LOGICAL PROPOSITIONS

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INTRODUCTION

- *Propositional logic* (aka *propositional calculus* or *Boolean algebra*) is concerned with statements (aka *propositions*) which may be *false* or *true* (but never both!)

  e.g. *Northampton is in Scotland*
  Tony Williams is a bellringer

- Simple propositions can be combined to form more complex propositions using *logical connectives*

- Typical logical connectives are:

  AND (symbolized by $\wedge$)
  OR (symbolized by $\vee$)
  NOT (symbolized by $\neg$)

- Logical connectives are also known as logical *operators* and the effect of such operators may be shown by a *truth table*

TRUTH TABLES - AND

- $\wedge$ or AND is often called the *conjunction* operator

- If P and Q are logical propositions then the truth table showing the effect of conjunction on P and Q is:

P	Q	P$\wedge$Q
false	*false*	false
false	*true*	false
true	*false*	false
true	*true*	true

- The table shows P$\wedge$Q is *true* if and only if P is *true* **and** Q is also *true*; otherwise it is *false*

  e.g. ***(July is the seventh month)*** $\wedge$ ***(July has 31 days)***

  is *true* because **each** of the component propositions is, separately, *true*

TRUTH TABLES - OR

- $\vee$ or OR is often called the *disjunction* operator

- If P and Q are logical propositions then the truth table showing the effect of disjunction on P and Q is:

P	Q	P$\vee$Q
false	*false*	false
false	*true*	true
true	*false*	true
true	*true*	true

- This table shows that P$\vee$Q is *true* if either P is *true* **or** Q is *true* (**or both** P and Q are *true*); otherwise it is *false*

  - P$\vee$Q is **false** only if **both** P and Q are false

TRUTH TABLES - NOT

- ¬ or NOT is often called the *negation* operator

- If P is a logical proposition then the truth table showing the effect of negation on P is:

P	¬P
false	true
true	false

- ¬P always has the opposite "truth value" to P

e.g. suppose we have the following propositions P, Q and R:

P : *the book is on the library shelf*
Q : *the student is allowed to borrow books*
R : *the book is a reference copy*

If $P \wedge Q \wedge (\neg R)$ is *true* (because each conjoined component is true) then it is also true that the student can borrow the book

OPERATOR PRIORITIES

- As in the example just given it is possible to use parentheses to clarify operator precedence (anything inside parentheses is evaluated *first*)

- Parentheses can always be used to clarify meaning but there is an agreed order of evaluation if a number of operators occur in a single proposition:

$\neg$	has highest priority
$\wedge$	has next highest priority
$\vee$	has next highest priority

e.g. for propositions P, Q and R

$\neg P \vee Q \wedge R$ is evaluated as $(\neg P) \vee (Q \wedge R)$
and $\neg P \wedge Q \vee R$ is evaluated as $((\neg P) \wedge Q) \vee R$

LOGICAL IMPLICATION

- If the truth of one proposition implies that another is also *true* we can use the implication operator $\Rightarrow$

- If P and Q are propositions then $P \Rightarrow Q$ is read as either
  - *if P then Q* or as
  - *P implies Q*

- the truth table for implication is:

P	Q	$P \Rightarrow Q$
false	*false*	true
false	*true*	true
true	*false*	false
true	*true*	true

- $P \Rightarrow Q$ can only be *false* when P is *true* and Q is *false*

LOGICAL IMPLICATION

- Example of possible use:
  - quadrilateral is a square $\Rightarrow$ two adjacent sides are equal

    which may be read as:

    The quadrilateral may not be a square
    but, if it is,
    then two adjacent sides must be equal

  - the bulb lights $\Rightarrow$ the power supply is connected

    Which may be read as:

    The bulb may not light
    but, if it does,
    then the power supply must be connected

- It is possible to show with a truth table that the propositions $P \Rightarrow Q$ and $\neg P \vee Q$ are equivalent and, therefore, the symbol $\Rightarrow$ can, if preferred, always be eliminated from logical expressions

LOGICAL EQUIVALENCE

- The *equivalence* operator behaves as a *logical equality* operator and has the meaning of the English phrases *"exactly when"* or *"only when"* or *"if, and only if"*

