Elliptic curves over function fields

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Contents

1 Introduction

Elliptic curves have been an important tool for advances in algebraic and number theory. A recent example of their use was the proof of Fermat's Last theorem by Wiles, Taylor *et al.*. Given an elliptic curve E defined over a field K, it is customary to consider the set of points on E, which turn out to form a commutative group. Additionally, the K-rational points of E form a subgroup, usually referred to as the Mordell-Weil (MW) group.

The celebrated Mordell-Weil theorem states that the MW group is finitely generated when $K = \mathbb{Q}$. This result was later extended to number fields, which are extensions of $\mathbb{Q}$ of finite degree. Moreover, the theorem has also been generalized to function fields $K = k(\mathcal{C})$ of dimension one, which will be the main interest of this work. In this setting, the formal variable of the function fields is interpreted as the affine coordinate of the base of an elliptic fibration. As a result, the K-rational points of the elliptic fibers become global sections of this fibration, and can be interpreted as divisors on the elliptic surface.

Several quotients can be defined from the group of divisors of an algebraic variety, using equivalence relations. Of main importance are the Picard group and the Néron-Severi group, which encodes the discrete part of the former group. A theorem due to Lang and Néron states that the Néron-Severi group is finitely generated for an algebraic variety and, when restricting to elliptic surfaces, it also becomes torsion free. At this stage, it might not come as a surprise that this group is related to the Mordell-Weil group of the elliptic curve over the function field $K = k(\mathcal{C})$, which we hope to get to by the end of the review.

Our main reference will be the beautiful book 'Mordell-Weil lattices' of Schütt and Shioda [1]. We will, at times, use [2] for certain aspects of elliptic curves, as well as other works preceeding [1], such as [3, 4]. Certain aspects of algebraic geometry will be presented following [5, 6].

2 Elliptic surfaces

2.1 Function fields and elliptic fibrations

Let us consider an elliptic curve E defined over a field K. Recall that when $\mathrm{char}(K) \neq 2, 3$, the elliptic curve is birationally equivalent to a cubic:

$$y^2 = x^3 + Ax + B \ , \qquad A, B \in K \ , \tag{2.1}$$

where the above should be regarded as a projective curve in affine coordinates. Such curves always have a K-rational point, conventionally chosen as $O = (0, 1, 0)$ in projective coordinates. As stated in the introduction, the focus of this review will be the case when K is a function field $K = k(\mathcal{C})$ of a curve $\mathcal{C}$ over some field k. In such cases, it is natural to associate to such elliptic curves an elliptic surface. To see how this construction arrises, let us consider a simple example, that is the Hesse pencil:

$$E_t \qquad X^3 + Y^3 + Z^3 - tXYZ = 0 \ . \tag{2.2}$$

with the parameter $t \in k$. For a fixed value of t, E_t describes a projective cubic. However, viewing t as the affine coordinate of $\mathcal{C} \cong \mathbb{P}^1$, with $[t_0 : t_1] \in \mathbb{P}^1$ homogeneous coordinates, we can define the surface:

$$\mathcal{S} : \qquad t_0(X^3 + Y^3 + Z^3) - t_1 XYZ = 0 \ . \tag{2.3}$$

We then have a morphism $\pi : \mathcal{S} \to \mathbb{P}^1$, defined by the projection onto $\mathbb{P}^1$, which induces an elliptic fibration over $\mathcal{C}$. That is, the fibres above a point $[t_0 : t_1] \in \mathbb{P}^1$ are precisely the elliptic curves E_t. We have already pointed out that elliptic curves come with a special point O; this will be reflected at the level of the elliptic fibration as a global section, which we discuss momentarily. Moreover, there exist points $t \in \mathcal{C}$ for which the fibre is not an elliptic curve, in the sense that the discriminant vanishes. The fibres above these points are singular, and we briefly review them in the next section.

Having discussed an example of an elliptic surface, let us now formally define these objects.

Definition 2.1. *An elliptic surface is a smooth projective surface $\mathcal{S}$ with a surjective morphism $f : \mathcal{S} \to \mathcal{C}$ to a smooth projective curve $\mathcal{C}$, and a section $\sigma_0 : \mathcal{C} \to \mathcal{S}$, such that all but finitely many fibers are elliptic curves. Moreover, one requires that no fibre contains an exceptional curve of the first kind.*

As anticipated before, not all fibers of an elliptic surface are elliptic curves. In particular, some of the fibers have vanishing discriminant $\Delta = 0$, being thus singular. We call the fiber over a generic point in the base curve $\mathcal{C}$ the *generic fiber*, which is an elliptic curve defined over the function field $k(\mathcal{C})$. Note also that the global section marks a point $\sigma_0(v)$ on each fibre F_v, with $v \in \mathcal{C}$. σ_0 is called the *zero-section*, as it marks the origin of each elliptic curve.

