Unit - IV

Vector Differential Calculus

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Algebra of Vectors:-

Let $\bar{a} = a_1\bar{i} + a_2\bar{j} + a_3\bar{k}$

$\bar{b} = b_1\bar{i} + b_2\bar{j} + b_3\bar{k}$

$\bar{c} = c_1\bar{i} + c_2\bar{j} + c_3\bar{k}$

The unit vector in the direction of $\bar{a}$ is given by $\hat{a}$

$$\hat{a} = \frac{\bar{a}}{|\bar{a}|}, \quad |\bar{a}| = \sqrt{a_1^2 + a_2^2 + a_3^2}$$

where $\bar{i}, \bar{j}$ & $\bar{k}$ are the unit vectors in the direction of x, y & z axis respectively.

1) Dot Product of two vectors:-

$$\bar{a} \cdot \bar{b} = ab\cos\theta$$

where θ is an angle between $\bar{a}$ & $\bar{b}$

Note:

① $\bar{i} \cdot \bar{i} = \bar{j} \cdot \bar{j} = \bar{k} \cdot \bar{k} = 1$ $\quad \because \cos 0 = 1$

② $i \cdot \bar{j} = \bar{j} \cdot \bar{k} = \bar{k} \cdot \bar{i} = 0$ $\quad \because \cos 90 = 0$

③ $\bar{a} \cdot \bar{b} = a_1b_1 + a_2b_2 + a_3b_3$

④ $\bar{a} \cdot \bar{b} = \bar{b} \cdot \bar{a}$

④ $\bar{a} \cdot \bar{b}$ is a scalar.

⑤ Angle between two vectors $\bar{a}$ & $\bar{b}$ is given by

$$\cos\theta = \frac{\bar{a} \cdot \bar{b}}{ab}$$

⑥ If two vectors $\bar{a}$ & $\bar{b}$ are perpendicular then $\bar{a} \cdot \bar{b} = 0$

② Cross Product of two vectors:

$$\bar{a} \times \bar{b} = ab\sin\theta\,\hat{n}$$

where $\hat{n}$ is a unit vector perpendicular to the plane of $\bar{a}$ and $\bar{b}$ is

$$\hat{n} = \frac{\bar{a} \times \bar{b}}{|\bar{a} \times \bar{b}|}$$

Note:

① $\bar{i} \times \bar{i} = \bar{j} \cdot \bar{j} = \bar{k} \cdot \bar{k} = 0$ $\quad \because \sin\theta = 0$

② $\bar{i} \times \bar{j} = \bar{k}$, $\bar{j} \times \bar{k} = \bar{i}$, $\bar{k} \times \bar{i} = \bar{j}$ $\quad \because \sin 90 = 1$

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③ $\bar{a} \times \bar{b} = \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$

④ $\bar{a} \times \bar{b} = -(\bar{b} \times \bar{a})$

⑤ cross product of two vectors is a vector

⑥ If two vectors are parallel then $\bar{a} \times \bar{b} = 0$

③ Scalar Triple product (Box product)

$$\bar{a} \cdot [\bar{b} \times \bar{c}] = [\bar{a}\ \bar{b}\ \bar{c}]$$

Now

$$\bar{a} \cdot [\bar{b} \times \bar{c}] = [\bar{a}\ \bar{b}\ \bar{c}] = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$

If $[\bar{a}\ \bar{b}\ \bar{c}] = 0$ then $\bar{a}, \bar{b}, \bar{c}$ are co-planer.

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★ Differentiation of vector :-

If $\bar{r} = x\bar{i} + y\bar{j} + z\bar{k}$ then

$$\frac{d\bar{r}}{dt} = \frac{dx}{dt}\bar{i} + \frac{dy}{dx}\bar{j} + \frac{dz}{dx}\bar{k}$$

and,

$$\frac{d^2\bar{r}}{dx^2} = \frac{d^2x}{dt^2}\bar{i} + \frac{d^2y}{dt^2}\bar{j} + \frac{d^2z}{dt^2}\bar{k}$$

Standard Results:-

1) $\frac{d}{dt}(\bar{u} \pm \bar{v}) = \frac{d\bar{u}}{dt} \pm \frac{d\bar{v}}{dt}$

2) $\frac{d}{dt}(\phi\bar{u}) = \frac{d\phi}{dt}\bar{u} + \phi\frac{d\bar{u}}{dt}$

3) $\frac{d}{dt}(\bar{u}\cdot\bar{v}) = \frac{d\bar{u}}{dt}\bar{v} + \bar{u}\frac{d\bar{v}}{dt}$

4) $\frac{d}{dt}(\bar{u}\times\bar{v}) = \frac{d\bar{u}}{dt}\times\bar{v} + \bar{u}\times\frac{d\bar{v}}{dt}$

5) $\frac{d}{dt}[\bar{u}\cdot(\bar{v}\times\bar{w})] = \frac{d\bar{u}}{dt}\cdot(\bar{v}\times\bar{w}) + \bar{u}\cdot\left(\frac{d\bar{v}}{dt}\times w\right) + \bar{u}\cdot\left(\bar{v}\times\frac{d\bar{w}}{dt}\right)$

6) $\frac{d}{dt}[\bar{u}\times(\bar{v}\times\bar{w})] = \frac{d\bar{u}}{dt}\times(\bar{v}\times\bar{w}) + \bar{u}\times\left(\frac{d\bar{v}}{dt}\times\bar{w}\right) + \bar{u}\times\left(\bar{v}\times\frac{d\bar{w}}{dt}\right)$

★ Some important deductions :-

Theorem:- 01

" The neccessary and sufficient condition for vector $\bar{F}(t)$ to have constant magnitude is

$$\bar{F}(t)\cdot\frac{d\bar{F}(t)}{dt} = 0 \text{ "}$$

Proof:

The proof of this theorem consist of two parts:
1) neccessary 2) sufficient.

Neccesary condition:-

Let $\bar{F}(t)$ be a vector of constant magnetude F.

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we know that,

$$\bar{F}(t)\cdot\bar{F}(t) = F\cdot F\cdot\cos 0 = F^2$$

$$\therefore \bar{F}(t)\cdot\bar{F}(t) = F^2 \quad \text{(constant)}$$

Differentiate w.r.to t

$$\frac{d\bar{F}(t)}{dt}\cdot\bar{F}(t) + \bar{F}(t)\cdot\frac{d\bar{F}(t)}{dt} = \frac{d}{dt}F^2$$

But $\bar{a}\cdot\bar{b} = \bar{b}\cdot\bar{a}$

$$\bar{F}(t)\cdot\frac{d\bar{F}(t)}{dt} + \bar{F}(t)\cdot\frac{d\bar{F}(t)}{dt} = 0$$

$$2\bar{F}(t)\cdot\frac{d}{dt}\bar{F}(t) = 0$$

$$\Rightarrow \bar{F}(t)\cdot\frac{d}{dt}\bar{F}(t) = 0$$

Which proves the neccessary condition.

Sufficient condition:-

Let $\bar{F}(t)\frac{d}{dt}\bar{F}(t) = 0$

we know that, $\bar{F}(t)\cdot\bar{F}(t) = F^2$

Diff. w.r.to t

$$\frac{d\bar{F}(t)}{dt}\cdot\bar{F}(t) + \bar{F}(t)\frac{d}{dt}\bar{F}(t) = 2F\cdot\frac{dF}{dt}$$

ie $$2\bar{F}(t)\frac{d}{dt}\bar{F}(t) = 2F\frac{dF}{dt}$$

$$0 = 2F\frac{dF}{dt}$$

$$\Rightarrow 2F\frac{dF}{dt} = 0$$

$$\Rightarrow F\frac{dF}{dt} = 0$$

$$\Rightarrow \frac{dF}{dt} = 0$$

$$\therefore F = \text{constant}$$

$\therefore \bar{F}(t)$ well have constant magnetude.

