Unit - III

Linear Differential Equation With constant coefficients

* **Linear differential Equation with constant Coefficient :-**

The general form of linear differential equation with constant coefficient is,

$$a_0 \frac{d^n y}{dx^n} + a_1 \frac{d^{n-1}y}{dx^{n-1}} + a_2 \frac{d^{n-2}y}{dx^{n-2}} + \cdots\cdots + a_n y = X \quad \text{--- (1)}$$

where $a_1, a_2, a_3, \cdots a_n$ are constant and X is a function of x.

By using differential operator in eq^n (2) as

$$\frac{d}{dx} = D, \quad \frac{d^2}{dx^2} = D^2, \quad \frac{d^3}{dx^3} = D^3 \cdots\cdots \frac{d^n}{dx^n} = D^n$$

Also,

$$\frac{dy}{dx} = Dy, \quad \frac{d^2y}{dx^2} = D^2y, \quad \frac{d^3y}{dx^3} = D^3y \cdots \frac{d^ny}{dx^n} = D^ny$$

$\therefore$ eq^n (2) becomes

$$a_0 D^n y + a_1 D^{n-1} y + a_2 D^{n-2} y + \cdots\cdots + a_n y = X$$

$$(a_0 D^n + a_1 D^{n-1} + a_2 D^{n-2} + \cdots + a_n) y = X$$

$$F(D) y = X$$

where,

$$F(D) = a_0 D^n + a_1 D^{n-1} + a_2 D^{n-2} + \cdots + a_n.$$

The Solution of eq^n (1) involves two parts

1) C.F. (complementary function)
2) P.I (Perticular integral)

$\therefore$ complete solution is given by

$$y = C.F + P.I.$$

* **Rules of finding complementary function:-**

To find C.F., find the roots of auxillary equation $f(m) = 0$.

1) If roots are real and distinct:

If m_1, m_2, m_3 are real and distinct roots then,

$$C.F. = c_1 e^{m_1 x} + c_2 e^{m_2 x} + c_3 e^{m_3 x}$$

2) If roots are real and repeated

If roots are repeated $m_1 = m_2$ then

$$C.F = (c_1 + x c_2) e^{m_1 x}$$

If m_1, m_1, m_1 are the roots of auxillary equati $f(m) = 0$ then,

$$C.F. = (c_1 + x c_2 + x^2 c_3) e^{m_1 x}$$

3) If the roots are complex and distinct:-

Complex roots always occurs in conjugate pair.

Let $m_1 = \alpha + i\beta$, $m_2 = \alpha - i\beta$ are the two roots then

$$C.F. = e^{\alpha x} \{ (c_1 + \cancel{x c_2}) \cos\beta x + (c_3 + \cancel{x c_4}) \sin\beta x \}$$

4) If the roots are complex and repeated.

Let $m_1 = m_2 = \alpha + i\beta$ and $m_3 = m_4 = \alpha - i\beta$ then

$$C.F. = e^{\alpha x} \{ (c_1 + x c_2) \cos\beta x + (c_3 + x c_4) \sin\beta x \}$$

5) If the roots are irrational:

Let $m_1 = \alpha + i\sqrt{\beta}$ and $m_2 = \alpha - i\sqrt{\beta}$

then,

$$C.F. = e^{\alpha x} \{ c_1 \cosh\sqrt{\beta} x + c_2 \sinh\sqrt{\beta} x \}$$

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Ex. 1 Solve: $\frac{d^3y}{dx^3} - 7\frac{dy}{dx} - 6y = 0$

$\Rightarrow$ Given,

$$\frac{d^3y}{dx^3} - 7\frac{dy}{dx} - 6y = 0$$

Auxullary equation is,

$$m^3 - 7m - 6 = 0$$

$$\begin{array}{c|cccc} 3 & 1 & 0 & -7 & -6 \\ & \downarrow & 3 & 9 & 6 \\ \hline & 1 & 3 & 2 & \boxed{0} \end{array}$$

$\therefore (m-3)(m^2+3m+2)$

$\therefore m^3 - 7m - 6 = 0$

$\therefore (m-3)(m+1)(m+2) = 0$

$\Rightarrow m = -1, -2, +3$

$\therefore$ Roots of axullary eq^n are real and distinct.

$$C.F = c_1 e^{m_1 x} + c_2 e^{m_2 x} + c_3 e^{m_3 x}$$

$$C.F. = c_1 e^{-x} + c_2 e^{-2x} + c_3 e^{3x}$$

$\therefore$ complete solution is,

$$y = C.F + P.I$$

$$y = c_1 e^{-x} + c_2 e^{-2x} + c_3 e^{3x} + 0$$

$$\therefore y = c_1 e^{-x} + c_2 e^{-2x} + c_3 e^{3x}.$$

Ex. 2 Solve: $(D^3 - 3D^2 + 4)y = 0$

Given,

$$(D^3 - 3D^2 + 4)y = 0$$

$\therefore$ Auxullary equation is

$$m^3 - 3m^2 + 4 = 0$$

$$\begin{array}{c|cccc} 2 & 1 & -3 & 0 & 4 \\ & \downarrow & 2 & -2 & -4 \\ \hline & 1 & -1 & -2 & \boxed{0} \end{array}$$

$\therefore (m-2)(m^2 - m - 2) = 0$

$(m-2)(m-2)(m+1) = 0$

$\Rightarrow m = -1, 2, 2$

Roots are real and repeated.

$\therefore$ C.F $= c_1 e^{m_1 x} + (c_2 + x c_3) e^{m_2 x}$

C.F. $= c_1 e^{-x} + (c_2 + x c_3) e^{2x}$

But P.I $= 0$

$\therefore$ complete solution is given by

$y =$ C.F + P.I

$y = c_1 e^{-x} + (c_2 + x c_3) e^{2x}$.

Ex 3 Solve: $(D^4 - n^4) = 0$, where $D \equiv \frac{d}{dx}$

$\Rightarrow$ Given,

$$(D^4 - n^4) y = 0$$

Auxullary equation is

$$(m^4 - n^4) = 0$$

$$[(m^2)^2 - (n^2)^2] = 0$$

$$(m^2 - n^2)(m^2 + n^2) = 0$$

$(m^2 - n^2) = 0$ or $m^2 + n^2 = 0$

$m^2 = n^2$ or $m^2 = -n^2$

$m = \pm n$ or $m = \pm ni$

$= 0 \pm ni = \alpha \pm i\beta$ $(\alpha = 0, \beta = n)$

$\therefore$ C.F $= c_1 e^{m_1 x} + c_2 e^{m_2 x} + e^{\alpha x}\{c_3 \cos\beta x + c_4 \sin\beta x\}$

$= c_1 e^{nx} + c_2 e^{-nx} + e^{0x}\{c_3 \cos nx + c_4 \sin nx\}$

C.F. $= c_1 e^{nx} + c_2 e^{-nx} + c_3 \cos nx + c_4 \sin \cancel{x} nx$

But,

P.I $= 0$

complete solution is

$y =$ C.F + P.I

$y = c_1 e^{nx} + c_2 e^{-nx} + c_3 \cos nx + c_4 \sin nx$

Ex. 4 Solve: $(D^4+8D^2+16)y=0$

$\Rightarrow$ Given,

$$(D^4+8D^2+16)y=0$$

Auxillary equation is,

$$m^4+8m^2+16=0$$

$$(m^2+4)^2=0$$

$$(m^2+4)(m^2+4)=0$$

$m^2+4=0$ or $m^2+4=0$

$m=\pm 2i$ or $m=\pm 2i$

$=0\pm 2i$ $=0\pm 2i$

$=\alpha\pm i\beta$ $=\alpha+i\beta$ $(\alpha=0,\ \beta=2)$

Roots are complex and repeated.

$$\therefore C.F = e^{\alpha x}\{(c_1+c_2x)\cos\beta x+(c_3+c_4x)\sin\beta x\}$$

$$= e^{0x}\{(c_1+c_2x)\cos 2x+(c_3+c_4x)\sin 2x\}$$

$$C.F. = (c_1+c_2x)\cos 2x+(c_3+c_4x)\sin 2x$$

But $P.I=0$

$\therefore$ complete solution is,

$$y=C.F+P.I$$

$$y=(c_1+c_2x)\cos 2x+(c_3+c_4x)\sin 2x.$$

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	Examples	Answer
1	Solve: $(D^3+1)^3 y = 0$	$C.F = (c_1+c_2x+c_3x^2)\cos x + (c_4+c_5x+c_6x^2)\sin x$
2	Solve: $(25D^2 - 20D + 4)y = 0$	$C.F = (c_1+c_2x)e^{2/5\,x}$
3	Solve: $(D^2+4)y = 0$, $y(0) = 2$, $y(\pi/2) = -2$	$C.F = y = 2(\cos x - \sin x)$
4	Solve: $(D^6-1)y = 0$	$C.F = c_1e^{x} + c_2e^{-x} + e^{x/2}(c_3\cos\frac{\sqrt{3}}{2} + c_4\sin\frac{\sqrt{3}}{2}x) + e^{-x/2}(c_5\cos\frac{\sqrt{3}}{2}x + c_6\sin\frac{\sqrt{3}}{2}x)$
5)	Solve: $\frac{d^4x}{dt^2} = n^4x$	

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★ Inverse operator:-

We know that,

$$F(D)\left\{\frac{1}{F(D)}X\right\} = X$$

Thus, $y = \frac{1}{F(D)}X$ satiesfies the equation $F(D)y = X$

Hence, $y = \frac{1}{F(D)}X$ is a solution of equation $F(D)y = X$.

This solution is called perticular solution.

Note:-

1) $F(D)$ and $1/F(D)$ are called inverse operator.

2) If $F(D) = D$, then $\frac{1}{F(D)}X = \frac{1}{D}X = \int X\,dx$.

