LU Decomposition Method or Factorization:

Consider three equations:

$$a_{11}x_1 + a_{12}x_2 + a_{13}x_3 = b_1$$

$$a_{21}x_1 + a_{22}x_2 + a_{23}x_3 = b_2$$

$$a_{31}x_1 + a_{32}x_2 + a_{33}x_3 = b_3$$

This can be written as $Ax = B$

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$$

Let matrix A divided into 2 parts, there are L→ Lower triangular matrix and U→ Upper triangular matrix.

$$L = \begin{bmatrix} 1 & 0 & 0 \\ l_{21} & 1 & 0 \\ l_{31} & l_{32} & 1 \end{bmatrix} \quad \text{and} \quad U = \begin{bmatrix} u_{11} & u_{12} & u_{13} \\ 0 & u_{22} & u_{23} \\ 0 & 0 & u_{33} \end{bmatrix}$$

Now the equation is,

$$LUx = B$$

Let $Ux = y$ there $y = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}$

Now, $Ly = B$

$$\Rightarrow \begin{bmatrix} 1 & 0 & 0 \\ l_{21} & 1 & 0 \\ l_{31} & l_{32} & 1 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$$

We will get the value of y and then solve the ~~the~~ $Ux = y$ equation and get the result of x.

LU decomposition

$$x + 5y + z = 14$$
$$2x + y + 3z = 13$$
$$3x + y + 4z = 17$$

$$\begin{bmatrix} 1 & 5 & 1 \\ 2 & 1 & 3 \\ 3 & 1 & 4 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 14 \\ 13 \\ 17 \end{bmatrix}$$

$$Ax = b$$

$$A = LU$$

$$\Rightarrow \begin{bmatrix} 1 & 5 & 1 \\ 2 & 1 & 3 \\ 3 & 1 & 4 \end{bmatrix} = LU$$

$$\Rightarrow \begin{bmatrix} 1 & 5 & 1 \\ 2 & 1 & 3 \\ 3 & 1 & 4 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ l_{21} & 1 & 0 \\ l_{31} & l_{32} & 1 \end{bmatrix} \begin{bmatrix} U_{11} & U_{12} & U_{13} \\ 0 & U_{22} & U_{23} \\ 0 & 0 & U_{33} \end{bmatrix}$$

$\cancel{ALU = b}$

$$Ax = b$$

$$\Rightarrow LUx = b$$

Let $Ux = y$ where, $y = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}$

$\therefore Ly = b$

$\Rightarrow$

$$LU = \begin{bmatrix} 1 & 0 & 0 \\ l_{21} & 1 & 0 \\ l_{31} & l_{32} & 1 \end{bmatrix} \begin{bmatrix} u_{11} & u_{12} & u_{13} \\ 0 & u_{22} & u_{23} \\ 0 & 0 & u_{33} \end{bmatrix}$$

$$= \begin{bmatrix} u_{11} & u_{12} & u_{13} \\ l_{21}u_{11} & l_{21}u_{12}+u_{22} & l_{21}u_{13}+u_{23} \\ l_{31}u_{11} & l_{31}u_{12}+l_{32}u_{22} & l_{31}u_{13}+l_{32}u_{23}+u_{33} \end{bmatrix}$$

$u_{11} = 1$, $u_{12} = 5$, $u_{13} = 1$

$l_{21}u_{11} = 2$
$\Rightarrow l_{21} \cdot 1 = 2$
$\therefore l_{21} = 2$

$l_{21}u_{12} + u_{22} = 1$
$\Rightarrow 2 \cdot u_{12} + u_{22} = 1$
$\Rightarrow 2 \cdot 5 + u_{22} = 1$
$\therefore u_{22} = -9$

$l_{31}u_{11} = 3$
$\Rightarrow l_{31} \cdot 1 = 3$
$\therefore l_{31} = 3$

$l_{21}u_{13} + u_{23} = 3$
$\Rightarrow 2 \cdot 1 + u_{23} = 3$
$\therefore u_{23} = 1$

$l_{31}u_{11} = 3$
$\Rightarrow l_{31} \cdot 1 = 3$
$\therefore l_{31} = 3$

$l_{31}u_{12} + l_{32}u_{22} = 1$
$\Rightarrow 3 \cdot 5 + l_{32}(-9) = 1$
$\Rightarrow -9l_{32} = -14$
$\therefore l_{32} = \frac{14}{9}$

$l_{31}u_{13} + l_{32}u_{23} + u_{33} = 4$
$\Rightarrow 3 \cdot 1 + \frac{14}{9} \cdot 1 + u_{33} = 4$
$\Rightarrow u_{33} = 1 - \frac{14}{9}$
$\therefore u_{33} = \frac{-5}{9}$

$$A = LU = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 3 & \frac{14}{9} & 1 \end{bmatrix} \begin{bmatrix} 1 & 5 & 1 \\ 0 & -9 & 1 \\ 0 & 0 & -\frac{5}{9} \end{bmatrix}$$

$Ly = B$

$$\Rightarrow \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 3 & \frac{14}{9} & 1 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} 14 \\ 13 \\ 17 \end{bmatrix}$$

$\Rightarrow y_1 = 14$

$2y_1 + y_2 = 13$

$\Rightarrow 2.14 + y_2 = 13$

$\Rightarrow y_2 = -15$

$3y_1 + \frac{14}{9}y_2 + y_3 = 17$

$\Rightarrow 3.14 + \frac{14}{9}\cdot(-15) + y_3 = 17$

$\Rightarrow y_3 = -\frac{5}{3}$

Now,

$Ux = y$

$$\Rightarrow \begin{bmatrix} 1 & 5 & 1 \\ 0 & -9 & 1 \\ 0 & 0 & -\frac{5}{9} \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 14 \\ -15 \\ -\frac{5}{3} \end{bmatrix}$$

$x + 5y + z = 14$

$-9y + z = -15$

$-\frac{5}{9}z = -\frac{5}{3}$

$\therefore z = 3$

$-9y + 3 = -15$

$\Rightarrow y = 2$

$x + 5y + z = 14$

$\Rightarrow x + 5.2 + 3 = 14$

$\therefore x = 1$

$\therefore (x, y, z) = (1, 2, 3)$