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Theorem :- 02

Statement :-

" The neccessary and sufficient condition for a vector $\bar{F}(t)$ to have constant direction is

$$\bar{F}(t) \times \frac{d\bar{F}(t)}{dt} = 0$$

⇒ Proof :-

Necessary condition :-

Let us consider that $\bar{F}(t)$ to have constant direction.

is ~~$\bar{F}(t) \cdot \frac{d\bar{F}(t)}{dt} \neq 0$~~

Then $\hat{F}(t)$ is the unit vector of $\bar{F}(t)$ then

$$\hat{F}(t) = \frac{\bar{F}(t)}{|\bar{F}(t)|} = \frac{\bar{F}(t)}{F(t)}, \quad \text{where } F(t) = |\bar{F}(t)| \quad \text{——— (1)}$$

$$\text{ie} \quad \bar{F}(t) = F(t) \cdot \hat{F}(t) \quad \text{——— (2)}$$

From (1), the direction of $\bar{F}(t)$ & $\hat{F}(t)$ are same.

Since the direction of $\bar{F}(t)$ is constant, the direction of $\hat{F}(t)$ well also be constant.

Also, $|\hat{F}(t)| = 1$ = constant

ie magnetude of $\hat{F}(t)$ is constant

$$\therefore \frac{d}{dt}\hat{F}(t) = 0$$

Now,

$$\bar{F}(t) \times \frac{d}{dt}\bar{F}(t) = F(t)\hat{F}(t) \times \frac{d}{dt}\{F(t) \cdot \hat{F}(t)\}$$

$$= F(t)\hat{F}(t) \times \left\{\frac{d}{dt}F(t) \cdot \hat{F}(t) + F(t)\frac{d}{dt}\hat{F}(t)\right\}$$

$$= F(t)\hat{F}(t) \times \left\{\frac{d}{dt}F(t) \cdot \hat{F}(t) + 0\right\}$$

$$= F(t)\hat{F}(t) \times \left\{\frac{d}{dt}F(t) \cdot \hat{F}(t)\right\}$$

$$= F(t)\frac{dF(t)}{dt}\{\hat{F}(t) \times \hat{F}(t)\}$$

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$$= F(t) \frac{d F(t)}{dt} (\bar{0})$$

$$= 0$$

$$\boxed{\therefore \quad \bar{F}(t) \times \frac{d\bar{F}(t)}{dt} = 0}$$

<u>Sufficient condition:-</u>

Let $\bar{F}(t) \times \frac{d\bar{F}(t)}{dt} = 0$

then, $F(t)\hat{F}(t) \times \frac{d}{dt}\{F(t) \cdot \hat{F}(t)\} = 0$

$$\Rightarrow \quad F(t)\hat{F}(t) \times \left\{ \frac{dF(t)}{dt} \cdot \hat{F}(t) + F(t) \frac{d\hat{F}(t)}{dt} \right\} = 0$$

$$\Rightarrow \quad F(t) \cdot \frac{dF(t)}{dt} \{\hat{F}(t) \times \hat{F}(t)\} + [F(t)]^2 \hat{F}(t) \times \frac{d\hat{F}(t)}{dt} = 0$$

$$\Rightarrow \quad [F(t)]^2 \hat{F}(t) \times \frac{d\hat{F}(t)}{dt} = 0 \quad \text{——— (3)}$$

$$\Rightarrow \quad \hat{F}(t) \times \frac{d\hat{F}(t)}{dt} = 0 \quad \text{——— (4)}$$

From (3) & (4), we conclude that $\frac{d\hat{F}(t)}{dt} = 0$

$\therefore$ $\hat{F}(t)$ is a constant vector, so the direction of $\hat{F}(t)$ is also constant.

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★ Operator ∇ :-

The operator ∇, defined by,

$$\nabla \equiv \hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}$$

is called the vector differential operator and read as del or nabla.

1) Gradient of a scalar point function :-

Let $\phi = \phi(x, y, z)$ be a scalar point function and differentiable at each point (x, y, z) of the Region R, the gradient of scalar point function ϕ, denoted as $\nabla\phi$ or Grd ϕ is defined as

$$\nabla\phi = \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)\phi$$

$$\nabla\phi = \hat{i}\frac{\partial\phi}{\partial x} + \hat{j}\frac{\partial\phi}{\partial y} + \hat{k}\frac{\partial\phi}{\partial z}$$

Properties of Gradient :-

If ϕ and ψ are two scalar point functions, c is a constant, then

1) $\nabla(\phi \pm \psi) = \nabla\phi \pm \nabla\psi$
2) $\nabla(\phi\psi) = \phi\nabla\psi + (\nabla\phi)\psi$
3) $\nabla\left(\frac{\phi}{\psi}\right) = \frac{(\nabla\phi)\psi - \phi(\nabla\psi)}{\psi^2}, \quad \psi \neq 0$
4) $\nabla(c\phi) = c\,\nabla\phi$
5) $\nabla f(\phi) = f'(u)\nabla\phi, \quad f'(u) = \frac{df(u)}{du}$

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Ex.1 If $\bar{r}$ is the position vector $\bar{r} = x\hat{i} + y\hat{j} + z\hat{k}$ with $|\bar{r}| = r$, then evaluate: a) ∇r^m b) ∇r c) $\nabla 1/r$ d) $\nabla(1/r^2)$

Ans: → Given, $\bar{r} = x\hat{i} + y\hat{j} + z\hat{k}$

and $r = |\bar{r}| = \sqrt{x^2+y^2+z^2}$

$\therefore \; r^m = (x^2+y^2+z^2)^{m/2}$

$$\frac{\partial r^m}{\partial x} = \frac{m}{2}(x^2+y^2+z^2)^{m/2-1} \cdot \frac{\partial}{\partial x}(x^2+y^2+z^2)$$

$$= \frac{m}{2}(x^2+y^2+z^2)^{\frac{m-2}{2}} \cdot (2x)$$

$$= mx\,[r^2]^{\frac{m-2}{2}}$$

$$\frac{\partial r^m}{\partial x} = mx\,r^{m-2}$$

Simullaly,

$$\frac{\partial r^m}{\partial y} = my\,r^{m-2} \quad \& \quad \frac{\partial r^m}{\partial z} = mz\,r^{m-2}$$

1) $$\nabla r^m = \left(\frac{\partial}{\partial x}\hat{i} + \frac{\partial}{\partial y}\hat{j} + \frac{\partial}{\partial z}\hat{k}\right) r^m$$

$$= \frac{\partial r^m}{\partial x}\hat{i} + \frac{\partial r^m}{\partial y}\hat{j} + \frac{\partial r^m}{\partial z}\hat{k}$$

$$= mx\,r^{m-2}\hat{i} + my\,r^{m-2}\hat{j} + mz\,r^{m-2}\hat{k}$$

$$= m\,r^{m-2}(x\hat{i} + y\hat{j} + z\hat{k})$$

$$\nabla r^m = m\,r^{m-2} \cdot \bar{r} \quad \text{——— (1)}$$

2) Put $m = 1$ in eq^n (1)

$$\nabla r^1 = (1)\,r^{1-2} \cdot \bar{r}$$

$$\nabla r = r^{-1} \cdot \bar{r}$$

$$\nabla r = \frac{\bar{r}}{r}$$

3) Put $m = -1$ in eq^n (1)

$$\nabla r^{-1} = (-1)\,r^{-1-2} \cdot \bar{r}$$

$$\nabla\left(\frac{1}{r}\right) = -r^{-3} \cdot \bar{r} = -\frac{\bar{r}}{r^3}$$