3) If $F(D) = D - a$, then $\frac{1}{F(D)}X = \frac{1}{D-a}X = e^{ax}\int e^{-ax}X\,dx$

4) If $F(D) = D + a$, then $\frac{1}{F(D)}X = \frac{1}{D+a}X = e^{-ax}\int e^{ax}X\,dx$

★ Rules for Finding the Perticular Integral:-

Case:- I : When $X = e^{ax}$ then

$$\frac{1}{F(D)}X = \frac{1}{F(D)}e^{ax} = \frac{1}{F(a)}e^{ax}, \text{ provided } F(a) \neq 0$$

If $F(a) = 0$, then $\frac{1}{F(D)}e^{ax} = x\frac{1}{F'(a)}e^{ax}$, provided $F'(a) \neq 0$

If $F'(a) = 0$, then $\frac{1}{F(D)}e^{ax} = x^2\frac{1}{F''(a)}e^{ax}$, provided $F''(a) \neq 0$

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Ex. 1 Solve: $(D^2 - 3D + 2)y = e^{3x}$

$\Rightarrow$ Given, $(D^2 - 3D + 2)y = e^{3x}$

The given equation is $F(D)y = X$

where,

$F(D) = D^2 - 3D + 2$ & $X = e^{3x}$

Auxullary equation is

$$m^2 - 3m + 2 = 0$$

$$(m-1)(m-2) = 0$$

$$\Rightarrow m = 1, 2$$

$\therefore$ C.F $= c_1 e^{m_1 x} + c_2 e^{m_2 x}$

C.F $= c_1 e^{x} + c_2 e^{2x}$

Now,

$$P.I. = \frac{1}{F(D)} X$$

$$= \frac{1}{D^2 - 3D + 2} e^{3x}$$

$$= \frac{1}{(3)^2 - 3(3) + 2} e^{3x}$$

$$= \frac{1}{9 - 9 + 2} e^{3x}$$

$$P.I. = \frac{1}{2} e^{3x}$$

$\therefore$ Complete solution is given by

$$y = C.F + P.I$$

$$y = c_1 e^{x} + c_2 e^{2x} + \frac{1}{2} e^{3x}$$

Ex. 2 Solve: $(D^3 - 6D^2 + 11D - 6)y = e^{-2x} + e^{-3x}$

$\Rightarrow$ Given,

$$(D^3 - 6D^2 + 11D - 6)y = e^{-2x} + e^{-3x}$$

Given equation is $F(D)y = X$

where,

$F(D) = D^3 - 6D^2 + 11D - 6$ & $X = e^{-2x} + e^{-3x}$

Auxullary equation is

$$m^3 - 6m^2 + 11m - 6 = 0$$

$$\therefore (m-1)(m^2 - 5m + 6) = 0$$

$$(m-1)(m-2)(m-3) = 0$$

$$\Rightarrow m = 1, 2, 3.$$

1	1	-6	11	-6
	↓	1	-5	6
	1	-5	6	$\boxed{0}$

$$\therefore C.F = c_1 e^{m_1 x} + c_2 e^{m_2 x} + c_3 e^{m_3 x}.$$

$$C.F = c_1 e^{x} + c_2 e^{2x} + c_3 e^{3x}$$

Now,

$$P.I = \frac{1}{F(D)} x$$

$$= \frac{1}{D^3 - 6D^2 + 11D - 6} \cdot (e^{-2x} + e^{-3x})$$

$$= \frac{1}{D^3 - 6D^2 + 11D - 6} e^{-2x} + \frac{1}{D^3 - 6D^2 + 11D - 6} e^{-3x}$$

$$= \frac{1}{(-2)^3 - 6(-2)^2 + 11(-2) - 6} e^{-2x} + \frac{1}{(-3)^3 - 6(-3)^2 + 11(-3) - 6} e^{-3x}$$

$$= \frac{1}{-8 - 24 - 22 - 6} e^{-2x} + \frac{1}{-27 - 54 - 33 - 6} e^{-3x}$$

$$= \frac{-1}{60} e^{-2x} + \frac{-1}{120} e^{-3x}$$

$$P.I = \frac{-1}{60} e^{-2x} - \frac{1}{120} e^{-3x}$$

$\therefore$ Complete solution is given by

$$y = C.F + P.I$$

$$= c_1 e^{x} + c_2 e^{2x} + c_3 e^{3x} - \frac{1}{60} e^{-2x} - \frac{1}{120} e^{-3x}.$$

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Ex. 3 Solve: $(D^2-a^2)y = e^{ax} - e^{-ax}$; $D = \frac{d}{dx}$

$\Rightarrow$ Given,

$$(D^2-a^2)y = e^{ax} - e^{-ax}$$

Given equation is $f(D)y = X$

where,

$$f(D) = D^2-a^2 \quad \& \quad X = e^{ax} - e^{-ax}$$

Auxillary equation is

$$m^2 - a^2 = 0$$

$$m^2 = a^2$$

$$m = \pm a$$

$$\therefore C.F = C_1 e^{m_1 x} + C_2 e^{m_2 x}$$

$$C.F. = C_1 e^{ax} + C_2 e^{-ax}$$

Now,

$$P.I. = \frac{1}{f(D)} X$$

$$= \frac{1}{D^2-a^2}(e^{ax} - e^{-ax})$$

$$= \frac{1}{D^2-a^2} e^{ax} - \frac{1}{D^2-a^2} e^{-ax}$$

$$= x \cdot \frac{1}{2D} e^{ax} - x \cdot \frac{1}{2D} e^{-ax} \quad \left(\text{if } f(a)=0,\ \frac{x \cdot 1}{f'(a)} X\right)$$

$$= \frac{x}{2}\int e^{ax}dx - \frac{x}{2}\int e^{-ax}dx$$

$$= \frac{x}{2}\left(\frac{e^{ax}}{a}\right) - \frac{x}{2}\left(\frac{e^{-ax}}{-a}\right)$$

$$= \frac{x}{a}\left(\frac{e^{ax}+e^{-ax}}{2}\right)$$

$$P.I = \frac{x}{a}\cosh(ax)$$

$\therefore$ complete Solution is given by

$$y = C.F. + P.I$$

$$y = C_1 e^{ax} + C_2 e^{-ax} + \frac{x}{a}\cosh(ax).$$

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Ex. 4 Solve: $y'' + 4y' + 13y = 18e^{-2x}$; $y(0) = 0$, $y'(0) = 9$

Given, $y'' + 4y' + 13y = 18e^{-2x}$

Given equation is $F(D)y = X$

where,

$$F(D) = D^2 + 4D + 13 \quad \& \quad X = 18e^{-2x}$$

Auxiliary equation is

$$m^2 + 4m + 13 = 0$$

$$\Rightarrow \quad a = 1, \; b = 4, \; c = 13$$

$$m = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$= \frac{-4 \pm \sqrt{16 - 4 \times 1 \times 13}}{2(1)} = \frac{-4 \pm \sqrt{-36}}{2}$$

$$= \frac{-4 \pm 6i}{2} = -2 \pm 3i$$

$$m = -2 \pm 3i = \alpha + i\beta$$

$$\alpha = -2, \quad \beta = 3.$$

$$\therefore C.F = e^{\alpha x}(c_1 \cos\beta x + c_2 \sin\beta x)$$

$$C.F. = e^{-2x}(c_1 \cos 3x + c_2 \sin 3x)$$

Now,

$$P.I. = \frac{1}{F(D)} X$$

$$= \frac{1}{D^2 + 4D + 13}(18e^{-2x})$$

$$= 18 \frac{1}{D^2 + 4D + 13} e^{-2x}$$

$$= 18 \cdot \frac{1}{(-2)^2 + 4(-2) + 13} e^{-2x}$$

$$= 18 \cdot \frac{1}{4 - 8 + 13} e^{-2x}$$

$$= \frac{18}{9} e^{-2x}$$

$$P.I = 2e^{-2x}.$$

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$\therefore$ complete solution is given by

$$y = C.F. + P.I.$$

$$y = e^{-2x}(c_1 \cos 3x + c_2 \sin 3x) + 2e^{-2x} \quad \text{--- (1)}$$

But

$y(0) = 0 \Rightarrow$ put $x=0,\ y=0$ in eq^n (1)

$$0 = 1\,(c_1(1) + c_2(0)) + 2(1)$$

$$0 = c_1 + 2$$

$$c_1 = -2$$

Also, $y'(0) = 9 \Rightarrow$

From (1),

$$y' = -2e^{-2x}(c_1 \cos 3x + c_2 \sin 3x) + e^{-2x}(-3c_1 \sin 3x + 3c_2 \cos 3x) + (-4)e^{-2x}$$

put $y' = 9,\ x = 0$

$$9 = -2\,(c_1(1) + c_2(0)) + (1)(-3c_1(0) + 3c_2(1)) - 4(1)$$

$$= -2c_1 + 3c_2 - 4$$

$$9 = -2(-2) + 3c_2 - 4$$

$$9 = -4 + 3c_2 - 4$$

$$3c_2 = 9$$

$$\Rightarrow c_2 = 3$$

$\therefore$ Eq^n (1) becomes

$$y = e^{-2x}(-2\cos 3x + 3\sin 3x) + 2e^{-2x}.$$

Case - II : When $X = \sin(ax+b)$ or $\cos(ax+b)$

$$\frac{1}{F(D^2)} \sin(ax+b) = \frac{1}{F(-a^2)} \sin(ax+b) \quad ; \quad F(-a^2) \neq 0$$

$$\frac{1}{F(D^2)} \cos(ax+b) = \frac{1}{F(-a^2)} \cos(ax+b) \quad ; \quad F(-a^2) \neq 0$$

if $F(-a^2) = 0$, then

$$\frac{1}{F(D^2)} \sin(ax+b) = x \cdot \frac{1}{F'(-a^2)} \sin(ax+b) \; ; \quad F'(-a^2) \neq 0$$

$$\frac{1}{F(D^2)} \cos(ax+b) = x \cdot \frac{1}{F'(-a^2)} \cos(ax+b) \; ; \quad F'(-a^2) \neq 0$$

if $F'(-a^2) = 0$, then

$$\frac{1}{F(D^2)} \sin(ax+b) = x^2 \cdot \frac{1}{F''(-a^2)} \sin(ax+b) \; ; \quad F''(-a^2) \neq 0$$

$$\frac{1}{F(D^2)} \cos(ax+b) = x^2 \cdot \frac{1}{F''(-a^2)} \cos(ax+b) \; ; \quad F''(-a^2) \neq 0$$

Ex. 1

Solve: $\frac{d^3y}{dx^3} - 3\frac{d^2y}{dx^2} + 4\frac{dy}{dx} - 2y = e^x + \cos x$.