- The symbol for *equivalence* is $\Leftrightarrow$ and the truth table for *equivalence* is:

P	Q	$P \Leftrightarrow Q$
false	*false*	true
false	*true*	false
true	*false*	false
true	*true*	true

- $P \Leftrightarrow Q$ is equivalent to $(P \Rightarrow Q) \land (Q \Rightarrow P)$

- Example of use:

$$\text{valid\_user\_number} \land \text{matching\_password} \Leftrightarrow \text{login\_successful}$$

AIRCRAFT SPECIFICATION REVISITED

- We shall now illustrate a typical use of the logical operators by extending our specification of the *aircraft passenger system* considered previously

- In that system, we considered the operations for
  - *boarding* and
  - *disembarking*

- Both operations required *preconditions*; but we did not consider what was to happen if the preconditions were not satisfied

- This will now be remedied by introducing a free type FEEDBACK, where:

  FEEDBACK ::= OK | on board | full | not on board | two errors

  to provide a response from the system to clarify the outcome of each operation

BOARDING OPERATION

- As before, when a passenger, *p*, boards the aircraft the value of *onboard* will change, and the value of *onboard* after such a change is written as *onboard′*

- As previously, we have: p : PERSON
  and now: $reply$: FEEDBACK

  Using our previous definitions, the total 'Boarding' operation may be specified as:

$$
\begin{aligned}
&(p \notin onboard \wedge \#onboard < capacity \wedge \\
&\quad onboard' = onboard \cup \{p\} \wedge reply = \text{OK}) \\
&\vee \\
&(p \in onboard \wedge \#onboard = capacity \wedge \\
&\quad onboard' = onboard \wedge reply = \text{two errors}) \\
&\vee \\
&(p \in onboard \wedge \#onboard < capacity \wedge \\
&\quad onboard' = onboard \wedge reply = \text{on board}) \\
&\vee \\
&(p \notin onboard \wedge \#onboard = capacity \wedge \\
&\quad onboard' = onboard \wedge reply = \text{full})
\end{aligned}
$$

  which caters for the original preconditions not being satisfied!

DISEMBARKING OPERATION

- A similar specification is possible to describe a passenger disembarking:

$$p : \mathrm{PERSON}$$
$$reply : \mathrm{FEEDBACK}$$

$$
\begin{aligned}
&(p \in onboard \land \\
&\qquad onboard' = onboard \setminus \{p\} \land reply = \mathrm{OK}) \\
&\lor \\
&(p \notin onboard \land \\
&\qquad onboard' = onboard \land reply = \text{not on board})
\end{aligned}
$$

which, again, caters for the preconditions not being satisfied!

- The above operation specifications are somewhat complicated but, later, we shall see that the Z specification language offers a more concise approach!

SUMMARY OF SYMBOLS

- Summary of logical symbols introduced:

false, true	*logical constants*
$\neg P$	*negation* : 'not P'
$P \wedge Q$	*conjunction* : 'P and Q'
$P \vee Q$	*disjunction* : 'P or Q'
$P \Rightarrow Q$	*implication* : 'P implies Q' or 'if P then Q'
$P \Leftrightarrow Q$	*equivalence* : 'P is logically equivalent to Q'

- The precedence order for the operators is (highest first):

$$\neg, \ \wedge, \ \vee, \ \Rightarrow, \ \Leftrightarrow$$

EXERCISES

In what follows, P, Q and R represent logical propositions

1. Write out the truth tables for the following expressions:
   - (a) $P \lor \neg Q$
   - (b) $\neg P \land Q$
   - (c) $P \Rightarrow \neg Q$
   - (d) $P \land (Q \lor \neg R)$
   - (e) $P \Rightarrow (Q \Rightarrow R)$

2. Use a truth table to demonstrate that the proposition $(P \land Q) \lor R$ is logically equivalent to the proposition $(P \lor R) \land (Q \lor R)$ (i.e. $(P \land Q) \lor R \Leftrightarrow (P \lor R) \land (Q \lor R)$)

