Quadratic forms

The Gauss–Jordan elimination completes the square

Gyula Magyarkuti

November 10, 2025

Department of Mathematics
Corvinus University of Budapest
S208/B

email: `magyarkuti@uni-corvinus.hu`
web: `https://magyarkuti.github.io/linear-algebra`

Outline

Definition of quadratic forms

A quadratic form is a special multi-variable function.

Definition

Let $A \in \mathbb{R}^{n\times n}$ be a symmetric matrix. Assume that $a_{k,j}$ denotes the appropriate element of that matrix. The function $Q : \mathbb{R}^n \to \mathbb{R}$

$$Q\left(x_1, x_2, \cdots, x_n\right) = \sum_{k=1}^{n}\sum_{j=1}^{n} a_{k,j}x_kx_j$$

is called n-variable quadratic function.

If $n = 1$ then $Q(x_1) = ax_1^2$.
If $n = 2$ then

$$Q\left(x_1, x_2\right) = a_{1,1}x_1^2 + a_{1,2}x_1x_2 + a_{2,1}x_2x_1 + a_{2,2}x_2^2.$$

Compute the case when $n = 3$.

Theorem

If $A \in \mathbb{R}^{n\times n}$ is a symmetric matrix then the quadratic form determined by A is

$$Q(x) = x \cdot Ax$$

for every $x \in \mathbb{R}^n$.

Example

Write the symmetric matrix of quadratic form given before. Write the quadratic form of a symmetric matrix given before.

Theorem

The quadratic form determined by $A + B$ is the same as the sum of quadratic forms determined by A and B respectively. Similarly, the quadratic form determined by αA is the same as the form determined by A multiplied by α.

Our goal is to classify a quadratic form with respect to its range!

Definition

Let $Q : \mathbb{R}^n \to \mathbb{R}$ be a non-zero quadratic form. It is called

positive definite, if $Q(x) > 0$ for every $x \in \mathbb{R}^n, x \neq 0$;

positive semidefinite, if $Q(x) \geq 0$ for every $x \in \mathbb{R}^n$, but there exists a $z \in \mathbb{R}^n, z \neq 0$ such that $Q(z) = 0$;

negative definite, if $Q(x) < 0$ for every $x \in \mathbb{R}^n, x \neq 0$;

negative semidefinite, if $Q(x) \leq 0$ for every $x \in \mathbb{R}^n$, but there exists a $z \in \mathbb{R}^n, z \neq 0$ such that $Q(z) = 0$;

indefinite, if there exist two vectors $x, y \in \mathbb{R}^n$ for which $Q(x) > 0$ but $Q(y) < 0$.

Dyadic decomposition

Dyadic decomposition

Dyad and completing the square

The dyad is a special symmetric matrix representing the perfect squares.

Definition

A symmetric matrix $A \in \mathbb{R}^{n \times n}$ is called *dyad*, if there exists a vector $a \in \mathbb{R}^{n \times 1}$ for which $A = a \cdot a^T$

Theorem

A quadratic form is a complete square if and only if it is determined by a dyad. If $A = a \cdot a^T$ where $a = (a_1, a_2, \cdots, a_n)$, then $Q(x_1, x_2, \cdots, x_n) = (a_1 x_1 + a_2 x_2 + \cdots + a_n x_n)^2$, here Q is the quadratic form determined by the dyad A.

Dyadic decomposition

High school method for a few variables only

Theorem

Every symmetric matrix is a linear combination of dyads. Thus every quadratic form is a linear combination of complete squares.

Example

Completing the square, rewrite the following quadratic forms as a linear combination of perfect squares.

$Q(x_1, x_2) = x_1^2 + 4x_1x_2 - 5x_2^2,$
$Q(x_1, x_2) = 4x_1^2 + 6x_1x_2 + 9x_2^2,$
$Q(x_1, x_2) = 5x_1^2 - 6x_1x_2 + x_2^2,$
$Q(x_1, x_2, x_3) = 5x_1^2 - 6x_1x_2 + x_1x_3,$

Dyadic decomposition

Professional method

Dyadic decomposition using Gauss–Jordan elimination

Every n-variable quadratic form can be reduced by subtracting a multiple of a dyad such that the resulting quadratic form has one fewer variable:

Theorem

Consider a symmetric matrix $A \in \mathbb{R}^{n\times n}$. Assume that $a_{t,t} \neq 0$ and denote $d = \frac{1}{a_{t,t}} a_t$, where a_t is the t-th column of the matrix A. Then

$$R = A - a_{t,t} d \cdot d^T$$

is a symmetric matrix whose t-th row (and column) is zero vector.