The last condition in the above definition might appear unusual at first sight, but it is required in order to ensure that the fibres cannot be deformed arbitrarily. Recall that (-1)–curves are curves that appear as a result of blow-ups of surfaces. Thus, if this condition is not imposed, one would be free two change the 'shape' of the fibres, by blowing-up the surface.

An important aspect of the elliptic surfaces constructed from elliptic curves over function fields E/K, for $K = k(\mathcal{C})$, is the following.

Proposition 2.2. *Given an elliptic curve E over a function field $K = k(\mathcal{C})$, the global sections of the associated elliptic surface $f : \mathcal{S} \to \mathcal{C}$ are in a natural one-to-one correspondence with the K-rational points of E/K.*

Proof. Let us sketch the proof of this statement here. For a complete proof, see *e.g.* [1]. First, by definition, a section $\sigma : \mathcal{C} \to \mathcal{S}$ of the elliptic fibration defines a curve $\sigma(\mathcal{C}) \cong \mathcal{C}$ inside the elliptic surface $\mathcal{S}$. This curve meets every fibre transversally at a single point, which also turns out to be a K-rational point of the generic fibre.

Similarly, a K-rational point $P \in E(K)$ on the generic fiber F_v defines a curve in $D \subset \mathcal{S}$, as, by construction, the points vary rationally with $v \in \mathcal{C}$. This curve restricts to the base curve $\mathcal{C}$ using the morphism of the elliptic fibration :

$$f|_D : D \to \mathcal{C} \ . \tag{2.4}$$

Furthermore, this turns out to be an isomorphism, and thus the inverse of the above map gives a unique section $\sigma : \mathcal{C} \to \mathcal{S}$ with $\sigma(\mathcal{C}) \cong D$. □

This statement will be of central importance for the rest of this review. We will also denote the group of sections by $E(K)$; moreover, given a K-rational point $P \in E(K)$, we denote by (P) the curve on the elliptic surface. Let us note, however, that the group law of K-rational points in the Mordell-Weil group needs to be modified for curves on $\mathcal{S}$. In this context, it is natural to talk about divisors and the group of divisors, which we shall do next.

2.2 Group of divisors

The group law for elliptic curves can be based on the notion of divisors on a smooth curve $\mathcal{C}$. We will start with the case of elliptic curves and later generalize the construction to elliptic surfaces.

Divisors on smooth curves. Consider the formal (finite) sums of $\bar{K}$-rational points on a smooth curve $\mathcal{C}$, where $\bar{K}$ is the algebraic closure of K. Then, it is easy to see that this leads to an abelian group:

$$\mathcal{D}(\mathcal{C}) = \left\{ \sum_{i=1}^{r} n_i P_i \ : \ r \in \mathbb{N},\, n_i \in \mathbb{Z},\, P_i \in \mathcal{C}(\bar{K}) \right\} \ , \tag{2.5}$$

i.e. the divisor group of $\mathcal{C}$. Recall that the degree of a divisor $D = \sum_{i=1}^{r} n_i P_i$ is defined as:

$$\deg(D) = \sum_{i=1}^{r} n_i \ , \tag{2.6}$$

with the divisors of vanishing degree being called principal divisors. This construction can be also defined for generic fields K, with a divisor $D \in \mathcal{D}(\mathcal{C})$ being K-rational if it is fixed under the action of the Galois group $\mathrm{Gal}(\bar{K}/K)$ (see *e.g.* [7]). Let us denote the group of K-rational divisors by $\mathcal{D}(\mathcal{C})_K$.

Instead of using the divisor group, it is customary to define an equivalence relation, called *linear equivalence*, as follows. Two divisors are said to be linearly equivalent if they differ by a principal divisor. The quotient of $\mathcal{D}(\mathcal{C})$ by this equivalence relation is the Picard group. Specialising to an elliptic curve $\mathcal{C} = E$ over K, let us denote by $J(K)$ the divisor class group:

$$J(K) = \mathcal{D}^0(E)_K / \sim \, , \tag{2.7}$$

where the superscript indicates that we restrict attention to principal K-rational divisors. We then have the following bijection.

Lemma 2.3. *Given an elliptic curve E/K, the map $\alpha : E(K) \to J(K)$ sending a K-rational point $P \in E(K)$ to the divisor class of $[P] - [O]$ is bijective.*

