Quadratic forms

The Gauss–Jordan elimination completes the square

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November 4, 2024

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Outline

Definition of quadratic forms

A quadratic form is a special multi-variable function.

Definition

Let $A \in \mathbb{R}^{n \times n}$ be a symmetric matrix. Assume that $a_{k,j}$ denotes the appropriate element of that matrix. The function $Q : \mathbb{R}^n \to \mathbb{R}$

$$Q\left(x_1, x_2, \cdots, x_n\right) = \sum_{k=1}^{n} \sum_{j=1}^{n} a_{k,j} x_k x_j$$

is called n variables quadratic function.

If $n = 1$ then $Q(x_1) = a x_1^2$.
If $n = 2$ then

$$Q\left(x_1, x_2\right) = a_{1,1} x_1^2 + a_{1,2} x_1 x_2 + a_{2,1} x_2 x_1 + a_{2,2} x_2^2.$$

Compute the case when $n = 3$.

Theorem

If $A \in \mathbb{R}^{n\times n}$ *is a symmetric matrix then the quadratic form determined by* A *is*

$$Q(x) = x \cdot Ax$$

for every $x \in \mathbb{R}^n$.

Example

Write the symmetric matrix of quadratic form given before. Write the quadratic form of a symmetric matrix given before.

Theorem

The quadratic form determined by $A + B$ *is the same as the sum of quadratic forms determined by* A *and* B *respectively. Similarly, the quadratic form determined by* αA *is the same as the form determined by* A *multiplied by* α.

Our goal is to classify a quadratic form with respect to its range!

Definition

Let $Q : \mathbb{R}^n \to \mathbb{R}$ be a non-zero quadratic form. It is called

positive definite, if $Q(x) > 0$ for every $x \in \mathbb{R}^n, x \neq 0$;

positive semidefinite, if $Q(x) \geq 0$ for every $x \in \mathbb{R}^n$, but there exists a $z \in \mathbb{R}^n, z \neq 0$ such that $Q(z) = 0$;

negative definite, if $Q(x) < 0$ for every $x \in \mathbb{R}^n, x \neq 0$;

negative semidefinite, if $Q(x) \leq 0$ for every $x \in \mathbb{R}^n$, but there exists a $z \in \mathbb{R}^n, z \neq 0$ such that $Q(z) = 0$;

indefinite, if there exist two vectors $x, y \in \mathbb{R}^n$ for which $Q(x) > 0$ but $Q(y) < 0$.

Dyadic decomposition

Dyadic decomposition

Dyad and completing the square

The dyad is a special symmetric matrix representing the perfect squares.

Definition

A symmetric matrix $A \in \mathbb{R}^{n \times n}$ is called *dyad*, if there exists a vector $a \in \mathbb{R}^{n \times 1}$ for which $A = a \cdot a^T$

Theorem

A quadratic form is a complete square if and only if it is determined by a dyad. If $A = a \cdot a^T$ where $a = (a_1, a_2, \cdots, a_n)$, then $Q(x_1, x_2, \cdots, x_n) = (a_1 x_1 + a_2 x_2 + \cdots + a_n x_n)^2$, here Q is the quadratic form determined by the dyad A.

Dyadic decomposition

High school method for a few variables only

Theorem

Every symmetric matrix is a linear combination of dyads. Thus every quadratic form is a linear combination of complete squares.

Example

Completing the square, rewrite the following quadratic forms as a linear combination of perfect squares.

$Q(x_1, x_2) = x_1^2 + 4x_1x_2 - 5x_2^2,$

$Q(x_1, x_2) = 4x_1^2 + 6x_1x_2 + 9x_2^2,$

$Q(x_1, x_2) = 5x_1^2 - 6x_1x_2 + x_2^2,$

$Q(x_1, x_2, x_3) = 5x_1^2 - 6x_1x_2 + x_1x_3,$

Dyadic decomposition

Professional method

Dyadic decomposition using Gauss–Jordan elimination

Every n-variable quadratic form can be decreased by a multiply of a dyad such that the remainder quadratic form has one variable less:

Theorem

Consider a symmetric matrix $A \in \mathbb{R}^{n \times n}$. Assume, that $a_{1,1} \neq 0$ and denote $d = \frac{1}{a_{1,1}} a_1$, where a_1 is the first column of matrix A. Then

$$R = A - a_{1,1} d \cdot d^T$$

is a symmetric matrix, where the first row (and column) is the zero vector.

Proof.

The sum of symmetric matrices is a symmetric matrix. If δ_k denotes the k-th coordinate of d, then $r_{k,j} = a_{k,j} - a_{1,1} \delta_k \delta_j$. If $k = 1$ we obtain $r_{1,j} = a_{1,j} - a_{1,1} \cdot 1 \cdot \frac{a_{1,j}}{a_{1,1}} = 0$. $\square$

The k, j term of the matrix of the remainder quadratic form is

$$r_{k,j} = a_{k,j} - a_{1,1}\delta_k\delta_j = a_{k,j} - a_{1,1}\frac{a_{k,1}}{a_{1,1}}\delta_j = a_{k,j} - a_{k,1}\delta_j.$$

If $k > 1$ and $j > 1$, then the above formula just computes the new coordinates of the j-th column after the first column has entered to the base at the first position. Thus we proved the following theorem.

Theorem

Let $A \in \mathbb{R}^{n\times n}$ be a symmetric matrix and assume that $a_{1,1} \neq 0$. After separating the first complete square, as in the theorem above, the matrix of the remainder quadratic form is the bottom right $n-1 \times n-1$ matrix of the transformation table, when $a_{1,1}$ is the pivot term.

Problems

Write the following quadratic forms as a linear combination of complete squares.

1. $Q(x_2, x_3) = 13x_2^2 - 4x_2x_3 + 8x_3^2,$
2. $Q(x_1, x_2, x_3) = 5x_1^2 + 6x_2^2 + 4x_3^2 - 4x_1x_2 - 4x_1x_3,$
3. $Q(z_1, z_2, z_3) = 2z_1^2 + \frac{3}{2}z_3^2 + 2z_1z_2 - 4z_1z_3 + 2z_2z_3,$
4. $Q(x_1, x_2, x_3) = 3x_2^2 + 2x_1x_2 + 2x_1x_3 + 6x_2x_3,$
5. $Q(x_1, x_2, x_3, x_4) = 2x_1^2 + 2x_2^2 + 2x_3^2 + 2x_4^2 - 4x_1x_2 + 2x_1x_4 + 2x_2x_3 - 4x_3x_4$
6. $Q(x_1, x_2, x_3) = 2x_1x_2 + 2x_2x_3$

Let Q be a quadratic form defined as

$$Q\left(x_1, x_2, x_3, x_4\right) = \left(x_1 + x_2 + 2x_3 + x_4\right)^2 + \left(x_1 + x_4\right)^2 + \left(x_2 + 2x_3\right)^2$$

for real numbers x_1, x_2, x_3, x_4.

1. Characterize the definiteness of Q.
2. Write the matrix of the quadratic form above.
3. Compute the rank of the matrix above.
4. Write the matrix as a linear combinations of r dyads, where r is the rank above.

Definiteness of a quadratic form

Definiteness

Classify a quadratic form with respect to its range!

Definition

Let $Q : \mathbb{R}^n \to \mathbb{R}$ be a non-zero quadratic form. It is called

positive definite, if $Q(x) > 0$ for every $x \in \mathbb{R}^n, x \neq 0$;

positive semidefinite, if $Q(x) \geq 0$ for every $x \in \mathbb{R}^n$, but there exists a $z \in \mathbb{R}^n, z \neq 0$ such that $Q(z) = 0$;

negative definite, if $Q(x) < 0$ for every $x \in \mathbb{R}^n, x \neq 0$;

negative semidefinite, if $Q(x) \leq 0$ for every $x \in \mathbb{R}^n$, but there exists a $z \in \mathbb{R}^n, z \neq 0$ such that $Q(z) = 0$;

indefinite, if there exist two vectors $x, y \in \mathbb{R}^n$ for which $Q(x) > 0$ but $Q(y) < 0$.

Theorem

Consider a square matrix A having rank r. Assume that

$$A = B \cdot C$$

where the number of the columns of B (and the number of the rows of C) is the same as the rank r.
Then the row system of C is a linearly independent system.

Proof.

Every row of A is a linear combination of the rows of C. It means that the row system of C is a spanning set of the row space of A. But the number of the rows of C is the same as the dimension of the row space of A, thus it is a minimal spanning set. We proved, that the rows of C form an independent set. $\square$

We also proved that the the row system of C is a base of the row space of A and similarly the column system of B forms a base of the column space of A.

Definiteness of a quadratic form

Summary

Definiteness after completing the square

It is not hard to decide the definiteness of a quadratic form if it is given with its dyad decomposition. Let Q be an n-variable non-zero quadratic form.
If the dyad decomposition

- includes positive and negative pivot terms then Q is an *indefinite* quadratic form.

Now assume the dyad decomposition includes exactly n complete squares. If

- all pivot terms are positive numbers then Q is *positive definite*;
- all pivot terms are negative numbers then Q is *negative definite*.

Definiteness after completing the square ...

If the number of dyads is less than n, the same characterization holds true but the quadratic form is semidefinite only:
Thus if

- all pivot terms are positive numbers then Q is *positive semidefinite*;
- all pivot terms are negative numbers then Q is *negative semidefinite*.