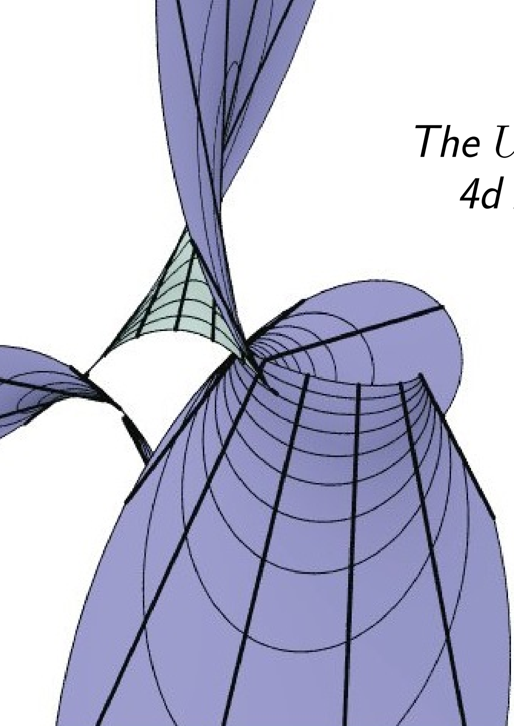

The U-plane of rank-one 4d $\mathcal{N}=2$ KK theories

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Based on

2107.03509 with C. Closset

2203.03755

Motivation

- Seiberg and Witten determined the exact low-energy solutions of $\mathcal{N} = 2$ supersymmetric gauge theories using 'Seiberg-Witten curves'. [Seiberg, Witten 1994]
- Simplest 4d $\mathcal{N} = 2$ SCFTs were found on the Coulomb branches of $SU(2)$ gauge theories with N_f fundamentals, based on their Seiberg-Witten geometries. [Argyres, Douglas 1995; Argyres, Plesser, Seiberg, Witten 1996]

$$
\begin{tikzcd}
N_f = 4 \arrow[r] \arrow[dr] & N_f = 3 \arrow[r] \arrow[d] & N_f = 2 \arrow[r] \arrow[d] & N_f = 1 \arrow[r] \arrow[d] & N_f = 0 \\
 & \mathrm{AD}(H_2) \arrow[r] & \mathrm{AD}(H_1) \arrow[r] & \mathrm{AD}(H_0) &
\end{tikzcd}
$$

5d SCFTs

- Consider a 5d Yang-Mills theory:

$$S = \int d^5x \left(\frac{1}{g_{5d}^2} Tr(F \wedge *F) + \dots \right) \ .$$

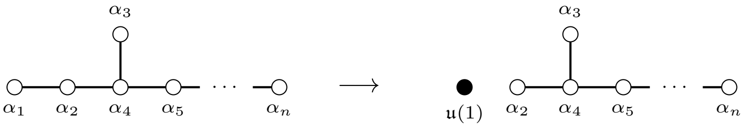
  The scaling dimension of the gauge coupling is negative, so action is non-renormalizable: $[g_{5d}^2] = -1$.
- UV completion of certain $\mathcal{N} = 1$ theories to strongly coupled SCFTs has been argued to exist in string theory. [Seiberg 1996]
- Various ways to engineer 5d SCFTs: systems of five-branes in IIB [Aharony, Hanany, Kol 1997], or M-theory compactifications on *canonical threefold singularities* [Witten 1996]:

$$\mathcal{T}_{\boldsymbol{X}}^{5d} \text{ on } \mathbb{R}^5 \longleftrightarrow \text{ M-theory on } \mathbb{R}^5 \times \boldsymbol{X} \ .$$

5d SCFTs

- Focus for today will be on the simplest 5d SCFTs - the E_n Seiberg theories. [Seiberg 1996; Morrison, Seiberg 1996]
- These theories admit mass deformations to 5d $\mathcal{N}=1$ gauge theories in IR, with $N_f = n-1$ fundamentals, which break the flavour symmetry:

$$E_n \qquad \longrightarrow \qquad U(1)_T \times SO\big(2(n-1)\big) \ .$$



- Are there non-trivial RG flows from 5d $\mathcal{N}=1$ SCFTs to 4d $\mathcal{N}=2$ SCFTs? Partial answer given by [Ganor, Morrison, Seiberg 1996]