Here we denoted by $[P]$ the divisor class of degree one corresponding to a point $P \in E(K)$. The result is perhaps not surprising given the above construction, but emphasizes the need for linear equivalence classes. Moreover, the lemma also shows that the group law needs to be modified for divisor classes. For instance, for three collinear K-rational points on E/K, P_i, with $i = 1, 2, 3$, we have:

$$\sum_{i=1}^{3} P_i = O \, . \tag{2.8}$$

In $J(K)$, however, $\sum [P_i]$ is not the same as $[O]$, as this does not match the degree of the divisors. One has instead:

$$\sum_{i=1}^{3} [P_i] \sim 3[O] \, . \tag{2.9}$$

Néron-Severi group. Having discussed divisors on smooth curves, it is straightforward to generalise these notions to smooth surfaces $\mathcal{S}$ defined over algebraically closed fields k. We define the divisor group as before:

$$\mathcal{D}(\mathcal{S}) = \left\{ \sum_{i=1}^{r} n_i \mathcal{C}_i \, : \, r \in \mathbb{N}, \, n_i \in \mathbb{Z}, \, \mathcal{C}_i \subset \mathcal{S}_i \text{ irreducible} \right\} , \tag{2.10}$$

with the degree of the divisor:

$$\deg(D) = \sum_{i=1}^{r} n_i \deg \mathcal{C}_i \, . \tag{2.11}$$

Linear equivalence and the Picard group are defined as before. The Picard group has an important subgroup called the Néron-Severi group. This is based on a different equivalence relation called *algebraic equivalence*, which is more appropriate for fibrations. The idea is that two divisors are algebraically equivalent if they belong to the same 'algebraic family' of divisors on $\mathcal{S}$, with a more formal definiton given below, based on [5].

Definition 2.4. *Let X, T be arbitrary irreducible varieties. A family of divisors on X with base T is a map $f : T \to \mathcal{D}(X)$. A family is said to be algebraic if there exists a divisor $D \in \mathcal{D}(X \times T)$ such that the induced divisor D_t^* on X is defined for each $t \in T$ and $D_t^* = f(t)$. Two divisors D_1, D_2 on X are algebraically equivalent if there is an algebraic family f with two points $t_i \in T$ such that $f(t_i) = D_i$.*

This definition can be reformulated more elegantly in terms of schemes, but the above presentation shall suffice for our purposes. While this might not appear related to linear equivalence, we note the following.

Lemma 2.5. *Linear equivalence implies algebraic equivalence.*

Proof. To prove the above statement, we can restrict attention to equivalence to 0. Hence, let $D \sim 0$, so $D = (g)$ is the divisor arising from some non-zero rational function $g \in k(X)$, and consider $T = \mathbb{A}^2 \setminus (0,0)$, with coordinates $(u,v) \in T$. Then, viewing u, v and g as functions on $X \times T$, we have an algebraic family of divisors $f : T \to \mathcal{D}(X)$ defined by the divisor $C = (u + vg)$ on $X \times T$. We then have $f(1,0) = (0) = C|_{(1,0)}$, while $f(0,1) = D = C|_{(0,1)}$, and thus $D \approx 0$, where by '$\approx$' we denote algebraic equivalence. $\square$

At this stage, let us emphasize the difference between linear and algebraic equivalence for a fibration $f : \mathcal{S} \to \mathcal{C}$. Viewing the fibres as divisors on $\mathcal{S}$, two fibres are linearly equivalent if and only if they are the same, or, if the base curve is a genus zero curve. However, any two fibres are algebraically equivalent regardless of the base curve, as they are part of the same family. We claim that algebraic equivalence is also an equivalence relation, in which case we can define the quotient $\mathcal{D}(X)/\approx$, which is called the *Néron-Severi group*. Furthermore, one has the following, due to Lang.

Theorem 2.6. *Given a smooth projective variety X, the Néron-Severi group $\mathrm{NS}(X) = \mathcal{D}(X)/\approx$, is a finitely generated abelian group.*

While this result is remarkable on its own, it turns out that when one restricts attention to elliptic surfaces $\mathcal{S}$, we have a further simplification. We will not prove these statements here, and refer to [1] for details. However, we will comment on the restriction to elliptic surfaces in the next section. This final result is what leads to the connection to lattice theory, which we shall expand on in the next section.

3 (Towards the) Mordell-Weil lattices

3.1 Lattice theory basics

Before continuing our journey on elliptic surfaces, let us review some basics of lattice theory which will turn out to be useful in the next subsection.

Definition 3.1. *A lattice is a $\mathbb{Z}$-module L, together with a non-degenerate symmetric bilinear pairing with values in $\mathbb{R}$:*

$$\langle \cdot\, \cdot \rangle : L \times L \to \mathbb{R} \ .$$

Here, by non-degenerate we mean that the $\mathbb{R}$-linear extension of the pairing to $L_{\mathbb{R}} = L \otimes \mathbb{R}$ is non-degenerate.

