Basics

- dot product: $\underline{a} \cdot \underline{b} = a_1 b_1 + a_2 b_2 + ... = |\underline{a}||\underline{b}| \cos\theta$, $W = \underline{F} \cdot \underline{d}$ where F is force and d is distance
- Cross Product: $\underline{v} \times \underline{w} = \begin{vmatrix} i & -j & k \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & W_3 \end{vmatrix}$
- The area of the quadrilateral which the vectors are enclosing is the determinant of the cross product
- Vector equation of line : $(x, y, z) = (x_0, y_0, z_0) + t(a, b, c)$; where (a, b, c) is a vector parallel to the line and (x_0, y_0, z_0) is a point on the line
- Standard equation of line/plane: $\underline{n} \cdot ((x, y, z) - (x_0, y_0, z_0)) = 0$, where $\underline{n}$ is a vector normal to the line/plane
- **projection a onto b**: $Proj_{\underline{b}}(\underline{a}) = \underline{a_b} = (\underline{a} \cdot \frac{\underline{a}}{|\underline{b}|})\frac{\underline{b}}{|\underline{b}|})$

Parametrization and Tangents to planes ...

- **Curve**: $R^2 : \underline{r} = (x(t), y(t))$; $R^3 : \underline{r} = (x(t), y(t), z(t))$
- **Tangent line at** $t = t_0$: $L(s) = \underline{r}(t_0) + s\underline{r}'(t_0)$
- **Tangent plane of graph at** $(a, b, f(a, b))$: $z = f(a, b) + f_x(a, b)(x - a) + f_x(a, b)(y - b)$

Parametrized Surface ($\underline{r}(u, v)$) and Curve and regular parametrization

- $\underline{r}(u, v) = (x(u, v), y(u, v), z(u, v))$
- $\underline{r_u}(u_0, v_0)$ and $\underline{r_v}(u_0, v_0)$ are two vectors parallel to the plane tangent to the surface at $\underline{r}(u_0, v_0)$
- $\underline{n} = \underline{r_u}(u_0, v_0) \times \underline{r_v}(u_0, v_0)$ is a vector normal to the above tangent plane, if $\underline{n} \neq \underline{0}$ then the parametrization is regular at $\underline{r}(u_0, v_0)$
- Tangent plane can be written in two ways:
  - As a parametrization: $(x, y, z) = \underline{r}(u_0, v_0) + a\underline{r_u}(u_0, v_0) + b\underline{r_v}(u_0, v_0)$
  - As a level set (Standard equation): $\underline{n} \cdot ((x, y, z) - \underline{r}(u_0, v_0)) = 0$
- **Parametrization of special surfaces**
  - **cylinder**: $\underline{r}(\theta, z) = (R\cos(\theta), R\sin(\theta), z)$
  - **sphere**: $\underline{r}(\theta, \phi) = (R\sin(\phi)\cos(\theta), R\sin(\phi)\sin(\theta), R\cos(\phi))$
  - **graph** $y = f(x)$: $\underline{r}(x) = (x, f(x))$
- **Parametrization of curves**
  - **line segment**: $\underline{r}(t) = \underline{a} + t(\underline{b} - \underline{b})$
  - **circle in** R^2: $\underline{r}(t) = (R\cos(t), R\sin(t))$
  - **graph** $z = f(x, y)$: $\underline{r}(x, y) = (x, y, f(x, y))$

Surfaces and Gradient vectors

- **Common surfaces**:
  - **Bowl/cup**: $z = x^2 + y^2$
  - **Saddle**: $z = x^2 - y^2$
  - **Sphere**: $x^2 + y^2 + z^2 = R^2$
  - **Cone**: $z = \pm c\sqrt{x^2 + y^2}$
  - **Cylinder**: $x^2 + y^2 = R^2$
  - **Plane**: $ax + by + cz = d$

Level sets

- **level set of** $f(x, y)$ **is a curve in the xy-plane:** $f(x, y) = const$
- **level set of** $f(x, y, z)$ **is a graph in the xyz-plane:** $f(x, y, z) = const$

Gradient Vector $\nabla f = (f_{x_1}, f_{x_2}, ..., f_{x_n})$

- Maximum increase: the direction: direction of ∇f ; the rate: $|\nabla f|$
- Minimum increase: opposite direction of ∇f ; the rate: $-|\nabla f|$
- No change: the normal to ∇f
- Rate of change in direction $\hat{u}$: **directional derivative** $D_{\hat{u}} f(\underline{p}) = \nabla f(\underline{p}) \cdot \hat{u}$
- ∇f is normal to level sets
- Tangent set to level set at $\underline{p}$: $\nabla f(\underline{p}) \cdot (\underline{x} - \underline{p}) = 0$

Linear approximation, tangent plane, Local minimum and Maximum (optimization)

- **Linearization**: $L(\underline{x}) = f(\underline{p}) + \nabla f(\underline{p}) \cdot (\underline{x} - \underline{p})$
- **Linear Approximation** : $f(\underline{x}) = L(\underline{x})$
- **Critical Points**: $\nabla f(\underline{p}) = \underline{0}$
- **Second derivative test**: $D = f_{xx} f_{yy} - (f_{xy})^2$ at point $\underline{p}$
  - If $D > 0$ and ($f_{xx} > 0$ or $f_{yy} > 0$), then $\underline{p}$ is a local min
  - If $D > 0$ and ($f_{xx} < 0$ or $f_{yy} < 0$), then $\underline{p}$ is a local max
  - If $D < 0$ then $\underline{p}$ is a saddle point
- **chain rule**: $f = f(\underline{x}), x_i = x_i(\underline{t})$ then $\frac{\partial f}{\partial t_i} = \nabla f(\underline{x}) \cdot \frac{\partial \underline{x}}{\partial t_i}$

