergravity solution is to be trusted when $N \gg 1$. That is, on the field theory side, we are looking at gauge groups $U(N)$ with a large N value.

[3]See https://en.wikipedia.org/wiki/Anti-de_Sitter_space.

The ABJM solution. We have seen that the gravitational dual of a stack of M2-branes is M-theory on $AdS_4 \times S^7$. Meanwhile, the ABJM theory is constructed as the CFT living on a stack of M2-branes probing a local $\mathbb{C}^4/\mathbb{Z}_k$ singularity. What is its gravitational dual? The $\mathbb{Z}_k$ quotient acts on the complex coordinates as:

$$z_I \to e^{\frac{2\pi i}{k}} z_I \ , \tag{3.15}$$

which thus preserves an $SU(4)$ isometry. Hence, it is more natural to use different coordinates for the $AdS_4 \times S^7$ solution. In particular, the S^7 sphere can be realized as one of the Hopf fibrations:

$$S^1 \to S^7 \to \mathbb{CP}^3 \ , \tag{3.16}$$

with the metric given in terms of the $\mathbb{C}^4$ coordinates z_i:

$$ds^2_{S^7} = (d\varphi' + \omega)^2 + ds^2_{\mathbb{CP}^3} \ , \qquad ds^2_{\mathbb{CP}^3} = \frac{\sum_i dz_i d\bar{z}_i}{\rho^2} - \frac{|\sum_i z_i d\bar{z}_i|^2}{\rho^4} \ , \tag{3.17}$$

where φ' is the coordinate along the S^1, $\rho^2 = \sum_i |z_i|^2$ and ω the connection on $\mathbb{CP}^3$, such that $d\omega = J$ is the Kähler form on $\mathbb{CP}^3$. In these coordinates, the quotient on the z_i coordinates is easier to interpret and turns out to lead to the new metric:

$$ds^2_{S^7/\mathbb{Z}_k} = \frac{1}{k^2}(d\varphi + k\omega)^2 + ds^2_{\mathbb{CP}^3} \ , \qquad \varphi \sim \varphi + 2\pi \ , \tag{3.18}$$

where $\varphi' = \varphi/k$. Hence, the ABJM theory is dual to the near horizon geometry $AdS_4 \times S^7/\mathbb{Z}_k$.

4 ABJM and 4d S-folds

We started these notes by mentioning that 3d $\mathcal{N} = 6$ theories have the same amount of supersymmetry as 4d $\mathcal{N} = 3$ theories. We won't dwelve into many details here, but let us present some curious results due to [1].

Consider a stack of N D3-branes in IIB, positioned along the (0123) coordinates. The remaining coordinates parametrise a $\mathbb{C}^3$, with $z_1 = (45)$, $z_2 = (67)$, $z_3 = (89)$. The worldvolume theory of the stack is simply given by 4d $\mathcal{N} = 4$ $U(N)$ SYM. Next, we add an O3 plane to the geometry, as follows:

	0	1	2	3	4	5	6	7	8	9
N D3	×	×	×	×						
O3	×	×	×	×						

(4.1)

The O3 action can be decomposed into $\mathcal{I}(-1)^{F_L}\Omega$, where:

$$\mathcal{I} : (z_1, z_2, z_3) \to (-z_1, -z_2, -z_3) \ , \tag{4.2}$$

while Ω is the worldsheet reversal and $(-1)^{F_L}$ acts as a ± 1 on the RR and RNS states, respectively. Together, one has:

$$(-1)^{F_L}\Omega :\ (B_2, C_2) \to (-B_2, -C_2)\ , \tag{4.3}$$

on the B_2 and C_2 fields of IIB. This setup is better understood when considering the uplift to F-theory, where this action becomes purely geometrical. In particular, the B_2 and C_2 forms combine in the τ parameter of the F-theory T^2, and the $(-1)^{F_L}\Omega$ can be interpreted as:

$$(-1)^{F_L}\Omega :\ \tau \to \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} \tau\ . \tag{4.4}$$

