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Generating random density matrices

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We study various methods to generate ensembles of random density matrices of a fixed size N , obtained by partial trace of pure states on composite systems. Structured ensembles of random pure states, invariant with respect to local unitary transformations are introduced. To analyze statistical properties of quantum entanglement in bi-partite systems we analyze the distribution of Schmidt coefficients of random pure states. Such a distribution is derived in the case of a superposition of k random maximally entangled states. For another ensemble, obtained by performing selective measurements in a maximally entangled basis on a multi-partite system, we show that this distribution is given by the Fuss-Catalan law and find the average entanglement entropy. A more general class of structured ensembles proposed, containing also the case of Bures, forms an extension of the standard ensemble of structureless random pure states, described asymptotically, as $N \rightarrow \infty$, by the Marchenko-Pastur distribution. © 2011 American Institute of Physics. [doi:10.1063/1.3595693]

I. INTRODUCTION

Random states are often used in various problems in quantum mechanics and the theory of quantum information. On one hand, to describe properties of a quantum state affected by noise present in the system, one may assume that a given state is subjected to random interaction. On the other hand, random quantum states emerge in a natural way due to time evolution of arbitrary initial states of quantum analogues of classically chaotic systems.¹ Furthermore, not knowing much about a given physical state one can ask about their generic properties, characteristic of a “typical” state. For instance, a key conjecture in the theory of quantum information concerning the additivity of minimal output entropy was recently shown to be false by investigating properties of random states obtained by random operations applied to the maximally entangled state of a composed system.²

A standard ensemble of random pure states of size N is induced by the Haar measure over the unitary group $U(N)$. The same construction works for quantum composite systems. For instance, random pure states of an $N \times K$ quantum system corresponds to the natural Fubini-Study measure, invariant with respect to the unitary group $U(NK)$. Thus, such ensembles of random pure states

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are structureless, as the probability measure is determined by the total dimension of the Hilbert space and does not depend on the tensor product structure.^{3,4}

In this work we are going to analyze *structured ensembles* of random states on composite systems, in which such a tensor product structure plays a crucial role. For instance, in the case of an $N \times K$ system, it is natural to consider ensembles of random states invariant with respect to local unitary transformations, described by the product unitary group, $U(N) \times U(K)$. Well known constructions of random product states and random maximally entangled states can thus serve as simplest examples of the structured ensembles of random states. Other examples of structured ensembles of random pure states, which correspond to certain graphs were recently studied in Ref. 5.

The main aim of this paper is to introduce physically motivated ensembles of structured random states and to characterize the quantum entanglement of such states. For this purpose we analyze the spectral density $P(x)$ of the reduced density matrix of size N , where x denotes the rescaled eigenvalue, $x = N\lambda$. In some cases we evaluate also the average entanglement entropy, defined as the Shannon entropy of the vector $\vec{\lambda}$ of the Schmidt coefficients.

We treat in detail two cases of a direct importance in the theory of quantum information. The first ensemble is obtained by taking a coherent superposition of a given number of k independent, random maximally entangled states of an $N \times N$ system. We derive explicit formulae for the asymptotic spectral density $P_k(x)$ of the reduced state and show that in the limit of large k , the density converges to the *Marchenko-Pastur distribution* (MP).^{6,7} This feature is characteristic to the case of the structureless ensemble of random pure states, which induces probability measures in the set of mixed quantum states.^{4,8,9}

To introduce the second example of a structured ensemble we need to consider a four-partite system. We start with an arbitrary product state and by allowing for a generic bi-partite interaction we create two random states on subsystems AB and CD . The key step is now to perform an orthogonal selective measurement in the maximally entangled basis on subsystems B and C . Even though the resulting pure state on the remaining subsystems A and D does depend on the outcome of the measurement, its statistical properties do not, and we show that the corresponding level density is given by the Fuss-Catalan (FC) distribution.

Note that the above two constructions could be experimentally accessible, at least in the two-qubit case for $k = 2$. The second construction can be easily generalized for a system consisting of $2s$ subsystems. An initially product state is then transformed by a sequence of bi-partite interactions into a product of s bi-partite random states. Performing an orthogonal projection into a product of $(s - 1)$ maximally entangled bases we arrive with a random bi-partite state of the structured ensemble defined in this way. The distribution of its Schmidt coefficients is shown to be asymptotically described by the *Fuss-Catalan distribution* of order s , since its moments are given by the generalized Fuss-Catalan numbers.^{10–12} This distribution can be considered as a generalization of the MP distribution which is obtained for $s = 1$. We conclude this paper proposing a generalized, two-parameter ensemble of structured random states, which contains, as particular cases, all the ensembles analyzed earlier in this work.

This paper is organized as follows. In Sec. II we recall necessary definitions and describe a general scheme of generating mixed states by taking random pure states on a composite system of a given ensemble and taking an average over selected subsystems. In Sec. III we use 2 independent random unitary operators to construct the arcsine ensemble obtained by superposition of two random maximally entangled states. More general structured ensembles are obtained by superposing k random maximally entangled states. Other ensembles of random pure states are discussed in Sec. IV. For completeness we review here some older results on ensembles which lead to the Hilbert-Schmidt (HS) and the Bures measures in the space of density matrices.

