Elements of Linear Algebra

Vectors, Linear combination

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Outline

References

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Vectors

It is well known, that there is a one-to-one correspondence between the elements of the set of the real numbers and the points of a straight line. It is also known, that we can make a similar one-to-one correspondence between the points of the plane and the elements of set of the of ordered pairs of real numbers $\mathbb{R}^2$. Similarly, we can identify the points of the 3-dimensional space and the elements of ordered real number triples $\mathbb{R}^3$.

We generalize and introduce the n-dimensional space as the set $\mathbb{R}^n$ containing all the ordered real n-tuples, that is, n-element real sequences.

Definition

The elements of $\mathbb{R}^n$ will be written in the form $\mathbf{x} = (x_1, x_2, \dots, x_n)$ $x_i \in \mathbb{R}$, $i = 1, \dots, n$ and are said to be vectors. The real numbers $x_1, \dots, x_n$ are the *components* of $\mathbf{x}$.

Remark: The zero vector in $\mathbb{R}^n$ is obviously the vector $\mathbf{0}$, all of whose components are zero, and the opposite of a vector is obtained if each of its component is replaced with its opposite.

If $\mathbf{y} = (y_1, y_2, \dots, y_n)$ is another vector in $\mathbb{R}^n$ then we define the sum:

$$\mathbf{x} + \mathbf{y} = (x_1 + y_1, x_2 + y_2, \dots, x_n + y_n)$$

and if λ is a real number we also define the scalar multiplication:

$$\lambda\mathbf{x} = (\lambda x_1, \lambda x_2, \dots, \lambda x_n).$$

Thus $\mathbf{x} + \mathbf{y} \in \mathbb{R}^n$ and also $\lambda\mathbf{x} \in \mathbb{R}^n$.

Example

Let $\mathbf{a} = (2, 3, 4, 1)$ and $\mathbf{b} = (-1, 0, 1, 2)$ be two vectors in $\mathbb{R}^4$. Determine the vector $\mathbf{a}+2\mathbf{b}$.

$$\begin{aligned}\mathbf{a} + 2\mathbf{b} &= (2, 3, 4, 1) + (2 \cdot (-1), 2 \cdot 0, 2 \cdot 1, 2 \cdot 2) = \\ &= (2 - 2, 3 + 0, 4 + 2, 1 + 4) = (0, 3, 6, 5)\end{aligned}$$

Since the addition of vectors is led back to the addition of their coordinates, which are real numbers, the following properties hold true.

Theorem

Let $\mathbf{x}$, $\mathbf{y}$ *and* $\mathbf{z}$ *be arbitrary vectors in* $\mathbb{R}^n$, *then*

1. $\mathbf{x} + \mathbf{y} = \mathbf{y} + \mathbf{x}$, *(the addition is commutative,)*
2. $(\mathbf{x} + \mathbf{y}) + \mathbf{z} = \mathbf{x} + (\mathbf{y} + \mathbf{z})$ *(associative,)*
3. $\exists \mathbf{0} \in \mathbb{R}^n : \forall \mathbf{x} \in \mathbb{R}^n : \mathbf{0} + \mathbf{x} = \mathbf{x}$, *(there exists zero vector,)*
4. $\forall \mathbf{x} \in \mathbb{R}^n : \exists (-\mathbf{x}) \in \mathbb{R}^n : \mathbf{x} + (-\mathbf{x}) = \mathbf{0}$. *(each element has an opposite.)*

Furthermore, if $\lambda, \mu \in \mathbb{R}$, *then*

1. $\lambda(\mathbf{x} + \mathbf{y}) = \lambda\mathbf{x} + \lambda\mathbf{y}$,
2. $(\lambda + \mu)\mathbf{x} = \lambda\mathbf{x} + \mu\mathbf{x}$,
3. $(\lambda \cdot \mu)\mathbf{x} = \lambda(\mu\mathbf{x})$,
4. $1 \cdot \mathbf{x} = \mathbf{x}$.

