SETS

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INTRODUCTION

- Why *Mathematics*?

    - Natural language is not always *precise* and *unambiguous*

    - *Meaning* often depends on *context*

      Consider:

      He drives a red car
      He drives a hard bargain

      or:

      She sang like her sister
      She sang like a nightingale
      He sang like a canary

INTRODUCTION

- Mathematics
  - has a *proven track record* in science and engineering
  - is *precise*
  - is *concise* and self-contained
  - has *clarity* with little scope for misunderstanding
  - helps us concentrate on the *essentials*
  - is independent of natural language
  - may prove *correctness*

- We shall be concerned with the mathematics of *sets* and *logic* rather than numbers (though sets of *integer numbers* will be of interest)

DEFINITION & NOTATION

- A *set* is (informally) a:

    - well-defined,
    - unordered

  collection of ***similar*** items where each item is

    - *identifiable*, and
    - *distinct* from the other items

- A set may be defined by listing (or *enumerating*) its *members* or *elements* inside curly braces:

  {a, e, i, o, u} is the *set of vowels*, and

  {England, France, Ireland, Italy, Scotland, Wales}

  is *the set of countries which participate in Rugby Union's six-nations' championship*

SPECIAL SETS

- A set with just **one** member is a *singleton* set:

  {February} is *the set of months with less than 30 days*

- A set with no members is called the *null* set or *empty* set and is denoted either by {} or ∅

  e.g. the *set of all humans over twenty feet tall* is empty or null (i.e. = {})

- Certain sets of integers are denoted by generally accepted special symbols:

  $\mathbb{N}$ represents the set of *natural numbers* (≥ 0)

  $\mathbb{N}_1$ represents the non-zero *natural numbers* (≥ 1)

  and

  $\mathbb{Z}$ represents the set of positive and negative *integers* (i.e. whole numbers)

EQUIVALENCE & EQUALITY

- Two sets are *equal* or *equivalent* if, and only if, they have the same members

  e.g. {1, 2, 3, 5, 7, 11} and {3, 1, 2, 11, 5, 7} are equal/equivalent sets

- Set *definition* (i.e. *syntactic equivalence*) will be shown by = =

  e.g. *Vowels* = = {a, e, i, o, u}

  - Here *Vowels* is a shorthand 'name' for the set enumerated to the right of the = = sign

- The single = sign is often used to show equivalence between two sets but it may also be used to 'define' a set where the members of that set may change

  e.g. *CourseTeam* = {Smith, Jones, Patel}

SET MEMBERSHIP

- Set *membership* is denoted by $\in$ which is read as

    - *is a member of*, or
    - *is an element of*, or
    - *belongs to*

    e.g. u $\in$ *Vowels* (the set defined above)
    $0 \in \mathbb{N}$ and $-7 \in \mathbb{Z}$

- *Non-membership* is denoted by $\notin$ which is read as

    - *is not a member of*, or
    - *is not an element of*, or
    - *does not belong to*

    e.g. p $\notin$ *Vowels*
    $0 \notin \mathbb{N}_1$ and $3.14 \notin \mathbb{Z}$

OPERATIONS ON SETS - CARDINALITY

- The number of unique elements in a set is denoted by #

  e.g. $\#Vowels = 5$ and $\#\{\} = 0$

- For a set P, #P is often called the *size* or *cardinality* of the set P

- If #P is a finite number then P is said to be a *finite* set (otherwise it is an *infinite* set)

- Examples of infinite sets are: $\mathbb{Z}$, $\mathbb{N}$ and $\mathbb{N}_1$

- The arithmetic of infinite sets can seem 'weird':

  If $A == \{1, 2, 8\}$ and $B == \{5, 7, 9, 17\}$ then we can see $\#A = 3$ and $\#B = 4$ and so $\#B > \#A$

  What, though, of $\#\mathbb{N}$ and $\#\mathbb{N}_1$?