In Sec. V we present a scheme to generate random states by considering a state defined on a $2s$ -partite system and performing measurement onto the product of $(s - 1)$ maximally entangled states. The distribution of the Schmidt coefficients of the resulting random pure state is then given by the Fuss-Catalan distribution of order s . An explicit form of this distribution is presented in Sec. VI for any s . In Sec. VII we list various ensembles of random states, characterize their statistical properties, and introduce a generalized ensemble of structured states, see Eqs. (42) and (43). Details concerning the properties of the sum of k independent random unitary matrices are presented in Appendix A.

Brief discussion of real random pure states and real density matrices is provided in Appendix B, in which we state that the distribution of singular values of a product of s real random Ginibre matrices can be asymptotically described by the Fuss–Catalan distribution.

II. ENSEMBLES OF RANDOM STATES

To construct an ensemble of random states of a given size N one needs to specify a probability measure in the set of all density operators of this size. Interestingly, there is no single, distinguished probability measure in this set, so various ensembles of random density operators are used.

A possible way to define such an ensemble is to take random pure states of a given ensemble on a bi-partite system and to average over a chosen subsystem. The structureless ensemble of random pure states on $N \times N$ system distributed according to unitarily invariant measure leads then to the Hilbert–Schmidt measure.^{3,4} A similar construction to generate random states according to the Bures measure^{9,13,14} was recently proposed in Ref. 15. These algorithms to generate random quantum states are already implemented in a recent MATHEMATICA package devoted to quantum information.^{16,17}

Let us recall here the necessary notions and definitions. Consider the set of quantum states \mathcal{M}_N which contains all Hermitian, positive operators $\rho = \rho^\dagger \geq 0$ of size N normalized by the trace condition $\text{Tr} \rho = 1$. Any hermitian matrix can be diagonalized, $\rho = V \Lambda V^\dagger$. Here V is a unitary matrix consisting of eigenvectors of ρ , while Λ is the diagonal matrix containing the eigenvalues $\{\lambda_1, \dots, \lambda_N\}$.

In order to describe an ensemble of random density matrices, one needs to specify a concrete probability distribution in the set \mathcal{M}_N of quantum states. In this work we are going to analyze ensembles of random states, for which the probability measure has a product form and may be factorized as^{4,13}

$$d\mu_\rho = d\nu(\lambda_1, \lambda_2, \dots, \lambda_N) \times d\mu_V, \quad (1)$$

so the distribution of eigenvalues and eigenvectors are independent. It is natural to assume that the eigenvectors are distributed according to the unique, unitarily invariant, Haar measure $d\mu_V$ on $U(N)$. Taking this assumption as granted, the measure in the space of density matrices will be determined by the first factor $d\nu$ describing the joint distribution of eigenvalues $P(\lambda_1, \dots, \lambda_N)$.

In quantum theory mixed states arise due to an interaction of the system investigated with an external environment. One may then make certain assumptions concerning the distribution of pure states describing the extended system. The desired mixed state ρ on the principal system A of size N can be then obtained by a partial trace over the subsystem B of an arbitrary size K .

Consider an arbitrary orthonormal product basis $|i\rangle \otimes |j\rangle \in \mathcal{H}_N \otimes \mathcal{H}_K$, with $i = 1, \dots, N$ and $j = 1, \dots, K$. Any pure state $|\psi\rangle$ of the bi-partite system (not necessarily normalized) can be expanded in this base,

$$|\psi\rangle = \sum_{i=1}^N \sum_{j=1}^K X_{ij} |i\rangle \otimes |j\rangle, \quad (2)$$

where X is a given complex rectangular matrix of size $N \times K$.

Let us then consider an arbitrary complex matrix X . It leads to a (weakly) positive matrix XX^\dagger . Thus, normalizing it one obtains a legitimate quantum state, which corresponds to the partial trace of the initial pure state $|\psi\rangle$ over the subsystem B ,

$$\rho = \frac{\text{Tr}_B |\psi\rangle \langle \psi|}{\langle \psi | \psi \rangle} = \frac{XX^\dagger}{\text{Tr} XX^\dagger}. \quad (3)$$

For instance, taking X from the *Ginibre ensemble*^{18,19} of complex square matrices we get the Hilbert–Schmidt ensemble of quantum states, while a more general family of induced measures corresponds²⁰ to the ensemble of rectangular Ginibre matrices.

The spectrum of a density matrix ρ is thus equivalent to the set of Schmidt coefficient of the initially pure state $|\psi\rangle$, which is equal to squared singular values of a matrix X , normalized in such a way that $\text{Tr} \rho = 1$. The degree of mixing of the reduced matrix ρ can be characterized by

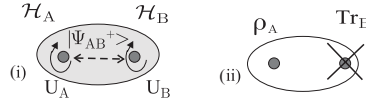


FIG. 1. Generating mixed states according to $\pi^{(0)}$: step (i) a local operation $U_A \otimes U_B$ on $|\Psi_{AB}^+\rangle$ creates a random entangled state $|\phi_{\text{ent}}\rangle$, while the partial trace over an auxiliary subsystem B leads in step (ii) to the maximally mixed state on the system A .

its von Neumann entropy, $S(\rho) = -\text{Tr} \rho \ln \rho$, equal to the Shannon entropy of the Schmidt vector, $E(|\psi\rangle) = -\sum_i \lambda_i \ln \lambda_i$. This quantity is also called *entropy of entanglement* of the pure state $|\psi\rangle$, as it is equal to zero iff the state has a tensor product structure and is separable. We are also going to use the Chebyshev entropy, which depends only on the largest eigenvalue $S_\infty(\rho) = -\ln \lambda_{\text{max}}$, and determines the geometric measure of entanglement of the pure state $|\psi\rangle$.