Definition

Let an addition be defined on a set V satisfying the first four properties above and for any real number α let a map $\alpha : V \longrightarrow V$ – called scalar multiplication – be defined satisfying the second four properties above. Then V is called a real vector space.

- The directed line segments in the plane with fixed initial point form a real vector space, with the addition according to the parallelogram rule and scalar multiplication defined as follows: multiply the length of the vector by the absolute value of the number and change its direction to the opposite if the number is negative.
- $\mathbb{R}^n$ is a real vector space with the addition and scalar multiplication defined above.
- The set of all functions of the form $f : X \longrightarrow \mathbb{R}$ is a real vector space, where the addition of two functions f and g is defined by $(f+g)(x) := f(x) + g(x)\ \forall x \in X$, and for any $\lambda \in \mathbb{R}$, $(\lambda f)(x) := \lambda \cdot f(x)$ defines the scalar multiplication.

Definition

A non-empty subset M of a vector space V is called to be a subspace of V if

- $x + y \in M$ for all $x, y \in M$;
- $\alpha x \in M$ for all $x \in M$ and $\alpha \in \mathbb{R}$.

Linear combination

Linearly independent vector set

Let $\mathbf{x}_1, \ldots, \mathbf{x}_k$ be arbitrary vectors in $\mathbb{R}^n$ and $\lambda_1, \ldots, \lambda_k$ be real numbers. Then the vector

$$\lambda_1 \mathbf{x}_1 + \cdots + \lambda_k \mathbf{x}_k$$

is called a *linear combination* of the vectors $\mathbf{x}_1, \ldots, \mathbf{x}_k$.

Definition

The set of vectors $\mathbf{x}_1, \ldots, \mathbf{x}_k$ is said to be linearly independent if $\lambda_1 \mathbf{x}_1 + \cdots + \lambda_k \mathbf{x}_k = \mathbf{0} \Longrightarrow \lambda_1 = \ldots = \lambda_k = 0$.

The only way to obtain the zero vector as a linear combination of a linearly independent set of vectors is the trivial one, that is, if we choose all the scalar coefficients equal zero.

Definition

The system of vectors $\mathbf{v}_1, \ldots, \mathbf{v}_k$ is called linearly dependent if it is not independent.

Example

Decide if the system of vectors $\{\mathbf{a} = (1, 2, 3), \mathbf{b} = (-1, 0, 1), \mathbf{c} = (-2, -1, 0)\}$ is linearly independent or dependent.

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We have to find out if the equation $\alpha\mathbf{a} + \beta\mathbf{b} + \gamma\mathbf{c} = \mathbf{0}$ implies that each of the coefficients α, β, γ are zero. Comparing the coordinates of both sides of the equation we obtain the system of linear equations:

$$\begin{aligned} \alpha - \beta - 2\gamma &= 0 \\ 2\alpha - \gamma &= 0 \\ 3\alpha + \beta &= 0 \end{aligned}$$

From the second equation it follows that $\gamma = 2\alpha$ and from the third one we have that $\beta = -3\alpha$ and the first equation is a consequence of these. Thus for any real number τ the triple $\alpha = \tau, \beta = -3\tau, \gamma = 2\tau$ satisfies the system of equations, therefore the vector system $\{\mathbf{a} = (1, 2, 3), \mathbf{b} = (-1, 0, 1), \mathbf{c} = (-2, 0, 2)\}$ is linearly dependent.

Problems

Decide if the systems of the following vectors form a linear independent or a dependent set

- $$\{\mathbf{b}_1 = (1,0,0), \mathbf{b}_2 = (0,1,0), \mathbf{b}_3 = (0,0,1)\}$$
- $$\{\mathbf{b}_1 = (1,1,0), \mathbf{b}_2 = (0,1,1), \mathbf{b}_3 = (1,0,1)\}$$
- $$\{\mathbf{b}_1 = (1,1,1), \mathbf{b}_2 = (0,1,1), \mathbf{b}_3 = (1,1,0)\}$$
- $$\{\mathbf{a}_1 = (1,2,3), \mathbf{a}_2 = (3,4,5), \mathbf{a}_3 = (5,6,7)\}$$
- $$\{\mathbf{a}_1 = (2,1,3), \mathbf{a}_2 = (4,1,5), \mathbf{a}_3 = (6,1,7)\}$$

Important properties of system of vectors

The subsequent statements are easily verified:

- A set of vectors containing the zero vector is linearly dependent.
- A system of vectors having two equal vectors is linearly dependent.
- Any one element set containing only a non-zero vector is linearly independent.