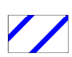
OPERATIONS ON SETS - UNION

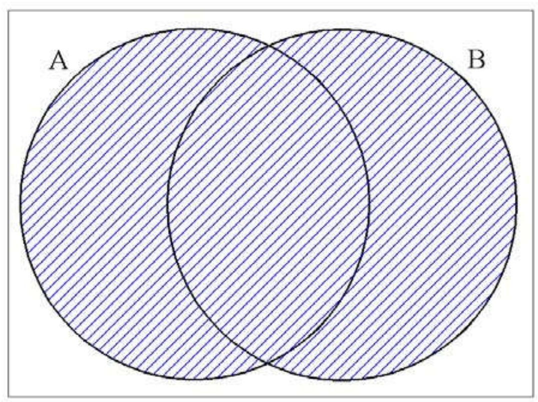
- The *union* of two sets A and B is the set of all elements contained in both A and B with any element occurring in **both** A and B being listed **once only** in the union

- The *union* of A and B is written $A \cup B$.

- If A = = {p, q, u, v} and B = = {g, h, k, u, v, y}

  then $A \cup B = \{g, h, k, p, q, u, v, y\}$

- A *Venn* diagram provides a graphic illustration with the union of sets A and B (i.e. $A \cup B$) depicted by the whole area shaded like 



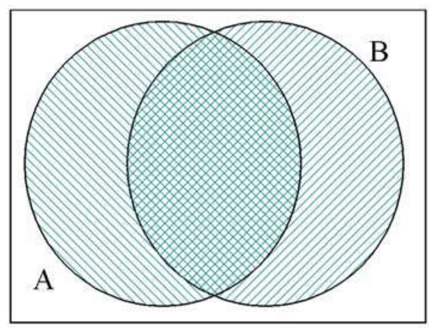
- The enclosing rectangle represents the *universal* set (i.e. **all** the elements in the domain in which we are interested)

OPERATIONS ON SETS - INTERSECTION

- The *intersection* of sets A and B is the set of those elements **common to both A and B** and is written $A \cap B$

- If A = = {p, q, u, v} and B = = {g, h, q, t, v, y}
  then $A \cap B = \{q, v\}$

- The corresponding *Venn* diagram might be as shown with the intersection of the two sets A and B (i.e. $A \cap B$) represented by the area shaded like 



- If A and B have no members in common they are said to be *disjoint* and we can then write:

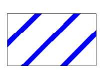
  $A \cap B = \{\}$ or $A \cap B = \varnothing$
  or, less often, *disjoint* $\langle A,B \rangle$

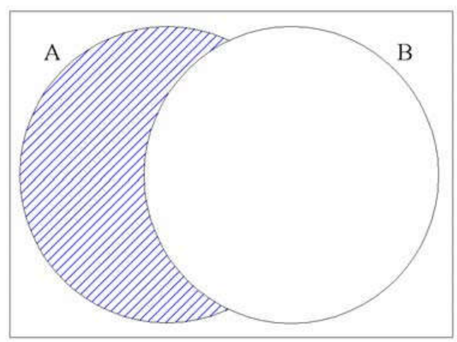
OPERATIONS ON SETS - DIFFERENCE

- The *difference* (or *relative complement*) of sets A and B is the set of all those elements which occur in A but not in B

- The *difference* of A and B is written A \ B

  e.g. if $A = = \{p, q, u, v\}$ and
  $B = = \{g, h, k, u, v, y\}$
  then $A \setminus B = \{p, q\}$

- The corresponding *Venn* diagram might be as shown where A \ B is depicted by the area shaded similar to 



SUBSETS

- Suppose A = = {d, f, h, p, t} and B = = {h, t} then we notice that all members of set B are also members of set A. In such a case we say:

    - B is a *subset* of A, or
    - set B *is included in* set A

- If we know all members of set B are also in set A we can write $B \subseteq A$

- $B \subseteq A$ allows that the two sets *may*
    - be equivalent and
    - have exactly the same members

- An obvious corollary is that any set *must* be a subset of itself (i.e. for any set A, $A \subseteq A$)

