Diagonal matrices

The eignevalues of a matrix

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Outline

Important

Taking part on the written exam at 4.00 pm. of December 16, 2025 is compulsory!

Notion of Eigenvalue and Eigenvector

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Definition

Eigenvalue and Eigenvector

Definition

The scalar $\lambda \in \mathbb{R}$ is called an eigenvalue of the matrix $\mathbf{A} \in \mathbb{R}^{n \times n}$, if there exists a nonzero vector $\mathbf{s} \in \mathbb{R}^n$ for which

$$\mathbf{A}\mathbf{s} = \lambda \mathbf{s}$$

holds. The vector $\mathbf{s}$ is said to be the eigenvector corresponding to the eigenvalue λ of matrix $\mathbf{A}$.

The set of eigenvectors form a subspace

If both $\mathbf{s}$ and $\mathbf{t}$ are eigenvectors corresponding to the eigenvalue λ then their any nontrivial linear combination $\alpha\mathbf{s} + \beta\mathbf{t}$ is also an eigenvector with the same eigenvalue λ:

$$\mathbf{A}\left(\alpha\mathbf{s} + \beta\mathbf{t}\right) = \alpha\mathbf{A}\mathbf{s} + \beta\mathbf{A}\mathbf{t} = \alpha\lambda\mathbf{s} + \beta\lambda\mathbf{t} = \lambda\left(\alpha\mathbf{s} + \beta\mathbf{t}\right).$$

Dependent system

The eigenvectors corresponding to the eigenvalue λ are the nonzero solutions of the homogeneous system of linear equations

$$(\mathbf{A} - \lambda\mathbf{E})\,\mathbf{x} = \mathbf{0}.$$

A homogeneous system of linear equations with a square coefficient matrix has nonzero solution if and only if its columns form a dependent system.

It follows that λ is an eigenvalue of $\mathbf{A}$ if and only if the columns of matrix

$$\mathbf{A} - \lambda\mathbf{E}$$

is a linearly dependent system.

Determine the eigenvalues and the eigenvectors

Example

Find the eigenvalues and the corresponding eigenvectors of the following matrix:

$$\mathbf{A} = \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 1 \\ 2 & 0 & -2 \end{bmatrix}$$

We find the values of the parameter t for which the matrix $\mathbf{A} - t\mathbf{E}$ is singular and with the obtained values we find the nonzero solutions of the homogeneous system of linear equations $(\mathbf{A} - t\mathbf{E})\mathbf{x} = \mathbf{0}$.

Determine the eigenvalues and the eigenvectors

Basis transformation

Find the eigenvalues and the eigenvector with basis transformation

Using elementary basis transformation we solve the homogeneous system of linear equations $(\mathbf{A} - t\mathbf{E})\mathbf{x} = \mathbf{0}$ for each value of the parameter t for which the coefficient matrix is singular:

$$
\begin{array}{c|ccc}
 & x_1 & x_2 & x_3 \\
\hline
 & 1-t & 0 & \boxed{-1} \\
 & 0 & 1-t & 1 \\
 & 2 & 0 & -2-t
\end{array}
\quad \rightarrow \quad
\begin{array}{c|cc}
 & x_1 & x_2 \\
\hline
x_3 & t-1 & 0 \\
 & 1-t & 1-t \\
 & t^2+t & 0
\end{array}
$$

If $t = 1$ then we can continue as follows:

$$\begin{array}{c|cc} & x_1 & x_2 \\ \hline x_3 & 0 & 0 \\ & 0 & 0 \\ & \boxed{2} & 0 \end{array} \quad \to \quad \begin{array}{c|c} & x_2 \\ \hline x_3 & 0 \\ & 0 \\ x_1 & 0 \end{array}$$

From this table we obtain that at $t = 1$ the coefficient matrix is singular and the solution of the homogeneous system of linear equations $(\mathbf{A} - \mathbf{E})\mathbf{x} = \mathbf{0}$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \tau \quad \tau \in \mathbb{R},$$

that is 1 is an eigenvalue and the above vectors are the corresponding eigenvectors for all $\tau \neq 0$.