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- Any non-empty subset of a linearly independent set of vectors is linearly independent.
- A system of vectors is linearly dependent if and only if one of its vector is the linear combination of the other vectors in the system.
- If $\mathbf{v} = \lambda_1 \mathbf{x}_1 + \cdots + \lambda_k \mathbf{x}_k$ and $\{\mathbf{x}_1, \ldots, \mathbf{x}_k\}$ is linearly independent, then the scalars $\lambda_1, \ldots, \lambda_k$ are unique.

Linear span

Definition

Let S be a subset of a vector space. The span of S is defined as the set of all of linear combinations of the vectors of S.

$$\operatorname{span}(S) = \left\{ \sum_{i=1}^{k} \lambda_i x_i : k \in \mathbb{N}, \lambda_i \in \mathbb{R}, x_i \in S \right\}.$$

Assume that M is a subspace of the vector space and $S \subseteq M$ is a subset. If $\operatorname{span}(S) = M$, then S is a spanning set of M.

The commonly used terminologies are the following: S spans M; S generates M; S is a spanning set of M; S is a generating set of M. Sometimes, the set $\operatorname{span}(S)$ is denoted by $\operatorname{lin}(S)$ and the spans of S is called the linear hull of S.

Observe that $\operatorname{span}(S)$ is always a subspace.

Example

Consider the vector space $\mathbb{R}^3$. Decide if the set S is a spanning set of the vector space or not. $S = \{a, b, c\}$.

1. $a = (-1, 0, 0); b = (0, 1, 0)\ ; c = (0, 0, -1)$.
2. $a = (1, 0, 0); b = (0, 1, 0)\ ; c = (1, 1, 0)$.
3. $a = (1, 0, 0); b = (0, 1, 0)\ ; c = (1, 1, 1)$.

Example

Decide if the vector $\mathbf{x} = (1, 0, 3)$ is in the linear hull of vector system $\{\mathbf{a}_1 = (1, 1, -1), \mathbf{a}_2 = (2, 1, 2), \mathbf{a}_3 = (0, 1, -4)\}$.

Basis

Definition

The rank of a system of vectors $\{\mathbf{x}_1, \ldots, \mathbf{x}_k\}$ is the integer r if it has an r element linearly independent subset, but any at least $r+1$ element subsystem is linearly dependent.

Basis

Minimal spanning set and maximal independent set

Minimal spanning set

Consider a vector space including a finite spanning set $\{x_1, \dots, x_s\}$. It is clear that if one of the vectors–for example x_1–is a linear combination of the rest of the vectors $\{x_2, \dots, x_s\}$, then after dropping this vector from the system, the set $\{x_2, \dots, x_s\}$ remains a spanning set.

Let us call a spanning set H minimal iff there is no proper spannig subset of the set of H.

Thus we proved that every minimal spanning set is an independent set.

Assume that we have an $\{x_1, \dots, x_s\}$ independent set for which $y \notin \operatorname{span}\{x_1, \dots, x_s\}$. One can see that only the trivial linear combinations of the system

$$\{x_1, \dots, x_s, y\}$$

provides 0, thus it is a linearly independent set.

Let us call a linear independent set H maximal iff there is no proper independent superset of H.

Thus we proved that every maximal linear independent set is a spanning set.

The existence of a minimal spanning set

Definition

A vector space having finite spanning set is called finite dimensional or *finitely generated* vector space.