Definition 3.2. *Two lattices are said to be isomorphic if there is a module isomorphism $\varphi : L \to L'$ that preserves the pairings:*

$$\langle \varphi(x), \varphi(y) \rangle' = \langle x, y \rangle \ , \qquad \forall x, y \in L \ .$$

Let L be a rank-r lattice, with a $\mathbb{Z}$-basis $\{\xi_i\}$, such that any $x \in L$ becomes:

$$x = \sum_{i=1}^{r} x_i \xi_i \ , \qquad x_i \in \mathbb{Z} \ . \tag{3.1}$$

Then, the matrix:

$$I = (\langle \xi_i, \xi_j \rangle) \ , \tag{3.2}$$

called the Gram matrix of the lattice L completely determines the lattice. That follows from the fact that the isomorphism classes of rank r lattices are in one-to-one correspondence with equivalence classes of non-degenerate quadratic forms in r variables. We also define the determinant of a lattice as the determinant of its Gram matrix. Moreover, we call a lattice positive-definite if its Gram matrix is positive-definite and define the signature of the lattice as the signature of I.

We will denote by L^- the opposite lattice of L, which is the module L with the pairing $-\langle \cdot, \cdot \rangle$. Moreover, the lattice $L(n)$, for $n \in \mathbb{Q}$, is the lattice obtained by multiplying the Gram matrix by n. For example, $L^- = L(-1)$. A sublattice T of L is a submodule of L such that the restriction of the pairing to $T \times T$ is also non-degenerate.

If the pairing is $\mathbb{Z}$-valued the lattice is called integral, while if it also satisfies $\langle x, x \rangle = 2\mathbb{Z}$ for all $x \in L$, then it is an even lattice, or odd otherwise. An integral lattice with $\det L = \pm 1$ is called unimodular.

The dual lattice $L^\vee$ of an integral lattice is defined as:

$$L^\vee = \{x \in L \otimes \mathbb{Q} : \langle x, y \rangle \in \mathbb{Z} \, , \, \forall y \in L\} \ , \tag{3.3}$$

with pairing naturally extended from L. Note that the index of L in $L^\vee$ is given by $|\det L|$. A sublattice T of a lattice L is primitive if the quotient L/T is torsion-free. The orthogonal complement of a sublattice T of L, defined by:

$$T^\perp = \{x \in L : \langle x, y \rangle = 0 \, , \, \forall y \in T\} \ , \tag{3.4}$$

which, by definition, is a primitive sublattice of L.

3.2 Néron-Severi lattice

Having discussed some basic aspects of lattice theory, let us return to the groups of divisors. We claimed that these groups are related to lattices and, for this reason, we must accompany them by a pairing. Let us start with the Picard group of some smooth algebraic surface X. To define a suitable pairing, let us recall some aspects of intersection theory.

Definition 3.3. *For two irreducible curves $\mathcal{C}$, $\mathcal{C}'$ on a smooth algebraic surface $\mathcal{S}$ described by local equations f, g in a neighburhood of a point $p \in X$, the intersection multiplicity at a p is defined as:*

$$\text{mult}_p(\mathcal{C}, \mathcal{C}') = \text{dim}\mathcal{O}_{X,p}/(f, g) \ .$$

Here $\mathcal{O}_{X,p}$ is the algebra of germs of regular functions at p. This definition formalizes the naive notion of multiplicity, which we can understand through a simple example. Consider, for instance, two curves $\mathcal{C}, \mathcal{C}'$ in $\mathbb{C}$ given by the equations $x^2 - y = 0$ and $y - a^2 = 0$, respectively (i.e. a parabola and a straight line). When $a \neq 0$, the curves intersect at two points $(\pm a, a^2)$, both with multiplicity 1, but when $a = 0$, there is a single point, with multiplicity 2. Alternatively, using the above definition, the coordinate ring of the intersection of the two curves is:

$$\frac{\mathbb{C}[x, y]}{(x^2 - y, y - a^2)} = \frac{\mathbb{C}[x]}{(x^2 - a^2)} = \begin{cases} \frac{\mathbb{C}[x]}{(x-a)} \oplus \frac{\mathbb{C}[x]}{(x+a)} & \text{if } a \neq 0 \ , \\ \frac{\mathbb{C}[x]}{(x^2)} & \text{if } a = 0 \ . \end{cases} \tag{3.5}$$

Hence, for $a \neq 0$, $\dim(\mathbb{C}[x]/(x \pm a)) = 1$, but it is 2 at $a = 0$, which is the result we intuitively expected. We then define the intersection number of $\mathcal{C}$ and $\mathcal{C}'$ as:

$$(\mathcal{C}.\mathcal{C}') = \sum_{p \in X} \text{mult}_p(\mathcal{C}, \mathcal{C}') \ . \tag{3.6}$$

Our goal is to define a bilinear symmetric pairing on the Picard group of X which is compatible with these intersection numbers. In fact, we have the following theorem [6].

Theorem 3.4. *The intersection number on a smooth surface X extends to a symmetric bilinear pairing* $\text{Pic}(X) \times \text{Pic}(X) \to \mathbb{Z}$.

