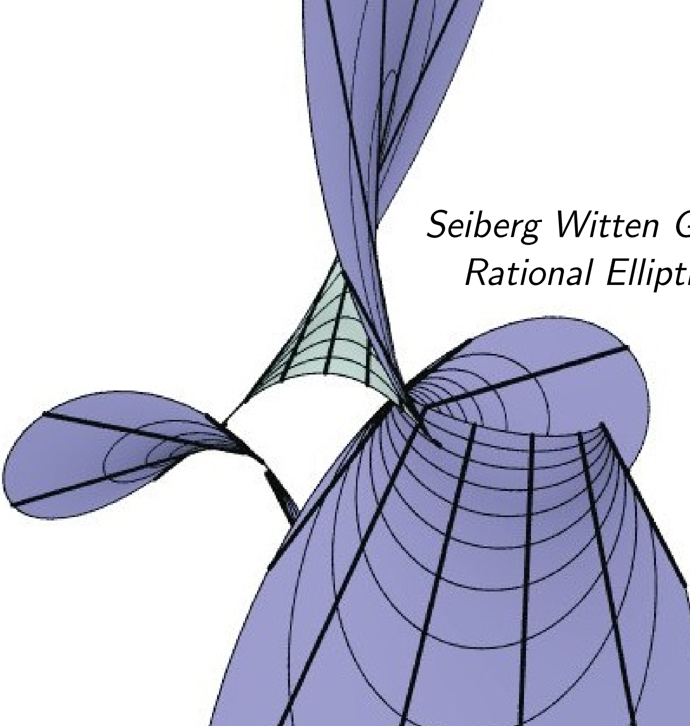
Seiberg Witten Geometry and Rational Elliptic Surfaces



Motivation

- The understanding of the strongly coupled regime of a QFT is in general a difficult task. The existence of supersymmetry allows for a better understanding of such regimes.
- Seiberg and Witten determined the exact low-energy solutions of $\mathcal{N} = 2$ supersymmetric gauge theories using 'Seiberg-Witten curves'.
- These theories also turn out to be related to Donaldson invariants and can thus lead to a better understanding of 4-manifolds.
- The classification of the rank one $4d$ $\mathcal{N} = 2$ SCFTs is based on Kodaira's classification of singular fibres.

Outline

1. Seiberg-Witten Theory
2. Rational Elliptic Surfaces
3. Coulomb branch as a Rational Elliptic Surface

Seiberg-Witten Theory

A Lighting Review of Supersymmetry

- Start with a relativistic QFT in four-dimensions. By assumption, the space-time symmetry is the Poincaré group, with generators:

$$M_{\mu\nu} \ , \qquad P_\mu \ .$$

satisfying the usual Poincaré algebra.

- By Noether's theorem these symmetries are associated with energy and momentum conservation.
- Supersymmetry is a symmetry of a QFT in which bosons and fermions are transformed into each other. Formally, we introduce a *superalgebra* which is a $\mathbb{Z}_2$ graded vector space: $A = B \oplus F$, with a bilinear multiplication $A \times A \to A$ such that:

$$bb' \in B \ , \qquad ff' \in B \ , \qquad bf \in F \ .$$

A Lighting Review of Supersymmetry

- Then, a *supersymmetry algebra* is a superalgebra whose bosonic subalgebra is the d-dimensional Poincaré algebra. Schematically, denote the bosonic/fermionic generators by: $X \in B$ and $Q \in F$, such that:

$$[X, X'] = X'' , \qquad [X, Q] = Q' , \qquad \{Q, Q'\} = X .$$

- Here we use the *super-commutator* of the superalgebra which takes into account the $\mathbb{Z}_2$ grading. The supersymmetry generators Q transform as spinors under the Lorentz group.
- Today we focus on $4d$ $\mathcal{N} = 2$ supersymmetry, i.e. $\mathcal{N}$ Weyl spinors:

$$Q^I_\alpha , \qquad \overline{Q}^I_{\dot{\alpha}} ,$$

where $I = 1, 2$, with the SUSY algebra:

$$\{Q^I_\alpha, \overline{Q}_{\dot{\beta}J}\} = 2\sigma^\mu_{\alpha\dot{\beta}} P_\mu \delta^I_J , \qquad \{Q^I_\alpha, Q^J_\beta\} = 2\epsilon_{\alpha\beta}\epsilon^{IJ} Z ,$$

