Unit - VI

Fourier Series

★ Fourier Series :-

$$F(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx)$$

is called Fourier series and a_0, a_n, b_n is called fourier coefficient.

★ Euler's formula :-

The Fourier Series is given by

$$F(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx)$$

where

$$a_0 = \frac{1}{\pi} \int_{\alpha}^{\alpha+2\pi} F(x)\,dx \quad \text{—— (1)}$$

$$a_n = \frac{1}{\pi} \int_{\alpha}^{\alpha+2\pi} F(x) \cos nx\,dx \quad \text{—— (2)}$$

$$b_n = \frac{1}{\pi} \int_{\alpha}^{\alpha+2\pi} F(x) \sin nx\,dx \quad \text{—— (3)}$$

Formula or calculation of (1), (2) & (3) are called Euler's Formula.

★ Some important Formula for Integration:

1) $\int_{0}^{2\pi} \sin nx\,dx = 0$

2) $\int_{0}^{2\pi} \cos nx\,dx = 0$

3) $\int_{-\pi}^{\pi} \cos nx\,dx = 0$

4) $\int_{-\pi}^{\pi} \sin nx\,dx = 0$

5) $\int_0^{2\pi} \cos^2 nx\,dx = 0$

6) $\int_0^{2\pi} \sin^2 nx\,dx = 0$

7) $\int_0^{2\pi} \sin nx \cos nx\,dx = 0$

8) $\int_{-\pi}^{\pi} \sin nx \cos nx\,dx = 0$

9) $\int e^{ax} \sin bx\,dx = \frac{e^{ax}}{a^2+b^2}\{a \sin bx - b\cos bx\}$

10) $\int e^{ax} \cos bx\,dx = \frac{e^{ax}}{a^2+b^2}\{a\cos bx + b\sin bx\}$

11) Generalize rule of integration by parts

$\int uv\,dx = u(v_1) - u'(v_2) + u''(v_3) - u'''(v_4) + \cdots$

12) $\cos n\pi = (-1)^n$

13) $\sin n\pi = 0$

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★ Fourier series Expansion in the range $(0, 2\pi)$

$$F(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx)$$

where, $a_0 = \frac{1}{\pi}\int_0^{2\pi} F(x)dx$, $a_n = \frac{1}{\pi}\int_0^{2\pi} F(x)\cos nx\,dx$, $b_n = \frac{1}{\pi}\int_0^{2\pi} F(x)\sin nx\,dx$

Ex. 1 Find fourier series of $f(x) = x^2$ in the interval $(0, 2\pi)$ and hence deduct that

$$\frac{\pi^2}{12} = \frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \cdots$$

⇒ Given, $F(x) = x^2$ and range is $(0, 2\pi)$

Fourier series is given by

$$F(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx)$$

where,

$$a_0 = \frac{1}{\pi}\int_0^{2\pi} F(x)dx = \frac{1}{\pi}\int_0^{2\pi} x^2 dx = \frac{1}{\pi}\left(\frac{x^3}{3}\right)_0^{2\pi} = \frac{1}{\pi}\left(\frac{8\pi^3}{3} - 0\right)$$

$$\therefore\ a_0 = \frac{8\pi^2}{3}$$

$$a_n = \frac{1}{\pi}\int_0^{2\pi} F(x)\cos nx\,dx = \frac{1}{\pi}\int_0^{2\pi} x^2 \cos nx\,dx$$

$$= \frac{1}{\pi}\left\{x^2\left(\frac{\sin nx}{n}\right) - 2x\left(-\frac{\cos nx}{n^2}\right) + 2\left(-\frac{\sin nx}{n^3}\right)\right\}_0^{2\pi}$$

$$= \frac{1}{\pi}\left\{\left[\frac{(2\pi)^2 \sin 2n\pi}{n} + \frac{2(2\pi)\cos 2n\pi}{n^2} - \frac{2\sin 2n\pi}{n^3}\right] - \left[0 - 0 - \frac{2\sin 0}{n^3}\right]\right\}$$