Different assumptions concerning the distributions of pure states of the bi-partite system lead to different ensembles of mixed states on the system A . A natural assumption that $|\psi\rangle$ belongs to the standard, structureless ensemble of random pure state distributed with respect to the unitarily invariant measure leads to the induced measures⁴ in the space of mixed states.

In Sec. III we are going to present simple examples of structured ensembles of random states, in which the tensor structure plays a key role. For instance, we discuss first ensembles of random states of a bi-partite systems, which are invariant with respect to local unitary operations.

III. BI-PARTITE SYSTEMS

A. Random separable and maximally entangled pure states

For completeness we shall start the discussion with the somewhat trivial case of generating random separable states. Consider an arbitrary product state on a bi-partite system, $|0, 0\rangle = |0\rangle_A \otimes |0\rangle_B$. A local unitary operation, $U = U_A \otimes U_B$, cannot produce quantum entanglement, so the state defined by two random unitary matrices, $|\psi_{AB}\rangle = U|0, 0\rangle = U_A|0\rangle_A \otimes U_B|0\rangle_B$ is also separable. Hence, a random separable state is just a product of two random states, $|\psi_{AB}\rangle = |\psi_A\rangle \otimes |\psi_B\rangle$.

Consider now a generalized Bell state defined on an $N \times N$ system A, B ,

$$|\Psi^+\rangle = \frac{1}{\sqrt{N}} \sum_{i=1}^N |i\rangle_A \otimes |i\rangle_B. \quad (4)$$

As the entanglement entropy of this state is maximal, $E(|\Psi^+\rangle) = \ln N$, this state is called maximally entangled.

Any local unitary operation, $U_A \otimes U_B$, preserves quantum entanglement, so the locally transformed state $|\phi_{\text{ent}}\rangle := (U_A \otimes U_B)|\Psi^+\rangle$ remains maximally entangled. Hence, taking random unitary matrices $U_A, U_B \in U(N)$ according to the Haar measure we obtain an ensemble of random entangled states, such that their partial trace is equal to the maximally mixed state,

$$\rho = \text{Tr}_B |\phi_{\text{ent}}\rangle \langle \phi_{\text{ent}}| = \frac{1}{N} \mathbb{1}_N =: \rho_*. \quad (5)$$

This Dirac distribution will be denoted by $\pi^{(0)}(\rho) = \delta(\rho - \rho_*)$ and a scheme of generating it by averaging over an auxiliary subsystem is shown in Fig. 1.

B. Arcsine ensemble

Consider now a more general case of a coherent superposition of two maximally entangled states. To be precise we fix a given maximally entangled state $|\psi_1\rangle$, and use such a local basis that it is represented by (4). Taking a local random unitary matrix $U_A \in U(N)$ we generate another maximally entangled state $|\psi_2\rangle = U_A \otimes \mathbb{1} |\psi_1\rangle$. As shown in Fig. 2 we construct their symmetric

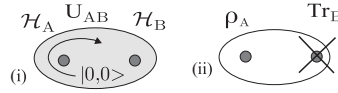


FIG. 2. To generate states from the arcsine ensemble described by (8) one has to (i) construct a superposition of a maximally entangled state $|\Psi_{AB}^+\rangle$ with another maximally entangled state $(U_A \otimes \mathbb{1})|\Psi_{AB}^+\rangle$, and (ii) perform partial trace over an auxiliary subsystem B .

superposition,

$$|\phi\rangle = (|\psi_1\rangle + |\psi_2\rangle) = [\mathbb{1}_{N^2} + (U \otimes \mathbb{1}_N)]|\Psi^+\rangle. \quad (6)$$

Note that this ensemble is invariant with respect to local unitary operations, $U(N) \times U(N)$. Let us specify a subsystem B and average over it to obtain the reduced state

$$\rho = \frac{\text{Tr}_B |\phi\rangle\langle\phi|}{\langle\phi|\phi\rangle} = \frac{2\mathbb{1} + U + U^\dagger}{2N + \text{Tr}(U + U^\dagger)}. \quad (7)$$

If the matrix U is generated according to the Haar measure on $U(N)$ its eigenphases are distributed according to the uniform distribution,¹⁹ $P(\alpha) = 1/2\pi$ for $\alpha \in [0, 2\pi)$. Thus, the term $\text{Tr}(U + U^\dagger)$ present in the normalization constant becomes negligible for large N . Therefore, eigenvalues of a random density matrix ρ by (7) have the form $\lambda_i = (1 + \cos \alpha_i)/N$, for $i = 1, \dots, N$. Making use of the rescaled variable $x = \lambda N$ we arrive at a conclusion that the spectral level density of random density matrices (7) is asymptotically described by the *arcsine* distribution,²¹