Assume that the vector space has a finite spanning set H_1. If it is minimal then, the set H_1 is linearly independent. Otherwise we are able to find a proper subset $H_2 \subseteq H_1$ which remains spanning set.

If the system H_2 is minimal then, the set H_2 is linearly independent. Otherwise we are able to find a proper subset $H_3 \subseteq H_2$ which remains spanning set.

Using this process–after finite steps–we obtain a minimal spanning set: It is a subset of the original system $\{x_1, \dots, x_n\}$ which is spanning set and linearly independent together.

Thus the following proposition is proven.

Proposition

Every finitely generated vector space does have a linearly independent spanning set.

Definition

A set of vectors $\{v_1, \dots, v_n\}$ is called a basis of the vector space V, if

- it is linearly independent and
- it is a spanning set of V.

We have just proved that every finitely generated vector space has a basis.

Verify that the set of vectors $\{\mathbf{e}_1, \dots, \mathbf{e}_n\}$, where $\mathbf{e}_i$ is the vector with all 0s except for a 1 in the ith component, is a basis of the vector space $\mathbb{R}^n$.

1) The linear combination $\lambda_1 \mathbf{e}_1 + \dots + \lambda_n \mathbf{e}_n = (\lambda_1, \dots, \lambda_n) = \mathbf{0}$ implies that $\lambda_1 = \dots = \lambda_n = 0$, that is, $\{\mathbf{e}_1, \dots, \mathbf{e}_n\}$ is linearly independent.
2) If $\mathbf{x} = (x_1, \dots, x_n)$ is an arbitrary vector in $\mathbb{R}^n$, then obviously $\mathbf{x} = x_1 \mathbf{e}_1 + \dots + x_n \mathbf{e}_n$, therefore $\operatorname{lin}(\mathbf{e}_1, \dots, \mathbf{e}_n) = \mathbb{R}^n$.

It is important to emphasize that there are many bases in the vector spaces.

But if we have two bases in a vector space then the number of the vectors of the two systems are the same. We are going to prove this fact.

First, we realize that the number of the vectors of an independent system can not be greater than the number of the vectors of a spanning set.

Lemma (Steinitz)

Assume that a vector space has a finite spanning set having exactly n vectors. Then every system of vectors including more than n vector is linearly dependent.

Let $\{y_1, \dots, y_n\}$ be a spanning set and assume that the system $\{x_1, \dots, x_m\}$ has more vector than the spanning set has. Thus $m > n$. By the definition of the spanning set: for all $1 \leq k \leq m$ there exist coefficients $\alpha_{j,k}$, $1 \leq j \leq n$, such that

$$x_k = \sum_{j=1}^{n} \alpha_{j,k} y_j .$$

Write a linear combination of the system $\{x_1, \dots, x_m\}$ using the multipliers $\xi_1, \dots, \xi_m$ specified later.

$$\sum_{k=1}^{m} \xi_k x_k = \sum_{k=1}^{m} \sum_{j=1}^{n} \xi_k \alpha_{j,k} y_j = \sum_{j=1}^{n} \left(\sum_{k=1}^{m} \alpha_{j,k} \xi_k \right) y_j \tag{1}$$

Consider the homogeneous system of linear equations belonging to the coefficients $(\alpha_{j,k})$. Here $j = 1, \dots, n$ and $k = 1, \dots, m$. The assumption $m > n$ indicates the number of unknowns is greater than the number of equations of this linear system. Remember: this fact implicates the existence of a nontrivial solution of this system. Thus there exist numbers $\xi_1, \dots, \xi_m$, for which not all of them is zero and at the same time for every $j = 1, \dots, n$ the equations

$$\sum_{k=1}^{m} \alpha_{j,k}\xi_k = 0$$

hold. Thus if we use the above found unknowns (ξ_k) for the multipliers of (1), then the right hand side of (1) becomes zero. Consider now the left hand side of (1). It is a non-trivial linear combination of the system $\{x_1, \dots, x_m\}$.

It was to be proved.

