Unit - II

Ordinary Differential Equations of first order and first degree.

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Ordinary differential Equation :-

A differential equation is a relation between the dependent and independent variables, provided that in the relation there occurs at least one derivative term of the dependent variable with respect to independent variable.

Ex. 1) $\frac{dy}{dx} + y = 0$

2) $\frac{d^2y}{dx^2} + \left(\frac{dy}{dx}\right)^2 + y = 0$

Order of differential Equation :-

The order of differential Equation is the order of highest order derivative occuring in that differential Equation.

Degree of differential Equation :-

The degree of differential is the degree of the degree of the highest order derivative occurring in the equation which is free from radical sign and fractions.

Ex. order, Degree

$$\left(\frac{d^2y}{dx^2}\right)^2 + \left(\frac{dy}{dx}\right)^4 + 3y = 0$$

$\therefore$ Order = 2, Degree = 2

General Solution :-

The general solution of a differential Equation is that in which the number of independent constant (arbitrary) is equal to the order of differential equation

Ex.

$y = c_1e^{x} + c_2e^{-x}$ is the general solution of the differential equation $\frac{d^2y}{dx^2} - y = 0$

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Perticular solution :-

By assiguing the arbitrary constant value in general solution we get perticular solution.

Ex.

$y = ~~y~~\, e^{x} - e^{-x}$ is the perticular solution of differential equation $\frac{d^2y}{dx^2} - y = 0$

★ Formation of differential Equations of first order and first degree by elimenating orbitrary constant.

Ex.1 Obtain the differential Equation by elimenating orbitrary constant $y = Ae^{3x} + Be^{+2x}$

⇒ Given,

$$y = Ae^{3x} + Be^{2x} \quad \text{——— (1)}$$

No. of arbitrary constant = 02 (A & B)

∴ Diff. eqⁿ (1) w.r.to x

$$\frac{dy}{dx} = Ae^{3x}(3) + Be^{2x}(2)$$

$$= 3Ae^{3x} + 2Be^{2x} \quad \text{——— (2)}$$

Again diff eqⁿ (2) w.r.to x

$$\frac{d^2y}{dx^2} = 9Ae^{3x} + 4Be^{2x} \quad \text{——— (3)}$$

from (1), (2) & (3)

$$Ae^{3x} + Be^{2x} - y = 0$$

$$A \cdot 3e^{3x} + 2Be^{2x} - dy/dx = 0$$

$$A \cdot 9e^{3x} + B \cdot 4e^{2x} - d^2y/dx^2 = 0$$

$$\begin{vmatrix} e^{3x} & e^{2x} & -y \\ 3e^{3x} & 2e^{2x} & -dy/dx \\ 9e^{3x} & 4e^{2x} & -d^2y/dx^2 \end{vmatrix} = 0$$

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$$e^{3x}\cdot e^{2x}\begin{vmatrix} 1 & 1 & -y \\ 3 & 2 & -dy/dx \\ 9 & 4 & -d^2y/dx^2 \end{vmatrix} = 0$$

$$e^{5x}\left\{1\left(-2\frac{d^2y}{dx^2}+4\frac{dy}{dx}\right)-1\left(-3\frac{d^2y}{dx^2}+9\frac{dy}{dx}\right)-y(12-18)\right\}=0$$

$$\therefore -2\frac{d^2y}{dx^2}+4\frac{dy}{dx}+3\frac{d^2y}{dx^2}-9\frac{dy}{dx}+6y=0$$

$$\therefore \frac{d^2y}{dx^2}-5\frac{dy}{dx}+6y=0$$

Ex. 2 Prove that $Ax^2+By^2=1$ is the solution of

$$x\left\{y\frac{d^2y}{dx^2}+\left(\frac{dy}{dx}\right)^2\right\}=y\frac{dy}{dx}$$

$\Rightarrow$ Given, $Ax^2+By^2=1$ —— ①

No. of arbitrary constant = 02 (A & B)

Differentiate eq^n ① w.r.to x

$$2Ax+2By\frac{dy}{dx}=0 \quad \text{—— ②}$$

Again diff. eq^n ② w.r.to x

$$2A+2B\left(y\frac{d^2y}{dx^2}+\frac{dy}{dx}\cdot\frac{dy}{dx}\right)=0$$

$$A+B\left(y\frac{d^2y}{dx^2}+\left(\frac{dy}{dx}\right)^2\right)=0 \quad \text{—— ③}$$

From ①, ② & ③

$$\begin{vmatrix} x^2 & y^2 & -1 \\ x & y\frac{dy}{dx} & 0 \\ 1 & y\frac{d^2y}{dx^2}+\left(\frac{dy}{dx}\right)^2 & 0 \end{vmatrix}=0$$

$$x^2(0)-y^2(0)-1\left\{x\left(y\frac{d^2y}{dx^2}+\left(\frac{dy}{dx}\right)^2\right)-y\frac{dy}{dx}\right\}=0$$

$$x\left(y\frac{dy}{dx}+\left(\frac{dy}{dx}\right)^2\right)-y\frac{dy}{dx}=0$$

$$\Rightarrow \quad y\frac{dy}{dx} = x\left\{y\frac{dy}{dx}+\left(\frac{dy}{dx}\right)^2\right\}$$

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⑤ Linear differential Equation (LDE)

A differential equation in the form of $\frac{dy}{dx} + Py = Q$

where P & Q are functions of x or constant is called linear differential equation.