$$P_{\text{arc}}(x) = \frac{1}{\pi \sqrt{x(2-x)}} \quad (8)$$

defined on the compact support $[0, 2]$. This is a particular case of the beta distribution (with parameters $\alpha = \beta = 1/2$) and the Dirichlet distribution. It describes the Jeffreys prior of a Bernoulli trial²² and is related to the statistical distance between classical probability distributions.⁹ The name “arcsine” is due to the fact that the corresponding cumulative distribution is proportional to $\sin^{-1} \sqrt{x/2}$. Thus, the ensemble of random density matrices constructed according to the procedure shown in Fig. 2 will be called *arcsine ensemble*. The average entropy for the arcsine distribution reads $\int_0^2 [-x \ln x P_{\text{arc}}(x)] dx = \ln 2 - 1 \approx -0.307$, so the average entropy of entanglement of a random pure state on the $N \times N$ system generated from this ensemble reads $\langle E \rangle_\psi \approx \ln N - 1 + \ln 2$.

It is clear that one may define a family of interpolating ensembles by taking a convex combination of the mixed states defined by (5) and (7). In other words, one can take a family of random matrices parametrized by a real number $a \in [0, 1]$ and write $W_a = a\mathbb{1} + (1-a)U$. Plugging this expression in place of X into (3) we construct a family of ensembles of density matrices which gives the Dirac mass for $a = 0$ and $a = 1$, while the arcsine ensemble is obtained for $a = 1/2$.

C. Superposition of k maximally entangled states

The arcsine ensemble introduced above can be obtained by superimposing $k = 2$ random maximally entangled states. It is straightforward to generalize this construction for an arbitrary number of k maximally entangled states. Each of them can be written by an action of a local unitary on the fixed maximally entangled state $|\Psi^+\rangle$. More precisely we set $|\psi_i\rangle = (U_i \otimes \mathbb{1})|\Psi^+\rangle$ for $i = 1, \dots, k$ and construct their equi-probable superposition $|\phi\rangle = \sum_{i=1}^k |\psi_i\rangle = [(\sum_{i=1}^k U_i) \otimes \mathbb{1}_N]|\Psi^+\rangle$.

As before this ensemble is invariant with respect to local transformations. The random mixed state is obtained by taking the partial trace over the subsystem B and normalizing the outcome $\rho = \text{Tr}_B |\phi\rangle\langle\phi| / \langle\phi|\phi\rangle$ as in Eq. (7). This procedure leads now to the following expression for a mixed state, generated by k independent random unitaries:

$$\rho = \frac{(U_1 + \dots + U_k)(U_1^\dagger + \dots + U_k^\dagger)}{\text{Tr}(U_1 + \dots + U_k)(U_1^\dagger + \dots + U_k^\dagger)}. \quad (9)$$

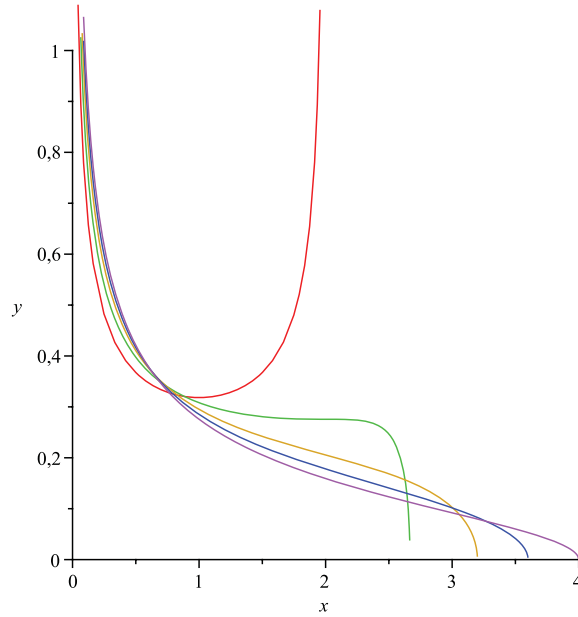


FIG. 3. (Color online) Family of spectral distributions $v_k(x)$ plotted for $k = 2$ (red curve), $k = 3$ (green curve), $k = 5$ (yellow curve), $k = 10$ (blue curve) and $k \rightarrow \infty$ (pink curve). The support of the measures is an increasing function of the parameter k . They describe the level density of mixed states generated by superposition of k random maximally entangled states and performing the partial trace. The case $k = 2$ corresponds to the arcsine ensemble, while in the case $k \rightarrow \infty$, the sum of k random unitaries has properties of a random Ginibre matrix, so the spectral distribution converges to the Marchenko-Pastur law.

As shown in Appendix A the spectral density of random states defined above converges, for large system size N to the following distribution $v_k(x)$, defined for $x \in [0, 4\frac{k-1}{k}]$:

$$v_k(x) = \frac{1}{2\pi} \frac{\sqrt{4k(k-1)x - k^2x^2}}{kx - x^2}. \quad (10)$$

The shape of these distributions is presented for some values of k in Fig. 3. It is easy to see that for a large number k the above measures converge weakly to the limit $v_\infty = \pi^{(1)} = \frac{\sqrt{4x-x^2}}{2\pi x}$ which is the Marchenko-Pastur distribution $\pi^{(1)}$. In other words, the superposition of a large number of random maximally entangled states destroys the structure of the ensemble, as the resulting state becomes typical of the structureless ensemble.