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4) Put $m = -2$ in equation ①

$$\nabla \bar{r}^{2} = (-2)\, \bar{r}^{2-2} \cdot \bar{r}$$

$$= -2 \cdot \bar{r}^{4} \cdot \bar{r}$$

$$\nabla\left(\frac{1}{r^2}\right) = \frac{-2\bar{r}}{r^4}$$

Ex. 2 Find $\nabla\phi$, if

1) $\phi = \log(x^2+y^2+z^2)$

2) $\phi = x\sin z - y\cos z$

3) $\phi = r^2 e^{-r}$

Ans:-

1) Given $\phi = \log(x^2+y^2+z^2)$

$$\frac{\partial\phi}{\partial x} = \frac{1}{x^2+y^2+z^2} \cdot (2x)$$

$$\frac{\partial\phi}{\partial x} = \frac{2x}{x^2+y^2+z^2}$$

Similarly, $\frac{\partial\phi}{\partial y} = \frac{2y}{x^2+y^2+z^2}$, $\frac{\partial\phi}{\partial z} = \frac{2z}{x^2+y^2+z^2}$

Now,

$$\nabla\phi = \frac{\partial\phi}{\partial x}\hat{i} + \frac{\partial\phi}{\partial y}\hat{j} + \frac{\partial\phi}{\partial z}\hat{k}$$

$$= \left(\frac{2x}{x^2+y^2+z^2}\right)\hat{i} + \left(\frac{2y}{x^2+y^2+z^2}\right)\hat{j} + \left(\frac{2z}{x^2+y^2+z^2}\right)\hat{k}$$

$$= \frac{2}{x^2+y^2+z^2}(x\hat{i}+y\hat{j}+z\hat{k})$$

$$= \frac{2}{r^2} \cdot \bar{r}$$

$$\nabla\phi = \left(\frac{2}{r^2}\right)\bar{r}$$

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2) Given, $\phi = x\sin z - y\cos z$

$$\frac{\partial \phi}{\partial x} = \sin z, \quad \frac{\partial \phi}{\partial y} = -\cos z, \quad \frac{\partial \phi}{\partial z} = y\sin z + x\cos z$$

$$\nabla\phi = \frac{\partial \phi}{\partial x}\hat{i} + \frac{\partial \phi}{\partial y}\hat{j} + \frac{\partial \phi}{\partial z}\hat{k}$$

$$= \sin z\,\hat{i} + (-\cos z)\hat{j} + (x\cos z + y\sin z)\hat{k}$$

$$\nabla\phi = (\sin z)\hat{i} - (\cos z)\hat{j} + (x\cos z + y\sin z)\hat{k}$$

3) Given, $\phi = r^2 e^{-r}$

$$\frac{\partial \phi}{\partial x} = \frac{\partial}{\partial x}(r^2 e^{-r}) = \frac{\partial}{\partial r}(r^2 \cdot e^{-r}) \cdot \frac{\partial r}{\partial x}$$

$$= r^2 \cdot e^{-r}(-1) + e^{-r}(2r) \cdot \frac{\partial}{\partial x}(x^2+y^2+z^2)^{1/2}$$

$$= (2re^{-r} - r^2 e^{-r}) \cdot \frac{1}{2}(x^2+y^2+z^2)^{-1/2}(2x)$$

$$= (2re^{-r} - r^2 e^{-r})\, x\, (x^2+y^2+z^2)^{-1/2}$$

$$= r(2e^{-r} - re^{-r})\frac{x}{\sqrt{x^2+y^2+z^2}}$$

$$= r(2e^{-r} - re^{-r}) \cdot \frac{x}{r}$$

$$\frac{d\phi}{dx} = x(2e^{-r} - re^{-r})$$

Simullarly, $\frac{d\phi}{dy} = y(2e^{-r} - re^{-r})$ & $\frac{d\phi}{dz} = z(2e^{-r} - re^{-r})$

$$\therefore \nabla\phi = \frac{\partial \phi}{\partial x}\hat{i} + \frac{\partial \phi}{\partial y}\hat{j} + \frac{\partial \phi}{\partial z}\hat{k}$$

$$= x(2e^{-r} - re^{-r})\hat{i} + y(2e^{-r} - re^{-r})\hat{j} + (2e^{-r} - re^{-r})\hat{k}$$

$$= (2e^{-r} - re^{-r})(x\hat{i} + y\hat{j} + z\hat{k})$$

$$\nabla\phi = (2e^{-r} - re^{-r})\,\bar{r}$$

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* Directional Derivatives :-

Theorem:-

" The directional derivative of a scalar point function ϕ along the direction of a unit vector $\hat{a}$ is $\nabla\phi \cdot \hat{a}$ "

Proof:-

The unit vector $\hat{a}$ along a line whose direction cosines are l, m & n is

$$\hat{a} = l\hat{i} + m\hat{j} + n\hat{k}$$

$$\therefore \nabla\phi \cdot \hat{a} = \left(\hat{i}\frac{\partial\phi}{\partial x} + \hat{j}\frac{\partial\phi}{\partial y} + \hat{k}\frac{\partial\phi}{\partial z}\right) \cdot (l\hat{i} + m\hat{j} + \hat{k}n)$$

$$\nabla\phi \cdot \hat{a} = l\frac{\partial\phi}{\partial x} + m\frac{\partial\phi}{\partial y} + n\frac{\partial\phi}{\partial z}.$$

Ex. 1 Find the directional derivatives of $\phi = e^{2x}\cos(yz)$ at $(0,0,0)$ in the direction of the tangent to the curves $x = a\sin t$, $y = a\cos t$, $z = at$ at $t = \pi/4$

Ans:- Given $\phi = e^{2x}\cos(yz)$ ——— (1)

Now, $\frac{\partial\phi}{\partial x} = 2e^{2x}\cos(yz)$

$$\frac{\partial\phi}{\partial y} = e^{2x}(-\sin(yz)) \cdot z = -ze^{2x}\sin(yz)$$

$$\frac{\partial\phi}{\partial z} = e^{2x}(-\sin(yz)) \cdot y = -ye^{2x}\sin(yz)$$

$$\therefore \nabla\phi = \frac{\partial\phi}{\partial x}\hat{i} + \frac{\partial\phi}{\partial y}\hat{j} + \frac{\partial\phi}{\partial z}\hat{k}$$

$$= (2e^{2x}\cos(yz))\hat{i} + (-ze^{2x}\sin(yz))\hat{j} + (-ye^{2x}\sin(yz))\hat{k}$$

$$\nabla\phi = 2e^{2x}\cos(yz)\hat{i} - ze^{2x}\sin(yz)\hat{j} - ye^{2x}\sin(yz)\hat{k}$$

$$\nabla\phi\Big|_{(0,0,0)} = 2(1)(1)\hat{i} - 0\hat{j} - 0\hat{k}$$

$$= 2\hat{i} + 0\hat{j} + 0\hat{k} = \bar{a} \text{ (say)}$$

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Also, $\bar{r} = x\hat{i} + y\hat{j} + z\hat{k}$

$= (a\sin t)\hat{i} + (a\cos t)\hat{j} + (at)\hat{k}$

$\therefore \frac{\partial \bar{r}}{\partial t} = a\cos t\,\hat{i} - a\sin t\,\hat{j} + a\hat{k}$

$\left.\frac{d\bar{r}}{dt}\right|_{t=\pi/4} = a\cos\pi/4\,\hat{i} - a\sin\pi/4\,\hat{j} + a\hat{k}$

$= \frac{a}{\sqrt{2}}\hat{i} - \frac{a}{\sqrt{2}}\hat{j} + a\hat{k}$

$\therefore \hat{b} = \frac{\bar{b}}{|\bar{b}|} = \frac{\frac{a}{\sqrt{2}}\hat{i} - \frac{a}{\sqrt{2}}\hat{j} + a\hat{k}}{\sqrt{\frac{a^2}{2} + \frac{a^2}{2} + a^2}} = \frac{\frac{1}{\sqrt{2}}\hat{i} - \frac{1}{\sqrt{2}}\hat{j} + \hat{k}}{\sqrt{2}}$

$\hat{b} = \frac{1}{2}\hat{i} - \frac{1}{2}\hat{j} + \frac{1}{\sqrt{2}}\hat{k}$

Now,

$\nabla\phi \cdot \hat{b} = (2\bar{i} + 0\bar{j} + 0\bar{k}) \cdot \left(\frac{1}{2}\hat{i} + \frac{1}{2}\hat{j} + \frac{1}{\sqrt{2}}\bar{k}\right)$

$= 1 + 0 + 0$

$\underline{\underline{\nabla\phi \cdot \hat{b} = 1}}$

which is the required directional derivatives.