- Note that (unlike $\in$) *set inclusion* ($\subseteq$) is transitive

SUBSETS

- If we know all of the members of B are also in A (i.e. $B \subseteq A$) but that A also has members which are not in B, then we should strictly write $B \subset A$

- For example:

$\{g, m\} \subseteq \{f, g, k, m, p\}$	is *true*
$\{a, p, t\} \subseteq \{p, t, a\}$	is *true*
$\varnothing \subseteq \{k, y\}$	is *true*
$\{a, p, t\} \subset \{p, t, a\}$	is *false*
$\{a, t\} \subset \{p, t, a\}$	is *true*

and, in particular:

$\mathbb{N}_1 \subset \mathbb{N}$	is *true*
$\mathbb{N} \subset \mathbb{Z}$	is *true*

POWERSETS

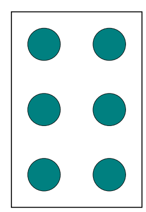
- The set of all possible subsets of a set A is called the *powerset* of A and is written $\mathbb{P}A$

- Since { } (i.e. $\varnothing$) is a valid subset of *any* set, the powerset of {a, b, c} is:

  $$\{\{\,\}, \{a\}, \{b\}, \{c\}, \{a, b\}, \{b, c\}, \{c, a\}, \{a, b, c\}\}$$

- An example of one of the uses of a powerset is the construction of *Braille* characters:

  - Each *Braille* character is based upon a 'cell' of 6 dots:

  - Each *Braille* "cell" can be regarded as a set of 6 possible dot-positions
    - The set of possible characters is the powerset of C (i.e. all possible subsets selected from C)
    - How *many* are there?

SET COMPREHENSION

- *Enumeration* is, generally, used to define sets only when there are not many members

- When *enumeration* is used
  - the set members can be analysed only by inspecting each and every member, and
  - even careful inspection does not always clarify the characteristic(s) shared by members of a set (e.g. consider $\{1, 3, 5\}$)

- Sets are collections of objects which share similar characteristics and this fact provides a better mechanism (*abstraction*) for defining what a set contains

SET COMPREHENSION

- If *Big_Countries* is the set of countries with more than 100 million people and *C* represents the set of all countries, we could write:

  Big_Countries = = {c:*C* | c *has more than 100 million people*}

  - 'c:*C'* (the *signature*) means the values of c are drawn from the set *C*, and
  - the vertical bar '|*'* (called the *constraint* bar) is read as 'such that*'*

- The 'rule', 'condition' or 'constraint', appearing to the right of the *constraint bar* '|' is called a *predicate* and is either *false* or *true*

- The above way of specifying a set according to the characteristics shared by its members rather than by *enumeration* is called *set comprehension*

SET COMPREHENSION

- An alternative form of set comprehension specifies set elements by using a *pattern*

- If *Evens* is the set {0, 2, 4, 6, 8, ...} we can specify *Evens* as:

  $$Evens == \{ x:\mathbb{N} \bullet 2x \}$$

  which is read as:

  'the set Evens is defined to comprise elements generated by the pattern or term "2 multiplied by x" where x is taken from the set of natural numbers'

- Similarly, if $Non\_Zero\_Evens == \{2, 4, 6, ...\}$

  we may write: $Non\_Zero\_Evens == \{x:\mathbb{N} \mid x>0 \bullet 2x\}$

  where the constraining predicate ($x > 0$) acts as a filter to ensure non-zero values. Do we need it?