Motivation

$$
\begin{tikzcd}
 & & & & & & & E_1 \arrow[dd, bend left] & \\
E_8 \arrow[r] \arrow[d] & E_7 \arrow[r] \arrow[d] & E_6 \arrow[r] \arrow[d] & E_5 \arrow[r] \arrow[d] & E_4 \arrow[r] \arrow[d] & E_3 \arrow[r] \arrow[d] \arrow[ddl, dotted] & E_2 \arrow[r] \arrow[d] \arrow[ur] & \widetilde{E}_1 \arrow[r] \arrow[d] & E_0 \\
E_8 \arrow[r] & E_7 \arrow[r] & E_6 \arrow[r] & D_4 \arrow[r] \arrow[dr] & D_3 \arrow[r] \arrow[d] & D_2 \arrow[r] \arrow[d] & U(1) \arrow[r] \arrow[d] & \emptyset & \\
 & & & & A_2 \arrow[r] & A_1 \arrow[r] & \emptyset & &
\end{tikzcd}
$$

- Can we find different RG flows? Seiberg-Witten geometry should be able to tell us! Expect more 'flows' from relation to q-Painleve equations. [Bonelli, Del Monte, Tanzini, Grassi ...]
- Here $E_5 = D_5$, $E_4 = A_4$, $E_3 = A_2 \oplus A_1$, $E_2 = A_1 \oplus \mathfrak{u}(1)$, $E_1 = A_1$, $\widetilde{E}_1 = \mathfrak{u}(1)$ and $E_0 = \emptyset$, so more breaking patterns allowed purely from a group theory perspective.

Motivation

- Part of motivation has to do with localization and the so called u-plane integral. [Moore, Witten 1997; Korpas, Manschot, Nidaiev, Aspman, Furrer...]
- In our setting, more natural to consider the 5d theories on $\mathbb{R}^4 \times S^1$, leading to 4d $\mathcal{N} = 2$ KK theories:

$$D_{S^1}\mathcal{T}^{5d}_{\boldsymbol{X}} \text{ on } \mathbb{R}^4 \quad \cong \quad \mathcal{T}^{5d}_{\boldsymbol{X}} \text{ on } \mathbb{R}^4 \times S^1_\beta \ .$$

- We introduce a scale $\mu = \frac{1}{\beta}$, so $D_{S^1}\mathcal{T}^{5d}_{\boldsymbol{X}}$ are not conformal. Can we determine the BPS spectra of these theories? [Eager, Selmani, Walcher, 2016; Banerjee, Longhi, Romo, 2019, 2020; Closset, Del Zotto, 2019; Longhi, 2020; Mozgovoy, Pioline, 2020]

Outline

1. Seiberg-Witten theory and rational elliptic surfaces
2. U-plane and local mirror symmetry
3. Modularity and BPS quivers

Seiberg-Witten theory and rational elliptic surfaces

4d $\mathcal{N} = 2$ SYM

- Classical vacua of 4d $\mathcal{N} = 2$ given by vanishing of scalar potential:

$$V = \frac{1}{2} \mathrm{Tr}\, [\bar{\phi}, \phi]^2 \overset{!}{=} 0 \ .$$

- In such vacua, the scalar field ϕ belongs to the Cartan subalgebra $\mathfrak{h}$ of G, e.g. for $G = SU(2)$:

$$\phi = \begin{pmatrix} a & 0 \\ 0 & -a \end{pmatrix} , \qquad a \in \mathbb{C} \ .$$

- The VEV of ϕ breaks the gauge group to $SU(2) \longrightarrow U(1)$ by Higgs mechanism, leading to a $U(1)$ LEET. This moduli space is the *Coulomb Branch*, being parametrized by the gauge invariant operator:

$$u = \mathrm{Tr}\, \phi^2 = 2a^2 \big(+ \dots \big) \ .$$

- The LEEA is obtained by integrating out the massive degrees of freedom, with the Coulomb branch having a special-Kähler structure:

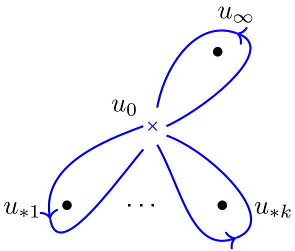
$$\tau(a) = \frac{\partial^2 \mathcal{F}}{\partial a^2} = \frac{\partial a_D}{\partial a} \ , \qquad a_D = \frac{\partial \mathcal{F}}{\partial a} \ .$$

- $\mathrm{SL}(2,\mathbb{Z})$ duality of $U(1)$ LEET acts on τ or, equivalently, on (a_D, a):

$$\tau \to \gamma\tau \ , \qquad \begin{pmatrix} a_D \\ a \end{pmatrix} \to \gamma \begin{pmatrix} a_D \\ a \end{pmatrix} \ , \qquad \gamma \in \mathrm{SL}(2,\mathbb{Z}) \ .$$

- LEEA breaks down where states become massless, leading to u-plane singularities. Due to $\mathrm{SL}(2,\mathbb{Z})$ duality, paths around singularities induce monodromies:

$$\textstyle\prod_{\mathsf{sing}} \mathbb{M}_* = 1 \ .$$



Seiberg-Witten solution

- Seiberg and Witten proposed that τ should be interpreted as the complex structure modulus of an elliptic curve, such that:

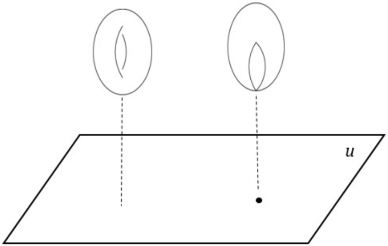
$$a = \oint_A \lambda_{SW} \ , \qquad a_D = \oint_B \lambda_{SW} \ .$$

- One can generally bring the curves to Weierstrass form:

$$y^2 = 4x^3 - g_2(u)x - g_3(u) \ ,$$

with the singularities given be the zeroes of the discriminant locus:

$$\Delta(u) = g_2(u)^3 - 27g_3(u)^2 \ .$$



Elliptic Surfaces

Definition. An *elliptic surface* is a genus one fibration $f : S \to C$ from a smooth projective surface S to a smooth projective curve C, with a section $\sigma_0 : C \to S$.

- All but finitely many fibres F_v are smooth genus one curves. The *singular* fibers can be resolved through blow-ups:

$$F_v = \sum_{i=0}^{m_v - 1} \mu_{v,i} \Theta_{v,i} \ ,$$

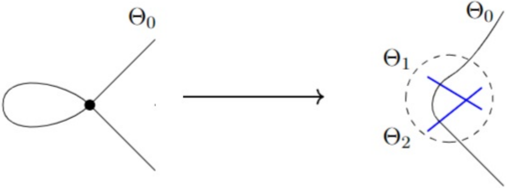
  where m_v is the number of (distinct) irreducible components, $\Theta_{v,i}$ the irreducible components and $\mu_{v,i}$ the multiplicity of $\Theta_{v,i}$.

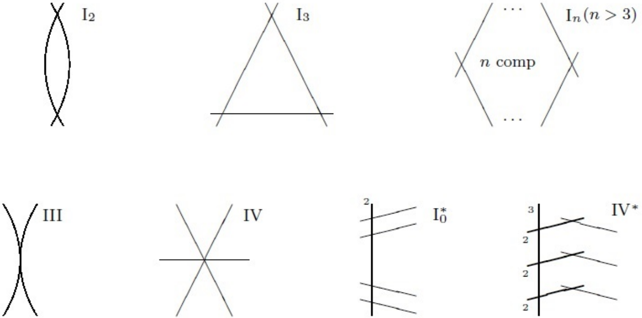
- The component denoted by $\Theta_{v,0}$ is the unique component of F_v which intersects the zero section $[\sigma_0]$.