If $t \neq 1$ then

$$\begin{array}{c|cc} & x_1 & x_2 \\ \hline x_3 & t-1 & 0 \\ & 1-t & \boxed{1-t} \\ & t^2+t & 0 \end{array} \rightarrow \begin{array}{c|c} & x_1 \\ \hline x_3 & t-1 \\ x_2 & 1 \\ & t^2+t \end{array}$$

and we obtain that the coefficient matrix is also singular if $t = 0$ or $t = -1$.

One can see, from the table, that the eigenvectors corresponding to the eigenvalue 0 are

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} \tau \quad \tau \in \mathbb{R}\, \tau \neq 0$$

and the eigenvectors corresponding to the eigenvalue -1 are

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix} \tau \quad \tau \in \mathbb{R}\, \tau \neq 0.$$

There are real matrices which has no real eigenvalue, for instance the matrix

Matrix having no real eigenvalue

$$\mathbf{A} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$$

has no real eigenvalue, because the columns of matrix $A - tE$ always form an independent system.

Characteristic polynomial

Definition

If A is a square matrix then the polynomial

$$k(t) = \det(A - tI)$$

is called the *characteristic polynomial* of the matrix A.

Theorem

Let A be a square matrix and consider a real number λ. The number λ is an eigenvalue of matrix A if and only if $k(\lambda) = 0$.

Verify that similar matrices have the same characteristic polynomial.

Now there are two ways to find the eigenvalues and the corresponding eigenvectors:

First We find the values of the parameter t for which the matrix $\mathbf{A} - t\mathbf{E}$ is singular and with the obtained values we find the nonzero solutions of the homogeneous system of linear equations $(\mathbf{A} - t\mathbf{E})\mathbf{x} = \mathbf{0}$, or

Two alternatives looking for eigenvalues and eigenvectors

Now there are two ways to find the eigenvalues and the corresponding eigenvectors:

First We find the values of the parameter t for which the matrix $\mathbf{A} - t\mathbf{E}$ is singular and with the obtained values we find the nonzero solutions of the homogeneous system of linear equations $(\mathbf{A} - t\mathbf{E})\mathbf{x} = \mathbf{0}$, or

Second We determine the roots of the characteristic polynomial of $\mathbf{A}$, those are the eigenvalues of $\mathbf{A}$ and then solving the homogeneous system of linear equations $(\mathbf{A} - \lambda\mathbf{E})\,\mathbf{x} = \mathbf{0}$ for each root we obtain the corresponding eigenvectors.

2nd method to find the eigenvalues of a matrix

The characteristic equation:

$$|\mathbf{A} - t\mathbf{E}| = \begin{vmatrix} 1-t & 0 & -1 \\ 0 & 1-t & 1 \\ 2 & 0 & -2-t \end{vmatrix} = -t^3 + t = 0$$

Its solutions: $t_1 = 0$, $t_2 = -1$ and $t_3 = 1$ are the eigenvalues of the matrix $\mathbf{A}$.

The eigenvectors belonging to the eigenvalue 0 are the nonzero solutions of the homogeneous system of linear equations $(A - 0 \cdot \mathbf{E})\,\mathbf{x} = \mathbf{0}$. Thus the system to be solved:

$$\begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 1 \\ 2 & 0 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$

The solutions: $x_1 = \tau, x_2 = -\tau, x_3 = \tau \quad (\tau \in \mathbb{R})$. Therefore the eigenvectors belonging to the eigenvalue $t_1 = 0$ are

$$\begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} \tau, \quad (\tau \in \mathbb{R} \setminus \{0\})$$

