Characteristic classes and YM Instantons

October 28, 2024

Outline

References

- Labastida, Marino - TQFT and Four Manifolds (see also Marino - Phd Thesis)
- Daniel, Viallet - The geometrical setting of gauge theories of the YM type (1980)
- Nakahara - Geometry, Topology and Physics
- José Figueroa-O'Farrill (University of Edinbourgh) - Course on Spin Geometry

Associated Bundles

- Given a principal G-bundle P, we can form an associated vector bundle:

$$\text{Principal } G\text{-bundle} \xleftrightarrow{\text{representation } \rho} \text{Vector bundle}$$

- Let V be a vector space and ρ a representation of G. Define the right action of G on $P \times V$ by:

$$(p, v) \cdot g = (p \cdot g,\ \rho(g^{-1})\ v)\ , \qquad p \in P\ ,\ v \in V\ ,\ g \in G\ .$$

- The quotient: $E = (P \times V)/G$ consists of the equivalence classes:

$$[p \cdot g, v] = [p, \rho(g)v], \forall g \in G\ .$$

- The projection $\pi_\rho : E \to M$ is defined by $\pi_\rho\left([p, v])\right) = \pi(p)$. The fiber of E is thus V and the structure group is G.

Associated Bundles

- Given a vector bundle, we can also construct an associated principal bundle. Consider a real rank k vector bundle E with fibre at $x \in M$ given by the (vector space) E_x.
- Denote by F_x the set of all frames at x. This has a natural right action by $GL(k, \mathbb{R})$ - i.e. a 'change of basis'.
- This action leads to a principal $GL(k, \mathbb{R})$-bundle given by the disjoint union of all F_x.
- Correspondence can be extended to other groups by adding more structure on the vector bundle E.

Adjoint Bundle

- We will consider a principal G-bundle P, with $\mathfrak{g}$ the Lie algebra of the group G.
- The Adjoint bundle is the bundle of Lie algebras associated to the Adjoint representation of G:

$$\mathfrak{g}_P = P \times_{Ad} \mathfrak{g} \ ,$$

consisting of the equivalence classes:

$$[p \cdot g, x] = [p, Ad_g(x)] \ ,$$

for $p \in P$, $g \in G$ and $x \in \mathfrak{g}$.

- Sections of associated bundles $\sigma_\alpha : U_\alpha \to V$ satisfy:

$$\sigma_\alpha(x) = \rho(t_{\alpha\beta}(x)) \ \sigma_\beta(x) \ , \qquad \forall \ x \in U_\alpha \cap U_\beta \ .$$

Conventions

- Denote by A the connection on the principal bundle P, which is a section of $T^*P \otimes \mathfrak{g}$.
- Denote by $\Omega^p_M(E)$ the space of sections of $\Lambda^p T^*M \otimes E$, and similarly for the principle bundle and adjoint bundle.
- In terms of a local trivialization $\{U_\alpha\}$, the connection A_α is a $\mathfrak{g}$-valued 1-form satisfying the compatibility condition:

$$A_\beta = t_{\alpha\beta}^{-1}\ A_\alpha\ t_{\alpha\beta} + i\ t_{\alpha\beta}^{-1}\ d\ t_{\alpha\beta}\ ,$$

on $U_\alpha \cap U_\beta$, with $t_{\alpha\beta}$ transition functions. The connection on the different U_α is obtained using the pullback of the local sections:

$$A_\alpha = \sigma_\alpha^*(A)\ .$$

Gauge Transformations

- Focus on U_α and consider transformation: $\sigma'_\alpha(x) = \sigma_\alpha(x) \cdot g(x)$. This leads to a 'transformation formula':

$$A'_\alpha = Ad_g\ A_\alpha + g^{-1} dg$$

- A gauge transformations is an automorphism $f : P \to P$ such that:
  1. $\forall\ p \in P$, $\exists\ \gamma : P \to G$, such that $f(p) = p \cdot \gamma(p)$
  2. $\gamma(pg) = g^{-1} \cdot \gamma(p) \cdot g$, $\forall\ p \in P$, $\forall\ g \in G$
- The map γ also satisfies a local compatibility condition:

$$\gamma_\beta(x) = t^{-1}_{\alpha\beta}(x)\ \gamma_\alpha(x)\ t_{\alpha\beta}(x)\ , \qquad x \in U_\alpha \cap U_\beta\ .$$

- The group $\mathcal{G} = Aut(P)$ of gauge transformations is an infinite-dimensional group.