3. Show, using a truth table: $(P \Rightarrow Q) \Leftrightarrow (\neg P \lor Q)$

4. Another logical operator may be introduced which is usually called *exclusive-or* (written *XOR*) or *exclusive-disjunction* (written $\lor_e$). The disjunction operator ($\lor$) is an *inclusive-or* operator in the sense that $P \lor Q$ is true when P is true or Q is true or **both** P and Q are true, whereas the compound expression (P xor Q) is true only if P is true or Q is true but **not** if **both** P and Q are true. Use truth tables to demonstrate:

   - (a) $P \text{ XOR } Q \Leftrightarrow (P \lor Q) \land \neg(P \land Q)$
   - (b) $P \text{ XOR } Q \Leftrightarrow (P \land \neg Q) \lor (\neg P \land Q)$

5. The proposition $P \Leftrightarrow Q$ states that the propositions P and Q are equivalent. This may, or may not, be true. If, however, we know that P and Q will always be equivalent no matter what the circumstances, then it is, perhaps, more appropriate to use $\equiv$ (the identity symbol) rather than $\Leftrightarrow$. Use truth tables to demonstrate the following identities:

(a)	$P \land Q \equiv Q \land P$	Shows $\land$ is a *commutative* operator
(b)	$P \lor Q \equiv Q \lor P$	Shows $\lor$ is a *commutative* operator
(c)	$(P \land Q) \land R \equiv P \land (Q \land R)$	Shows $\land$ is an *associative* operator
(d)	$(P \lor Q) \lor R \equiv P \lor (Q \lor R)$	Shows $\lor$ is an *associative* operator
(e)	$P \lor (Q \land R) \equiv (P \lor Q) \land (P \lor R)$	Shows $\lor$ is a *distributive* operator
(f)	$P \land (Q \lor R) \equiv (P \land Q) \lor (P \land R)$	Shows $\land$ is a *distributive* operator
(g)	$\neg (P \land Q) \equiv \neg P \lor \neg Q$	Known as *De Morgan's Law*
(h)	$\neg (P \lor Q) \equiv \neg P \land \neg Q$	Another of *De Morgan's Laws*
(i)	$\neg \neg P \equiv P$	*Double negation* property

Propositions which are always true (such as those just considered) are called *tautologies*.

6. Let P, Q and R be the propositions: P: the membership is less than 20
Q: all the members are men
R: the maximum number of members is 50

(a) Describe, in *English*, the meaning of the following propositions:

(i) $P \wedge Q$
(ii) $\neg R$

(b) Using P, Q and R as defined, represent, *symbolically*, the proposition:

There are at least 20 members and some of them are women

7. Suppose P represents the proposition "Claire is happy" and Q represents the proposition "Claire is rich". Write, in symbolic form:

(a) Claire is poor but happy
(b) Claire is neither rich nor happy
(c) Claire is either rich or unhappy
(d) Claire is either poor or is both rich and unhappy.

8. Show that the following are *tautologies* (see comment at end of question 5 above):
(a) $P \Rightarrow (P \vee Q)$
(b) $P \wedge Q \Rightarrow P$
(c) $((\neg P) \wedge (P \vee Q)) \Rightarrow Q$
(d) $((\neg P) \vee Q) \Leftrightarrow (P \Rightarrow Q)$

9. A college provides a multi-user computer system for its members. All members must *register* with the college's IT Services unit before they are allowed access to the computer system. To use the system, each registered user must *log_in*. At any given time a registered user will either be *logged-in* or not *logged-in* and it is not possible for a user to be *logged-in* more than once concurrently. In the following, express your solutions both symbolically (using sets and propositional logic), in the style outlined towards the end of the preceding notes, *and* in narrative form using **plain** English. Your solutions should cater for the necessary preconditions **not** being satisfied and be based upon a free type of the form:

RESPONSE ::= OK | Already a user | Not a user | Logged in | Not logged in

(a) Define an operation to register a new user.
(b) Define an operation to cancel a user's registration.
(c) Define an operation to *log-in*.
(d) Define an operation to *log-out*.