$\mathcal{N} = 2$ SYM

- Fields combine into representations of the superalgebra called *supermultiplets*. The 'basic' multiplet of the $\mathcal{N} = 2$ superalgebra is the *vector multiplet*: $(A_\mu, \phi, \psi, D_{IJ})$.
- Consider the SYM theory in four-dimensions, with the symmetry group in Euclidean signature:

$$SU(2)_+ \times SU(2)_- \times SU(2)_R \times U(1)_R \ .$$

- The supercharges and fields transform as follows:

$$\begin{array}{cclccl}
\overline{Q}^I_{\dot\alpha} & : & (1,2,2)_1 \ , & \overline{\psi}^I_\alpha & : & (2,1,2)_{-1} \ , \\
Q^I_\alpha & : & (2,1,2)_{-1} \ , & \psi^I_{\dot\alpha} & : & (1,2,2)_1 \ , \\
A_\mu & : & (2,2,1)_0 \ , & \phi & : & (1,1,1)_2 \ , \\
D_{IJ} & : & (1,1,3)_0 \ , & \bar\phi & : & (1,1,1)_{-2} \ .
\end{array}$$

$\mathcal{N} = 2$ SYM

- For a gauge group G, all fields in the vector multiplet will transform in the adjoint representation. Then, the $\mathcal{N} = 2$ SYM theory is the 'supersymmetric completion' of the usual YM action:

$$\mathcal{L}_{SYM} = \frac{1}{g^2} tr\Big(-\frac{1}{4} F_{\mu\nu}F^{\mu\nu} - D_\mu \bar{\phi} D^\mu \phi - i\bar{\lambda}_I \bar{\sigma}^\mu D_\mu \lambda^I \\ -\frac{1}{2}[\bar{\phi}, \bar{\phi}]^2 - \frac{i}{\sqrt{2}} \bar{\phi} \epsilon_{IJ} [\lambda^I, \lambda^J] + \frac{i}{\sqrt{2}} \epsilon^{IJ} [\bar{\lambda}_I, \bar{\lambda}_J] \phi \Big) \ .$$

- The classical analysis of this theory is performed by looking at the scalar potential: $V = \frac{1}{2} Tr[\bar{\phi}, \phi]^2$. In particular, the classical vacua are determined by setting $V = 0$, which thus leads to $[\bar{\phi}, \phi] = 0$.
- In such vacua, the scalar field ϕ belongs to the Cartan subalgebra of G, e.g. for $G = SU(2)$:

$$\phi = \begin{pmatrix} a & 0 \\ 0 & -a \end{pmatrix} .$$

Coulomb Branch

- The VEV of ϕ breaks the gauge group by the Higgs mechanism to $G \longrightarrow U(1)$. One is left with one $U(1)$ $\mathcal{N}=2$ supersymmetric multiplet.
- This supersymmetric vacuum moduli space is caled the *Coulomb Branch*: $\mathcal{M}_C = \mathfrak{h}_{\mathbb{C}}/W_G$. It is parametrized by the gauge invariant operator $u = Tr\phi^2 = 2a^2$ for $G = SU(2)$, where $W_{SU(2)} = \mathbb{Z}_2$.
- The next question is what is the effective quantum theory for any 'u'. In principle, the 'LEEA' is obtained by integrating out the massive degrees of freedom above a low-energy cutoff.
- This is generally a very difficult task, but here we can use the constraining power of $\mathcal{N}=2$ supersymmetry: the effective action is completely determined by a holomorphic function $\mathcal{F}(a)$ called the *prepotential*.

- In SYM theory, coupling constants are naturally seen as complex couplings, which enter holomorphically. It is thus customary to define:

$$\tau = \frac{4\pi i}{g^2} + \frac{\theta}{2\pi} \ .$$

where the θ-term gives rise to the topological term:

$$\mathcal{L}_{SYM} \supset \theta F \wedge F \ .$$

- This can be non-trivial in the presence of a non-trivial gauge-field configuration, as it reproduces the Pontryagin class of a G-bundle over a four-manifold. In an instanton-background we have the weigths in the path integral:

$$e^{S_{YM}+S_{top}} = e^{-\frac{8\pi^2}{g^2}|k|} e^{i\theta k} \ .$$

- These give non-perturbative contributions to the path integral when g grows under RG flow.

LEEFT

- As mentioned before, due to $\mathcal{N} = 2$ supersymmetry, the $U(1)$ EFT fully described by the *prepotential*. One has:

$$\tau(a) = \frac{\partial^2 \mathcal{F}}{\partial a^2} \ , \qquad a_D = \frac{\partial \mathcal{F}}{\partial a} \ .$$

- The gauge coupling, and thus the prepotential, receive perturbative (only at one-loop due to supersymmetry) and non-perturbative corrections.
- Importantly, the gauge coupling τ and the prepotential $\mathcal{F}$ should be viewed as well-defined functions only locally. A priori, these are defined only in the semi-classical coordinate patch near infinity and thus the 'quantum' moduli space must be different as compared to the classical moduli space.

