# Continuous-time Markov chains (CTMCs)

**Horia Magureanu**

*E-mail:* magureanuhoria@gmail.com

## Contents



## 1 Continuous-time Markov chains

Similar to the definition of a discrete Markov chain, we say that the process $\{X_{t\geq 0}\}$ with state space $\mathcal{S}$ is a CTMC if for all times $t, s \geq 0$ and non-negative integers $i, j, x(u)$, with $0 \leq u < s$, we have:

$$\mathbb{P}(X_{t+s} = j |\, X_s = i, X_u = x(u)) = \mathbb{P}(X_{t+s} = j |\, X_s = i) \ . \tag{1.1}$$

That is, the conditional distribution of the future $X_{t+s}$ given the present $X_s$ and the past $X_u$ will only depend on the present. A CTMC is said to be *time-homogeneous* if:

$$\mathbb{P}(X_{t+s} = j |\, X_s = i) = \mathbb{P}(X_{t+s} = j |\, X_0 = i) = \mathbb{P}_{ij}(t) \ . \tag{1.2}$$

We will focus here on time-homogeneous CTMCs. The memoryless property of a CTMC can be also understood as follows. Suppose that a CTMC enters state $i$ at some time, say, time $s$, and suppose that the process does not leave state $i$ (that is, a transition does not occur) during the next $t$ minutes. Let $T_i$ be the amount of time that the process stays in state $i$ before making a transition into a different state. We then have:

$$\mathbb{P}(T_i > s + t |\, T_i > s) = \mathbb{P}(T_i > t) \ , \tag{1.3}$$

for all $s, t \geq 0$. Thus, the random variable $T_i$ is memoryless, and must be exponentially distributed:[1]

$$T_i \sim \mathrm{Exp}(\lambda_i) \ . \tag{1.4}$$

$\lambda_i$ are called the transition rates of the process. As an example, consider a Birth-Death process, with the transition rates outlined in figure 1.

That is:

$$\begin{aligned} &\text{Time until next `birth':} \quad && B_i \sim \mathrm{Exp}(\lambda_i) \ , \quad i \geq 0 \ . \\ &\text{Time until next `death':} \quad && D_i \sim \mathrm{Exp}(\mu_i) \ , \quad i \geq 1 \ . \end{aligned} \tag{1.5}$$

[1] This gives an alternative way of defining a CTMC, in terms of the distribution of the amount of time spent in a state.

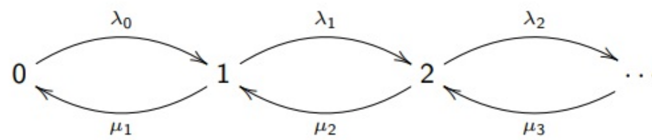


**Figure 1**: Transition rates in Birth-Death CTMC.

Then, the time until the next transition is given by $T_0 \sim \text{Exp}(\lambda_0)$, while:

$$\mathbb{P}(T_i \le x) = \begin{cases} 1 - \lambda_i \mu_i e^{-(\lambda_i + \mu_i)x} \ , & x > 0 \ , \\ 0 \ , & x \le 0 \ . \end{cases} \tag{1.6}$$

To find the transition probabilities $\mathbb{P}_{i,i+1}(t)$, we need to find the probability that the birth $B_i$ occurs before death $D_i$. For some small $\Delta t$, we have:

$$\begin{aligned} \mathbb{P}_{i,i+1}(\Delta t) &= \lambda_i \Delta t + \mathcal{O}(\Delta t) \ , \\ \mathbb{P}_{i,i-1}(\Delta t) &= \mu_i \Delta t + \mathcal{O}(\Delta t) \ , \\ \mathbb{P}_{i,i}(\Delta t) &= 1 - (\lambda_i + \mu_i)\Delta t + \mathcal{O}(\Delta t) \ . \end{aligned} \tag{1.7}$$

These infinitesimal probabilities are called the *transition rates* of the CTMC. It is customary to encode them in a *generator matrix* $\boldsymbol{Q}$, with entries $Q_{i,j}$ being the transition rate from state $i$ to state $j$, with $Q_{i,i} = -\nu_i$, where $\nu_i$ being the transition rates from state $i$.