Ex. 2 Find the directional derivative of $\phi = x^2 - y^2 + 2z^2$ at the Point $P(1,2,3)$ in the direction of the line PQ where Q is the point $(5,0,4)$. Also calculate the magnetude of the maximum directional derivatives.

$\Rightarrow$ Given, $\phi = x^2 - y^2 + 2z^2$ —— ①

$\frac{\partial\phi}{\partial x} = 2x, \quad \frac{\partial\phi}{\partial y} = -2y, \quad \frac{\partial\phi}{\partial z} = 4z$

$\nabla\phi = \frac{\partial\phi}{\partial x}\hat{i} + \frac{\partial\phi}{\partial y}\hat{j} + \frac{\partial\phi}{\partial z}\hat{k}$

$\nabla\phi = 2x\hat{i} - 2y\hat{j} + 4z\hat{k}$

$\left.\nabla\phi\right|_{(1,2,3)} = 2\hat{i} - 4\hat{j} + 12\hat{k}$

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Also, $PQ = OQ - OP$

$$= (5\hat{i} + 0\hat{j} + 4\hat{k}) - (\hat{i} + 2\hat{j} + 3\hat{k})$$

$$= 4\hat{i} - 2\hat{j} + \hat{k} = \bar{a} \text{ (say)}$$

$$\therefore \hat{a} = \frac{\bar{a}}{|\bar{a}|}$$

$$= \frac{4\hat{i} - 2\hat{j} + \hat{k}}{\sqrt{16+4+1}}$$

$$\hat{a} = \frac{4\hat{i} - 2\hat{j} + \hat{k}}{\sqrt{21}}$$

The required directional derivative given by

$$\nabla\phi_{(1,2,3)} \cdot \hat{a} = (2\hat{i} - 4\hat{j} + 12\hat{k}) \cdot \frac{(4\hat{i} - 2\hat{j} + \hat{k})}{\sqrt{21}}$$

$$= \frac{1}{\sqrt{21}} (8 + 8 + 12)$$

$$= \frac{28}{\sqrt{21}}$$

The magnetude of maximum directional derivative $= |\nabla\phi|$

$$= \sqrt{(2)^2 + (-4)^2 + (12)^2}$$

$$= \sqrt{4 + 16 + 144}$$

$$= \sqrt{164}$$

Ans:- Given, $\phi = 5x^2y - 5y^2z + 2.5x^2z$ —— (1)

$$\frac{\partial\phi}{\partial x} = 10xy + 5xz, \quad \frac{\partial\phi}{\partial y} = -10yz, \quad \frac{\partial\phi}{\partial z} = 5xz.$$

$$\nabla\phi = \frac{\partial\phi}{\partial x}\hat{i} + \frac{\partial\phi}{\partial y}\hat{j} + \frac{\partial\phi}{\partial z}\hat{k}$$

$$= 10xy\hat{i} - 10yz\hat{j} + 5xz\hat{k}$$

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Ex. 3 Find the directional derivatives of $\phi = 5x^2y - 5y^2z + 2.5z^2x$ at the point $P(1,1,1)$ in the direction of line

$$\frac{x-1}{2} = \frac{y-3}{-2} = z.$$

$\Rightarrow$ Given, $\phi = 5x^2y - 5y^2z + 2.5z^2x$ —— (1)

$$\frac{\partial \phi}{\partial x} = 10xy + 2.5z^2$$

$$\frac{\partial \phi}{\partial y} = -10yz + 5x^2$$

$$\frac{\partial \phi}{\partial z} = -5y^2 + 5xz$$

Now, $\nabla \phi = \frac{\partial \phi}{\partial x}\hat{i} + \frac{\partial \phi}{\partial y}\hat{j} + \frac{\partial \phi}{\partial z}\hat{k}$

$$\nabla \phi = (10xy + 2.5z^2)\hat{i} + (5x^2 - 10yz)\hat{j} + (5xz - 5y^2)\hat{k}$$

$$\nabla \phi\big|_{(1,1,1)} = (12.5)\hat{i} - 5\hat{j} + 0\hat{k}$$
$$= 12.5\hat{i} - 5\hat{j} + 0\hat{k}$$

Given line is

$$\frac{x-1}{2} = \frac{y-3}{-2} = \frac{z}{1}$$

The vector along a line whose direction cosine are $(2,-2,1)$ is

$$\bar{a} = 2\hat{i} - 2\hat{j} + \hat{k}$$

$$\hat{a} = \frac{\bar{a}}{|\bar{a}|} = \frac{2\hat{i} - 2\hat{j} + \hat{k}}{3}$$

Required directional derivative is

$$\nabla \phi\big|_{(1,1,1)} \cdot \hat{a} = \left((12.5)\hat{i} - 5\hat{j} + 0\hat{k}\right) \cdot \frac{(2\hat{i} - 2\hat{j} + \hat{k})}{3}$$

$$= \frac{(12.5)(2) + (-5)(-2) + 0\cdot(1)}{3}$$

$$= \frac{25 + 10}{3} = \frac{35}{3}$$

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Ex.4 If the directional derivative of $\phi = axy^2 + byz + cz^2x^3$ at $(1,2,-1)$ has a maximum magnetude of 64 in the direction parallel to z-axis, find the values of a, b, c.

Given, $\phi = axy^2 + byz + cz^2x^3$

$$\frac{\partial \phi}{\partial x} = ay^2 + 3cz^2x^2$$

$$\frac{\partial \phi}{\partial y} = 2axy + bz$$

$$\frac{\partial \phi}{\partial z} = by + 2czx^2$$

$$\nabla\phi = \frac{\partial \phi}{\partial x}\hat{i} + \frac{\partial \phi}{\partial y}\hat{j} + \frac{\partial \phi}{\partial z}\hat{k}$$

$$\nabla\phi = (ay^2 + 3cz^2x^2)\hat{i} + (2axy + bz)\hat{j} + (by + 2czx^2)\hat{k}$$

$$\nabla\phi\Big|_{(1,2,-1)} = (4a + 3c)\hat{i} + (4a - b)\hat{j} + (2b - 2c)\hat{k}$$

But, maximum magnetude $= \nabla\phi|_{(1,2,-1)} = 64$

$$64\hat{k} = (4a+3c)\hat{i} + (4a-b)\hat{j} + (2b-2c)\hat{k}$$

$\Rightarrow$ $4a + 3c = 0$ —— ①

$4a - b = 0$ —— ②

$2b - 2c = 64$ —— ③

From ① & ②, $4a + 3c = 0$

$$\underline{-4a \mp b = \mp 0}$$

$b + 3c = 0$ —— ④

From ③ & ④,

$$2b - 2c = 64$$
$$\underline{-2b \pm 6c = 0}$$
$$-8c = 64$$
$$\underline{c = -8}$$

From ①, $4a = -3(-8) = 24$

$$\underline{a = 6}$$

From ②, $4a = b$

$$b = 4(6) = 24$$

$\therefore$ $\underline{a = 6,\ b = 24,\ c = -8}$

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★ Divergence of a vector point function:-

Let $\bar{F} = f_1\hat{i} + f_2\hat{j} + f_3\hat{k}$ be any vector point function defined in a certain field, then

$$\nabla\cdot\bar{F} = \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)\cdot\left(f_1\hat{i} + f_2\hat{j} + f_3\hat{k}\right)$$

$$\nabla\cdot\bar{F} = \frac{\partial f_1}{\partial x} + \frac{\partial f_2}{\partial y} + \frac{\partial f_3}{\partial z}$$

is called the divergence of $\bar{F}$.

