

QUADRATIC FORMS

THE GAUSS–JORDAN ELIMINATION COMPLETES THE SQUARE



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OUTLINE

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DEFINITION OF QUADRATIC FORMS

A quadratic form is a special multi-variable function.

Definition

Let $A \in \mathbb{R}^{n \times n}$ be a symmetric matrix. Assume that $a_{k,j}$ denotes the appropriate element of that matrix. The function $Q : \mathbb{R}^n \rightarrow \mathbb{R}$

$$Q(x_1, x_2, \dots, x_n) = \sum_{k=1}^n \sum_{j=1}^n a_{k,j} x_k x_j$$

is called n -variable quadratic function.

If $n = 1$ then $Q(x_1) = ax_1^2$.

If $n = 2$ then

$$Q(x_1, x_2) = a_{1,1}x_1^2 + a_{1,2}x_1x_2 + a_{2,1}x_2x_1 + a_{2,2}x_2^2.$$

Compute the case when $n = 3$.

Theorem

If $A \in \mathbb{R}^{n \times n}$ is a symmetric matrix then the quadratic form determined by A is

$$Q(x) = x \cdot Ax$$

for every $x \in \mathbb{R}^n$.

Example

Write the symmetric matrix of quadratic form given before. Write the quadratic form of a symmetric matrix given before.

Theorem

The quadratic form determined by $A + B$ is the same as the sum of quadratic forms determined by A and B respectively. Similarly, the quadratic form determined by αA is the same as the form determined by A multiplied by α .

DEFINITENESS

Our goal is to classify a quadratic form with respect to its range!

Definition

Let $Q : \mathbb{R}^n \rightarrow \mathbb{R}$ be a non-zero quadratic form. It is called

positive definite, if $Q(x) > 0$ for every $x \in \mathbb{R}^n, x \neq 0$;

positive semidefinite, if $Q(x) \geq 0$ for every $x \in \mathbb{R}^n$, but there exists a $z \in \mathbb{R}^n, z \neq 0$ such that $Q(z) = 0$;

negative definite, if $Q(x) < 0$ for every $x \in \mathbb{R}^n, x \neq 0$;

negative semidefinite, if $Q(x) \leq 0$ for every $x \in \mathbb{R}^n$, but there exists a $z \in \mathbb{R}^n, z \neq 0$ such that $Q(z) = 0$;

indefinite, if there exist two vectors $x, y \in \mathbb{R}^n$ for which $Q(x) > 0$ but $Q(y) < 0$.

DYADIC DECOMPOSITION

DYADIC DECOMPOSITION

DYAD AND COMPLETING THE SQUARE

The dyad is a special symmetric matrix representing the perfect squares.

Definition

A symmetric matrix $A \in \mathbb{R}^{n \times n}$ is called **dyad**, if there exists a vector $a \in \mathbb{R}^{n \times 1}$ for which $A = a \cdot a^T$

Theorem

A quadratic form is a complete square if and only if it is determined by a dyad. If $A = a \cdot a^T$ where $a = (a_1, a_2, \dots, a_n)$, then $Q(x_1, x_2, \dots, x_n) = (a_1 x_1 + a_2 x_2 + \dots + a_n x_n)^2$, here Q is the quadratic form determined by the dyad A .

DYADIC DECOMPOSITION

HIGH SCHOOL METHOD FOR A FEW
VARIABLES ONLY

DECOMPOSITION

Theorem

Every symmetric matrix is a linear combination of dyads. Thus every quadratic form is a linear combination of complete squares.

Example

Completing the square, rewrite the following quadratic forms as a linear combination of perfect squares.

$$Q(x_1, x_2) = x_1^2 + 4x_1x_2 - 5x_2^2,$$

$$Q(x_1, x_2) = 4x_1^2 + 6x_1x_2 + 9x_2^2,$$

$$Q(x_1, x_2) = 5x_1^2 - 6x_1x_2 + x_2^2,$$

$$Q(x_1, x_2, x_3) = 5x_1^2 - 6x_1x_2 + x_1x_3,$$

DYADIC DECOMPOSITION

PROFESSIONAL METHOD

DYADIC DECOMPOSITION USING GAUSS–JORDAN ELIMINATION

Every n -variable quadratic form can be reduced by subtracting a multiple of a dyad such that the resulting quadratic form has one fewer variable:

Theorem

Consider a symmetric matrix $A \in \mathbb{R}^{n \times n}$. Assume that $a_{t,t} \neq 0$ and denote $d = \frac{1}{a_{t,t}}a_t$, where a_t is the t -th column of the matrix A . Then

$$R = A - a_{t,t}d \cdot d^T$$

is a symmetric matrix whose t -th row (and column) is zero vector.

Proof.

The sum of symmetric matrices is symmetric. If δ_k denotes the k -th coordinate of d , then $a_t^T - a_{t,t}\delta_t d^T$ is the t -th row of matrix R . It is the zero row because $\delta_t = 1$ and $a_{t,t}d = a_t$. By the symmetry of R , the t -th column is also the zero column. □

Understanding the second step of the algorithm guaranteeing dyadic decomposition relies on the observation that the matrix of the residual quadratic form, namely

$$R = A - a_{t,t}d \cdot d^T,$$

can be calculated far more efficiently than by the original formula. It was already seen that every element in the t -th row is zero. Now let us write down the k -th row of matrix R for $k \neq t$. It is important to note that the row d^T is the auxiliary row of the Gauss–Jordan elimination when choosing the pivot element $a_{t,t}$. Consider the k -th row of the dyad $d \cdot d^T$ as $\delta_k d^T$. Then it is clear that the k -th row of matrix R is the difference of the k -th row of matrix A and the row $a_{t,k}d^T$ because

$$a_{t,t}\delta_k d^T = a_{t,t} \frac{a_{k,t}}{a_{t,t}} d^T.$$

Recognize that this is exactly the computational rule of the elimination algorithm in the case when the diagonal t -th element is the pivot and we compute the entries of the rows not containing the pivot element.