Space of Connections

- Denote by $\mathcal{A}$ the space of all connections. This is an affine space with tangent space at A given by:

$$T_A\mathcal{A} = \Omega^1(\mathfrak{g}_E) = \Gamma(T^*M \otimes \mathfrak{g}_E) \ .$$

- To see this, note that $\tau_\alpha = A_\alpha - A'_\alpha$ is a section of the adjoint bundle since:

$$\tau_\alpha = t_{\alpha\beta} \ \tau_\beta \ t_{\alpha\beta}^{-1} \ .$$

- Curvature associated to connection A is an element of $\Omega^2(\mathfrak{g}_E)$:

$$F_\beta = t_{\alpha\beta}^{-1} \ F_\alpha \ t_{\alpha\beta}$$

- Finally, the Lie algebra of the group of gauge transformations $\mathcal{G}$ is $Lie(\mathcal{G}) = \Omega^0(\mathfrak{g}_E)$.

ASD Connections

- A connection is ASD if $F_A^+ = 0$. For $\mathbb{S}^4$, this is the system of PDEs:

$$F_{12} + F_{34} = 0 \ ,$$
$$F_{14} + F_{23} = 0 \ ,$$
$$F_{13} + F_{42} = 0 \ .$$

for $F = \frac{1}{2} F_{\mu\nu} dx^\mu \wedge dx^\nu$.

- Chern-Weil theory (last week):

$$k \longrightarrow \frac{1}{8\pi^2} Tr(F \wedge F) = \begin{cases} c_2(E) - \frac{1}{2} c_1(E)^2 \\ - \frac{1}{4} p_1(V) \end{cases}$$

- ASD connections minimize the YM action: $S_{YM} \geq 4\pi^2 |k|$

Moduli Space of ASD Connections

- The ASD condition defines a subspace of $\mathcal{A}$, which can be viewed as the zero locus of the (equivariant) section: $s : \mathcal{A} \to \Omega^{2,+}(\mathfrak{g}_E)$:

$$s(A) = F_A^+ \ .$$

- Define the moduli space of ASD connections as:

$$\mathcal{M}_{ASD} = \{[A] \in \mathcal{A}/\mathcal{G} \mid s(A) = 0\} \ .$$

Instanton Deformation Complex

- We want a local model for the ASD moduli space - i.e. consider tangent space at an ASD connection A in $\mathcal{A}$ and find the directions that preserve the ASD condition but are not gauge orbits. We do this in two steps.

- *Mod out gauge orbits*

  - $[A]$ contains $A' \in \mathcal{A}$ for which $\exists\ u \in \mathcal{G}$ s.t. $A' = u(A)$.

  $$u(A_\alpha) = A_\alpha + i(d_A u_\alpha)u_\alpha^{-1}$$

  - Gauge orbits: $Im(d_A)$

- *Enforce ASD Condition*

  - for $a \in \Omega^1(\mathfrak{g}_E)$:

  $$\left.\begin{aligned} s(A) &= 0 \\ s(A+a) &= 0 \end{aligned}\right\} \Rightarrow ds(a) = 0\ .$$

Instanton Deformation Complex

- The short exact sequence:

$$0 \to \Omega^0(\mathfrak{g}_E) \xrightarrow{d_A} \Omega^1(\mathfrak{g}_E) \xrightarrow{ds} \Omega^2(\mathfrak{g}_E) \to 0$$

  is in fact an *elliptic complex*, which proves that $\mathcal{M}_{ASD}$ is finite dimensional. One has:

$$T_{[A]}\mathcal{M}_{ASD} = H^1_A = \frac{Ker\ ds}{Im\ d_A}$$

- The AS index theorem leads to:

$$dim\ \mathcal{M}_{ASD} = 4Nc_2(E) - \frac{N^2-1}{2}(\chi + \sigma)\ .$$

ADHM Construction

- For ASD connections on $\mathbb{R}^4$, for $G = SU(2)$, $k = 1$, one has the solution:

$$A_m = \frac{2(\boldsymbol{x} - \boldsymbol{x}_0)_n}{(\boldsymbol{x} - \boldsymbol{x}_0)^2 + \rho^2}\sigma_{nm} \ , \qquad \sigma_{mn} = \frac{1}{4}(\sigma_m \overline{\sigma}_n - \sigma_n \bar{\sigma}_m)$$

- The 4 positions $\boldsymbol{x}_0$ and the size ρ parametrize the one-instanton moduli space: $\mathbb{R}^4 \times \mathbb{R}^+$.