For the proof we refer to [6]. Let us note that intersection numbers are compatible with linear equivalence and thus extend to a pairing on the Picard group, rather than just the group of divisors. Moreover, this pairing is also compatible with algebraic equivalence, inducing thus a symmetric bilinear pairing on the Néron-Severi group. To see this, we first introduce another equivalence relation called *numerical equivalence*. In particular, two divisors D, D' are said to be *numerically equivalent* if and only if they have the same intersection numbers with any curve on X:

$$D \equiv D' \iff (D.C) = (D'.C) \ , \qquad \forall C \subset x \ . \tag{3.7}$$

As a group, the Néron-Severi group is a subgroup of the Picard group, which encodes the discrete part of $\text{Pic}(X)$. Its rank, $\rho(X)$, is called the Picard number. Let us also note the following.

Lemma 3.5. *Algebraic equivalence implies numerical equivalence.*

This statement is similar to lemma 2.5. The statement can be proved by eliminating torsion from the Néron-Severi lattice, which is achieved by tensoring with $\mathbb{R}$. This essentially leads to a subgroup NS/torsion, which turns out to be defined based on numerical equivalence. Numerical equivalence is particularly useful for the following reason.

Theorem 3.6. *For an elliptic surface $\mathcal{S}$, algebraic and numerical equivalence coincide.*

To prove the above theorem, we will need to introduce some more technology, which we shall do momentarily.

The canonical divisor. We will follow the exposition of [3], which uses the Riemann-Roch theorem, combined with Serre duality. For this, we first note that there exists an alternative definition for the Picard group, in terms of invertible sheaves. Recall first that a sheaf $\mathcal{F}$ on a topological space X associates an abelian group $\mathcal{F}(U)$ to every open set $U \subset X$, with some additional 'gluing' constraints. In our case, there exists a canonical bijection that associates to each divisor class D an invertible sheaf $\mathcal{O}_{\mathcal{S}}(D)$ of functions on $\mathcal{S}$ that are regular outside D, but may have poles along the divisor D:

$$\mathcal{O}_{\mathcal{S}}(D) = \{f \in k(\mathcal{S})^{\times} : (f) + D \geq 0\} \ . \tag{3.8}$$

In this case, the identity of the Picard group becomes the structure sheaf $\mathcal{O}_{\mathcal{S}}$ (*i.e.* the sheaf of regular functions on $\mathcal{S}$), while multiplication is replaced by tensor product on sheaves.

Given this bijection, it will be useful to consider sheaf cohomology.[1] The Hodge numbers satisfy *Serre duality*:

$$h^i(\mathcal{O}_{\mathcal{S}}(D)) = h^{2-i}(\mathcal{O}_{\mathcal{S}}(\mathcal{K}_{\mathcal{S}} - D)) \ , \tag{3.9}$$

where $\mathcal{K}_{\mathcal{S}}$ is the canonical divisor of $\mathcal{S}$. In this formalism, the canonical divisor is defined as the divisor associated to the wedge product $\wedge^2\Omega^1_{\mathcal{S}}$, where $\Omega^1_{\mathcal{S}}$ is the invertible sheaf of differential 1-forms on $\mathcal{S}$. The canonical divisor will play a key role in the proof of theorem 3.6. Let as also note that when the divisor D is effective, *i.e.* all $n_i \geq 0$, then $h^0(\mathcal{O}_{\mathcal{S}}(D)) > 0$. Based on the Hodge numbers, we define the following quantity.

Definition 3.7. *The Euler characteristic of an invertible sheaf $\mathcal{L}$ on a smooth algebraic surface $\mathcal{S}$ is defined as:*

$$\chi(\mathcal{L}) = h^0(\mathcal{L}) - h^1(\mathcal{L}) + h^2(\mathcal{L}) \ .$$

Theorem 3.8. *(Riemann-Roch) Consider an invertible sheaf $\mathcal{L}$ on a smooth algebraic surface $\mathcal{S}$, with associated divisor class $D_{\mathcal{L}}$. Then:*

$$\chi(\mathcal{L}) = \chi(\mathcal{O}_{\mathcal{S}}) + \frac{1}{2}((D_{\mathcal{L}} - \mathcal{K}_{\mathcal{S}}).D_{\mathcal{L}}) \ .$$

The Riemann-Roch theorem can be used to prove the 'canonical bundle formula', which states that the canonical divisor of an elliptic surface $\mathcal{S}$ with generic fibre F is given by:

$$\mathcal{K}_{\mathcal{S}} \approx (2g - 2 + \chi)F \ , \tag{3.10}$$

where g is the genus of the base curve and $\chi = \chi(\mathcal{O}_{\mathcal{S}})$. We have already seen that any two fibers are algebraically equivalent irrespective of the base curve. Thus, we have:

$$(F.F) = 0 \ , \tag{3.11}$$

[1] Recall that sheaf cohomology is defined as the right derived functor of the $\mathrm{Hom}(\mathcal{O}_X, -)$ functor.