**Probabilities from transition rates.** Now, given the transition rates, it is possible to compute the transition probabilities $\mathbb{P}_{i,j}(t)$, for arbitrary $t$. Let $\nu_i$ be the rate at which the process makes a transition when in state $i$.[2] Then, the rate $q_{i,j}$ at which the process makes a transition into state $j$ from state $i$ depends on the probability $P_{i,j}$ that the transition is into state $j$:

$$q_{i,j} = \nu_i \mathbb{P}_{i,j} \ . \tag{1.8}$$

As a result, we have:

$$\mathbb{P}_{i,j} = \frac{q_{i,j}}{\nu_i} = \frac{q_{i,j}}{\sum_j q_{i,j}} \ . \tag{1.9}$$

We then have the following two lemmas.

**Lemma 1.1**

$$\lim_{h \to 0} \frac{1 - \mathbb{P}_{i,i}(h)}{h} = \nu_i \ , \qquad \textit{and} \qquad \lim_{h \to 0} \frac{\mathbb{P}_{i,j}(h)}{h} = q_{i,j} \ , \quad \textit{when } i \neq j \ .$$

**Lemma 1.2** *(Chapman-Kolmogorov) For all $s, t \ge 0$, we have:*

$$\mathbb{P}_{i,j}(t+s) = \sum_{k \in \mathcal{S}} \mathbb{P}_{ik}(t) \mathbb{P}_{kj}(s) \ .$$

[2]For the above birth-death chain, we have $\nu_i = \mu_i + \lambda_i$.

From these, one can find:

**Theorem 1.3** *(Kolmogorov's Backward equations) For all states $i, j$ and times $t > 0$, with initial conditions $\mathbb{P}_{i,i}(0) = 1$ and $\mathbb{P}_{i,j}(0) = 0$ for all $j \neq i$, we have:*

$$\mathbb{P}'_{ij}(t) = \sum_{k \neq i} q_{ik} \mathbb{P}_{kj}(t) - \nu_i \mathbb{P}_{i,j}(t) \ .$$

*(Kolmogorov's Forward equations) Under suitable regularity conditions:*

$$\mathbb{P}'_{ij}(t) = \sum_{k \neq j} \mathbb{P}_{ik}(t) q_{kj} - \nu_j \mathbb{P}_{i,j}(t) \ .$$

**Limiting probabilities.** Assuming that the limit $\lim_{t\to\infty} \mathbb{P}_{i,j}(t) \equiv P_j$ exists, we can define it by taking the limit on the Kolmogorov forward equations. This yields:

$$\nu_j P_j = \sum_{k \neq j} q_{kj} P_k \ . \tag{1.10}$$

This set of equations, together with the normalization equation $\sum_j P_j = 1$ can be solved to obtain the limiting probabilities. When they exist, we say that the MC is *ergodic*. The $P_j$ are sometimes also called *stationary distributions*: if the initial state is chosen according to the distribution $\{P_j\}$, then the probability of being in state $j$ at time $t$ is $P_j$, for all $t \geq 0$.

More elegantly, we can write (1.10) in matrix form:

$$\boldsymbol{PQ} = 0 \ , \qquad \boldsymbol{P} = (P_0, P_1, \dots) \ . \tag{1.11}$$

For the above birth-death chain in figure 1, with an upper limit $n$, we find that the invariant probability density function is:

$$P_i = \frac{1}{\kappa_n} \frac{\lambda_0 \dots \lambda_{i-1}}{\mu_1 \dots \mu_i} \ , \qquad \kappa_n = \sum_{i=0}^{n} \frac{\lambda_0 \dots \lambda_{i-1}}{\mu_1 \dots \mu_i} \ . \tag{1.12}$$

**Absorption.** Often times, in such birth-death processes we can have absorbing states. Consider a scenario where state $\{0\}$ is absorbing. Then, the population will either be extinct at some point, or it will 'explode'. Let $T = \min\{t \in [0, \infty) : X_t = 0\}$ be the time that the chain reaches the absorbing state $\{0\}$. The absorption probability is given by:

$$v(i) = \mathbb{P}(T \leq \infty) = \mathbb{P}(X_t = 0 \text{ for some } t \in [0, \infty) | X_0 = i] \ . \tag{1.13}$$

The result of this expression can be found in [1]. The idea is to reduce the computation to a discrete-time analogue. Additionally, we can also define the mean time to extinction as $\tau_i = \mathbb{E}[T|X_0 = i]$, which can be computed from first principles. We refer again to [1].

## References


[1] "Random Services: Continuous-Time Birth-Death Chains." https://www.randomservices.org/random/markov/BirthDeath2.html#int/.