Viewing the torus as $T^2 \cong \mathbb{C}/\mathbb{L}$, for some lattice $\mathbb{L}$, with u the $\mathbb{C}$ coordinate, we thus have:

$$\mathcal{I}(-1)^{F_L}\Omega :\ (z_1, z_2, z_3, u) \to (-z_1, -z_2, -z_3, -u)\ . \tag{4.5}$$

This action has four fixed points, at $z_1 = z_2 = 0 = z_3 = 0$ and $u = -u \bmod \mathbb{L}$. Close to any of those fixed points, we have a $\mathbb{C}^4/\mathbb{Z}_2$ geometry. From a field theoretic perspective, this action preserves all $\mathcal{N} = 4$ SUSY in 4d, but breaks the gauge group to a subgroup $G \subset U(N)$, depending on the orientifold action on the CP factors.

The generalized geometries considered in [1] involve orbifolds of the form:

$$\mathbb{R}^{1,2} \times (\mathbb{C}^3 \times T^2)/\mathbb{Z}_k\ , \tag{4.6}$$

for $k \in \{2, 3, 4, 6\}$, with the $\mathbb{Z}_k$ action:

$$z^i \to e^{2\pi i v_k^i} z^i\ , \qquad v_k = (1, -1, 1, -1)/k\ . \tag{4.7}$$

These values of k are the only possibilities that are well-defined on the torus, and, for $k \neq 2$, such a $\mathbb{Z}_k$ symmetry acts as an involution only for fixed values of the complex structure parameter τ. Moreover, these geometries turn out to preserve only 12 supercharges.

Now, for the holographic description, it is instructive to start with the stack of D3-branes, which are dual to $AdS_5 \times S^5$. Then, under the quotient described above, the new near-horizon geometry becomes $AdS_5 \times S^5/\mathbb{Z}_k$, with a non-trivial $\mathrm{SL}(2,\mathbb{Z})$ bundle over $S^5/\mathbb{Z}_k$. So how do we relate this system to the ABJM theory? The answer lies in the S^1 compactification of the 4d theory. In particular, after this circle compactification, the system admits a dual description as a stack of $M2$-branes moving on a background with singularities of the type $\mathbb{C}^4/\mathbb{Z}_k$. Thus, one should expect that flows to ABJM theories can be realised explicitly [1].

References

[1] I. n. García-Etxebarria and D. Regalado, $\mathcal{N} = 3$ *four dimensional field theories*, *JHEP* **03** (2016) 083, [1512.06434].

[2] O. Aharony, O. Bergman, D. L. Jafferis and J. Maldacena, *N=6 superconformal Chern-Simons-matter theories, M2-branes and their gravity duals*, JHEP **10** (2008) 091, [0806.1218].

[3] O. Aharony, O. Bergman and D. L. Jafferis, *Fractional M2-branes*, JHEP **11** (2008) 043, [0807.4924].

[4] Y. Tachikawa and G. Zafrir, *Reflection groups and 3d $\mathcal{N} \geq 6$ SCFTs*, JHEP **12** (2019) 176, [1908.03346].

[5] C. Closset and H. Kim, *Three-dimensional $\mathscr{N} = 2$ supersymmetric gauge theories and partition functions on Seifert manifolds: A review*, Int. J. Mod. Phys. A **34** (2019) 1930011, [1908.08875].

[6] A. Hanany and E. Witten, *Type IIB superstrings, BPS monopoles, and three-dimensional gauge dynamics*, Nucl. Phys. B **492** (1997) 152–190, [hep-th/9611230].

[7] P. K. Townsend, *The eleven-dimensional supermembrane revisited*, Phys. Lett. B **350** (1995) 184–187, [hep-th/9501068].

[8] J. M. Maldacena, *The Large N limit of superconformal field theories and supergravity*, Adv. Theor. Math. Phys. **2** (1998) 231–252, [hep-th/9711200].

[9] M. J. Duff, G. W. Gibbons and P. K. Townsend, *Macroscopic superstrings as interpolating solitons*, Phys. Lett. B **332** (1994) 321–328, [hep-th/9405124].

[10] C. Closset, *Studies of fractional D-branes in the gauge/gravity correspondence & flavored Chern-Simons quivers for M2-branes*. PhD thesis, U. Brussels, Brussels U., PTM, 2010.