as we can compute instead the intersection number of two distinct fibers, which vanishes as the two fibers do not intersect. Hence, we also have:

$$(\mathcal{K}_{\mathcal{S}}.F) = (\mathcal{K}_{\mathcal{S}}.\mathcal{K}_{\mathcal{S}}) = 0 \ , \tag{3.12}$$

which is equivalent to saying that the canonical divisor is a vertical divisor. Given this technology, let us attempt the proof of theorem 3.6. The last important piece of information that we require is that $\chi(\mathcal{S}) = \chi(\mathcal{O}_{\mathcal{S}}) > 0$ for any elliptic surface. This is not difficult to prove and it follows from Noether's formula, but it requires some more machinery that we have not developed yet. Thus, we refer to *e.g.* [1] for an explicit proof.

Proof. (*Thm 3.6*) We have already claimed in lemma 2.5 that one direction of the proof holds for any algebraic variety. The converse, however, is not usually true. Let us thus start with a divisor that is algebraically equivalent to 0, i.e. $D \equiv 0$. Then, from the Riemann-Roch theorem, we have:

$$\chi(\mathcal{O}_{\mathcal{S}}(D)) = \chi + \frac{1}{2}\mathcal{D}^2 - \frac{1}{2}D.\mathcal{K}_{\mathcal{S}} \ , \tag{3.13}$$

with the last two terms vanishing, due to algebraic equivalence to 0. Then, using the definition of the Euler characteristic, as well as $\chi \geq 0$, we have:

$$h^0(\mathcal{O}_{\mathcal{S}}(D)) + h^2(\mathcal{O}_{\mathcal{S}}(D)) > 0 \ . \tag{3.14}$$

The first possibility, $h^0(\mathcal{O}_{\mathcal{S}}(D)) > 0$ implies that D is an effective divisor and, together with $D \equiv 0$, leads to the linear equivalence $D \sim 0$. Thus, since linear equivalence implies algebraic equivalence, one automatically has $D \approx 0$ as well.

For the second case, $h^0(\mathcal{O}_{\mathcal{S}}(D)) > 0$, by Serre duality it follows that $D' \sim \mathcal{K}_{\mathcal{S}} - D$ is effective instead. Then, we note that the intersection number of D' with the generic fiber F vanishes, as both $(F.\mathcal{K}_{\mathcal{S}})$ and $(F.D)$ vanish, and thus D' is a vertical divisor. Thus, D' must be algebraically equivalent to some multiple of the fibre:

$$D \approx mF \ , \qquad m \in \mathbb{Z} \ . \tag{3.15}$$

But since the intersection with the zero section $(D.(O))$ does also vanish (as $D \equiv 0$), this can only happen for $m = 0$, so $D \approx 0$. □

Finally, having proven this remarkable result, let us state the following theorem.

Theorem 3.9. *For an elliptic surface $\mathcal{S}$, the Néron-Severi group* $\mathrm{NS}(\mathcal{S})$ *is finitely generated and torsion-free.*

At this stage, let us briefly recap the main results obtained so far. We have seen that there is a natural isomorphism from the K-rational points of the generic fiber of an elliptic surface to global sections of the elliptic fibration. Thus, it becomes natural to talk about divisors on $\mathcal{S}$, which, up to linear equivalence, form the Picard group. The discrete part of this group is the Néron-Severi group, which is always finitely generated, but, when $\mathcal{S}$ is an elliptic surface it also becomes torsion free. Moreover, the Néron–Severi group $\mathrm{NS}(\mathcal{S})$

becomes an integral lattice with respect to the intersection pairing. The remaining question is how is the Néron-Severi group related to the Mordell-Weil group of the elliptic curve over the function field $k(\mathcal{C})$, which we shall answer in the remaining part of the review.

We should point out that the finiteness part holds for generic algebraic varieties, as shown by Lang (theorem 2.6). The proof of this statement reduces to the Mordell-Weil theorem for the higher-dimensional Jacobian variety of an auxiliary curve on the variety in question. For elliptic surfaces, the proof simplifies, reducing to the statement that algebraic and numerical equivalence are the same equivalence relation.

3.3 Singular fibers

For the rest of this section let $\mathcal{S} \to \mathcal{C}$ be an elliptic surface with a section. Let us first discuss some aspects of the singular fibers. Recall that our definition of an elliptic surface involved singular fibers, that is elliptic curves with vanishing discriminant. In the Weierstrass model (2.1), vanishing discriminant leads to one of the following possibilities:

$$y^2 = (x-\alpha)^2(x-\beta) \ . \tag{3.16}$$

Then, if $\alpha \neq \beta$ the singularity of the curve is a node, while if $\alpha = \beta$, it is a cusp. Thus, returning to elliptic curves over function fields, the Weierstrass model has codimension-one singularities along the discriminant locus $\Delta(t) = 0$, with $t \in \mathcal{C}$. Each singular fiber can be resolved in a canonical fashion by blowing-up the singular points. The resulting smooth surface:

$$\pi : \tilde{\mathcal{S}} \longrightarrow \mathcal{S} \ , \tag{3.17}$$

called the Kodaira-Néron model, is birational to the original Weierstrass model $\mathcal{S}$. Moreover, assuming that the singular fibers F_v of the Weierstrass model lie above points $t_v \in \mathcal{C}$, we can decompose them in the resolved surface in terms of the exceptional divisors:

$$\pi^{-1}(t_v) = F_v \cong \sum_{i=0}^{m_v-1} \hat{m}_{v,i}\Theta_{v,i} \ , \tag{3.18}$$

with $\Theta_{v,i}$ the irreducible fiber components of multiplicity $\hat{m}_{v,i}$.

Theorem 3.10. *There exists a unique component of F_v which intersects the zero section O, called the identity component.*

The identity component is usually denoted by Θ_0. Recall that singular fibres are, in fact, curves with a cusp or with a node, in the Weierstrass model. Their singularities can be resolved through blow-ups, but it is the original component of the smoothed curve that turns out to be the identity component.

Corollary 3.11. *The lattice T_v generated by the pairing $-(\Theta_i.\Theta_j)$ is a positive-definite lattice isomorphic to the root lattice of type A, D, E.*

The above corollary essentially states that the fiber components of some singular fiber intersect according to the Dynkin diagram of the A, D, E groups. We will exemplify this statement through an example. For this, we will consider the following Weierstrass model:

$$y^2 = x^3 - 12(t^2 - t + 1)x - 2(8t^3 + 15t^2 - 12t + 8) \ , \tag{3.19}$$

which is an elliptic surface with base curve $\mathcal{C} = \mathbb{P}^1$, with discriminant:

$$\Delta(t) = 2916t^3(16t^2 - 13t + 8) \ . \tag{3.20}$$

We thus see that the curve is smooth for $t \neq 0$, as well as for t not being a root of the quadratic. This corresponds to an elliptic surface with three singular fibers.[2] In all cases, the elliptic curve becomes:

$$y^2 = (x-\alpha)^2(x-\beta) \ , \qquad \alpha \neq \beta \ , \tag{3.21}$$

which is a curve with a node. However, using the Jacobi criterion, only one of the three nodes does correspond to a surface singularity, that is $(x, y, t) = (2, 0, 0)$. In particular, the singular fibers at $t = t_{1,2}$, for $t_{1,2}$ the roots of the quadratic term in the discriminant, are already irreducible, being usually referred to as I_1 singular fibers in Kodaira's classification of singular fibers.

The node of the fiber at $t = 0$ is, however, a true surface singularity, and, thus, a reducible curve. We call the smooth part of (3.19) the Θ_0 component of the reducible fiber. To resolve the surface singularity, we view (x, y, t) as coordinates of the affine space $\mathbb{A}^3$ and blow-up the ambient space:

$$B_0\mathbb{A}^3 = \{((x, y, z), [u : v : w]) \in \mathbb{A}^3 \times \mathbb{P}^2 : (x-2)v = yu,\ (x-2)w = zu,\ yw = zv\} \ . \tag{3.22}$$

We then consider the $u = 1$ patch of the $\mathbb{P}^2$, where the strict transform of (3.19) becomes:

$$v^2 = 16(x-2)w^3 + 6(5-2x)w^2 + 4 + x - 12w \ , \tag{3.23}$$

with the exceptional divisor:

$$v^2 = 6(1+w)^2 \ , \qquad x = 2 \ . \tag{3.24}$$

Using the Jacobi criterion, we immediately see that the strict transform (3.23) is smooth in the $u = 1$ patch. This analysis can be repeated for the other two patches, and one can check that the surface is smooth everywhere. We also note that the above exceptional divisor splits into to distinct components which we will call $\Theta_{1,2}$:

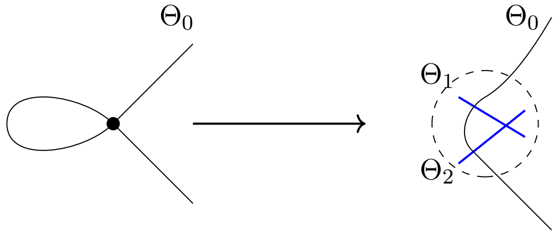
$$\begin{aligned} \Theta_1 &: \ x = 2 \ , \qquad v = \sqrt{6}(1+w) \ , \\ \Theta_2 &: \ x = 2 \ , \qquad v = -\sqrt{6}(1+w) \ , \end{aligned} \tag{3.25}$$

which intersect at one point $\big((x, y, t), [u : v : w]\big) = \big((2, 0, 0), [1 : 0 : -1]\big)$, with multiplicity one. Moreover, the Θ_0 component in this coordinate patch reads:

$$\Theta_0 : \quad w = 0 \ , \qquad v^2 = 4 + x \ , \tag{3.26}$$

which intersects Θ_1 and Θ_2 at $\big((x, y, t), [u : v : w]\big) = \big((2, 0, 0), [1 : \pm\sqrt{6} : 0]\big)$, respectively, again with multiplicities one. Schematically, we have the following diagram:

[2] The surface also has a singular fiber at 'infinity'.