Proof.

The sum of symmetric matrices is symmetric. If δ_k denotes the k-th coordinate of d, then $a_t^T - a_{t,t}\delta_t d^T$ is the t-th row of matrix R. It is the zero row because $\delta_t = 1$ and $a_{t,t} d = a_t$. By the symmetry of R, the t-th column is also the zero column. $\square$

Understanding the second step of the algorithm guaranteeing dyadic decomposition relies on the observation that the matrix of the residual quadratic form, namely

$$R = A - a_{t,t} d \cdot d^T,$$

can be calculated far more efficiently than by the original formula. It was already seen that every element in the t-th row is zero. Now let us write down the k-th row of matrix R for $k \neq t$. It is important to note that the row d^T is the auxiliary row of the Gauss–Jordan elimination when choosing the pivot element $a_{t,t}$. Consider the k-th row of the dyad $d \cdot d^T$ as $\delta_k d^T$. Then it is clear that the k-th row of matrix R is the difference of the k-th row of matrix A and the row $a_{t,k} d^T$ because

$$a_{t,t} \delta_k d^T = a_{t,t} \frac{a_{k,t}}{a_{t,t}} d^T.$$

Recognize that this is exactly the computational rule of the elimination algorithm in the case when the diagonal t-th element is the pivot and we compute the entries of the rows not containing the pivot element.

This means that the operation

$$A - a_{t,t} d \cdot d^T$$

can be calculated by Gauss–Jordan elimination instead of dyadic multiplication and matrix subtraction.

We have thus proven the following proposition.

Proposition

Let $A \in \mathbb{R}^{n \times n}$ be a symmetric matrix such that $a_{t,t} \neq 0$. Let R be the matrix of the residual quadratic form after removing the first dyad, that is,

$$R = A - a_{t,t} d \cdot d^T,$$

where d is the t-th column of matrix A multiplied by the reciprocal of $a_{t,t}$.[a] *We know that the t-th column and row of matrix R contain only zeros. The rest of matrix R can be obtained by Gauss–Jordan elimination on matrix A with the pivot element $a_{t,t}$.*

[a] That is, d^T is the auxiliary row of the elimination.

Problems

Write the following quadratic forms as a linear combination of complete squares.

1. $Q(x_2, x_3) = 13x_2^2 - 4x_2x_3 + 8x_3^2$,
2. $Q(x_1, x_2, x_3) = 5x_1^2 + 6x_2^2 + 4x_3^2 - 4x_1x_2 - 4x_1x_3$,
3. $Q(z_1, z_2, z_3) = 2z_1^2 + \frac{3}{2}z_3^2 + 2z_1z_2 - 4z_1z_3 + 2z_2z_3$,
4. $Q(x_1, x_2, x_3) = 3x_2^2 + 2x_1x_2 + 2x_1x_3 + 6x_2x_3$,
5. $Q(x_1, x_2, x_3, x_4) = 2x_1^2 + 2x_2^2 + 2x_3^2 + 2x_4^2 - 4x_1x_2 + 2x_1x_4 + 2x_2x_3 - 4x_3x_4$
6. $Q(x_1, x_2, x_3) = 2x_1x_2 + 2x_2x_3$

Let Q be a quadratic form defined as

$$Q\left(x_1, x_2, x_3, x_4\right) = \left(x_1 + x_2 + 2x_3 + x_4\right)^2 + \left(x_1 + x_4\right)^2 + \left(x_2 + 2x_3\right)^2$$

for real numbers x_1, x_2, x_3, x_4.

1. Characterize the definiteness of Q.
2. Write the matrix of the quadratic form above.
3. Compute the rank of the matrix above.
4. Write the matrix as a linear combinations of r dyads, where r is the rank above.

Definiteness of a quadratic form

Classify a quadratic form with respect to its range!