- If F_v is irreducible, then it must be either a rational curve with a node (type I_1), or a rational curve with a cusp (type II).
- All possible reducible fibers have been classified by Kodaira:

fiber	ord(g_2)	ord(g_3)	ord(Δ)	m_v	$\mathbb{M}_*$	$\mathfrak{g}$
I_k	0	0	k	k	T^k	$\mathfrak{su}(k)$
I_k^*	2	3	$k+6$	$k+5$	$-T^k$	$\mathfrak{so}(2k+8)$
I_0^*	≥ 2	≥ 3	6	5	$-\mathbb{I}$	$\mathfrak{so}(8)$
II	≥ 1	1	2	1	$(ST)^{-1}$	-
II^*	≥ 4	5	10	9	(ST)	E_8
III	1	≥ 2	3	2	S^{-1}	$\mathfrak{su}(2)$
III^*	3	≥ 5	9	8	S	E_7
IV	≥ 2	2	4	3	$(ST)^{-2}$	$\mathfrak{su}(3)$
IV^*	≥ 3	4	8	7	$(ST)^2$	E_6

Singular fibres





Rational Elliptic Surfaces

- An elliptic surface S is *rational* if it is birationally equivalent to $\mathbb{P}^2$. In this case, the base curve C is the projective line $\mathbb{P}^1$. Can view SW geometry as a one-parameter family of elliptic curves over the u-plane:

$$E \hookrightarrow S \longrightarrow \overline{\mathcal{M}}_{CB} \cong \mathbb{P}^1 \ ,$$

  where $\overline{\mathcal{M}}_{CB}$ is the u-plane with the point at infinity added and the fiber E is the 'Seiberg-Witten' curve.
- The configurations of singular fibres of rational elliptic surfaces have been classified by Persson and Miranda.
- Important constraint:

$$\sum_v \mathrm{ord}(\Delta) = 12 \ .$$

Fixing the fibre at infinity

- 'UV physics' fixed by fiber at infinity: e.g. for pure $SU(2)$, from one-loop computation: $\tau \to \tau - 4$, i.e. the monodromy is T^{-4}, or $(-T^4)$, and thus corresponds to an I_4^* singularity. [Caorsi, Cecotti 2018]
- One can add matter (*hypermultiplets*) in the fundamental representation of the gauge group, which will lead to $I^*_{4-N_f}$ singularities. Note that for $N_f > 4$ the theories are IR free:

$$SU(2)\, N_f \text{ flavours } \longrightarrow F_\infty = I^*_{4-N_f} \ .$$

- Only one configuration with a I_4^* singularity in Persson's list, namely:

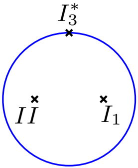
$$(I_4^*, I_1, I_1) \ .$$

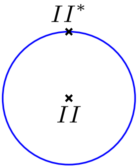
Persson's list

- Consider configurations with one flavour. There are two such configurations with a $I^*_{4-N_f} = I^*_3$ singularity, namely:

$$(I^*_3, I_1, I_1, I_1)\ , \qquad (I^*_3, I_1, II)\ .$$

- How do we interpret physically the type II singularity? 'Zoom in' around this singularity:





- This is one of the configurations for which the value of the complex structure parameter τ is fixed and, in physical terms, the coupling is pinned at the strongly-coupled value. This corresponds to one of the Argyres-Douglas $\mathcal{N} = 2$ SCFTs.

U-plane and local mirror symmetry

The U-plane at last

- We established that the fiber at infinity fixes the 4d $\mathcal{N}=2$ theory: SQCD with I_n^*, MN with II, III or IV and AD with II^*, III^*, IV^*. What about the simplest I_n fibers?
- Consider the E_n theories on a circle, engineered on $\boldsymbol{X} = \mathrm{Tot}(\mathcal{K} \to dP_n)$:

$$\text{M-theory on } \mathbb{R}^4 \times S^1 \times \boldsymbol{X} \longleftrightarrow \text{ IIA on } \mathbb{R}^4 \times \boldsymbol{X} \ .$$