$SL(2,\mathbb{Z})$ Duality

- The prepotential receives quantum corrections:

$$\mathcal{F} = \frac{1}{2}\tau_0 a^2 + \frac{i}{\pi} a^2 Log\left(\frac{a^2}{\Lambda^2}\right) + \frac{a^2}{2\pi i}\sum_{k=1} c_k \left(\frac{\Lambda}{a}\right)^{4k} .$$

- From the perturbative one-loop correction, one finds that for the $SU(2)$ theory with no matter multiplets, a loop around $u \sim \infty$ will produce a shift in τ because of the logarithmic branch cut:

$$\tau \to \tau - 4 \ .$$

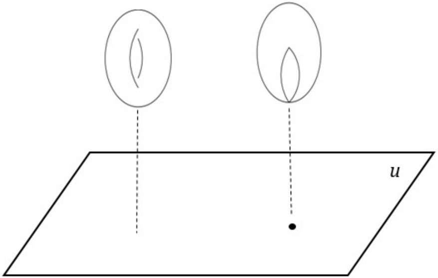
- One can perform an $SL(2,\mathbb{Z})$ transformation to obtain another description of the theory. These 'duality' transformations are a generalization of the usual duality of abelian Maxwell theory:

$$\tau \to \frac{a\tau + b}{c\tau + d} \ .$$

Seiberg-Witten solution

- Seiberg and Witten proposed that the gauge coupling τ could be interpreted as the complex structure modulus of an elliptic curve, with the right monodromies. The elliptic curve encodes information about the physical theory:

$$a = \oint_A \lambda_{SW} \ , \qquad a_D = \oint_B \lambda_{SW} \qquad \frac{d\lambda_{SW}}{du} = \omega.$$



Seiberg-Witten solution

- The SW curve for the pure $SU(2)$ theory is given by:

$$y^2 = (x^2 - u)^2 - 1 \ .$$

- More generally, one should view the SW curve as a genus-one Riemann surface with punctures, where the SW differential has simple poles, with residues given by the masses of the hypermultiplets.
- One can generally bring the curves to Weierstrass form:

$$y^2 = 4x^3 - g_2(u)x - g_3(u) \ ,$$

with the singularities given be the zeroes of the discriminant locus:

$$\Delta(u) = g_2(u)^3 - 27g_3(u)^2 \ .$$

Rational Elliptic Surfaces

Elliptic Surfaces

Definition. An *elliptic surface* is a genus one fibration $f : S \to C$ from a smooth projective surface S to a smooth projective curve C, with a section $\sigma_0 : C \to S$.

- All but finitely many fibres F_v are smooth genus one curves. The fibres that are not smooth are called *singular* and we can write them as divisors on S with multiplicities:

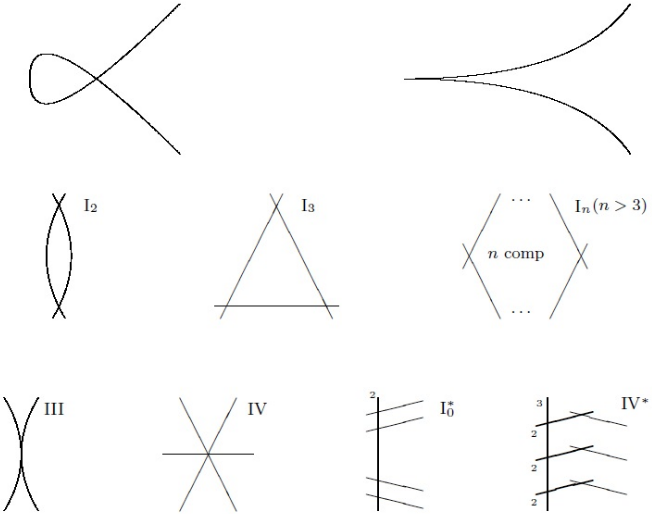
$$F_v = \sum_{i=0}^{m_v-1} \mu_{v,i}\Theta_{v,i} \ ,$$

  where m_v is the number of (distinct) irreducible components, $\Theta_{v,i}$ the irreducible components and $\mu_{v,i}$ the multiplicity of $\Theta_{v,i}$.
- The component denoted by $\Theta_{v,0}$ is the unique component of F_v which intersects the zero section $[\sigma_0]$.