Note:-

1) $\text{div}\,\bar{F} = \nabla\cdot\bar{F} = \frac{\partial f_1}{\partial x} + \frac{\partial f_2}{\partial y} + \frac{\partial f_3}{\partial z}$

2) $\nabla\cdot\bar{F} = 0$ then $\bar{F}$ is said to be Solenoidal.

Ex. 1 Find $\nabla\cdot\bar{F}$ where $\bar{F} = \cancel{\text{div}}\ \text{grad}\,(x^3+y^3+z^3-3xyz)$.

Given,

Let $\phi = x^3+y^3+z^3-3xyz$

$\bar{F} = \text{grad}\,\phi = \nabla\phi$

$$= \frac{\partial\phi}{\partial x}\hat{i} + \frac{\partial\phi}{\partial y}\hat{j} + \frac{\partial\phi}{\partial z}\hat{k}$$

$\frac{\partial\phi}{\partial x} = 3x^2-3yz$, $\frac{\partial\phi}{\partial y} = 3y^2-3xz$, $\frac{\partial\phi}{\partial z} = 3z^2-3xy$

$\therefore \bar{F} = (3x^2-3yz)\hat{i} + (3y^2-3xz)\hat{j} + (3z^2-3xy)\hat{k}$

$\bar{F} = f_1\hat{i} + F_2\hat{j} + F_3\hat{k}$

$\therefore f_1 = 3x^2-3yz$, $F_2 = 3y^2-3xz$, $F_3 = 3z^2-3xy$

$\frac{\partial f_1}{\partial x} = 6x$, $\frac{\partial F_2}{\partial y} = 6y$, $\frac{\partial f_3}{\partial z} = 6z$

$$\therefore \nabla\cdot\bar{F} = \frac{\partial f_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial f_3}{\partial z}$$

$$= 6x + 6y + 6z$$

$$\nabla\cdot\bar{F} = 6(x+y+z)$$

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Ex 2 Find $\nabla \cdot \bar{F}$, where $\bar{F} = \left(\frac{x}{r}\right)\hat{i} + \left(\frac{y}{r}\right)\hat{j} + \left(\frac{z}{r}\right)\hat{k}$

$\Rightarrow$ we know that

$$\nabla \cdot \bar{F} = \frac{\partial f_1}{\partial x} + \frac{\partial f_2}{\partial y} + \frac{\partial f_3}{\partial z} \quad \text{——— (1)}$$

Given, $\bar{F} = f_1\hat{i} + f_2\hat{j} + f_3\hat{k} = \left(\frac{x}{r}\right)\hat{i} + \left(\frac{y}{r}\right)\hat{j} + \left(\frac{z}{r}\right)\hat{k}$

$$\therefore f_1 = \frac{x}{r} = \frac{x}{(x^2+y^2+z^2)^{1/2}} \qquad \left(\because \bar{r} = x\hat{i} + y\hat{j} + z\hat{k},\ r = \sqrt{x^2+y^2+z^2}\right)$$

$$f_2 = \frac{y}{r} = \frac{y}{(x^2+y^2+z^2)^{1/2}}$$

$$f_3 = \frac{z}{r} = \frac{z}{(x^2+y^2+z^2)^{1/2}}$$

Now,

$$\frac{\partial f_1}{\partial x} = \frac{(x^2+y^2+z^2)^{1/2}(1) - x(1/2)(x^2+y^2+z^2)^{-1/2}(2x)}{(x^2+y^2+z^2)}$$

$$= \frac{(x^2+y^2+z^2)^{1/2} - x^2(x^2+y^2+z^2)^{-1/2}}{(x^2+y^2+z^2)}$$

$$= \frac{x^2+y^2+z^2-x^2}{(x^2+y^2+z^2)(x^2+y^2+z^2)^{1/2}}$$

$$= \frac{y^2+z^2}{(x^2+y^2+z^2)^{3/2}} = \frac{(y^2+z^2)}{(r^2)^{3/2}}$$

$$\frac{\partial f_1}{\partial x} = \frac{y^2+z^2}{r^3}$$

Simillarly,

$$\frac{\partial f_2}{\partial y} = \frac{x^2+z^2}{r^3} \quad \& \quad \frac{\partial f_3}{\partial z} = \frac{x^2+y^2}{r^3}$$

$\therefore$ eq^n (1) becomes.

$$\nabla \cdot \bar{F} = \frac{y^2+z^2}{r^3} + \frac{x^2+z^2}{r^3} + \frac{x^2+y^2}{r^3}$$

$$= \frac{2(x^2+y^2+z^2)}{r^3} = \frac{2 \cdot r^2}{r^3}$$

$$\nabla \cdot \bar{F} = \underline{2}$$

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Ex. Find the value of the constant λ such that the vector field defined by $\bar{F} = (2x^2y^2 + z^2)\hat{i} + (3xy^3 - x^2z)\hat{j} + (\lambda xy^2z + xy)\hat{k}$ is solenoidal

$\Longrightarrow$

Given that, $\bar{F}$ is a solenoidal

$$\therefore \nabla \cdot \bar{F} = 0$$

$$\frac{\partial f_1}{\partial x} + \frac{\partial f_2}{\partial y} + \frac{\partial f_3}{\partial z} = 0 \quad \text{——} \quad (1)$$

Also,

$$\bar{F} = (2x^2y^2 + z^2)\hat{i} + (3xy^3 - x^2z)\hat{j} + (\lambda xy^2z + xy)\hat{k}$$

comparing with $\bar{F} = f_1\hat{i} + f_2\hat{j} + f_3\hat{k}$

$$\therefore f_1 = 2x^2y^2 + z^2$$
$$f_2 = 3xy^3 - x^2z$$
$$f_3 = \lambda xy^2z + xy$$

$$\therefore \frac{\partial f_1}{\partial x} = 4xy^2$$

$$\frac{\partial f_2}{\partial y} = 6xy^2$$

$$\frac{\partial f_3}{\partial z} = \lambda xy^2$$

$\therefore$ Eq^n (1) becomes

$$\frac{\partial f_1}{\partial x} + \frac{\partial f_2}{\partial y} + \frac{\partial f_3}{\partial z} = 0$$

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$$\therefore 4xy^2 + 9xy^2 + \lambda xy^2 = 0$$
$$xy^2(4+9+\lambda) = 0$$
$$\Rightarrow 4+9+\lambda = 0$$
$$\Rightarrow 13+\lambda = 0$$
$$\Rightarrow \boxed{\lambda = -13}$$

Ex.5 Prove that: $\operatorname{div}(r^n \bar{r}) = (n+3)r^n$. Hence prove that $\frac{\bar{r}}{r^3}$ is solenoidal.