- Similar to usual 4d $\mathcal{N}=2$ $SU(2)$ gauge theories, we have a low-energy $U(1)$ scalar:

$$a = i(\varphi + iA_5) \ , \qquad e^{2\pi i A_5} = e^{\int_{S^1} A} \ ,$$

  and define the gauge invariant order parameter:

$$U = \frac{1}{2}\left(e^{2\pi i a} + e^{-2\pi i a}\right) + \dots \ .$$

- Complexified mass parameters: $M = e^{2\pi i \mu}$. [Closset, del Zotto, Saxena 2018]

Local mirror symmetry

- For toric geometries, SW geometry is encoded in the mirror:

$$\text{IIA on } \tilde{\boldsymbol{X}} \longleftrightarrow \text{ IIB on } \hat{\boldsymbol{Y}} \ ,$$

where the mirror given by Hori-Vafa construction:

$$E \times \mathbb{C}^* \longrightarrow \hat{\boldsymbol{Y}} \longrightarrow \mathbb{C} \cong \{W\} \ ,$$

with:

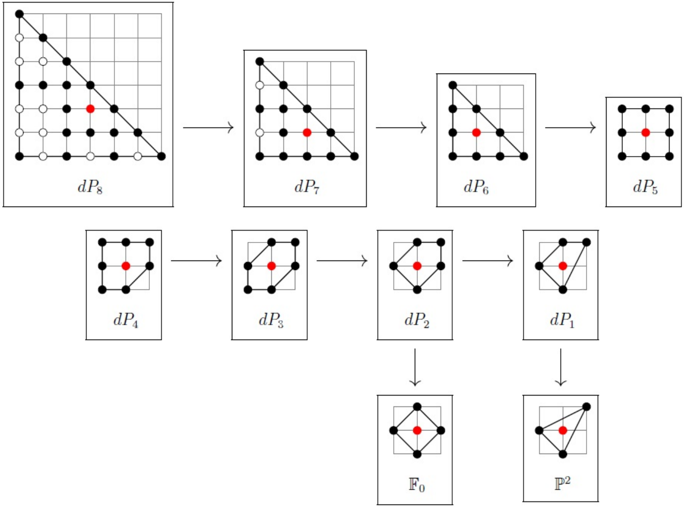
$$F(w, t; W) = 0, \qquad v_1 v_2 = U - W \ ,$$

where $F(w, t)$ is the Newton polynomial associated with the toric diagram. Their SW geometry is the mirror curve:

$$F(w, t; U) = 0 \ .$$

- For non-toric del Pezzos ($n \geq 4$), mirror curves were found as limits of the E-string theory SW curve. [Eguchi, Sakai 2002; Ganor, Morrison, Seiberg, Hanany...]

E_n theories from dP_n

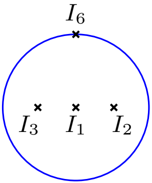


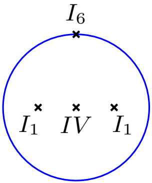
The missing piece

- Computing prepotential from gauge theory perspective for $D_{S^1}\mathcal{T}^{5d}_{\boldsymbol{X}}$, one finds monodromy: [Nekrasov 1998]

$$a_D \longrightarrow a_D + (9-n)a + \dots \ ,$$

  or $\mathbb{M}_\infty = T^{9-n}$, *i.e.* an I_{9-n} singularity!
- Look at E_3 theory, i.e. $\mathbb{M}_\infty = T^6$. In 5d, this is the UV completion of 5d $\mathcal{N}=1$ $SU(2)$ with $N_f = 2$, so we have 3 parameters in the SW curve: 2 flavour masses M_i and the exponentiated gauge coupling λ.





$(\lambda, M_1, M_2) = (1,1,1)$ is the massless limit. For $(\lambda, M_1, M_2) = (1, i, -i)$ we get a IV singularity!

