Division A

NUTAN MAHARASHTRA VIDYA PRASARAK MANDAL'S

NUTAN COLLEGE OF ENGINEERING & RESEARCH (NCER)

Department of First Year Engineering

BTBS 201

Engineering Mathematics-II

Teaching Notes

Lecture Number	Topic to be covered
1	**Unit 1: Complex Numbers (07 hrs.)** ➢ Introduction and Definition of complex number ➢ Geometrical representation of complex number
2	➢ De-Moivre's theorem(without proof) with Examples
3	➢ Roots of complex numbers by using De-Moivre's theorem with Examples
4	➢ Roots of complex numbers by using De-Moivre's theorem with Examples .
5	➢ Circular functions of complex variable – definition; Hyperbolic functions .
6	➢ Relations between circular and hyperbolic functions ; Real and imaginary parts of circular and hyperbolic functions with Examples
7	➢ Logarithm of Complex quantities

Submitted by:
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Engineering Mathematics – II

Unit-1. Complex Numbers

Total Hours : 07

1) Definition of complex number:-

A number in the form of $z = x+iy$, where x and y are real numbers and i is the imaginary number, where $i=\sqrt{-1}$, is called complex number.

2) Definition of conjugate of complex number:-

Let $z = x+iy$ be a complex number then conjugate of complex number is denoted by $\bar{z}$ and given by

$$\bar{z} = x-iy$$

3) Modulus of complex number:

Let $z = x+iy$ be a complex number then modulus of complex number is denoted by $|z|$ and given by

$$r = |z| = \sqrt{x^2+y^2}$$

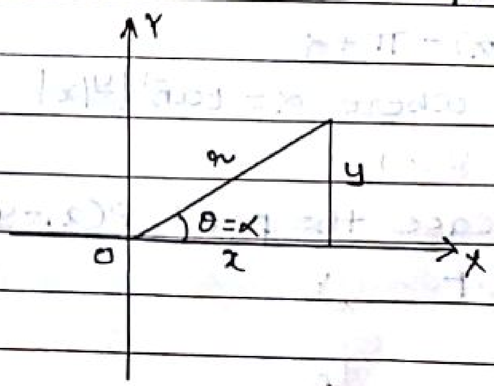
4) Equality of complex number:

If two complex numbers are equal, then their real and imaginary parts will respectively be equal.

5) Amplitude or Argument of complex number:

1) $z = x+iy$ $(x>0, y>0)$:

In this case $P(x,y)$ lie in the first quadrant.



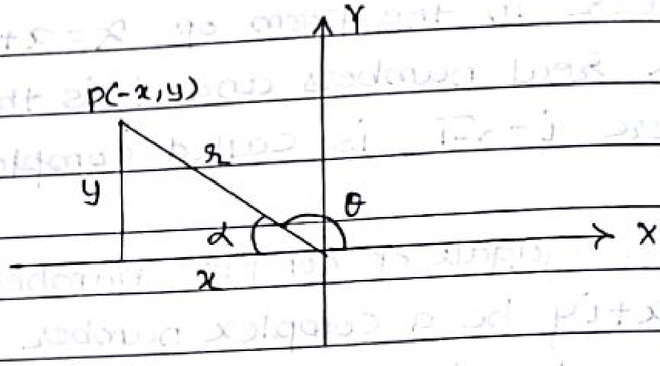
$z = x+iy$

$$\text{Modulus} = r = \sqrt{x^2+y^2}$$

$$\text{Amp}(z) = \text{Arg}(z) = \alpha = \theta = \tan^{-1}|y/x|$$

2) $z = -x+iy$: $(x<0,\ y>0)$:

In this case $P(x,y)$ lies in Second quadrant.



$z = -x+iy$

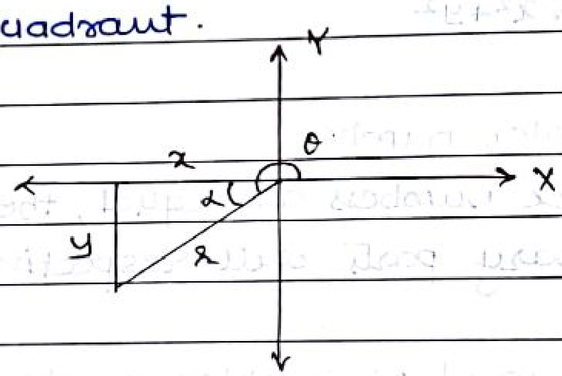
Modulus $= r = \sqrt{x^2+y^2}$

$\text{Amp}(z) = \text{Arg}(z) = \pi - \alpha$

where $\alpha = \tan^{-1}\left|\frac{y}{x}\right|$

3) $z = -x-iy$ $(x<0,\ y<0)$:

In this case the point $P(-x,-y)$ lies in the third quadrant.



$z = -x+iy$

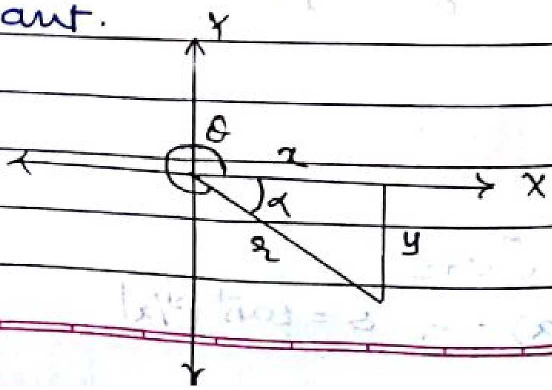
Modulus $= r = \sqrt{x^2+y^2}$

$\text{Amp}(z) = \text{Arg}(z) = \pi + \alpha$

where $\alpha = \tan^{-1}|y/x|$

4) $z = x-iy$ $(x>0,\ y<0)$:

In this case the point $P(x,-y)$ lies in the fourth quadrant.



$z = x - iy$

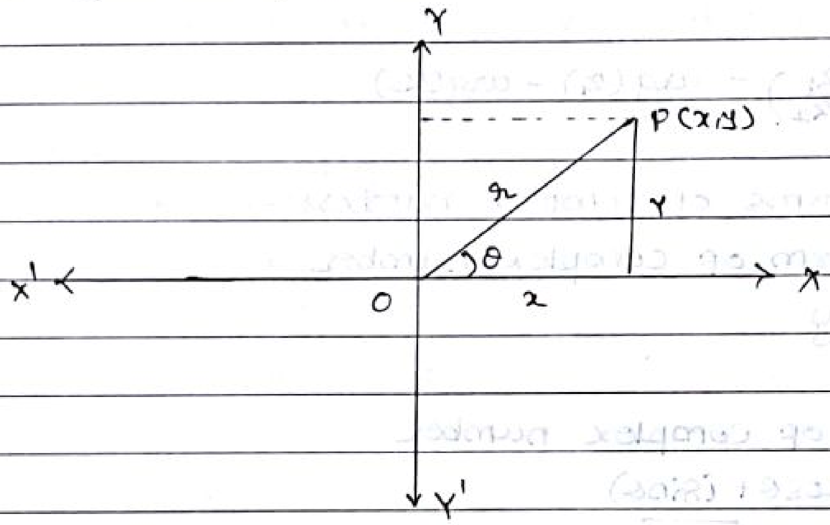
Modulus $= r = \sqrt{x^2+y^2}$

$Amp(z) = Arg(z) = 2\pi - \alpha$ OR

$Amp(z) = Arg(z) = -\alpha$

where $\alpha = \tan^{-1}(y/x)$

* Geometric Representation of a complex number: (Argand digram).



Mathematician Argan represented a complex number in a diagram known as Argand diagram.

A Complex number $z = x+iy$ can be represented by a point $P(x,y)$ in XY-plane. Where x is real axis & y is a imaginary axis. The distance op is the modulus and the angle OP makes with the x-axis is called argument of $x+iy$.

* Properties of complex number:

- Let $z = x+iy$ be a complex number and $\bar{z} = x-iy$ be a conjugate of complex number then

1) $Re(z) = x = \frac{1}{2}(z+\bar{z})$

2) $Im(z) = y = \frac{1}{2i}(z-\bar{z})$

3) $\overline{z_1 + z_2} = \bar{z}_1 + \bar{z}_2$

4) $\overline{z_1 \cdot z_2} = \bar{z}_1 \cdot \bar{z}_2$

5) $\overline{(z_1/z_2)} = \dfrac{\bar{z}_1}{\bar{z}_2}$

6) $z_1 \cdot \bar{z}_1 = |z_1|^2 = |\bar{z}_1|^2$

7) $|z_1 \cdot z_2| = |z_1||z_2|$

and $\text{Arg}(z_1 \cdot z_2) = \arg(z_1) + \arg(z_2)$

8) $\left|\dfrac{z_1}{z_2}\right| = \dfrac{|z_1|}{|z_2|}$

and $\arg\left(\dfrac{z_1}{z_2}\right) = \arg(z_1) - \arg(z_2)$

★ Different forms of complex numbers:-

1) Cartesian form of complex number

$z = x + iy$

2) Polar form of complex number

$z = r(\cos\theta + i\sin\theta)$

where $r = \sqrt{x^2+y^2}$

3) Exponential form of complex number

$z = re^{i\theta}$

where $e^{i\theta} = \cos\theta + i\sin\theta$ (Eulers formula)

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① Find the modulus and argument of complex $z = -\frac{1}{2} + \frac{\sqrt{3}}{2} i$

$\Rightarrow$ Given,

$$z = -\frac{1}{2} + \frac{\sqrt{3}}{2} i$$

Comparing with $z = x+iy$

Here $x = -1/2 < 0$, $y = \sqrt{3}/2 > 0$

$\therefore$ (x,y) lies in the second quadrant.

1) Modulus:-

$$|z| = r = \sqrt{x^2+y^2}$$

$$= \sqrt{\left(-\frac{1}{2}\right)^2 + \left(\frac{\sqrt{3}}{2}\right)^2}$$

$$= \sqrt{\frac{1}{4} + \frac{3}{4}}$$

$$r = 1$$

2) Argument of z

$$Arg(z) = \pi - \alpha$$

where $\alpha = \tan^{-1}\left|\frac{y}{x}\right|$

$$= \tan^{-1}\left|\frac{\sqrt{3}/2}{-1/2}\right|$$

$$= \tan^{-1}(\sqrt{3})$$

$$\alpha = \pi/3$$

$$\therefore \ Arg(z) = \pi - \frac{\pi}{3}$$

$$Arg(z) = 2\pi/3$$

2) If $\arg(z+1) = \pi/6$ and $\arg(z-1) = 2\pi/3$ find z.

$\Rightarrow$

Let $z = x + iy$

Given, $\arg(z+1) = \pi/6$

$\arg(x+iy+1) = \pi/6$

$\arg(x+1+iy) = \pi/6$

$\tan^{-1}\left(\frac{y}{x+1}\right) = \pi/6$ $\quad (\because \text{Arg}(z) = \tan^{-1}(y/x))$

$\frac{y}{x+1} = \tan\pi/6 = \frac{1}{\sqrt{3}}$

$\sqrt{3}y = x+1$

$x - \sqrt{3}y = -1$ —— ①

Also,

$\arg(z-1) = 2\pi/3$

$\arg(x+iy-1) = 2\pi/3$

$\arg(x-1+iy) = 2\pi/3$

$\tan^{-1}\left(\frac{y}{x-1}\right) = \frac{2\pi}{3}$

$\frac{y}{x-1} = \tan\left(\frac{2\pi}{3}\right) = -\sqrt{3}$

$y = -\sqrt{3}x + \sqrt{3}$

$\sqrt{3}x + y = \sqrt{3}$ —— ②

Solve eq^n ① & ②

Multiply eq^n ② by $\sqrt{3}$ & add in eq^n ①

$3x + \sqrt{3}y = 3$

$x - \sqrt{3}y = -1$

$4x = 2$

$\underline{x = 1/2}$

from ①

$1/2 - \sqrt{3}y = -1$

$\sqrt{3}y = 1/2 + 1 = 3/2$

$\underline{y = \frac{\sqrt{3}}{2}}$

$\therefore \quad z = \frac{1}{2} + \frac{\sqrt{3}}{2}i$

3) If $|z+i| = |z|$ and $\arg\left(\frac{z+i}{z}\right) = \frac{\pi}{4}$ find z.

$\Rightarrow$ Given $|z+i| = |z|$

Let $z = x+iy$

$$|x+iy+i| = |x+iy|$$

$$|x+(y+1)i| = |x+iy|$$

$$\sqrt{x^2+(y+1)^2} = \sqrt{x^2+y^2}$$

$$x^2+(y+1)^2 = x^2+y^2$$

$$x^2+y^2+2y+1 = x^2+y^2$$

$$2y+1 = 0$$

$$2y = -1$$

$$\underline{y = -1/2}$$

Also,

$$\arg\left(\frac{z+i}{z}\right) = \frac{\pi}{4}$$

$$\arg(z+i) - \log z = \pi/4$$

$$\arg(x+iy+i) - \log(x+iy) = \pi/4$$

$$\arg(x+(y+1)i) - \log(x+iy) = \pi/4$$

$$\tan^{-1}\left(\frac{y+1}{x}\right) - \tan^{-1}\left(\frac{y}{x}\right) = \pi/4$$

$$\tan^{-1}\left\{\frac{\frac{y+1}{x} - \frac{y}{x}}{1+\frac{y+1}{x}\cdot\frac{y}{x}}\right\} = \pi/4$$

$$\frac{x(y+1)-xy}{x^2+y(y+1)} = \tan(\pi/4)$$

$$\frac{xy+x-xy}{x^2+y^2+y} = 1$$

$$x = x^2+y^2+y$$

$$x = x^2+\frac{1}{4}-\frac{1}{2}$$

$$x^2-x-1/4 = 0$$

$$4x^2-4x-1 = 0$$

Comparing with $ax^2+bx+c=0$

$$a = 4, \ b = -4, \ c = -1$$

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$= \frac{4 \pm \sqrt{16 - 4(4)(-1)}}{2(4)}$$

$$= \frac{4 \pm \sqrt{32}}{8} = \frac{4 \pm 4\sqrt{2}}{8}$$

$$x = \frac{1 \pm \sqrt{2}}{2}$$

$$\therefore \ z = x + iy = \left(\frac{1 \pm \sqrt{2}}{2}\right) - \frac{1}{2} i$$

4) If z_1 and z_2 are two complex numbers such that $|z_1 + z_2| = |z_1 - z_2|$, then prove that $\operatorname{amp}\left(\frac{z_1}{z_2}\right) = \frac{\pi}{2}$ OR

Prove that $\arg(z_1) - \arg(z_2) = \pi/2$

$\Rightarrow$ Given

$$|z_1 + z_2| = |z_1 - z_2|$$

Let $z_1 = x_1 + iy_1$ and $z_2 = x_2 + iy_2$

$$|x_1 + iy_1 + x_2 + iy_2| = |x_1 + iy_1 - (x_2 + iy_2)|$$

$$|(x_1 + x_2) + i(y_1 + y_2)| = |(x_1 - x_2) + i(y_1 - y_2)|$$

$$\sqrt{(x_1 + x_2)^2 + (y_1 + y_2)^2} = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}$$

$$(x_1 + x_2)^2 + (y_1 + y_2)^2 = (x_1 - x_2)^2 + (y_1 - y_2)^2$$

$$x_1^2 + 2x_1x_2 + x_2^2 + y_1^2 + 2y_1y_2 + y_2^2 = x_1^2 - 2x_1x_2 + x_2^2 + y_1^2 - 2y_1y_2 + y_2^2$$

$$2x_1x_2 + 2y_1y_2 = -2x_1x_2 - 2y_1y_2$$

$$4x_1x_2 + 4y_1y_2 = 0$$

$$x_1x_2 + y_1y_2 = 0 \longleftarrow (1)$$

$$\operatorname{Amp}\left(\frac{z_1}{z_2}\right) = \operatorname{Amp}(z_1) - \operatorname{Amp}(z_2)$$

$$= \operatorname{Amp}(x_1 + iy_1) - \operatorname{Amp}(x_2 + iy_2)$$

$$= \tan^{-1}\left(\frac{y_1}{x_1}\right) - \tan^{-1}\left(\frac{y_2}{x_2}\right)$$

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$$= \tan^{-1}\left\{\frac{\frac{y_1}{x_1} - \frac{y_2}{x_2}}{1 + \frac{y_1}{x_1}\cdot\frac{y_2}{x_2}}\right\}$$

$$= \tan^{-1}\left\{\frac{\frac{x_2y_1 - x_1y_2}{x_1x_2}}{\frac{x_1x_2 + y_1y_2}{x_1x_2}}\right\}$$

$$= \tan^{-1}\left\{\frac{x_2y_1 - x_1y_2}{0}\right\} \quad \text{from } ①$$

$$= \tan^{-1}(\infty)$$

$$= \frac{\pi}{2}$$

$\therefore \ Amp(z_1) - Amp(z_2) = \pi/2$

5) If $\alpha - i\beta = \frac{1}{a-ib}$ prove that $(\alpha^2+\beta^2)(a^2+b^2) = 1$

$\Rightarrow$ Given that,

$$\alpha - i\beta = \frac{1}{a-ib}$$

$$= \frac{1}{a+ib} \times \frac{a+ib}{a+ib}$$

$$= \frac{a+ib}{a^2+b^2}$$

$$\alpha - i\beta = \left(\frac{a}{a^2+b^2}\right) + i\left(\frac{b}{a^2+b^2}\right)$$

Here $\alpha = \frac{a}{a^2+b^2}$, $\beta = \frac{-b}{a^2+b^2}$

Now, $\alpha^2+\beta^2 = \left(\frac{a}{a^2+b^2}\right)^2 + \left(\frac{-b}{a^2+b^2}\right)^2$

$$= \frac{a^2}{(a^2+b^2)^2} + \frac{b^2}{(a^2+b^2)^2} = \frac{a^2+b^2}{(a^2+b^2)^2}$$

$$\alpha^2+\beta^2 = \frac{1}{a^2+b^2}$$

$\therefore \ (\alpha^2+\beta^2)(a^2+b^2) = 1$

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6) If $x+iy = \sqrt[3]{a+ib}$ Prove that $\frac{a}{x} + \frac{b}{y} = 4(x^2-y^2)$

$\Rightarrow$ Given that

$$x+iy = \sqrt[3]{a+ib}$$

$$(x+iy)^3 = a+ib$$

$$a+ib = x^3 + 3x^2.iy + 3x(iy)^2 + (iy)^3$$

$$= x^3 + 3x^2yi - 3xy^2 - y^3 i$$

$$a+ib = (x^3 - 3xy^2) + (3x^2y - y^3)i$$

Here,

$$a = x^3 - 3xy^2 \quad \& \quad b = 3x^2y - y^3$$

Now,

$$\frac{a}{x} + \frac{y}{b} = \frac{x^3-3xy^2}{x} + \frac{3x^2y-y^3}{y}$$

$$= x^2 - 3y^2 + 3x^2 - y^2$$

$$= 4x^2 - 4y^2$$

$$\frac{a}{x} + \frac{b}{y} = 4(x^2-y^2)$$

7) Prove that $e^{2ai\cot^{-1}(b)} \left(\frac{bi-1}{bi+1}\right)^{-a} = 1.$

$\Rightarrow$ we know that

$$\frac{bi-1}{bi+1} = \frac{i(bi-1)}{i(bi+1)} = \frac{bi^2-i}{bi^2+i} = \frac{-b-i}{-b+i}$$

$$\therefore \frac{bi-1}{bi+1} = \frac{b+i}{b-i}$$

Let $b+i = r(\cos\theta + i\sin\theta)$

$$= re^{i\theta}$$

Here $b = r\cos\theta$ & $1 = r\sin\theta$

$$\therefore r = \sqrt{b^2+1^2} = \sqrt{b^2+1}$$

$$\theta = \tan^{-1}\left(\frac{y}{x}\right) = \tan^{-1}\left(\frac{1}{b}\right) = \cot^{-1}(b)$$

Also, $b-i = r(\cos\theta - i\sin\theta)$

$$= re^{-i\theta}.$$

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Now $\frac{bi-1}{bi+1} = \frac{b+i}{b-i}$

$$= \frac{re^{i\theta}}{re^{-i\theta}}$$

$$= e^{2i\theta}$$

$$= e^{2i\cot^{-1}(b)}$$

$$\therefore \left(\frac{bi-1}{bi+1}\right)^{-a} = \left\{e^{2i\cot^{-1}(b)}\right\}^{-a}$$

$$= e^{-2ai\cot^{-1}(b)}$$

$$= \frac{1}{e^{2ai\cot^{-1}(b)}}$$

$$\therefore e^{2ai\cot^{-1}(b)}\left(\frac{bi-1}{bi+1}\right)^{-a} = 1$$

8) If $i^{i^{i\cdots\infty}} = A+iB$, Prove that $A^2+B^2 = e^{-\pi B}$ & $\tan\left(\frac{\pi A}{2}\right) = \frac{B}{A}$

$\Rightarrow$ We know that

$$i^{A+iB} = A+iB$$

$$\Rightarrow \left(\cos\frac{\pi}{2} + i\sin\frac{\pi}{2}\right)^{A+iB} = A+iB$$

$$\Rightarrow \left(e^{i\pi/2}\right)^{A+iB} = A+iB \quad \text{(Eulers thm}^{m}\text{)}$$

$$e^{i\frac{\pi}{2}(A+iB)} = A+iB$$

$$\Rightarrow e^{i\frac{\pi}{2}A} \cdot e^{i^2\frac{\pi}{2}B} = A+iB$$

$$e^{-\frac{\pi}{2}B}\left(\cos\frac{A\pi}{2} + i\sin\frac{A\pi}{2}\right) = A+iB$$

$$e^{-\frac{\pi}{2}B}\cos\left(\frac{A\pi}{2}\right) + i\,e^{-\frac{\pi}{2}B}\sin\left(\frac{A\pi}{2}\right) = A+iB$$

$$\therefore A = e^{-\frac{\pi}{2}B}\cos\left(\frac{A\pi}{2}\right), \quad B = e^{-\frac{\pi}{2}B}\sin\left(\frac{A\pi}{2}\right)$$

i) $A^2+B^2 = \left\{ e^{-\pi B/2} \cos\frac{\pi A}{2} \right\}^2 + \left\{ e^{-\pi B/2} \sin\left(\frac{\pi A}{2}\right) \right\}^2$

$= \left(e^{-\pi B/2}\right)^2 \left\{ \cos^2\frac{\pi A}{2} + \sin^2\frac{\pi A}{2} \right\}$

$= e^{-\pi B} \; (1)$

$\therefore \; A^2+B^2 = e^{-\pi B}$

ii) $\frac{B}{A} = \frac{e^{-iB/2}\sin(A\pi/2)}{e^{-iB/2}\cos(A\pi/2)}$

$= \frac{\sin(A\pi/2)}{\cos(A\pi/2)}$

$\frac{B}{A} = \tan(A\pi/2)$

9) If $a = \cos\alpha + i\sin\alpha$, $b = \cos\beta + i\sin\beta$, $c = \cos\gamma + i\sin\gamma$ then Prove that,

$$\frac{(b+c)(c+a)(a+b)}{abc} = 8\cos\left(\frac{\beta-\gamma}{2}\right)\cos\left(\frac{\alpha-\beta}{2}\right)\cos\left(\frac{\gamma-\alpha}{2}\right)$$

$\Rightarrow$ Given, $a = \cos\alpha + i\sin\alpha$, $b = \cos\beta + i\sin\beta$, $c = \cos\gamma + i\sin\gamma$

Now

$b+c = \cos\beta + i\sin\beta + \cos\gamma + i\sin\gamma$

$= (\cos\beta + \cos\gamma) + i(\sin\beta + \sin\gamma)$

$= 2\cos\left(\frac{\beta+\gamma}{2}\right)\cos\left(\frac{\beta-\gamma}{2}\right) + i\,2\sin\left(\frac{\beta+\gamma}{2}\right)\cos\left(\frac{\beta-\gamma}{2}\right)$

$= 2\cos\left(\frac{\beta+\gamma}{2}\right)\left\{\cos\left(\frac{\beta+\gamma}{2}\right) + i\sin\left(\frac{\beta+\gamma}{2}\right)\right\}$

$b+c = 2\cos\left(\frac{\beta-\gamma}{2}\right) e^{i\left(\frac{\beta+\gamma}{2}\right)}$

Similarly,

$c+a = 2\cos\left(\frac{\gamma-\alpha}{2}\right) e^{i\left(\frac{\gamma+\alpha}{2}\right)}$

& $a+b = 2\cos\left(\frac{\alpha-\beta}{2}\right) e^{i\left(\frac{\alpha+\beta}{2}\right)}$

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Also, $abc = (\cos\alpha + i\sin\alpha)(\cos\beta + i\sin\beta)(\cos\gamma + i\sin\gamma)$

$$= e^{i\alpha} \cdot e^{i\beta} \cdot e^{i\gamma}$$

$$= e^{i(\alpha+\beta+\gamma)}$$

$$LHS = \frac{(b+c)(c+a)(a+b)}{abc}$$

$$= \frac{2\cos\left(\frac{\beta-\gamma}{2}\right) e^{i\left(\frac{\beta+\gamma}{2}\right)} \cdot 2\cos\left(\frac{\gamma-\alpha}{2}\right) e^{i\left(\frac{\gamma+\alpha}{2}\right)} \cdot 2\cos\left(\frac{\alpha-\beta}{2}\right) e^{i\left(\frac{\alpha+\beta}{2}\right)}}{e^{i(\alpha+\beta+\gamma)}}$$

$$= \frac{8\, e^{i\left\{\frac{\beta+\gamma+\gamma+\alpha+\alpha+\beta}{2}\right\}} \cdot \cos\left(\frac{\beta-\gamma}{2}\right)\cos\left(\frac{\gamma-\alpha}{2}\right)\cos\left(\frac{\alpha-\beta}{2}\right)}{e^{i(\alpha+\beta+\gamma)}}$$

$$= \frac{8\, e^{i(\alpha+\beta+\gamma)} \cos\left(\frac{\beta-\gamma}{2}\right)\cos\left(\frac{\gamma-\alpha}{2}\right)\cos\left(\frac{\alpha-\beta}{2}\right)}{e^{i(\alpha+\beta+\gamma)}}$$

$$= 8\cos\left(\frac{\beta-\gamma}{2}\right)\cos\left(\frac{\gamma-\alpha}{2}\right)\cos\left(\frac{\alpha-\beta}{2}\right)$$

LHS = RHS

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Exercise - 01

1) Express $1+\sin\alpha+i\cos\alpha$ in the modulus and amplitude form.

2) Find the complex number z if $\arg(z+1)=\pi/6$, $\arg(z-1)=2\pi/3$

3) If $x=\cos\theta+i\sin\theta$, $y=\cos\phi+i\sin\phi$ Prove that
$$\frac{x-y}{x+y}=i\tan\left(\frac{\theta-\phi}{2}\right)$$

4) If $|z+1|=|z-1|$, then prove that real $z=0$.

5)

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* De-Moivre's theorem:-

Statement:-

"If n is any real number, one of the values of $(\cos\theta + i\sin\theta)^n$ is $\cos n\theta + i\sin n\theta$."

ie. $(\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta$.

Different cases:-

1) If n is possitive integers.

Let $n = k+1$

$$(\cos\theta + i\sin\theta)^{k+1} = \cos(k+1)\theta + i\sin(k+1)\theta.$$

2) If n is an negative integers.

Let $n = -m$

$$(\cos\theta + i\sin\theta)^{-m} = \cos(-m)\theta + i\sin(-m)\theta$$

$$= \cos m\theta - i\sin m\theta.$$

3) If n is a fraction.

Let $n = p/q$

$$(\cos\theta + i\sin\theta)^{p/q} = \cos\left(\frac{p}{q}\right)\theta + i\sin\left(\frac{p}{q}\right)\theta.$$

* Application of De-Moivre's theorem:

1) If $z = \cos\theta + i\sin\theta$ then $\frac{1}{z} = \cos\theta - i\sin\theta$.

$\Rightarrow$ Given, $z = \cos\theta + i\sin\theta$

Now, $\frac{1}{z} = \frac{1}{\cos\theta + i\sin\theta}$

$$= (\cos\theta + i\sin\theta)^{-1}$$

$$= \cos(-1)\theta + i\sin(-1)\theta$$

$$\frac{1}{z} = \cos\theta - i\sin\theta$$

2) $(\cos\theta - i\sin\theta)^n = \cos n\theta - i\sin n\theta$.

$\Rightarrow$ $(\cos\theta - i\sin\theta)^n = \{\cos(-\theta) + i\sin(-\theta)\}^n$

$$= \cos n(-\theta) + i\sin n(-\theta)$$

$$= \cos(-n\theta) + i\sin(-n\theta)$$

$$= \cos n\theta - i\sin n\theta.$$

③ If $z_1 = \cos\theta + i\sin\theta$ & $z_2 = \cos\phi + i\sin\phi$ then

$$\frac{z_1}{z_2} = \cos(\theta-\phi) + i\sin(\theta-\phi)$$

$$\Rightarrow \frac{z_1}{z_2} = \frac{\cos\theta + i\sin\theta}{\cos\phi + i\sin\phi}$$

$$= (\cos\theta + i\sin\theta)(\cos\phi + i\sin\phi)^{-1}$$

$$= (\cos\theta + i\sin\theta)(\cos\phi - i\sin\phi)$$

$$= (\cos\theta\cos\phi + \sin\theta\sin\phi) + i(\sin\theta\cos\phi - \cos\theta\sin\phi)$$

$$\frac{z_1}{z_2} = \cos(\theta-\phi) + i\sin(\theta-\phi)$$

④ $(\sin\theta + i\cos\theta)^n = \cos n\left(\frac{\pi}{2}-\theta\right) + i\sin n\left(\frac{\pi}{2}-\theta\right)$

$$\Rightarrow (\sin\theta + i\cos\theta)^n = \left\{\cos\left(\frac{\pi}{2}-\theta\right) + i\sin\left(\frac{\pi}{2}-\theta\right)\right\}^n$$

$$= \cos n\left(\frac{\pi}{2}-\theta\right) + i\sin n\left(\frac{\pi}{2}-\theta\right)$$

Examples:-

1) Prove that $\left(\frac{1+\sin\alpha + i\cos\alpha}{1+\sin\alpha - i\cos\alpha}\right)^n = \cos\left(\frac{n\pi}{2} - n\alpha\right) + i\sin\left(\frac{n\pi}{2} - n\alpha\right)$

$$LHS = \left(\frac{1+\sin\alpha + i\cos\alpha}{1+\sin\alpha - i\cos\alpha}\right)^n$$

$$= \left\{\frac{1+\cos\left(\frac{\pi}{2}-\alpha\right) + i\sin\left(\frac{\pi}{2}-\alpha\right)}{1+\cos\left(\frac{\pi}{2}-\alpha\right) - i\sin\left(\frac{\pi}{2}-\alpha\right)}\right\}^n$$

$$= \left\{\frac{2\cos^2\left(\frac{\pi}{4}-\frac{\alpha}{2}\right) + 2i\sin\left(\frac{\pi}{4}-\frac{\alpha}{2}\right)\cos\left(\frac{\pi}{4}-\frac{\alpha}{2}\right)}{2\cos^2\left(\frac{\pi}{4}-\frac{\alpha}{2}\right) - 2i\sin\left(\frac{\pi}{4}-\frac{\alpha}{2}\right)\cos\left(\frac{\pi}{4}-\frac{\alpha}{2}\right)}\right\}^n$$

$$= \left\{\frac{2\cos\left(\frac{\pi}{4}-\frac{\alpha}{2}\right)\left[\cos\left(\frac{\pi}{4}-\frac{\alpha}{2}\right) + i\sin\left(\frac{\pi}{4}-\frac{\alpha}{2}\right)\right]}{2\cos\left(\frac{\pi}{4}-\frac{\alpha}{2}\right)\left[\cos\left(\frac{\pi}{4}-\frac{\alpha}{2}\right) - i\sin\left(\frac{\pi}{4}-\frac{\alpha}{2}\right)\right]}\right\}^n$$

$$= \left\{\frac{\cos\left(\frac{\pi}{4}-\frac{\alpha}{2}\right) + i\sin\left(\frac{\pi}{2}-\frac{\alpha}{2}\right)}{\cos(\pi/4-\alpha/2) - i\sin(\pi/2-\alpha/2)}\right\}^n$$

$$= \left\{ \frac{e^{i\left(\frac{\pi}{4}-\frac{\alpha}{2}\right)}}{e^{-i\left(\frac{\pi}{4}-\frac{\alpha}{2}\right)}} \right\}^n$$

$$= \left\{ e^{2i\left(\frac{\pi}{4}-\frac{\alpha}{2}\right)} \right\}^n$$

$$= e^{2ni\left(\frac{\pi}{4}-\frac{\alpha}{2}\right)}$$

$$= e^{ni\left(\frac{\pi}{2}-\alpha\right)}$$

$$= \cos n\left(\frac{\pi}{2}-\alpha\right) + i\sin n\left(\frac{\pi}{2}-\alpha\right)$$

$$= \cos\left(\frac{n\pi}{2}-n\alpha\right) + i\sin\left(\frac{n\pi}{2}-n\alpha\right)$$

LHS. = RHS.

2) If α and β are the roots of the equation $x^2\sin^2\theta - x\sin 2\theta + 1 = 0$ then prove that $\alpha^n + \beta^n = 2\cos n\theta \operatorname{cosec}^n\theta$, where n is possetive integer.

$\Rightarrow$ Given,

$$x^2\sin^2\theta - x\sin 2\theta + 1 = 0$$

Comparing with, $ax^2+bx+c = 0$

$$a = \sin^2\theta,\ b = -\sin 2\theta,\ c = 1$$

$$\therefore \quad x = \frac{-b \pm \sqrt{b^2-4ac}}{2a}$$

$$= \frac{\sin 2\theta \pm \sqrt{(-\sin 2\theta)^2 - 4\sin^2\theta}}{2\sin^2\theta}$$

$$= \frac{2\sin\theta\cos\theta \pm \sqrt{4\sin^2\theta\cos^2\theta - 4\sin^2\theta}}{2\sin^2\theta}$$

$$= 2\sin\theta \left\{ \frac{\cos\theta \pm \sqrt{\cos^2\theta - 1}}{2\sin^2\theta} \right\}$$

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$$= \frac{\cos\theta \pm \sqrt{-1}\sin\theta}{\sin\theta}$$

$$x = \frac{\cos\theta \pm i\sin\theta}{\sin\theta}$$

$$x = e^{\pm i\theta}\cdot \operatorname{cosec}\theta \qquad (\because e^{i\theta} = \cos\theta + i\sin\theta)$$

Let $\alpha = e^{i\theta}\operatorname{cosec}\theta$, $\beta = e^{-i\theta}\operatorname{cosec}\theta$.

Now,

$$\alpha^n + \beta^n = (e^{i\theta}\operatorname{cosec}\theta)^n + (e^{-i\theta}\operatorname{cosec}\theta)^n$$

$$= e^{ni\theta}\operatorname{cosec}^n\theta + e^{-ni\theta}\operatorname{cosec}^n\theta$$

$$= \operatorname{cosec}^n\theta\left(e^{ni\theta} + e^{-in\theta}\right)$$

$$= 2\operatorname{cosec}^n\theta\left(\frac{e^{in\theta} + e^{-in\theta}}{2}\right)$$

$$= 2\operatorname{cosec}^n\theta\cdot\cos n\theta. \qquad \because\left(\frac{e^{i\theta}+e^{-i\theta}}{2} = \cos\theta\right)$$

$$\alpha^n + \beta^n = 2\cos n\theta\cdot\operatorname{cosec}^n\theta.$$

3) If $\alpha = 1+i$, $\beta = 1-i$ and $\cot\theta = 1+x$, prove that $(x+\alpha)^n + (x+\beta)^n = 2\cos n\theta\operatorname{cosec}^n\theta$

⇒ Given,

$$\alpha = 1+i, \quad \beta = 1-i$$

& $\cot\theta = 1+x$

$$x = \cot\theta - 1$$

Now $x+\alpha = \cot\theta - 1 + \alpha$

$$= \cot\theta - 1 + 1 + i$$

$$= \cot\theta + i$$

$$= \frac{\cos\theta}{\sin\theta} + i$$

$$= \frac{\cos\theta + i\sin\theta}{\sin\theta}$$

$$x+\alpha = e^{i\theta}\operatorname{cosec}\theta$$

$$(x+\alpha)^n = e^{in\theta}\operatorname{cosec}^n\theta.$$

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Also, $x+\beta = \cot\theta - 1 + \beta$

$= \cot\theta - 1 + 1 - i$

$= \cot\theta - i$

$= \frac{\cos\theta}{\sin\theta} - i$

$= \frac{\cos\theta - i\sin\theta}{\sin\theta}$

$x+\beta = e^{-i\theta}\operatorname{cosec}\theta$

$\therefore (x+\beta)^n = e^{-in\theta}\operatorname{cosec}^n\theta$.

LHS $= (x+\alpha)^n + (x+\beta)^n$

$= e^{in\theta}\operatorname{cosec}^n\theta + e^{-in\theta}\operatorname{cosec}^n\theta$

$= \operatorname{cosec}^n\theta\,(e^{in\theta} + e^{-in\theta})$

$= 2\operatorname{cosec}^n\theta \left(\frac{e^{in\theta} + e^{-in\theta}}{2}\right)$

$= 2\operatorname{cosec}^n\theta \cos n\theta$

$= 2\cos n\theta \cdot \operatorname{cosec}^n\theta$.

$=$ RHS.

4) If α & β are the roots of equation $x^2 - 2\sqrt{3}x + 4 = 0$, prove that $\alpha^3 + \beta^3 = 0$

$\Rightarrow$ Given,

$x^2 - 2\sqrt{3}x + 4 = 0$

Comparing with, $ax^2 + bx + c = 0$

$a = 1,\ b = -2\sqrt{3},\ c = 4$

$\therefore\ x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

$= \frac{2\sqrt{3} \pm \sqrt{12 - 16}}{2(1)}$

$= \frac{2\sqrt{3} \pm \sqrt{-4}}{2} = \frac{2\sqrt{3} \pm 2i}{2}$

$x = \sqrt{3} \pm i$

Let $\alpha = \sqrt{3} + i$ and $\beta = \sqrt{3} - i$

Let $\alpha = \sqrt{3} + i = r(\cos\theta + i\sin\theta)$

$\sqrt{3} + i = r\cos\theta + ri\sin\theta$.

$\therefore r\cos\theta = \sqrt{3}$ and $r\sin\theta = 1$

$\Rightarrow r = \sqrt{3+1} = \sqrt{4} = 2$

$\theta = \tan^{-1}\left(\frac{y}{x}\right) = \tan^{-1}\left(\frac{1}{\sqrt{3}}\right) = \frac{\pi}{6}$

$\therefore \alpha = 2\left(\cos\frac{\pi}{6} + i\sin\frac{\pi}{6}\right)$

Let $\beta = \sqrt{3} - i = r(\cos\theta + i\sin\theta)$

$\sqrt{3} - i = r\cos\theta + ri\sin\theta$

$\therefore r\cos\theta = \sqrt{3}$, $r\sin\theta = -1$

$\Rightarrow r = \sqrt{3+1} = \sqrt{4} = 2$

$\theta = \tan^{-1}\left(\frac{y}{x}\right) = \tan^{-1}\left(\frac{-1}{\sqrt{3}}\right) = -\frac{\pi}{6}$

$\therefore \beta = 2\left(\cos\left(-\frac{\pi}{6}\right) + i\sin\left(-\frac{\pi}{6}\right)\right)$

$= 2\left\{\cos\frac{\pi}{6} - i\sin\frac{\pi}{6}\right\}$

Now,

$\alpha^3 + \beta^3 = \left\{2\left(\cos\frac{\pi}{6} + i\sin\frac{\pi}{6}\right)\right\}^3 + \left\{2\left(\cos\frac{\pi}{6} - i\sin\frac{\pi}{6}\right)\right\}^3$

$= 2^3\left(\cos 3\left(\frac{\pi}{6}\right) + i\sin 3\left(\frac{\pi}{6}\right)\right) + 2^3\left[\cos 3\left(\frac{\pi}{6}\right) - i\sin 3\left(\frac{\pi}{6}\right)\right]$

$= 2^3\left\{\cos\frac{\pi}{2} + i\sin\frac{\pi}{2}\right\} + 2^3\left\{\cos\frac{\pi}{2} - i\sin\frac{\pi}{2}\right\}$

$= 2^3\{0 + i(1)\} + 2^3\{0 - i(1)\}$

$= 8i + (-8i)$

$= 8i - 8i$

$\therefore \alpha^3 + \beta^3 = 0$

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(5) Prove that $(4n)^{th}$ power of $\frac{1+7i}{(2-i)^2}$ is equal to $(-4)^n$, where n is possitive integer.

$\Rightarrow$ Given,

$$\frac{1+7i}{(2-i)^2} = \frac{1+7i}{4-4i+i^2} = \frac{1+7i}{4-4i-1}$$

$$= \frac{1+7i}{3-4i}$$

$$= \frac{1+7i}{3-4i} \times \frac{3+4i}{3+4i}$$

$$= \frac{3+4i+21i+28i^2}{9-16(-1)}$$

$$= \frac{3+25i+28(-1)}{9+16}$$

$$= \frac{-25+25i}{25}$$

$$\frac{1+7i}{(2-i)^2} = -1+i$$

$$\therefore \left\{\frac{1+7i}{(2-i)^2}\right\}^{4n} = (-1+i)^{4n}$$

Let $-1+i = r(\cos\theta + i\sin\theta)$

$$r = \sqrt{(-1)^2+(1)^2} = \sqrt{2}$$

$$\theta = \tan^{-1}\left(\frac{1}{-1}\right) = \tan^{-1}(-1) = \pi - \tan^{-1}(1)$$

$$= \pi - \frac{\pi}{4} = \frac{3\pi}{4}$$

$$\therefore -1+i = \sqrt{2}\{\cos(3\pi/4) + i\sin(3\pi/4)\}$$

$$(-1+i)^{4n} = \{\sqrt{2}[\cos(3\pi/4) + i\sin(3\pi/4)]\}^{4n}$$

$$= (\sqrt{2})^{4n}\left\{\cos 4n\left(\frac{3\pi}{4}\right) + i\sin 4n\left(\frac{3\pi}{4}\right)\right\}$$

$$= 2^{2n}\{\cos(3n\pi) + i\sin(3n\pi)\}$$

$$= (2^2)^n\{\cos 3\pi + i\sin 3\pi\}^n$$

$$= (4)^n\{-1+i(0)\}^n$$

$$= (4)^n(-1)^n$$

$$= (-4)^n$$

$$\therefore \quad (-1+i)^{4n} = \left[\frac{1+7i}{(2-i)^2}\right]^{4n} = (-4)^n.$$

⑥ If $\cos\alpha + 2\cos\beta + 3\cos\gamma = \sin\alpha + 2\sin\beta + 3\sin\gamma = 0$

Prove that:

1) $\cos 3\alpha + 8\cos 3\beta + 27\cos 3\gamma = 18\cos(\alpha+\beta+\gamma)$

2) $\sin 3\alpha + 8\sin 3\beta + 27\sin 3\gamma = 18\sin(\alpha+\beta+\gamma)$

$\Rightarrow$ Let $x = \cos\alpha + i\sin\alpha$

$y = 2(\cos\beta + i\sin\beta)$

$z = 3(\cos\gamma + i\sin\gamma)$

Now,

$$x+y+z = (\cos\alpha + i\sin\alpha) + 2(\cos\beta + i\sin\beta) + 3(\cos\gamma + i\sin\gamma)$$

$$= (\cos\alpha + 2\cos\beta + 3\cos\gamma) + i(\sin\alpha + 2\sin\beta + 3\sin\gamma)$$

$$= 0 + i(0)$$

$x+y+z = 0$ —— ①

But, we know that

$$(x+y+z)^3 = x^3+y^3+z^3+3(x+y)(y+z)(z+x)$$

From ①

$$(0)^3 = x^3+y^3+z^3+3(-z)(-x)(-y)$$

$$0 = x^3+y^3+z^3-3xyz$$

$$\therefore \quad x^3+y^3+z^3 = 3xyz.$$

$$\Rightarrow (\cos\alpha + i\sin\alpha)^3 + \{2(\cos\beta + i\sin\beta)\}^3 + \{3(\cos\gamma + i\sin\gamma)\}^3$$

$$= 3(\cos\alpha + i\sin\alpha)\cdot 2(\cos\beta + i\sin\beta)\cdot 3(\cos\gamma + i\sin\gamma)$$

$$\Rightarrow (\cos 3\alpha + i\sin 3\alpha) + 8(\cos 3\beta + i\sin 3\beta) + 27(\cos 3\gamma + i\sin 3\gamma)$$

$$= 18\{\cos(\alpha+\beta+\gamma) + i\sin(\alpha+\beta+\gamma)\}$$

$$\Rightarrow (\cos 3\alpha + 8\cos 3\beta + 27\cos 3\gamma) + i(\sin 3\alpha + 8\sin 3\beta + 27\sin 3\gamma)$$

$$= 18\cos(\alpha+\beta+\gamma) + 18i\sin(\alpha+\beta+\gamma)$$

$\therefore$ 1) $\cos 3\alpha + 8\cos 3\beta + 27\cos 3\gamma = 18\cos(\alpha+\beta+\gamma)$

2) $\sin 3\alpha + 8\sin 3\beta + 27\sin 3\gamma = 18\sin(\alpha+\beta+\gamma)$

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⑦ If $x + 1/x = 2\cos\theta$, $y + 1/y = 2\cos\phi$, $z + 1/z = 2\cos\psi$ then

Prove that: 1) $xyz + \frac{1}{xyz} = 2\cos(\theta+\phi+\psi)$

2) $\frac{x^m}{y^n} + \frac{y^n}{x^m} = 2\cos(m\theta - n\phi)$

3) $x^m y^n + \frac{1}{x^m y^n} = 2\cos(m\theta + n\phi)$

$\Rightarrow$ Given, $x + \frac{1}{x} = 2\cos\theta$

$\therefore x^2 - 2x\cos\theta + 1 = 0$

$\therefore x = \frac{-b \pm \sqrt{b^2-4ac}}{2a}$ $\quad \because a = 1, b = -2\cos\theta, c = 1$

$= \frac{2\cos\theta \pm \sqrt{4\cos^2\theta - 4}}{2(1)}$

$= \frac{2\cos\theta \pm 2\sqrt{-(1-\cos^2\theta)}}{2}$

$= \cos\theta \pm \sin\theta\sqrt{-1}$

$x = \cos\theta \pm i\sin\theta$

$\therefore$ Let $x = \cos\theta + i\sin\theta$ ——— ①

Similarly, $y + \frac{1}{y} = 2\cos\phi$

$\therefore y = \cos\phi + i\sin\phi$ ——— ②

& $z + \frac{1}{z} = 2\cos\psi$

$\therefore z = \cos\psi + i\sin\psi$ ——— ③

1) $xyz = (\cos\theta + i\sin\theta)(\cos\phi + i\sin\phi)(\cos\psi + i\sin\psi)$

$xyz = \cos(\theta+\phi+\psi) + i\sin(\theta+\phi+\psi)$ ——— ④

$\therefore \frac{1}{xyz} = \frac{1}{\cos(\theta+\phi+\psi) + i\sin(\theta+\phi+\psi)}$

$= \{\cos(\theta+\phi+\psi) + i\sin(\theta+\phi+\psi)\}^{-1}$

$= \cos(\theta+\phi+\psi) - i\sin(\theta+\phi+\psi)$ ——— ⑤

Adding ④ & ⑤

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$$xyz + \frac{1}{xyz} = \cos(\theta+\phi+\psi) + i\sin(\theta+\phi+\psi) + \cos(\theta+\phi+\psi) - i\sin(\theta+\phi+\psi)$$

$$xyz + \frac{1}{xyz} = 2\cos(\theta+\phi+\psi)$$

2) $$\frac{x^m}{y^n} + \frac{y^n}{x^m} = \frac{(\cos\theta + i\sin\theta)^m}{(\cos\phi + i\sin\phi)^n} + \frac{(\cos\phi + i\sin\phi)^n}{(\cos\theta + i\sin\theta)^m}$$

$$= \frac{(\cos m\theta + i\sin m\theta)}{(\cos n\phi + i\sin n\phi)} + \frac{(\cos n\phi + i\sin n\phi)}{(\cos m\theta + i\sin m\theta)}$$

$$= (\cos m\theta + i\sin m\theta)(\cos n\phi + i\sin n\phi)^{-1} + (\cos n\phi + i\sin n\phi)(\cos m\theta + i\sin m\theta)^{-1}$$

$$= (\cos m\theta + i\sin m\theta)[\cos(-n\phi) + i\sin(-n\phi)] + (\cos n\phi + i\sin n\phi)[\cos(-m\theta) + i\sin(-m\theta)]$$

$$= \cos(m\theta - n\phi) + i\sin(m\theta - n\phi) + \cos(n\phi - m\theta) + i\sin(n\phi - m\theta)$$

$$= \cos(m\theta - n\phi) - i\sin(n\phi - m\theta) + \cos(m\theta - n\phi) + i\sin(n\phi - m\theta)$$

$$\frac{x^m}{y^n} + \frac{y^n}{x^m} = 2\cos(m\theta - n\phi)$$

3) $$x^m y^n = (\cos\theta + i\sin\theta)^m (\cos\phi + i\sin\phi)^n$$

$$= (\cos m\theta + i\sin m\theta)(\cos n\phi + i\sin n\phi)$$

$$x^m y^n = \cos(m\theta + n\phi) + i\sin(m\theta + n\phi)$$

$$\therefore \frac{1}{x^m y^n} = \frac{1}{\cos(m\theta + n\phi) + i\sin(m\theta + n\phi)}$$

$$= \{\cos(m\theta + n\phi) + i\sin(m\theta + n\phi)\}^{-1}$$

$$= \cos(m\theta + n\phi) - i\sin(m\theta + n\phi)$$

$$\therefore x^m y^n + \frac{1}{x^m y^n} = \cos(m\theta + n\phi) + i\sin(m\theta + n\phi) + \cos(m\theta + n\phi) - i\sin(m\theta + n\phi)$$

$$= 2\cos(m\theta + n\phi).$$

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⑧ If $x_r = \cos\left(\frac{\pi}{2^r}\right) + i\sin\left(\frac{\pi}{2^r}\right)$, show that $x_1, x_2, x_3 \cdots \infty = -1$

⇒ Given,

$$x_r = \cos\left(\frac{\pi}{2^r}\right) + i\sin\left(\frac{\pi}{2^r}\right)$$

putting $r = 1, 2, 3, \cdots$

$$x_1 = \cos\left(\frac{\pi}{2}\right) + i\sin\left(\frac{\pi}{2}\right) = e^{i\pi/2}$$

$$x_2 = \cos\left(\frac{\pi}{2^2}\right) + i\sin\left(\frac{\pi}{2^2}\right) = e^{i\pi/2^2}$$

$$x_3 = \cos\left(\frac{\pi}{2^3}\right) + i\sin\left(\frac{\pi}{2^3}\right) = e^{i\pi/2^3}$$

$\vdots$

Consider, $x_1 \cdot x_2 \cdot x_3 \cdots \infty = e^{i\pi/2} \cdot e^{i\pi/2^2} \cdot e^{i\pi/2^3} \cdots$

$$= e^{i\left(\frac{\pi}{2} + \frac{\pi}{2^2} + \frac{\pi}{2^3} + \cdots\right)}$$

The bracket terms is an infinite geometric series with first term $= \frac{\pi}{2}$ & Common ratio $= \frac{1}{2}$

$$\therefore \text{Sum} = \frac{\text{first term}}{1 - \text{common ratio}}$$

$$= \frac{\pi/2}{1 - 1/2} = \frac{\pi/2}{1/2}$$

$$= \pi$$

$$\therefore x_1 \cdot x_2 \cdot x_3 \cdots \infty = e^{i\pi}$$

$$= \cos\pi + i\sin\pi$$

$$= (-1) + i(0)$$

$$= -1$$

$$\therefore x_1 \cdot x_2 \cdot x_3 \cdots \infty = -1$$

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Exercise - 02

1) If $\sin\alpha + \sin\beta + \sin\gamma = 0$ & $\cos\alpha + \cos\beta + \cos\gamma = 0$ then prove that:

1) $\sin(\alpha+\beta) + \sin(\beta+\gamma) + \sin(\gamma+\alpha) = 0$ and
$\cos(\alpha+\beta) + \cos(\beta+\gamma) + \cos(\gamma+\alpha) = 0$

2) $\cos 2\alpha + \cos 2\beta + \cos 2\gamma = 0$ and $\sin 2\alpha + \sin 2\beta + \sin 2\gamma = 0$

3) $\cos^2\alpha + \cos^2\beta + \cos^2\gamma = 3/2$

4) $\sin 3\alpha + \sin 3\beta + \sin 3\gamma = 3\sin(\alpha+\beta+\gamma)$ and
$\cos 3\alpha + \cos 2\beta + \cos 3\gamma = 3\cos(\alpha+\beta+\gamma)$

2) If $x_r = \cos\left(\frac{\pi}{3^r}\right) + i\sin\left(\frac{\pi}{3^r}\right)$ Prove that

1) $x_1 x_2 x_3 \cdots \infty = i$

2) $x_0 x_1 x_2 \cdots \infty = -i$

3) Show that:
$\{(\cos\theta + \cos\phi) + i(\sin\theta + \sin\phi)\}^n + \{(\cos\theta + \cos\phi) - i(\sin\theta + \sin\phi)\}^n$
$= 2^{n+1}\cos^n\left(\frac{\theta-\phi}{2}\right)\cos\left(\frac{n\theta+n\phi}{2}\right)$

4) If α and β are roots of $x^2 - 2x + 2 = 0$. Prove that
$\alpha^n + \beta^n = 2^{2n+1}\cos\left(\frac{n\pi}{4}\right)$

5) Prove that:
$(1+\cos\theta + i\sin\theta)^n + (1+\cos\theta - i\sin\theta)^n = 2^{n+1}\cos^n(\theta/2)\cos(n\theta/2)$

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* Expansion of $\sin n\theta$, $\cos n\theta$ in the powers of $\sin\theta$ & $\cos\theta$.

By De-Moivres thm

$$(\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta \quad \text{——— (1)}$$

By Bionomial theorem,

$$(\cos\theta + i\sin\theta)^n = {}^nC_0 \cos^n\theta (i\sin\theta)^0 + {}^nC_1 \cos^{n-1}\theta (i\sin\theta)^1 + {}^nC_2 \cos^{n-2}\theta (i\sin\theta)^2 + {}^nC_3 \cos^{n-3}\theta (i\sin\theta)^3 + \dots + {}^nC_n \cos^0\theta \cdot (i\sin\theta)^n.$$

$$= \cos^n\theta + i({}^nC_1 \cos^{n-1}\theta \sin\theta) - {}^nC_2 \cos^{n-2}\theta \sin^2\theta - i({}^nC_3 \cos^{n-3}\theta \sin^3\theta) + {}^nC_4 \cos^{n-4}\theta \sin^4\theta + \dots (-1)^{n/2} \sin^n\theta \quad \text{——— (2)}$$

Equating real and imaginary parts from eq^n (1) & (2)

$$\cos n\theta = \cos^n\theta - {}^nC_2 \cos^{n-2}\theta \sin^2\theta + {}^nC_4 \cos^{n-4}\theta \sin^4\theta - \dots$$

$$\sin n\theta = {}^nC_1 \cos^{n-1}\theta \sin\theta - {}^nC_3 \cos^{n-3}\theta \sin^3\theta + \dots$$

Note :-
1) ${}^nC_n = 1$
2) ${}^nC_1 = n$
3) ${}^5C_2 = \frac{5\times 4}{2\times 1} = 10$
4) ${}^nC_0 = 1$
5) ${}^7C_5 = \frac{7\times6\times5\times4\times3}{5\times4\times3\times2\times1} = 21$

Examples :-

1) Prove that

1) $\sin 5\theta = 5\sin\theta - 20\sin^3\theta + 16\sin^5\theta$

2) $\cos 5\theta = 5\cos\theta - 20\cos^3\theta + 16\cos^5\theta$.

⇒ By De-Moivre's theorem,

$$(\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta$$

$$\therefore (\cos\theta + i\sin\theta)^5 = \cos 5\theta + i\sin 5\theta \quad \text{——— (1)}$$

By Bionomial theorem,

$$(\cos\theta + i\sin\theta)^5 = {}^5C_0 \cos^5\theta (i\sin\theta)^0 + {}^5C_1 \cos^4\theta (i\sin\theta) + {}^5C_2 \cos^3\theta (i\sin\theta)^2 + {}^5C_3 \cos^2\theta (i\sin\theta)^3 + {}^5C_4 \cos\theta (i\sin\theta)^4 + {}^5C_5 (\cos\theta)^0 (i\sin\theta)^5$$

$= \cos^5\theta + i(5\cos^4\theta\sin\theta) - 10\cos^3\theta\sin^2\theta$

$+ \frac{5\times4\times3}{3\times2\times1}\cos^2\theta \cdot i^3\sin^3\theta + \frac{5\times4\times3\times2}{4\times3\times2\times1}\cos\theta \cdot i^4\sin^4\theta$

$+ 1 \cdot i^5\sin^5\theta$

$= \cos^5\theta + i(5\cos^4\theta\sin\theta) - 10(\cos^3\theta\sin^2\theta)$
$- i(10\cos^2\theta\sin^3\theta) + 5\cos\theta\sin^4\theta + i\sin^5\theta$ —— ②

Equating real & imaginary parts from eqn ① & ②

$\cos 5\theta = \cos^5\theta - 10\cos^3\theta\sin^2\theta + 5\cos\theta\sin^4\theta$ ......

$\sin 5\theta = 5\cos^4\theta\sin\theta - 10\cos^2\theta\sin^3\theta + \sin^5\theta$ ......

Now,

1) $\cos 5\theta = \cos^5\theta - 10\cos^3\theta(1-\cos^2\theta) + 5\cos\theta(1-\cos^2\theta)^2$
$= \cos^5\theta - 10\cos^3\theta + 10\cos^5\theta + 5\cos\theta(1 - 2\cos^2\theta + \cos^4\theta)$
$= \cos^5\theta - 10\cos^3\theta + 10\cos^5\theta + 5\cos\theta - 10\cos^3\theta$
$+ 5\cos^5\theta$
$= 16\cos^5\theta - 20\cos^3\theta + 5\cos\theta$
$= 5\cos\theta - 20\cos^3\theta + 16\cos^5\theta.$

2) $\sin 5\theta = 5\sin\theta(1-\sin^2\theta)^2 - 10\sin^3\theta(1-\sin^2\theta) + \sin^5\theta$
$= 5\sin\theta(1 - 2\sin^2\theta + \sin^4\theta) - 10\sin^3\theta + 10\sin^5\theta + \sin^5\theta$
$= 5\sin\theta - 10\sin^3\theta + 5\sin^5\theta - 10\sin^3\theta + 10\sin^5\theta + \sin^5\theta$
$= 5\sin\theta - 20\sin^3\theta + 16\sin^5\theta.$

2) Prove that: $\tan 7\theta = \frac{7\tan\theta - 35\tan^3\theta + 21\tan^5\theta - \tan^7\theta}{1 - 21\tan^2\theta + 35\tan^4\theta - 7\tan^6\theta}$.

⇒ By De-Moivre's theorem,

$(\cos\theta + i\sin\theta)^7 = \cos 7\theta + i\sin 7\theta$ —— ①

By Binomial theorem,

$(\cos\theta + i\sin\theta)^7 = {}^7C_0\cos^7\theta(i\sin\theta)^0 + {}^7C_1\cos^6\theta(i\sin\theta)^1 +$
${}^7C_2\cos^5\theta(i\sin\theta)^2 + {}^7C_3\cos^4\theta(i\sin\theta)^3 + {}^7C_4\cos^3\theta(i\sin\theta)^4$
$+ {}^7C_5\cos^2\theta(i\sin\theta)^5 + {}^7C_6\cos\theta(i\sin\theta)^6 + {}^7C_7(\cos\theta)^0(i\sin\theta)^7$

$= \cos^7\theta + i(7\cos^6\theta\sin\theta) - 21\cos^5\theta\sin^2\theta - i(35\cos^4\theta\sin^3\theta)$
$+ 35\cos^3\theta\sin^4\theta + i(21\cos^2\theta\sin^5\theta) - 7\cos\theta\sin^6\theta - i\sin^7\theta + \cdots$ —— ②

Equating real & imaginary parts of eqⁿ ① & ②

$\cos 7\theta = \cos^7\theta - 21\cos^5\theta\sin^2\theta + 35\cos^3\theta\sin^4\theta - 7\cos\theta\sin^6\theta$

$\sin 7\theta = 7\cos^6\theta\sin\theta - 35\cos^4\theta\sin^3\theta + 21\cos^2\theta\sin^5\theta - \sin^7\theta$

But,

$$\tan 7\theta = \frac{\sin 7\theta}{\cos 7\theta}$$

$$= \frac{7\cos^6\theta\sin\theta - 35\cos^4\theta\sin^3\theta + 21\cos^2\theta\sin^5\theta - \sin^7\theta}{\cos^7\theta - 21\cos^5\theta\sin^2\theta + 35\cos^3\theta\sin^4\theta - 7\cos\theta\sin^6\theta}.$$

$$= \frac{\cos^7\theta\{7\tan\theta - 35\tan^3\theta + 21\tan^5\theta - \tan^7\theta\}}{\cos^7\theta\{1 - 21\tan^2\theta + 35\tan^4\theta - 7\tan^6\theta\}}$$

$$= \frac{7\tan\theta - 35\tan^3\theta + 21\tan^5\theta - \tan^7\theta}{1 - 21\tan^2\theta + 35\tan^4\theta - 7\tan^6\theta}.$$

3) Using De-Moivre's theorem, express $\frac{\sin 7\theta}{\sin\theta}$ in powers of $\sin\theta$.

By De-Moivre's theorem,

$(\cos\theta + i\sin\theta)^7 = \cos 7\theta + i\sin 7\theta$ —— ①

By Bionomial theorem,

$(\cos\theta + i\sin\theta)^7 = {}^7C_0\cos^7\theta(i\sin\theta)^0 + {}^7C_1\cos^6\theta(i\sin\theta) + {}^7C_2\cos^5\theta(i\sin\theta)^2$
$+ {}^7C_3\cos^4\theta(i\sin\theta)^3 + {}^7C_4\cos^3\theta(i\sin\theta)^4 + {}^7C_5\cos^2\theta(i\sin\theta)^5$
$+ {}^7C_6\cos\theta(i\sin\theta)^6 + {}^7C_7(\cos\theta)^0(i\sin\theta)^7$

$= \cos^7\theta + i(7\cos^6\theta\sin\theta) - 21\cos^5\theta\sin^2\theta - i(35\cos^4\theta\sin^3\theta)$
$+ 35\cos^3\theta\sin^4\theta + i(21\cos^2\theta\sin^5\theta) - 7\cos\theta\sin^6\theta - i\sin^7\theta \cdots$ —— ②

From eqⁿ ① & ②, Equating imaginary part,

$\sin 7\theta = 7\cos^6\theta\sin\theta - 35\cos^4\theta\sin^3\theta + 21\cos^2\theta\sin^5\theta - \sin^7\theta$

Now

$$\frac{\sin 7\theta}{\sin\theta} = \frac{7\cos^6\theta\sin\theta - 35\cos^4\theta\sin^3\theta + 21\cos^2\theta\sin^5\theta - \sin^7\theta}{\sin\theta}$$

$$= 7\cos^6\theta - 35\cos^4\theta\sin^2\theta + 21\cos^2\theta\sin^4\theta - \sin^6\theta$$

$$= 7(1-\sin^2\theta)^3 - 35(1-\sin^2\theta)^2\sin^2\theta + 21(1-\sin^2\theta)\sin^4\theta - \sin^6\theta.$$

$$= 7(1-3\sin^2\theta+3\sin^4\theta-\sin^6\theta) - 35(1-2\sin^2\theta+\sin^4\theta)\cdot\sin^2\theta + 21\sin^4\theta - 21\sin^6\theta - \sin^6\theta$$

$$= 7 - 21\sin^2\theta + 21\sin^4\theta - 7\sin^6\theta - 35\sin^2\theta + 70\sin^4\theta - 35\sin^6\theta + 21\sin^4\theta - 21\sin^6\theta - \sin^6\theta$$

$$\frac{\sin 7\theta}{\sin\theta} = 7 - 56\sin^2\theta + 112\sin^4\theta - 64\sin^6\theta$$

* Expansion of $\sin^n\theta$ and $\cos^n\theta$ in terms of $\sin\theta$ & $\cos\theta$ of multiples of θ.

Let $x = \cos\theta + i\sin\theta$ then $\frac{1}{x} = \cos\theta - i\sin\theta$.

$\therefore\ x + \frac{1}{x} = 2\cos\theta$ & $x - \frac{1}{x} = 2i\sin\theta$

$\Rightarrow \left(x+\frac{1}{x}\right)^n = 2^n\cos^n\theta$ & $\left(x-\frac{1}{x}\right)^n = 2^n i^n \sin^n\theta$.

1) Show that: $\sin^5\theta = \frac{1}{16}(\sin 5\theta - 5\sin 3\theta + 10\sin\theta)$

Let $x = \cos\theta + i\sin\theta$, then $\frac{1}{x} = \cos\theta - i\sin\theta$

$\therefore \left(x - \frac{1}{x}\right) = 2i\sin\theta$

$2^5 i^5 \sin^5\theta = \left(x - \frac{1}{x}\right)^5$

By Bionomial thm^m

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$$32i\sin^5\theta = x^5 + {}^nC_1 x^4\left(-\frac{1}{x}\right) + {}^nC_2 x^3\left(-\frac{1}{x}\right)^2 + {}^nC_3 x^2\left(-\frac{1}{x}\right)^3$$

$$+ {}^nC_4 x(-1/x)^4 + {}^5C_5 x^0(-1/x)^5$$

$$= x^5 - 5x^3 + 10x - \frac{10}{x} + \frac{5}{x^3} - \frac{1}{x^5}$$

$$= \left(x^5 - \frac{1}{x^5}\right) - 5\left(x^3 - \frac{1}{x^3}\right) + 10\left(x - \frac{1}{x}\right)$$

$$= 2i\sin 5\theta - 5(2i\sin 3\theta) + 10(2i\sin\theta)$$

$$32i\sin^5\theta = i(2\sin 5\theta - 10\sin 3\theta + 20\sin\theta)$$

$$32\sin^5\theta = 2\sin 5\theta - 10\sin 3\theta + 20\sin\theta$$

$$\sin^5\theta = \frac{1}{32}(2\sin 5\theta - 10\sin 3\theta + 20\sin\theta)$$

$$\sin^5\theta = \frac{1}{16}(\sin 5\theta - 5\sin 3\theta + 10\sin\theta)$$

2) Expand $\sin^7\theta$ in a series of sines of multiple of θ.

Let $x = \cos\theta + i\sin\theta$, then $\frac{1}{x} = \cos\theta - i\sin\theta$.

$$\therefore \left(x - \frac{1}{x}\right) = 2i\sin\theta$$

Taking 7^{th} power bothside

$$2^7 i^7 \sin^7\theta = \left(x - \frac{1}{x}\right)^7$$

$$= x^7 + {}^7C_1 x^6\left(-\frac{1}{x}\right) + {}^7C_2 x^5\left(-\frac{1}{x}\right)^2 + {}^7C_3 x^4\left(-\frac{1}{x}\right)^3$$

$$+ {}^7C_4 x^3\left(-\frac{1}{x}\right)^4 + {}^7C_5 x^2\left(-\frac{1}{x}\right)^5 + {}^7C_6 x\left(-\frac{1}{x}\right)^6 + \left(-\frac{1}{x}\right)^7$$

$$= x^7 - 7x^5 + 21x^3 - 35x + \frac{35}{x} - \frac{21}{x^3} + \frac{7}{x^5} - \frac{1}{x^7}$$

$$-128i\sin^7\theta = \left(x^7 - \frac{1}{x^7}\right) - 7\left(x^5 - \frac{1}{x^5}\right) + 21\left(x^3 - \frac{1}{x^3}\right) - 35\left(x - \frac{1}{x}\right)$$

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$-128 i \sin^7\theta = 2i\sin 7\theta - 7(2i\sin 5\theta) + 21(2i\sin 3\theta) - 35(2i\sin\theta)$

$$\sin^7\theta = \frac{-2i\{-\sin 7\theta + 7\sin 5\theta - 21\sin 3\theta + 35\sin\theta\}}{-128i}$$

$$\sin^7\theta = \frac{1}{64}(35\sin\theta - 21\sin 3\theta + 7\sin 5\theta - \sin 7\theta)$$

3) If $\sin^4\theta\cos^3\theta = a\cos\theta + b\cos 3\theta + c\cos 5\theta + d\cos 7\theta$, find the values of a, b, c & d.

Let $x = \cos\theta + i\sin\theta$, then $\frac{1}{x} = \cos\theta - i\sin\theta$

$\therefore \left(x + \frac{1}{x}\right) = 2\cos\theta$

$\Rightarrow 2^3\cos^3\theta = \left(x + \frac{1}{x}\right)^3$

and $\left(x - \frac{1}{x}\right)^4 = 2^4 \cdot i^4 \sin^4\theta$

$\Rightarrow 2^4\sin^4\theta = \left(x - \frac{1}{x}\right)^4$

$\therefore \sin^4\theta\cos^3\theta(2^3)(2^4) = \left(x + \frac{1}{x}\right)^3 \cdot \left(x - \frac{1}{x}\right)^4$

$8 \cdot 16 \sin^4\theta\cos^3\theta = \left(x + \frac{1}{x}\right)^3\left(x - \frac{1}{x}\right)^3\left(x - \frac{1}{x}\right)$

$128\sin^4\theta\cos^3\theta = \left(x^2 - \frac{1}{x^2}\right)^3\left(x - \frac{1}{x}\right)$

$$= \left(x^6 - \frac{1}{x^6} - 3x^2 + \frac{3}{x^2}\right)\left(x - \frac{1}{x}\right)$$

$$= x^7 - \frac{1}{x^5} - 3x^2 + \frac{3}{x^2} - x^5 + \frac{1}{x^7} + 3x - \frac{3}{x^3}$$

$$= \left(x^7 + \frac{1}{x^7}\right) - \left(x^5 + \frac{1}{x^5}\right) - 3\left(x^2 + \frac{1}{x^3}\right) + 3\left(x + \frac{1}{x}\right)$$

$$= 2\cos 7\theta - 2\cos 5\theta - 3(\cos 3\theta \cdot 2) + 3(2\cos\theta)$$

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$$128 \sin^4\theta \cos^3\theta = 2(\cos7\theta - \cos5\theta - 3\cos3\theta + 3\cos\theta)$$

$$\therefore \sin^4\theta \cos^3\theta = \frac{1}{64}\cos7\theta - \frac{1}{64}\cos5\theta - \frac{3}{64}\cos3\theta + \frac{3}{64}\cos\theta.$$

comparing with

$$\sin^4\theta \cos^3\theta = a\cos\theta + b\cos3\theta + c\cdot\cos5\theta + d\cos7\theta$$

we get

$$a = \frac{3}{64}, \quad b = -\frac{3}{64}, \quad c = -\frac{1}{64}, \quad d = \frac{1}{64}.$$

Exercise-03

1) Using De-Moivres theorem, prove that

$$\frac{\sin 6\theta}{\sin 2\theta} = 16\cos^4\theta - 16\cos^2\theta + 3$$

2) Prove that, $\cos^6\theta - \sin^6\theta = \frac{1}{16}(\cos 6\theta + 15\sin 2\theta)$

3) If $\tan^{-1}x + \tan^{-1}y + \tan^{-1}z = \pi$, Prove that $x+y+z = xyz$

4) Prove that:

1) $\sin^8\theta = \frac{1}{2^7}(\cos 8\theta - 8\cos 6\theta + 28\cos 4\theta - 56\cos 2\theta + 35)$

5) $\cos^8\theta + \sin^8\theta = \frac{1}{64}(\cos 8\theta + 28\cos 4\theta + 35)$

6) $\sin^6\theta + \cos^6\theta = \frac{1}{8}(3\cos 4\theta + 5)$

7) If $\sin 6\theta = a\cos^5\theta \cdot \sin\theta + b\sin^3\theta \cdot \cos^3\theta + c\sin^5\theta\cos\theta$ then find the values a, b and c.

* Roots of a Complex number :-

Statement :

"If p & q are integers relatively prime to each other then $(\cos\theta + i\sin\theta)^{p/q}$ has exactly q distinct values that can be arranged in geometrical progression."

⇒ $\sin\theta$ and $\cos\theta$ are periodic function of 2π

$$\therefore (\cos\theta + i\sin\theta)^{p/q} = \{\cos(2n\pi+\theta) + i\sin(2n\pi+\theta)\}^{p/q}$$

$$= \cos \frac{p(2n\pi+\theta)}{q} + i\sin \frac{p(2n\pi+\theta)}{q}$$

By putting $n = 0, 1, 2, \cdots q-1$ distinct roots of complex numbers.

Examples:-

1) Find all values of $(i)^{1/4}$

⇒ we have,

$$i = 0 + i(1)$$

$$= \left(\cos\frac{\pi}{2} + i\sin\frac{\pi}{2}\right)$$

$$= \cos\left(2n\pi + \frac{\pi}{2}\right) + i\sin\left(2n\pi + \frac{\pi}{2}\right)$$

$$i = \cos\left(\frac{4n\pi+\pi}{2}\right) + i\sin\left(\frac{4n\pi+\pi}{2}\right)$$

$$(i)^{1/4} = \left\{\cos\left(\frac{4n\pi+\pi}{2}\right) + i\sin\left(\frac{4n\pi+\pi}{2}\right)\right\}^{1/4}$$

$$= \left\{\cos\frac{1}{4}\left(\frac{4n\pi+\pi}{2}\right) + i\sin\frac{1}{4}\left(\frac{4n\pi+\pi}{2}\right)\right\}$$

$$(i)^{1/4} = \cos\left(\frac{4n\pi+\pi}{8}\right) + i\sin\left(\frac{4n\pi+\pi}{8}\right) \quad \text{--- (1)}$$

putting $n = 0, 1, 2, 3$, we get required values

For $n = 0$, $\alpha_0 = \cos\left(\frac{\pi}{8}\right) + i\sin\left(\frac{\pi}{8}\right)$

For $n=1$, $\alpha_1 = \cos\left(\frac{5\pi}{8}\right) + i\sin\left(\frac{5\pi}{8}\right)$

For $n=2$, $\alpha_2 = \cos\left(\frac{9\pi}{8}\right) + i\sin\left(\frac{9\pi}{8}\right)$

For $n=3$, $\alpha_3 = \cos\left(\frac{13\pi}{8}\right) + i\sin\left(\frac{13\pi}{8}\right)$

2) Find the all values $(1+i)^{1/3}$ and represents these values on Argand's diagram.

$\Rightarrow$ Let $1+i = r(\cos\theta + i\sin\theta)$

$\therefore r\cos\theta = 1$ & $r\sin\theta = 1$

squaring and adding, we get

$r^2 = 2$

$\therefore r = \sqrt{2}$

and $\frac{r\sin\theta}{r\cos\theta} = \frac{1}{1} \Rightarrow \tan\theta = 1$

$\therefore \theta = \tan^{-1}(1) = \pi/4$

$\therefore 1+i = \sqrt{2}\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right)$

$$(1+i)^{1/3} = \left\{\sqrt{2}\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right)\right\}^{1/3}$$

$$= (2^{1/2})^{1/3}\left(\cos\cdot\frac{1}{3}\cdot\frac{\pi}{4} + i\sin\frac{1}{3}\cdot\frac{\pi}{4}\right)$$

$$= 2^{1/6}\left(\cos\frac{\pi}{12} + i\sin\frac{\pi}{12}\right)$$

$$= 2^{1/6}\left(\cos\left(2n\pi + \frac{\pi}{2}\right) + i\sin\right.$$

$$= \left\{2^{1/2}\left(\cos\left(2n\pi + \frac{\pi}{4}\right) + i\sin\left(2n\pi + \frac{\pi}{4}\right)\right)\right\}^{1/3}$$

$$= 2^{1/6}\left\{\cos\left(\frac{2n\pi}{3} + \frac{\pi}{12}\right) + i\sin\left(\frac{2n\pi}{3} + \frac{\pi}{12}\right)\right\}$$

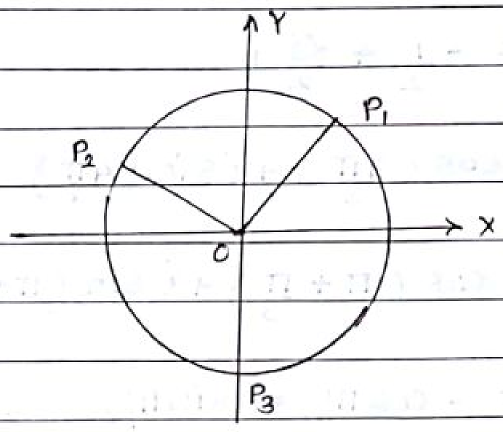
$$(1+i)^{1/3} = 2^{1/6}\left\{\cos\left(\frac{8n\pi + \pi}{12}\right) + i\sin\left(\frac{8n\pi + \pi}{12}\right)\right\}$$

putting, $n = 0, 1, 2$

For $n=0$, $\alpha_0 = 2^{1/6}\left\{\cos\frac{\pi}{12} + i\sin\frac{\pi}{12}\right\}$

For $n=1$, $\alpha_1 = 2^{1/6}\left\{\cos\frac{9\pi}{12} + i\sin\frac{9\pi}{12}\right\}$

For $n=2$, $\alpha_2 = 2^{1/6}\left\{\cos\frac{17\pi}{12} + i\sin\frac{17\pi}{12}\right\}$



On the above Argand diagram, these three points P_1, P_2 & P_3 well lie on the circle of radius $2^{1/6}$ and each separated from other two by an angle $\theta = 2\pi/3$

3) Find the cube roots of unity.

We have,

$$1 = 1 + 0i$$

$$1 = \cos 0 + i\sin 0$$

$$= \cos(2n\pi + 0) + i\sin(2n\pi + 0)$$

$$1 = \cos(2n\pi) + i\sin(2n\pi)$$

$$(1)^{1/3} = \{\cos(2n\pi) + i\sin(2n\pi)\}^{1/3}$$

$$(1)^{1/3} = \cos\left(\frac{2n\pi}{3}\right) + i\sin\left(\frac{2n\pi}{3}\right) \quad \text{—— } (1)$$

putting $n=0,1,2$, we get cube roots of unity

For $n=0$, $\alpha_0 = \cos 0 + i\sin 0$

$$= 1 + 0$$

$$\alpha_0 = 1$$

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For $n=1$, $\alpha_1 = \cos\left(\frac{2\pi}{3}\right) + i\sin\left(\frac{2\pi}{3}\right)$

$$= \cos\left(\pi - \frac{\pi}{3}\right) + i\sin\left(\pi - \frac{\pi}{3}\right)$$

$$= -\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}$$

$$\alpha_1 = -\frac{1}{2} + \frac{\sqrt{3}}{2}i$$

For $n=2$, $\alpha_2 = \cos\left(\frac{4\pi}{3}\right) + i\sin\left(\frac{4\pi}{3}\right)$

$$= \cos\left(\pi + \frac{\pi}{3}\right) + i\sin\left(\pi + \frac{\pi}{3}\right)$$

$$= -\cos\pi/3 - \sin\pi/3$$

$$\alpha_2 = -\frac{1}{2} - \frac{\sqrt{3}}{2}i$$

We observed that, $\alpha_0 + \alpha_1 + \alpha_3 = 0$ or $1 + \alpha_1 + \alpha_2 = 0$

4) If ω is a complex cube root of unity, Prove that $(1-\omega)^6 = -27$.

⇒ Given, ω is a complex cube root of unity.

$\therefore x^3 = 1$

$\Rightarrow (x^3 - 1) = 0$

$\Rightarrow (x-1)(x^2+x+1) = 0$

But ω is a root of $x^2+x+1=0$

$\therefore \omega^2 + \omega + 1 = 0$ —— (1)

& $\omega^3 = 1$ —— (2)

Now,

$(1-\omega)^2 = 1 - 2\omega + \omega^2 = (1+\omega^2) - 2\omega$

$= -\omega - 2\omega = -3\omega$

$\therefore (1-\omega)^6 = [(1-\omega)^2]^3$

$= (-3\omega)^3 = (-3)^3\omega^3$

$= -27(1)$

$= -27$

5) Show that the roots of $z^5 = 1$ can be written as $1, \alpha, \alpha^2, \alpha^3, \alpha^4$. Hence prove that $(1-\alpha)(1-\alpha^2)(1-\alpha^3)(1-\alpha^4) = 5$.

$\Rightarrow$ Given,

$$z^5 = 1 = 1 + 0i$$
$$= \cos 0 + i \sin 0$$
$$z^5 = \cos(2n\pi) + i\sin(2n\pi)$$
$$(z^5)^{1/5} = \{\cos(2n\pi) + i\sin(2n\pi)\}^{1/5}$$

$$z = \cos\left(\frac{2n\pi}{5}\right) + i\sin\left(\frac{2n\pi}{5}\right) \quad \text{---} \quad (1)$$

Putting $n = 0, 1, 2, 3, 4$, we get required roots.

For $n=0$, $z_0 = \cos 0 + i\sin 0 = 1 + 0 = 1$

For $n=1$, $z_1 = \cos\left(\frac{2\pi}{3}\right) + i\sin\left(\frac{2\pi}{3}\right) = \alpha$ (say)

For $n=2$, $z_2 = \cos\left(\frac{4\pi}{5}\right) + i\sin\left(\frac{4\pi}{5}\right) = \alpha^2 = \left(\cos\frac{2\pi}{3} + i\sin\frac{2\pi}{5}\right)^2$

For $n=3$, $z_3 = \cos\left(\frac{6\pi}{5}\right) + i\sin\left(\frac{6\pi}{5}\right) = \alpha^3 = \left(\cos\frac{2\pi}{5} + i\sin\frac{2\pi}{5}\right)^3$

For $n=4$, $z_4 = \cos\left(\frac{8\pi}{5}\right) + i\sin\left(\frac{8\pi}{5}\right)$, $\alpha^4 = \left(\cos\frac{2\pi}{3} + i\sin\frac{2\pi}{3}\right)^4$

$\therefore$ The required roots are $1, \alpha, \alpha^2, \alpha^3, \alpha^4$.

We know that $1, \alpha, \alpha^2, \alpha^3$ & α^4 are the roots of the equation $z^5 - 1 = 0$

$$\therefore z^5 - 1 = (z-1)(z-\alpha)(z-\alpha^2)(z-\alpha^3)(z-\alpha^4)$$
$$(z-1)(z^4+z^3+z^2+z+1) = (z-1)(z-\alpha)(z-\alpha^2)(z-\alpha^3)(z-\alpha^4)$$
$$\therefore z^4+z^3+z^2+z+1 = (z-\alpha)(z-\alpha^2)(z-\alpha^3)(z-\alpha^4)$$

But $z = 1$

$$1+1+1+1+1 = (1-\alpha)(1-\alpha^2)(1-\alpha^3)(1-\alpha^4)$$
$$\therefore (1-\alpha)(1-\alpha^2)(1-\alpha^3)(1-\alpha^4) = 5$$

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6) Use De-Moivre's theorem, solve $x^8+x^5+x^3+1=0$

$\Rightarrow$ Given,

$$x^8+x^5+x^3+1=0$$

$$x^5(x^3+1)+1(x^3+1)=0$$

$$(x^5+1)(x^3+1)=0$$

$$\Rightarrow x^5+1=0 \text{ or } x^3+1=0$$

1) Now,

$$x^5+1=0$$

$$x^5=-1=\cos\pi+i\sin\pi$$

$$=\cos(2n\pi+\pi)+i\sin(2n\pi+\pi)$$

$$x=\{\cos(2n\pi+\pi)+i\sin(2n\pi+\pi)\}^{1/5}$$

$$x=\cos\left(\frac{2n\pi+\pi}{5}\right)+i\sin\left(\frac{2n\pi+\pi}{5}\right) \quad \text{—— (1)}$$

putting $n=0,1,2,3,4$ we get the roots of $x^5+1=0$

For $n=0$, $x_0=\cos\left(\frac{\pi}{5}\right)+i\sin\left(\frac{\pi}{5}\right)$

For $n=1$, $x_1=\cos\left(\frac{3\pi}{5}\right)+i\sin\left(\frac{3\pi}{5}\right)$

For $n=2$, $x_2=\cos\left(\frac{5\pi}{5}\right)+i\sin\left(\frac{5\pi}{5}\right)$

For $n=3$, $x_3=\cos\left(\frac{7\pi}{5}\right)+i\sin\left(\frac{7\pi}{5}\right)$

For $n=4$, $x_4=\cos\left(\frac{9\pi}{5}\right)+i\sin\left(\frac{9\pi}{5}\right)$

2) And $x^3+1=0$

$$x^3=-1=\cos\pi+i\sin\pi$$

$$=\cos(2n\pi+\pi)+i\sin(2n\pi+\pi)$$

$$x=(-1)^{1/3}=\{\cos(2n\pi+\pi)+i\sin(2n\pi+\pi)\}^{1/3}$$

$$=\cos\left(\frac{2n\pi+\pi}{3}\right)+i\sin\left(\frac{2n\pi+\pi}{3}\right) \quad \text{—— (2)}$$

putting $n=0,1,2$ we get the roots of $x^3+1=0$

For $n=0$, $x_0 = \cos\left(\frac{\pi}{3}\right) + i\sin\left(\frac{\pi}{3}\right)$

For $n=1$, $x_1 = \cos\left(\frac{3\pi}{3}\right) + i\sin\left(\frac{3\pi}{3}\right) = \cos\pi + i\sin\pi$

For $n=2$, $x_2 = \cos\left(\frac{5\pi}{3}\right) + i\sin\left(\frac{5\pi}{3}\right)$

7) Solve the equation $x^3 = (x+1)^3$ and hence show that
$x = -\frac{1}{2} + \frac{i}{2}\cot(\theta/2)$; where $\theta = \frac{2n\pi}{3}$

$\Rightarrow$ Given $x^3 = (x+1)^3$

$$\frac{x^3}{(x+1)^3} = 1 = 1 + 0i$$
$$= \cos 0 + i\sin 0$$
$$= \cos(2n\pi) + i\sin(2n\pi)$$

$$\therefore \frac{x}{x+1} = \{\cos(2n\pi) + i\sin(2n\pi)\}^{1/3}$$

$$= \cos\left(\frac{2n\pi}{3}\right) + i\sin\left(\frac{2n\pi}{3}\right)$$

$$\therefore \frac{x}{x+1} = \frac{\cos\theta + i\sin\theta}{1}, \quad \text{where } \theta = \frac{2n\pi}{3}$$

Subtract x bothside in denomenator

$$\frac{x}{x+1-x} = \frac{\cos\theta + i\sin\theta}{1-(\cos\theta + i\sin\theta)}$$

$$x = \frac{\cos\theta + i\sin\theta}{1-\cos\theta - i\sin\theta}$$

$$= \frac{\cos\theta + i\sin\theta}{2\sin^2\theta/2 - 2i\sin\theta/2\cos\theta/2}$$

$$= \frac{\cos\theta + i\sin\theta}{2\sin\theta/2\,(\sin\theta/2 - i\cos\theta/2)}$$

$$= \frac{\cos\theta + i\sin\theta}{2\sin\theta/2\,(\sin\theta/2 - i\cos\theta/2)} \times \frac{(\sin\theta/2 + i\cos\theta/2)}{(\sin\theta/2 + i\cos\theta/2)}$$

$$= \frac{(\cos\theta + i\sin\theta)(\sin\theta/2 + i\cos\theta/2)}{2\sin\theta/2}$$

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$$= \frac{\cos\theta \sin\theta/2 + i\cos\theta\cos\theta/2 + i\sin\theta\sin\theta/2 - \sin\theta\cos\theta/2}{2\sin\theta/2}$$

$$= \frac{(\cos\theta\sin\theta/2 - \sin\theta\cos\theta/2) + i(\cos\theta\cos\theta/2 + \sin\theta\sin\theta/2)}{2\sin\theta/2}$$

$$= \frac{\cancel{\sin(\theta-\theta/2)}\ \sin(\theta/2-\theta) + i\cos(\theta-\theta/2)}{2\sin\theta/2}$$

$$= \frac{-\sin(\theta/2) + i\cos(\theta/2)}{2\sin\theta/2}$$

$$x = -\frac{1}{2} + \frac{i}{2}\cot(\theta/2)$$

8) Show that all the roots of $(x+1)^6 + (x-1)^6 = 0$ are given by $x = -i\cot\frac{(2K+1)\pi}{12}$, $K = 0,1,2,3,4,5$.

$\Rightarrow$ Given, $(x+1)^6 + (x-1)^6 = 0$

$$(x+1)^6 = -(x-1)^6$$

$$\frac{(x+1)^6}{(x-1)^6} = -1$$

$$= -1 + 0i$$

$$= \cos\pi + i\sin\pi$$

$$= \cos(2\pi K + \pi) + i\sin(2K\pi + \pi)$$

$$\frac{x+1}{x-1} = \{\cos(2K\pi+\pi) + i\sin(2K\pi+\pi)\}^{1/6}$$

$$\frac{x+1}{x-1} = \cos\left(\frac{2K\pi+\pi}{6}\right) + i\sin\left(\frac{2K\pi+\pi}{6}\right) \quad ; K = 0,1,2,3,4,5.$$

put $\theta = \frac{2K\pi+\pi}{6} = \frac{(2K+1)\pi}{6}$

$$\therefore \frac{x+1}{x-1} = \cos\theta + i\sin\theta$$

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$$\frac{(x+1)+(x-1)}{(x+1)-(x-1)} = \frac{\cos\theta + i\sin\theta + 1}{\cos\theta + i\sin\theta - 1}$$

$$\frac{2x}{2} = \frac{(1+\cos\theta)+i\sin\theta}{-(1-\cos\theta)+i\sin\theta}$$

$$x = \frac{2\cos^2\theta/2 + 2i\sin\theta/2\cos\theta/2}{-2\sin^2\theta/2 + 2i\sin\theta/2\cos\theta/2}$$

$$= \frac{2\cos\theta/2\,(\cos\theta/2 + i\sin\theta/2)}{2\sin\theta/2\,(-\sin\theta/2 + i\cos\theta/2)}$$

$$= \cot\theta/2\,\frac{(\cos\theta/2 + i\sin\theta/2)(-\sin\theta/2 - i\cos\theta/2)}{(-\sin\theta/2 + i\cos\theta/2)(-\sin\theta/2 - i\cos\theta/2)}$$

$$= \cot\theta/2 \left\{ \begin{matrix} -\sin\theta/2\cos\theta/2 - i\cos^2\theta/2 - i\sin^2\theta/2 \\ + \sin\theta/2\cos\theta/2 \end{matrix} \right\}$$

$$= \cot\theta/2 \left\{ \begin{matrix} (\sin\theta/2\cos\theta/2 - \sin\theta/2\cos\theta/2) \\ - i\,(\cos^2\theta/2 + \sin^2\theta/2) \end{matrix} \right\}$$

$$= \cot\theta/2\,\{0 - i(1)\}$$

$$x = -i\cot\theta/2$$

$$= -i\cot\left(\frac{(2\pi k+\pi)}{6} \Big/ 2\right)$$

$$= -i\cot\left(\frac{(2k+1)\pi}{12}\right)$$

9) Solve the equation: $x^7 + x^4 + i(x^3+1) = 0$

$\Rightarrow$ Given,

$$x^7 + x^4 + i(x^3+1) = 0$$

$$x^4(x^3+1) + i(x^3+1) = 0$$

$$(x^3+1)(x^4+i) = 0$$

$$\Rightarrow x^4 + i = 0 \text{ or } x^3 + 1 = 0$$

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$x^4 = -i$

$= \cos \pi/2 - i \sin \pi/2$

$= \cos\left(2k\pi + \frac{\pi}{2}\right) - i \sin\left(2k\pi + \frac{\pi}{2}\right)$

$x = \left\{\cos\left(2k\pi + \frac{\pi}{2}\right) - i \sin\left(2k\pi + \frac{\pi}{2}\right)\right\}^{1/4}$

$= \left\{\cos\left(\frac{4k\pi + \pi}{2}\right) - i \sin\left(\frac{4k\pi + \pi}{2}\right)\right\}^{1/4}$

$x = \cos \frac{(4k+1)\pi}{8} - i \sin \frac{(4k+1)\pi}{8}$ —— (1)

Putting $k = 0, 1, 2, 3$, we gets all roots of $(x^4 + i) = 0$

$k = 0$, $x_1 = \cos\left(\frac{\pi}{8}\right) - i \sin\left(\frac{\pi}{8}\right)$

$k = 1$, $x_2 = \cos\left(\frac{5\pi}{8}\right) - i \sin\left(\frac{5\pi}{8}\right)$

$k = 2$, $x_3 = \cos\left(\frac{9\pi}{8}\right) - i \sin\left(\frac{9\pi}{8}\right)$

$k = 3$, $x_4 = \cos\left(\frac{13\pi}{8}\right) - i \sin\left(\frac{13\pi}{8}\right)$

Also, $x^3 + 1 = 0$

$\Rightarrow x^3 = -1 = \cos \pi + i \sin \pi$

$x^3 = \cos(2k\pi + \pi) + i \sin(2k\pi + \pi)$

$x = \{\cos(2k\pi + \pi) + i \sin(2k\pi + \pi)\}^{1/3}$

$x = \cos\left(\frac{2k\pi + \pi}{3}\right) + i \sin\left(\frac{2k\pi + \pi}{3}\right)$

putting, $k = 0, 1, 2$ we get all roots of $x^3 + 1 = 0$

$k = 0$, $x_1 = \cos\left(\frac{\pi}{3}\right) + i \sin\left(\frac{\pi}{3}\right) = \frac{1}{2} + \frac{\sqrt{3}}{2} i$

$k = 1$, $x_2 = \cos \pi + i \sin \pi = -1 + i(0) = -1$

$k = 2$, $x_3 = \cos\left(\frac{5\pi}{3}\right) + i \sin\left(\frac{5\pi}{3}\right) = \frac{1}{2} - \frac{\sqrt{3}}{2} i$

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10) Solve the equation: $x^{10}+11x^5+10=0$

$\Rightarrow$ Given,

$$x^{10}+11x^5+10=0 \quad \text{——— (1)}$$

$$(x^5)^2+11x^5+10=0$$

put $x^5=y$

$$\therefore \; y^2+11y+10=0$$

$$\Rightarrow (y+10)(y+1)=0$$

$$\Rightarrow y+10=0 \text{ or } y+1=0$$

$$\Rightarrow y=-10,-1$$

Now,

$y=-10$

$$x^5=-10$$
$$=10(-1)$$
$$=10(\cos\pi+i\sin\pi)$$

$$x=\{10(\cos\pi+i\sin\pi)\}^{1/5}=\{10\cos(2k\pi+\pi)+i\sin(2k\pi+\pi)\}^{1/5}$$

$$x=10^{\frac{1}{5}}\left\{\cos\left(\frac{2k\pi+\pi}{5}\right)+i\sin\left(\frac{2k\pi+\pi}{5}\right)\right\}$$

putting $k=0,1,2,3,4$ we get all roots of $x^5+10=0$

$k=0$, $x_1=10^{1/5}\left\{\cos\left(\frac{\pi}{5}\right)+i\sin\left(\frac{\pi}{5}\right)\right\}$

$k=1$, $x_2=10^{1/5}\left\{\cos\left(\frac{3\pi}{5}\right)+i\sin\left(\frac{3\pi}{5}\right)\right\}$

$k=2$, $x_3=10^{1/5}\left\{\cos\left(\frac{5\pi}{5}\right)+i\sin\left(\frac{5\pi}{5}\right)\right\}=10^{1/5}(-1+i(0))=-10^{1/5}$

$k=3$, $x_4=10^{1/5}\left\{\cos\left(\frac{7\pi}{5}\right)+i\sin\left(\frac{7\pi}{5}\right)\right\}$

$k=4$, $x_5=10^{1/5}\left\{\cos\left(\frac{9\pi}{5}\right)+i\sin\left(\frac{9\pi}{5}\right)\right\}$

Also,

$y=-1$

$$x^5=-1=\cos\pi+i\sin\pi$$

$$x^5=\cos(2k\pi+\pi)+i\sin(2k\pi+\pi)$$

$$x = \left\{ \cos(2k\pi + \pi) + i\sin(2k\pi + \pi) \right\}^{1/5}$$

$$x = \cos\left(\frac{2k\pi + \pi}{5}\right) + i\sin\left(\frac{2k\pi + \pi}{5}\right)$$

Putting, $k = 0, 1, 2, 3, 4$ we get all roots of $x^5 + 1 = 0$

$k = 0$, $x_1 = \cos\left(\frac{\pi}{5}\right) + i\sin\left(\frac{\pi}{5}\right)$

$k = 1$, $x_2 = \cos\frac{3\pi}{5} + i\sin\frac{3\pi}{5}$

$k = 2$, $x_3 = \cos\left(\frac{5\pi}{5}\right) + i\sin\left(\frac{5\pi}{5}\right) = -1$

$k = 3$, $x_4 = \cos\left(\frac{7\pi}{5}\right) + i\sin\left(\frac{7\pi}{5}\right)$

$k = 4$, $x_5 = \cos\left(\frac{9\pi}{5}\right) + i\sin\left(\frac{9\pi}{5}\right)$

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Exercise - 04

1) Find all the value of $\left(\frac{1}{2}+i\frac{\sqrt{3}}{2}\right)^{3/4}$ and show that their continued product is unity.

2) Solve the equation: $x^6 - i = 0$

3) Solve the equation $z^3 = (z+1)^3$ then show that $z = -\frac{1}{2} + \frac{1}{2}\cot(\theta/2)$, where $\theta = 2n\pi/3$.

4) Solve the equation : $x^9 + x^5 - x^4 - 1 = 0$

5) Show that all the roots of $(x+1)^7 = (x-1)^7$ are given by $x = \pm i\cot\left(\frac{k\pi}{7}\right)$, $k = 1, 2, 3, \cdots$

6) Solve the equation : $x^6 + 1 = 0$

7) If ω is the 7^{th} root of unity, Prove that $S = 1 + w^n + w^{2n} + w^{3n} + w^{4n} + w^{5n} + w^{6n} = 7$, when n is a multiple of 7 and is equal to zero otherwise.

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★ Circular functions of complex number :-

We know that

$$e^{iy} = \cos y + i\sin y \quad \text{——} \ ①$$

$$e^{-iy} = \cos y - i\sin y \quad \text{——} \ ②$$

Adding and subtracting eqn ① & ② we get

$$\cos y = \frac{e^{iy} + e^{-iy}}{2}$$

$$\sin y = \frac{e^{iy} - e^{-iy}}{2i}$$

★ Hyperbolic functions of complex number :-

1) If z is a complex number, then sine hyperbolic z is denoted by $\sinh z$ and given by

$$\sinh z = \frac{e^{z} - e^{-z}}{2}$$

2) If z is a complex number, then cosine hyperbolic z is denoted by $\cosh z$ and given by

$$\cosh z = \frac{e^{z} + e^{-z}}{2}$$

★ Relation between circular and hyperbolic function :-

1) $\sin(ix) = i\sinh x$
2) $\cos(ix) = \cosh x$
3) $\tan(ix) = i\tan x$
4) $\sinh(ix) = i\sin x$
5) $\cosh(ix) = \cos x$
6) $\tanh(ix) = i\tan x$

★ Formula for Hyperbolic function :-

1) Fundamental formulae

a) $\cosh^2 x - \sinh^2 x = 1$

b) $\operatorname{sech}^2 x + \tanh^2 x = 1$

c) $\coth^2 x - \operatorname{cosech}^2 x = 1$

2) Addition or subtraction formulae:

1) $\sinh(x \pm y) = \sinh x \cdot \cosh y \pm \cosh x \sinh y$

2) $\cosh(x \pm y) = \cosh x \cosh y \pm \sinh x \sinh y$

3) $\tanh(x \pm y) = \dfrac{\tan x \pm \tan y}{1 \mp \tan x \cdot \tan y}$

3) Function of $2x$

1) $\sinh 2x = 2\sinh x \cdot \cosh x$

2) $\cosh 2x = \cosh^2 x + \sinh^2 x = 2\cosh^2 x - 1 = 1 + 2\sinh^2 x$

3) $\tanh 2x = \dfrac{2\tanh x}{1 + \tanh^2 x}$

4) Function of $3x$

1) $\sinh 3x = 3\sinh x + 4\sinh^3 x$

2) $\cosh 3x = 4\cosh^3 x - 3\cosh x$

3) $\tanh 3x = \dfrac{3\tanh x + \tanh^3 x}{1 + 3\tanh^2 x}$

5) Factorization formulae

1) $\sinh x + \sinh y = 2\sinh\left(\frac{x+y}{2}\right)\cosh\left(\frac{x-y}{2}\right)$

2) $\sinh x - \sinh y = 2\cosh\left(\frac{x+y}{2}\right)\sinh\left(\frac{x-y}{2}\right)$

3) $\cosh x + \cosh y = 2\cosh\left(\frac{x+y}{2}\right)\cosh\left(\frac{x-y}{2}\right)$

4) $\cosh x - \cosh y = 2\sinh\left(\frac{x+y}{2}\right)\sinh\left(\frac{x-y}{2}\right)$

1) If $\log(\tan x) = y$, Prove that

1) $\sinh y = \frac{1}{2}(\tan^n x - \cot^n x)$

2) $\cosh(n+1)y + \cosh(n-1)y = 2\cosh ny . \operatorname{cosec} 2x$.

$\Rightarrow$ Given,

$$\log_e(\tan x) = y$$

$$\Rightarrow \tan x = e^y$$

$$\Rightarrow \cot x = \frac{1}{\tan x} = \frac{1}{e^y} = e^{-y}$$

$\therefore\ e^y = \tan x$ & $e^{-y} = \cot x$

1) we know that

$$\sinh ny = \frac{e^{ny} - e^{-ny}}{2}$$

$$= \frac{1}{2}\left\{(e^y)^n - (e^{-y})^n\right\}$$

$$= \frac{1}{2}\left\{\tan^n x - \cot^n x\right\}$$

2) $\cosh(n+1)y + \cosh(n-1)y$

$$= 2\cosh\left(\frac{n+1+n-1}{2}\right)y . \sinh\left(\frac{n+1-n+1}{2}\right)y$$

$$= 2\cosh(ny) . \sinh y$$

$$= 2\cosh ny . \left(\frac{e^y + e^{-y}}{2}\right)$$

$$= \cosh ny\ (e^y + e^{-y})$$

$$= \cosh ny\ (\tan x + \cot x)$$

$$= \cosh ny\left(\frac{\sin x}{\cos x} + \frac{\cos x}{\sin x}\right)$$

$$= \cosh ny\left(\frac{\sin^2 x + \cos^2 x}{\sin x \cos x}\right) = \cosh ny\left(\frac{1}{\sin x \cos x}\right)$$

$$= 2\cosh ny\left(\frac{1}{2\sin x \cos x}\right) = 2\cosh ny . \frac{1}{\sin 2x}$$

$$= 2\cosh ny . \operatorname{cosec} 2x$$

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2) Solve the equation $7\cosh x + 8\sinh x = 1$ For real values.

$\Rightarrow$ Given

$$7\cosh x + 8\sinh x = 1$$

we know that,

$$\cosh x = \frac{e^x + e^{-x}}{2} \quad \& \quad \sinh x = \frac{e^x - e^{-x}}{2}$$

$$7\left(\frac{e^x + e^{-x}}{2}\right) + 8\left(\frac{e^x - e^{-x}}{2}\right) = 1$$

$$7(e^x + e^{-x}) + 8(e^x - e^{-x}) = 2$$

$$7e^x + 7e^{-x} + 8e^x - 8e^{-x} = 2$$

$$15e^x - e^{-x} = 2$$

$$15e^x - \frac{1}{e^x} = 2$$

$$15(e^x)^2 - 1 = 2e^x$$

$$15(e^x)^2 - 2e^x - 1 = 0$$

This is quadratic equation in e^x

Comparing with, $ax^2 + bx + c = 0$

$\Rightarrow a = 15,\ b = -2,\ c = -1$

$$\therefore e^x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$= \frac{2 \pm \sqrt{4 + 60}}{2(15)}$$

$$= \frac{2 \pm \sqrt{64}}{30}$$

$$= \frac{2 \pm 8}{30}$$

$$= \frac{10}{30} \text{ or } \frac{-6}{30}$$

$$e^x = \frac{1}{3} \text{ or } \frac{-1}{5}$$

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3) If $\tanh x = 2/3$. find the values of x and then $\cosh(2x)$.

$\Rightarrow$ Given,

$$\tanh x = \frac{2}{3}$$

$$\frac{e^x - e^{-x}}{e^x + e^{-x}} = \frac{2}{3}$$

$$\frac{(e^x - e^{-x}) + (e^x + e^{-x})}{(e^x + e^{-x}) - (e^x - e^{-x})} = \frac{2+3}{3-2}$$

$$\frac{2e^x}{2e^{-x}} = \frac{5}{1}$$

$$e^{2x} = 5 \quad \text{—————} \quad (1)$$

$$2x = \log_e 5$$

$$x = \frac{1}{2}\log_e 5.$$

Also from (1)

$$e^{-2x} = \frac{1}{e^{2x}} = \frac{1}{5}$$

$$\therefore \cosh 2x = \frac{e^{2x} + e^{-2x}}{2}$$

$$= \frac{5 + 1/5}{2}$$

$$= \frac{25+1}{10}$$

$$\cosh 2x = \frac{26}{10}$$

4) If $\tan x/2 = \tanh u/2$ then show that $u = \log \tan\left(\frac{\pi}{4} + \frac{x}{2}\right)$

$\Rightarrow$ Given,

$$\tanh u/2 = \tan x/2$$

$$\frac{e^{u/2} - e^{-u/2}}{e^{u/2} + e^{-u/2}} = \frac{\tan x/2}{1}$$

$$\frac{(e^{u/2} - e^{-u/2}) + (e^{u/2} + e^{-u/2})}{(e^{u/2} + e^{-u/2}) - (e^{u/2} - e^{-u/2})} = \frac{\tan x/2 + 1}{1 - \tan x/2}$$

$$\frac{2e^{u/2}}{2e^{-u/2}} = \frac{1 + \tan x/2}{1 - \tan x/2}$$

$$e^{(u/2 + u/2)} = \frac{\tan \pi/4 + \tan x/2}{1 - \tan\pi/4 \tan x/2}$$

$$e^{u} = \tan\left(\frac{\pi}{4} + \frac{x}{2}\right)$$

$$u = \log \tan\left(\frac{\pi}{4} + \frac{x}{2}\right)$$

* Real and Imaginary parts of Circular and hyperbolic Functions:

1) To saparate the real and imaginary parts of:
1) $\sin(x+iy)$ 2) $\cos(x+iy)$ 3) $\tan(x+iy)$

1) $\sin(x+iy) = \sin x \cos iy + \cos x \sin iy$
$= \sin x \cosh y + \cos x \, (i \sinh y)$
$\sin(x+iy) = \sin x \cosh y + i \cos x \cosh y$
$\therefore$ Real part $= \sin x \cosh y$
& Imaginary part $= \cos x \cosh y$

2) $\cos(x+iy) = \cos x \cos(iy) - \sin x \sin(iy)$
$= \cos x \cosh y - \sin x \, (i \sinh y)$
$\cos(x+iy) = \cos x \cosh y - i \sin x \sinh y$
Real part $= \cos x \cosh y$
Imaginary part $= -\sin x \sinh y$

3) Let $\alpha + i\beta = \tan(x+iy)$ —— ①
$\alpha - i\beta = \tan(x-iy)$ —— ②
Adding ① & ②
$2\alpha = \tan(x+iy) + \tan(x-iy)$

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$$= \frac{\sin(x+iy)}{\cos(x+iy)} + \frac{\sin(x-iy)}{\cos(x-iy)}$$

$$= \frac{\sin(x+iy)\cos(x-iy) + \cos(x+iy)\sin(x-iy)}{\cos(x+iy)\cos(x-iy)}$$

$$= \frac{\sin\{(x+iy)+(x-iy)\}}{\cos(x+iy)\cos(x-iy)}$$

$$= \frac{\sin 2x}{\cos(x+iy)\cos(x-iy)}$$

$$= \frac{2\sin 2x}{2\cos(x+iy)\cos(x-iy)}$$

$$= \frac{2\sin 2x}{\cos 2x + \cos(2iy)}$$

$$2\alpha = \frac{2\sin 2x}{\cos 2x + \cosh y}.$$

$$\therefore \alpha = \text{Real part} = \frac{\sin 2x}{\cos 2x + \cosh y}$$

Subtracting ① \$ ②

$$2i\beta = \tan(x+iy) - \tan(x-iy)$$

$$= \frac{\sin(x+iy)}{\cos(x+iy)} - \frac{\sin(x-iy)}{\cos(x-iy)}$$

$$= \frac{\sin(x+iy)\cos(x-iy) - \sin(x-iy)\cos(x+iy)}{\cos(x+iy)\cos(x-iy)}$$

$$= \frac{\sin\{(x+iy)-(x-iy)\}}{\cos(x+iy)\cos(x-iy)}$$

$$= \frac{2\sin 2iy}{2\cos(x+iy)\cos(x-iy)}$$

$$= \frac{2i\sinh y}{\cos 2x + \cos(2iy)}$$

$$2\beta = \frac{2i\sinh y}{\cos 2x + \cosh y}$$

$$\beta = i\frac{\sinh y}{\cos 2x + \cosh y}$$

$$\beta = \text{Imaginary part} = \frac{\sinh y}{\cos 2x + \cosh y}$$

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2) To saparate real and imaginary parts of

1) $\sinh(x+iy)$ 2) $\cosh(x+iy)$ 3) $\tanh(x+iy)$

1) We know that

$$\sin(ix) = i\sinh x$$

$$\sinh x = \frac{1}{i}\sin(ix)$$

Replace x by $x+iy$

$$\sinh(x+iy) = \frac{1}{i}\sin\{i(x+iy)\}$$

$$= \frac{1}{i}\sin(ix-y)$$

$$= -i\sin(xi-y)$$

$$= -i(\sin(ix)\cos y - \cos(ix)\sin y)$$

$$= -i(i\sinh x\cos y - \cosh x\sin y)$$

$$= -i^2\sinh x\cos y + i\cosh x\sin y$$

$$\sin(x+iy) = \sinh x\cos y + i\cosh x\sin y$$

Real part $= \sinh x\cos y$

Imaginary part $= \cosh x\sin y$.

2) We know that,

$$\cos(ix) = \cosh x$$

$$\cosh x = \cos(ix)$$

Replace by x to $x+iy$

$$\cosh(x+iy) = \cos\{i(x+iy)\}$$

$$= \cos(ix-y)$$

$$= \cos(ix)\cos y + \sin(ix)\sin y$$

$$\cosh(x+iy) = \cosh y\cos y + i\sinh x\sin y$$

Real part $= \cosh y\cos y$

Imaginary part $= \sinh x\sin y$.

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3) We know that

$$\tan(ix) = i\tanh x$$

$$\tanh x = \frac{1}{i}\tan(ix) = -i\tan(ix)$$

Replace x by $(x+iy)$

$$\tanh(x+iy) = -i\tan\{i(x+iy)\}$$

Let $\alpha + i\beta = \tan(x+iy) = -i\tan(ix-y)$ — ① then

$$\alpha - i\beta = \tan(x-iy) = -(-i)\tan(-ix-y)$$

$$\alpha - i\beta = -i\tan(ix+y) \quad \text{— ②}$$

Adding ① & ②

$$2\alpha = -i\{\tan(ix-y) + \tan(ix+y)\}$$

$$= -i\left\{\frac{\sin(ix-y)}{\cos(ix-y)} + \frac{\sin(ix+y)}{\cos(ix+y)}\right\}$$

$$= -i\left\{\frac{\sin(ix-y)\cos(ix-y) + \sin(ix+y)\cos(ix-y)}{\cos(ix-y)\cos(ix+y)}\right\}$$

$$= -i\left\{\frac{\sin\{(ix-y)+(ix+y)\}}{\cos(ix-y)\cos(ix+y)}\right\}$$

$$\alpha = \frac{-i\sin(2ix)}{2\cos(ix-y)\cos(ix+y)}$$

$$= \frac{-i\cdot i\sinh 2x}{\cos(2ix) + \cos 2y}$$

$$= \frac{\sinh 2x}{\cosh 2x + \cos 2y}$$

$\therefore$ Real part $= \alpha = \dfrac{\sinh 2x}{\cosh 2x + \cos 2y}$

Subtracting ① & ②

$$2i\beta = -i\tan(ix-y) + i\tan(ix+y)$$

$$= i[\tan(ix+y) - \tan(ix-y)]$$

$$= i\left\{\frac{\sin(ix+y)}{\cos(ix+y)} - \frac{\sin(ix-y)}{\cos(ix-y)}\right\}$$

$$= i\left\{\frac{\sin(ix+y)\cos(ix-y) - \sin(ix-y)\cos(ix+y)}{\cos(ix+y)\cos(ix-y)}\right\}$$

$$= i\left\{ \frac{\sin[(ix+y)-(ix-y)]}{\cos(ix+y)\cos(ix-y)} \right\}$$

$$\beta = \frac{i \sin 2y}{2i\cos(ix+y)\cos(ix-y)} \}$$

$$= \frac{\sin 2y}{\cos(2ix) + \cos 2y}$$

$$\beta = \frac{\sin 2y}{\cosh 2x + \cos 2y}$$

$\therefore$ Imaginary part $= \dfrac{\sin 2y}{\cosh 2x + \cos 2y}$

3) If $\cosh(u+iv) = x+iy$. prove that

1) $\dfrac{x^2}{\cosh^2 u} + \dfrac{y^2}{\sinh^2 u} = 1$ 2) $\dfrac{x^2}{\cos^2 v} - \dfrac{y^2}{\sin^2 v} = 1$

$\Rightarrow$ Given that

$$\cosh(u+iv) = x+iy$$

But $\cos ix = \cosh x$

$$\therefore \cos i(u+iv) = \cosh(u+iv)$$

$$\therefore \cos i(u+iv) = x+iy$$

$$\cos(iu-v) = x+iy$$

$$\cos iu \cos v + \sin iu \cdot \sin v = x+iy$$

$$\cosh u \cos v + i\sinh u \sin v = x+iy$$

$\Rightarrow$ $x = \cosh u \cos v$ —— ①

$y = \sinh u \sin v$ —— ②

From ①, $\dfrac{x}{\cosh u} = \cos v$ $\therefore \dfrac{x^2}{\cosh^2 u} = \cos^2 v$

From ② $\dfrac{y}{\sinh u} = \sin v$ $\therefore \dfrac{y^2}{\sinh^2 u} = \sin^2 v$

Adding, we get

$$\frac{x^2}{\cosh^2 u} + \frac{y^2}{\sinh^2 u} = \cos^2 v + \sin^2 v$$

$$= 1$$

From 1) $\frac{x}{\cos v} = \cosh u \quad \therefore \frac{x^2}{\cos^2 v} = \cosh^2 u$

From 2) $\frac{y}{\sin v} = \sinh u \quad \therefore \frac{y^2}{\sin^2 v} = \sinh^2 u$

Subtracting, we get

$$\frac{x^2}{\cos^2 v} - \frac{y^2}{\sin^2 v} = \cosh^2 u - \sinh^2 u = 1$$

4) If $\sin(\theta + i\phi) = \cos\alpha + i\sin\alpha$, Prove that $\cos^2\theta = \pm\sin\alpha$

Given,

$$\sin(\theta + i\phi) = \cos\alpha + i\sin\alpha$$

$$\sin\theta \cos i\phi + \cos\theta \sin i\phi = \cos\alpha + i\sin\alpha$$

$$\sin\theta \cosh\phi + i\cos\theta \sinh\phi = \cos\alpha + i\sin\alpha$$

$\Rightarrow \sin\theta \cosh\phi = \cos\alpha$

$\cos\theta \sinh\phi = \sin\alpha$

$\Rightarrow \cosh\phi = \frac{\cos\alpha}{\sin\theta}$ & $\sinh\phi = \frac{\sin\alpha}{\cos\theta}$

Now, $\cosh^2\phi - \sinh^2\phi = \frac{\cos^2\alpha}{\sin^2\theta} - \frac{\sin^2\alpha}{\cos^2\theta}$

$$1 = \frac{\cos^2\alpha \cos^2\theta - \sin^2\alpha \sin^2\theta}{\sin^2\theta \cos^2\theta}$$

$$\sin^2\theta \cos^2\theta = \cos^2\alpha \cos^2\theta - \sin^2\alpha \sin^2\theta$$

$$(1 - \cos^2\theta)\cos^2\theta = \cos^2\alpha \cos^2\theta - \sin^2\alpha (1 - \cos^2\theta)$$

$$\cos^2\theta - \cos^4\theta = \cos^2\alpha \cos^2\theta - \sin^2\alpha + \sin^2\alpha \cos^2\theta$$

$$\cos^4\theta + \cos^2\alpha \cos^2\theta - \sin^2\alpha + \sin^2\alpha \cos^2\theta - \cos^2\theta = 0$$

$$\cos^4\theta - \sin^2\alpha + \cos^2\theta(\cos^2\alpha + \sin^2\alpha - 1) = 0$$

$$\cos^4\theta - \sin^2\alpha + \cos^2\theta(1 - 1) = 0$$

$$\cos^4\theta - \sin^2\alpha = 0$$

$$\cos^4\theta = \sin^2\alpha$$

$$\therefore \underline{\cos^2\theta = \pm\sin\alpha}$$

5) If $\cos(\theta + i\phi) = Re^{i\alpha}$, show that $\phi = \frac{1}{2}\log\frac{\sin(\theta-\alpha)}{\sin(\theta+\alpha)}$

$\Rightarrow$ Given,

$$\cos(\theta + i\phi) = Re^{i\alpha}$$

$$\cos\theta \cos i\phi - \sin\theta \sin i\phi = R(\cos\alpha + i\sin\alpha)$$

$$\cos\theta \cosh\phi - i\sin\theta \sinh\phi = R\cos\alpha + iR\sin\alpha$$

$$\Rightarrow \quad R\cos\alpha = \cos\theta \cosh\phi$$

$$R\sin\alpha = -\sin\theta \sinh\phi$$

Now,

$$\frac{R\sin\alpha}{R\cos\alpha} = \frac{-\sin\theta \sinh\phi}{\cos\theta \cosh\phi}$$

$$\tan\alpha = -\tan\theta \tanh\phi$$

$$\therefore \tanh\phi = -\frac{\tan\alpha}{\tan\theta}$$

$$\frac{e^{\phi} + e^{-\phi}}{e^{\phi} - e^{-\phi}} = -\frac{\sin\alpha\cos\theta}{\cos\alpha\sin\theta}$$

$$\Rightarrow \frac{(e^{\phi} + e^{-\phi}) + (e^{\phi} - e^{-\phi})}{(e^{\phi} + e^{-\phi}) - (e^{\phi} - e^{-\phi})} = \frac{-\sin\alpha\cos\theta + \cos\alpha\sin\theta}{\cos\alpha\sin\theta + \sin\alpha\cos\theta}$$

$$\frac{2e^{\phi}}{2e^{-\phi}} = \frac{\sin(\theta-\alpha)}{\sin(\theta+\alpha)}$$

$$e^{\phi} \cdot e^{\phi} = \frac{\sin(\theta-\alpha)}{\sin(\theta+\alpha)}$$

$$e^{2\phi} = \frac{\sin(\theta-\alpha)}{\sin(\theta+\alpha)}$$

$$2\phi = \log\frac{\sin(\theta+\alpha)}{\sin(\theta+\alpha)}$$

$$\therefore \phi = \frac{1}{2}\log\frac{\sin(\theta-\alpha)}{\sin(\theta+\alpha)}$$

Exercise - 05

1) Prove that $\left(\frac{1+\tanh x}{1-\tanh x}\right)^3 = \cosh 6x + \sinh 6x$

2) If $u = \log \tan\left(\frac{\pi}{4}+\frac{\theta}{2}\right)$ Prove that

1) $\cosh u = \sec\theta$ 2) $\sinh u = \tan\theta$
3) $\tanh u = \sin\theta$ 4) $\tan(u/2) = \tan(\theta/2)$

3) If $\cos(x+iy)\cos(u+iv) = 1$, where x, y, u & v are real, show that: $\tanh^2 y \cosh^2 v = \sin^2 u$.

4) To separate Real and Imaginary parts of:
1) $\cot(x+iy)$ 2) $\sec(x+iy)$ 3) $\operatorname{cosec}(x+iy)$

5) If $\sin(A+iB) = x+iy$ then prove that:
1) $\frac{x^2}{\cosh^2 B} + \frac{y^2}{\sinh^2 B} = 1$ 2) $\frac{x^2}{\sin^2 A} - \frac{y^2}{\cos^2 A} = 1$

6) If $\tan(A+iB) = x+iy$, Prove that
1) $\tan 2A = \frac{2x}{1-x^2-y^2}$ 2) $\tanh 2B = \frac{2y}{1+x^2+y^2}$

7) If $\tan(\theta+i\phi) = \cos\alpha + i\sin\alpha$, Prove that
1) $\theta = \frac{n\pi}{2}+\frac{\pi}{4}$ 2) $\phi = \frac{1}{2}\log\tan\left(\frac{\pi}{4}+\frac{\alpha}{2}\right)$

8) Separate $\sin^{-1}(\cos\theta + i\sin\theta)$ into real and imaginary parts, where θ is possitive acute angle.

* Logarithms of complex quantities:

Let $z = \log(x+iy)$

where $x = r\cos\theta$, $y = r\sin\theta$

and $r = \sqrt{x^2+y^2}$, $\theta = \tan^{-1}(y/x)$

$$\therefore z = \log(r\cos\theta + ir\sin\theta)$$
$$= \log r(\cos\theta + i\sin\theta)$$
$$= \log r e^{i\theta}$$
$$= \log r + \log e^{i\theta}$$
$$= \log r + i\theta$$
$$z = \log\sqrt{x^2+y^2} + i\tan^{-1}(y/x)$$

This is known as principal value of $\log(x+iy)$

The general value of $\log(x+iy)$ is given by

$$\text{Log}(x+iy) = \log r + i(\theta + 2n\pi)$$

1) Find the general value of: $\log(-3)$

$\Rightarrow$ We know that

$$-3 = 3(-1)$$
$$= 3(\cos\pi + i\sin\pi)$$
$$-3 = 3e^{i\pi}$$
$$\Rightarrow \log(-3) = \log(3e^{(\pi i)})$$
$$= \log 3 + \log e^{\pi i}$$
$$= \log 3 + i\pi$$
$$= \log 3 + (2n\pi i + \pi i)$$
$$\log(-3) = \log 3 + (2n+1)\pi i$$

2) Prove that $\log(1+e^{2i\theta}) = \log(2\cos\theta) + i\theta$

$$\Rightarrow \log(1+e^{2i\theta}) = \log(1+\cos 2\theta + i\sin 2\theta)$$
$$= \log(1 + 2\cos^2\theta - 1 + 2i\sin\theta\cos\theta)$$
$$= \log(2\cos^2\theta + 2i\sin\theta\cos\theta)$$
$$= \log\{2\cos\theta(\cos\theta + i\sin\theta)\}$$
$$= \log\{2\cos\theta\, e^{i\theta}\}$$
$$= \log(2\cos\theta) + \log e^{i\theta}$$
$$\log(1+e^{2i\theta}) = \log(2\cos\theta) + i\theta$$

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3) Prove that $\log(e^{i\alpha}+e^{i\beta}) = \log\left(2\cos\frac{\alpha-\beta}{2}\right) + i\left(\frac{\alpha+\beta}{2}\right)$

$$\Rightarrow \log(e^{i\alpha}+e^{i\beta}) = \log\{(\cos\alpha+i\sin\alpha)+(\cos\beta+i\sin\beta)\}$$
$$= \log\{(\cos\alpha+\cos\beta)+i\sin(\alpha)+i\sin\beta\}$$
$$= \log\left\{2\cos\frac{\alpha+\beta}{2}\cos\frac{\alpha-\beta}{2} + 2i\sin\frac{\alpha+\beta}{2}\cos\frac{\alpha-\beta}{2}\right\}$$
$$= \log\left\{2\cos\frac{\alpha-\beta}{2}\left(\cos\frac{\alpha+\beta}{2}+i\sin\frac{\alpha+\beta}{2}\right)\right\}$$
$$= \log\left\{2\cos\frac{\alpha-\beta}{2}\cdot e^{i\left(\frac{\alpha+\beta}{2}\right)}\right\}$$
$$= \log\left(2\cos\frac{\alpha-\beta}{2}\right) + \log e^{i\left(\frac{\alpha+\beta}{2}\right)}$$
$$= \log\left(2\cos\frac{\alpha-\beta}{2}\right) + i\left(\frac{\alpha+\beta}{2}\right)$$

4) Prove that $\tan\left\{i\log\left(\frac{a-ib}{a+ib}\right)\right\} = \frac{2ab}{a^2-b^2}$

We know that,

$$\log(a+ib) = \log\sqrt{a^2+b^2} + i\tan^{-1}(b/a)$$
$$\& \quad \log(a-ib) = \log\sqrt{a^2+b^2} - i\tan^{-1}(b/a)$$

$$\therefore \log(a+ib)-\log(a-ib) = 2i\tan^{-1}(b/a)$$

$$-\log\left(\frac{a-ib}{a+ib}\right) = 2i\tan^{-1}(b/a)$$

$$i\log\left(\frac{a-ib}{a+ib}\right) = 2\tan^{-1}(b/a)$$

$$\Rightarrow \tan\left\{i\log\left(\frac{a-ib}{a+ib}\right)\right\} = \tan\{2\tan^{-1}(b/a)\}$$

$$= \tan 2\theta, \quad \text{where } \theta = \tan^{-1}(b/a)$$
$$= \frac{2\tan\theta}{1-\tan^2\theta} = \frac{2(b/a)}{1-b^2/a^2}$$

$$\tan\left(i\log\frac{a-ib}{a+ib}\right) = \frac{2ab}{a^2-b^2}$$

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5) If $\log\log(x+iy) = p+iq$ then prove that
$y = x\tan[\tan q \log\sqrt{x^2+y^2}]$.

$\Rightarrow$ Given

$$\log\log(x+iy) = p+iq$$
$$\log(x+iy) = e^{p+iq}$$
$$= e^{p}\cdot e^{iq}$$
$$\log(x+iy) = e^{p}(\cos q + i\sin q)$$

But $\log(x+iy) = \log\sqrt{x^2+y^2} + i\tan^{-1}(y/x)$

$$\therefore \log\sqrt{x^2+y^2} + i\tan^{-1}(y/x) = e^{p}(\cos q + i\sin q)$$
$$= e^{p}\cos q + ie^{p}\sin q.$$

$\Rightarrow$ $e^{p}\cos q = \log\sqrt{x^2+y^2}$

& $e^{p}\sin q = \tan^{-1}(y/x)$

$$\therefore \frac{e^{p}\sin q}{e^{p}\cos q} = \frac{\tan^{-1}(y/x)}{\log\sqrt{x^2+y^2}}$$

$$\tan q = \frac{\tan^{-1}(y/x)}{\log\sqrt{x^2+y^2}}$$

$$\tan q \cdot \log\sqrt{x^2+y^2} = \tan^{-1}(y/x)$$

$$y/x = \tan\{\tan q \cdot \log\sqrt{x^2+y^2}\}$$

$$y = x\tan\{\tan q \cdot \log\sqrt{x^2+y^2}\}$$

6) If $i^{i^{\cdots\infty}} = A+iB$, prove that $\tan\frac{\pi A}{2} = \frac{B}{A}$ & $A^2+B^2 = e^{-\pi B}$

$\Rightarrow$ Given $i^{i^{\cdots\infty}} = A+iB$

$\Rightarrow$ $i^{A+iB} = A+iB$

$$A+iB = e^{(A+iB)\log i}$$
$$= e^{(A+iB)\log(\cos\pi/2 + i\sin\pi/2)}$$
$$= e^{(A+iB)\cdot\log e^{i\pi/2}}$$
$$= e^{(A+iB)i\pi/2}$$
$$= e^{Ai\pi/2}\cdot e^{-B\pi/2}$$

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$$A+iB = e^{-B\pi/2}(\cos \pi A/2 + i \sin \pi A/2)$$

Equating real & imaginary parts, we get

$$A = e^{-B\pi/2} \cos \frac{\pi A}{2} \quad \text{——} \quad (1)$$

$$B = e^{-B\pi/2} \sin \frac{\pi A}{2} \quad \text{——} \quad (2)$$

1) Dividing (2) & (1)

$$\frac{B}{A} = \frac{\sin \pi A/2}{\cos \pi A/2}$$

$$\frac{B}{A} = \tan \frac{\pi A}{2}$$

2) Squaring & adding eqn (1) & (2)

$$A^2+B^2 = (e^{-\pi B/2})^2 \{\cos^2 \pi A/2 + \sin^2 \pi A/2\}$$

$$= e^{-\pi B} (1)$$

$$\therefore \quad A^2+B^2 = e^{-\pi B}$$

7) Separate into real and imaginary parts $\log \sin(x+iy)$

$\Rightarrow \log \sin(x+iy) = \log(\sin x \cos iy + \cos x \sin iy)$

$\log r(\cos\theta + i\sin\theta) = \log(\sin x \cosh y + i \cos x \sinh y)$

where,

$r\cos\theta = \sin x \cosh y$

& $r \sin\theta = \cos x \sinh y$

where

$$r = \sqrt{\sin^2 x \cosh^2 y + \cos^2 x \sinh^2 y}$$

$$= \sqrt{\frac{1-\cos 2x}{2} \cdot \frac{1+\cosh 2y}{2} \cdot \frac{1+\cos 2x}{2} \cdot \frac{\cosh 2y - 1}{2}}$$

$$= \frac{1}{2}\sqrt{1+\cosh 2y - \cos 2x - \cos 2x \cosh 2y + \cosh 2y - 1 + \cos 2x \cosh 2y - \cos 2}$$

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$$= \frac{1}{2}\sqrt{2\cosh 2y - 2\cos 2x}$$

$$r = \sqrt{\frac{\cosh 2y - \cos 2x}{2}}$$

and $\theta = \tan^{-1}(\cot x \tanh y)$

$$\therefore \log\sin(x+iy) = \log(re^{i\theta})$$

$$= \log r + i\theta$$

$$= \log\sqrt{\frac{\cosh 2y - \cos 2x}{2}} + i\tan^{-1}(\cot x \tanh y)$$

$$= \frac{1}{2}\log\left(\frac{\cosh 2y - \cos 2x}{2}\right) + i\tan^{-1}(\cot x \cdot \tanh y)$$

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Exercise :- 06

1) Find the General values of : 1) $\log(-i)$ 2) $\log(4+3i)$

2) Prove that : $\frac{1}{1-e^{i\theta}} = \log\left(\frac{1}{2}\operatorname{cosec}\left(\frac{\theta}{2}\right)\right) + i\left(\frac{\pi}{2} - \frac{\theta}{2}\right)$

3) Prove that : $\log(-\log i) = \log \pi/_2 - i\pi/_2$

4) If $p\log(a+ib) = (x+iy)\log m$, prove that

$$\frac{y}{x} = \frac{2\tan^{-1}(b/a)}{\log(a^2+b^2)}$$

5) Prove that : 1) $i^i = e^{-(4n+1)\pi/2}$ & $\log i^i = -(2n+1/2)\pi$

2) $(\sqrt{i})^{\sqrt{i}} = e^{-\alpha}\cos\alpha$ where $\alpha = \frac{\pi}{4\sqrt{2}}$

6) Prove that : $\tan\log(x^2+y^2) = \frac{2a}{1-a^2-b^2}$

where $\tan\log(x+iy) = a+ib$, where $a^2+b^2 \neq 1$