- If F_v is irreducible, then it must be either a rational curve with a node (type I_1), or a rational curve with a cusp (type II).
- All possible reducible fibers have been classified by Kodaira:

fiber	ord(g_2)	ord(g_3)	ord(Δ)	$\mathbb{M}_*$	$\mathfrak{g}$
I_k	0	0	k	T^k	$\mathfrak{su}(k)$
I_k^*	2	3	$k+6$	$-T^k$	$\mathfrak{so}(2k+8)$
I_0^*	≥ 2	≥ 3	6	$-\mathbb{I}$	$\mathfrak{so}(8)$
II	≥ 1	1	2	$(ST)^{-1}$	-
II^*	≥ 4	5	10	(ST)	$\mathfrak{e}_8$
III	1	≥ 2	3	S^{-1}	$\mathfrak{su}(2)$
III^*	3	≥ 5	9	S	$\mathfrak{e}_7$
IV	≥ 2	2	4	$(ST)^{-2}$	$\mathfrak{su}(3)$
IV^*	≥ 3	4	8	$(ST)^2$	$\mathfrak{e}_6$

Singular fibres



Rational Elliptic Surfaces

- The elliptic surface S is *rational* if it is birationally equivalent to $\mathbb{P}^2$. In this case, the base curve C is the projective line $\mathbb{P}^1$.
- The relevant aspect is the the SW geometry can be viewed as a one-parameter family of elliptic curves over the u-plane:

$$E \longrightarrow S \longrightarrow \overline{\mathcal{M}}_{CB} \cong \mathbb{P}^1 \ ,$$

  where $\overline{\mathcal{M}}_{CB}$ is the u-plane with the point at infinity added and the fiber E is the 'Seiberg-Witten' curve.
- The configurations of singular fibres of rational elliptic surfaces have been classified by Persson and Miranda.

Coulomb branch as a Rational Elliptic Surface

Fixing the fibre at infinity

- For a given theory we will be interested in the class of all rational elliptic surfaces with a fixed singularity at $u \to \infty$. This fixes the 'UV physics': e.g. for pure $SU(2)$, we have seen that semi-classically $\tau \to \tau - 4$, i.e. the monodromy is T^{-4}, or $(-T^4)$, and thus corresponds to an I_4^* singularity.
- One can add matter (*hypermultiplets*) in the fundamental representation of the gauge group, which will lead to $I_{4-N_f}^*$ singularities. Note that for $N_f > 4$ the theories are IR free.

Persson's list

- Looking through Persson's list, there is only one configuration with a I_4^* singularity, namely:

$$(I_4^*, I_1, I_1) \ .$$

- This corresponds to the pure $SU(2)$ theory that we have just discussed. The I_1 singularities are the 'strong coupling' singularities that appear on the Coulomb branch of the theory and are interpreted physically as loci where certain states become massless.

Persson's list

- Consider configurations with one flavour. There are two such configurations with a $I^*_{4-N_f} = I^*_3$ singularity, namely:

$$(I_3^*, I_1, I_1, I_1) \ , \qquad (I_3^*, I_1, II) \ .$$

- For this theory there is an additional parameter: the mass of the hypermultiplet. The one-parameter family of elliptic curves over the u-plane:

$$E \to S \to \overline{\mathcal{M}}_{CB} \ ,$$

  should be interpreted as family for a fixed values of the mass parameter. Depending on this value, we find that there are two allowed configurations of singular fibres.

Rank one SCFTs

- How do we interpret physically the type II singularity? Let's 'zoom in' around this singularity by sending the additional I_1 to infinity. The allowed configuration is:

$$(II, II^*) \ ,$$

  where we view the type II as the singularity at the origin of the Coulomb branch.

- This is one of the configurations for which the value of the complex structure parameter τ is fixed and, in physical terms, the coupling is pinned at the strongly-coupled value. This corresponds to one of the Argyres-Douglas $\mathcal{N} = 2$ SCFTs.

Conclusions

- We have discussed how the Seiberg-Witten solution can be interpreted in terms of Rational elliptic surfaces.
- A similar analysis can be done for $N_f = 2, 3$ flavours, leading to richer structures.
- An important simplification in this analysis occurs when the rational elliptic surface is *modular*, i.e. when the monodromy group is a congruence subgroup of $SL(2, \mathbb{Z})$.

Thank you!

References

- Many references for SW theory: original work of Seiberg, Witten; Tachikawa, Martone, Bilal reviews etc.
- Shioda, Schutt - *Mordell-Weil Lattices*