Definition

The dimension of a vector space is m if it has an m element basis.

The above concept of dimension is well defined.

Assume that systems $\{x_1, \dots, x_m\}$ and $\{y_1, \dots, y_n\}$ form bases.

On the one hand, $m \leq n$ because $\{x_1, \dots, x_m\}$ is independent and $\{y_1, \dots, y_n\}$ is a spanning set.

On the other hand, $n \leq m$ because $\{y_1, \dots, y_n\}$ is independent and $\{x_1, \dots, x_m\}$ is a spanning set.

Thus, we proved, that any two bases have the same number of terms in a vector space (in which there exist finitely many vectors forming a spanning set).

The other consequence of the Steinitz lemma is as follows.

Proposition

Let V be an n dimensional vector space.

1. *If the system $\{x_1, \cdots, x_n\}$ is an exactly n element independent set then it is a maximal independent set, thus it is a basis.*
2. *If the system $\{x_1, \cdots, x_n\}$ is an exactly n element spanning set then it is a minimal spanning set, thus it is a basis.*

Definition

Let $V = \{v_1, \ldots, v_k\}$ be a basis of the vector space V and $x \in V$. Then the scalars in the linear combination $x = \xi_1 v_1 + \cdots + \xi_k v_k$ are called the coordinates of the vector x. In particular, ξ_j is the coordinate of x with respect to the basis vector v_j. The column vector

$$\mathbf{x}_V = \begin{bmatrix} \xi_1 \\ \vdots \\ \xi_k \end{bmatrix}$$

is said to be the coordinate vector of x with respect to the basis V.

The coordinate vector depends on the basis. The same vector has different coordinate vectors with respect to different bases.

Basis transformation

Elementary basis transformation

The problem is to find out how the coordinate vector is changing if the basis is changed. Let $V = \{v_1, \dots, v_i, \dots, v_k\}$ be a basis of the vector space V and $w \in V$ a nonzero vector, $w = \alpha_1 v_1 + \dots + \alpha_i v_i + \dots + \alpha_k v_k$ such that $\alpha_i \neq 0$. Then we have $v_i = -\frac{\alpha_1}{\alpha_i} v_1 - \dots + \frac{1}{\alpha_i} w - \dots - \frac{\alpha_k}{\alpha_i} v_k$, Therefore if the vector x as a linear combination of basis vectors in V is

$$x = \xi_1 v_1 + \dots + \xi_i v_i + \dots + \xi_k v_k,$$

then it can be written as the linear combination of the system $V' = \{v_1, \dots, w, \dots, v_k\}$ as follows

$$\begin{aligned} x &= \xi_1 v_1 + \dots + \xi_i v_i + \dots + \xi_k v_k = \\ &= \xi_1 v_1 + \dots + \xi_i \Big(-\frac{\alpha_1}{\alpha_i} v_1 - \dots + \frac{1}{\alpha_i} w - \dots - \frac{\alpha_k}{\alpha_i} v_k\Big) + \dots + \xi_k v_k = \\ &= \Big(\xi_1 - \alpha_1 \frac{\xi_i}{\alpha_i}\Big) v_1 + \dots + \frac{\xi_i}{\alpha_i} w + \dots + \Big(\xi_k - \alpha_k \frac{\xi_i}{\alpha_i}\Big) v_k \,. \end{aligned}$$

We started from a basis $\{v_1, \ldots, v_k\}$ and a vector w for which the i-th coordinate of w (with respect to the basis above) is non-zero. We proved that any vector $x \in \operatorname{span}\{v_1, \cdots, v_k\}$ can be expressed as a linear combination of the system $\{v_1, \ldots, w, \ldots, v_k\}$. It means that the new system $\{v_1, \ldots, w, \ldots, v_k\}$ is also a spanning set.

Realize that this new system is a k element spanning set of a k dimensional space, thus it becomes a new basis.

Proposition

Let the system $\{v_1, \ldots, v_k\}$ be a basis and assume that the i-th coordinate of the vector w–with respect to the basis above–is non-zero. Then the changed system

$$\{v_1, \ldots, w, \ldots v_k\}$$

is also a basis.