First Stiefel Whitney Class

- Consider a real smooth n-dimensional Riemannian manifold M and tangent bundle TM. The structure group is $O(n)$.
- Given the transition functions $t_{\alpha\beta} : U_\alpha \cap U_\beta \to O(n)$, define:

$$f_{\alpha\beta}(x) = det\ t_{\alpha\beta}(x) \in \{\pm 1\}\ , \qquad x \in M$$

- These $f_{\alpha\beta}$ define a bundle over M with structure group $\mathbb{Z}_2$. If this bundle is trivial, then the manifold is orientable.
- It turns out that iff $f_{\alpha\beta}(x) = f_\alpha(x) f_\beta(x)$, for some $f_\alpha : U_\alpha \to \mathbb{Z}_2$ we can in fact find new transition functions $\widetilde{t}_{\alpha\beta} : U_\alpha \cap U_\beta \to SO(n)$:

$$\widetilde{t}_{\alpha\beta}(x) = h_\alpha(x) t_{\alpha\beta}(x) h_\beta^{-1}(x)\ ,$$

for $h_\alpha : U_\alpha \to O(n)$, with $f_\alpha(x) = det\ h_\alpha(x)$.

First Stiefel Whitney Class

- $\{f_{\alpha\beta}\}$ can be viewed as a Čech cochain, defining a cohomology class $w_1(M) \in H^1(M, \mathbb{Z}_2)$, which is the first Stiefel-Whitney class of M.
- The short exact sequence of groups:

$$1 \longrightarrow SO(n) \longrightarrow O(n) \xrightarrow{det} \mathbb{Z}_2 \longrightarrow 1$$

  induces an exact sequence of sheaves, which has an associated long exact sequence of cohomology:

$$H^1(M, SO(n)) \longrightarrow H^1(M, O(n)) \longrightarrow H^1(M, \mathbb{Z}_2)$$

- The obstruction of orientability can be viewed as the image of the class in $H^1(M, O(n))$ that corresponds to the orthonormal frame bundle under the last map of the sequence.

Second Stiefel Whitney Class

- Given that M is orientable, with transition functions $t_{\alpha\beta}$ valued in $SO(n)$, can we find lifts to $\widetilde{t}_{\alpha\beta} : U_\alpha \cap U_\beta \to Spin(n)$?
- We can do this on each overlap $U_\alpha \cap U_\beta$ using the homomorphism $\varphi : Spin(n) \to SO(n)$, with $\varphi(\ \widetilde{t}_{\alpha\beta}\ ) = t_{\alpha\beta}$, but the cocycle condition might fail on $U_\alpha \cap U_\beta \cap U_\gamma$:

$$\widetilde{t}_{\alpha\beta}(x)\ \widetilde{t}_{\beta\gamma}(x)\ \widetilde{t}_{\gamma\alpha}(x) = f_{\alpha\beta\gamma} \in \{\pm 1\}\ .$$

- $f_{\alpha\beta\gamma}$ define a Čech cocycle and thus a cohomology class $w_2(X)$ in $H^2(X, \mathbb{Z}_2)$, which is the obstruction to the existence of a spin structure.

Second Stiefel Whitney Class

- The short exact sequence of groups:

$$1 \longrightarrow \mathbb{Z}_2 \longrightarrow Spin(n) \longrightarrow SO(n) \longrightarrow 1$$

  'leads' to a long exact cohomology sequence:

$$H^1(M, \mathbb{Z}_2) \to H^1(M, Spin(n)) \to H^1(M, SO(n)) \to H^2(M, \mathbb{Z}_2)$$

- The obstruction of the existence of a spin structure can be viewed as the image of the class in $H^1(M, SO(n))$ corresponding to the $SO(n)$ frame under the last map.

Thank you for your attention!

- Total Chern class for a G-bundle over n-dim complex manifold M is defined as:

$$c(E) = det\left(1 + \frac{i\mathcal{F}}{2\pi}\right) = 1 + c_1(E) + c_2(E) + \dots$$

- Note that $c_j(E) \in \Omega^{2j}(M)$, so $c_j = 0$ for $2j > n$.
- Total Chern class of a Whitney sum bundle: $E \oplus F$ is:

$$c(E \oplus F) = c(E) \wedge c(F).$$

- Computationally, diagonalize curvature form to $diag(x_1, \dots x_k)$, with x_i two-forms, and find:

$$c(E) = \prod_{i=1}^{k}(1 + x_i)$$

This is reffered to as the **Splitting principle**, since it allows to 'view' E as a Whitney sum of k complex line bundles.