Coordinate systems

- **Rectangular Coordinates** (x, y, z)
  - **to cylindrical** : $x = r\cos\theta$, $y = r\sin\theta$, $z = z$
  - **from cylindrical** : $r = \sqrt{x^2 + y^2}$, $\theta = \tan^{-1}(\frac{y}{x})$, $z = z$
- **Cylinder Coordinates** (θ, r, z)
  - **from spherical** : $r = \rho\cos\phi$, $z = \rho\sin\phi$, $\theta = \theta$
  - **to spherical** : $\rho = \sqrt{r^2 + z^2}$, $\phi \tan^{-1}(\frac{r}{z}$, $\theta = \theta$
- **Spherical Coordinates**(ϕ, θ, ρ)
  - **from rectangular** : $\rho = \sqrt{x^2 + y^2 + z^2}$, $\phi = \tan^{-1}(\frac{\sqrt{x^2+y^2}}{z})$, $\theta = \tan^{-1}(\frac{y}{x})$
  - **to rectangular** : $x = \rho\sin\phi\cos\theta$ $y = \rho\sin\phi\sin\theta$ $z = \rho\cos\phi$

Surface Area, Area, Volume, Mass, ...

- R^2 **Area elements** dA
  - **Cartesian**(x, y) $dA = dxdy$
  - **Polar**(θ, r) $dA = rdrd\theta$
- R^3 **Volume elements** dV
  - **Cartesian**$(x, y), z$ $dV = dxdydz$
  - **Cylindrical**(θ, r, z) $dV = rdrd\theta dz$
  - **Spherical**(θ, ρ, ϕ) $dV = \rho^2 \sin\phi d\rho d\phi d\theta$
- **Surface Area element** (dS) **/ Surface Area**
  - **Surface Area element**: $dS = |\underline{r_u} \times \underline{r_u}|dudv$
    - **Graph** $z = f(x, y)$: $\underline{r}(x, y) = (x, y, f(x, y))$ $dS = \sqrt{{f_x}^2 + {f_y}^2 + 1}$
    - **Sphere** $\underline{r}(\theta, \phi) = (R\sin(\phi)\cos(\theta), R\sin(\phi)\sin(\theta), R\cos(\phi))$: $dS = R^2 \sin\phi d\phi d\theta$
    - **Cylinder** $\underline{r}(\theta, z) = (R\cos(\theta), R\sin(\theta), z)$: $dS = Rdzd\theta$
  - **Surface Area**: $\int\int_D dS = \int\int_D dS \int\int_D |\underline{r_u} \times \underline{r_v}|dudv$
- **Mass** given a density $\delta_a r(x, y)$ in R^2 or $\delta(x, y, z)$ in R^3 the mass is:
  - $M = \int\int_R \delta_a r(x, y)dA$ $M = \int\int\int_R \delta(x, y, z)dV$

Vector Fields, Line Integral (work), Conservative Field, Flux

- **Vector Field** : R^2 : $\underline{F}(\underline{x}) = (P(x, y), Q(x, y))$ R^3 : $\underline{F}(\underline{x}) = (P(x, y, z), Q(x, y, z), R(x, y, z))$
- **Divergence**: $div\underline{F} = \nabla \cdot \underline{F}$, **curl**: $curl\underline{F} = \nabla \times \underline{F}$ in 2d : $curl = Q_x - P_y$
- **Line integral (work)** of $\underline{\text{F}}$ = (P, Q, R) along a path (C) and parametrized by $\underline{r}(t) = (x(t), y(t))$ $t : a \to b$

$$\int_c \underline{F} \cdot d\underline{r} = \int_a^b \underline{F}(\underline{r}(t)) \cdot \underline{r}'(t)dt$$

- **Conservative Field** $\nabla \cdot \underline{F} = \underline{0}$ **(curl is 0)** :
  * $\underline{F} = \nabla f$ where f is called the potential function , find f by integrating
  * **line integral** becomes path independent (fundamental theorem of line integrals) $\int_c \underline{F} \cdot d\underline{r} = f(\underline{b}) - f(\underline{a})$ where a and b are the points that path starts and ends from
- **Flux of a vector field** $\underline{\textbf{F}}$ **through surface M**:

$$Flux = \int\int_M \underline{V} \cdot d\underline{S} = \int\int_M \underline{V} \cdot \hat{n} dS = \int\int_D \underline{V}(\underline{r}(u, v)) \cdot |\underline{r_u} \times \underline{r_v}|dudv$$

The three theorems for calculating work (line integral) and flux (all assume closed integrals)

Green's Theorem (line integral) : $\int_{\partial R} \underline{F} \cdot d\underline{r} = \int\int_R (Q_x - P_y)dA$

Divergence Theorem (flux) : $\int\int_{\partial E} \underline{F} \cdot \hat{n}dS = \int\int\int_E (\nabla \cdot \underline{F})dV$

Stoke's Theorem (line integral) : $\int_{\partial R} \underline{F} \cdot d\underline{r} = \int\int_M (\nabla \times \underline{F}) \cdot \hat{n}dS$

- For green's theorem, may have to add paths to close the path, for divergence theorem, may have to add surfaces
- Corollary to stoke's theorem: two surfaces sharing the same boundary have the same line integral