The eigenvectors belonging to the eigenvalue -1 are the nonzero solutions of the homogeneous system of linear equations $(A + 1 \cdot \mathbf{E})\,\mathbf{x} = \mathbf{0}$. Thus the system to be solved:

$$\begin{bmatrix} 2 & 0 & -1 \\ 0 & 2 & 1 \\ 2 & 0 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$

The solutions: $x_1 = \tau, x_2 = -\tau, x_3 = 2\tau \quad (\tau \in \mathbb{R})$. Therefore the eigenvectors belonging to the eigenvalue $t_2 = -1$:

$$\begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix} \tau, \quad (\tau \in \mathbb{R} \setminus \{0\}).$$

The eigenvectors belonging to the eigenvalue 1 are the nonzero solutions of the homogeneous system of linear equations $(A - 1 \cdot \mathbf{E})\,\mathbf{x} = \mathbf{0}$, that is,

$$\begin{bmatrix} 0 & 0 & -1 \\ 0 & 0 & 1 \\ 2 & 0 & -3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$

Solving the system the eigenvectors :

$$\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \tau, \quad (\tau \in \mathbb{R} \setminus \{0\}).$$

There are real matrices which have no real eigenvalue, for instance the matrix

Matrix having no real eigenvalue

$$\mathbf{A} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$$

has the characteristic polynomial $t^2 + 1$, therefore it has no real root.

Diagonal matrix

Diagonal matrix

Definition

A diagonal matrix is a square matrix, whose entries outside of the main diagonal are zeros. They will be denoted by

$$\mathbf{D} = \operatorname{diag}\langle d_1, \dots, d_n\rangle,$$

$d_1, \dots, d_n$ are the entries in the main diagonal.

The eigenvalues of a diagonal matrix are just the entries in the main diagonal.

Definition

A square matrix $\mathbf{A}$ of order n is said to be diagonalizable if there exists a regular matrix $\mathbf{P}$ and a diagonal matrix $\mathbf{D}$ such that

$$\mathbf{P}^{-1}\mathbf{A}\mathbf{P} = \mathbf{D} = \operatorname{diag}\langle d_1, \dots, d_n \rangle .$$

For example the matrix

$$\mathbf{A} = \begin{pmatrix} 3 & 1 \\ 2 & 4 \end{pmatrix}$$

is diagonalizable, because

$$\mathbf{P}^{-1}\mathbf{A}\mathbf{P} = -\frac{1}{3}\begin{bmatrix} 2 & -1 \\ -1 & -1 \end{bmatrix}\begin{bmatrix} 3 & 1 \\ 2 & 4 \end{bmatrix}\begin{bmatrix} -1 & 1 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} 2 & 0 \\ 0 & 5 \end{bmatrix}$$

Diagonalizable matrix

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It is fascinating. :) How can we find the matrix $\mathbf{P}$, above?

A necessary and sufficient condition

Theorem

A square matrix $\mathbf{A}$ of order n is diagonalizable if and only if it has n linearly independent eigenvectors $\mathbf{x}_1, \dots, \mathbf{x}_n$. In this case

$$\mathbf{P}^{-1}\mathbf{A}\mathbf{P} = \begin{bmatrix} \lambda_1 & 0 & \dots & 0 \\ 0 & \lambda_2 & \dots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \dots & \lambda_n \end{bmatrix},$$

where the columns of $\mathbf{P}$ are $\mathbf{x}_1, \dots, \mathbf{x}_n$, and $\lambda_1, \dots, \lambda_n$ scalars are the corresponding eigenvalues.

Proof

If $\mathbf{x}_1, \dots, \mathbf{x}_n$ are linearly independent eigenvectors of $\mathbf{A}$ and $\lambda_1, \dots, \lambda_n$ are the corresponding eigenvalues, that is,

$$\mathbf{A}\mathbf{x}_1 = \lambda_1 \mathbf{x}_1, \quad \mathbf{A}\mathbf{x}_2 = \lambda_2 \mathbf{x}_2, \quad \dots, \quad \mathbf{A}\mathbf{x}_n = \lambda_n \mathbf{x}_n.$$

then let $\mathbf{P} = [\mathbf{x}_1, \dots, \mathbf{x}_n]$. Thus

$$\begin{aligned}\mathbf{A}\mathbf{P} = \mathbf{A}[\mathbf{x}_1, \dots, \mathbf{x}_n] = [\mathbf{A}\mathbf{x}_1, \dots, \mathbf{A}\mathbf{x}_n] = [\lambda_1 \mathbf{x}_1, \dots, \lambda_n \mathbf{x}_n] = \\ [\mathbf{x}_1, \dots, \mathbf{x}_n] \operatorname{diag}\langle \lambda_1, \dots, \lambda_n \rangle = \mathbf{P} \operatorname{diag}\langle \lambda_1, \dots, \lambda_n \rangle,\end{aligned}$$

therefore multiplying both sides by the inverse of $\mathbf{P}$, we obtain the identity required:

$$\mathbf{P}^{-1}\mathbf{A}\mathbf{P} = \operatorname{diag}\langle \lambda_1, \dots, \lambda_n \rangle.$$

Conversely, if

$$\mathbf{P}^{-1}\mathbf{A}\mathbf{P} = \operatorname{diag}\langle\lambda_1, \dots, \lambda_n\rangle\,,$$

then

$$\mathbf{A}\mathbf{P} = \mathbf{P}\operatorname{diag}\langle\lambda_1, \dots, \lambda_n\rangle\,,$$

that is, for each column $\mathbf{x}_i$ $(i = 1, \dots, n)$ of $\mathbf{P}$ the equation $\mathbf{A}\mathbf{x}_i = \lambda_i\mathbf{x}_i$ holds, verifying that λ_i is an eigenvalue of $\mathbf{A}$ and a corresponding eigenvector is $\mathbf{x}_i$.

Example

Decide if the matrix $\mathbf{A} = \begin{pmatrix} 3 & 1 \\ 2 & 4 \end{pmatrix}$ is diagonalizable! If it is diagonalizable then give an invertible matrix $\mathbf{P}$ for which $\mathbf{P}^{-1}\mathbf{A}\mathbf{P}$ is a diagonal matrix.

Example

Decide if the matrix $\mathbf{A} = \begin{pmatrix} 3 & 1 \\ 2 & 4 \end{pmatrix}$ is diagonalizable! If it is diagonalizable then give an invertible matrix $\mathbf{P}$ for which $\mathbf{P}^{-1}\mathbf{A}\mathbf{P}$ is a diagonal matrix.

The eigenvalues of $\mathbf{A}$ are $\lambda_1 = 2$ and $\lambda_2 = 5$, furthermore the corresponding linearly independent eigenvectors:

$$\mathbf{v}_1 = \begin{bmatrix} -1 \\ 1 \end{bmatrix} \quad \text{and} \quad \mathbf{v}_2 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

Example

Decide if the matrix $\mathbf{A} = \begin{pmatrix} 3 & 1 \\ 2 & 4 \end{pmatrix}$ is diagonalizable! If it is diagonalizable then give an invertible matrix $\mathbf{P}$ for which $\mathbf{P}^{-1}\mathbf{A}\mathbf{P}$ is a diagonal matrix.

The eigenvalues of $\mathbf{A}$ are $\lambda_1 = 2$ and $\lambda_2 = 5$, furthermore the corresponding linearly independent eigenvectors:

$$\mathbf{v}_1 = \begin{bmatrix} -1 \\ 1 \end{bmatrix} \quad \text{and} \quad \mathbf{v}_2 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

Hence by the previous theorem $\mathbf{A}$ is a diagonalizable matrix.

If $\mathbf{P} = \begin{bmatrix} -1 & 1 \\ 1 & 2 \end{bmatrix}$, then

$$\mathbf{P}^{-1}\mathbf{A}\mathbf{P} = -\frac{1}{3}\begin{bmatrix} 2 & -1 \\ -1 & -1 \end{bmatrix}\begin{bmatrix} 3 & 1 \\ 2 & 4 \end{bmatrix}\begin{bmatrix} -1 & 1 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} 2 & 0 \\ 0 & 5 \end{bmatrix}$$

An important special case

The necessary and sufficient condition for a matrix is being diagonalizable in theorem (7) can not be easily checked. Therefore it is an important result – can be proved using mathematical induction – that eigenvectors of a square matrix $\mathbf{A}$ corresponding to different eigenvalues are linearly independent, from which it immediately follows that an $n \times n$ real matrix $\mathbf{A}$ having n different real eigenvalues is diagonalizable. Of course the above condition is not a necessary one.

Example

Consider the matrix:

$$\mathbf{A} = \begin{bmatrix} 2 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 2 \end{bmatrix}$$

This matrix has only two different eigenvalues $\lambda_1 = 2$ and $\lambda_2 = 1$.

The linearly independent eigenvectors corresponding to the eigenvalue $\lambda_1 = 2$ are:

$\mathbf{v}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ and $\mathbf{v}_2 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$, while the eigenvector belonging to the eigenvalue $\lambda_2 = 1$ is $\mathbf{v}_3 = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$. Thus the matrix has three linearly independent eigenvectors, therefore it is diagonalizable.

Symmetric matrices

Symmetric matrices

The square matrices occurring in economical applications are often symmetric, therefore it seems to be reasonable to study the question if the symmetric matrices are diagonalizable.

Recall that a real matrix $\mathbf{A}$ is said to be symmetric if $\mathbf{A}^\top = \mathbf{A}$ and it is orthogonal if $\mathbf{A}^\top = \mathbf{A}^{-1}$.

Theorem

- *The characteristic equation of an $n \times n$ symmetric matrix has n real roots counting their multiplicity as well.*
- *The eigenvectors corresponding to different eigenvalues of a symmetric matrix are orthogonal.*

Theorem

The symmetric matrices are diagonalizable, furthermore to each symmetric matrix $\mathbf{A}$ there exists an orthogonal matrix $\mathbf{U}$, such that

$$\mathbf{U}^{-1}\mathbf{A}\mathbf{U} = diag\left\langle \lambda_1, \dots, \lambda_n \right\rangle,$$

where $\lambda_1, \dots, \lambda_n$ are the eigenvalues of $\mathbf{A}$.

Example

Find the orthogonal matrix $\mathbf{U}$ to the symmetric matrix $\mathbf{A}$ for which the matrix $\mathbf{U}^\top\mathbf{A}\mathbf{U}$ is a diagonal one.

$$\mathbf{A} = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$$

The eigenvalues of the matrix are $\lambda_1 = -1,\ \lambda_2 = 1$ and $\lambda_3 = 3$, and the corresponding eigenvectors are

$$\mathbf{v}_1 = \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix}, \mathbf{v}_2 = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} \quad \text{and} \quad \mathbf{v}_3 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}.$$

The normed eigenvectors are the columns of the orthogonal matrix $\mathbf{U}$. Thus

$$\mathbf{U} = \begin{bmatrix} 0 & 0 & 1 \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \end{bmatrix}$$

and $\mathbf{U}^\top \mathbf{A} \mathbf{U} = \operatorname{diag} \langle -1, 1, 3 \rangle$.

Back to the quadratic forms

Classificiation with eigenvalues

Theorem

Consider the quadratic form Q, and let B denote the corresponding symmetric matrix, i.e. $Q(x) = \langle x, Bx \rangle$ for every $x \in \mathbb{R}^N$ n. Examine the eigenvalues of B.

- *If all eigenvalues are positive, then Q is positive definite.*
- *If all eigenvalues are nonnegative and at least one of them is zero, then Q is positive semidefinite.*
- *If all eigenvalues are negative, then Q is negative definite.*
- *If all eigenvalues are nonpositive and at least one of them is zero, then Q is negative semidefinite.*
- *If there are both positive and negative eigenvalues, then Q is indefinite.*

Classification with leading principal minors

Let B denote an $n \times n$ real matrix. The determinant of the top left $k \times k$ matrices $(k = 1, \cdots, n)$ are called leading principal minors.

Theorem (Sylvester)

Consider the quadratic form Q, and let B denote the corresponding symmetric matrix, i.e. $Q(x) = \langle x, Bx \rangle$ for every $x \in \mathbb{R}^N$.
The quadratic form is positive definite if and only if all of the leading principal minors of B is positive.

Proof

Assume first, that Q is positive definite. If λ is an eigenvalue, and x is an eigenvector such that $Bx = \lambda x$, then

$$0 < Q(x) = \langle x, Bx \rangle = \langle x, \lambda x \rangle = \lambda \langle x, x \rangle = \lambda \|x\|^2$$

which means that all of the eigenvalues of B are positive.

Realize, that the determinant of a symmetric matrix is the product of all of the eigenvalues. We obtained that the determinant of any positive definite matrix is positive.

Obviously, the top left $k \times k$ submatrix of a positive definite matrix is also positive definite. Thus we proved, that all of the principal minors of B is a positive number.

Proof

Vice versa, assume now that the sequence of the principal minors $\Delta_1, \cdots, \Delta_n$ are all positive numbers. At this case all of the top left $k \times k$ submatrix is regular, thus all of the columns of any $k \times k$ submatrix form a linear independent system. This fact implicates, that during the dyadic decomposition process all of the top left pivot terms are nonzero numbers.

If $\alpha_1, \cdots, \alpha_n$ is the sequence of the pivot terms, then the dyadic decompositon is

$$B = \sum_{k=1}^{n} \alpha_k d_k \cdot d_k^T$$

We know that the derminant can be computed as the product of the pivot terms (at this case) thus $\Delta_1 = \alpha_1$, and $\Delta_k = \alpha_k \Delta_{k-1}$ if $k > 1$. This fact implicates that all of the pivot terms α_k are positive numbers, thus the quadratic form is positive definite, which was to be demonstrated.

Problems

Problem 1

Diagonalize the matrix

$$A = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}$$

Problem 2

Diagonalize the matrix

$$A = \begin{pmatrix} 4 & -3 & -3 \\ 3 & -2 & -3 \\ -1 & 1 & 2 \end{pmatrix}$$

Compute the matrix A^{10}.

Problem 3

Diagonalize the matrix

$$A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$$

Problem 4

Consider the matrix

$$A = \begin{pmatrix} 1 & 2 \\ 4 & 3 \end{pmatrix}$$

Compute A^{10}.

Write A^{10} as a linear combination of dyads.

Problem 5

Diagonalize the matrix

$$A = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 2 \end{pmatrix}$$

Problem 6

Diagonalize the matrix

$$A = \begin{pmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{pmatrix}$$

Definition

If A is a square matrix, then the sum of the diagonal terms of A is called the *trace* of matrix A. Notation: $\operatorname{tr} A$.

Prove that the trace of similar matrices are the same. That is for any regular matrix P

$$\operatorname{tr}\left(P^{-1}AP\right) = \operatorname{tr} A.$$

Hint: Let us prove first, that $\operatorname{tr}(AB) = \operatorname{tr}(BA)$ for any square matrices A and B.

Problem 7

Diagonalize the matrix

$$A = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}$$

Problem 8

Diagonalize the matrix

$$A = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 \end{pmatrix}$$

Upper triangular matrices

Prove that an upper triangular matrix is regular if and only if the diagonal terms are non zero numbers.

Prove that λ is an eigenvalue of an upper triangular matrix if and only if λ is one of the terms of the diagonal of the matrix.