TYPED SETS

- Recall our (informal) definition of a set:

  A set is a well-defined, unordered collection of similar items where each item is clearly identifiable and distinct from the other items

- *Well-defined* means that given a ‘value’ we are able to decide whether it is a member of the set

- Sets are *homogeneous* in the sense that all members of a set are in some way similar

- All possible values that a set may have as members is said to define the *type* of the set

TYPED SETS

- If we attempted to model a *Library* system we would deal with sets of books and sets of people

- Suppose PERSON represents the set of all people that might ever be associated with our library, then

- At any given time, the set of Library staff

  - would be one particular *subset* of PERSON, and, hence

  - the set of Library staff, would be **one** of the sets defined by the powerset of PERSON

    Remember: *The powerset of a set A (i.e. $\mathbb{P}A$) is the set of* ***all*** *possible subsets of A*

- Similarly, if BOOK represents the set of all books that might ever be associated with our library, then

- At any given time, the set of books actually on loan (or, similarly, the set of books available for loan)
  - would be one particular *subset* of BOOK, and, hence
  - at that time, the set of books on loan, say, would be one of the sets defined by the powerset of BOOK

TYPED SETS

- When using sets to specify systems we start by declaring *basic set types* (or *given sets*) which characterize the universal sets of objects we anticipate having to deal with

- These *basic types* are declared by writing them using upper-case letters in *square brackets*:

  [BOOK] and [PERSON] for a *Library* system
  [STUDENT] and [COURSE] for a *College Admin* system

  and, several types can be given in one declaration: [BOOK, PERSON]

- If each *member* of a set, which is based upon a given set, is of type T, then that set has type $\mathbb{P}$ T (remember, the *type* of the set is the set of *all possible* values the set may contain)

TYPED SETS

- In an academic *course-administration* system,
  - a **given set** could be [STUDENT]
  - *enrolled* could be the set of students who enrol on the BSc *Computing*, and
  - *graduated* could be the set of students who successfully complete the course
  - any *member* of each of the sets *enrolled* and *graduated* will be of type STUDENT
  - the type of **both** of the sets *enrolled* and *graduated* will be $\mathbb{P}$ STUDENT
  - we write: $\quad enrolled, graduated : \mathbb{P}\ \mathrm{STUDENT}$

- Equivalent statements to

  $$enrolled, graduated : \mathbb{P}\ \mathrm{STUDENT}$$

  are:

  $enrolled \in \mathbb{P}\ \mathrm{STUDENT}\ ;\ graduated \in \mathbb{P}\ \mathrm{STUDENT}$

  or:

  $enrolled \subseteq \mathrm{STUDENT}\ ;\ graduated \subseteq \mathrm{STUDENT}$

FREE TYPES

- *Free types* or *enumerated types* can also be declared by enumerating the allowed identifiers for each of their elements:

  e.g. RESPONSE ::= yes | no
  STUDENT_GRADE ::= distinction | merit | pass | refer | fail

  where the vertical bar, "|", is read as "or"

- "RESPONSE ::= yes | no" is a shorthand for the following declarations and predicates:

[RESPONSE]	RESPONSE is a given set
yes : RESPONSE	*yes* is a value of the set
no : RESPONSE	*no* is a value of the set
yes ≠ no	*yes* and *no* are distinct
RESPONSE = {yes, no}	*yes* and *no* are the **only** values of the type

WELL-FORMED EXPRESSIONS

- When dealing with *typed* sets, the set operations considered previously (such as $\in$, #, $\cap$, etc) must only be applied to sets of compatible types

- If $Benelux == \{Belgium, Holland, Luxembourg\}$ and *Reference* is a set of books which may **not** be borrowed, then the members of the sets *Benelux* and *Reference* (and hence the sets themselves) are of *different* types

- It is, therefore, meaningless to write expressions such as

$$Holland \notin Reference$$

  since *Holland* is **not of the same type** as the members of the set *Reference*

- Expressions involving **incompatible** types are said to be **not** *well-formed*