The measure (10) can be connected to the free Meixner measures of Ref. 23: v_k is the free Meixner measure of parameters $a = 0$ and $b = -1/k$. The measure $v_k(x)$ can be characterized by its second moment, which gives the asymptotic average purity of random mixed states obtained by a superposition of k maximally entangled states, $\langle \text{Tr} \rho^2 \rangle_{v_k} \approx (2 - 1/k) \frac{1}{N}$.

D. Induced measures

Consider a quantum system composed of two subsystems. Let N denote the dimension of the principal system, which interacts with a K dimensional environment. Assume first that the state $|\psi\rangle \in \mathcal{H}_N \otimes \mathcal{H}_K$ is taken from the structureless ensemble, so it is distributed uniformly according to the Fubini–Study measure on the complex projective space. In other words, the random pure state can be represented by $|\psi\rangle = U|0, 0\rangle$, where U is a global random unitary matrix distributed according to the Haar measure—see Fig. 4. The initial state is arbitrary, so for concreteness we may choose it as a given product state $|0, 0\rangle$ in $\mathcal{H}_N \otimes \mathcal{H}_K$. The mixed state ρ obtained by the partial trace

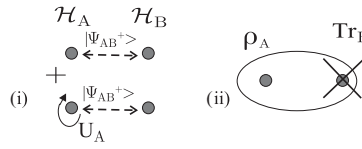


FIG. 4. Taking (i) a random pure state $U_{AB}|0, 0\rangle$ on a bi-partite system AB and performing (ii) the partial trace over subsystem B leads random state distributed according to an induced measure, with spectral density described by the Marchenko-Pastur distribution.

over the K dimensional environment

$$\rho = \text{Tr}_K |\psi\rangle\langle\psi|, \quad \text{with } |\psi\rangle \in \mathcal{H}_N \otimes \mathcal{H}_K, \quad (11)$$

is distributed according to the measure $\mu_{N,K}$ in the space of density matrices of size N , induced by the Haar measure on the unitary group $U(NK)$. This procedure yields the following probability distribution:

$$d\mu(\rho) \propto \Theta(\rho) \delta(\text{Tr}\rho - 1) \det \rho^{K-N}. \quad (12)$$

The Θ step-function and the Dirac δ reflect key properties of density matrices: positivity, $\rho \geq 0$ and normalization, $\text{Tr}\rho = 1$.

If $|\psi\rangle$ is a random pure state distributed uniformly according to the unitarily invariant Fubini-Study measure in the space of pure states, all elements of G are independent complex Gaussian variables with the same variance, so this matrix belongs to the Ginibre ensemble.¹⁸ Hence, the reduced state (11) obtained by the partial trace has the form (3) with $X = G$. The eigenvalues of a random matrix ρ generated with respect to the measure $\mu_{N,K}$ are thus equal to the squared singular eigenvalues of a normalized rectangular $N \times K$ complex random matrix G from the Ginibre ensemble.

Any density matrix is Hermitian and can be diagonalized by a unitary rotation. Integrating out the eigenvectors of a random state ρ defined by (12) one reduces $d\mu$ to the measure on the simplex of eigenvalues $\{\lambda_1, \dots, \lambda_N\}$ of the density operator,²⁴

$$P_{N,K}(\lambda_1, \dots, \lambda_N) = C_{N,K} \prod_i \lambda_i^{K-N} \prod_{i < j}^{1 \dots N} (\lambda_i - \lambda_j)^2. \quad (13)$$

In this expression all eigenvalues are assumed to be non-negative, $\lambda_i \geq 0$ and they sum to unity, $\sum_i \lambda_i = 1$. The normalization constant can be expressed⁴ by the Gamma function:

$$C_{N,K} = \frac{\Gamma(KN)}{\prod_{j=0}^{N-1} \Gamma(K-j) \Gamma(N-j+1)}. \quad (14)$$

Induced measures are stable with respect to the partial trace, what can be formulated as follows.

Proposition 1: Consider a random state ρ generated according to the induced measures $\mu_{M,K}$. Assume that the dimension is composed, $M = N \times L$, and defines the partial trace over the L dimensional subsystem, $\sigma = \text{Tr}_L(\rho)$. Then the reduced matrix σ is generated according to the measure $\mu_{N,KL}$.

Proof: The state ρ can be considered as a random pure state $|\psi\rangle \in \mathcal{H}_M \otimes \mathcal{H}_K$ averaged over K dimensional environment. Averaging the projector $|\psi\rangle\langle\psi|$ over a composed subsystem of size KL we arrive at state σ of size N , which shows²⁵ that it is distributed according to the induced measure $\mu_{N,KL}$.

IV. MEASURES DEFINED BY A METRIC

Let d denote any distance in the space of the normalized density operators of a fixed size N . With respect to this distance one can define unit spheres and unit balls. Attributing the same weight to any ball of radius equal to unity one defines the measure corresponding to a given distance.

Consider, for instance, the Hilbert-Schmidt distance between any two mixed states $D_{\text{HS}}(\rho, \sigma) = [\text{Tr}(\rho - \sigma)^2]^{1/2}$. This distance induces in the space of density operators an Euclidean geometry: the space of one-qubit mixed states has a form of the three-ball bounded by the Bloch sphere.