Chern Characters

- The total Chern Character is defined by:

$$ch(E) = Tr\ exp\left(\frac{i\mathcal{F}}{2\pi}\right) = \sum_{j=1}^{k} exp(x_j)\ .$$

- The Todd class is defined by:

$$Td(E) = \prod_j \frac{x_j}{1 - e^{-x_j}}.$$

- Properties:

$$ch(E \oplus F) = ch(E) \oplus ch(F).$$

$$Td(E \oplus F) = Td(E) \wedge Td(F).$$

Real Vector Bundles

- Total Pontrjagin class:

$$p(F) = det\left(1 + \frac{\mathcal{F}}{2\pi}\right) = \prod_{i=1}^{[k/2]} (1 + x_i^2) \ ,$$

- For $2n$ dimensional orientable Riemannian manifold, define the Euler class:

$$e(M)e(M) = p_n(M)$$

- Formally:

$$e(M) = \prod_{i=1}^{l} x_i$$

AS Index Theorem

[AHS '78]

- Let (E, D) be an elliptic complex over an n-dimensional compact manifold M:

$$\dots \xrightarrow{D_{i-2}} \Gamma(M, E_{i-1}) \xrightarrow{D_{i-1}} \Gamma(M, E_i) \xrightarrow{D_i} \Gamma(M, E_{i+1}) \xrightarrow{D_{i+1}} \dots$$

- For this to be true, D has to be nilpotent: $D_{i-i} \circ D_i = 0$. Define:

$$H^i(E, D) = ker\ D_i / Im\ D_{i-1}.$$

- The analytic index is defined as:

$$Index(E, D) = \sum_{i=0}^{m} (-1)^i\ dim\ H^i(E, D)$$

AS Index Theorem

- **Atiyah-Singer theorem.** The index is given by:

$$Index(E, D) = (-1)^{\frac{n(n+1)}{2}} \int_M ch\left(\bigoplus_r (-1)^r E_r\right) \frac{Td\left(TM^{\mathbb{C}}\right)}{e(TM)}.$$

de Rham complex

- For an m-dimensional compact orientable manifold, consider the complex:

$$\ldots \xrightarrow{d} \Omega^{r-1}(M)^{\mathbb{C}} \xrightarrow{d} \Omega^{r}(M)^{\mathbb{C}} \xrightarrow{d} \Omega^{r}(M)^{\mathbb{C}} \xrightarrow{d} \ldots$$

- The Chern character splits as:

$$ch\left(\bigoplus_{r}^{4}(-1)^r \bigwedge^r T^*M^{\mathbb{C}}\right) = \sum_{r=0}^{4} ch\left(\bigwedge^r T^*M^{\mathbb{C}}\right)$$
$$= 1 - \sum_{i=1}^{4} e^{-x_i} + \sum_{i<j} e^{-x_i}e^{-x_j} + \ldots = \prod_{i=1}^{n=4}\left(1 - e^{-x_i}\right)\left(TM^{\mathbb{C}}\right),$$

- where $x_i\left(T^*M^{\mathbb{C}}\right) = -x_i\left(TM^{\mathbb{C}}\right)$ are two-forms.

$$Td\left(TM^{\mathbb{C}}\right) = \prod_{i=1}^{n=4} \frac{x_i}{1-e^{-x_i}}\left(TM^{\mathbb{C}}\right),$$

$$e(TM) = \prod_{i=1}^{n/2=2} x_i\left(TM^{\mathbb{C}}\right).$$

- In the last line we used the fact that $TM^{\mathbb{C}} = TM \otimes \mathbb{C} = TM \oplus TM$ as a real vector bundle. The AS index theorem then reduces to:

$$Index(d) = (-1)^{10} \int_M \prod_{i=1}^{2} x_i\left(TM^{\mathbb{C}}\right)$$

$$= \int_M e(TM) = \chi(M),$$

where in the last part we used the Gauss-Bonnet theorem.

Showing $T_A\mathcal{A} = \Omega^1_M(\mathfrak{g}_E)$

- **Lemma.** Let E, F be vector bundles over M and $\mathcal{C} : \Gamma(E) \to \Gamma(F)$ a linear map. $\mathcal{C}$ is $C^\infty(M)$-linear if and only if there exists a bundle map $\alpha : E \to F$, such that:

$$\mathcal{C}(f)(x) = \alpha\,(f(x)) \ ,$$

$$\begin{tikzcd} M \arrow[r, "f"] \arrow[dr, "\alpha\circ f"'] & E \arrow[d, "\alpha"] \\ & F \end{tikzcd}$$

where $f \in \Gamma(E)$ and $x \in M$.

- For our problem, let $\mathcal{C} = \nabla_1 - \nabla_2 : \Omega^0_M(E) \to \Omega^1_M(E)$. This is a $C^\infty(M)$-linear map as can be checked from:

$$(\nabla_1 - \nabla_2)(fs) = f(\nabla_1 - \nabla_2)s$$

for $f \in C^\infty(M)$, $s \in \Gamma(E)$. Hence, $\mathcal{C}$ is associated to a bundle map $\alpha : E \to T^*M \otimes E$ and is thus a tensor in $\Gamma\left(T^*M \otimes End(E)\right)$. Restriction to $\Omega^1_M(\mathfrak{g}_E)$ ensures compatibility with structure group.