EXERCISES

1. Decide whether the following functions are *partial* or *total*:
   (a) A function called *halve* which maps integers to integers so that any value which is in **ran** *halve* is exactly half of the corresponding value in **dom** *halve*.
   (b) A function called *passport* that maps people to passport numbers.
   (c) A function called *square* that maps integers to integers so that each value which is in **ran** *square* is the square of the corresponding value in **dom** *square*.

2. Decide what category of function best models each of the following:
   (a) The relationship between all the countries of the world and their capital cities.
   (b) The relationship between the countries of Europe and their capital cities.
   (c) The relationship between countries and their reigning monarchs.
   (d) The relationship between countries and their currencies.
   (e) The relationship between a month and its predecessor.
   (f) The relationship between a month and its successor.
   (g) The relationship between national flags and the countries to which they belong.

3. Decide whether any of the following relations may be functions:
   (a) *anagram* : letter_sequence $\leftrightarrow$ letter_sequence
   (b) *road_to* : town $\leftrightarrow$ town
   (c) *greater_than* : number $\leftrightarrow$ number
   (d) *has_number* : person $\leftrightarrow$ phone_number
   (e) *studies* : student $\leftrightarrow$ subject
   (f) *author_of* : person $\leftrightarrow$ book

4. Categorize each of the following relations as either a total function, a partial function or a relation which is not a function.

   (a) $much\_less\_than == \{ x, y : \mathbb{Z} \mid x < y - 99 \bullet x \mapsto y \}$
   (b) The size of the population of each country of the world as a relation from countries of the world to the set of integers.
   (c) The number of cars *owned* by a person as a relation from the set of people to $\mathbb{N}_1$.

5.
   (a) If function f is a *bijection*, describe the kind of function that is the inverse of f.
   (b) If function f is a *total injection*, describe the kind of function that is the inverse of f.

6. Suppose COMPANY is a given set: COMPANY = = {bt, ici, glaxo, shell}
and *share_price* is a function: $share\_price : \text{COMPANY} \rightarrow \mathbb{N}$
which models the share price of those companies which are members of COMPANY as a mapping from the set COMPANY to the set of natural numbers (in effect, the **range** of *share_price* gives the company share values in pence).

(a) Suppose a regulatory committee bars *bt* from providing entertainment services over its phone-lines and the share price of *bt* consequently drops to 76 pence. If *sharePrices* is the value of the share_price function before the fall in *bt* shares and *newSharePrices* is the value after, write down an expression connecting *sharePrices* and *newSharePrices*.

(b) If *double* is a function that maps any natural number onto a value which is twice the original value, write down a similar expression which will yield a doubling of the *bt* share price.

7. Explain using an example why $f \oplus g \neq (f \setminus g) \cup g$, for all f and g in $X \rightarrow\!\!\!\!\!+ Y$

8. A vending-machine offers the following selections:

Drink	Price (pence)
Orange	25
Coffee	30
Cola	20
Tea	15

(a) If f is the function mapping *Drink* to *Price*, categorize f.
(b) Write a formal expression for the price of *Cola* being increased to 35p.
(c) If the price of *Cola* is changed to 25p, how will this affect the functional model?

9. Suppose f and g are functions given by:

$f = \{(a, x), (b, y), (c, z)\}$ and $g = \{(1, a), (2, a), (3, c)\}$

(a) Determine $g \fatsemi f$ as a set of ordered pairs.
(b) If $h = g \fatsemi f$, does h^{-1} exist?

10.

(a) A Library system is to be modelled using the given sets:

[BOOK]	which contains, as elements, all the possible copies of books which are likely to appear on the Library shelves;
[PERSON]	which contains, as elements, all people ever likely to be members of the Library;
[AUTHOR]	which contains, as elements, all people who are ever likely to be the authors of the books owned by the Library;
[TITLE]	which contains, as elements, all book titles likely to appear in the Library catalogue;

and the derived sets:

books	representing the set of book copies owned by a Library;
on_loan	representing the set of books currently on loan;
on_shelves	representing the set of books currently on the shelves;
borrowers	representing the set of people with books on loan from the Library;
members	representing the set of Library members.

Using these definitions, write expressions in set notation which are equivalent to the statements:

(i) *A book owned by the Library is either on the shelves or on loan*
(ii) *Only books owned by the Library can be on loan*
(iii) *Only Library members are allowed to borrow books*
(iv) *Borrowed books cannot still be on the shelves*

(b) Suppose *wrote* is a relation on the given sets AUTHOR and TITLE:

i.e. $\_wrote\_ : \text{AUTHOR} \leftrightarrow \text{TITLE}$

(i) What sort of values would be contained in the sets **dom** *wrote* and **ran** *wrote*?
(ii) *Thomas Hardy* wrote a book entitled *The Woodlanders*. Express that fact symbolically.
(iii) If *lent_to* is a (partial) function declared by: $lent\_to : \text{BOOK} \rightarrow\!\!\!\!\!\!+\; \text{PERSON}$ write down an expression for the set of library books on loan to a particular person p. If p is not a member of *borrowers*, what does this expression denote?

11. At a particular bank, a person is allowed to open no more than one account. Suppose *accounts* is a function relating each customer at the bank to their account, while *balance* is a function relating accounts and account-balances (which are always in whole numbers of pounds).

If we have given sets [PERSON] and [ACCOUNT] representing, respectively, all possible people who may ever open an account and all possible accounts that may ever be opened, then we may define:

$$accounts : \text{PERSON} \rightarrow\!\!\!\!\!\!+\;\, \text{ACCOUNT}$$
$$balances : \text{ACCOUNT} \rightarrow\!\!\!\!\!\!+\;\, \mathbb{Z}$$

(a) Explain why *accounts* and *balances* are **partial** functions.

(b)

(i) Write a symbolic expression to state that all customer accounts will have balances.

(ii) Create an expression which will yield a set containing ordered pairs where the first value in a pair is a *customer* and the second value is that customer's account balance.

(c) Write symbolic expressions to give:

(i) all customers with accounts;

(ii) the account which belongs to customer *c*.

(d) If the bank changes its rules so that any customer may have more than one account and customers can have joint accounts, modify the definition of *accounts* to reflect the change in the rules.

(e) For the revised model, write symbolic expressions to give:

(i) the **number** of accounts owned by customer *c*;

(ii) the **set** of customers who have at least one account *overdrawn*.

12. A university awards degrees with classifications defined by the free type DEGREE_CLASS where:

DEGREE_CLASS ::= *ordinary* | *pass* | *third* | *lower second* | *upper second* | *first*

If [STUDENT] defines the given set of all possible students, *comp_sci* denotes the set of final-year students graduating in *Computer Science* and the relationship showing the degree classification obtained by each student is modelled by:

$$final\_deg\_results : \text{STUDENT} \rightarrow\!\!\!\!\!\!+\;\, \text{DEGREE\_CLASS}$$

(a) Write a symbolic expression which shows, for those graduating in *Computer Science*, who obtained which degree classification.

(b) Write a symbolic expression giving the **number** of non-*Computer Science* graduating students who were awarded *first* class degrees.

(c) Write a symbolic expression giving those students graduating in *Computer Science* who *failed* to get a *lower second* or better.

(d) Write a symbolic expression giving the complete set of final year *Computer Science* results after the External Examiner persuades the Examination Board to upgrade to a *third* class degree all those *Computer Science* students previously recommended for *pass* degrees.