A. Hilbert–Schmidt ensemble

The Hilbert–Schmidt measure defined in this way belongs to the class of induced measures⁴ and can be obtained by a reduction of random pure states defined on a bi-partite quantum system. Looking at expression (13) we see that in the special case $K = N$ the term with the determinant is equal to unity, so measure reduces to the Hilbert-Schmidt measure.⁴ This observation leads to a simple algorithm to generate a Hilbert-Schmidt random matrix:⁴ (i) Take a square complex random matrix A of size N pertaining to the Ginibre ensemble^{18,19} (with real and imaginary parts of each element being independent normal random variables); (ii) Write down the random matrix

$$\rho_{\text{HS}} = \frac{GG^\dagger}{\text{Tr}GG^\dagger}, \quad (15)$$

which is by construction Hermitian, positive definite, and normalized, so it forms a legitimate density matrix.

To characterize spectral properties of random density matrices one considers the joint distribution of eigenvalues (13), integrates out $N - 1$ variables $\lambda_1, \dots, \lambda_N$ to obtain the level density denoted by $P(\lambda)$. This problem was discussed by Page,⁸ who found the asymptotic distribution for the rescaled variable $x = N\lambda$. The result depends on the ratio of both dimensions $c = K/N$ and is given by the Marchenko–Pastur distribution,⁶

$$\pi_c^{(1)}(x) = \max(1 - c, 0)\delta(0) + \frac{\sqrt{4c - (x - c - 1)^2}}{2\pi x}. \quad (16)$$

This expression is valid for $x \in [x_-, x_+]$, where $x_\pm = 1 + c \pm 2\sqrt{c}$. In the case $c = 1$ corresponding to the Hilbert–Schmidt ensemble this distribution reads

$$P_{\text{HS}}(x) = \frac{1}{2\pi} \sqrt{\frac{4}{x} - 1}, \quad \text{for } x \in [0, 4]. \quad (17)$$

It diverges as $x^{-1/2}$ for $x \rightarrow 0$ and in the rescaled variable, $y = \sqrt{x}$, it becomes a quarter-circle law, $P(y) = \frac{1}{\pi} \sqrt{4 - y^2}$.

The average von Neumann entropy of random states distributed with respect to the HS measure reads^{8,20}

$$\langle S(\rho) \rangle_{\text{HS}} = \ln N - \frac{1}{2} + O\left(\frac{\ln N}{N}\right). \quad (18)$$

As the rescaled eigenvalue is $x = N\lambda$, this result is consistent with the fact that the average entropy of the asymptotic distribution (17) reads $\int_0^4 [-x \ln x P_{\text{HS}}(x)] dx = -1/2$. Note that the mean entropy of a random mixed state is close to the maximal entropy in the N dimensional system, which equals $\ln N$ for the maximally mixed state $\rho_* = \mathbb{1}/N$. The asymptotic average purity $\langle \text{Tr} \rho^2 \rangle_{\text{HS}} \approx 2/N$ which is consistent with the second moment of the MP distribution, $\int_0^4 x^2 P_{\text{HS}}(x) dx = 2$.

B. Bures ensemble

Another distinguished measure in the space Ω of quantum mixed states, is induced by the Bures distance,^{26,27} written here for two arbitrary states ρ and σ

$$D_B(\rho, \sigma) = \sqrt{2 - 2\text{Tr}(\sqrt{\rho\sigma\sqrt{\rho}})^{1/2}}. \quad (19)$$

The Bures distance plays an important role in the investigation of the set of quantum states.⁹ The Bures metric, related to quantum distinguishability,²² is known to be the minimal monotone metric²⁸ and applied to any two diagonal matrices it gives their statistical distance. These special features

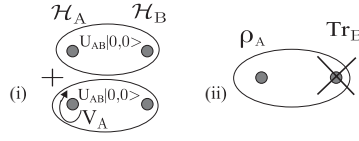


FIG. 5. To generate states according to the Bures distribution one (i) constructs a superposition of a random bi-partite state $|\psi_1\rangle = U_{AB}|0, 0\rangle$ with a locally transformed random state $|\psi_2\rangle = (V_A \otimes \mathbb{1})|\psi_1\rangle$, and (ii) performs partial trace over an auxiliary subsystem B .

support the claim that without any prior knowledge on a certain state acting on \mathcal{H}_N , the optimal way to mimic it is to generate a random density operator with respect to the Bures measure.

The Bures measure is characterized by the following joint probability of eigenvalues:¹³

$$P_B(\lambda_1, \dots, \lambda_N) = C_N^B \prod_i \lambda_i^{-1/2} \prod_{i < j} \frac{(\lambda_i - \lambda_j)^2}{\lambda_i + \lambda_j}, \quad (20)$$

where all eigenvalues are non-negative, $\lambda_i \geq 0$ and they sum to unity, $\sum_i \lambda_i = 1$. The normalization constant for this measure

$$C_N^B = 2^{N^2-N} \frac{\Gamma(N^2/2)}{\pi^{N/2} \prod_{j=1}^N \Gamma(j+1)} \quad (21)$$

was obtained in Refs. 13 and 14 for small N and in Ref. 29 for the general case.

To generate random states with respect to the Bures ensemble one can proceed according to the following algorithm¹⁵ illustrated in Fig. 5.