The change can be better followed in the subsequent tables:

$$
\begin{array}{c|c|c}
 & w & x \\
\hline
v_1 & \alpha_1 & \xi_1 \\
\vdots & \vdots & \vdots \\
v_i & \boxed{\alpha_i} & \xi_i \\
\vdots & \vdots & \vdots \\
v_k & \alpha_k & \xi_k
\end{array}
\quad \longrightarrow \quad
\begin{array}{c|c}
 & x \\
\hline
v_1 & \xi_1 - \alpha_1 \frac{\xi_i}{\alpha_i} \\
\vdots & \vdots \\
w & \frac{\xi_i}{\alpha_i} \\
\vdots & \vdots \\
v_k & \xi_k - \alpha_k \frac{\xi_i}{\alpha_i}
\end{array}
$$

In the left heading of the tables the base vectors and in the top heading the vectors are listed and below in the jth row of the tables the coordinate of the corresponding vector with respect to the jth base vector. Replacing the ith basis vector by the vector w is denoted by the frame around the ith coordinate of w. This coordinate is called pivot element. Notice that the pivot element is nonzero, this makes possible to exchange the v_i basis vector with the vector w.

In the second table the coordinates of the x vector is shown with respect to the new basis $V' = \{v_1, \dots, w, \dots v_k\}$. The coordinate of x with respect to the new basis vector w is the quotient of its original coordinate and the pivot element, while any other coordinate is obtained if from the original coordinate we subtract the product of this quotient and the corresponding coordinate of w. Thus the coordinate vectors of the x vector are

$$\mathbf{x}_V = \begin{bmatrix} \xi_1 \\ \vdots \\ \xi_i \\ \vdots \\ \xi_k \end{bmatrix} \quad \text{and} \quad \mathbf{x}_{V'} = \begin{bmatrix} \xi_1 - \alpha_1 \frac{\xi_i}{\alpha_i} \\ \vdots \\ \frac{\xi_i}{\alpha_i} \\ \vdots \\ \xi_k - \alpha_k \frac{\xi_i}{\alpha_i} \end{bmatrix}$$

with respect to the basis V and V' respectively.

Some applications

Using elementary basis transformation actually any problem arising in linear algebra can be solved, that is, the linear algebra problems are easily solvable in appropriate coordinate system. To illustrate this statement we provide some examples.

Example

Decide if the system of vectors

$$\{\mathbf{y}_1 = (1, -1, 2, 0), \mathbf{y}_2 = (-1, 2, -2, 3), \mathbf{y}_3 = (3, -4, 6, -3)\}$$

in $\mathbb{R}^4$ is linearly independent or dependent!

The set of vectors $E = \{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3, \mathbf{e}_4\}$ is a basis in $\mathbb{R}^4$. The coordinate vectors of $\mathbf{y}_1 = (1, -1, 2, 0), \mathbf{y}_2 = (-1, 2, -2, 3), \mathbf{y}_3 = (3, -4, 6, -3)$ with respect to the basis E are

$$\mathbf{y}_{1E} = \begin{bmatrix} 1 \\ -1 \\ 2 \\ 0 \end{bmatrix}, \quad \mathbf{y}_{2E} = \begin{bmatrix} -1 \\ 2 \\ -2 \\ 3 \end{bmatrix} \quad \text{and} \quad \mathbf{y}_{3E} = \begin{bmatrix} 3 \\ -4 \\ 6 \\ -3 \end{bmatrix}$$

Using a sequence of elementary basis transformations we change the original E basis taking as many $\mathbf{y}_i$ vectors as we can into the new basis.