$$= \frac{1}{\pi}\left\{\left(0 + \frac{4\pi(1)}{n^2} - 0\right) - 0\right\}$$

$$= \frac{1}{\pi}\left(\frac{4\pi}{n^2}\right)$$

$$a_n = \frac{4}{n^2}$$

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$$b_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\sin nx\,dx = \frac{1}{\pi}\int_0^{2\pi} x^2 \sin nx\,dx$$

$$= \frac{1}{\pi}\left\{ x^2\left(-\frac{\cos nx}{n}\right) - 2x\left(-\frac{\sin nx}{n^2}\right) + 2\left(\frac{\cos nx}{n^3}\right)\right\}_0^{2\pi}$$

$$= \frac{1}{\pi}\left\{\left[-\frac{(2\pi)^2\cos 2n\pi}{n} + \frac{2(2\pi)\sin 2n\pi}{n^2} + \frac{2\cos 2n\pi}{n^3}\right] - \left[-0 + 0 + \frac{2\cos 2n\pi}{n^3}\right]\right\}$$

$$= \frac{1}{\pi}\left\{\left(-\frac{4\pi^2(1)}{n} + 0 + \frac{2(1)}{n^3}\right) - \frac{2(1)}{n^3}\right\}$$

$$= \frac{1}{\pi}\left(-\frac{4\pi^2}{n} + \frac{2}{n^3} - \frac{2}{n^3}\right)$$

$$= \frac{1}{\pi}\left(-\frac{4\pi^2}{n}\right)$$

$$b_n = -\frac{4\pi}{n}$$

$\therefore$ Eqn ① becomes

$$f(x) = x^2 = \frac{1}{2}\left(\frac{8\pi^2}{3}\right) + \sum_{n=1}^{\infty}\left(\frac{4}{n^2}\cos nx - \frac{4\pi}{n}\sin nx\right) \quad \text{——} \quad ②$$

which is required fourier series.

Putting, $x = \pi$

$$\pi^2 = \frac{4\pi^2}{3} + \sum_{n=1}^{\infty}\frac{4\cos n\pi}{n^2}$$

$$\sum_{n=1}^{\infty}\frac{\cos n\pi}{n^2} = \frac{1}{4}\left(\pi^2 - \frac{4\pi^2}{3}\right) = \frac{1}{4}\left(-\frac{\pi^2}{3}\right) = -\frac{\pi^2}{12}$$

$$\frac{\cos\pi}{1^2} + \frac{\cos 2\pi}{2^2} + \frac{\cos 3\pi}{3^2} + \frac{\cos 4\pi}{4^2} + \cdots = -\frac{\pi^2}{12}$$

$$-\frac{1}{1^2} + \frac{1}{2^2} - \frac{1}{3^2} + \frac{1}{4^2} - \cdots = -\frac{\pi^2}{12}$$

$$\therefore \frac{\pi^2}{12} = \frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \cdots$$

Ex. 1

Find the fourier series of the function $F(x) = x$ in the interval $(0, 2\pi)$

$\Rightarrow$ Given,

$F(x) = x$ and the range $(0, 2\pi)$

Fourier series is given by

$$F(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx) \quad \text{--- (1)}$$

where,

$$a_0 = \frac{1}{\pi} \int_0^{2\pi} F(x)\,dx = \frac{1}{\pi} \int_0^{2\pi} x\,dx$$

$$= \frac{1}{\pi} \left(\frac{x^2}{2} \right)_0^{2\pi} = \frac{1}{\pi} \left(\frac{4\pi^2 - 0}{2} \right)$$

$$= \frac{1}{\pi} (2\pi^2)$$

$$a_0 = 2\pi$$

$$a_n = \frac{1}{\pi} \int_0^{2\pi} F(x) \cos nx\,dx$$

$$= \frac{1}{\pi} \int_0^{2\pi} x \cos nx\,dx$$

$$= \frac{1}{\pi} \left\{ \left[x \frac{\sin nx}{n} \right]_0^{2\pi} - \int_0^{2\pi} \frac{\sin nx}{n} \cdot (1)\,dx \right\}$$

$$= \frac{1}{\pi} \left\{ \left[x \frac{\sin nx}{n} \right]_0^{2\pi} - \frac{1}{n} \times \left(- \frac{\cos nx}{n} \right)_0^{2\pi} \right\}$$