(i) Take a complex random matrix G of size N pertaining to the Ginibre ensemble and a random unitary matrix U distributed according to the Haar measure on $U(N)$.³⁰

(ii) Write down the random matrix

$$\rho_B = \frac{(\mathbb{1} + U)GG^\dagger(\mathbb{1} + U^\dagger)}{\text{Tr}[(\mathbb{1} + U)GG^\dagger(\mathbb{1} + U^\dagger)]} \quad (22)$$

which is distributed according to the Bures measure. In the analogy to the Hilbert-Schmidt ensemble we can write this state as reduction of a pure state on the composed system,

$$\rho_B = \frac{\text{Tr}_N |\phi\rangle\langle\phi|}{\langle\phi|\phi\rangle}, \quad \text{where } |\phi\rangle := [(\mathbb{1} + V_A) \otimes \mathbb{1}]|\psi_1\rangle. \quad (23)$$

Here $|\psi_1\rangle = U_{AB}|0, 0\rangle$ is a random state on the bipartite system used in Eq. (11) and $V_A \in U(N)$.

The asymptotic probability distribution for the rescaled eigenvalue $x = N\lambda$ of a random density matrix generated according to the Bures ensemble reads²⁰

$$P_B(x) = C \left[\left(\frac{a}{x} + \sqrt{\left(\frac{a}{x}\right)^2 - 1} \right)^{2/3} - \left(\frac{a}{x} - \sqrt{\left(\frac{a}{x}\right)^2 - 1} \right)^{2/3} \right], \quad (24)$$

where $C = 1/4\pi\sqrt{3}$ and $a = 3\sqrt{3}$. This distribution is defined on a support larger than the standard MP distribution, $x \in [0, a]$ and it diverges for $x \rightarrow 0$ as $x^{-2/3}$.

The average entropy of a random state from the Bures ensemble reads²⁰

$$\langle S(\rho) \rangle_B = \ln N - \ln 2 + O\left(\frac{\ln N}{N}\right). \quad (25)$$

This value is smaller than the average entropy (18), which shows that the Bures states are typically less mixed than the states from the Hilbert-Schmidt ensemble. A similar conclusion follows from the comparison of average purity for the Bures ensemble, $\langle \text{Tr} \rho^2 \rangle_B \approx 5/2N$,²⁰ with the average purity for the HS measure.

By considering random matrices of the form $W = (a\mathbb{1} + (1-a)U)G$, one may construct a continuous family of measures interpolating between the Bures and the Hilbert-Schmidt ensembles¹⁵

and labeled by a real parameter $a \in [0, 1/2]$. A more general class of interpolating ensembles is proposed in Sec. VII.

V. PROJECTION ONTO MAXIMALLY ENTANGLED STATES

A. Four-partite systems and measurements in a maximally entangled basis

Consider a system consisting of four subsystems, labeled as A , B , C , and D . For simplicity assume here that the dimensions of all subsystems are equal, $N_1 = N_2 = N_3 = N_4 = N$. Consider an arbitrary four-partite product state, say $|\psi_0\rangle = |0\rangle_A \otimes |0\rangle_B \otimes |0\rangle_C \otimes |0\rangle_D =: |0, 0, 0, 0\rangle$. Taking two independent random unitary matrices U_{AB} and U_{CD} of size N^2 , which act on the first and the second pair of subsystems, respectively, we define a random state $|\psi\rangle = U_{AB} \otimes U_{CD} |\psi_0\rangle$. By construction, it is a product state with respect to the partition into two parts: (A, B) and (C, D) . In the analogy to (2) it can be expanded in the product basis:

$$|\psi\rangle = \sum_{i,j=1}^N \sum_{k,l=1}^N G_{ij} E_{kl} |i\rangle_A \otimes |j\rangle_B \otimes |k\rangle_C \otimes |l\rangle_D. \quad (26)$$

Consider now a maximally entangled state on the second and the third subsystem:

$$|\Psi_{BC}^+\rangle = \frac{1}{\sqrt{N}} \sum_{\mu=1}^N |\mu\rangle_B \otimes |\mu\rangle_C, \quad (27)$$

and the corresponding projector $P_{BC} := |\Psi_{BC}^+\rangle \langle \Psi_{BC}^+|$. One can extend it into a four-partite operator and define

$$P := \mathbb{1}_A \otimes \left(\frac{1}{N} \sum_{\mu,v} |\mu, \mu\rangle_{BC} \langle v, v| \right) \otimes \mathbb{1}_D \quad (28)$$

Let us assume that the random pure state $|\psi\rangle$ defined in (26) is subjected to a projective measurement performed onto the second and third subsystem, and that the result is post-selected to be associated to the projector P . This leads to a non-normalized pure state $|\phi\rangle$ describing the remaining two subsystems,

$$|\phi\rangle = P|\psi\rangle = \frac{1}{N} \sum_{i=1}^N \sum_{l=1}^N \sum_{k=1}^N G_{ik} E_{kl} |i\rangle_A \otimes |l\rangle_D. \quad (29)$$

Note that the projection on the entangled state $|\Psi_{BC}^+\rangle$ introduces a coupling between the subsystems B and C . A similar idea was used in analysis of entanglement swapping,³⁴ and in studies of “matrix product states”³¹ and “projected entangled pair states.”^{32,33} Constructing the matrix product states an $N \times N$ entangled state is projected down into a subspace of an arbitrary dimension d , while in our approach a projection onto the maximally entangled state of BC takes place, (which formally corresponds to $d = 1$), and only two edge systems labeled by A and D survive the projection (see Fig. 6).