The calculation is shown by the subsequent tables:

$$
\begin{array}{c|rrr}
 & \mathbf{y}_1 & \mathbf{y}_2 & \mathbf{y}_3 \\
\hline
\mathbf{e}_1 & \boxed{1} & -1 & 3 \\
\mathbf{e}_2 & -1 & 2 & -4 \\
\mathbf{e}_3 & 2 & -2 & 6 \\
\mathbf{e}_4 & 0 & 3 & -3 \\
\hline
\end{array}
\longrightarrow
\begin{array}{c|rr}
 & \mathbf{y}_2 & \mathbf{y}_3 \\
\hline
\mathbf{y}_1 & -1 & 3 \\
\mathbf{e}_2 & \boxed{1} & -1 \\
\mathbf{e}_3 & 0 & 0 \\
\mathbf{e}_4 & 3 & -3 \\
\hline
\end{array}
\longrightarrow
\begin{array}{c|r}
 & \mathbf{y}_3 \\
\hline
\mathbf{y}_1 & 2 \\
\mathbf{y}_2 & -1 \\
\mathbf{e}_3 & 0 \\
\mathbf{e}_4 & 0 \\
\hline
\end{array}
$$

The last table shows, that the new basis is $\{\mathbf{y}_1, \mathbf{y}_2, \mathbf{e}_3, \mathbf{e}_4\}$ and that $\mathbf{y}_3$ can be expressed as a linear combination of $\mathbf{y}_1$ and $\mathbf{y}_2$, namely $\mathbf{y}_3 = 2\mathbf{y}_1 - 1\mathbf{y}_2$. Therefore the system $\{\mathbf{y}_1, \mathbf{y}_2, \mathbf{y}_3\}$ is linearly dependent.

Example

Decide if the vector $\mathbf{x} = (1, 0, 3)$ is in the linear hull of vector system $\{\mathbf{a}_1 = (1, 1, -1), \mathbf{a}_2 = (2, 1, 2), \mathbf{a}_3 = (0, 1, -4)\}$.

Example

Decide if the vector $\mathbf{x} = (1, 0, 3)$ is in the linear hull of vector system $\{\mathbf{a}_1 = (1, 1, -1), \mathbf{a}_2 = (2, 1, 2), \mathbf{a}_3 = (0, 1, -4)\}$.

Again begin with the basis of unit vectors $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ of $\mathbb{R}^3$ because the coordinates of the vectors in $\mathbb{R}^3$ with respect to this basis is the same as their components. Now the calculations are:

$$
\begin{array}{c|rrr|r}
 & \mathbf{a}_1 & \mathbf{a}_2 & \mathbf{a}_3 & \mathbf{x} \\
\hline
\mathbf{e}_1 & \boxed{1} & 2 & 0 & 1 \\
\mathbf{e}_2 & 1 & 1 & 1 & 0 \\
\mathbf{e}_3 & -1 & 2 & -4 & 3
\end{array}
\longrightarrow
\begin{array}{c|rr|r}
 & \mathbf{a}_2 & \mathbf{a}_3 & \mathbf{x} \\
\hline
\mathbf{a}_1 & 2 & 0 & 1 \\
\mathbf{e}_2 & \boxed{-1} & 1 & -1 \\
\mathbf{e}_3 & 4 & -4 & 4
\end{array}
\longrightarrow
\begin{array}{c|r|r}
 & \mathbf{a}_3 & \mathbf{x} \\
\hline
\mathbf{a}_1 & 2 & -1 \\
\mathbf{a}_2 & -1 & 1 \\
\mathbf{e}_3 & 0 & 0
\end{array}
$$

From the last table it can be concluded that $\mathbf{x} = -1\mathbf{a}_1 + 1\mathbf{a}_2$, that is $\mathbf{x} \in \mathrm{lin}(\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3)$ Notice that the subspace $\mathrm{lin}(\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3)$ is already spanned by the vectors $\mathbf{a}_1$ and $\mathbf{a}_2$ as well, since $\mathbf{a}_3$ is linear combination of them, that is, $\{\mathbf{a}_1, \mathbf{a}_2\}$ is a basis of $\mathrm{lin}(\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3)$ and therefore it is a 2-dimensional subspace of $\mathbb{R}^3$.