SUMMARY OF SET SYMBOLS

$\mathbb{Z}$	*Set of integers (positive or negative whole numbers)*
$\mathbb{N}$	*Set of natural numbers (≥ 0)*
$\mathbb{N}_1$	*Set of positive natural numbers (≥ 1)*
$t \in S$	*t is an element of set S*
$t \notin S$	*t is not an element of set S*
$S \subseteq T$	*Set S is contained in set T*
$S \nsubseteq T$	*Set S is not contained in set T*
$S \subset T$	*Set S is strictly contained in set T (S ≠ T)*
$\varnothing$ or $\{\}$	*empty set*
$\{t_1, t_2, \ldots t_n\}$	*the set containing elements t_1, t_2, ... t_n*
$\mathbb{P}S$	*Powerset of set S : the set of all possible subsets of S*
$S \cup T$	*Union of sets S and T : the set of elements which are in S or in T or in both*
$S \cap T$	*Intersection of sets S and T : the set of elements which are both in S and in T*
$S \setminus T$	*Difference of sets S and T : the set of elements which are in S but not in T*
$\#S$	*Size or cardinality of set S : the number of elements contained in set S*
$\{D \mid P \bullet t\}$	*Set of elements t such that declarations D and P hold true*

EXERCISES

1. Translate the following *symbolic* statements into *English*:
   (a) $x \in S$
   (b) $x \notin S$
   (c) $X \subseteq Y$
   (d) $A \subset B$
   (e) $A \in B$
   (f) $B \notin C$

2. If NATO == {*Belgium, Canada, Denmark, Iceland, Italy, Luxembourg, Holland, Norway, Portugal, United Kingdom, United States, Greece, Turkey, Spain, Germany*}
   EC == {*Belgium, France, Germany, Italy, Luxembourg, Holland, Denmark, Greece, Ireland, United Kingdom, Spain, Portugal*}
   Scandinavia == {*Denmark, Finland, Norway, Sweden, Iceland*}
   Benelux == {*Belgium, Holland, Luxembourg*}
   Central America == {*Costa Rica, Honduras, El Salvador, Guatemala, Nicaragua, Belize, Panama*},

   enumerate the following sets:

   (a) EC $\cup$ NATO
   (b) EC $\cap$ NATO
   (c) NATO \ EC
   (d) EC \ NATO
   (e) Scandinavia \ NATO
   (f) EC $\cup$ Benelux
   (g) EC \ Benelux
   (h) Benelux \ EC
   (i) NATO $\cap$ Scandinavia
   (j) Central America $\cap$ Benelux
   (k) EC $\cap$ (NATO $\cap$ Scandinavia)
   (l) (EC $\cap$ NATO) $\cap$ Scandinavia
   (m) EC $\cup$ (NATO $\cup$ Scandinavia)
   (n) (EC $\cup$ NATO) $\cup$ Scandinavia

3. Suppose A, B and C are three sets such that $A \subseteq B$ and $B \cap C$ is the empty set.
   (a) Draw a Venn diagram that illustrates this situation.
   (b) Draw another Venn diagram for the case when $A \subseteq B$ and $A \cap C$ is the empty set.

4. Use Venn diagrams to demonstrate
   (a) $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$
   (b) $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$

5. In what circumstances will $A \setminus B = B \setminus A$?

6. The concepts of *union* and *intersection* of two sets can be extended to any number of sets. The symbols used are like those already encountered but are *larger* and and are written in front of the set of sets on which it operates. For example, the *union* of the sets {2, 5}, {2, 7, 8} and {3, 7, 8, 11} would be written:

   $$\bigcup \{\{2, 5\}, \{2, 7, 8\}, \{3, 7, 8, 11\}\} = \{2, 3, 5, 7, 8, 11\}$$

   Similarly the intersection of the three sets is written:

   $$\bigcap \{\{2, 5\}, \{2, 7, 8\}, \{3, 7, 8, 11\}\} = \{\} = \varnothing$$

   If A = {1, 2, 3, 4}, B = {2, 4, 6, 8, 19} and C = {2, 3, 5, 7} write down the sets:
   (a) $\bigcup\{A, B, C\}$
   (b) $\bigcap\{A, B, C\}$
   (c) $A \setminus (B \cap C)$

7. Given A = {a, b, c}, B = {b, c, d, e} and C = {a, b, c, d, e, f}, find:
   (a) #A
   (b) #C
   (c) $A \cup B$
   (d) $A \cap B$
   (e) $A \setminus B$
   (f) $A \cap (B \cup C)$
   (g) $A \cap (B \cap C)$