$$= \frac{1}{\pi} \left\{ \left[\frac{2\pi \sin 2n\pi}{n} - 0 \right] + \frac{1}{n^2} \left[\cos 2n\pi - \cos 0 \right] \right\}$$

$$= \frac{1}{\pi} \left\{ 0 + \frac{1}{n^2} (1-1) \right\}$$

$$= \frac{1}{\pi} (0)$$

$$a_n = 0$$

$$b_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\sin nx\,dx$$

$$= \frac{1}{\pi}\int_0^{2\pi} x\sin nx\,dx$$

$$= \frac{1}{\pi}\left\{\left(\frac{-x\cos nx}{n}\right)_0^{2\pi} - \int_0^{2\pi}\left(-\frac{\cos nx}{n}\right)dx\right\}$$

$$= \frac{1}{\pi}\left\{\left(-\frac{x\cos nx}{n}\right)_0^{2\pi} + \left(\frac{\sin nx}{n^2}\right)_0^{2\pi}\right\}$$

$$= \frac{1}{\pi}\left\{\left(-\frac{2\pi\cos 2n\pi}{n} - 0\right) + \frac{1}{n^2}\left(\sin 2n\pi - \sin n0\right)\right\}$$

$$= \frac{1}{\pi}\left\{-\frac{2\pi(1)}{n} + 0\right\}$$

$$b_n = -\frac{2}{n}$$

$\therefore$ Fourier series is given by

$$f(x) = \frac{1}{2}(2\pi) + \sum_{n=1}^{\infty}(0)\cos nx + \left(-\frac{2}{n}\right)\sin nx$$

$$f(x) = \pi + \sum_{n=1}^{\infty} -\frac{2}{n}\sin nx$$

Ex. 2 Obtain the fourier series for $f(x) = e^{-x}$ in the range $(0, 2\pi)$

$\Rightarrow$ Given,

$f(x) = e^{-x}$ and the range is $(0, 2\pi)$

Fourier Series is given by

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}(a_n\cos nx + b_n\sin nx)$$

where,

$$a_0 = \frac{1}{\pi}\int_0^{2\pi} f(x)dx = \frac{1}{\pi}\int_0^{2\pi} e^{-x}dx$$

$$= \frac{1}{\pi}\left(\frac{e^{-x}}{-1}\right)_0^{2\pi} = \frac{-1}{\pi}\left(e^{-2\pi} - e^{0}\right)$$

$$a_0 = \frac{1}{\pi}\left(1 - e^{-2\pi}\right)$$

$$a_n = \frac{1}{\pi}\int_0^{2\pi} F(x)\cos nx\,dx$$

$$= \frac{1}{\pi}\int_0^{2\pi} e^{-x}\cos nx\,dx$$

$$= \frac{1}{\pi}\left\{\frac{e^{-x}}{(-1)^2+n^2}\left(-\cos nx + n\sin nx\right)\right\}_0^{2\pi}$$

$$\cancel{= \frac{1}{\pi}\left\{\frac{e^{-x}}{1+n^2}\left[-\cos n(2\pi) + n\sin 2n\pi - \left(-\cos 0 + n\sin n 0\right)\right]\right\}}$$

$$\cancel{= \frac{1}{\pi}\left\{\frac{e^{-x}}{1+n^2}\left[-1+0+1-0\right.\right.}$$

$$= \frac{1}{\pi(1+n^2)}\left\{e^{-2\pi}\left(-\cos 2n\pi + n\sin 2n\pi\right) - e^{0}\left(-\cos 0 + n\sin n0\right)\right\}$$

$$= \frac{1}{\pi(1+n^2)}\left\{e^{-2\pi}(-1+0) - 1(-1+0)\right\}$$

$$= \frac{1}{\pi(1+n^2)}\left\{-e^{-2\pi} + 1\right\}$$

$$\therefore\ a_n = \frac{(1-e^{-2\pi})}{\pi(1+n^2)}$$

$$b_n = \frac{1}{\pi}\int_0^{2\pi} F(x)\sin nx\,dx$$

$$= \frac{1}{\pi}\int_0^{2\pi} e^{-x}\sin nx\,dx$$

$$= \frac{1}{\pi}\left\{\frac{e^{-x}}{(-1)^2+n^2}\left(-\sin nx - n\cos nx\right)\right\}_0^{2\pi}$$