Complete Picture

In the massless limit $\lambda = 1$, $M_i = 1$, we get the (bulk) singularities:

$$\begin{aligned}
E_8 &: \quad II^* \oplus I_1 \ , \\
E_7 &: \quad III^* \oplus I_1 \ , \\
E_6 &: \quad IV^* \oplus I_1 \ , \\
D_5 = E_5 &: \quad I_1^* \oplus I_1 \ , \\
A_4 = E_4 &: \quad I_5 \oplus I_1 \oplus I_1 \ , \\
A_2 \oplus A_1 = E_3 &: \quad I_3 \oplus I_2 \oplus I_1 \ , \\
A_1 \oplus \mathfrak{u}(1) = E_2 &: \quad I_2 \oplus I_1 \oplus I_1 \oplus I_1 \ , \\
A_1 = E_1 &: \quad I_2 \oplus I_1 \oplus I_1 \ , \\
\mathfrak{u}(1) = \widetilde{E}_1 &: \quad I_1 \oplus I_1 \oplus I_1 \oplus I_1 \ , \\
\emptyset = E_0 &: \quad I_1 \oplus I_1 \oplus I_1 \ .
\end{aligned}$$

Non-abelian flavour symmetry is manifest in the mirror curve. Note that this is the maximal flavour symmetry, as massless limit is the origin of the extended CB, so corresponds to S^1 reduction of the SCFT. [Ganor, Morrison, Seiberg 1996]

Complete picture

- Since configurations of singular fibers are completely classified, just look for configurations containing a I_{9-n} fiber for each E_n theory. Recall:

$$\sum_v \mathrm{ord}(\Delta(F_v)) = 12 \ .$$

$$\begin{tikzcd}
 & & & & & & & E_1 \arrow[dd, bend left] & \\
E_8 \arrow[r] \arrow[d] & E_7 \arrow[r] \arrow[d] & E_6 \arrow[r] \arrow[d] & E_5 \arrow[r] \arrow[d] & E_4 \arrow[r] \arrow[d] & E_3 \arrow[r] \arrow[d] \arrow[ddl, dotted] & E_2 \arrow[r] \arrow[d] \arrow[ur] \arrow[ddl, dotted] & \widetilde{E}_1 \arrow[r] \arrow[d] \arrow[ddl, dotted] & E_0 \\
{\color{brown}E_8} \arrow[r] & {\color{brown}E_7} \arrow[r] & {\color{brown}E_6} \arrow[r] & {\color{purple}D_4} \arrow[r] \arrow[dr] & D_3 \arrow[r] \arrow[d] & D_2 \arrow[r] \arrow[d] & U(1) \arrow[r] \arrow[d] & \emptyset & \\
 & & & & {\color{red}A_2} \arrow[r] & {\color{red}A_1} \arrow[r] & {\color{red}\emptyset} & &
\end{tikzcd}$$

Flavour symmetry

- Recall mirror threefold is a double fibration over a W-plane, at fixed U and mass parameters: [Hori, Vafa 2002]

$$E \times \mathbb{C}^* \hookrightarrow \hat{\boldsymbol{Y}} \to \mathbb{C} \cong \{W\} \ .$$

  Then, from the IIB picture:

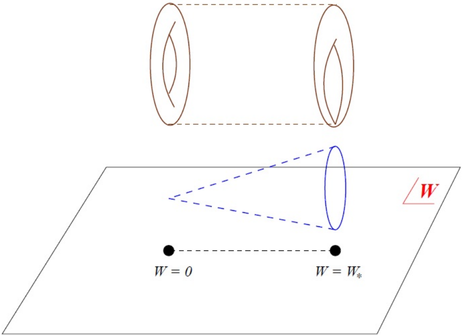
$$\text{F-theory on } \mathbb{R}^4 \times \hat{\boldsymbol{Y}} \times T^2 \ ,$$

  we interpret the elliptic fiber as the axio-dilaton instead of the T^2 factor!
- Non-abelian flavour symmetry algebra is encoded in the Kodaira fibers [Sen, Banks, Douglas, Seiberg, Aspinwall, Fukae, Yamada, Yang ...]
- Fiber at infinity does not contribute to flavour symmetry!