$$\Rightarrow \operatorname{div}(r^n \bar{r}) = \nabla \cdot (r^n \bar{r})$$

Let $\bar{r} = x\hat{i} + y\hat{j} + z\hat{k}$

$$\therefore r = |\bar{r}| = \sqrt{x^2+y^2+z^2} = (x^2+y^2+z^2)^{1/2}$$
$$\therefore r^n = (x^2+y^2+z^2)^{n/2}$$

Now,

$$\operatorname{div}(r^n \bar{r}) = \nabla \cdot (r^n \bar{r})$$
$$= \nabla \cdot \{(x^2+y^2+z^2)^{n/2}(x\hat{i}+y\hat{j}+z\hat{k})\}$$
$$= \nabla \cdot \{x(x^2+y^2+z^2)^{n/2}\hat{i} + y(x^2+y^2+z^2)^{n/2}\hat{j} + z(x^2+y^2+z^2)^{n/2}\hat{k}$$
$$= \nabla \cdot \bar{F}(F_1\hat{i} + F_2\hat{j} + F_3\hat{k})$$
$$= \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z} \quad \text{——— (1)}$$

$$F_1 = x(x^2+y^2+z^2)^{n/2}, \quad F_2 = y(x^2+y^2+z^2)^{n/2}, \quad F_3 = z(x^2+y^2+z^2)^{n/2}$$

$$\frac{\partial F_1}{\partial x} = x \cdot \frac{n}{2}(x^2+y^2+z^2)^{\frac{n}{2}-1} \cdot (2x) + (x^2+y^2+z^2)^{n/2}$$

$$\frac{\partial F_1}{\partial x} = nx^2(x^2+y^2+z^2)^{\frac{n-2}{2}} + (x^2+y^2+z^2)^{n/2}$$

Similarly,

$$\frac{\partial F_2}{\partial y} = ny^2(x^2+y^2+z^2)^{\frac{n-2}{2}} + (x^2+y^2+z^2)^{n/2}$$

$$\frac{\partial F_3}{\partial z} = nz^2(x^2+y^2+z^2)^{\frac{n-2}{2}} + (x^2+y^2+z^2)^{n/2}$$

From (1)

$$\operatorname{div}(r^n \bar{r}) = nx^2(x^2+y^2+z^2)^{\frac{n-2}{2}} + ny^2(x^2+y^2+z^2)^{\frac{n-2}{2}} + nz^2(x^2+y^2+z^2)^{\frac{n-2}{2}} + 3(x^2+y^2+z^2)^{n/2}$$

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$$= n(x^2+y^2+z^2)^{\frac{n-2}{2}}(x^2+y^2+z^2) + 3(x^2+y^2+z^2)^{n/2}$$

$$= n(x^2+y^2+z^2)^{n/2} + 3(x^2+y^2+z^2)^{n/2}$$

$$= (x^2+y^2+z^2)^{n/2}(n+3)$$

$$= (r^2)^{n/2}(n+3)$$

$$= (n+3)r^n$$

$$\therefore \underline{\operatorname{div}(r^n\bar{r}) = \nabla\cdot(r^n\bar{r}) = (n+3)r^n}$$

put $n = -3$

$$\nabla\cdot(r^{-3}\bar{r}) = (-3+3)r^n$$

$$\nabla\cdot\left(\frac{\bar{r}}{r^3}\right) = 0\cdot r^n$$

$$\nabla\cdot\left(\frac{\bar{r}}{r^3}\right) = 0$$

$\therefore$ By property $\frac{\bar{r}}{r^3}$ is a solenoidal.

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* Curl of a vector point function :-

Let $\bar{F} = f_1\hat{i} + f_2\hat{j} + f_3\hat{k}$ be any vector point function in a certain field, then

$$\nabla \times \bar{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ f_1 & f_2 & f_3 \end{vmatrix}$$

is called the curl of $\bar{F}$.

Note :-

$\nabla \times \bar{F} = 0$, then F is said to be irrotational or conservative.

In this case there exists a scalar point function ϕ such that

$$\bar{F} = \nabla \phi$$

where ϕ is called a scalar potential of $\bar{F}$

Also,

$$\phi = \underset{\substack{(y,z \\ \text{const})}}{\int} f_1 dx + \underset{\substack{(x\text{-const} \\ x\text{-free})}}{\int} f_2 dy + \underset{(x,y\text{-free})}{\int} f_3 dz$$

work done $= [Q]_P^Q$.

Ex. 1 If $\bar{r} = x\hat{i} + y\hat{j} + z\hat{k}$, then show that

1) $\nabla \cdot \bar{r} = 3$

2) $\nabla \times \bar{r} = 0$

$\Rightarrow$

1) we know that

$$\nabla \cdot \bar{r} = \frac{\partial f_1}{\partial x} + \frac{\partial f_2}{\partial y} + \frac{\partial f_3}{\partial z} \quad \text{——— (1)}$$

Given, $\bar{F} = f_1\hat{i} + f_2\hat{j} + f_3\hat{k} = x\hat{i} + y\hat{j} + z\hat{k}$

$\therefore \bar{r} = \bar{F}$, $f_1 = x$, $f_2 = y$, $f_3 = z$

$$\therefore \frac{\partial f_1}{\partial x} = 1, \quad \frac{\partial f_2}{\partial y} = 1, \quad \frac{\partial f_3}{\partial z} = 1$$

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$\therefore$ Eqn ① becomes

$\nabla \cdot \bar{r} = 1+1+1$

$\nabla \cdot \bar{r} = 3$

② let $\bar{r} = \bar{F} = f_1\hat{i} + f_2\hat{j} + f_3\hat{k} = x\hat{i} + y\hat{j} + z\hat{k}$

$$\nabla \times \bar{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ f_1 & f_2 & f_3 \end{vmatrix} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x & y & z \end{vmatrix}$$

$$= \hat{i}\left(\frac{\partial z}{\partial y} - \frac{\partial y}{\partial z}\right) - \hat{j}\left(\frac{\partial z}{\partial x} - \frac{\partial x}{\partial z}\right) + \hat{k}\left(\frac{\partial y}{\partial x} - \frac{\partial x}{\partial y}\right)$$

$$= \hat{i}(0) - \hat{j}(0) + \hat{k}(0)$$

$$= 0$$

$\therefore \nabla \times \bar{F} = 0$

Ex. 2 Find curl $\bar{F}$, where $\bar{F} = \nabla(x^3+y^3+z^3-3xyz)$

$\Rightarrow$ Given,

$\bar{F} = \nabla(x^3+y^3+z^3-3xyz)$

$\bar{F} = \nabla\phi$

where $\phi = x^3+y^3+z^3-3xyz$

$$\therefore \bar{F} = \frac{\partial\phi}{\partial x}\hat{i} + \frac{\partial\phi}{\partial y}\hat{j} + \frac{\partial\phi}{\partial z}\hat{k} \quad \text{——— ①}$$

$\therefore \frac{\partial\phi}{\partial x} = 3x^2 - 3yz, \quad \frac{\partial\phi}{\partial y} = 3y^2 - 3xz, \quad \frac{\partial\phi}{\partial z} = 3z^2 - 3xy$

$\therefore \bar{F} = (3x^2 - 3yz)\hat{i} + (3y^2 - 3xz)\hat{j} + (3z^2 - 3xy)$

$= f_1\hat{i} + f_2\hat{j} + f_3\hat{k}$

$\therefore f_1 = 3x^2 - 3yz, \quad f_2 = 3y^2 - 3xz, \quad f_3 = 3z^2 - 3xy$

Now

$$\nabla \times \bar{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ f_1 & f_2 & f_3 \end{vmatrix}$$