Given $(D^3 - 3D^2 + 4D - 2)y = e^x + \cos x$

given equation is $F(D)y = X$

where, $F(D) = D^3 - 3D^2 + 4D - 2$ & $X = e^x + \cos x$

Auxiliary equation is

$$m^3 - 3m^2 + 4m - 2 = 0$$

1	1	-3	4	-2
	↓	1	-2	2
	1	-2	2	$\boxed{0}$

$\therefore (m-1)(m^2 - 2m + 2) = 0$

$\Rightarrow m = 1$ or $m^2 - 2m + 2 = 0$

$a = 1, \; b = -2, \; c = 2$

$$m = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$= \frac{2 \pm \sqrt{4 - 4 \times 1 \times 2}}{2(1)}$$

$$= \frac{2 \pm \sqrt{-4}}{2} = \frac{2 \pm 2i}{2}$$

$$m = 1 \pm i = \alpha + i\beta$$

$\alpha = 1, \; \beta = 1$

$$C.F = c_1 e^{m_1 x} + e^{\alpha x}(c_2 \cos\beta x + c_3 \sin\beta x)$$

$$C.F. = c_1 e^{x} + e^{x}(c_2 \cos x + c_3 \sin x)$$

Now,

$$P.I = \frac{1}{F(D)} X$$

$$= \frac{1}{D^3 - 3D^2 + 4D - 2}(e^x + \cos x)$$

$$= \frac{1}{D^3 - 3D^2 + 4D - 2} e^x + \frac{1}{D^3 - 3D^2 + 4D - 2} \cos x$$

$$= x \cdot \frac{1}{3D^2 - 6D + 4} e^x + \frac{1}{D^2 \cdot D - 3D^2 + 4D - 2} \cos x$$

$$= x \cdot \frac{1}{3(1)^2 - 6(1) + 4} e^x + \frac{1}{(-1)D - 3(-1) + 4D - 2} \cos x$$

$$= \frac{x \cdot e^x}{1} + \frac{1}{-D + 3 + 4D - 2} \cos x$$

$$= xe^x + \frac{1}{3D + 1} \cos x$$

$$= xe^x + \frac{(3D - 1)}{9D^2 - 1} \cos x$$

$$= xe^x + \frac{(3D - 1)}{9(-1) - 1} \cos x$$

$$= xe^x + \frac{(3D - 1)\cos x}{-10}$$

$$= xe^x - \frac{1}{10}\{3D.\cos x - \cos x\}$$

$$= xe^x - \frac{1}{10}(-3\sin x - \cos x)$$

$$P.I = xe^x + \frac{3}{10}\sin x + \frac{1}{10}\cos x$$

$\therefore$ complete solution is given by

$$y = C.F + P.I$$

$$y = c_1 e^x + e^x(c_2 \cos x + c_3 \sin x) + xe^x + \frac{3}{10}\sin x + \frac{1}{10}\cos x.$$

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Ex. 2 Solve: $(D^4+2D^3-3D^2)y = 3e^{2x}+4\sin x$

Given, $(D^4+2D^3-3D^2)y = 3e^{2x}+4\sin x$

given equation is $f(D)y = X$

$$F(D) = D^4+2D^3-3D^2, \quad X = 3e^{2x}+4\sin x$$

Auxillary equation is

$$m^4+2m^3-3m^2=0$$

$$m^2(m^2+2m-3)=0$$

$$m^2(m-1)(m+3)=0$$

$\Rightarrow m^2=0, \; m-1=0$ or $m=-3$

$$m = 0, 0, 1, -3$$

$$\therefore C.F = c_1e^{m_1x}+c_2e^{m_2x}+c_3e^{m_3x}+c_4e^{m_4x}$$

$$= c_1e^0+c_2x\cdot e^0+c_3e^x+c_4e^{-3x}$$

$$C.F. = c_1+c_2x+c_3e^x+c_4e^{-3x}$$

Now,

$$P.I = \frac{1}{F(D)}X$$

$$= \frac{1}{D^4+2D^3-3D^2}(3e^{2x}+4\sin x)$$

$$= 3\cdot\frac{1}{D^4+2D^3-3D^2}e^{2x}+4\cdot\frac{1}{D^4+2D^3-3D^2}\sin x$$

$$= 3\frac{1}{(2)^4+2(2)^3-3(2)^2}e^{2x}+4\cdot\frac{1}{D^2\cdot D^2+2\cdot D\cdot D^2-3D^2}\sin x$$

$$= \frac{3}{20}e^{2x}+4\cdot\frac{1}{(-1)(-1)+2D(-1)-3(-1)}\sin x$$

$$= \frac{3}{20}e^{2x}+\frac{4}{1-2D+3}\sin x$$

$$= \frac{3}{20}e^{2x}+\frac{4}{4-2D}\sin x$$

$$= \frac{3}{20}e^{2x}+\frac{4(4+2D)}{16-4D^2}\sin x$$

$$= \frac{3}{20}e^{2x}+\frac{4(4+2D)}{16-4(-1)}\sin x$$

$$= \frac{3}{20}e^{2x}+\frac{4}{20}(4+2D)\sin x$$

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$$= \frac{3}{20}e^{2x} + \frac{4}{20}\left\{4\sin x + 2\cdot D\sin x\right\}$$

$$P.I. = \frac{3}{20}e^{2x} + \frac{4}{20}(4\sin x + 2\cos x)$$

$\therefore$ Complete solution is given by

$y = C.F + P.I.$

$$y = c_1 + c_2 x + c_3 e^{x} + c_4 e^{-3x} + \frac{3}{20}e^{2x} + \frac{4}{20}(4\sin x + 3\cos x)$$

Ex.3 Solve: $\frac{d^2y}{dx^2} - 2\frac{dy}{dx} + y = xe^{x}\sin x$.

$\Rightarrow$ Given, $(D^2 - 2D + 1)y = xe^{x}\sin x$

Given equation is $f(D)y = X$

where,

$f(D) = D^2 - 2D + 1$, $X = xe^{x}\sin x$

Auxiliary equation is

$m^2 - 2m + 1 = 0$

$(m-1)(m-1) = 0$

$m = 1, 1.$

$C.F = (c_1 + c_2 x)e^{m_1 x}$

$C.F. = (c_1 + c_2 x)e^{x}$

Now,

$$P.I. = \frac{1}{f(D)}X$$

$$= \frac{1}{D^2 - 2D + 1}(xe^{x}\sin x)$$

$$= \frac{1}{(D-1)^2}(xe^{x}\sin x)$$

$$= e^{x}\cdot\left\{\frac{1}{D^2}x\sin x\right\}$$

$$= e^{x}\left\{\frac{1}{D}\cdot\left[\int x\sin x\,dx\right]\right\}$$

$$= e^{x}\left\{\frac{1}{D}\left[-x\cos x - \int -\cos x\cdot 1\,dx\right]\right\}$$

$$= e^{x}\left\{\frac{1}{D}(-x\cos x+\sin x)\,\cancel{dx}\right\}$$

$$= e^{x}\left\{-\int x\cos x\,dx+\int \sin x\,dx\right\}$$

$$= e^{x}\left\{-x\sin x-\int \sin x\,(-1)\,dx-\cos x\right\}$$

$$= e^{x}\left\{-x\sin x-\cos x-\cos x\right\}$$

$$= -e^{x}(x\sin x+2\cos x)$$

$\therefore$ Complete solution is given by

$$y = C.F + P.I$$

$$y = (c_1+c_2x)e^{x} - e^{x}(x\sin x+2\cos x)$$

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	Examples	Answers.
1	Solve: $\frac{d^3y}{dx^3} - 3\frac{d^2y}{dx^2} + 3\frac{dy}{dx} - y = e^x + 2$	$C.F = (c_1 + c_2x + c_3x^2)e^x$ $P.I = \frac{x^3}{6}e^x - 2$
2	Solve: $\frac{d^2y}{dx^2} + 4\frac{dy}{dx} + 5y = -2\cosh x$	$C.F = e^{-2x}(c_1\cos x + c_2\sin x)$ $P.I = -\frac{1}{10}e^x - \frac{1}{2}e^{-x}$
3	Solve: $\frac{d^3y}{dx^3} + y = 3 + 5e^x$	$C.F = c_1e^{-x} + e^{x/2}\left(c_2\cos\frac{\sqrt{3}}{2}x + c_3\sin\frac{\sqrt{3}}{2}x\right)$ $P.I. = 3 + \frac{5}{2}e^x$
4	Solve: $\frac{d^2y}{dx^2} - 4y = (1+e^x)^2$	$C.F = c_1e^{2x} + c_2e^{-2x}$ $P.I = -\frac{1}{4} - \frac{2}{3}e^x + \frac{1}{4}xe^{2x}$
5	Solve: $(D^3 - 3D^2 + 4)y = e^{2x}$	$C.F = c_1e^{-x} + e^{2x}(c_2 + c_3x)$ $P.I = \frac{x^2}{6}e^{2x}$
6	Solve: $(D^3+1)y = \cos 2x$	$C.F = c_1e^{-x} + e^{x/2}\left(c_2\cos\frac{\sqrt{3}}{2}x + c_3\sin\frac{\sqrt{3}}{2}x\right)$ $P.I. = \frac{1}{65}(\cos 2x - 8\sin 2x)$
7	Solve: $(D^2 - 4D + 4)y = e^{-4x} + 5\cos 3x$	$C.F. = (c_1 + c_2x)e^{2x}$ $P.I = \frac{1}{30}e^{-4x} + \frac{5}{169}(-12\sin 3x + 5\cos 3x)$
8	Solve: $(D^2 - 4D + 3)y = \sin 3x \cos 2x$	$C.F. = c_1e^x + c_2e^{3x}$ $P.I = \frac{-1}{884}(-10\cos 5x + 11\sin 5x)$ $+ 1/20(\sin x + 2\cos x)$
9	Solve: $(D^4 + 2D^2n^2 + n^4)y = \cos mx$	$C.F = (c_1 + c_2x)\cos nx + (c_3 + c_4x)\sin nx$ $P.I = \frac{1}{(m^2-n^2)^2}\cos mx$.

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Case-III : When $X = x^m$

$$P.I. = \frac{1}{F(D)} X = \frac{1}{F(D)} x^m = [F(D)]^{-1} x^m$$

Expand $[F(D)]^{-1}$ in the ascending powers of D by Bionomial therm.

Note:-

1) $\frac{1}{(1-x)} = (1-x)^{-1} = 1 + x + x^2 + \cdots$

2) $\frac{1}{(1+x)} = (1+x)^{-1} = 1 - x + x^2 - x^3 + \cdots$

Ex:1 Solve: $(D^2+5D+4)y = x^2+7x+9$

Given, $(D^2+5D+4)y = x^2+7x+9$

Given equation is $F(D)y = X$

where,

$F(D) = D^2+5D+4, \quad X = x^2+7x+9$

Auxullary equation is

$m^2+5m+4 = 0$

$(m+1)(m+4) = 0$

$\Rightarrow m = -1, -4$

$\therefore C.F = c_1 e^{m_1 x} + c_2 e^{m_2 x}$

$C.F. = c_1 e^{-x} + c_2 e^{-4x}$

Now,

$P.I. = \frac{1}{F(D)} X$

$= \frac{1}{D^2+5D+4} (x^2+7x+9)$

$= \frac{1}{4} \frac{1}{\left[1+\left(\frac{5D}{4}+\frac{D^2}{4}\right)\right]} (x^2+7x+9)$

$= \frac{1}{4} \left[1+\left(\frac{5D}{4}+\frac{D^2}{4}\right)\right]^{-1} (x^2+7x+9)$

$$= \frac{1}{4}\left\{1-\left(\frac{5D}{4}+\frac{D^2}{4}\right)+\left(\frac{5D}{4}+\frac{D^2}{4}\right)^2\right\}(x^2+7x+9)$$

$$= \frac{1}{4}\left\{1-\frac{5D}{4}-\frac{D^2}{4}+\frac{25D^2}{16}+\frac{10D^3}{16}+\frac{D^4}{16}\right\}(x^2+7x+9)$$

$$= \frac{1}{4}\left\{1-\frac{5D}{4}-\frac{D^2}{4}+\frac{25D^2}{4}\right\}(x^2+7x+9)$$

$$= \frac{1}{4}\left\{1-\frac{5D}{4}+\frac{21}{16}D^2\right\}(x^2+7x+9)$$

$$= \frac{1}{4}\left\{(x^2+7x+9)-\frac{5}{4}D(x^2+7x+9)+\frac{21}{16}D^2(x^2+7x+9)\right\}$$

$$= \frac{1}{4}\left\{x^2+7x+9-\frac{5}{4}(2x+7)+\frac{21}{16}(2)\right\}$$

$$= \frac{1}{4}\left\{x^2+7x+9-\frac{5x}{2}-\frac{35}{4}+\frac{42}{16}\right\}$$

$$P.I. = \frac{1}{4}\left\{x^2+\frac{9}{2}x+\frac{23}{8}\right\}$$

$\therefore$ Complete solution is given by

$y = C.F + P.I.$

$$y = c_1e^{-x}+c_2e^{-4x}+\frac{1}{4}\left(x^2+\frac{9}{2}x+\frac{23}{8}\right)$$

Ex. 2 Solve: $\frac{d^2y}{dx^2}-2\frac{dy}{dx}+5y = 25x^2$

Given, $(D^2-2D+5)y = 25x^2$

Given equation is $F(D)y = X$

where,

$$F(D) = D^2-2D+5, \quad X = 25x^2$$

Auxillary equation is,

$$m^2-2m+5 = 0$$

$$a=1, \; b=-2, \; c=5$$

$$m = \frac{-b\pm\sqrt{b^2-4ac}}{2a}$$

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$$= \frac{2 \pm \sqrt{4 - 4\times1\times5}}{2(1)}$$

$$= \frac{2 \pm \sqrt{-16}}{2}$$

$$= \frac{2 \pm 4i}{2}$$

$$m = 1 \pm 2i = \alpha + i\beta \qquad (\because \alpha = 1, \beta = 2)$$

$$\therefore C.F. = e^{\alpha x}(c_1 \cos\beta x + c_2 \sin\beta x)$$

$$C.F. = e^{x}(c_1 \cos 2x + c_2 \sin 2x)$$

Now,

$$P.I. = \frac{1}{F(D)} x$$

$$= \frac{1}{D^2 - 2D + 5}(25x^2)$$

$$= \frac{1}{5} \cdot \frac{1}{\left(1 - \frac{2D}{5} + \frac{D^2}{5}\right)}(25x^2)$$

$$= \frac{1}{5}\left[1 + \left(\frac{D^2}{5} - \frac{2D}{5}\right)\right]^{-1}(25x^2)$$

$$= \frac{1}{5}\left\{1 - \left(\frac{D^2}{5} - \frac{2D}{5}\right) + \left(\frac{D^2}{5} - \frac{2D}{5}\right)^2 - \cdots\right\}(25x^2)$$

$$= \frac{1}{5}\left\{1 - \frac{D^2}{5} + \frac{2D}{5} + \frac{D^4}{25} - \frac{4D^3}{25} + \frac{4D^2}{25}\right\}(25x^2)$$

$$= \frac{1}{5}\left\{1 - \frac{D^2}{5} + \frac{2D}{5} + \frac{4D^2}{25}\right\}(25x^2)$$

$$= \frac{1}{5}\left\{1 + \frac{2D}{5} - \frac{D^2}{25}\right\}(25x^2)$$

$$= \frac{1}{5}\left\{25x^2 + \frac{2}{5}D(25x^2) - \frac{1}{25}D^2(25x^2)\right\}$$

$$= \frac{1}{5}\left\{25x^2 + \frac{(2)(25)(2x)}{5} - \frac{1}{25}\cdot 25(2)\right\}$$

$$= \frac{1}{5}\left\{25x^2 + 20x - 2\right\} = 5x^2 + 4x - \frac{2}{5}$$

$\therefore$ Complete Solution is given by

$$y = C.F + P.I$$

$$y = e^{x}(c_1 \cos 2x + c_2 \sin 2x) + 5x^2 + 4x - 2/5$$

Ex. 3 Solve: $(D^3 - D^2 - 6D)y = 1 + x^2$, $\frac{d}{dx} \equiv D$.

$\Rightarrow$ Given, $(D^3 - D^2 - 6D)y = 1 + x^2$

Given equation is $F(D)y = X$

where,

$$F(D) = D^3 - D^2 - 6D, \quad X = 1 + x^2$$

Auxulary equation is

$$m^3 - m^2 - 6m = 0$$

$$m(m^2 - m - 6) = 0$$

$$m(m-3)(m+2) = 0$$

$$\Rightarrow m = 0, -2, 3$$

$$\therefore C.F. = c_1 e^{m_1 x} + c_2 e^{m_2 x} + c_3 e^{m_3 x}$$

$$= c_1 e^{0x} + c_2 e^{-2x} + c_3 e^{3x}$$

$$C.F. = c_1 + c_2 e^{-2x} + c_3 e^{3x}$$

Now,

$$P.I. = \frac{1}{F(D)} X$$

$$= \frac{1}{D^3 - D^2 - 6D}(1+x^2)$$

$$= \frac{1}{D(D^2 - D - 6)}(1+x^2)$$

$$= \frac{1}{D} \cdot \frac{1}{-6} \frac{1}{\left[1 + \frac{D}{6} - \frac{D^2}{6}\right]}(1+x^2)$$

$$= -\frac{1}{6D}\left[1 + \left(\frac{D}{6} - \frac{D^2}{6}\right)\right]^{-1}(1+x^2)$$

$$= -\frac{1}{6D}\left\{1 - \left(\frac{D}{6} - \frac{D^2}{6}\right) + \left(\frac{D}{6} - \frac{D^2}{6}\right)^2 + \dots\right\}(1+x^2)$$

$$= \frac{-1}{6D}\left\{1 - \frac{D}{6} + \frac{D^2}{6} + \frac{D^2}{36} - \frac{2D^3}{36} + \frac{D^4}{36}\right\}(1+x^2)$$

$$= \frac{-1}{6D}\left\{1-\frac{D}{6}+\frac{7}{36}D^2\right\}(1+x^2)$$

$$= -\frac{1}{6D}\left\{(1+x^2)-\frac{1}{6}D(1+x^2)+\frac{7}{36}D^2(1+x^2)\right\}$$

$$= \frac{-1}{6D}\left\{1+x^2-\frac{1}{6}(2x)+\frac{7}{36}(2)\right\}$$

$$= \frac{-1}{6D}\left\{1+x^2-\frac{1}{3}x+\frac{7}{18}\right\}$$

$$= \frac{-1}{6D}\left\{x^2-\frac{x}{3}+\frac{25}{18}\right\}$$

$$= -\frac{1}{6}\left\{\frac{x^3}{3}-\frac{1}{3}\cdot\frac{x^2}{2}+\frac{25}{18}x\right\}$$

$$P.I. = \frac{x^2}{36}-\frac{x^3}{18}-\frac{25}{108}x$$

Complete solution is given by

$y = C.F + P.I.$

$$y = c_1 + c_2 e^{-2x} + c_3 e^{3x} + \frac{x^2}{36} - \frac{x^3}{18} - \frac{25}{108}x$$

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Case - IV : When $X = e^{ax} V$, where V is a function of x.

$\Rightarrow$ $P.I. = \frac{1}{F(D)} X = \frac{1}{F(D)} e^{ax} V$

$= e^{ax}\left[\frac{1}{F(D+a)} V\right]$

Ex.1 Obtain the general solution of the differential equation,

$$\frac{d^2y}{dx^2} + 4\frac{dy}{dx} - 12y = e^{2x}(x-1)$$

$\Rightarrow$ Given, $(D^2+4D-12)y = e^{2x}(x-1)$

Given equation is $F(D)y = X$

where,

$F(D) = D^2+4D-12$, $X = e^{2x}(x-1)$

Auxillary equation is

$m^2+4m-12 = 0$

$(m+6)(m-2) = 0$

$\Rightarrow$ $m = 2, -6$

$C.F. = c_1 e^{m_1 x} + c_2 e^{m_2 x}$

$C.F. = c_1 e^{2x} + c_2 e^{-6x}$

Now,

$P.I. = \frac{1}{F(D)} X$

$= \frac{1}{D^2+4D-12} e^{2x}(x-1)$

$= e^{2x}\left[\frac{1}{(D+2)^2+4(D+2)-12}(x-1)\right]$

$= e^{2x}\left\{\frac{1}{D^2+4D+4+4D+8-12}(x-1)\right]$

$= e^{2x}\left\{\frac{1}{D^2+8D-0}(x-1)\right\}$

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$$= e^{2x}\left\{\frac{1}{D(D+8)}(x-1)\right\}$$

$$= e^{2x}\left\{\frac{1}{8D\left[1+\frac{D}{8}\right]}(x-1)\right\}$$

$$= e^{2x}\cdot\frac{1}{8D}\left\{\left(1+\frac{D}{8}\right)^{-1}(x-1)\right\}$$

$$= e^{2x}\cdot\frac{1}{8D}\left\{\left(1-\frac{D}{8}+\frac{D^2}{64}\right)(x-1)\right\}$$

$$= e^{2x}\cdot\frac{1}{8D}\left\{\left(1-\frac{D}{8}\right)(x-1)\right\}$$

$$= e^{2x}\cdot\frac{1}{8D}\left\{(x-1)-\frac{D(x-1)}{8}\right\}$$

$$= e^{2x}\cdot\frac{1}{8D}\left\{(x-1)-\frac{1}{8}\right\}$$

$$= \frac{e^{2x}}{8}\cdot\frac{1}{D}\left(x-\frac{9}{8}\right)$$

$$= \frac{e^{2x}}{8}\int\left(x-\frac{9}{8}\right)dx$$

$$P.I = \frac{e^{2x}}{8}\left\{\frac{x^2}{2}-\frac{9}{8}x\right\}$$

Complete solution is given by

$y = C.F + P.I$

$$y = c_1 e^{2x} + c_2 e^{-6x} + \frac{e^{2x}}{8}\left\{\frac{x^2}{2}-\frac{9}{8}x\right\}$$

Ex. 2 Find the complete solution of

$$\frac{d^2y}{dx^2} - 3\frac{dy}{dx} + 2y = xe^{3x} + \sin 2x$$

$\Rightarrow$ Given, $(D^2-3D+2)y = xe^{3x} + \sin 2x$

Given equation is $F(D)y = X$

where,

$F(D) = D^2 - 3D + 2$, $X = xe^{3x} + \sin 2x$

Auxillary equation is

$$m^2 - 3m + 2 = 0$$

$$(m-1)(m-2) = 0$$

$$\therefore m = 1, 2$$

$$C.F. = c_1 e^{m_1 x} + c_2 e^{m_2 x}$$

$$C.F. = c_1 e^{x} + c_2 e^{2x}$$

Now,

$$P.I. = \frac{1}{F(D)} x$$

$$= \frac{1}{D^2 - 3D + 2} (xe^{3x} + \sin 2x)$$

$$= \frac{1}{D^2 - 3D + 2} xe^{3x} + \frac{1}{D^2 - 3D + 2} \sin 2x$$

$$= e^{3x} \left\{ \frac{1}{(D+3)^2 - 3(D+3) + 2} x \right\} + \frac{1}{-4 - 3D + 2} \sin 2x$$

$$= e^{3x} \left\{ \frac{1}{D^2 + 6D + 9 - 3D - 9 + 2} x \right\} + \frac{1}{-2 - 3D} \sin 2x$$

$$= e^{3x} \left\{ \frac{1}{D^2 + 3D + 2} x \right\} + \frac{(-2 + 3D)}{4 - 9D^2} \sin 2x$$

$$= e^{3x} \left\{ \frac{1}{2\left(1 + \frac{D^2}{2} + \frac{3D}{2}\right)} x \right\} - \frac{(2 - 3D)}{4 - 9(-4)} \sin 2x$$

$$= \frac{e^{3x}}{2} \left\{ \left[1 + \left(\frac{D^2}{2} + \frac{3D}{2}\right)\right]^{-1} x \right\} - \frac{(2 - 3D)}{40} \sin 2x$$

$$= \frac{e^{3x}}{2} \left\{ \left[1 - \left(\frac{D^2}{2} + \frac{3D}{2}\right)\right] x \right\} - \frac{1}{40} \left\{ 2\sin 2x - 3D(\sin 2x) \right.$$

$$= \frac{e^{3x}}{2} \left\{ \left(1 - \frac{3}{2} D\right) x \right\} - \frac{1}{40} \left(2\sin 2x - 3(\cos 2x)(2) \right)$$

$$= \frac{e^{3x}}{2} \left\{ x - \frac{3(1)}{2} \right\} - \frac{1}{40} (2\sin 2x - 6\cos 2x)$$

$$P.I. = \frac{(2x-3)}{4} e^{3x} - \frac{1}{20} (\sin 2x - 3\cos 2x)$$

$$\therefore y = C.F + P.I. = c_1 e^{x} + c_2 e^{2x} + \frac{(2x-3)}{4} e^{3x} - \frac{1}{20} (\sin 2x - 3\cos 2x)$$

Ex. 3 Find the complete solution of

$$\frac{d^4y}{dx^4} - y = \cos x \cdot \cosh x.$$

$\Rightarrow$ Given, $(D^4-1)y = \cos x \cdot \cosh x$

$$= \cos x \cdot \left(\frac{e^x + e^{-x}}{2}\right)$$

$$(D^4-1)y = \frac{1}{2}\left(e^x \cos x + e^{-x}\cos x\right)$$

Given equation is $f(D)y = X$

where,

$$f(D) = D^4 - 1, \quad X = \frac{1}{2}\left(e^x\cos x + e^{-x}\cos x\right)$$

Auxullary equation is

$$m^4 - 1 = 0$$

$$(m^2)^2 - 1 = 0$$

$$(m^2-1)(m^2+1) = 0$$

$$\Rightarrow m^2 - 1 = 0 \quad \text{or} \quad m^2 + 1 = 0$$

$$m^2 = 1 \quad \text{or} \quad m^2 = -1$$

$$m = \pm 1 \quad \text{or} \quad m = \pm i$$

$$m = 1, -1 \quad \text{or} \quad m = 0 \pm i = \alpha \pm i\beta$$

$$\alpha = 0, \ \beta = 1$$

$$C.F = c_1 e^{m_1 x} + c_2 e^{m_2 x} + e^{\alpha x}\left[c_3 \cos\beta x + c_4 \sin\beta x\right]$$

$$= c_1 e^{x} + c_2 e^{-x} + e^{0x}\left[c_3 \cos x + c_4 \sin x\right]$$

$$C.F. = c_1 e^x + c_2 e^{-x} + c_3 \cos x + c_4 \sin x$$

Now,

$$P.I. = \frac{1}{f(D)} X$$

$$= \frac{1}{(D^4-1)} \frac{1}{2}\left(e^x \cos x + e^{-x}\cos x\right)$$

$$= \frac{1}{D^4-1}\left(\frac{1}{2} e^x \cos x\right) + \frac{1}{D^4-1}\left(\frac{1}{2} e^{-x}\cos x\right)$$

$$= \frac{e^x}{2}\left\{\frac{1}{(D+1)^4 - 1}\cos x\right\} + \frac{1}{2} e^{-x}\left\{\frac{1}{(D-1)^4-1}\cos x\right\}$$

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$$= \frac{e^x}{2}\left\{\frac{1}{D^4+4D^3+6D^2+4D}\cos x\right\} + \frac{e^{-x}}{2}\left\{\frac{1}{D^4-4D^3+6D^2-4D}\cos x\right\}$$

$$= \frac{e^x}{2}\left\{\frac{1}{D^2 \cdot D^2+4D^2 \cdot D+6D^2+4D}\cos x\right\} + \frac{e^{-x}}{2}\left\{\frac{1}{D^2 \cdot D^2-4D^2 \cdot D+6D^2-4D}\cos x\right\}$$

$$= \frac{e^x}{2}\left\{\frac{1}{(-1)(-1)+4(-1)D+6(-1)+4D}\cos x\right\} + \frac{e^{-x}}{2}\left\{\frac{1}{(-1)(-1)-4(-1)D+6(-1)-4D}\cos x\right\}$$

$$= \frac{e^x}{2}\left\{\frac{1}{1-4D-6+4D}\cos x\right\} + \frac{e^{-x}}{2}\left\{\frac{1}{1+4D-6-4D}\cos x\right\}$$

$$= \frac{e^x}{2}\left\{-\frac{1}{5}\cos x\right\} + \frac{e^{-x}}{2}\left\{-\frac{1}{5}\cos x\right\}$$

$$= -\frac{1}{5}\cos x\left(\frac{e^x+e^{-x}}{2}\right)$$

$$P.I. = -\frac{1}{5}\cos x \cosh x$$

Complete solution is given by

$$y = C.F + P.I$$

$$y = c_1 e^x + c_2 e^{-x} + c_3 \cos x + c_4 \sin x - \frac{1}{5}\cos x \cosh x.$$

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	Examples	Answer
1	Solve: $(D-2)^2 y = 8(e^{2x}+\sin 2x + x^2)$	$C.F = (c_1+c_2x)e^{2x}$ $P.I = 4x^2e^{2x} + \cos 2x + 2x^2 + 4x + 3$
2	Solve: $(D^3-D)y = 2x+1+4\cos x + 2e^x$	$C.F = c_1 + c_2e^x + c_3e^{-x}$ $P.I = xe^x - 2\sin x - x^2 - x - 2$
3	Solve: $\frac{d^2y}{dx^2} - 6\frac{dy}{dx} + 25y = e^{2x} + \sin x + x$	$C.F = e^{3x}(c_1\cos 4x + c_2\sin 4x)$ $P.I = \frac{e^{2x}}{17} + \frac{1}{102}(\cos x + 4\sin x) + \frac{x}{25} + \frac{6}{125}$
4	Solve: $(D^2+2)y = x^3+x^2+e^{-2x}+\cos 3x$	$C.F = c_1\cos\sqrt{2}x + c_2\sin\sqrt{2}x$ $P.I = \frac{1}{2}(x^3+x^2-3x-1) + \frac{e^{-2x}}{6} - \frac{\cos 3x}{7}$
5	Solve: $\frac{d^3y}{dx^3} + 2\frac{d^2y}{dx} + \frac{dy}{dx} = e^{2x}+x^2+x$	$C.F = c_1 + (c_2+c_3x)e^{-x}$ $P.I = \frac{1}{3}x^3 - \frac{3}{2}x^2 + 4x + \frac{e^{2x}}{18}$
6	Solve: $(D^2-2D+1)y = xe^x\sin x$	$C.F = (c_1+c_2x)e^x$ $P.I = -e^x(x\sin x + 2\cos x)$
7	Solve: $\frac{d^2y}{dx^2} + 2y = x^2e^{3x} + e^x\cos x$	$C.F = c_1\cos\sqrt{2}x + c_2\sin\sqrt{2}x$ $P.I = \frac{e^{3x}}{11}\left(x^2 - \frac{12x}{11} + \frac{50}{121}\right) + \frac{e^x}{4}(\sin x + \cos x)$
8	Solve: $\frac{d^2y}{dx^2} + 5\frac{dy}{dx} + 6y = e^{-2x}\sin 2x$	$C.F = c_1e^{-2x} + c_2e^{-3x}$ $P.I = -\frac{e^{-2x}}{10}(\cos 2x + 2\sin 2x)$
9	Solve: $\frac{d^2y}{dx^2} - y = e^x + x^2e^x$	$C.F = c_1e^x + c_2e^{-x}$ $P.I = e^x\left(\frac{x^3}{6} - \frac{x^2}{4} + \frac{3}{4}x\right)$

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★ Method of Variation of Parameters:-

Let consider Linear differential equation of Second order with constant coefficients:

$$\frac{d^2y}{dx^2} + a_1\frac{dy}{dx} + a_2y = X$$

$P.I. = uy_1 + vy_2$

where:

$$u = -\int\frac{y_2X}{W}, \quad v = \int\frac{y_1X}{W}$$

$$W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$$

Ex. 1 Solve by the method of variation of parameters

$$\frac{d^2y}{dx^2} + y = \operatorname{cosec}x.$$

$\Rightarrow$ Given, $(D^2+1)y = \operatorname{cosec}x$

Given equation is $F(D)y = X$

$F(D) = D^2+1$, $X = \operatorname{cosec}x$

Auxillary equation is

$$m^2+1=0$$

$$\Rightarrow m^2 = -1$$

$$\Rightarrow m = \pm i$$

$$= 0 \pm i = \alpha \pm i\beta$$

$C.F. = e^{\alpha x}[c_1\cos\beta x + c_2\sin\beta x]$

$= e^{0x}[c_1\cos x + c_2\sin x]$

$C.F. = c_1\cos x + c_2\sin x$

$= c_1y_1 + c_2y_2$

$\therefore y_1 = \cos x$ & $y_2 = \sin x$

$y_1' = -\sin x$ & $y_2' = \cos x$

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$$W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} \cos x & \sin x \\ -\sin x & \cos x \end{vmatrix}$$

$$= \cos^2 x + \sin^2 x$$

$$\underline{\underline{W = 1}}$$

$$P.I = u y_1 + v y_2$$

where $u = -\int \frac{y_2 X}{W} dx$

$$u = -\int \frac{\sin x \cdot \operatorname{cosec} x}{1} = -\int 1\, dx = -x$$

$$v = \int \frac{y_1 X}{W} dx = \int \frac{\cos x \cdot \operatorname{cosec} x}{1} dx$$

$$v = \int \cot x\, dx = \log(\sin x)$$

$\therefore$ $P.I = \cos x(-x) + \log(\sin x)(\sin x)$

$P.I = -x\cos x + \sin x \log(\sin x)$

complete solution is given by

$y = C.F + P.I.$

$y = c_1 \cos x + c_2 \sin x - x\cos x + \sin x \log(\sin x)$

Ex. 2 Solve by the method of variation of parameters, the equation : $y'' - 2y' + 2y = e^x \tan x$.

$\Rightarrow$ Given,

$$(D^2 - 2D + 2)y = e^x \tan x$$

Given equation is $f(D)y = X$

$f(D) = D^2 - 2D + 2$, $X = e^x \tan x$

Auxiliary equation is

$$m^2 - 2m + 2 = 0$$

$$m = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$= \frac{2 \pm \sqrt{4-8}}{2} = \frac{2 \pm 2i}{2}$$

$$= 1 \pm i = \alpha \pm i\beta$$

$$\alpha = 1, \ \beta = 1$$

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$$C.F. = e^{\alpha x}(c_1 \sin\beta x + c_2 \cos\beta x)$$

$$C.F. = e^{x}(c_1 \sin x + c_2 \cos x)$$

$$C.F = c_1 e^{x}\cos x + c_2 e^{x}\sin x$$

$$C.F. = c_1 y_1 + c_2 y_2$$

$$y_1 = e^{x}\cos x \quad \& \quad y_2 = e^{x}\sin x$$

Diff. w. r. to x

$$y_1' = e^{x}(-\sin x) + \cos x \cdot e^{x}$$

$$= e^{x}(\cos x - \sin x)$$

$$y_2' = e^{x}\cos x + \sin x\, e^{x}$$

$$= e^{x}(\cos x + \sin x)$$

Now

$$W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} e^{x}\cos x & e^{x}\sin x \\ e^{x}(\cos x - \sin x) & e^{x}(\cos x + \sin x) \end{vmatrix}$$

$$= e^{2x}\cos x(\cos x + \sin x) - e^{2x}\sin x(\cos x - \sin x)$$

$$= e^{2x}\{\cos^2 x + \sin x\cos x - \sin x\cos x + \sin^2 x)$$

$$= e^{2x}(1)$$

$$W = e^{2x}$$

$$P.I. = uy_1 + vy_2$$

where

$$u = -\int \frac{y_2 X}{W}\,dx = -\int \frac{e^{x}\sin x \cdot e^{x}\tan x}{e^{2x}}$$

$$= -\int \sin x \cdot \frac{\sin x}{\cos x}\,dx$$

$$= -\int \frac{(1-\cos^2 x)}{\cos x}\,dx$$

$$= -\int (\sec x - \cos x)\,dx$$

$$= \log(\sec x + \tan x) + \sin x$$

$$v = \int \frac{y_1 X}{W}\,dx$$

$$= \int e^{x} \cancel{\sin x} \frac{\cos x \cdot e^{x} \tan x}{e^{2x}}$$

$$= \int \cos x \cdot \frac{\sin x}{\cos x}\, dx$$

$$= \int \sin x\, dx$$

$$v = -\cos x$$

$\therefore$ P.I. $= uy_1 + vy_2$

P.I $= e^{x}\cos x\,[-\log(\sec x + \tan x) + \sin x] - e^{x}\sin x \cos x.$

Complete solution is given by

$y = C.F + P.I$

$= c_1 e^{x}\cos x + c_2 e^{x}\sin x - e^{x}\cos x \log(\sec x + \tan x) -$

$e^{x}\cos x \sin x - e^{x}\sin x \cos x$

$\therefore y = c_1 e^{x}\cos x + c_2 e^{x}\sin x - e^{x}\cos x\,(\log(\sec x + \tan x))$

Ex. 3 Use method of variation of parameters to solve:

$y'' + 3y' + 2y = e^{e^{x}}$

$\Rightarrow$ Given,

$(D^2 + 3D + 2)y = e^{e^{x}}$

Given equation is $f(D)y = X$

$F(D) = D^2 + 3D + 2, \quad X = e^{e^{x}}$

Auxillary equation is

$m^2 + 3m + 2 = 0$

$(m+1)(m+2) = 0$

$\Rightarrow m = -1, -2$

C.F. $= c_1 e^{m_1 x} + c_2 e^{m_2 x}$

C.F. $= c_1 e^{-x} + c_2 e^{-2x}$

$= c_1 y_1 + c_2 y_2$

where, $y_1 = e^{-x}$ & $y_2 = e^{-2x}$

$y_1' = -e^{-x}$ & $y_2' = -2e^{-2x}$

P.I $= uy_1 + vy_2$

where, $u = -\int \frac{y_2 X}{W}$

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$$W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} e^{-x} & e^{-2x} \\ -e^{-x} & -2e^{-2x} \end{vmatrix}$$

$$= -e^{-x} \cdot 2e^{-2x} + e^{-x} \cdot e^{-2x}$$

$$= -2e^{-3x} + e^{-3x}$$

$$= -e^{-3x}$$

$$\therefore u = -\int \frac{e^{-2x} \cdot e^{e^x}}{-e^{-3x}} dx$$

$$= \int e^{x} \cdot e^{e^x} dx$$

put $e^x = t$

$e^x dx = dt$

$$u = \int e^t dt = e^t = e^{e^x}$$

$$v = \int \frac{y_1 x}{w} dx = \int \frac{e^{-x} \cdot e^{e^x}}{-e^{-3x}} dx$$

$$= -\int e^{2x} \cdot e^{e^x} dx$$

put $e^x = t$

$e^x dx = dt$

$$v = -\int t \cdot e^t dt$$

$$= -\left\{ t \cdot e^t - \int e^t \cdot (1) dt \right\} = -te^t + e^t$$

$$= e^t(1-t)$$

$$v = e^{e^x}(1 - e^x)$$

$$\therefore \text{P.I.} = e^{-x} \cdot e^{e^x} + e^{-2x} \cdot e^{e^x}(1 - e^x)$$

$$= e^{-x} \cdot e^{e^x} + e^{-2x} \cdot e^{e^x} - e^{-x} \cdot e^{e^x}$$

$$\text{P.I.} = e^{-2x} \cdot e^{e^x}$$

$$y = \text{C.F} + \text{P.I}$$

$$y = c_1 e^{-x} + c_2 e^{-2x} + e^{-2x} \cdot e^{e^x}$$

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Ex. 4 Solve: $(D^2+2D+1)y = e^{-x}\log x$ by method of variation of parameter.

$\Rightarrow$ Given $(D^2+2D+1)y = e^{-x}\log x$

Given equation is $f(D)y = X$

where,

$$f(D) = D^2+2D+1 \quad , \quad X = e^{-x}\log x$$

Auxiliary equation is

$$m^2+2m+1 = 0$$

$$(m+1)(m+1) = 0$$

$$\Rightarrow m = -1, -1$$

$$C.F. = (c_1+c_2x)e^{m_1x}$$

$$C.F. = (c_1+c_2x)e^{-x}$$

$$= c_1e^{-x} + c_2x \cdot e^{-x}$$

$$C.F. = c_1y_1 + c_2y_2$$

where $y_1 = e^{-x}$, $y_2 = x \cdot e^{-x}$

$$y_1' = -e^{-x} \quad , \quad y_2' = e^{-x} - xe^{-x} = (1-x)e^{-x}$$

Now,

$$P.I = uy_1 + vy_2$$

$$u = -\int \frac{y_2 X}{w} dx$$

$$w = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} e^{-x} & xe^{-x} \\ -e^{-x} & (1-x)e^{-x} \end{vmatrix}$$

$$= e^{-2x}(1-x) + xe^{-2x}$$

$$= e^{-2x} - xe^{-2x} + xe^{-2x}$$

$$= e^{-2x}$$

$$\therefore u = -\int \frac{xe^{-x} \cdot e^{-x}\log x}{e^{-2x}} dx$$

$$= -\int x\log x \, dx.$$

$$= -\left\{ \log x \cdot \frac{x^2}{2} - \int \frac{x^2}{2} \cdot \frac{1}{x} dx \right\} = -\frac{x^2\log x}{2} + \frac{1}{2}\frac{x^2}{2}$$

$$= -\frac{x^2\log x}{2} + \frac{x^2}{4}$$

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$$v = \int \frac{y_1 X}{W}\, dx$$

$$= \int \frac{e^{-x} \cdot e^{-x} \log x}{e^{-2x}}\, dx$$

$$= \int \log x \, dx$$

$$= \log x \cdot x - \int x \cdot \frac{1}{x}\, dx$$

$$= x \log x - x$$

$$v = x(\log x - 1)$$

$$P.I = e^{-x}\left(-\frac{x^2 \log x}{2} + \frac{x^2}{4}\right) + x e^{-x} \cdot x(\log x - 1)$$

Complete solution is given by

$y = C.F + P.I$

$$y = (c_1 + c_2 x)e^{-x} + x^2 e^{-x}(\log x - 1) + e^{-x}\left(-\frac{x^2 \log x}{2} + \frac{x^2}{4}\right)$$

Ex.5 By using method of variation of parameter Solve: $(D^2 - 4D + 4)y = e^{2x} \cdot \sec^2 x$.

$\Rightarrow$ Given, $(D^2 - 4D + 4)y = e^{2x} \cdot \sec^2 x$

Given equation is $F(D)y = X$

$F(D) = D^2 - 4D + 4$, $X = e^{2x} \cdot \sec^2 x$

Auxillary equation is

$$m^2 - 4m + 4 = 0$$

$$(m-2)(m-2) = 0$$

$$\Rightarrow m = 2, 2$$

$$C.F. = (c_1 + c_2 x) e^{m_1 x}$$

$$C.F. = (c_1 + c_2 x) e^{2x}$$

$$C.F. = c_1 e^{2x} + c_2 x e^{2x}$$

$$= c_1 y_1 + c_2 y_2$$

where,

$y_1 = e^{2x}$, $y_2 = x e^{2x}$

$y_1' = 2e^{2x}$, $y_2' = e^{2x} + 2x e^{2x}$

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Now,

P.I. $= uy_1 + vy_2$

where, $u = -\int \frac{y_2 X}{W} dx$

$$W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} e^{2x} & xe^{2x} \\ 2e^{2x} & e^{2x}(1+2x) \end{vmatrix}$$

$$= e^{4x}(1+2x) - 2xe^{4x}$$

$$= e^{4x} + 2xe^{4x} - 2xe^{4x}$$

$$= e^{4x}(\cancel{1-x})$$

$$\therefore u = -\int \frac{xe^{2x} \cdot e^{2x} \cdot \sec^2 x}{e^{4x}(\cancel{1-x})}$$

$$= -\int \frac{x \cdot \sec^2 x}{\cancel{1-x}}$$

$$= -\int x \sec^2 x \, dx$$

$$= -\left\{ x \cdot \tan x - \int \tan x \cdot 1 \, dx \right\}$$

$$= -x\tan x + \log(\sec x)$$

$$v = \int \frac{y_1 X}{W} dx$$

$$= \int \frac{e^{2x} \cdot e^{2x} \cdot \sec^2 x}{e^{4x}} dx$$

$$= \int \sec^2 x \, dx$$

$$= \tan x$$

P.I. $= e^{2x}(-x\tan x + \log(\sec x)) + xe^{2x} \cdot \tan x$

$= e^{2x}[-x\tan x + \log(\sec x) + x\tan x]$

P.I. $= e^{2x} \cdot \log(\sec x)$

Complete solution is given by

$y =$ C.F. + P.I.

$y = (c_1 + c_2 x)e^{2x} + e^{2x} \cdot \log(\sec x)$

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	Examples	Answer
	Solve the following examples by method of variation of parameters:	
1	Solve: $(D^2-1)y = \dfrac{2}{1+e^x}$	$C.F = c_1e^x + c_2e^{-x}$ $W = 2,\ u = -(e^x+1)+\log(e^{-x}+1)$ $v = -\log(e^x+1)$
2	Solve: $(D^2+1)y = \operatorname{cosec}x\cdot\cot x$	$C.F = c_1\cos x + c_2\sin x$ $W = 1,\ v = -\cot x - x$ $u = -\log(\sin x)$
3	Solve: $\dfrac{d^2y}{dx^2} + 4y = \tan 2x$	$C.F = c_1\cos 2x + c_2\sin 2x$ $W = 2,\ u = -\frac{1}{4}\log(\sec 4x + \tan 4x$ $+ \frac{\sin 2x}{4},\ v = -\frac{1}{4}\cos 2x$
4	Solve: $\dfrac{d^2y}{dx^2} + y = x\sin x$	$C.F = c_1\cos x + c_2\sin x$ $W = 1$

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* Cauchy's Homogeneous linear Equation :-

The differential equation of the form:

$$x^n \frac{d^n y}{dx^n} + a_1 x^{n-1} \frac{d^{n-1}y}{dx^{n-1}} + a_2 x^{n-2}\frac{d^{n-2}y}{dx^{n-2}} + \dots + a_{n-1} x\frac{dy}{dx} + a_n y = X$$

are called cauchy's homogeneous linear equation, where a_i's are constant and X is a function of x.

To reduce linear differential Equations with constant coefficients by taking substitution,

$$x = e^z \Rightarrow z = \log x.$$

then,

$$x\frac{dy}{dx} = \frac{dy}{dz} = Dy \text{ , where } D = \frac{d}{dz}$$

$$x^2 \frac{d^2y}{dx^2} = D(D-1)y$$

$$x^3 \frac{d^3y}{dx^3} = D(D-1)(D-2)y$$

$$\vdots$$

Substituting these values in above equation, we get differential equation with constant coefficient.

Ex.1 Solve: $x^3 \frac{d^3y}{dx^3} + 3x^2\frac{d^2y}{dx^2} + x\frac{dy}{dx} + y = x + \log x$

Given, $x^3\frac{d^3y}{dx^3} + 3x^2\frac{d^2y}{dx^2} + x\frac{dy}{dx} + y = x + \log x$ —— ①

Given equation is chauchy's homogeneous linear Equation,

put $x = e^z \Rightarrow z = \log x$

& $x^3\frac{d^3y}{dx^3} = D(D-1)(D-2)y$

$$x^2\frac{d^2y}{dx^2} = D(D-1)y$$

$$x\frac{dy}{dx} = Dy$$

$\therefore$ Equation ① becomes.

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$$D(D-1)(D-2)y + 3D(D-1)y + Dy + y = e^x + x$$

$$(D^2 - D)(D-2)y + (3D^2 - 3D)y + Dy + y = e^x + x$$

$$[D^3 - 2D^2 - D^2 + 2D + 3D^2 - 3D + D + 1]y = e^x + x$$

$$\cancel{D^3 - 2D^2}\ (D^3 + 1)y = e^x + x \quad \text{——} \quad (2)$$

Above equation in the form of $F(D)y = X$

$$F(D) = D^3 + 1, \quad X = e^x + x$$

Auxillary equation is

$$m^3 + 1 = 0$$

$$(m+1)(m^2 - m + 1) = 0$$

$$\Rightarrow m + 1 = 0 \text{ or } m^2 - m + 1 = 0$$

$$m = -1 \text{ or } m = \frac{+1 \pm \sqrt{1-4}}{2} = \frac{1 \pm \sqrt{-3}}{2}$$

$$m = -1 \text{ or } m = \frac{1 \pm \sqrt{3}i}{2} = \frac{1}{2} \pm \frac{\sqrt{3}}{2}i = \alpha \pm \beta$$

$$C.F. = c_1 e^{m_1 x} + e^{\alpha x}(c_1 \cos\beta x + c_2 \sin\beta x)$$

$$C.F. = c_1 e^{-x} + e^{x/2}\left(c_1 \cos\frac{\sqrt{3}x}{2} + c_2 \sin\frac{\sqrt{3}x}{2}\right)$$

$$P.I. = \frac{1}{F(D)} X$$

$$= \frac{1}{D^3+1} X = \frac{1}{D^3+1}(e^x + x)$$

$$= \frac{1}{D^3+1} e^x + \frac{1}{D^3+1} x$$

$$= \frac{1}{(1)^3+1} e^x + (1+D^3)^{-1} x$$

$$= \frac{1}{2} e^x + (1 - D^3) x$$

$$= \frac{1}{2} e^x + (x - 0) = \frac{e^x}{2} + x$$

$\therefore$ Complete solution is

$$y = C.F. + P.I. = c_1 e^{m_1 x} + e^{x/2}\left\{c_1 \cos\frac{\sqrt{3}x}{2} + c_2 \sin\frac{\sqrt{3}x}{2}\right\} + \frac{e^x}{2} + x$$

$$y = c_1\left(\frac{1}{x}\right) + e^{\frac{\log x}{2}}\left\{c_1 \cos\left(\frac{\sqrt{3}}{2}\log x\right) + c_3 \sin\left(\frac{\sqrt{3}}{2}\log x\right) + \frac{x}{2} + \log x\right.$$

Ex.2

Solve: $x^2\frac{d^2y}{dx^2} - 3x\frac{dy}{dx} + 5y = x^2\sin(\log x)$

Given, $x^2\frac{d^2y}{dx^2} - 3x\frac{dy}{dx} + 5y = x^2\sin(\log x)$ —— ①

put $x = e^z \Rightarrow z = \log x$

$$x\frac{dy}{dx} = Dy$$

$$x^2\frac{d^2y}{dx^2} = D(D-1)y$$

∴ Equation ① becomes.

$$D(D-1)y - 3Dy + 5y = e^{2z}\sin z$$

$$(D^2 - D - 3D + 5)y = e^{2z}\sin z$$

$$(D^2 - 4D + 5)y = e^{2z}\sin z \quad \text{—— ②}$$

Equation ② is in the form of

$$F(D)y = Z$$

$$\therefore F(D) = D^2 - 4D + 5, \quad Z = e^{2z}\sin z$$

Auxillary equation is

$$m^2 - 4m + 5 = 0$$

$$m = \frac{-b \pm \sqrt{b^2-4ac}}{2a}$$

$$= \frac{4 \pm \sqrt{16-20}}{2(1)} = \frac{4 \pm \sqrt{-4}}{2}$$

$$= \frac{4 \pm 2i}{2}$$

$$m = 2 \pm i = \alpha \pm i\beta$$

$$\therefore \alpha = 2, \ \beta = 1$$

$$C.F. = e^{\alpha z}(c_1\cos\beta z + c_2\sin\beta z)$$

$$C.F. = e^{2z}(c_1\cos z + c_2\sin z)$$

Now,

$$P.I. = \frac{1}{F(D)} Z$$

$$= \frac{1}{D^2 - 4D + 5} e^{2z}\sin z$$

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$$= e^{2z}\left\{\frac{1}{(D+2)^2-4(D+2)+5}\sin z\right\}$$

$$= e^{2z}\left\{\frac{1}{D^2+4D+4-4D-8+5}\sin z\right\}$$

$$= e^{2z}\left\{\frac{1}{D^2+1}\sin z\right\}$$

$$= e^{2z}\cdot z\cdot\frac{1}{2D}\sin z$$

$$= e^{2z}\cdot\frac{z}{2}\int \sin z\,dz$$

$$= \frac{e^{2z}\cdot z}{2}(-\cos z)$$

$$\text{P.I.} = -\frac{z}{2}e^{2z}\cos z$$

Complete solution is given by

$$y = \text{C.F} + \text{P.I.}$$

$$= e^{2z}(c_1\cos z + c_2\sin z) - \frac{z}{2}e^{2z}\cos z$$

$$y = x^2(c_1\cos(\log x) + c_2\sin(\log x)) - \frac{x^2\cdot\log x}{2}\cos(\log x)$$

Ex.3 Solve: $x^2\frac{d^2y}{dx^2} - x\frac{dy}{dx} + y = \log x$

$\Rightarrow$ Given, $x^2\frac{d^2y}{dx^2} - x\frac{dy}{dx} + y = \log x$ —— ①

Given equation is cauchy's differential equation, then, put

$$x = e^z \Rightarrow z = \log x$$

$$x^2\frac{d^2y}{dx^2} = D(D-1)y$$

$$x\frac{dy}{dx} = Dy$$

Equation ① becomes

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$$D(D-1)y - Dy + y = x$$
$$(D^2 - D - D + 1)y = x$$
$$(D^2 - 2D + 1)y = x$$

Given equation in the form of $F(D)y = x$

$$F(D) = D^2 - 2D + 1, \quad x = x$$

Auxullary equation is

$$m^2 - 2m + 1 = 0$$
$$(m-1)(m-1) = 0$$
$$\Rightarrow m = 1, 1$$

$$C.F. = \cancel{c_1 e^{m_1 x} + c_2 e^{m_2 x}}$$
$$= (c_1 + c_2 x) e^{m_1 x}$$
$$C.F. = (c_1 + c_2 x) e^{x}$$

Now,

$$P.I. = \frac{1}{F(D)} x$$
$$= \frac{1}{D^2 - 2D + 1} x$$
$$= [1 + (D^2 - 2D)]^{-1} x$$
$$= (1 - (D^2 - 2D)) x$$
$$= (1 + 2D) x$$
$$= x + 2Dx$$
$$= x + 2(1)$$
$$= x + 2$$

$\therefore$ Complete Solution is given by

$$y = C.F + P.I.$$
$$= (c_1 + c_2 x) e^{x} + x + 2$$
$$y = (c_1 + c_2 \log x) x + \log x + 2$$

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★ Legendre's Linear Equation :-

The equation in the form of

$$(a+bx)^n \frac{d^n y}{dx^n} + a_1 (a+bx)^{n-1} \frac{d^{n-1}y}{dx^{n-1}} + \cdots + a_{n-1}(a+bx)\frac{dy}{dx}$$

$$+ a_n y = X \quad \text{—— (1)}$$

is called Legendre's linear equation, where a_i's are constants and X is function of x.

Such equation can be converted to LDE with constant coefficients by substituting,

$$a+bx = e^z \Rightarrow z = \log(a+bx)$$

So that,

$$(a+bx)\frac{dy}{dx} = bDy \quad , \quad D \equiv \frac{d}{dz}$$

$$(a+bx)^2 \frac{d^2y}{dx^2} = b^2 D(D-1)y$$

Simullarly, $(a+bx)^3 \frac{d^3y}{dx^3} = b^3 D(D-1)(D-2)y$

Substituting these values in eq^n (1), equation (1) becomes LDE with constant coefficient.

Ex.1

Solve: $(2+3x)^2 \frac{d^2y}{dx^2} + 3(2+3x)\frac{dy}{dx} - 36y = 3x^2+4x+1$

$\Rightarrow$ Given, $(2+3x)^2 \frac{d^2y}{dx^2} + 3(2+3x)\frac{dy}{dx} - 36y = 3x^2+4x+1$ —— (1)

Given equation is Legendre's linear equation, so, put $2+3x = e^z \Rightarrow z = \log(2+3x)$

So that,

$$(2+3x)^2 \frac{d^2y}{dx^2} = (3)^2 D(D-1)y$$
$$= 9D(D-1)y.$$

& $$(2+3x)\frac{dy}{dx} = 3Dy$$

$\therefore$ Eq^n (1) becomes,

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$$9D(D-1)y + 3D(3)y - 36y = 3\left(\frac{e^x - 2}{3}\right)^2 + 4\left(\frac{e^x - 2}{3}\right) + 1$$

$$9[D^2 - D + D - 4]y = \frac{1}{3}(e^{2x} - 4x + 4) + \frac{4}{3}e^x - \frac{8}{3} + 1$$

$$9(D^2 - 4)y = \frac{e^{2x} - 4e^x + 4 + 4e^x - 8 + 3}{3}$$

$$(D^2 - 4)y = \frac{1}{27}(e^{2x} - 1)$$

Above equation is LDE with constant coefficient of

$F(D)y = X$

where,

$$F(D) = D^2 - 4, \quad X = \frac{1}{27}(e^{2x} - e^{0x})$$

Auxillary equation is

$$m^2 - 4 = 0$$

$$m^2 = 4$$

$$m = \pm 2$$

$$\therefore C.F = C_1 e^{m_1 x} + C_2 e^{m_2 x}$$

$$C.F. = C_1 e^{2x} + C_2 e^{-2x}$$

Now,

$$P.I. = \frac{1}{F(D)} X$$

$$= \frac{1}{D^2 - 4} \frac{1}{27}(e^{2x} - e^{0x})$$

$$= \frac{1}{27}\left[\frac{1}{D^2 - 4} e^{2x} - \frac{1}{D^2 - 4} e^{0x}\right]$$

$$= \frac{1}{27}\left[x \cdot \frac{1}{2D} e^{2x} - \frac{1}{0 - 4} e^{0x}\right]$$

$$= \frac{1}{27}\left[\frac{x}{2} \cdot \frac{e^{2x}}{2} + \frac{1}{4}\right]$$

$$= \frac{1}{27 \cdot 4}(xe^{2x} + 1)$$

$$P.I. = \frac{1}{108}(xe^{2x} + 1)$$

Complete solution is given by

$$y = C.F + P.I.$$

$$= c_1 e^{2z} + c_2 e^{-2z} + \frac{1}{108}\left(z e^{2z} + 1\right)$$

$$= c_1(2+3x)^2 + c_2(2+3x)^{-2} + \frac{1}{108}\left\{\underset{\log(2+3x)}{\left(\frac{e^z-2}{3}\right)}(3x+2)^2 + 1\right\}$$

$$= c_1(2+3x)^2 + \frac{c_2}{(2+3x)^2} + \frac{1}{108}\left(\log(2+3x)\cdot(2+3x)^2 + 1\right)$$

Ex. 2 Solve: $(x-1)^3\frac{d^3y}{dx^3} + 2(x-1)^2\frac{d^2y}{dx^2} - 4(x-1)\frac{dy}{dx} + 4y = 4\log(x-1)$

Given,

$$(x-1)^3\frac{d^3y}{dx^3} + 2(x-1)^2\frac{d^2y}{dx^2} - 4(x-1)\frac{dy}{dx} + 4y = 4\log(x-1) \quad \text{--- (1)}$$

Eq^n (1) is Legendre's linear equation, then

put $x-1 = e^z \Rightarrow z = \log(x-1)$

so that,

$$(x-1)^3\frac{d^3y}{dx^3} = (1)^3 D(D-1)(D-2)y$$
$$= D(D-1)(D-2)y$$

$$(x-1)^2\frac{d^2y}{dx^2} = (1)^2 D(D-1)y$$
$$= D(D-1)y$$

$$(x-1)\frac{dy}{dx} = (1)Dy$$
$$= Dy$$

$\therefore$ Eq^n (1) becomes,

$$D(D-1)(D-2)y + 2D(D-1)y - 4Dy + 4y = 4z$$

$$[(D^2-D)(D-2) + 2D^2 - 2D - 4D + 4]y = 4z$$

$$[D^3 - 2D^2 - D^2 + 2D + 2D^2 - 2D - 4D + 4]y = 4z$$

$$(D^3 - D^2 - 4D + 4)y = 4z$$

ie $F(D)y = Z$

where

$$F(D) = D^3 - D^2 - 4D + 4, \quad Z = 4z$$

Auxiliary eq^n is

$$m^3 - m^2 - 4m + 4 = 0$$

$m^2(m-1)-4(m-1)=0$

$(m-1)(m^2-4)=0$

$(m-1)(m-2)(m+2)=0$

$\Rightarrow m=1,2,-2$

$\therefore C.F. = c_1 e^{m_1 x} + c_2 e^{m_2 x} + c_3 e^{m_3 x}$

$C.F. = c_1 e^{x} + c_2 e^{2x} + c_3 e^{-2x}$

Now,

$$P.I. = \frac{1}{F(D)} x$$

$$= \frac{1}{D^3 - D^2 - 4D + 4} 4x$$

$$= \frac{4}{4}\left[\frac{1}{1+\left(\frac{D^3}{4}-\frac{D^2}{4}-D\right)} x\right]$$

$$= \left\{\left[1+\left(\frac{D^3}{4}-\frac{D^2}{4}-D\right)\right]^{-1} x\right\}$$

$$= \left\{1-\left(\frac{D^3}{4}-\frac{D^2}{4}-D\right)\right\} x$$

$$= (1+D)x$$

$$= x + D(x)$$

$P.I. = x+1$

Complete solution is given by

$y = C.F. + P.I.$

$= c_1 e^{x} + c_2 e^{2x} + c_3 e^{-2x} + x + 1$

$y = c_1(x-1) + c_2(x-1)^2 + c_3(x-1)^{-2} + \log(x-1) + 1$

	Examples	Answers.
1	Solve: $x^2 \frac{d^2y}{dx^2} - x\frac{dy}{dx} + y = \log x$	$(c_1 + c_2 \log x)x + \frac{c_3}{x} + \frac{1}{4x} \log x$
2	Solve: $(x^2D^2 + 5xD + 3)y = \frac{\log x}{x^2}$	$\frac{c_1}{x} + \frac{c_2}{x^3} - \frac{\log x}{x^2}$
3	Solve: $(x^2D^2 + xD + 1)y = \log x \cdot \sin(\log x)$	$c_1 \cos(\log x) + c_2 \sin(\log x) - \frac{(\log x)^2}{4} \cos(\log x) + \frac{\log x}{4} \sin(\log$
4	$[(1+x)^2D^2 + (1+x)D + 1]y = 2\sin\log(x+1)$	$c_1 \cos\log(x+1) - \log(1+x) \cdot \cos\log(x+1) + c_2 \sin\log(x+1)$
5	$(2x+3)^2 \frac{d^2y}{dx^2} - 2(2x+3)\frac{dy}{dx} - 12y = 6x$	$\frac{c_1}{2x+3} + c_2(2x+3)^3 - \frac{3}{4}(2x+3) +$