   State, giving reasons, whether the following statements are true for the above sets:

   (h) $b \in B$
   (i) $h \in B$
   (j) $A \in A$

8. Enumerate the following sets (e.g. $\{n : \mathbb{N} \mid n < 2\} = \{0, 1\}$):
   (a) $\{n : \mathbb{N} \mid n^2 < 17\}$
   (b) $\{n : \mathbb{Z} \mid n^2 < 17\}$
   (c) $\{n : \mathbb{N} \bullet n + 2\}$
   (d) $\{n : \mathbb{N} \mid n = 4 \bullet 2n\}$
   (e) $\{n : \mathbb{N} \mid n < 4 \bullet 2n\}$

9. Describe the following sets using set comprehension (there may be more than one answer!) (e.g. $\{0, 1, 8, 27\} = \{n{:}\mathbb{N} \mid n \leq 3 \bullet n^3\}$):

(a) $\{0, 1, 2, 3, 4\}$
(b) $\{0, 3, 6, 9, 12\}$
(c) $\{1, 2, 4, 8, 16, 32\}$
(d) The set of natural numbers greater than 15
(e) The set of integers whose square is more than 40

10. Find the sets defined by
(a) $\{n{:}\mathbb{N} \mid n > 7\} \cap \{n{:}\mathbb{N} \mid n < 10\}$
(b) $\{n{:}\mathbb{N} \mid n > 7 \bullet n^2 + 4\} \cap \{\ \}$

11. Which of the following predicates are *true* and which are *false*?
(a) $\{3, 4, 5\} \cup \{\ \} = \{\ \}$
(b) $\{3, 4, 5\} \cap \{\ \} = \{\ \}$
(c) $\{n{:}\mathbb{N} \mid n^2 < 20\} \cap \{1, 2, 3\} = \{2, 3\}$
(d) $\{n{:}\mathbb{N} \mid n > 5 \text{ and } n < 10\} \cup \{n{:}\mathbb{N} \mid n \leq 5\} = \{n{:}\mathbb{N} \mid n < 10\}$

12. Set comprehension may be used to define *set union* as follows:
Suppose the elements of the sets A and B are of type T, then the *union* of the sets A and B (written $A \cup B$) is defined by: $A \cup B = \{ x : T \mid (x \in A) \text{ OR } (x \in B)\}$
or, if we use $\vee$ as a shorthand for OR: $A \cup B = \{ x : T \mid (x \in A) \vee (x \in B)\}$

Using, where necessary, $\wedge$ for AND with $\vee$ for OR write set comprehensions for

(a) set *intersection* ($\cap$)
(b) set *difference* ($\setminus$)

13. Given the declarations: $x, y, z : \mathbb{Z}$; a : AUTHOR; b : BOOK;
on_shelves : $\mathbb{P}$ BOOK; novelists : $\mathbb{P}$ AUTHOR
say whether the following notations are *well-formed*
(a) $x > y$
(b) $x \in$ on_shelves
(c) $a \in$ on_shelves
(d) $a \in$ novelists
(e) on_shelves $\subseteq$ novelists
(f) on_shelves $\subseteq$ Book
(g) {on_shelves, novelists}
(h) $\{a, b\}$

14. Given the following definitions: $s = \{ 1, 4, 6, 7, 8, 9 \}$
$t = \{ x : \mathbb{N} \mid x^2 < 10 \bullet x^2 \}$

write down the results of the following expressions:

(a) $s \cup t$
(b) $s \cap t$
(c) $s \setminus t$

15. Suppose *int_sets* is a set containing sets of integers (i.e. $int\_sets : \mathbb{P}\ (\mathbb{P}\ \mathbb{N}\ ))$, with:

$$int\_sets = \{\ \{1, 2, 3\}, \{\ \}, \{3, 4, 5\}, \{3, 4\}\ \}$$

(a) What is produced by $\bigcup int\_sets$?
(b) What is produced by $\bigcap int\_sets$?