This means that the operation

$$A - a_{t,t}d \cdot d^T$$

can be calculated by Gauss–Jordan elimination instead of dyadic multiplication and matrix subtraction.

We have thus proven the following proposition.

Proposition

Let $A \in \mathbb{R}^{n \times n}$ be a symmetric matrix such that $a_{t,t} \neq 0$. Let R be the matrix of the residual quadratic form after removing the first dyad, that is,

$$R = A - a_{t,t}d \cdot d^T,$$

where d is the t -th column of matrix A multiplied by the reciprocal of $a_{t,t}$.^a We know that the t -th column and row of matrix R contain only zeros. The rest of matrix R can be obtained by Gauss–Jordan elimination on matrix A with the pivot element $a_{t,t}$.

^aThat is, d^T is the auxiliary row of the elimination.

PROBLEMS

PROBLEMS

Write the following quadratic forms as a linear combination of complete squares.

1. $Q(x_2, x_3) = 13x_2^2 - 4x_2x_3 + 8x_3^2,$
2. $Q(x_1, x_2, x_3) = 5x_1^2 + 6x_2^2 + 4x_3^2 - 4x_1x_2 - 4x_1x_3,$
3. $Q(z_1, z_2, z_3) = 2z_1^2 + \frac{3}{2}z_3^2 + 2z_1z_2 - 4z_1z_3 + 2z_2z_3,$
4. $Q(x_1, x_2, x_3) = 3x_2^2 + 2x_1x_2 + 2x_1x_3 + 6x_2x_3,$
5. $Q(x_1, x_2, x_3, x_4) = 2x_1^2 + 2x_2^2 + 2x_3^2 + 2x_4^2 - 4x_1x_2 + 2x_1x_4 + 2x_2x_3 - 4x_3x_4$
6. $Q(x_1, x_2, x_3) = 2x_1x_2 + 2x_2x_3$

Let Q be a quadratic form defined as

$$Q(x_1, x_2, x_3, x_4) = (x_1 + x_2 + 2x_3 + x_4)^2 + (x_1 + x_4)^2 + (x_2 + 2x_3)^2$$

for real numbers x_1, x_2, x_3, x_4 .

1. Characterize the definiteness of Q .
2. Write the matrix of the quadratic form above.
3. Compute the rank of the matrix above.
4. Write the matrix as a linear combinations of r dyads, where r is the rank above.

DEFINITENESS OF A QUADRATIC FORM

DEFINITENESS

Classify a quadratic form with respect to its range!

Definition

Let $Q : \mathbb{R}^n \rightarrow \mathbb{R}$ be a non-zero quadratic form. It is called

positive definite, if $Q(x) > 0$ for every $x \in \mathbb{R}^n, x \neq 0$;

positive semidefinite, if $Q(x) \geq 0$ for every $x \in \mathbb{R}^n$, but there exists a $z \in \mathbb{R}^n, z \neq 0$ such that $Q(z) = 0$;

negative definite, if $Q(x) < 0$ for every $x \in \mathbb{R}^n, x \neq 0$;

negative semidefinite, if $Q(x) \leq 0$ for every $x \in \mathbb{R}^n$, but there exists a $z \in \mathbb{R}^n, z \neq 0$ such that $Q(z) = 0$;

indefinite, if there exist two vectors $x, y \in \mathbb{R}^n$ for which $Q(x) > 0$ but $Q(y) < 0$.

Theorem

Consider a square matrix A having rank r . Assume that

$$A = B \cdot C$$

where the number of the columns of B (and the number of the rows of C) is the same as the rank r .

Then the row system of C is a linearly independent system.

Proof.

Every row of A is a linear combination of the rows of C . It means that the row system of C is a spanning set of the row space of A . But the number of the rows of C is the same as the dimension of the row space of A , thus it is a minimal spanning set. We proved, that the rows of C form an independent set. □

We also proved that the the row system of C is a base of the row space of A and similarly the column system of B forms a base of the column space of A .

DEFINITENESS OF A QUADRATIC FORM

SUMMARY

DEFINITENESS AFTER COMPLETING THE SQUARE

It is not hard to decide the definiteness of a quadratic form if it is given with its dyad decomposition. Let Q be an n -variable non-zero quadratic form.

If the dyad decomposition

- includes positive and negative pivot terms then Q is an *indefinite* quadratic form.

Now assume the dyad decomposition includes exactly n complete squares. If

- all pivot terms are positive numbers then Q is *positive definite*;
- all pivot terms are negative numbers then Q is *negative definite*.

DEFINITENESS AFTER COMPLETING THE SQUARE ...

If the number of dyads is less than n , the same characterization holds true but the quadratic form is semidefinite only:

Thus if

- all pivot terms are positive numbers then Q is *positive semidefinite*;
- all pivot terms are negative numbers then Q is *negative semidefinite*.