Solution:-

1) Integrating factor : (IF) = $e^{\int P dx}$

Note: 1) $e^{\log x} = x$

2) $e^{-\log x} = 1/x$

Solution:- $y(IF) = \int Q(IF)dx + C$

Ex.1 Solve: $\cos^2 x \frac{dy}{dx} + y = \tan x$

$\Rightarrow$ Given, $\cos^2 x \frac{dy}{dx} + y = \tan x$

$$\frac{dy}{dx} + \frac{1}{\cos^2 x} y = \frac{\tan x}{\cos^2 x}$$

$$\frac{dy}{dx} + \sec^2 x \cdot y = \tan x \cdot \sec^2 x \quad \text{—— ①}$$

Comparing with, $\frac{dy}{dx} + Py = Q$

Here $P = \sec^2 x$, $Q = \tan x \cdot \sec^2 x$.

$$IF = e^{\int P dx} = e^{\int \sec^2 x dx}$$

$$IF = e^{\tan x}$$

Solution:-

$$y(IF) = \int Q(IF)dx + C$$

$$ye^{\tan x} = \int \sec^2 x \cdot \tan x \cdot e^{\tan x} dx + C$$

put $\tan x = t$

$\sec^2 x \, dx = dt$

$$ye^{\tan x} = \int t \cdot e^t dt + C$$

$$ye^{\tan x} = t \cdot e^{t} - \int e^{t} dt + c$$

$$= te^{t} - e^{t} + c$$

$$ye^{\tan x} = e^{t}(t-1) + c$$

$$ye^{\tan x} = e^{\tan x}(\tan x - 1) + c$$

Ex. 2 Solve: $2y' \cos x + 4y \sin x = \sin 2x$, given $y=0$ at x.

⇒ Given,

$$2y' \cos x + 4y \sin x = \sin 2x$$

$$2\cos x \frac{dy}{dx} + (4\sin x)y = 2\sin x \cos x$$

Comparing with $\frac{dy}{dx} + Py = Q$

$$\frac{dy}{dx} + (2\tan x)y = \sin x$$

$$P = 2\tan x, \quad Q = \sin x$$

$$IF = e^{\int P dx} = e^{\int 2\tan x dx}$$

$$= e^{2\log \sec x}$$

$$= e^{\log \sec^2 x}$$

$$IF = \sec^2 x$$

Solution,

$$y(IF) = \int Q(IF) dx + c$$

$$y\sec^2 x = \int \sin x \cdot \sec^2 x dx + c$$

$$= \int \frac{\sin x}{\cos^2 x} dx + c$$

$$= \int \sec x \cdot \tan x dx + c$$

$$y\sec^2 x = \sec x + c \quad \text{—— (2)}$$

Now,

$$y = 0 \text{ at } x = \pi/3$$

$$0(\sec^2 \pi/3) = \sec \pi/3 + c$$

$$0 = 2 + c$$

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$c = -2$

$\therefore \ y\sec^2 x = \sec x - 2$

Ex.3 Solve: $(1+y^2)dx + (x - e^{\tan^{-1}y})dy = 0$

Given,

$$(1+y^2)\,dx + (x - e^{\tan^{-1}y})\,dy = 0$$

$$(1+y^2)\,dx = -(x - e^{\tan^{-1}y})\,dy$$

$$\frac{dx}{dy} = -\frac{(x - e^{\tan^{-1}y})}{1+y^2}$$

$$= \frac{-1}{1+y^2}x + \frac{e^{\tan^{-1}y}}{1+y^2}$$

$$\frac{dx}{dy} + \frac{1}{1+y^2}x = \frac{e^{\tan^{-1}x}}{1+y^2}$$

Comparing with

$$\frac{dx}{dy} + P'x = Q'$$

$P' = 1/1+y^2, \quad Q' = e^{\tan^{-1}x}/1+y^2$

$$IF = e^{\int P'dy} = e^{\int \frac{1}{1+y^2}dy}$$

$$= e^{\tan^{-1}y}$$

Solution,

$$x(IF) = \int Q(IF)dy + c$$

$$xe^{\tan^{-1}y} = \int \frac{e^{\tan^{-1}y}}{1+y^2} \cdot e^{\tan^{-1}y}\,dy$$

$$= \int \frac{(e^{\tan^{-1}y})^2 dy}{1+y^2}$$

Put $\tan^{-1}y = t$

$$\frac{1}{1+y^2}dy = dt$$

$$xe^{\tan^{-1}y} = \int e^{2t}dt$$

$$= \frac{1}{2}e^{2t} + c$$

$$xe^{\tan^{-1}y} = \frac{1}{2}e^{2\tan^{-1}y} + c$$

⑥ Bernoullis differential Equation :-

A differential Equation $\frac{dy}{dx} + Py = Qy^n$

is called Bernoulli's equation, where P & Q are functions of x or constant.

⇒ Procedure :-

$$\frac{dy}{dx} + Py = Qy^n$$

Devide by y^n bothside

$$y^{-n}\frac{dy}{dx} + y^{-n+1}P = Q$$

put $y^{-n+1} = v$ and find value of $\frac{dy}{dx}$.

Ex. 1 Solve: $x\frac{dy}{dx} + y = x^3y^6$

⇒ Given, $x\frac{dy}{dx} + y = x^3y^6$

$$\Rightarrow \frac{dy}{dx} + \frac{y}{x} = x^2y^6$$

Devide bothside by y^6

$$y^{-6}\frac{dy}{dx} + y^{-5}\frac{1}{x} = x^2$$

put $y^{-5} = v$

$$\cancel{-5y^{-6}\frac{dy}{dx} = \frac{dv}{dy}}$$

$$\cancel{\frac{dy}{dx}y^{-6} = \frac{-1}{5}\frac{dv}{dy}}$$

$y^{-5} = v$

Diff. w.r.to x

$$-5y^{-6}\frac{dy}{dx} = \frac{dv}{dx}$$

$$y^{-6}\frac{dy}{dx} = \frac{-1}{5}\frac{dv}{dx}$$

$$\therefore -\frac{1}{5}\frac{dv}{dx} + \frac{v}{x} = x^2$$

$$\frac{dv}{dx} + \left(\frac{-5}{x}\right)v = -5x^2$$

Comparing with $\frac{dv}{dx} + Py = Q$.