$$= \frac{1}{\pi(1+n^2)}\left\{e^{-2\pi}\left(-\sin 2n\pi - n\cos 2n\pi\right) - e^{0}\left(-\sin 0 - n\cos 0\right)\right\}$$

$$= \frac{1}{\pi(1+n^2)}\left\{e^{-2\pi}\left(0 - n(1)\right) - 1\left(0 - n(1)\right)\right\}$$

$$= \frac{1}{\pi(1+n^2)}\left\{-ne^{-2\pi} + n\right\}$$

$$b_n = \frac{n(1-e^{-2\pi})}{\pi(1+n^2)}$$

$\therefore$ Fourier Series becomes.

$$F(x) = \frac{1}{2} \cdot \frac{(1-e^{-2\pi})}{\pi} + \sum_{n=1}^{\infty} \left(\frac{1-e^{-2\pi}}{\pi(1+n^2)} \right) \cos nx + \frac{n(1-e^{-2\pi})}{\pi(1+n^2)} \sin nx$$

$$F(x) = \frac{(1-e^{-2\pi})}{2\pi} + \frac{(1-e^{-2\pi})}{\pi} \sum_{n=1}^{\infty} \left(\frac{\cos nx + n \sin nx}{1+n^2} \right)$$

② Fourier series Expansion in the Range $(-\pi, \pi)$

The Fourier Series expansion in the range $(-\pi, \pi)$ is given by

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx)$$

where, $a_0 = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x)\,dx$

$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos nx\,dx$$

$$b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin nx\,dx$$

Case-I : If $f(x)$ is an Even function, then

$$a_0 = \frac{2}{\pi} \int_{0}^{\pi} f(x)\,dx$$

$$a_n = \frac{2}{\pi} \int_{0}^{\pi} f(x) \cos nx\,dx$$

& $b_n = 0$

Case-II : If $f(x)$ is an odd function, then

$$a_0 = 0$$

$$a_n = 0$$

$$b_n = \frac{2}{\pi} \int_{0}^{\pi} f(x) \sin nx\,dx$$

Ex.1 Obtain the fourier series for the function $f(x) = x^2$, $-\pi < x < \pi$, and hence show that

1) $\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \cdots = \frac{\pi^2}{6}$

2) $\frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \cdots = \frac{\pi^2}{12}$

3) $\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \frac{1}{7^2} + \cdots = \frac{\pi^2}{8}$

⇒ Given, $f(x) = x^2$ and the range is $(-\pi < x < \pi)$

~~Fourier series is given~~

$$f(-x) = x^2 = f(x)$$

∴ $f(x)$ is an even function

Fourier series is given by

$$F(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx) \quad \text{—— (1)}$$

where, $a_0 = \frac{2}{\pi} \int_0^{\pi} F(x)\,dx = \frac{2}{\pi} \int_0^{\pi} x^2\,dx$

$$= \frac{2}{\pi} \left(\frac{x^3}{3} \right)_0^{\pi}$$

$$= \frac{2}{\pi} \left(\frac{\pi^3 - 0}{3} \right) = \frac{2}{\pi} \left(\frac{\pi^3}{3} \right)$$

$$a_0 = \frac{2\pi^2}{3}$$

$$a_n = \frac{2}{\pi} \int_0^{\pi} F(x) \cos nx\,dx$$

$$= \frac{2}{\pi} \int_0^{\pi} x^2 \cos nx\,dx$$

$$= \frac{2}{\pi} \left\{ x^2 \left(\frac{\sin nx}{n} \right) - 2x \left(\frac{-\cos nx}{n^2} \right) + 2 \left(-\frac{\sin nx}{n^3} \right) \right\}_0^{\pi}$$

$$= \frac{2}{\pi} \left\{ \left[\frac{\pi^2 \sin n\pi}{n} + \frac{2\pi \cos n\pi}{n^2} - \frac{2 \sin n\pi}{n^3} \right] - \left[0 - 0 - \frac{2 \sin 0}{n^3} \right] \right\}$$

$$= \frac{2}{\pi} \left(0 + \frac{2\pi(-1)^n}{n^2} - 0 \right)$$

$$a_n = \frac{4(-1)^n}{n^2}$$

& $b_n = 0$

∴ Fourier series becomes

$$F(x) = x^2 = \frac{1}{2} \cdot \left(\frac{2\pi^2}{3} \right) + \sum_{n=1}^{\infty} \frac{(-1)^n \cos nx}{n^2}$$

$$x^2 = \frac{\pi^2}{3} + \sum_{n=1}^{\infty} \frac{4(-1)^n \cos nx}{n^2} \quad \text{—— (2)}$$

Deductions:-

(1) put $x = \pi$

$$\pi^2 = \frac{\pi^2}{3} + 4\sum_{n=1}^{\infty} \frac{(-1)^n \cos n\pi}{n^2}$$

$$\frac{1}{4}\left(\pi^2 - \frac{\pi^2}{3}\right) = \sum_{n=1}^{\infty} \frac{(-1)^n(-1)^n}{n^2}$$

$$\frac{1}{4}\left(\frac{2\pi^2}{3}\right) = \sum_{n=1}^{\infty} \frac{(-1)^{2n}}{n^2}$$

$$\frac{\pi^2}{6} = \sum_{n=1}^{\infty} \frac{1}{n^2}$$

$$\therefore \frac{\pi^2}{6} = \frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \cdots \quad \text{——} \; (3)$$

(2) put $x = 0$

$$0 = \frac{\pi^2}{3} + 4\sum_{n=1}^{\infty} \frac{(-1)^n \cos 0}{n^2}$$

$$-\frac{\pi^2}{3\times 4} = \sum_{n=1}^{\infty} \frac{(-1)^n}{n^2}$$

$$-\frac{\pi^2}{12} = \frac{-1}{1^2} + \frac{1}{2^2} - \frac{1}{3^2} + \frac{1}{4^2} - \cdots$$

$$\frac{\pi^2}{12} = \frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \cdots \quad \text{——} \; (4)$$

(3) Adding (3) & (4)

$$\frac{\pi^2}{6} + \frac{\pi^2}{12} = 2\left(\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \cdots\right)$$

$$\frac{3\pi^2}{12} = 2\left(\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \cdots\right)$$

$$\frac{\pi^2}{8} = \frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \cdots$$

Ex. 2 Obtain the Fourier series for the function $f(x)$ given by

$$f(x) = \begin{cases} 1 + \frac{2x}{\pi}, & -\pi \le x \le 0 \\ 1 - \frac{2x}{\pi}, & 0 \le x \le \pi \end{cases}$$

and prove that $\frac{\pi^2}{8} = \frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \cdots$

$\Rightarrow$ Let $f(x) = 1 + \frac{2x}{\pi}$, $-\pi \le x \le 0$

put $x = -x$

$f(-x) = 1 - \frac{2x}{\pi}$, $-\pi \le -x \le 0$

$f(-x) = 1 - \frac{2x}{\pi}$, $0 \le x \le \pi$

$f(-x) = f(x)$

$\therefore$ $f(x)$ is an even function

Fourier series is given by

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx)$$

where, $a_0 = \frac{2}{\pi} \int_0^{\pi} f(x)\,dx = \frac{2}{\pi} \int_0^{\pi} \left(1 - \frac{2x}{\pi}\right) dx$

$$= \frac{2}{\pi} \left(x - \frac{2}{\pi}\frac{x^2}{2}\right)_0^{\pi}$$

$$= \frac{2}{\pi} \left(\pi - \frac{2}{\pi} \cdot \frac{\pi^2}{2}\right)$$

$$= \frac{2}{\pi} (\pi - \pi)$$

$$\underline{a_0 = 0}$$

$$a_n = \frac{2}{\pi} \int_0^{\pi} f(x) \cos nx\,dx$$

$$= \frac{2}{\pi} \int_0^{\pi} \left(1 - \frac{2x}{\pi}\right) \cos nx\,dx$$

PAGE NO.
DATE :

$$= \frac{2}{\pi}\left\{\left(1-\frac{2x}{\pi}\right)\left(\frac{\sin nx}{n}\right)-\left(-\frac{2}{\pi}\right)\left(-\frac{\cos nx}{n^2}\right)\right\}_0^{\pi}$$

$$= \frac{2}{\pi}\left\{\left[\left(1-\frac{2\pi}{\pi}\right)\left(\frac{\sin n\pi}{n}\right)+\frac{2}{\pi}\left(-\frac{\cos n\pi}{n^2}\right)\right]\right.$$

$$\left.-\left[(1-0)\frac{\sin 0}{n}+\frac{2}{\pi}\left(-\frac{\cos 0}{n^2}\right)\right]\right\}$$

$$= \frac{2}{\pi}\left\{(-1)0-\frac{2}{\pi}\frac{(-1)^n}{n^2}-0+\frac{2}{n^2\pi}\right\}$$

$$= \frac{2}{\pi}\left(\frac{2}{n^2\pi}-\frac{2(-1)^n}{\pi n^2}\right)$$

$$a_n = \frac{4}{n^2\pi^2}\left(1-(-1)^n\right)$$

and $b_n = 0$

$\therefore$ Fourier Series becomes

$$f(x) = \frac{1}{2}(0) + \sum_{n=1}^{\infty}\frac{4}{n^2\pi^2}\left(1-(-1)^n\right)\cos nx$$

$$f(x) = \frac{4}{\pi^2}\sum_{n=1}^{\infty}\frac{(1-(-1)^n)\cos nx}{n^2}$$

put $x=0$

$$f(0) = \frac{4}{\pi^2}\sum_{n=1}^{\infty}\frac{(1-(-1)^n)\cos 0}{n^2}$$

$$\frac{\pi^2}{4}f(0) = \sum_{n=1}^{\infty}\frac{(1-(-1)^n)}{n^2}$$

$$\frac{\pi^2}{4}(1) = \frac{1-(-1)}{1^2}+\frac{1-1}{2^2}+\frac{1-(-1)}{3^2}+\frac{1-1}{4^2}+\cdots$$

$$\frac{\pi^2}{4} = \frac{2}{1^2}+\frac{2}{3^2}+\frac{2}{5^2}+\cdots$$

$$\therefore \frac{\pi^2}{8} = \frac{1}{1^2}+\frac{1}{3^2}+\frac{1}{5^2}+\cdots$$

PAGE NO.
DATE :

Ex. 3 Find Fourier Series of

$$f(x) = \begin{cases} \cos x, & -\pi \le x \le 0 \\ -\cos x, & 0 < x < \pi \end{cases}$$

$\Rightarrow$ Given,

$$f(x) = \begin{cases} \cos x & -\pi < x < 0 \\ -\cos x & 0 < x < \pi \end{cases}$$

Replace x by $-x$

$$f(-x) = \begin{cases} \cos(-x), & -\pi < -x < 0 \\ -\cos(-x) & 0 < -x < \pi \end{cases}$$

$$= \begin{cases} \cos x & 0 < x < \pi \\ -\cos x & -\pi < x < 0 \end{cases}$$

$f(-x) = -f(x)$

$\therefore$ $f(x)$ is an odd function

Fourier series is given by

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx)$$

where, $a_0 = a_n = 0$

$$b_n = \frac{2}{\pi} \int_0^{\pi} f(x) \sin nx \, dx$$

$$= \frac{2}{\pi} \int_0^{\pi} -\cos x \sin nx \, dx$$

$$= \frac{-1}{\pi} \int_0^{\pi} 2\cos x \sin nx \, dx$$

$$= \frac{-1}{\pi} \int_0^{\pi} \{\sin(x+nx) - \sin(x-nx)\} dx$$

$$= \frac{-1}{\pi} \int_0^{\pi} \{\sin(1+n)x - \sin(1-n)x\} dx$$

$$= \frac{-1}{\pi} \left[-\frac{\cos(1+n)x}{1+n} + \frac{\cos(1-n)x}{1-n} \right]_0^{\pi}$$

$$= \frac{-1}{\pi}\left\{\left[\frac{-\cos(1+n)\pi}{1+n} + \frac{\cos(1-n)\pi}{1+n}\right] - \left[\frac{-\cos 0}{1+n} + \frac{\cos 0}{1-n}\right]\right\}$$

$$= \frac{-1}{\pi}\left\{\frac{-(-1)^{1+n}}{1+n} + \frac{(-1)^{1+n}}{1-n} + \frac{1}{1+n} - \frac{1}{1-n}\right\}$$

$$= \cancel{\frac{-1}{\pi}\left\{\frac{(-1)^n}{1+n} - \frac{(-1)^n}{1-n} + \frac{1-n-1-n}{1-n^2}\right\}}$$

$$= \frac{-1}{\pi}\left\{\frac{1+(-1)^n}{n+1} + \frac{1+(-1)^n}{n-1}\right\}$$

$$= -\frac{[1+(-1)^n]}{\pi}\left(\frac{1}{n+1} + \frac{1}{n-1}\right)$$

$$= -\frac{[1+(-1)^n]}{\pi}\left(\frac{n-1+n+1}{n^2-1}\right)$$

$$= -\frac{[1+(-1)^n]}{\pi}\left(+\frac{2n}{n^2-1}\right)$$

$$b_n = \frac{2n(1+(-1)^n)}{\pi(1-n^2)}$$

$\therefore$ Fourier Series becomes.

$$f(x) = 0 + \sum_{n=1}^{\infty} \frac{2n[1+(-1)^n]}{\pi(1-n^2)} \sin nx$$

$$= \frac{2}{\pi}\sum_{n=1}^{\infty} \frac{n[1+(-1)^n]}{1-n^2} \sin nx$$

Find the fourier series for $f(x)$ in the interval $(-\pi, \pi)$, when

$$f(x) = \begin{cases} \pi + x, & -\pi < x < 0 \\ \pi - x, & 0 < x < \pi \end{cases}$$

$\Rightarrow$ Given.

$$f(x) = \begin{cases} \pi + x, & -\pi < x < 0 \\ \pi - x & 0 < x < \pi \end{cases} \quad \text{——— (1)}$$

Replace x by $-x$

$$f(-x) = \begin{cases} \pi - x, & -\pi < -x < 0 \\ \pi + x, & 0 < -x < \pi \end{cases}$$

$$= \begin{cases} \pi - x, & 0 < x < \pi \\ \pi + x, & -\pi < x < 0 \end{cases}$$

$f(-x) = f(x)$ (From (1))

$\therefore$ $f(x)$ is an even function

Fourier series is given by

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx) \quad \text{——— (2)}$$

where,

$$a_0 = \frac{2}{\pi} \int_0^{\pi} f(x) dx$$

$$= \frac{2}{\pi} \int_0^{\pi} (\pi - x) dx$$

$$= \frac{2}{\pi} \left\{ \pi x - \frac{x^2}{2} \right\}_0^{\pi}$$

$$= \frac{2}{\pi} \left\{ \pi^2 - \frac{\pi^2}{2} \right\} = \frac{2}{\pi} \times \frac{\pi^2}{2}$$

$$= \pi$$

$$a_n = \frac{2}{\pi} \int_0^{\pi} f(x) \cos nx dx$$

$$= \frac{2}{\pi} \int_0^{\pi} (\pi - x) \cos nx dx$$

$$= \frac{2}{\pi} \left\{ (\pi - x)\left(\frac{\sin nx}{n}\right) - (-1)\left(-\frac{\cos nx}{n^2}\right) \right\}_0^{\pi}$$

$$= \frac{2}{\pi}\left\{\left(0 - \frac{\cos n\pi}{n^2}\right) - \left(\pi\frac{\sin n0}{n} - \frac{\cos n0}{n^2}\right)\right\}$$

$$= \frac{2}{\pi}\left\{-\frac{(-1)^n}{n^2} + \frac{1}{n^2}\right\}$$

$$= \frac{2}{\pi n^2}\left(1 - (-1)^n\right)$$

$\therefore$ Equation (2) becomes.

$$F(x) = \frac{1}{2}(\pi) + \sum_{n=1}^{\infty} \frac{2}{n^2\pi}\left(1 - (-1)^n\right)$$

$$F(x) = \frac{\pi}{2} + \frac{2}{\pi}\sum_{n=1}^{\infty}\left(\frac{1-(-1)^n}{n^2}\right)$$

Find the fourier series for the function $F(x) = x + x^2$, $-\pi < x < \pi$.

$\Rightarrow$ Given,

$F(x) = x + x^2$, $-\pi < x < \pi$

Replace x by $-x$

$F(-x) = -x + x^2$, $-\pi < -x < \pi$

$F(-x) = -x + x^2$, $-\pi < x < \pi$

$\therefore$ $F(x)$ is neither even or nor odd.

Fourier series is given by

$$F(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}(a_n \cos nx + b_n \sin nx)$$

where,

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} F(x)\,dx = \frac{1}{\pi}\int_{-\pi}^{\pi}(x + x^2)\,dx$$

$$= \frac{1}{\pi}\left\{\frac{x^2}{2} + \frac{x^3}{3}\right\}_{-\pi}^{\pi}$$

$$= \frac{1}{\pi}\left\{\left(\frac{\pi^2}{2} + \frac{\pi^3}{3}\right) - \left(\frac{\pi^2}{2} - \frac{\pi^3}{3}\right)\right\}$$

$$= \frac{1}{\pi}\left\{\frac{\pi^2}{2} + \frac{\pi^3}{3} - \frac{\pi^2}{2} + \frac{\pi^3}{3}\right\}$$

$$= \frac{1}{\pi} \times \frac{2\pi^3}{3}$$

$$a_0 = \frac{2\pi^2}{3}$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} F(x)\cos nx\,dx = \frac{1}{\pi}\int_{-\pi}^{\pi}(x + x^2)\cos nx\,dx$$

$$= \frac{1}{\pi}\left\{(x+x^2)\left(\frac{\sin nx}{n}\right) - (1+2x)\left(-\frac{\cos nx}{n^2}\right) + 2\left(-\frac{\sin nx}{n^3}\right)\right\}_{-\pi}^{\pi}$$

$$= \frac{1}{\pi}\left\{\left[(\pi+\pi^2)(0) + (1+2\pi)\frac{\cos n\pi}{n^2} - 2(0)\right] - \left[(-\pi+\pi^2)(0) - (1-2\pi)\frac{\cos n\pi}{n^2} + 2(0)\right]\right\}$$

$$= \frac{1}{\pi}\left\{0 + (1+2\pi)\frac{\cos n\pi}{n^2} - (1-2\pi)\frac{\cos n\pi}{n^2}\right\}$$

$$= \frac{1}{\pi}\frac{\cos n\pi}{n^2}\left(1+2\pi-1+2\pi\right)$$

$$= \frac{1}{\pi}\frac{\cos n\pi}{n^2} \times 4\pi$$

$$a_n = \frac{4(-1)^n}{n^2}$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,dx = \frac{1}{\pi}\int_{-\pi}^{\pi}(x+x^2)\sin nx\,dx$$

$$= \frac{1}{\pi}\left\{(x+x^2)\left(-\frac{\cos nx}{n}\right) - (1+2x)\left(-\frac{\sin nx}{n^2}\right) + 2\left(\frac{\cos nx}{n^3}\right)\right\}_{-\pi}^{\pi}$$

$$= \frac{1}{\pi}\left\{\left(-(\pi+\pi^2)\frac{\cos n\pi}{n} + 0 + 2\frac{\cos n\pi}{n^3}\right) - \left(-(-\pi+\pi^2)\frac{\cos n\pi}{n} + 0 + 2\frac{\cos n\pi}{n^3}\right)\right\}$$

$$= \frac{1}{\pi}\left\{\frac{\cos n\pi}{n}\left(-\pi-\pi^2-\pi+\pi^2\right)\right\}$$

$$= \frac{1}{\pi}\frac{\cos n\pi}{n}(-2\pi)$$

$$= -\frac{2(-1)^n}{n}$$

∴ Equation ② becomes

$$f(x) = \frac{1}{2}\times\frac{2\pi^2}{3} + \sum_{n=1}^{\infty}\left\{\frac{4(-1)^n}{n^2}\cos nx + -\frac{2(-1)^n}{n}\sin nx\right\}$$

$$= \frac{\pi^2}{3} + \sum_{n=1}^{\infty}\left\{\frac{4(-1)^n}{n^2}\cos nx - \frac{2(-1)^n}{n}\sin nx\right\}$$