Computing the self-intersection numbers, we thus notice that the intersection matrix $-(\Theta_i.\Theta_j)$ is closely related to the A_2 Lie algebra, in agreement with the corollary 3.11. In fact, the *dual graph* of the reducible fibre obtained by drawing vertices for each component Θ_i and edges according to the intersection numbers $(\Theta_i.\Theta_j)$, that reproduces the affine A_2 Dynkin diagram. The T_v lattice from corollary 3.11, however, is generated by the components $\Theta_{v\neq 0}$, having Gram matrix:

$$I = \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix} \ , \tag{3.27}$$

with discriminant $\det T_{I_3} = 3$. More generally, the determinant is given by the number of components in (3.18) having multiplicity one, and thus these lattices are clearly positive-definite.

In the above example we showed explicitly how singular fibers generate a lattice that is isomorphic to the root lattice of one of the A, D, E algebras. Let us now return to the Néron-Severi lattice, and see how these root lattices are embedded in this larger lattice.

Definition 3.12. *The trivial lattice* $\mathrm{Triv}(\mathcal{S})$ *is the sublattice of* $\mathrm{NS}(\mathcal{S})$ *generated by the zero section and the fibre components.*

We have already mentioned that fibers are algebraically equivalent and, thus, the only fibre components one has to consider for $\mathrm{Triv}(\mathcal{S})$ are a general fibre F and the components of the singular fibres not met by the zero section. This leads to the decomposition:

$$\mathrm{Triv}(\mathcal{S}) = \langle O, F\rangle \oplus \bigoplus_{v\in R} T_v^- \ , \tag{3.28}$$

with R denoting the set of points in the base curve that have singular fibers. Moreover, we have the following proposition.

Proposition 3.13. *The divisor classes* $\{(O), (F), (\Theta_{v,i})\}$*, for* $v \in R$ *and* $1 \leq i \leq m_v - 1$ *form a* $\mathbb{Z}$*-basis of the trivial lattice.*

Proof. The proof of this proposition follows from the canonical bundle formula, applied to any section P of the elliptic surface $\mathcal{S}$, such that the self-intersection number reads:

$$(P)^2 = -\chi(\mathcal{S}) \ . \tag{3.29}$$

As a result, the first term in the trivial lattice has the Gram matrix:

$$\langle O, F\rangle \cong \begin{pmatrix} -\chi & 1 \\ 1 & 0 \end{pmatrix} \ . \tag{3.30}$$

This is a matrix of determinant -1 and signature $(1,1)$. For the second term of $\mathrm{Triv}(\mathcal{S})$, the generators are $(\Theta_{v,i})$, as seen in the example above and in corollary 3.11, leading thus to a lattice of signature $(1, 1+\sum_{v\in R}(m_v-1))$. □

3.4 MW lattices

We now get to the final theorem of this review, which relates the Mordell-Weil group of the generic fiber of an elliptic surface to the Néron-Severi group.

Theorem 3.14. *Let $f:\mathcal{S}\to\mathcal{C}$ be an elliptic surface with generic fiber E over the field of rational functions $K=k(\mathcal{C})$. Then, the map $P\mapsto(P) \bmod \mathrm{Triv}(\mathcal{S})$ defines an isomorphism of abelian groups:*

$$E(K)\cong \mathrm{NS}(\mathcal{S})/\mathrm{Triv}(\mathcal{S})\ .$$

This theorem was proved by Shioda, see *e.g.* [1], and implies the following result.

Theorem 3.15. *Let E be an elliptic curve over the function field $K=k(\mathcal{C})$, with $\mathcal{C}$ and algebraic curve. Then, the abelian group $E(K)$ is finitely generated.*

We have seen so far that both groups $\mathrm{Triv}(\mathcal{S})$ and $\mathrm{NS}(\mathcal{S})$ come with a natural 'pairing', forming thus lattices. The same is in fact true for the Mordell-Weil group $E(K)$, which forms the so-called Mordell-Weil lattice. The discussion here is slightly more subtle, due to non-trivial torsion of the Mordell-Weil group $E(K)$, and is based on the group homomorphism:

$$\varphi: E(K)\longrightarrow \mathrm{NS}(\mathcal{S})_{\mathbb{Q}}\ , \tag{3.31}$$

with $\ker(\varphi)=E(K)_{tors}$. This map, sometimes called the Shioda map, can be used to define the 'height pairing' on $E(K)$, inducing thus the lattice structure we were seeking.

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