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$$= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 3x^2-3yz & 3y^2-3xz & 3z^2-3xy \end{vmatrix}$$

$$= \hat{i}\left[\frac{\partial}{\partial y}(3z^2-3xy) - \frac{\partial}{\partial z}(3y^2-3xz)\right]$$

$$-\hat{j}\left[\frac{\partial}{\partial x}(3z^2-3xy) - \frac{\partial}{\partial z}(3x^2-3yz)\right]$$

$$+\hat{k}\left[\frac{\partial}{\partial x}(3y^2-3xz) - \frac{\partial}{\partial y}(3x^2-3yz)\right]$$

$$= \hat{i}[-3x+3x] - \hat{j}[-3y+3y] + \hat{k}[-3z+3z]$$

$$= \hat{i}(0) - \hat{j}(0) + \hat{k}(0)$$

$$= 0$$

$\therefore$ $\underline{\text{curl}\,\bar{F} = \nabla \times \bar{F} = 0}$

Ex. 3 If $\bar{F} = (x+y+z)\hat{i} + \hat{j} - (x+y)\hat{k}$, prove that $\bar{F}\cdot \text{curl}\,\bar{F} = 0$

$\Rightarrow$ Given,

$$\bar{F} = f_1\hat{i} + f_2\hat{j} + f_3\hat{k} = (x+y+z)\hat{i} + \hat{j} - (x+y)\hat{k}$$

$$f_1 = x+y+z, \quad f_2 = 1, \quad f_3 = -(x+y)$$

$$\nabla \times \bar{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x+y+z & 1 & -(x+y) \end{vmatrix}$$

$$= \hat{i}\left[\frac{\partial}{\partial y}-(x+y) - \frac{\partial}{\partial z}(1)\right] - \hat{j}\left[\frac{\partial}{\partial x}-(x+y) - \frac{\partial}{\partial z}(x+y+z)\right]$$

$$+\hat{k}\left[\frac{\partial}{\partial x}(1) - \frac{\partial}{\partial y}(x+y+z)\right]$$

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$$= \hat{i}[-1-0] - \hat{j}[-1-0] + \hat{k}[0-1]$$
$$= -\hat{i} + \hat{j} - \hat{k}$$
$$\therefore \nabla \times \bar{F} = -\hat{i} + \hat{j} - \hat{k}$$

Now

$$\bar{F} \cdot \text{curl}\,\bar{F} = \bar{F} \cdot (\nabla \times \bar{F})$$
$$= [(x+y+1)\hat{i} + \hat{j} - (x+y)\hat{k}] \cdot (-\hat{i} + \hat{j} - \hat{k})$$
$$= -1(x+y+1) + 1(1) + (x+y)(1)$$
$$= -x - y - 1 + 1 + x + y$$
$$= 0$$
$$\therefore \bar{F} \cdot \text{curl}\,\bar{F} = 0$$

Ex. 4 Prove that $\bar{F} = (2xy + z^3)\hat{i} + x^2\hat{j} + 3xz^2\hat{k}$ is conservative force field. Find its scalar potential and the work done in moving an object from $(1,-2,1)$ to $(3,1,4)$

$\Rightarrow$ Given,

$$\bar{F} = (2xy + z^3)\hat{i} + x^2\hat{j} + 3xz^2\hat{k} = f_1\hat{i} + f_2\hat{j} + f_3\hat{k}$$

$$\nabla \times \bar{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 2xy + z^3 & x^2 & 3xz^2 \end{vmatrix}$$

$$= \hat{i}\left[\frac{\partial}{\partial y}(3xz^2) - \frac{\partial}{\partial z}(x^2)\right] - \hat{j}\left[\frac{\partial}{\partial x}(3xz^2) - \frac{\partial}{\partial z}(2xy + z^3)\right]$$
$$+ \hat{k}\left[\frac{\partial}{\partial x}(x^2) - \frac{\partial}{\partial y}(2xy + z^3)\right]$$

$$= i(0-0) - \hat{j}(3z^2 - 3z^2) + \hat{k}(2x - 2x)$$
$$= i(0) - \hat{j}(0) + \hat{k}(0)$$
$$= 0$$
$$\therefore \nabla \times \bar{F} = 0$$

$\therefore \bar{F}$ is an irrotational and there exists scalar ϕ such that

$$\bar{F} = \nabla \phi$$

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The scalar potential is given by

$$\phi = \int f_1 dx + \int f_2 dy + \int f_3 dz$$

$$= \int (2xy + z^3) dx + \int_{\text{free } x} x^2 dy + \int_{\text{free}(x,y)} 3xz^2 dz$$

$$= 2y \cdot \frac{x^2}{2} + z^3 \cdot x + 0 + 0$$

$$\phi = x^2 y + xz^3$$

$\therefore$ Total work done $= [\phi]_P^Q$

$$= \left[x^2 y + xz^3\right]_{(1,-2,1)}^{(3,1,4)}$$

$$= [(9)(1) + (3)(4)^3] - [(1)^2(-2) + (1)(1)^3]$$

$$= (9 + 192) - (-2 + 1)$$

$$= 201 - (-1)$$

$$= 201 + 1$$

$$= 202 \text{ units.}$$

Ex. 5 Show that $\bar{F} = (x^2 - yz)\hat{i} + (y^2 - xz)\hat{j} + (z^2 - xy)\hat{k}$ is irrotational. Also find the ϕ such that $\bar{F} = \nabla\phi$

$\Rightarrow$ Given,

$$\bar{F} = (x^2 - yz)\hat{i} + (y^2 - xz)\hat{j} + (z^2 - xy)\hat{k} = f_1\hat{i} + f_2\hat{j} + f_3\hat{k}$$

Now

$$\nabla \times \bar{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x^2 - yz & y^2 - xz & z^2 - xy \end{vmatrix}$$

$$= \hat{i}[-x + z] - \hat{j}[-y + y] + \hat{k}[-z + z]$$

$$= \hat{i}(0) - \hat{j}(0) + \hat{k}(0)$$

$$= 0$$

$\therefore \nabla \times \bar{F} = 0$

$\therefore \bar{F}$ is irrotational

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To find scalar potential ϕ,

$$\phi = \int F_1 dx + \int F_2 dy + \int F_3 dz$$

$$= \int (x^2 - yz) dx + \int y^2 dy + \int z^2 dz$$

$$= \frac{x^3}{3} - yzx + \frac{y^3}{3} + \frac{z^3}{3} + c$$

$$\therefore \phi = \frac{x^3}{3} + \frac{y^3}{3} + \frac{z^3}{3} - xyz + c.$$

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* Vector Identities: -

1) $\text{Div}(\phi\bar{u}) = \nabla\cdot(\phi\bar{u}) = \phi(\nabla\cdot\bar{u}) + \nabla\phi\cdot\bar{u}$

2) $\text{Curl}(\phi\bar{u}) = \nabla\times(\phi\bar{u}) = \phi(\nabla\times\bar{u}) + \nabla\phi\times\bar{u}$

3) $\text{Div}(\bar{u}\times\bar{v}) = \nabla\cdot(\bar{u}\times\bar{v}) = \bar{v}\cdot(\nabla\times\bar{u}) - \bar{u}\cdot(\nabla\times\bar{v})$

4) $\text{Curl}(\bar{u}\times\bar{v}) = \nabla\times(\bar{u}\times\bar{v}) = \bar{u}(\nabla\cdot\bar{v}) - \bar{v}(\nabla\cdot\bar{u}) + (\bar{v}\cdot\nabla)\bar{u} - (\bar{u}\cdot\nabla)\bar{v}$

5) $\nabla\cdot\bar{r} = 3$

6) $\nabla r^n = n.r^{n-2}\bar{r}$

7) $\nabla F(r) = F'(r)\nabla r$ and $\nabla r = \dfrac{\bar{r}}{r}$

8) $\nabla\times\bar{r} = 0$

9) $\bar{r}\cdot\bar{r} = r^2$

10) $\bar{r}\times\bar{r} = 0$

Ex. If r and $\bar{r}$ have their usual meanings and $\bar{a}, \bar{b}$ are constant vectors then prove that