Local CY Mirror

$\mathbb{C}^*$ fiber degenerates at $W = U$, while cycle γ of elliptic fiber degenerates at $W = W_*$. BPS states arrise from D3 branes wrapping Lagrangian 3-cycles:

$$\gamma \times S^1_* \longrightarrow S^3_\gamma \longrightarrow \alpha_\gamma \ .$$



[Feng, He, Kennaway, Vafa]

Mordell-Weil Group

- (Rational) elliptic surfaces can be thought of as elliptic curves over function fields E/K, with $K = k(U)$.
- The K-rational points of E/K form a group (Mordell-Weil group), which, by the celebrated MW theorem, is finitely generated.

$$\Phi = \mathbb{Z}^r \oplus \Phi_{tor} \ .$$

- The K-rational points of E/K are isomorphically mapped to rational sections of the elliptic surface.

Flavour symmetry

- *Claim:* Abelian flavour symmetry is $U(1)^r$, with r the rank of the MW group. [Morrison, Park, Mayrhofer, Till, Weigand, Cvetic, Lin, Moore, Monnier ...]
- Torsion part of Φ restricts global form of the flavour symmetry. Define:

$$\mathcal{Z}^{[1]} = \{P \in \Phi_{tor} : (P) \text{ intersects } \Theta_{v,0} \text{ for all } F_{v\neq\infty}\} \ ,$$

and denote by $\mathcal{F}$ the cokernel of the inclusion map $\mathcal{Z}^{[1]} \to \Phi_{tor}$. Thus, we have the short exact sequence:

$$0 \to \mathcal{Z}^{[1]} \to \Phi_{tor} \to \mathcal{F} \to 0 \ .$$

Claim: The flavour symmetry group of the theory $\mathcal{T}_{F_\infty}$ is given by:

$$G_F = \widetilde{G}_F / \mathcal{F} \ ,$$

with $\widetilde{G}_F$ the simply connected group with algebra $\mathfrak{g}_F$ from Kodaira fibers.

Flavour symmetry

$\{F_v\}$	Φ_{tor}	*4d theory*	$\mathfrak{g}_F$	G_F
II^*, II	-	AD H_0	-	-
		E_8 MN	E_8	E_8
III^*, III	$\mathbb{Z}_2$	AD H_1	A_1	$SU(2)/\mathbb{Z}_2$
		E_7 MN	E_7	$E_7/\mathbb{Z}_2$
IV^*, IV	$\mathbb{Z}_3$	AD H_2	A_2	$SU(3)/\mathbb{Z}_3$
		E_8 MN	E_6	$E_6/\mathbb{Z}_3$
I_0^*, I_0^*	$\mathbb{Z}_2 \times \mathbb{Z}_2$	$SU(2)\,, N_f = 4$	D_4	$\text{Spin}(8)/\mathbb{Z}_2 \times \mathbb{Z}_2$

New observation for AD H_2, others also discussed recently elsewhere.

[Closset, Schafer-Nameki, Wang, Del Zotto, García Etxebarria, Hosseini, Bhardwaj, Hubner, Apruzzi, Cordova, Shao, Buican, Nishinaka]

Higher form symmetries

- Given the previous short exact sequence, it is natural to conjecture:

$$\mathcal{Z}^{[1]} \cong \text{ 1-form symmetry of } \mathcal{T}_{F_\infty} \ ,$$

 while when Φ_{tor} is a non-trivial extension, we have:

$$\Phi_{tor} \cong \text{ 2-group symmetry of } \mathcal{T}_{F_\infty} \ .$$

- There are not many examples of such theories. In fact, we only have:

$$\begin{aligned}
\text{4d pure } SU(2): &\quad (I_4^*; I_1, I_1), && \Phi = \mathcal{Z}^{[1]} = \mathbb{Z}_2 \ , \\
D_{S^1} E_0: &\quad (I_9; I_1, I_1, I_1), && \Phi = \mathcal{Z}^{[1]} = \mathbb{Z}_3 \ , \\
D_{S^1} E_1: &\quad (I_8; I_2, I_1, I_1), && \Phi = \mathbb{Z}_4 \ , \mathcal{Z}^{[1]} = \mathbb{Z}_2 \ , \\
&\quad (I_8; I_1, I_1, I_1, I_1), && \Phi_{tor} = \mathcal{Z}^{[1]} = \mathbb{Z}_2 \ ,
\end{aligned}$$

 which agree with known results. [Gaiotto, Kapustin, Seiberg, Willet 2014; Morrison, Schafer-Nameki, Willet 2020; Albertini, Del Zotto, García Etxebarria, Hosseini 2020]

