FUNCTIONS

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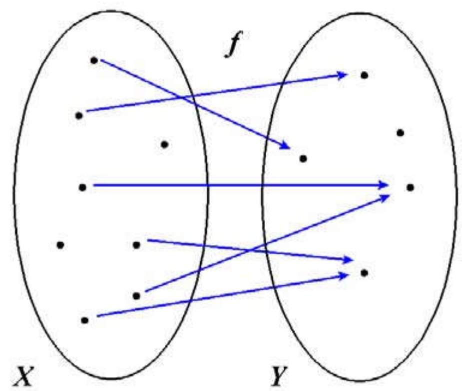
INTRODUCTION

- A *function* is a special type of *relation*, restricted so that any member of the *from-set* maps to, **at most**, one member of the *to-set*:



- The function f shown above from the set X to the set Y is declared by:

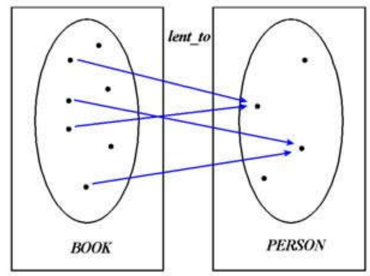
$$f : X \nrightarrow Y$$

  and read as "the function f from X to Y"

- This is equivalent to a relation f from X to Y (f:X$\leftrightarrow$Y) but **restricted** so that, for each $x \in X$, f relates that value x to, **at most**, one value y where $y \in Y$

PARTIAL FUNCTIONS

- It is usual for there to be a rule that, for a Library, any book may only be *on-loan* to one person at a time



- The *lent_to* function illustrated would be declared by:

  $$lent\_to : \mathrm{BOOK} \rightarrow\!\!\!\!\!\!+\ \mathrm{PERSON}$$

  which denotes the set of all functions from the set BOOK to the set PERSON

- The **type** of *lent_to* is: $\mathbb{P}$ (BOOK × PERSON)

- Note that the domain of *lent_to* is only a subset of BOOK and this makes *lent_to* a *partial* function

- Partial functions are the most general sort of function

TOTAL FUNCTIONS

- If the domain of a function is the whole of the source (or from-set), the function is said to be *total*

- The symbol for a total function is similar to that for a partial function but without the transverse bar

- The declaration: $f : X \rightarrow Y$
  is read as "f is the total function from X to Y"

- Examples of total functions:

$$age : \text{PERSON} \rightarrow \mathbb{N}$$
$$has\_mother : \text{PERSON} \rightarrow \text{PERSON}$$

- Note: $X \rightarrow Y == \{ f : X \nrightarrow Y \mid \mathbf{dom}\, f = X \}$

  - by implication total functions are special cases of partial functions

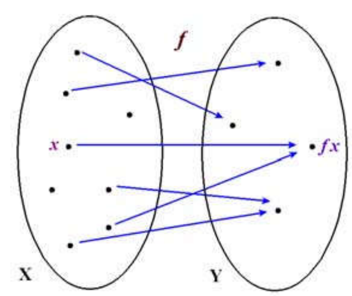
FUNCTION APPLICATION

- If (f : X $\rightarrow\!\!\!\!\!\!+$ Y) and ($x \in$ **dom** f) then we know there will be *at most* one value in **ran** f which will result from the mapping f being applied to the domain value x

- The *value* resulting from the application of a function f to x (its *argument*) is often written as:

  f x (or, sometimes, $f\,(x)$)

  f x is read as "f of x"
  or "f applied to x"



- For example, if *lent_to* : BOOK $\rightarrow\!\!\!\!\!\!+$ PERSON
  b : BOOK
  $b \in$ **dom** *lent_to*

  then *lent_to b* represents the person to whom the book has been lent

- NB: $f\,x = f\,(\!|\,\{\,x\,\}\,|\!)$

FUNCTIONAL OVERRIDING

- Suppose a value of the *lent_to* function is:

  $$\{029 \mapsto tom, 523 \mapsto sam, 109 \mapsto sam, 022 \mapsto jim\}$$

  where the first value in each pair is a unique book identifier and the second is the person to whom the book has been loaned