10. The operators of propositional logic ($\neg \wedge \vee \Rightarrow \Leftrightarrow$) obey recognized precedence rules (refer to the chapter summary). If P, Q, R and S represent logical propositions:

(a) Without altering their essential meaning, simplify the following predicates by omitting as many parentheses as possible:

(i) $(\neg (((\neg R) \wedge P) \vee Q)) \Rightarrow R$

(ii) $(\neg P) \Rightarrow (((P \Rightarrow Q) \Rightarrow R) \wedge S)$

(b) Insert parentheses to emphasise how the following predicates are interpreted according to the conventional precedence rules:

(i) $P \Rightarrow Q \Leftrightarrow \neg Q \Rightarrow \neg P$

(ii) $P \vee Q \wedge \neg R \vee Q \wedge P$

(c) Use a truth-table to establish the validity of De Morgan's law:

$$\neg (P \vee Q) \Leftrightarrow \neg P \wedge \neg Q$$

(d) Any logical proposition which is always *true* is called a *tautology*; any which is always *false* is called a *contradiction*. Identify which of the following propositions is a tautology, a contradiction or neither:

(i) $\neg (P \vee Q)$

(ii) *false* $\wedge \neg (P \vee Q)$

(iii) *false* $\vee$ *true*

(e) Consider the following information:

Oscar either cycles to work or uses his car. If it is not raining, Oscar cycles to work. If it is raining then Oscar uses his car unless the car does not start, in which case he has to cycle to work in the rain unless he can get a push-start from his neighbour.

If P, Q and R represent propositions as follows:

P : a push-start is available;
Q : the car starts;
R : it is raining

write down a logical expression involving P, Q and R which evaluates to *true* if Oscar cycles to work and *false* otherwise.

11. Suppose [PERSON] is a given type representing the set of all people there might ever be, past, present or future. Let the following sets be defined:

men	:	the set of all men
women	:	the set of all women

and suppose a company (which employs only men and women) comprises only the following departments:

marketing	:	the set of all people working in the marketing department of the company
personnel	:	the set of all people working in the personnel department of the company
production	:	is the set of all people working in the production department of the company

(a) Define a suitable **type** common to all five sets.

(b) Use set notation with appropriate logical connectives to represent each of the following statements symbolically:

(i) people at the company are either men or women but not both;
(ii) each employee of the company is in precisely **one** of the three departments, *marketing*, *personnel* or *production*;
(iii) the *personnel* department has a maximum of 10 employees;
(iv) all the employees in the *marketing* department are women;
(v) the company employs more men than women.

(c) If we assume, instead, that each employee of the company can be in more than one department, write symbolic expressions to represent the following:

(i) the number of women who work in all three departments;
(ii) the number of men who work in *marketing* and *personnel* but not *production*.

12.

(a) The disjunction operator ($\vee$) is an *inclusive-or* logical connective in the sense that $P \vee Q$ is true when P is true or Q is true or **both** P and Q are true. An *exclusive-or* logical connective, $\otimes$, say, may be defined so that $P \otimes Q$ is true only if P is true or Q is true but **not** if **both** P and Q are true.

(i) Construct an appropriate truth-table for $P \otimes Q$
(ii) Show $P \otimes Q \Leftrightarrow (P \vee Q) \wedge (\neg (P \wedge Q))$

(b) A special logical connective, represented by $\Diamond$, has the following truth table:

P	Q	P$\Diamond$Q
true	*true*	*false*
true	*false*	*false*
false	*true*	*false*
false	*false*	*true*

Show: $P \wedge Q \Leftrightarrow (P \Diamond P) \Diamond (Q \Diamond Q)$

13. All recognized modules which students at a university can study are modelled by the set *modules*. Modules that are taken in the first year are modelled by the set *firstYear*; those that are taken in the second year by the set *secondYear;* and those in the third year by the set *thirdYear*.

Express each of the following statements using set notation (note that the statements may not be consistent with each other).

(a) The total number of recognized modules that are available to be studied will never exceed 50.
(b) None of the available modules can be taken in different years of a course (i.e. every module can be taken only in the first year or only in the second year or only in the third year - where "or" is exclusive).
(c) The module *computing_fundamentals* is taught in the first year.
(d) The *computer_architecture* module may be taught in either year two or year three but never in both years.
(e) All modules taken in years one, two or three are recognized by the university.