$$P = -5/x, \quad Q = -5x^2$$

$$IF = e^{\int P\,dx} = e^{\int \frac{-5}{x} dx}$$

$$= e^{-5\log x}$$

$$= e^{-\log x^5}$$

$$= +\frac{1}{x^5} = \frac{1}{x^5}$$

Solution:-

$$v(IF) = \int Q(IF)\,dx + C$$

$$v \cdot \frac{1}{x^5} = \int -5x^2 \cdot \frac{1}{x^5}\,dx + C$$

$$\frac{v}{x^5} = -5\int x^{-3}\,dx + C$$

$$\frac{1}{x^5 . y^5} = -5\frac{x^{-2}}{-2} + C$$

$$\frac{1}{x^5 y^5} = \frac{5}{2x^2} + C$$

Ex. 2 Solve: $\frac{dy}{dx} + x\sin 2y = x^3\cos^2 y$

Given, $\frac{dy}{dx} + x\sin 2y = x^3\cos^2 y$

$$\frac{dy}{dx} + 2x\sin y\cos y = x^3\cos^2 y$$

Devide both side by $\cos^2 y$

$$\sec^2 y \frac{dy}{dx} + 2x \cdot \tan y = x^3 \quad \text{——— (1)}$$

put $\tan y = v$

$$\sec^2 y \frac{dy}{dx} = \frac{dv}{dx}$$

$$\frac{dv}{dx} + 2x \cdot v = x^3$$

Comparing with $\frac{dv}{dx} + Pv = Q$

$P = 2x$ & $Q = x^3$

$IF = e^{\int P dx}$

$= e^{\int 2x dx}$

$= e^{2 \cdot x^2/2}$

$= e^{x^2}$

$\therefore$ Solution is,

$y(IF) = \int Q(IF) dx + C$

$v e^{x^2} = \int x^3 \cdot e^{x^2} dx + C$

$= \int x^2 \cdot x e^{x^2} dx + C$

put $x^2 = t$

$2x dx = dt$

$x dx = dt/2$

$v e^{x^2} = \int t \cdot e^t \, dt/2 + C$

$= \frac{1}{2} \{ te^t - \int e^t dt \} + C$

$= \frac{1}{2} (te^t - e^t) + C$

$\tan y \, e^{x^2} = \frac{1}{2} (x^2 e^{x^2} - e^{x^2}) + C$

$\tan y \, e^{x^2} = \frac{e^{x^2}}{2} (x^2 - 1) + C$

Ex. 3 Solve: ~~$r\cos\theta$~~ $r\sin\theta - \cos\theta \frac{dr}{d\theta} = r^2$

$\Rightarrow$ Given,

$$r\sin\theta - \cos\theta \frac{dr}{d\theta} = r^2$$

$$-\cos\theta \frac{dr}{d\theta} + r\sin\theta = r^2$$

$$-\frac{dr}{d\theta} + r\tan\theta = \sec\theta \cdot r^2$$

Devide bothside by r^2

$$-\frac{1}{r^2}\frac{dr}{d\theta} + \frac{1}{r}\tan\theta = \sec\theta \quad \text{——— } (1)$$

put $\frac{1}{r} = v$

$$-\frac{1}{r^2}\frac{dr}{d\theta} = \frac{dv}{d\theta}$$

$$\therefore \quad \frac{dv}{d\theta} + \tan\theta v = \sec\theta$$

comparing with, $\frac{dv}{d\theta} + PQ = Q$

$$P = \tan\theta, \quad Q = \sec\theta$$

$$IF = e^{\int P dx} = e^{\int \tan\theta d\theta}$$

$$= e^{\log(\sec\theta)}$$

$$= \sec\theta$$

$\therefore$ Solution is,

$$v(IF) = \int Q(IF) d\theta + c$$

$$v\sec\theta = \int \sec\theta \cdot \sec\theta \, d\theta + c$$

$$= \int \sec^2\theta \, d\theta + c$$

$$v\sec\theta = \tan\theta + c$$

$$\frac{1}{r}\sec\theta = \tan\theta + c$$

	Examples	Answer
1)	$(x+2y^3)\frac{dy}{dx} = y$	$\frac{x}{y} = y^2 + c$
2)	$dr + (2r\cot\theta + \sin 2\theta)d\theta = 0$	$r\sin^2\theta = -\frac{\sin^4\theta}{2} + c$
3)	$(x+1)\frac{dy}{dx} - y = e^{3x}(x+1)^2$	$\frac{y}{x+1} = \frac{e^{3x}}{3} + c$
4)	$2xy' = 10x^3y^5 + y$	$\frac{x^2}{y^4} = -4x^2 + c$
5)	$\frac{dy}{dx} - \frac{\tan y}{1+x} = (1+x)e^x \sec y$	$\frac{\sin y}{1+x} = e^x + c$
6)	$\frac{dy}{dx} = y\tan x - y^2\sec x$	$\frac{\sec x}{y} = \tan x + c$
7)	$\tan y \frac{dy}{dx} + \tan x = \cos y \cos^2 x$	$\sec x \cdot \sec y = \sin x + c$

(7) Exact differential Equation :-

A differential equation $Mdx + Ndy = 0$ is said to be exact differential equation is

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}.$$

Solution:-

$$\int \underset{\substack{\downarrow \\ (\text{keeping } y \text{ constant})}}{M\,dx} + \int \underset{\substack{\downarrow \\ (\text{Taking only terms of } y \text{ \& , which is free from } x)}}{N\,dy} = c$$

Ex.1 Solve: $(y^2 e^{xy^2} + 4x^3)dx + (2xye^{xy^2} - 3y^2)dy = 0$

$\Rightarrow$ Given,

$$(y^2 e^{xy^2} + 4x^3)dx + (2xye^{xy^2} - 3y^2)dy = 0 \quad \text{---} \quad (1)$$

Comparing with, $Mdx + Ndy = 0$

$$M = y^2 e^{xy^2} + 4x^3$$
$$N = 2xye^{xy^2} - 3y^2$$

Now,

$$\frac{\partial M}{\partial y} = y^2 e^{xy^2} \frac{\partial}{\partial y}(xy^2) + e^{xy^2}(2y)$$
$$= y^2 e^{xy^2}(2xy) + e^{xy^2}(2y)$$
$$\frac{\partial M}{\partial y} = 2xy^3 e^{xy^2} + 2y e^{xy^2} \quad \text{---} \quad (2)$$

$$\frac{\partial N}{\partial x} = 2xy \cdot e^{xy^2}(y^2) + e^{xy^2}(2y)$$
$$\frac{\partial N}{\partial x} = 2xy^3 e^{xy^2} + 2y e^{xy^2} \quad \text{---} \quad (3)$$

from (2) & (3)

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

$\therefore$ Given differential equation is exact.

Solution:-

$$\int M dx + \int N dy = c$$

$$\int (y^2 e^{xy^2} + 4x^3) dx + \int (-3y^2) dy = c$$

$$y^2 \cdot \frac{e^{xy^2}}{y^2} + \frac{4x^4}{4} - \frac{3y^3}{3} = c$$

$$e^{xy^2} + x^4 - y^3 = c$$

Ex. 2 Solve: $(x^2 - 4xy - 2y^2)dx + (y^2 - 4xy - 2x^2)dy = 0$

$\Rightarrow$ Given,

$$(x^2 - 4xy - 2y^2)dx + (y^2 - 4xy - 2x^2)dy = 0 \quad \text{——} \ (1)$$

comparing with $Mdx + Ndy = 0$

$M = x^2 - 4xy - 2y^2$

$N = y^2 - 4xy - 2x^2$

Now,

$$\frac{\partial M}{\partial y} = 0 - 4x - 4y = -(4x + 4y) \quad \text{——} \ (2)$$

$$\frac{\partial N}{\partial x} = 0 - 4y - 4x = -(4y + 4x) \quad \text{——} \ (3)$$

From (2) & (3)

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

$\therefore$ Given differential equation is exact.

Solution:-

$$\int M dx + \int N dy = c$$

$$\int (x^2 - 4xy - 2y^2) dx + \int y^2 dy = c$$

$$\frac{x^3}{3} - 4y \cdot \frac{x^2}{2} - 2y^2 x + \frac{y^3}{3} = c$$

ie $$\frac{x^3}{3} - 2x^2 y - 2xy^2 + \frac{y^3}{3} = c$$

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Ex. 3 Solve: $\frac{dy}{dx} + \frac{y\cos x + \sin y + y}{\sin x + x\cos y + x} = 0$

Given,

$$\frac{dy}{dx} + \frac{y\cos x + \sin y + y}{\sin x + x\cos y + x} = 0$$

$$\Rightarrow \frac{dy}{dx} = -\frac{(y\cos x + \sin y + y)}{(\sin x + x\cos y + x)}$$

$$(\sin x + x\cos y + x)dy = -(y\cos x + \sin y + y)dx$$

$$(\sin x + x\cos y + x)dy + (y\cos x + \sin y + y)dx = 0 \quad \text{—— } (1)$$

comparing with $Mdx + Ndy = 0$

$N = \sin x + x\cos y + x$

$M = y\cos x + \sin y + y$

Now,

$$\frac{\partial M}{\partial y} = \cancel{0 + x(-\sin y) + 0}$$
$$= \cos x + \cos y + 1 \quad \text{—— } (2)$$

and,

$$\frac{\partial N}{\partial x} = \cos x + \cos y + 1 \quad \text{—— } (3)$$

From (2) & (3)

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

$\therefore$ Given differential equation is exact.

Solution,

$$\int M dx + \int N dy = c$$

$$\int (y\cos x + \sin y + y)dx + \int 0 dy = c$$

$$y\sin x + \sin y\,(x) + yx + 0 = c$$

$$y\sin x + x\sin y + xy = c$$

	Examples	Answer
1)	$\frac{2x}{y^3}\,dx + \frac{y^2-3x^2}{y^4}\,dy = 0$	$\frac{x^2}{y^3} - \frac{1}{y} = c$
2)	$\left\{y\left(1+\frac{1}{x}\right)+\cos y\right\}dx + (x+\log x - x\sin y)\,dy = 0$	$(x+\log x)y + x\cos y = c$
3)	$(2x^2+3y^2-7)x\,dx - (3x^2+2y^2-8)y\,dy = 0$	$x^2+y^2-3 = c(x^2-y^2-1)^5$
4)	$(x^2+y^2-a^2)x\,dx + (x^2-y^2-b^2)y\,dy = 0$	$\frac{x^4}{4} + \frac{x^2y^2}{2} - \frac{a^2x^2}{2} - \frac{y^4}{4} - \frac{b^2y^2}{2} =$

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8) Reduciable to ~~vari~~ Exact differential Equation :-

* Integrating factors :-

1) Integrating factor of homogeneous equation but not Exact :

$$I.F. = \frac{1}{Mx+Ny}$$

2) Integrating factor of the equation of the form:

$$f_1(xy)y\,dx + f_2(xy)x\,dy = 0$$

$$\therefore \quad I.F. = \frac{1}{Mx-Ny}$$

3) If $Mdx+Ndy=0$ is not exact, then the integrating factors can be calculated as :

1) If $\dfrac{\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}}{N} = f(x)$, then $I.F = e^{\int f(x)dx}$

2) If $\dfrac{\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}}{M} = f(y)$, then $I.F. = e^{\int f(y)dy}$.

Ex. 1 Solve : $(x^2+y^2)dx - xydy = 0$

⇒ Given,

$$(x^2+y^2)dx - xydy = 0 \quad \text{—— (1)}$$

Given equation is of the form $Mdx+Ndy=0$

$M = x^2+y^2$ & $N = -xy$

$\frac{\partial M}{\partial y} = 2y$ — (2) , $\frac{\partial N}{\partial x} = -y$ — (3)

from (2) & (3)

$$\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$$

∴ Given differential Equation is homogeneous but not exact.

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$\therefore$ I.F. $= \dfrac{1}{Mx + Ny}$

$= \dfrac{1}{(x^2+y^2)x + (-xy)y}$

$= \dfrac{1}{x^3 + xy^2 - xy^2}$

$= \dfrac{1}{x^3}$

Multiply $1/x^3$ to eq^n (1)

$\dfrac{1}{x^3}(x^2+y^2)dx - \dfrac{1}{x^3}(xy)dy = 0$

$\left(\dfrac{1}{x} + \dfrac{y^2}{x^3}\right)dx - \dfrac{y}{x^2}\,dy = 0$ —— (4)

$\therefore$ Eq^n (4) is exact diff eq^n of the form

$M'dx + N'dy = 0$

$M' = \dfrac{1}{x} + \dfrac{y^2}{x^3}$ & $N' = -\dfrac{y}{x^2}$

$\therefore$ Solution:-

$\int M'dx + \int N'dy = c$

$\int\left(\dfrac{1}{x} + \dfrac{y^2}{x^3}\right)dx + \int 0\,dy = c$

$\log x + \dfrac{x^{-2}}{-2}\cdot y^2 = c$

$\therefore \log x - \dfrac{1}{2x^2} = c$

Ex. 2 Solve: $(x^2y - 2xy^2)dx - (x^3 - 3x^2y)dy = 0$

$\Rightarrow$ Given,

$(x^2y - 2xy^2)dx - (x^3 - 3x^2y)dy = 0$ —— (1)

Given eq^n is of the form $Mdx + Ndy = 0$

$M = x^2y - 2xy^2$ & $N = -(x^3 - 3x^2y)$

$\dfrac{\partial M}{\partial y} = x^2 - 4xy$ —— (2) & $\dfrac{\partial N}{\partial x} = -3x^2 + 6xy$ —— (3)

From ② & ③

$$\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$$

$\therefore$ Given differential equation is homogeneous but not Exact.

$$I.F. = \frac{1}{Mx - Ny}$$

$$= \frac{1}{(x^2y - 2xy^2)x + (-x^3 + 3x^2y)y}$$

$$= \frac{1}{x^3y - 2x^2y^2 - x^3y + 3x^2y^2}$$

$$= \frac{1}{x^2y^2}$$

Multiply $1/x^2y^2$ to equation ① we get

$$\frac{1}{x^2y^2}(x^2y - 2xy^2)dx - \frac{1}{x^2y^2}(x^3 - 3x^2y)dy = 0$$

$$\left(\frac{1}{y} - \frac{2}{x}\right)dx - \left(\frac{x}{y^2} - \frac{3}{y}\right)dy = 0 \quad \text{——— ④}$$

$\therefore$ Equation ④ is exact in the form of $M'dx + N'dy = 0$

$M' = \frac{1}{y} - \frac{2}{x}$ and $N' = -\frac{x}{y^2} + \frac{3}{y}$

Solution :-

$$\int M'dx + \int N'dy = c$$

$$\int \left(\frac{1}{y} - \frac{2}{x}\right)dx + \int \frac{3}{y}\,dy = c$$

$$\frac{x}{y} - 2\log x + 3\log y = c$$

$$\frac{x}{y} + \log\left(\frac{y^3}{x^2}\right) = c$$

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Ex. 3 Solve: $(x^2y^2+xy+1)y\,dx + (x^2y^2-xy+1)x\,dy = 0$

$\Rightarrow$ Given

$$(x^2y^2+xy+1)y\,dx + (x^2y^2-xy+1)x\,dy = 0 \quad \text{—— (1)}$$

$$(x^2y^3+xy^2+y)\,dx + (x^3y^2-x^2y+x)\,dy = 0 \quad \text{—— (2)}$$

Eq^n (2) is in the form of $M\,dx + N\,dy = 0$

$M = x^2y^3+xy^2+y$ & $N = x^3y^2-x^2y+x$

$\frac{\partial M}{\partial y} = 3x^2y^2+2xy+1$ — (3) & $\frac{\partial N}{\partial x} = 3x^2y^2-2xy+1$ —— (4)

From (4) & (5)

$$\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$$

$\therefore$ Given diff. eq^n is not exact but in the form of

$$f_1(xy)y\,dx + f_2(xy)x\,dy = 0$$

$$\therefore \quad \text{I.F.} = \frac{1}{Mx-Ny}$$

$$= \frac{1}{(x^2y^3+xy^2+y)x-(x^3y^2-x^2y+x)y}$$

$$= \frac{1}{x^3y^3+x^2y^2+xy-x^3y^3+x^2y^2-xy}$$

$$\text{I.F.} = \frac{1}{2x^2y^2}$$

Multiply I.F to eq^n (2), we get

$$\frac{1}{2x^2y^2}(x^2y^3+xy^2+y)\,dx + \frac{1}{2x^2y^2}(x^3y^2-x^2y+x)\,dy = 0$$

$$\left(\frac{y}{2}+\frac{1}{2x}+\frac{1}{2x^2y}\right)dx + \left(\frac{x}{2}-\frac{1}{2y}+\frac{1}{2xy^2}\right)dy = 0$$

$\therefore$ Above diff eq^n is an exact.

Solution:-

$$\int M'\,dx + \int N'\,dy = c$$

$$\int\left(\frac{y}{2}+\frac{1}{2x}+\frac{1}{2x^2y}\right)dx + \int -\frac{1}{2y}\,dy = c$$

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$$\frac{xy}{2} + \frac{2}{2}\log x + \frac{1}{2y}\cdot\frac{x^{-1}}{-1} - \frac{2}{2}\log y = c$$

$$xy + \log x - \frac{1}{xy} - \log y = c'$$

$$xy + \log(x/y) - 1/xy = c'$$

Ex. 4 $y\,dx - x\,dy + \log x\,dx = 0$

Given,

$$(y + \log x)dx - x\,dy = 0 \quad \text{—— (1)}$$

Given eq^n is in the form of $M\,dx + N\,dy = 0$

$M = y + \log x$ & $N = -x$

$\frac{\partial M}{\partial y} = 1$ & $\frac{\partial N}{\partial x} = -1$

$\therefore \frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$

$\therefore$ Given diff. equation is not exact.

$$\therefore \text{I.F.} \quad \frac{\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}}{N} = f(x)$$

$$\frac{1-(-1)}{-x} = f(x)$$

$$\therefore \frac{2}{-x} = f(x)$$

$$\therefore \text{I.F.} = e^{\int f(x)dx} = e^{\int -\frac{2}{x}dx}$$

$$= e^{-2\log x}$$

$$= e^{\log x^{-2}}$$

$$= \frac{1}{x^2}$$

Multiply $1/x^2$ to eq^n (1)

$$\frac{1}{x^2}(y + \log x) - \frac{1}{x^2}x\,dy = 0$$

$$\left(\frac{y}{x^2} + \frac{\log x}{x^2}\right)dx - \frac{1}{x}dy = 0 \quad \text{—— (2)}$$

∴ Eq^n ② is an exact.

∴ Solution:-

$$\int M' dx + \int N' dy = c$$

$$\int \left(\frac{y}{x^2} + \frac{\log x}{x^2} \right) dx + \int 0 dy = c$$

$$y \int \frac{1}{x^2} dx + \int \frac{1}{x^2} \log x dx = c$$

$$y\left(-\frac{1}{x}\right) + \left\{ \log x \cdot \int x^{-2} dx - \int \left[\int x^{-2} dx \cdot \frac{d}{dx} \log x \right] dx \right\} = c$$

$$-\frac{y}{x} + \left\{ \log x \cdot \left(-\frac{1}{x}\right) - \int -\frac{1}{x} \cdot \frac{1}{x} dx \right\} = c$$

$$-\frac{y}{x} - \frac{1}{x} \log x + \int \frac{1}{x^2} dx = c$$

$$-\frac{y}{x} - \frac{1}{x} \log x - \frac{1}{x} = c$$

$$\frac{y}{x} + \frac{1}{x} \log x + \frac{1}{x} = -c$$

Ex.5 Solve: $(y^4 + 2y)dx + (xy^3 + 2y^4 - 4x)dy = 0$

Given,

$$(y^4 + 2y)dx + (xy^3 + 2y^4 - 4x)dy = 0 \quad \text{——} \ ①$$

Given eq^n is in the form of $Mdx + Ndy = 0$

$M = y^4 + 2y$ & $N = xy^3 + 2y^4 - 4x$

∴ $\frac{\partial M}{\partial y} = 4y^3 + 2$ — ② & $\frac{\partial N}{\partial x} = y^3 - 4$ —— ③

$$\therefore \frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$$

∴ Given diff. eq^n is not exact.

Now,

$$\frac{\left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right)}{M} = f(y)$$

$$\therefore f(y) = \frac{y^3 - 4 - (4y^3 + 2)}{y^4 + 2y}$$

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$$= \frac{y^3 - 4 - 4y^3 - 2}{y(y^3+2)}$$

$$= \frac{-3y^3 - 8}{y(y^3+2)}$$

$$= \frac{-3(y^2+2)}{y(y^3+2)}$$

$$f(y) = -\frac{3}{y}$$

$$\therefore \text{I.F.} = e^{\int f(y)dy} = e^{\int -\frac{3}{y} dy}$$

$$= e^{-3\log y}$$

$$= e^{\log y^{-3}}$$

$$= \frac{1}{y^3}$$

Multiply $1/y^3$ to eq^n ①

$$\frac{1}{y^3}(y^4+2y)dx + \frac{1}{y^3}(xy^3+2y^4-4x)dy = 0$$

$$\left(y + \frac{2}{y^2}\right)dx + \left(x + 2y - \frac{4x}{y^3}\right)dy = 0 \quad \text{— ④}$$

$\therefore$ Eq^n ④ is exact and in the form of $M'dx + N'dy = 0$

$M' = y + \frac{2}{y^2}$ & $N' = x + 2y - \frac{4x}{y^3}$

Solution:-

$$\int M'dx + \int N'dy = c$$

$$\int \left(y + \frac{2}{y^2}\right)dx + \int 2y\,dy = c$$

$$xy + \frac{2x}{y^2} + 2\cdot\frac{y^2}{2} = c$$

$$xy + \frac{2x}{y^2} + y^2 = c$$

	Examples	Answer
1)	$(1+xy)y\,dx + (1-xy)x\,dy = 0$	$-\frac{1}{xy} + \log\left(\frac{x}{y}\right) = c$
2)	$y(xy + 2x^2y^2)\,dx + x(xy - x^2y^2)\,dy = 0$ $\cancel{2(x^2y^2+x+y^4)}$	$-\frac{1}{xy} + 2\log x - \log y = c$
3)	$y\,dx - x\,dy + \log x\,dx = 0$	$\frac{y}{x} + \frac{1}{x} + \frac{1}{x}\log x = c$
4)	$(xy^2 - e^{1/x^3})\,dx - x^2y\,dy = 0$	$-\frac{y^2}{2x^2} + \frac{1}{3}e^{1/x^3} = c$
5)	$\frac{dy}{dx} = \frac{x^3+y^3}{xy^2}$	$\log x - \frac{y^3}{3x^3} = c$

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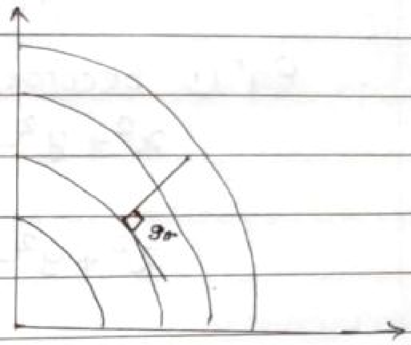
* Application of differential Equation :-

1) Orthogonal trajectories :-

Two families of curves are such that every curve of either family cuts each other family at right angles, they are called orthogonal trajectories of each other.



Orthogonal trajectories of are very useful in Engineering problems.

Working Rule :-

1) By differentiating the equation of curves find the differential equations in the form of $f(x, y, \frac{dy}{dx}) = 0$

2) Replace $\frac{dy}{dx} = -\frac{dx}{dy}$ $(\because m_1 m_2 = -1)$

3) Solve the DE of the orthogonal trajectories. ie $f(x, y, -dx/dy) = 0$

4) In polar form, Replace $\frac{dr}{d\theta} = -r^2 \frac{d\theta}{dr}$

Ex. 1 Find the orthogonal trajectories of the family of co-axial circles $x^2+y^2-2gx=0$

$\Rightarrow$ Given,

$$x^2+y^2-2gx=0 \quad \text{——— (1)}$$

Diff. w.r. to x

$$2x + 2y\frac{dy}{dx} - 2g = 0$$

$$x + y\frac{dy}{dx} - g = 0$$

Replace $\frac{dy}{dx} = -\frac{dx}{dy}$

$$x + y\left(-\frac{dx}{dy}\right) - g = 0$$

$$x - y\frac{dx}{dy} = g$$

$\therefore$ Eq^n ① becomes,

$$x^2 + y^2 - 2x\left(x - y\frac{dx}{dy}\right) = 0$$

$$x^2 + y^2 - 2x^2 + 2yx\frac{dx}{dy} = 0$$

$$-x^2 + y^2 + 2xy\frac{dx}{dy} = 0$$

$$\frac{dx}{dy} = \frac{x^2 - y^2}{2xy} \quad \text{—— ②}$$

Eq^n ② is homogeneous eq^n.

put $x = vy$

Diff. w.r.to y

$$\frac{dx}{dy} = v + y\frac{dv}{dy}$$

$$v + y\frac{dv}{dy} = \frac{(vy)^2 - y^2}{2(vy)y}$$

$$v + y\frac{dv}{dy} = \frac{v^2 - 1}{2v}$$

$$y\frac{dv}{dy} = \frac{v^2 - 1}{2v} - v$$

$$= \frac{v^2 - 1 - 2v^2}{2v}$$

$$= \frac{-1 - v^2}{2v}$$

$$\frac{2v}{v^2 + 1}\,dv = -\frac{dy}{y}$$

$$\int \frac{2v}{v^2 + 1}\,dv = -\int \frac{dy}{y}$$

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$$\log(v^2+1) = -\log y + \log c$$

$$\log(v^2+1) = \log(c/y)$$

$$v^2+1 = c/y$$

$$y(v^2+1) = c$$

$$y\left(\frac{x^2}{y^2}+1\right) = c$$

$$\underline{\underline{x^2+y^2 = cy}}$$

which is required orthogonal trajectories.

Ex. 2 Find the orthogonal trajectories of $\frac{x^2}{a^2} + \frac{y^2}{b^2+\lambda} = 1$ where λ is a parameter.

⇒ Given,

$$\frac{x^2}{a^2} + \frac{y^2}{b^2+\lambda} = 1 \quad \text{——— (1)}$$

Diff. w.r. to x

$$\frac{2x}{a^2} + \frac{1}{b^2+\lambda} \cdot 2y \cdot \frac{dy}{dx} = 0$$

Replace $\frac{dy}{dx} = -\frac{dx}{dy}$

$$\frac{x}{a^2} + \frac{y}{b^2+\lambda}\left(-\frac{dx}{dy}\right) = 0$$

$$\frac{x}{a^2} - \left(\frac{1}{b^2+\lambda}\right) \cdot y\frac{dx}{dy} = 0 \quad \text{——— (2)}$$

from (1),

$$\frac{y^2}{b^2+\lambda} = 1 - \frac{x^2}{a^2}$$

$$\frac{1}{b^2+\lambda} = \left(1 - \frac{x^2}{a^2}\right)\frac{1}{y^2}$$

$\therefore$ eqn (2) becomes

$$\frac{x}{a^2} - \left(1-\frac{x^2}{a^2}\right)\cdot\frac{1}{y^2}\cdot y\ \frac{dx}{dy} = 0$$

$$\frac{x}{a^2} = \left(1-\frac{x^2}{a^2}\right)\cdot\frac{1}{y}\ \frac{dx}{dy}$$

$$x = (a^2-x^2)\cdot\frac{1}{y}\ \frac{dx}{dy}$$

$$y\,dy = \frac{a^2-x^2}{x}\,dx$$

$$\int y\,dy = \int\frac{a^2}{x}\,dx - \int x\,dx$$

$$\frac{y^2}{2} = a^2\log x - \frac{x^2}{2} + c$$

which is the required orthogonal trajectories.

Ex. 3

Find the orthogonal trajectories of the cardioids $r = a(1-\cos\theta)$.

$\Rightarrow$ Given,

$$r = a(1-\cos\theta) \quad \text{——} \quad (1)$$

Diff. w.r.to θ

$$\frac{dr}{d\theta} = a\sin\theta$$

Replace, $\frac{dr}{d\theta} = -r^2\frac{d\theta}{dr}$

$$-r^2\frac{d\theta}{dr} = a\sin\theta$$

$$-\frac{d\theta}{\sin\theta} = \frac{a}{r^2}\,dr$$

$$-\operatorname{cosec}\theta\,d\theta = a r^{-2}\,dr$$

$$-\int\operatorname{cosec}\theta\,d\theta = a\int r^{-2}\,dr$$

$$-\log(\operatorname{cosec}\theta - \cot\theta) = a\frac{r^{-1}}{-1} + c$$

$$\log(\operatorname{cosec}\theta - \cot\theta) = a/r - c$$

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② Electric Circuits :-

① L-R Series circuits :-

Let i be the current flowing in the circuits containing resistant R and inductance L in series, with voltage source E, at any time t.

By voltage Law,

$$Ri + L\frac{di}{dt} = E$$

$$\Rightarrow \frac{di}{dt} + \frac{R}{L}i = \frac{E}{L}$$

② C-R Series Circuits :

Let i be current in the circuit containing resistance R, L and capacitance C in series with voltage source E at any time t.

By voltage Law,

$$Ri + \frac{q}{c} = E$$

$$R\frac{dq}{dt} + \frac{q}{c} = E \qquad (i = dq/dt)$$

Ex. When a switch is closed in a circuit containing a battery E, a resistance R and an inductance L, the current i builds up at the rate given by

$$L\frac{di}{dt} + Ri = E$$

find i as a function of t.

$\Rightarrow$ Given differential Equation is,

$$L\frac{di}{dt} + Ri = E$$

$$\therefore \frac{di}{dt} + \frac{R}{L}i = \frac{E}{L}$$

where $P = R/L$ & $Q = E/L$

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$$I.F = e^{\int P dt} = e^{\int \frac{R}{L} dt}$$
$$= e^{\frac{R}{L}t}$$

Solution:-

$$i\,(IF) = \int Q(IF)\,dt + C$$
$$i\,e^{Rt/L} = \int \frac{E}{L} \cdot e^{Rt/L}\,dt + C$$
$$= \frac{E}{L} \int e^{Rt/L}\,dt + C$$
$$= \frac{E}{L} \cdot \frac{e^{Rt/L}}{R/L} + C$$
$$i e^{Rt/L} = \frac{E}{R} e^{Rt/L} + C$$
$$\therefore i = \frac{E}{R} + C e^{-Rt/L} \quad \text{--- (2)}$$

when $i = 0$ at $t = 0$

$$0 = \frac{E}{R} + C(1)$$
$$\therefore C = -E/R$$

From (2),

$$i = \frac{E}{R} + \left(-\frac{E}{R}\right) e^{-Rt/L}$$
$$i = \frac{E}{R}\left(1 - e^{-Rt/L}\right)$$

Ex.2 A 20 ohms resistor is connected in a series with a capacitor of 0.01 farads, and the electromotive force E volts is given by $40e^{-3t} + 20e^{-6t}$. If $Q = 0$ at $t = 0$ show that the maximum charge on capacitor is 0.25 coulombs.

⇒ We know that

$$R\frac{dq}{dt} + \frac{q}{c} = E$$

$R = 20$, $c = 0.01$ & $E = 40e^{-3t} + 20e^{-6t}$

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$$20\frac{dq}{dt} + \frac{1}{0.01}q = 40e^{-3t} + 20e^{-6t}$$

$$\frac{dq}{dt} + 5q = 2e^{-3t} + e^{-6t}$$

where $p = 5$, $Q = 2e^{-3t} + e^{-6t}$

$$I.F. = e^{\int p\,dt} = e^{\int 5\,dt} = e^{5t}$$

Solution:-

$$q(IF) = \int Q(IF)\,dt + C$$

$$q e^{5t} = \int e^{5t}(2e^{-3t} + e^{-6t})\,dt + C$$

$$q e^{5t} = \int (2e^{2t} + e^{-t})\,dt + C$$

$$= \frac{2e^{2t}}{2} + \frac{e^{-t}}{-1} + C$$

$$q e^{5t} = e^{2t} - e^{-t} + C$$

$$q = e^{-3t} - e^{-6t} + C \quad \text{---} \quad (2)$$

when, $q = 0$ at $t = 0$

$$0 = 1 - 1 + C$$

$$\boxed{C = 0}$$

$$\therefore q = e^{-3t} - e^{-6t} \quad \text{---} \quad (3)$$

For maximum charge, q, put $\frac{dq}{dt} = 0$

$$\frac{dq}{dt} = -3e^{-3t} + 6e^{-6t} = 0$$

$$-e^{-3t} + 2e^{-6t} = 0$$

$$e^{-3t} = 2e^{-6t}$$

$$1 = 2e^{-3t}$$

$$e^{-3t} = 1/2$$

From (3), $q = e^{-3t} - (e^{-3t})^2 = \frac{1}{2} - \left(\frac{1}{2}\right)^2$

$$= \frac{1}{2} - \frac{1}{4}$$

$$= 1/4$$

$$q = 0.25$$

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Ex. 3 Determine the charge and current at any time t, in a series R-C Circuit with $R = 10\Omega$, $C = 2\times10^{-4}F$ and $E = 100V$, given that $q(0) = 0$

⇒ Given, R-C circuit,

$$R\frac{dq}{dt} + \frac{q}{c} = E$$

$$\frac{dq}{dt} + \frac{q}{RC} = \frac{E}{R} \quad \text{——— (1)}$$

where $P = \frac{1}{RC}$, $Q = \frac{E}{R}$

$$I.F. = e^{\int P dt}$$
$$= e^{\int 1/RC \, dt} = e^{t/RC}$$

Solution :-

$$q(IF) = \int Q(IF)dt + c'$$

$$q e^{t/RC} = \int \frac{E}{R} e^{t/RC} + c'$$

$$q e^{t/RC} = \frac{E}{R} \cdot \frac{e^{t/RC}}{1/RC} + c'$$

$$q \cdot e^{t/RC} = EC e^{t/RC} + c' \quad \text{——— (2)}$$

when, $q = 0$ at $t = 0$

$$0 = EC + c'$$
$$c' = -EC$$
$$\therefore q e^{t/RC} = EC e^{t/RC} - EC$$

$$q = EC(1 - e^{-t/RC})$$

But, $R = 10$, $c = 2\times10^{-4}$ & $E = 100$

$$q = 100\times2\times10^{-4}(1 - e^{-t/10\times2\times10^{-4}})$$
$$q = 0.02(1 - e^{-500t})$$

But $i = \frac{dq}{dt} = 0.02(0 - e^{-500t})(-500)$

$$i = 10 e^{-500t}$$

Examples:

1) Find orthogonal trajectory of $x^2 + cy^2 = 1$

Ans: $\log x - \frac{x^2}{2} = \frac{y^2}{2} + c$

2) Find orthogonal trajectories of $y^2 = 4ax$

Ans: $y^2 + 2x^2 = c$

3) Find orthogonal trajectories of $x^2 - ax + 4y = 0$

Ans: $x^2 = 4y + 8 + ce^{-y/2}$

4) Find the orthogonal trajectories of

1) $r = a(1 - \cos\theta)$

Ans: $r = c(1 + \cos\theta)$

2) $r^2 = a^2 \cos 2\theta$

$r^2 = c_1 \sin 2\theta$

5) A constant e.m.f. E volts is applied to a circuit containing a constant resistance R ohms in series and a constant inductance L henries. The current i at any time t is given by

$$L\frac{di}{dt} + Ri = E$$

if the initial current is zero. Show that the current builds up to half its theoretical maximum value in $L/R \cdot \log 2$ seconds.

6) An emf $200e^{-5t}$ is applied to a series circuit consisting of $20\,\Omega$ resister and 0.01 F capacitor. Find the charge and current at any time, assuming that there is no initial charge on capacitor.

Ans: $i = e^{-5t}(10 - 50t)$