1) $\nabla\cdot(r^n\bar{r}) = (n+3)r^n$

2) $\nabla\times(r^n\bar{r}) = 0$

3) $\nabla^2 r^n = n(n+1)r^{n-2}$

$\Rightarrow$

1) we know that

$\nabla\cdot(\phi\bar{u}) = \phi(\nabla\cdot\bar{u}) + \nabla\phi\cdot\bar{u}$

put $\phi = r^n$ & $\bar{r} = \bar{u}$

$$\begin{aligned}\nabla\cdot(r^n\bar{r}) &= r^n(\nabla\cdot\bar{r}) + \nabla r^n.\bar{r}\\ &= r^n(3) + nr^{n-2}\bar{r}\cdot\bar{r}\\ &= 3r^n + nr^{n-2}(\bar{r}\cdot\bar{r})\\ &= 3r^n + nr^{n-2}.r^2\\ &= 3r^n + nr^n\end{aligned}$$

$\nabla\cdot(r^n\bar{r}) = (3+n)r^n$

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(2) We know that,

$$\nabla \times (\phi \bar{u}) = \phi(\nabla \times \bar{u}) + \nabla\phi \times \bar{u}$$

put $\phi = r^n$ & $\bar{u} = \bar{r}$

$$\nabla \times (r^n \bar{r}) = r^n(\nabla \times \bar{r}) + \nabla r^n \times \bar{r}$$

$$= r^n(0) + n r^{n-2} \bar{r} \times \bar{r} \qquad (\nabla \times \bar{r} = 0)$$

$$= 0 + n r^{n-2}(\bar{r} \times \bar{r})$$

$$= n r^{n-2}(0)$$

$$= 0$$

$$\therefore \nabla \times (r^n \bar{r}) = 0$$

(3) We know that

$$\nabla^2 r^n = (\nabla \cdot \nabla) r^n$$

$$= \nabla \cdot \nabla r^n$$

$$= \nabla \cdot n r^{n-2} \bar{r}$$

$$= n(\nabla \cdot r^{n-2} \bar{r})$$

$$= n[r^{n-2}(\nabla \cdot \bar{r}) + \nabla r^{n-2} \cdot \bar{r}]$$

$$= n[r^{n-2}(3) + (n-2) r^{n-4} \bar{r} \cdot \bar{r}]$$

$$= n[3r^{n-2} + (n-2) r^{n-4} \cdot r^2]$$

$$= n[3r^{n-2} + (n-2) r^{n-2}]$$

$$= n[3 + n - 2] r^{n-2}$$

$$\nabla^2 r^n = n(n+1) r^{n-2}$$

Ex. 2 If $\bar{r}$ is a possition vector and $r = |\bar{r}|$, then prove that

1) $\nabla^2 f(r) = \frac{d^2 f}{dr^2} + \frac{2}{r}\frac{df}{dr} = f''(r) + \frac{2}{r} f'(r)$

2) $\nabla^4 e^r = e^r + \frac{4}{r} e^r$

(1) $\Rightarrow$ $\nabla^2 f(r) = \nabla \cdot \nabla f(r)$

$$= \nabla \cdot f'(r) \nabla r$$

$$= \nabla \cdot f'(r) \frac{\bar{r}}{r}$$

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$$= \nabla \cdot \left(\frac{F'(r)}{r} \right) \bar{r}$$

$$= \left(\nabla \cdot \frac{F'(r)}{r} \right) \cdot \bar{r} + \frac{F'(r)}{r} (\nabla \cdot \bar{r})$$

$$= \left\{ \frac{r \nabla F'(r) - F'(r) \nabla r}{r^2} \right\} \cdot \bar{r} + \frac{F'(r)}{r} (3)$$

$$= \frac{1}{r^2} \left\{ r F''(r) \nabla r - F'(r) \frac{\bar{r}}{r} \right\} \cdot \bar{r} + \frac{3F'(r)}{r}$$

$$= \frac{1}{r^2} \left\{ r F''(r) \cdot \frac{\bar{r}}{r} - F'(r) \frac{\bar{r}}{r} \right\} \cdot \bar{r} + \frac{3F'(r)}{r}$$

$$= \frac{1}{r^2} \left\{ \frac{r F''(r) - F'(r)}{r} \right\} (\bar{r} \cdot \bar{r}) + \frac{3F'(r)}{r}$$

$$= \frac{1}{r^2} \left\{ \frac{r F''(r) - F'(r)}{r} \right\} r^2 + \frac{3F'(r)}{r}$$

$$= \frac{r F''(r) - F'(r)}{r} + \frac{3F'(r)}{r}$$

$$= \frac{r F''(r) + 2F'(r)}{r}$$

$$= F''(r) + \frac{2}{r} F'(r)$$

$$\nabla^2 F(r) = \frac{d^2 F}{dr^2} + \frac{2}{r} \frac{dF}{dr}$$

(2) $\Rightarrow$ We know that

$$\nabla^4 e^r = \nabla^2 \nabla^2 e^r$$
$$= \nabla^2 (\nabla^2 e^r)$$

But $\nabla^2 r = \frac{dF^2}{dr^2} + \frac{2}{r} \frac{dF}{dr}$

$$= \nabla^2 \left\{ \frac{d^2 e^r}{dr^2} + \frac{2}{r} \frac{d e^r}{dr} \right\}$$

$$= \nabla^2 \left\{ e^r + \frac{2}{r} \cdot e^r \right\}$$

$$= \nabla^2 F(r), \quad \text{where } F(r) = e^r + \frac{2}{r} e^r$$

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$$= \frac{d^2 f(r)}{dr^2} + \frac{2}{r}\frac{df(r)}{dr}$$

$$= \frac{d^2}{dr^2}\left\{e^r + \frac{2}{r}e^r\right\} + \frac{2}{r}\frac{d}{dr}\left\{e^r + \frac{2}{r}e^r\right\}$$

$$= \frac{d}{dr}\left\{e^r + 2\left(\frac{re^r - e^r(1)}{r^2}\right)\right\} + \frac{2}{r}\left\{e^r + 2\left(\frac{re^r - e^r(1)}{r^2}\right)\right\}$$

$$= \frac{d}{dr}\left\{e^r + 2\left(\frac{re^r - e^r}{r^2}\right)\right\} + \frac{2}{r}\left(e^r + 2\left(\frac{re^r - e^r}{r^2}\right)\right)$$

$$= \frac{d}{dr}\left(e^r + \frac{2e^r}{r} - \frac{2e^r}{r^2}\right) + \frac{2e^r}{r} + \frac{4e^r}{r^2} - \frac{4e^r}{r^3}$$

$$= e^r + 2\left(\frac{re^r - e^r}{r^2}\right) - 2\left(\frac{r^2e^r - 2re^r}{r^4}\right) + \frac{2e^r}{r} + \frac{4e^r}{r^2} - \frac{4e^r}{r^3}$$

$$= e^r + \frac{2re^r}{r^2} - \cancel{\frac{2e^r}{r^2}} - \cancel{\frac{2e^r}{r^2}} + \cancel{\frac{4e^r}{r^3}} + \frac{2e^r}{r} + \cancel{\frac{4e^r}{r^2}} - \cancel{\frac{4e^r}{r^3}}$$

$$= e^r + \frac{2e^r}{r} + \frac{2e^r}{r}$$

$$= e^r + \frac{4}{r}e^r$$

$$\therefore \nabla^2\nabla^2 e^r = \nabla^4 e^r = e^r + \frac{4}{r}e^r.$$

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