- Suppose, further, that *updates* is a set of pairs to be incorporated into *lent_to*:

  $$updates = \{ 029 \mapsto jim, 427 \mapsto tom \}$$

  which means:

  - book *029* has been transferred to *jim*, and
  - book *427*, previously on the shelves, has been loaned to *tom*

FUNCTIONAL OVERRIDING

- If $lent\_to'$ is the value of the function after amendment by $updates$, then $lent\_to'$ should contain all the pairs from *updates* and any of the original pairs from *lent_to* that do not conflict with those in *updates* - meaning those pairs from *old* which do not have the same first member as a pair from *updates*

- Hence, we recover:

$$lent\_to' = \{029 \mapsto jim, 427 \mapsto tom, 523 \mapsto sam, 109 \mapsto sam, 022 \mapsto jim\}$$

- Symbolically: $lent\_to' = lent\_to \oplus updates$

  where $\oplus$ is the sign to represent "overriding" or updating

- Since *functions* are *relations* (and, hence, *sets*) the usual set operations apply and *overriding* may be expressed, more clumsily, by:

$$lent\_to \oplus updates = ((\mathbf{dom}\ updates) \lhd lent\_to) \cup updates$$

FUNCTION DEFINITION

- *Declaring* a function does **not** *define* its members

- Since a function is a set, we may use both set enumeration (provided there are not many maplet-pairs) and set comprehension to define a function explicitly

  e.g. $pos\_int\_sqrt == \{n : \mathbb{Z} \mid n \geq 0 \bullet n^2 \mapsto n\}$

- An alternative (and, often, shorter) form is provided by *lambda abstraction*

  e.g.: if $square == \{(0,0),(1,1),(2,4),(3,9),(4,16)\}$

  then, using *set comprehension*:

  $$square == \{n : \mathbb{N} \mid n \leq 4 \bullet n \mapsto n^2\}$$

  or, using *lambda abstraction*:

  $$square == \lambda\, n : \mathbb{N} \mid n \leq 4 \bullet n^2$$

FUNCTION DEFINITION

- By definition, a lambda expression describes a function, and, hence
  - the mapping is implicit
  - the set braces may be omitted

- Another example:

  if $total_{m<n}$ represents the function that accepts two non-negative integers and maps the pair to their sum, then we may define $total_{m<n}$ by:

$$total_{m<n} == \lambda\, m, n : \mathbb{N} \mid m < n \bullet m + n$$

- The constraint bar is used only if needed

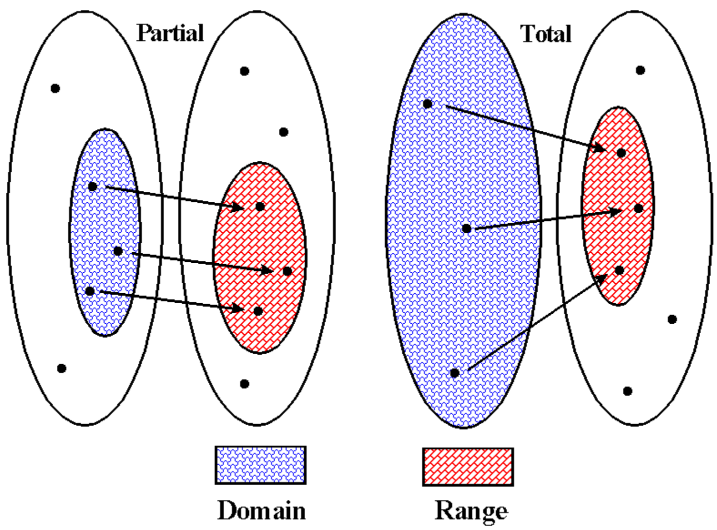
  e.g. $double == \lambda\, \mathrm{n} : \mathbb{N} \bullet 2n$

  defines the infinite set which maps each non-negative integer to its double

OTHER CLASSES OF FUNCTIONS

- As well as being partial or total, functions may be categorized as:

  - *injective* functions (or *injections*)

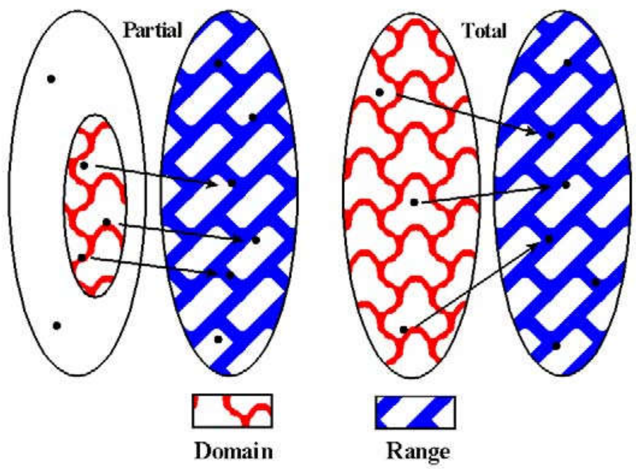


  - An *injection* is a 1:1 function: distinct elements of the domain map to distinct elements of the range (i.e. no two domain values map to the same range value)

  - A consequence is that an injective function will have an inverse which is also a function

OTHER CLASSES OF FUNCTIONS

- *surjective* functions (or *surjections*)

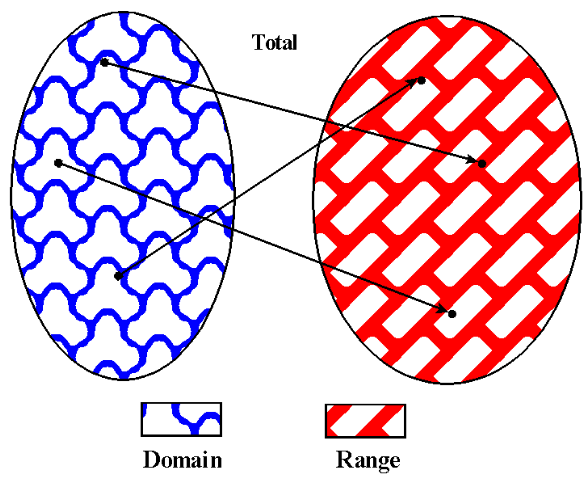


- Here, the range is the *whole* of the to-set (i.e. it 'fills' the to-set)

- *Surjective* functions are often referred to as *onto* functions

OTHER CLASSES OF FUNCTIONS

- *bijective* functions (or *bijections*)



- A *bijection* is a function that is a **total** 1:1 correspondence between the from-set and the to-set and, hence is both a *total injection* and a *total surjection*:
  - each member of the from-set maps to one and only one member of the to-set, and
  - every member of the to-set is associated with exactly one member of the from-set

CLOSING COMMENTS

- There are special arrow symbols associated with both partial and total versions of *injection*s, *surjections* and *bijections* but, in an effort to avoid confusion and 'symbol overload', we shall (at least for the present) not employ them

- Note that, in general, since functions are relations (and sets) the usual relation (and set) operations may be applied to functions BUT it should be observed that not all functions will have an inverse which is, itself, a function

- Whereas the set may be regarded as the fundamental simple atomic object of typed set theory, the relation is the fundamental complex object

- Simple sets, relations and functions form the cornerstones of formal specifications

SUMMARY OF SYMBOLS

$X \rightarrowtail Y$ *the set of partial functions from the set X to the set Y*

$X \rightarrow Y$ *the set of total functions from set X to set Y*

$== \{ f : X \rightarrowtail Y \mid \mathbf{dom}\, f = X \}$

$f\, x$ or $f(x)$ *the function f applied to x*

$f \oplus g$ *functional overriding*

$== (\mathbf{dom}\, g \lhd f) \cup g$

$\lambda D \mid P \bullet E$ *function definition where P is a predicate constraining the values declared in D and E is an expression giving the function in terms of the values declared in D*

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EXERCISES

1. Decide whether the following functions are *partial* or *total*:
   (a) A function called *halve* which maps integers to integers so that any value which is in **ran** *halve* is exactly half of the corresponding value in **dom** *halve*.
   (b) A function called *passport* that maps people to passport numbers.
   (c) A function called *square* that maps integers to integers so that each value which is in **ran** *square* is the square of the corresponding value in **dom** *square*.

2. Decide what category of function best models each of the following:
   (a) The relationship between all the countries of the world and their capital cities.
   (b) The relationship between the countries of Europe and their capital cities.
   (c) The relationship between countries and their reigning monarchs.
   (d) The relationship between countries and their currencies.
   (e) The relationship between a month and its predecessor.
   (f) The relationship between a month and its successor.
   (g) The relationship between national flags and the countries to which they belong.

3. Decide whether any of the following relations may be functions:
   (a) *anagram* : letter_sequence $\leftrightarrow$ letter_sequence
   (b) *road_to* : town $\leftrightarrow$ town
   (c) *greater_than* : number $\leftrightarrow$ number
   (d) *has_number* : person $\leftrightarrow$ phone_number
   (e) *studies* : student $\leftrightarrow$ subject
   (f) *author_of* : person $\leftrightarrow$ book

4. Categorize each of the following relations as either a total function, a partial function or a relation which is not a function.

   (a) $much\_less\_than == \{ x, y : \mathbb{Z} \mid x < y - 99 \bullet x \mapsto y \}$
   (b) The size of the population of each country of the world as a relation from countries of the world to the set of integers.
   (c) The number of cars *owned* by a person as a relation from the set of people to $\mathbb{N}_1$.

5.
   (a) If function f is a *bijection*, describe the kind of function that is the inverse of f.
   (b) If function f is a *total injection*, describe the kind of function that is the inverse of f.

6. Suppose COMPANY is a given set: COMPANY == {bt, ici, glaxo, shell}
and *share_price* is a function: $share\_price : \text{COMPANY} \rightarrow \mathbb{N}$
which models the share price of those companies which are members of COMPANY as a mapping from the set COMPANY to the set of natural numbers (in effect, the **range** of *share_price* gives the company share values in pence).

(a) Suppose a regulatory committee bars *bt* from providing entertainment services over its phone-lines and the share price of *bt* consequently drops to 76 pence. If *sharePrices* is the value of the share_price function before the fall in *bt* shares and *newSharePrices* is the value after, write down an expression connecting *sharePrices* and *newSharePrices*.

(b) If *double* is a function that maps any natural number onto a value which is twice the original value, write down a similar expression which will yield a doubling of the *bt* share price.

7. Explain using an example why $f \oplus g \neq (f \setminus g) \cup g$, for all f and g in $X \rightarrow\!\!\!\!\!\!+\; Y$

8. A vending-machine offers the following selections:

Drink	Price (pence)
Orange	25
Coffee	30
Cola	20
Tea	15

(a) If f is the function mapping *Drink* to *Price*, categorize f.
(b) Write a formal expression for the price of *Cola* being increased to 35p.
(c) If the price of *Cola* is changed to 25p, how will this affect the functional model?

9. Suppose f and g are functions given by:

$$f = \{(a, x), (b, y), (c, z)\} \text{ and } g = \{(1, a), (2, a), (3, c)\}$$

(a) Determine $g \mathbin{⨾} f$ as a set of ordered pairs.
(b) If $h = g \mathbin{⨾} f$, does h^{-1} exist?

10.

(a) A Library system is to be modelled using the given sets:

[BOOK]	which contains, as elements, all the possible copies of books which are likely to appear on the Library shelves;
[PERSON]	which contains, as elements, all people ever likely to be members of the Library;
[AUTHOR]	which contains, as elements, all people who are ever likely to be the authors of the books owned by the Library;
[TITLE]	which contains, as elements, all book titles likely to appear in the Library catalogue;

and the derived sets:

books	representing the set of book copies owned by a Library;
on_loan	representing the set of books currently on loan;
on_shelves	representing the set of books currently on the shelves;
borrowers	representing the set of people with books on loan from the Library;
members	representing the set of Library members.

Using these definitions, write expressions in set notation which are equivalent to the statements:

(i) *A book owned by the Library is either on the shelves or on loan*
(ii) *Only books owned by the Library can be on loan*
(iii) *Only Library members are allowed to borrow books*
(iv) *Borrowed books cannot still be on the shelves*

(b) Suppose *wrote* is a relation on the given sets AUTHOR and TITLE:

i.e. $\_wrote\_ : \text{AUTHOR} \leftrightarrow \text{TITLE}$

(i) What sort of values would be contained in the sets **dom** *wrote* and **ran** *wrote*?
(ii) *Thomas Hardy* wrote a book entitled *The Woodlanders*. Express that fact symbolically.
(iii) If *lent_to* is a (partial) function declared by: $lent\_to : \text{BOOK} \nrightarrow \text{PERSON}$ write down an expression for the set of library books on loan to a particular person *p*. If *p* is not a member of *borrowers*, what does this expression denote?

11. At a particular bank, a person is allowed to open no more than one account. Suppose *accounts* is a function relating each customer at the bank to their account, while *balance* is a function relating accounts and account-balances (which are always in whole numbers of pounds).

If we have given sets [PERSON] and [ACCOUNT] representing, respectively, all possible people who may ever open an account and all possible accounts that may ever be opened, then we may define:

$$accounts : \text{PERSON} \rightarrow\!\!\!\!\!\!+ \; \text{ACCOUNT}$$

$$balances : \text{ACCOUNT} \rightarrow\!\!\!\!\!\!+ \; \mathbb{Z}$$

(a) Explain why *accounts* and *balances* are **partial** functions.

(b)

(i) Write a symbolic expression to state that all customer accounts will have balances.

(ii) Create an expression which will yield a set containing ordered pairs where the first value in a pair is a *customer* and the second value is that customer's account balance.

(c) Write symbolic expressions to give:

(i) all customers with accounts;

(ii) the account which belongs to customer c.

(d) If the bank changes its rules so that any customer may have more than one account and customers can have joint accounts, modify the definition of *accounts* to reflect the change in the rules.

(e) For the revised model, write symbolic expressions to give:

(i) the **number** of accounts owned by customer c;

(ii) the **set** of customers who have at least one account *overdrawn*.

12. A university awards degrees with classifications defined by the free type DEGREE_CLASS where:

DEGREE_CLASS ::= *ordinary* | *pass* | *third* | *lower second* | *upper second* | *first*

If [STUDENT] defines the given set of all possible students, *comp_sci* denotes the set of final-year students graduating in *Computer Science* and the relationship showing the degree classification obtained by each student is modelled by:

$$final\_deg\_results : \text{STUDENT} \rightarrow\!\!\!\!\!\!+ \; \text{DEGREE\_CLASS}$$

(a) Write a symbolic expression which shows, for those graduating in *Computer Science*, who obtained which degree classification.

(b) Write a symbolic expression giving the **number** of non-*Computer Science* graduating students who were awarded *first* class degrees.

(c) Write a symbolic expression giving those students graduating in *Computer Science* who *failed* to get a *lower second* or better.

(d) Write a symbolic expression giving the complete set of final year *Computer Science* results after the External Examiner persuades the Examination Board to upgrade to a *third* class degree all those *Computer Science* students previously recommended for *pass* degrees.