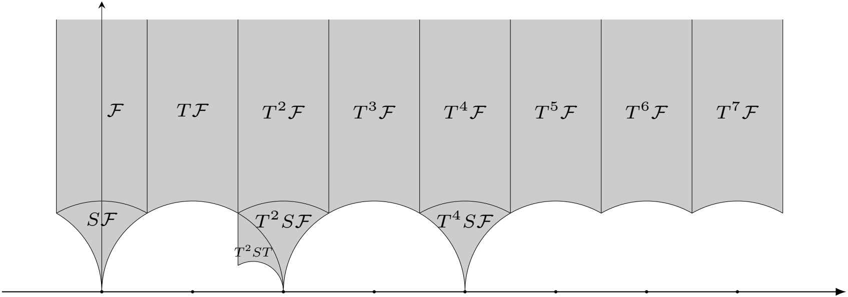
Modularity and BPS quivers

Modularity

- In certain cases, the configuration of interest is *modular*. That is, the rational elliptic surface is constructed from the quotient $\mathbb{H}/\Gamma$ with $\mathbb{H}$ the upper half-plane and $\Gamma \in \mathrm{PSL}(2,\mathbb{Z})$, together with a finite number of points.
- Subgroups of $\mathrm{PSL}(2,\mathbb{Z})$ have a set of special points: *elliptic points* - points with non-trivial stabilizer, and *cusps* - points in the $\mathrm{PSL}(2,\mathbb{Z})$-orbit of $\tau = i\infty$. These points are mapped to the U-plane singularities. [Shioda 1972]
- When RES is modular, $U = U(\tau)$ is a modular function for Γ and periods are modular forms. Can also map U-plane to upper half-plane isomorphically. [Aspman, Furrer, Manschot 2020-2021; HM, Closset 2021; HM 2022]
- Advantage is that upper-half plane knows about *monodromies*!

Example: the massless E_1

- For massless E_1, monodromy group is $\Gamma^0(8)$, having four cusps of widths $(8; 1, 2, 1)$, which correspond to fibers $(I_8; I_1, I_2, I_1)$.



- Monodromies are easy to read from fundamental domains:

$$\mathbb{M}_{\tau=0} = STS^{-1}\,, \qquad \mathbb{M}_{\tau=2} = (T^2S)T^2(T^2S)^{-1}\,, \qquad \mathbb{M}_{\tau=4} = (T^4S)T(T^4S)^{-1}\,,$$

from which we get the light BPS states:

$$(1,0), \qquad 2(-1,2), \qquad (1,-4)\ .$$

Conclusions and Outlook

- (Rank-one) Seiberg-Witten geometries have a natural interpretation as rational elliptic surfaces, where RG flows are 'trivial'. Non-abelian part of flavour symmetry algebra encoded in singular fibers, while abelian part and global aspects encoded in MW group.
- Classification of 4d $\mathcal{N} = 2$ SCFTs involves theories with *undeformable* singularities. How do we understand these geometries?
- Modularity allows one to determine light BPS states and BPS quivers. Can modularity shed some light on BPS spectra? Can we find the quiver super-potentials from the U-plane?
- Going past modularity: do fundamental domains retain information about monodromies even when configurations are not modular?

Thank you!

Persson's list

- Consider configurations with one flavour. There are two such configurations with a $I^*_{4-N_f} = I^*_3$ singularity, namely:

$$(I^*_3, I_1, I_1, I_1) \ , \qquad (I^*_3, I_1, II) \ .$$

- How do we interpret physically the type II singularity? 'Zoom in' around this singularity: