Unit - V

Vector Integral Calculus :-

* Line Integral :-

The integral of the tangential component of vector point function $\bar{F}$ along the curve C between some fixed points A & B is denoted by $\int_{CAB} \bar{F}\cdot\hat{T}\,ds$, where ds is an arc element. This integral is called the line integral of $\bar{F}$ along the curve C between A & B.

* Work done by $\bar{F}$:-

Total work done in moving the particle along C from A to B given by

$$\text{Work done} = \int_A^B \bar{F}\cdot d\bar{r}$$

Note :-

① In evaluation of line integrals we have to express the line integrals $\int_c \bar{F}\cdot d\bar{r}$ in ~~the~~ terms of one variable either x or y or z or t or θ.

② In polar coordinates :

a) Integral over circle $x^2+y^2=a^2$

put $x = a\cos\theta \quad \therefore dx = -a\sin\theta\, d\theta$

$y = a\sin\theta \quad \therefore dy = a\cos\theta\, d\theta$

Limits of θ are θ = 0 to 2π

b) Integral over ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$

put $x = a\cos\theta \quad \therefore dx = -a\sin\theta\, d\theta$

$y = b\sin\theta \quad \therefore dy = b\cos\theta\, d\theta$

Limits of θ are θ = 0 to 2π.

c) Equation of line joining ~~A(x₁,y₁,z₁)~~ . A(x_1,y_1,z_1) to B(x_2,y_2,z_2)

$$\frac{x-x_1}{x_2-x_1} = \frac{y-y_1}{y_2-y_1} = \frac{z-z_1}{z_2-z_1}$$

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Ex. 1 If $\bar{F} = x^2\hat{i} + (x-y)\hat{j} + (y+z)\hat{k}$ displaces a particle from A(1,0,1) to B(2,1,2) along the straight line AB. Find the work done.

$\Rightarrow$ Given,

$$\bar{F} = x^2\hat{i} + (x-y)\hat{j} + (y+z)\hat{k}$$

Also, $d\bar{r} = \hat{i}dx + \hat{j}dy + \hat{k}dz$

$$\therefore \bar{F}\cdot d\bar{r} = x^2dx + (x-y)dy + (y+z)dz$$

The equation of line joining A(1,0,1) to B(2,1,2) is

$$\frac{x-1}{2-1} = \frac{y-0}{1-0} = \frac{z-1}{2-1}$$

$$\frac{x-1}{+1} = \frac{y}{+1} = \frac{z-1}{1} = t \text{ (say)}$$

$$\therefore x = t+1, \quad y = t, \quad z = t+1$$

$$\therefore dx = dt, \quad dy = dt, \quad dz = dt$$

limits are from $t=0$ to $t=1$

$$\text{Work done} = \int_0^1 \bar{F}\cdot d\bar{r}$$

$$= \int_0^1 (x^2dx + (x-y)dy + (y+z)dz)$$

$$= \int_0^1 \{(1+t)^2dt + (t+1-t)dt + (t+t+1)dt\}$$

$$= \int_0^1 \{1+2t+t^2+1+2t+1\}dt$$

$$= \int_0^1 (t^2+4t+3)dt$$

$$= \left(\frac{t^3}{3} + \frac{4t^2}{2} + 3t\right)_0^1$$

$$= \frac{1}{3} + 2 + 3(1) - 0$$

$$= \frac{1}{3} + 5 = \frac{16}{3} \text{ units}$$

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Ex. 2 If $\bar{F} = (2x+y^2)\hat{i} + (3y-4x)\hat{j}$ then evaluate $\int_C \bar{F}\cdot d\bar{r}$ around the parabolic arc $y^2 = x$ joining $(0,0)$ to $(1,1)$

⇒ Given,

$$\bar{F} = (2x+y^2)\hat{i} + (3y-4x)\hat{j} + 0\hat{k}$$

& $$d\bar{r} = \hat{i}\,dx + \hat{j}\,dy + \hat{k}\,dz$$

$$\therefore \bar{F}\cdot d\bar{r} = (2x+y^2)\,dx + (3y-4x)\,dy$$

But $y^2 = x$

$$\therefore dx = 2y\,dy$$

∴ The limits of y are 0 & 1

$$\text{work done} = \int_0^1 \{(2x+y^2)\,dx + (3y-4x)\,dy\}$$

$$= \int_0^1 \{(2(y^2)+y^2)\,2y\,dy + (3y-4y^2)\,dy\}$$

$$= \int_0^1 \{(2y^2+y^2)\,2y\,dy + (3y-4y^2)\,dy\}$$

$$= \int_0^1 \{4y^3 + 2y^3 + 3y - 4y^2\}\,dy$$

$$= \int_0^1 (6y^3 - 4y^2 + 3y)\,dy$$

$$= \left\{ \frac{6y^4}{4} - \frac{4y^3}{3} + \frac{3y^2}{2} \right\}_0^1$$

$$= \frac{6}{4} - \frac{4}{3} + \frac{3}{2}$$

$$= \frac{12}{4} - \frac{4}{3}$$

$$= 3 - \frac{4}{3}$$

$$= \frac{9-4}{3}$$

$$= \frac{5}{3} \text{ units}$$

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Ex. 3 Find the total work done in moving a particle in the force field given by $\bar{F} = 3xy\hat{i} - 5z\hat{j} + 10x\hat{k}$ along the curve $x = t^2+1$, $y = 2t^2$, $z = t^3$ from $t=1$ to $t=2$

$\Rightarrow$ Given,

$$\bar{F} = 3xy\hat{i} - 5z\hat{j} + 10x\hat{k}$$

$$d\bar{r} = \hat{i}dx + \hat{j}dy + \hat{k}dz$$

$$\therefore \bar{F}\cdot d\bar{r} = 3xy\,dx - 5z\,dy + 10x\,dz$$

But,

$$x = t^2+1, \quad y = 2t^2, \quad z = t^3$$

$$dx = 2t\,dt \quad dy = 4t\,dt \quad dz = 3t^2\,dt$$

Limits are $t = 01$ to $t = 2$

$$\therefore \text{work done} = \int_1^2 (3xy\,dx - 5z\,dy + 10x\,dz)$$

$$= \int_1^2 \left\{\left[3(t^2+1)2t^2\cdot 2t\,dt\right] - 5t^3\cdot 4t\,dt + 10(t^2+1)\cdot 3t^2\,dt\right\}$$

$$= \int_1^2 \left\{12t^5 + 12t^3 - 20t^4 + 30t^4 + 30t^2\right\}dt$$

$$= \int_1^2 \left\{12t^5 + 10t^4 + 12t^3 + 30t^2\right\}dt$$

$$= \left\{\frac{12t^6}{6} + \frac{10t^5}{5} + \frac{12t^4}{4} + \frac{30t^3}{3}\right\}_1^2$$

$$= \left\{2(2)^6 + 2(2)^5 + 3(2)^4 + 10(2)^3\right\} - \left\{2(1)^6 + 2(1)^5 + 3(1)^4 + 10(1)^3\right.$$

$$= \left\{128 + 64 + 48 + 80\right\} - \left\{2 + 2 + 3 + 10\right\}$$

$$= \cancel{192 - 13} \qquad 320 - 17$$

$$= \cancel{179} \qquad = 303$$

$$= 303 \text{ units}$$

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Ex. 4 Find work done in moving a particle once around ellipse $\frac{x^2}{16}+\frac{y^2}{9}=1$ in plane $z=0$. where $\bar{F}=(3x-2y)\hat{i}+(2z+8y)\hat{j}+y^2\hat{k}$

$\Rightarrow$ Given

$$\bar{F}=(3x-2y)\hat{i}+(2x+8y)\hat{j}+y^2\hat{k}$$

$$d\bar{r}=\hat{i}dx+\hat{j}dy+\hat{k}dz$$

$$\therefore \bar{F}\cdot d\bar{r}=(3x-2y)dx+(2x+8y)dy+y^2dz$$

Given ellipse is $\frac{x^2}{16}+\frac{y^2}{9}=1$ ie $\frac{x^2}{4^2}+\frac{y^2}{3^2}=1$

$\therefore a=4$ & $b=3$

put $x=a\cos\theta$, $y=b\sin\theta$, $z=0$

$x=4\cos\theta$ $\quad y=3\sin\theta$ $\quad \therefore dz=0$

$\therefore dx=-4\sin\theta\, d\theta$ $\quad dy=3\cos\theta\, d\theta$

$\therefore$ limits of θ are 0 to 2π

$$\therefore \text{work done}=\int_0^{2\pi}\{(3x-2y)dx+(2x+8y)dy+y^2dz\}$$

$$=\int_0^{2\pi}\{[3\cdot 4\cos\theta-2\cdot 3\sin\theta](-4\sin\theta)d\theta+$$

$$[2\cdot 4\cos\theta+8\cdot 3\sin\theta]3\cos\theta d\theta+0$$

$$=\int_0^{2\pi}(-48\sin\theta\cos\theta+24\sin^2\theta+24\cos^2\theta+72\sin\theta\cos\theta\}d\theta$$

$$=\int_0^{2\pi}(24\sin^2\theta+24\cos^2\theta)d\theta+24\int_0^{2\pi}\sin\theta\cos\theta\, d\theta$$

$$=\int_0^{2\pi}24(\sin^2\theta+\cos^2\theta)d\theta+24(0)\quad \left\{\because \int_0^{2\pi}\sin^m\theta\cos^n\theta d\theta=0 \text{ if } m \,\&\, n \text{ are odd}\right\}$$

$$=24\int_0^{2\pi}d\theta$$

$$=24(\theta)_0^{2\pi}$$

$$=24(2\pi-0)$$

$$=48\pi \text{ units}$$

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* Green's Theorem :-

1) cartesian form :-

If R is a closed region of the XY-plane bounded by a simple closed curve C and if M and N are continuous function of X and Y having continuous derivation in R then

$$\oint_C M dx + N dy = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right) dx dy$$

where C is transvered in positive direction (clockwise)

2) Green's theorem in plane in vector notation :-

Green's theorem in XOY plane can be written as

$$\oint_C \bar{F} \cdot d\bar{r} = \iint_R (\nabla \times \bar{F}) \cdot \hat{k} \, dx dy$$

Ex. 1 Apply Green's theorem to evaluate :

$$\int_C (2x^2 - y^2) dx + (x^2 + y^2) dy$$

where C is the boundary of the area enclosed by the axis and upper half of the circle $x^2 + y^2 = 16$

$\Rightarrow$

we know that

$$\oint_C M dx + N dy = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right) dx dy$$

Given,

$$M = 2x^2 - y^2 \quad \& \quad N = x^2 + y^2$$

$$\frac{\partial M}{\partial y} = -2y \quad \& \quad \frac{\partial N}{\partial x} = 2x$$

$$\therefore \frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} = +2x + 2y$$
$$= 2(x+y)$$

$$\therefore I = \oint M dx + N dy = \iint_R 2(x+y) dx dy$$

Given, $x^2+y^2=16$

$\therefore$ put $x = r\cos\theta$, $y = r\sin\theta$

$\therefore dxdy = r\,dr\,d\theta$.

$\therefore$ limits of θ are 0 to π (half circle)

limits of r are 0 to 4

$$\therefore I = \int_0^{\pi}\int_0^{4} 2(r\cos\theta + r\sin\theta)\, r\,dr\,d\theta$$

$$= 2\int_0^{\pi}\int_0^{4} r^2(\cos\theta+\sin\theta)\,dr\,d\theta$$

$$= 2\int_0^{\pi}(\cos\theta+\sin\theta)d\theta \cdot \int_0^4 r^2 dr$$

$$= 2\int_0^{\pi}(\cos\theta+\sin\theta)d\theta \left(\frac{r^3}{3}\right)_0^4$$

$$= 2\int_0^{\pi}(\cos\theta+\sin\theta)d\theta \cdot \frac{64}{3}$$

$$= 2\times\frac{64}{3}\left[\sin\theta-\cos\theta\right]_0^{\pi}$$

$$= \frac{128}{3}\left\{[0-0]-[-1-1]\right\}$$

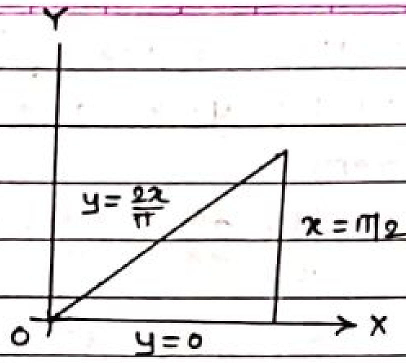
$$= \frac{128\times 2}{3}$$

$$I = \frac{256}{3}$$

Ex. 2 Using Green's theorem evaluate $\int_c ((y-\sin x)dx + \cos x\,dy)$ where c is the plane triangle enclosed by the lines $y=0$, $x=\pi/2$, $y = 2x/\pi$.

$\Rightarrow$ By Green's theorem

$$\oint M dx + N dy = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dxdy$$



$M = y - \sin x$ & $N = \cos x$

$\frac{\partial M}{\partial y} = 1$ & $\frac{\partial N}{\partial x} = -\sin x$

$$\therefore \quad I = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right) dx\,dy$$

$$= \iint_R (-\sin x - 1)\, dx\,dy$$

limits are $x = 0$ to $x = \pi/2$ & $y = 0$ to $y = 2x/\pi$

$$= \int_0^{\pi/2} \int_0^{2x/\pi} (-\sin x - 1)\, dx\,dy$$

$$= \int_0^{\pi/2} (-\sin x - 1) dx \int_0^{2x/\pi} dy$$

$$= -\int_0^{\pi/2} (\sin x + 1)\, dx \, (y)_0^{2x/\pi}$$

$$= -\int_0^{\pi/2} (\sin x + 1) \frac{2x}{\pi}\, dx$$

$$= -\frac{2}{\pi} \int_0^{\pi/2} x(\sin x + 1)\, dx$$

$$= -\frac{2}{\pi} \left\{ \left[x(-\cos x + x) \right]_0^{\pi/2} - \int_0^{\pi/2} (-\cos x + x)(1)\, dx \right.$$

$$= -\frac{2}{\pi} \left\{ -x\cos\frac{\pi}{2} + \left(\frac{\pi}{2}\right)^2 \right\} - \left\{ -\sin x + \frac{x^2}{2} \right\}_0^{\pi/2}$$

$$= -\frac{2}{\pi} \left\{ \left[0 + \frac{\pi^2}{4} \right] - \left[-1 + \frac{\pi^2}{4} \right] \right. = -\frac{2}{\pi} \left\{ \frac{\pi^2}{4} + 1 - \frac{\pi^2}{8} \right\}$$

$$= -\frac{2}{\pi} \left\{ 1 + \frac{\pi^2}{8} \right\} = -\frac{2}{\pi} - \frac{\pi}{4}$$

Ex. 3 Verify the Green's theorem for $\int_c \{(xy+y^2)dx + x^2 dy\}$ where c is bounded by $y=x$ and $y=x^2$

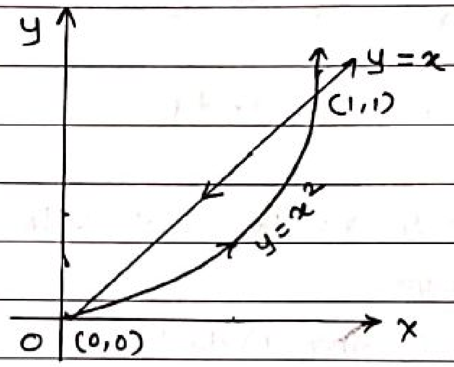
⇒ By Green's theorem

$$\oint_c M dx + N dy = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx dy \quad \text{——— (1)}$$

Given $M = xy + y^2$ & $N = x^2$

$\frac{\partial M}{\partial y} = x + 2y$ & $\frac{\partial N}{\partial x} = 2x$

Also, $y = x$ & $y = x^2$



$$LHS = \oint_c M dx + N dy = \int_{c_1} + \int_{c_2}$$

Now, for C_1, $y = x^2$ where x varies from 0 to 1 & $dy = 2x dx$

$$\int_{c_1} = \int_0^1 \{x \cdot x^2 + (x^2)^2\} dx + x^2 \cdot 2x dx$$

$$= \int_0^1 (x^3 + x^4 + 2x^3) dx$$

$$= \int_0^1 (3x^3 + 4x^4) dx$$

$$= \left(\frac{3x^4}{4} + \frac{x^5}{5}\right)_0^1$$

$$= \frac{3}{4} + \frac{1}{5} - 0$$

$$= \frac{15+4}{20}$$

$$= \frac{19}{20}$$

For c_2, $y = x \Rightarrow dy = dx$

x varies from 1 to 0

$$\therefore \int_{c_2} = \int_1^0 \{ (x \cdot x + x^2) dx + x^2 dx \}$$

$$= \int_1^0 (2x^2 + x^2)\, dx = \int_1^0 3x^2 dx$$

$$= 3\left(\frac{x^3}{3}\right)_1^0$$

$$= (x^3)_1^0 = 0 - 1^3$$

$$= -1$$

$$\therefore \quad LHS = \frac{19}{20} - 1$$

$$= \frac{-1}{20} \quad \text{———} \quad (1)$$

Also,

$$RHS = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx\, dy$$

limits are $x = 0$ to $x = 1$ and $y = x^2$ to $y = x$

$$= \int_0^1 \int_{x^2}^{x} (2x - x - 2y)\, dx\, dy$$

$$= \int_0^1 dx \int_{x^2}^{x} (x - 2y) dy$$

$$= \int_0^1 dx \left(xy - 2 \cdot \frac{y^2}{2}\right)_{x^2}^{x}$$

$$= \int_0^1 \{ (x \cdot x - x^2) - (x \cdot x^2 - x^4) \} dx$$

$$= \int_0^1 -(x^3 - x^4) dx$$

$$= \int_0^1 (x^4 - x^3) dx$$

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$$= \left(\frac{x^5}{5} - \frac{x^4}{4}\right)_0^1$$

$$= \left(\frac{1}{5} - \frac{1}{4}\right)$$

$$= \frac{4-5}{20}$$

$$RHS = \frac{-1}{20} \quad \text{———} \quad (2)$$

From (1) & (2)

LHS = RHS

$\therefore$ Green theorem verified.

Ex. 6 Verify Green's theorem in plane for $\int_C \{[3x^2-8y^2]dx + (4y-6xy)dy\}$ where C is the boundry of the region defined by $y=\sqrt{x}$, $y=x^2$

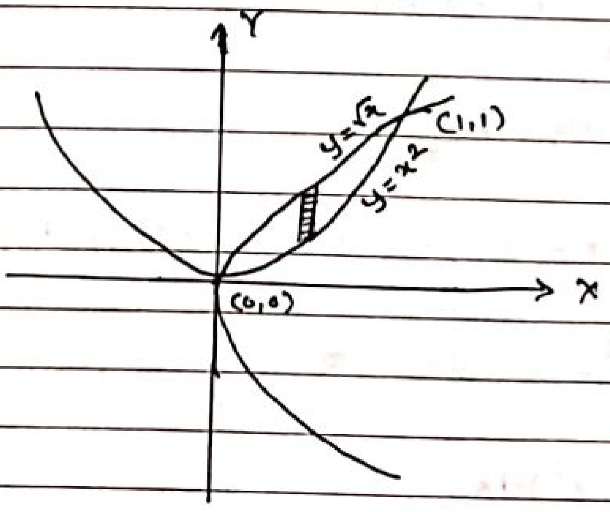
$\Rightarrow$ By Green's theorem

$$\oint_C Mdx + Ndy = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx\,dy$$

where, $M = 3x^2 - 8y^2$ & $N = 4y - 6xy$

$$\frac{\partial M}{\partial y} = -16y \quad \& \quad \frac{\partial N}{\partial x} = -6y$$

Given, $y=\sqrt{x} \Rightarrow y^2 = x$ & $x^2 = y$



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$$LHS = \oint_C M\,dx + N\,dy = \int_{C_1} + \int_{C_2}$$

for C_1, $y = x^2 \Rightarrow dy = 2x\,dx$

$\therefore$ x varies from 0 to 1

$$\int_{C_1} = \int_0^1 \{(3x^2 - 8y^2)dx + (4y - 6xy)dy\}$$

$$= \int_0^1 \{(3x^2 - 8x^4)dx + (4x^2 - 6x^3)\,2x\,dx\}$$

$$= \int_0^1 \{3x^2 - 8x^4 + 8x^3 - 12x^4\}dx$$

$$= \int_0^1 \{3x^2 + 8x^3 - 20x^4\}dx$$

$$= \left(\frac{3x^3}{3} + \frac{8x^4}{4} - \frac{20x^5}{5}\right)_0^1$$

$$= (1 + 2 - 4)$$

$$= -1$$

And,

for C_2, $y = \sqrt{x} \Rightarrow x = y^2$

$\therefore$ $dx = 2y\,dy$

$\therefore$ y varies from 1 to 0

$$\therefore \int_{C_2} = \int_1^0 \{(3x^2 - 8y^2)dx + (4y - 6xy)dy\}$$

$$= \int_1^0 \{(3y^4 - 8y^2)\,2y\,dy + (4y - 6y^3)dy\}$$

$$= \int_1^0 \{6y^5 - 16y^3 + 4y - 6y^3\}dy$$

$$= \int_1^0 \{6y^5 - 22y^3 + 4y\}dy$$

$$= \left(\frac{6y^6}{6} - \frac{22y^4}{4} + \frac{4y^2}{2}\right)_1^0$$

$$= 0 - \left(1 - \frac{22}{4} + 2\right) = -\left(3 - \frac{11}{2}\right) = -\left(-\frac{5}{2}\right) = \frac{5}{2}$$

$\therefore$ LHS $= -1 + \frac{5}{2} = \frac{-2+5}{2}$

LHS $= \frac{3}{2}$ ——— ①

Now,

RHS $= \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx\,dy$

$\therefore$ limits are $x=0$ to $x=1$ & $y=\sqrt{x}$ to $y=x^2$

$= \int_0^1 \int_{x^2}^{\sqrt{x}} (-6y+16y)\,dx\,dy$

$= \int_0^1 \int_{x^2}^{\sqrt{x}} 10y\,dx\,dy$

$= \int_0^1 dx \int_{x^2}^{\sqrt{x}} 10y\,dy$

$= \int_0^1 dx \left(\frac{10y^2}{2}\right)_{x^2}^{\sqrt{x}}$

$= 5\int_0^1 dx\,\{x - x^4\}$

$= 5\int_0^1 (x-x^4)\,dx$

$= 5\left(\frac{x^2}{2} - \frac{x^5}{5}\right)_0^1$

$= 5\left(\frac{1}{2} - \frac{1}{5}\right)$

$= 5\frac{(5-2)}{10}$

RHS $= \frac{3}{2}$ ——— ②

From ① & ②

LHS = RHS.

$\therefore$ Greens theorem verified.

Ex. 3 Verify Green's theorem for $\int_c \left(\frac{1}{y}dx + \frac{1}{x}dy\right)$, where c is the boundary of the region defined by $x=1$, $x=4$, $y=1$ and $y=\sqrt{x}$.

$\Rightarrow$ By Green's theorem

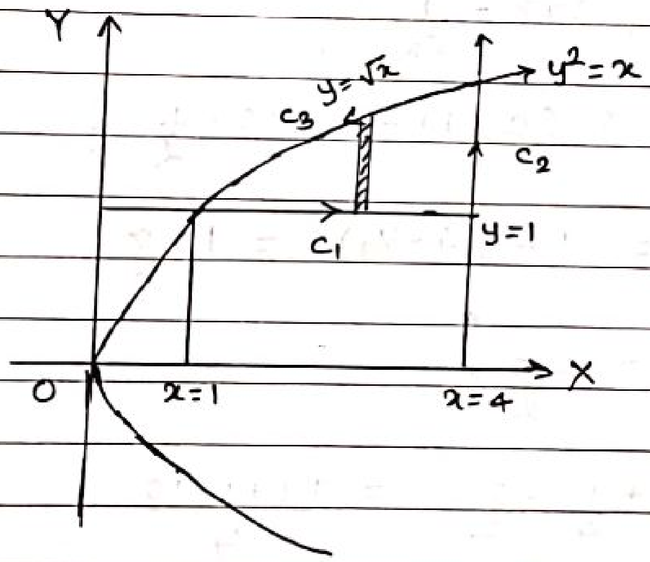
$$\oint_c M dx + N dy = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx\, dy$$

Here $M = \frac{1}{y}$ and $N = \frac{1}{x}$

$$\therefore \frac{\partial M}{\partial y} = \frac{-1}{y^2} \quad \& \quad \frac{\partial N}{\partial x} = \frac{-1}{x^2}$$

Given boundaries are

$x=1$, $x=4$, $y=1$ and $y=\sqrt{x} \Rightarrow y^2 = x$



$$LHS = \oint M dx + N dy = \int_{c_1} + \int_{c_2} + \int_{c_3}$$

For c_1, $y=1 \Rightarrow dy=0$ and x varies from $x=1$ to $x=4$

$$\int_{c_1} = \int_1^4 \left(\frac{1}{1}dx + \frac{1}{x}(0)\right) d$$

$$= \int_1^4 dx = (x)_1^4 = 4-1 = 3$$

For c_2, $x=4 \Rightarrow dx=0$

$\therefore$ y varies from $y=1$ to $y=2$

$$\int_{C_2} = \int_1^2 \left(\frac{1}{y}(0) + \frac{1}{4}\,dy\right) = \int_1^2 \frac{1}{4}\,dy$$

$$= \frac{1}{4}(y)_1^2 = \frac{1}{4}(2-1) = \frac{1}{4}$$

For C_3, $y = \sqrt{x} \Rightarrow dy = \frac{1}{2\sqrt{x}}\,dx$ and x varies from 4 to 1

$$\int_{C_3} = \int_4^1 \left(\cancel{\frac{x}{y}} \cdot \frac{1}{2\sqrt{x}}\,dx + \frac{1}{x}\cdot\frac{1}{2\sqrt{x}}\,dx\right)$$

$$= \int_4^1 \left(\frac{1}{\sqrt{x}} + \frac{1}{2x^{3/2}}\right)dx$$

$$= \left(2\sqrt{x} + \frac{1}{2}\cdot\frac{x^{-1/2}}{-1/2}\right)_4^1$$

$$= \left(2\sqrt{x} - \frac{1}{\sqrt{x}}\right)_4^1$$

$$= \left\{(2-1) - \left(2\times 2 - \frac{1}{2}\right)\right\}$$

$$= 1 - (4 - 1/2) = 1 - \frac{7}{2}$$

$$= -\frac{5}{2}$$

$$\therefore \text{LHS} = 3 + \frac{1}{4} - \frac{5}{2} = \frac{12+1-10}{4}$$

$$= \frac{3}{4} \longrightarrow (1)$$

Now,

$$\text{RHS} = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx\,dy$$

Limits of region R are

$x = 1$ to $x = 4$ & $y = 1$ to $y = \sqrt{x}$

$$= \int_1^4 \int_1^{\sqrt{x}} \left(\frac{-1}{x^2} + \frac{1}{y^2}\right) dx\,dy$$

$$= \int_1^4 dx \int_1^{\sqrt{x}} \left(-\frac{1}{x^2} + \frac{1}{y^2}\right) dy$$

$$= \int_1^4 dx \left(-\frac{y}{x^2} - \frac{1}{y}\right)_1^{\sqrt{x}}$$

$$= \int_1^4 dx \left[\left(-\frac{\sqrt{x}}{x^2} - \frac{1}{\sqrt{x}}\right) - \left(-\frac{1}{x^2} - 1\right)\right]$$

$$= \int_1^4 dx \left(-x^{-3/2} - \frac{1}{\sqrt{x}} + \frac{1}{x^2} + 1\right)$$

$$= \left(-\frac{x^{-1/2}}{-1/2} - 2\sqrt{x} - \frac{1}{x} + x\right)_1^4$$

$$= \left(\frac{2}{\sqrt{x}} - 2\sqrt{x} - \frac{1}{x} + x\right)_1^4$$

$$= \left(\frac{2}{2} - 2(2) - \frac{1}{4} + 4\right) - \left(2 - 2 - 1 + 1\right)$$

$$= \left(1 - 4 - \frac{1}{4} + 4\right) - 0$$

$$= 1 - \frac{1}{4}$$

$$= \frac{3}{4} \quad \text{———} \quad (2)$$

From (1) & (2)

LHS = RHS

$\therefore$ Greens theorem verified.

* Stokes theorem:-

If surface integral of the normal component of the curl of the vector point function $\bar{F}$ taken over an open surface S bounded by closed curve c is equal to the line integral of the tangential component of $\bar{F}$ taken around the curve c.

$$\int_C \bar{F}\cdot d\bar{r} = \iint_S (\nabla\times\bar{F})\cdot\hat{n}\, ds$$

Note:-

1) If surface is an XOY plane then,

$\hat{n} = \hat{k}$ and $ds = dx\,dy$.

2) If surface is an not XOY plane then

$$\hat{n} = \frac{\nabla\phi}{|\nabla\phi|}, \quad ds = \frac{dx\,dy}{|\hat{n}\cdot\hat{k}|}$$

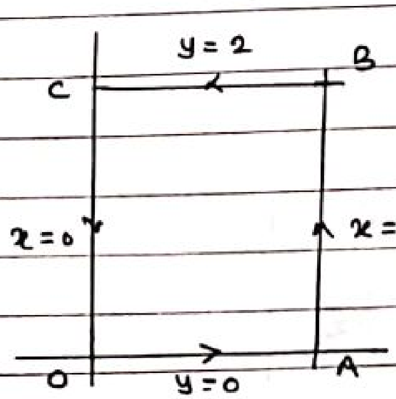
where $\phi = c$ is the surface for the projection of surface element ds on XOY plane.

Ex. 1 Verify Stoke's them for $\bar{F} = xy^2\hat{i} + y\hat{j} + z^2x\hat{k}$ for the surface of the rectangle bounded by $x=0, y=0, x=1, y=2, z=0$.

$\Rightarrow$ Given, $\bar{F} = xy^2\hat{i} + y\hat{j} + z^2x\hat{k}$

& $d\bar{r} = \hat{i}dx + \hat{j}dy$ $(z=0 \Rightarrow dz=0)$

$\therefore \bar{F}\cdot d\bar{r} = xy^2dx + ydy$



By Stoke's theorem

$$\int_C \bar{F}\cdot d\bar{r} = \iint_S (\nabla\times\bar{F})\cdot\hat{n}\, ds$$

$$LHS = \int_C \bar{F}\cdot d\bar{r} = \int_C (xy^2dx + ydy) \quad \text{---} \quad (1)$$

where c is the path OABCD as shown in fig.

1) Along OA, $y=0 \Rightarrow dy=0$

$\therefore$ limits of x are $x=0$ to $x=1$

2) Along AB, $x=1 \Rightarrow dx=0$

$\therefore$ limits of y are $y=0$ to $y=2$

3) Along BC, $y=2 \Rightarrow dy=0$

$\therefore$ limits of x are $x=1$ to $x=0$

4) Along CO, $x=0 \Rightarrow dx=0$

$\therefore$ limits of y are $y=2$ to $y=0$

$$\therefore LH = \int_{OA} 0\,dx + \int_{AB} y\,dy + \int_{BC} 4x\,dx + \int_{CO} y\,dy$$

$$= 0 + \int_0^2 y\,dy + \int_1^0 4x\,dx + \int_2^0 y\,dy$$

$$= \left(\frac{y^2}{2}\right)_0^2 + \left(\frac{4x^2}{2}\right)_1^0 + \left(\frac{y^2}{2}\right)_2^0$$

$$= \frac{1}{2}(4) + 2(0-1) + \frac{1}{2}(0-4)$$

$$= 2 - 2 - 2$$

$$LHS = -2 \quad \text{---} \quad (2)$$

Now,

$$RHS = \iint_S (\nabla\times\bar{F})\cdot\hat{n}\, ds$$

$$\nabla\times\bar{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ xy^2 & y & 0 \end{vmatrix}$$

$= \hat{i}(0-0) - \hat{j}(0-0) + \hat{k}(0-2xy)$

$= -2xy\hat{k}$

Here, surface is on XOY plane

$\therefore \hat{n} = \hat{k}$ & $ds = dxdy$

Limits are $x=0$ to $x=1$ & $y=0$ to $y=2$

$$\therefore RHS = \int_0^1 \int_0^2 (-2xy)\hat{k}\cdot\hat{k}\, dxdy$$

$$= \int_0^1 \int_0^2 (-2xy)\, dxdy$$

$$= -2 \int_0^1 x\,dx \int_0^2 y\,dy$$

$$= -2 \int_0^1 x \left(\frac{y^2}{2}\right)_0^2 dx$$

$$= -2\times 2 \int_0^1 x\,dx$$

$$= -4 \left(\frac{x^2}{2}\right)_0^1$$

$$= -\frac{4}{2}\cdot(1-0)$$

$RHS = -2$ ——— ③

From ② & ③

RHS = LHS.

$\therefore$ Stokes theorem verified.

Ex

Verify Stokes theorem for $\bar{F} = yz\hat{i} + zx\hat{j} + xy\hat{k}$ and C is boundary of the circle $x^2+y^2+z^2=1$, $z=0$

$\Rightarrow$ By Stokes theorem

$$\int_C \bar{F}\cdot d\bar{r} = \iint_S (\nabla\times\bar{F})\cdot\hat{n}\, ds$$

$$LHS = \int_C \bar{F}\cdot d\bar{r}$$

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$\bar{F} = yz\hat{i} + zx\hat{j} + xy\hat{k}$

$d\bar{r} = \hat{i}\,dx + \hat{j}\,dy + \hat{k}\,dz$

$\therefore \bar{F}\cdot d\bar{r} = yz\,dx + zx\,dy + xy\,dz$

The boundry is a unit circle in xy plane, so put $x = \cos\theta$, $y = \sin\theta$, $z = 0$

$dx = -\sin\theta \quad dy = \cos\theta, \quad dz = 0$

$\therefore \bar{F}\cdot d\bar{r} = 0 + 0 + 0$

$\therefore \bar{F}\cdot d\bar{r} = 0$

$\therefore \text{LHS} = \int_C \bar{F}\cdot d\bar{r} = 0$ ——— ①

$\text{RHS} = \iint_S (\nabla\times\bar{F})\cdot\hat{n}\,ds$

$$\nabla\times\bar{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ yz & zx & xy \end{vmatrix}$$

$= \hat{i}(x - x) - \hat{j}(y - y) + \hat{k}(z - z)$

$= 0$

$\therefore \text{RHS} = \iint_S 0\cdot\hat{n}\,ds = \iint_S 0\cdot ds = 0$ ——— ②

from ① & ②

LHS = RHS.

$\therefore$ stokes theorem verified.

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Ex. 3 Apply Stokes theorem to evaluate $\int_c \{4y\,dx + 2z\,dy + 6y\,dz\}$, where c is the curve of intersection of $x^2+y^2+z^2 = 6z$ and $z = x+3$.

$\Rightarrow$ Let $\bar{F} = 4y\hat{i} + 2z\hat{j} + 6y\hat{k}$ & $d\bar{r} = \hat{i}dx + \hat{j}dy + \hat{k}dz$

By Stokes theorem

$$\int_c \bar{F}\cdot d\bar{r} = \int_c (4y\,dx + 2z\,dy + 6y\,dz) = \iint_S (\nabla\times\bar{F})\cdot\hat{n}\,ds$$

where S is surface of circle $x^2+y^2+z^2 = 6z$, $z = x+3$.

Let $\hat{n}$ be the normal to the plane $x - z + 3 = 0$

Let $\phi = x - z + 3 = 0$

$$\nabla\phi = \hat{i}\frac{\partial\phi}{\partial x} + \hat{j}\frac{\partial\phi}{\partial y} + \hat{k}\frac{\partial\phi}{\partial z}$$

$$= \hat{i}(1) + \hat{j}(0) + \hat{k}(-1)$$

$$= \hat{i} - \hat{k}$$

But

$$\hat{n} = \frac{\nabla\phi}{|\nabla\phi|} = \frac{\hat{i}-\hat{k}}{\sqrt{1+1}} = \frac{1}{\sqrt{2}}(\hat{i}-\hat{k})$$

Also,

$$\nabla\times\bar{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 4y & 2z & 6y \end{vmatrix}$$

$$= \hat{i}(6-2) - \hat{j}(0-0) + \hat{k}(0-4)$$

$$= 4\hat{i} - 4\hat{k}$$

$$= 4(\hat{i}-\hat{k})$$

$$\therefore (\nabla\times\bar{F})\cdot\hat{n} = 4(\hat{i}-\hat{k})\cdot\frac{1}{\sqrt{2}}(\hat{i}-\hat{k})$$

$$= \frac{4}{\sqrt{2}}(1\cdot1 + (-1)(-1)) = \frac{4\cdot2}{\sqrt{2}}$$

$$= 4\sqrt{2}$$

$$\therefore \iint_S (\nabla\times\bar{F})\cdot\hat{n}\,ds = \iint_S 4\sqrt{2}\,ds$$

$$= 4\sqrt{2}\iint_S ds$$

$= 4\sqrt{2}$ (Area of Circle)

we have

$$x^2+y^2+z^2 = 6z$$

$$x^2+y^2+(z-3)^2 = 9$$

$$(x-0)^2+(y-0)^2+(z-3)^2 = 9$$

$\therefore$ Centre $= (0,0,3)$ & radius $= \sqrt{9} = 3$

$\therefore$ The plane $z = x+3$ passes through $(0,0,3)$

$\therefore$ Radius of circle = radius of shere $= 3$

$$\therefore \int_c \bar{F}\cdot d\bar{r} = 4\sqrt{2}(\pi r^2)$$

$$= 4\sqrt{2}(\pi \cdot 9)$$

$$= \underline{\underline{36\sqrt{2}\,\pi}}$$

Ex.4 Apply stoke theorem, evaluate : $\int_c (x+y)dx + (2x-z)dy + (y+z)dz$ where c is the curve given by $x^2+y^2+z^2-2ax-2ay=0$ and $x+y=2a$.

$\Rightarrow$ Let $\bar{F} = (x+y)\hat{i} + (2x-z)\hat{j} + (y+z)\hat{k}$

& $d\bar{r} = \hat{i}dx + \hat{j}dy + \hat{k}dz$.

By stokes theorem,

$$\int_c \bar{F}\cdot d\bar{r} = \int_c \{(x+y)dx + (2x-z)dy + (y+z)dz\} = \iint_S (\nabla\times\bar{F})\cdot\hat{n}\,ds$$

where S is surface of intersection of circle $x^2+y^2+z^2-2ax-2ay=0$ & $x+y=2a$

Let $\phi = x+y-2a = 0$

$$\therefore \nabla\phi = \hat{i}\frac{\partial\phi}{\partial x} + \hat{j}\frac{\partial\phi}{\partial y} + \hat{k}\frac{\partial\phi}{\partial z}$$

$$= \hat{i}(1) + \hat{j}(1) + \hat{k}(0)$$

$$\nabla\phi = \hat{i}+\hat{j}$$

But

$$\hat{n} = \frac{\nabla\phi}{|\nabla\phi|} = \frac{\hat{i}+\hat{j}}{\sqrt{1+1}} = \frac{1}{\sqrt{2}}(\hat{i}+\hat{j})$$

$$(\nabla \times \bar{F}) = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x+y & 2x-z & y+z \end{vmatrix}$$

$$= \hat{i}\,(1-(-1)) - \hat{j}\,(0-0) + \hat{k}\,(2-1)$$

$$= 2\hat{i} + \hat{k}$$

$$\therefore (\nabla \times \bar{F})\cdot\hat{n} = (2\hat{i}+\hat{k})\cdot\frac{(\hat{i}+\hat{j})}{\sqrt{2}}$$

$$= \frac{2+0+0}{\sqrt{2}} = \frac{2}{\sqrt{2}} = \sqrt{2}$$

$$\therefore \iint_S (\nabla \times \bar{F})\cdot\hat{n}\, ds = \iint_S \sqrt{2}\, ds$$

$$= \sqrt{2}\iint_S ds$$

$$= \sqrt{2}\,(\text{Area of given circle})$$

we have

$$x^2+y^2+z^2-2ax-2ay=0$$

$$(x-a)^2+(y-a)^2+(z-0)^2 = 2a^2$$

$\therefore$ centre $= (a,a,0)$ & radius $= \sqrt{2a^2} = \sqrt{2}\,a$

$\therefore$ The plane $x+y=2a$ passes through $(a,a,0)$

$\therefore$ Radius of circle = radius of sphere = $\sqrt{2}\,a$.

$$\therefore \iint_S (\nabla \times \bar{F})\cdot\hat{n}\, ds = \sqrt{2}\,(\pi r^2)$$

$$= \sqrt{2}\,(\pi\cdot 2a^2)$$

$$= \underline{\underline{2\sqrt{2}\,a^2\pi}}$$

Ex. 5 Use stokes theorem for $\int_C (4y\hat{i}+2z\hat{j}+6y\hat{k})\cdot d\bar{r}$ where C is intersection of $x^2+y^2+z^2=2z$ and $x=z-1$

(Ans $= \iint_S (\nabla\times\bar{F})\cdot\hat{n}\,ds = \iint_S \frac{8}{\sqrt{2}}(\text{Area of circle}) = \frac{8}{\sqrt{2}}\pi(1)^2 = \frac{8}{\sqrt{2}}\pi$

* Gauss Divergence theorem:-

$$\iint_S \bar{F}\cdot\hat{n}\,ds = \iiint_V \nabla\cdot\bar{F}\,dv$$

where $dv = dx\,dy\,dz$. & $\hat{n}$ is a unit vector.

Ex. 1 Show that $\iiint_V \frac{dv}{r^2} = \iint_S \frac{\bar{r}\cdot\hat{n}}{r^2}\,ds$.

$\Rightarrow$ By Gauss divergence theorem

$$\iint_S \bar{F}\cdot\hat{n}\,ds = \iiint_V \nabla\cdot\bar{F}\,dv$$

But $\bar{F} = \frac{\bar{r}}{r^2}$

$$\therefore \iint_S \frac{\bar{r}\cdot\hat{n}}{r^2}\,ds = \iiint_V \nabla\cdot\left(\frac{\bar{r}}{r^2}\right)dv \quad \text{———} \quad (1)$$

Now,

we know that

$$\nabla\cdot\left(\frac{\bar{r}}{r^2}\right) = \nabla\cdot(\bar{r}\,r^{-2})$$

$$= (\nabla\cdot\bar{r})r^{-2} + \bar{r}\cdot(\nabla r^{-2})$$

$$= (3)r^{-2} + \bar{r}\cdot(-2r^{-4})\bar{r} \qquad (\nabla r^n = n r^{n-2}\bar{r}) \;\&\; (\nabla\cdot\bar{r} = 3)$$

$$= 3r^{-2} - 2r^{-4}(\bar{r}\cdot\bar{r})$$

$$= 3r^{2} - 2r^{-4}r^{2}$$

$$= 3r^{2} - 2r^{-2}$$

$$= r^{-2}$$

$$= \frac{1}{r^2}$$

$\therefore$ (1) becomes

$$\iint_S \frac{\bar{r}\cdot\hat{n}}{r^2}\,ds = \iiint_V \frac{1}{r^2}\,dv$$

$$\iint_S \frac{\bar{r}\cdot\hat{n}}{r^2}\,ds = \iiint_V \frac{dv}{r^2}$$

Hence proved.

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Ex. 2 Using Gauss divergence theorem, prove that

$$\iiint_V \frac{2}{r}\, dV = \iint_S \frac{\bar{r}\cdot\hat{n}}{r}\, dS$$

$\Rightarrow$ By Gauss divergence theorem,

$$\iint_S \bar{F}\cdot\hat{n}\, dS = \iiint_V \nabla\cdot\bar{F}\, dV$$

put $\bar{F} = \dfrac{\bar{r}}{r}$

$$\therefore \iint_S \frac{\bar{r}}{r}\cdot\hat{n}\, dS = \iiint_V \nabla\cdot\left(\frac{\bar{r}}{r}\right) dV \quad \text{——— (1)}$$

But,

$$\begin{aligned}
\nabla\cdot\left(\frac{\bar{r}}{r}\right) &= \nabla\cdot(\bar{r}\, r^{-1}) \\
&= (\nabla\cdot\bar{r})\, r^{-1} + (\nabla r^{-1})\cdot\bar{r} \\
&= 3r^{-1} + (-1) r^{-3}\, \bar{r}\cdot\bar{r} \\
&= 3r^{-1} - r^{-3} r^{2} \\
&= 3r^{-1} - r^{-1} \\
&= (3-1)\, r^{-1} \\
&= 2r^{-1} \\
&= \frac{2}{r}
\end{aligned}$$

$(\nabla\cdot\bar{r} = 3$ & $\nabla r^n = n r^{n-2}\bar{r}$

$\therefore$ Eq^n (1) becomes

$$\iint_S \frac{\bar{r}\cdot\hat{n}}{r}\, dS = \iiint_V \frac{2}{r}\, dV$$

Hence proved.

Ex. 3 Using Gauss divergence, evaluate $\iint_S \bar{F}\cdot d\bar{S}$, where $\bar{F} = 4x\hat{i} - 2y^2\hat{j} + z^2\hat{k}$ and S is the region bounded by $y^2 = 4x$, $x = 1$, $z = 0$, $z = 3$

$\Rightarrow$

By Gauss divergence theorem

$$\iint_S \hat{n}\cdot\bar{F}\, dS = \iiint_V \nabla\cdot\bar{F}\, dV$$

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Given, $\bar{F} = 4x\hat{i} - 2y^2\hat{j} + z^2\hat{k}$

$$\nabla\cdot\bar{F} = \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)\cdot\left(4x\hat{i} - 2y^2\hat{j} + z^2\hat{k}\right)$$

$$= \frac{\partial}{\partial x}(4x) + \frac{\partial}{\partial y}(-2y^2) + \frac{\partial}{\partial z}(z^2)$$

$$\nabla\cdot\bar{F} = 4 - 4y + 2z$$

& $dv = dx\,dy\,dz$

$\therefore$ Limits are, $z=0$ to $z=3$, $x=0$ to $x=1$ & $y = 2\sqrt{x}$, $y = -2\sqrt{x}$

$$= \int_{x=0}^{x=1}\int_{y=-2\sqrt{x}}^{y=2\sqrt{x}}\int_{z=0}^{z=3} (4-4y+2z)\,dx\,dy\,dz$$

$$= \int_{x=0}^{x=1}\int_{y=-2\sqrt{x}}^{y=+2\sqrt{x}} \left(4z - 4yz + z^2\right)_0^3 dy\,dz$$

$$= \int_{x=0}^{x=1}\int_{y=-2\sqrt{x}}^{y=2\sqrt{x}} (12 - 12y + 9)\,dy\,dx$$

$$= \int_{x=0}^{x=1} \left(12y - 6y^2 + 9y\right)_{-2\sqrt{x}}^{2\sqrt{x}} dx$$

$$= \int_{x=0}^{x=1} \left\{(24\sqrt{x} - 24x + 18\sqrt{x}) - (-24\sqrt{x} - 24x - 18\sqrt{x})\right\} dx$$

$$= \int_{x=0}^{x=1} (84\sqrt{x})\,dx$$

$$= 84\left(\frac{x^{1/2+1}}{1/2+1}\right)_0^1 \cancel{dx}$$

$$= 84\left(\frac{x^{3/2}}{3/2}\right)_0^1$$

$$= 84\times\frac{2}{3}(1-0)$$

$$= 28\times 2$$

$$= 56 \,//$$

Ex. 4 Using Gauss divergence theorem, evaluate $\iint (2xy\hat{i} + yz^2\hat{j} + xz\hat{k})$ over the total surface area bounded by

$x=0,\ y=0,\ z=0,\ y=3$ & $x+2z=6$.

$\Rightarrow$ Let $\bar{F} = 2xy\hat{i} + yz^2\hat{j} + xz\hat{k}$

By Gauss divergence theorem,

$$\iint_S \bar{F}\cdot d\bar{s} = \iiint_V \nabla\cdot\bar{F}\, dv$$

Now,

$$\nabla\cdot\bar{F} = \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)\cdot(2xy\hat{i} + yz^2\hat{j} + xz\hat{k})$$

$$= \frac{\partial}{\partial x}(2xy) + \frac{\partial}{\partial y}(yz^2) + \frac{\partial}{\partial z}(xz)$$

$$= 2y + z^2 + x$$

& $dv = dx\,dy\,dz$

$\therefore$ Bounded by curves

$y=0$ to $y=3$, $z=0$ to $z=3$ & $x=0$ to $x=6-2z$

$$\therefore \iint_S \bar{F}\cdot d\bar{s} = \int_{z=0}^{z=3}\int_{y=0}^{y=3}\int_{x=0}^{x=6-2z} (2y+z^2+x)\,dx\,dy\,dz$$

$$= \int_0^3\int_0^3 \left(2xy + z^2x + \frac{x^2}{2}\right)_{x=0}^{x=6-2z} dy\,dz$$

$$= \int_0^3\int_0^3 \left\{2y(6-2z) + z^2(6-2z) + \frac{(6-2z)^2}{2}\right\}$$

$$= \int_0^3\int_0^3 \left\{12y - 4yz + 6z^2 - 2z^3 + 2(9-6z+z^2)\right\} dy\,dz$$

$$= \int_0^3\int_0^3 \{12y - 4yz + 6z^2 - 2z^3 + 18 - 12z + 2z^2)\, dy\,dz$$

$$= \int_0^3 \left\{6y^2 - 2y^2z + 8z^2y - 2z^3y + 18y - 12zy\right\}_0^3$$

$$= \int_0^3 \left\{36 - 18z + 18z^2 - 6z^3 + 54 - 36z\right\}$$

$$= \int_0^3 (18z^2 - 6z^3 - 54z + 90)\,dz$$

$\frac{351}{2}$

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$$= \left(6x^3 - \frac{6x^4}{4} - 27x^2 + 90x\right)_0^3$$

$$= 162 - \frac{243}{2} - 143 + 270$$

$$= 19 + 270 - \frac{243}{2}$$

$$= 289 - \frac{243}{2}$$

$$= \frac{578 - 243}{2}$$

$$= \frac{335}{2}$$