Definition

Let $Q : \mathbb{R}^n \to \mathbb{R}$ be a non-zero quadratic form. It is called

positive definite, if $Q(x) > 0$ for every $x \in \mathbb{R}^n, x \neq 0$;

positive semidefinite, if $Q(x) \geq 0$ for every $x \in \mathbb{R}^n$, but there exists a $z \in \mathbb{R}^n, z \neq 0$ such that $Q(z) = 0$;

negative definite, if $Q(x) < 0$ for every $x \in \mathbb{R}^n, x \neq 0$;

negative semidefinite, if $Q(x) \leq 0$ for every $x \in \mathbb{R}^n$, but there exists a $z \in \mathbb{R}^n, z \neq 0$ such that $Q(z) = 0$;

indefinite, if there exist two vectors $x, y \in \mathbb{R}^n$ for which $Q(x) > 0$ but $Q(y) < 0$.

Theorem

Consider a square matrix A having rank r. Assume that

$$A = B \cdot C$$

where the number of the columns of B (and the number of the rows of C) is the same as the rank r.
Then the row system of C is a linearly independent system.

Proof.

Every row of A is a linear combination of the rows of C. It means that the row system of C is a spanning set of the row space of A. But the number of the rows of C is the same as the dimension of the row space of A, thus it is a minimal spanning set. We proved, that the rows of C form an independent set. $\square$

We also proved that the the row system of C is a base of the row space of A and similarly the column system of B forms a base of the column space of A.

Definiteness of a quadratic form

Summary

Definiteness after completing the square

It is not hard to decide the definiteness of a quadratic form if it is given with its dyad decomposition. Let Q be an n-variable non-zero quadratic form.
If the dyad decomposition

- includes positive and negative pivot terms then Q is an *indefinite* quadratic form.

Now assume the dyad decomposition includes exactly n complete squares. If

- all pivot terms are positive numbers then Q is *positive definite*;
- all pivot terms are negative numbers then Q is *negative definite*.

If the number of dyads is less than n, the same characterization holds true but the quadratic form is semidefinite only:
Thus if

- all pivot terms are positive numbers then Q is *positive semidefinite*;
- all pivot terms are negative numbers then Q is *negative semidefinite*.

Proof.

Let us start with the case when, in the last tableau of the elimination algorithm, all elements on the diagonal are zero, but there exists at least one nonzero element.

Then the quadratic form Q can be decomposed as $Q = Q_1 + Q_R$, where Q_R is indefinite.

For simplicity, assume that $x_1, \dots, x_k$ are the separated variables, and Q_R depends only on the remaining variables. Accordingly, the quadratic form Q_1 is a linear combination of at most k perfect squares.

Consider the homogeneous linear system whose coefficient matrix is formed by stacking the auxiliary rows of the elimination algorithm in the order given by the elimination tableaux. It is clear that the last row contains the variable x_k.

Each higher row contains at least one more variable than the row below it. Thus, for arbitrary values of $x_{k+1}, \dots, x_n$, there exist values for $x_1, \dots, x_k$ such that the variables form a solution to the linear system, hence $Q_1(x_1, \dots, x_n) = 0$.

This shows that not only Q_R but also Q is indefinite.

Assume, now that Q is a linear combination of r perfect squares, where r is the rank of the matrix representing the original quadratic form.

Consider the case when there are both positive and negative pivot elements.

Denote by $Dx = b_+$ and $Dx = b_-$ the inhomogeneous systems where the rows of D are the auxiliary rows of the elimination algorithm, and the vector b_+ (respectively b_-) contains 1 or 0 depending on whether the pivot element corresponding to the row is positive or negative.

Similarly, the elements of b_- are 1 or zero, depending on whether the pivot element of the corresponding row is negative or not.

Here, $x = (x_1, \dots, x_n)$ are the variables of the quadratic form.

Since the row system of D is linearly independent system, therefore, both systems $Dx = b_+$ and $Dx = b_-$ have solutions, implying that Q is indeed indefinite.

Finally, consider the case when all pivot elements are positive (or all are negative).

It is clear that

$$Q(x_1, \dots, x_n) \geq 0 \quad (\text{or } \leq 0).$$

If $r = n$, then with the above notations, D is a regular matrix. Hence,

$$Q(x) = 0 \quad \text{if and only if} \quad Dx = 0, \quad \text{which holds only if} \quad x = 0.$$

This proves the statement about positive (or negative) definite quadratic forms.

If $r < n$, then the dimension of the nullspace is $n - r > 0$, so there exists a nonzero x such that $Q(x) = 0$. This proves the statement about positive (or negative) semi definite quadratic forms.