Normalizing the resulting state $|\phi\rangle$ and performing the partial trace over the fourth subsystem D we arrive at a compact expression for the resulting mixed state on the first subsystem A ,

$$\rho = \frac{\text{Tr}_D |\phi\rangle \langle \phi|}{\langle \phi | \phi \rangle} = \frac{G E E^\dagger G^\dagger}{\text{Tr } G E E^\dagger G^\dagger}. \quad (30)$$

For any matrices G and E this expression provides a valid quantum state, normalized and positive. If initial pure states are random, then the matrices G and E belong to the Ginibre ensemble and the spectrum of a random state ρ consists of squared singular values of the product GE of two random Ginibre matrices. Random states with the same statistical properties were recently found in ensembles associated with certain graphs.⁵

Observe that the statistical properties of the ensemble defined will change if the projection is performed with respect to an arbitrary maximally entangled state, $|\Psi_{BC}^+\rangle' = (U_B \otimes U_C) |\Psi_{BC}^+\rangle$, as

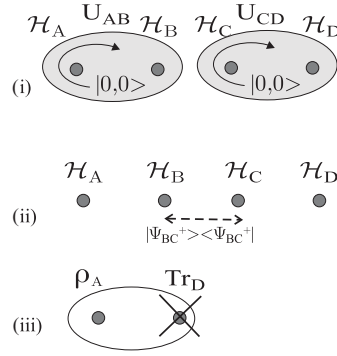


FIG. 6. To generate a random mixed state with spectral density $\pi^{(2)}$ one has to (i) take a product random pure state $|\psi_{AB}\rangle \otimes |\psi_{CD}\rangle$ on a four particle system A, B, C, D , (ii) measure subsystems B, C by a projection onto the maximally entangled state $|\Psi_{BC}^+\rangle$, (iii) average over the subsystem D .

the local unitaries U_B and U_C can be absorbed in the definition of random Ginibre matrices, G and E , respectively.

Furthermore, one may choose N^2 unitary matrices $U_i \in U(N)$, which are orthogonal in the sense of the Hilbert–Schmidt scalar product, $\text{Tr } U_i U_j^\dagger = N \delta_{ij}$. Then the set of N^2 maximally entangled states, $|\Psi_i^+\rangle = (U_i \otimes \mathbb{1})|\Psi_{BC}^+\rangle$, forms a maximally entangled basis, which are known to exist in any dimension.³⁵ Thus, one may consider another setup in which a selective measurement on subsystems B and C is performed in the maximally entangled basis. The outcome state on subsystems AD depends on the result of the quantum measurement of subsystems BC . However, these results are equivalent up to a unitary transformation, which again can be absorbed into the definition of the Ginibre ensemble. Thus, the statistical properties of the random state (30) on subsystem A obtained in consequence of the measurement in the maximally entangled basis in BC followed by the partial trace over subsystem D do not depend on the outcome of the measurement.

Furthermore, the same construction holds in a more general setup in which some dimensions of four subsystems are different. To use the maximally entangled state $|\Psi_{BC}^+\rangle$ we set $N_B = N_C$, but the dimensions $N = N_A$ and N_D can be different. This leads to two rectangular random Ginibre matrices, G of size $N \times N_B$ and E of size $N_B \times N_D$. As in the previous case, formula (30) provides a density matrix ρ of size N , but now the model is a function of two parameters: dimensions N_B and N_D . It is sometimes convenient to use two ratios, $c_1 = N_B/N$ and $c_2 = N_D/N$, so the standard version of the model corresponds to putting $c_1 = c_2 = 1$.

B. Multi-partite systems

Another possibility to generalize the model is to consider a larger system consisting of an even number $2s$ of subsystems. For simplicity assume first that their dimensions are set to N . In analogy to (26) we use s independent unitaries of size N^2 to generate a random pure state $|\psi\rangle$.

To work with an arbitrary even number of subsystems it is convenient to modify the notation and label the subsystems by integers $1, 2, \dots, 2s$ (see Fig. 7). Consider an arbitrary product state of a $2s$ – particle system, $|\psi_0\rangle = |0\rangle_1 \otimes \dots \otimes |0\rangle_{2s} =: |0, \dots, 0\rangle$. Taking s independent Haar random unitary matrices $U_{1,2}, U_{3,4}, \dots, U_{2s-1,2s}$ of size N^2 , we define a random state $|\psi\rangle = U_{1,2} \otimes \dots \otimes U_{2s-1,2s} |\psi_0\rangle$. Expanding this state in the product basis one obtains

$$\begin{aligned} |\psi\rangle &= (U_{1,2} \otimes \dots \otimes U_{2s-1,2s}) |0, \dots, 0\rangle \\ &= \sum_{i_1, \dots, i_{2s}} (G_1)_{i_1, i_2} \dots (G_s)_{i_{2s-1}, i_{2s}} |i_1, \dots, i_{2s}\rangle. \end{aligned} \quad (31)$$

Performing a projection onto a product of $s - 1$ maximally entangled states,

$$P_s := \mathbb{1}_1 \otimes |\Psi_{2,3}^+\rangle \langle \Psi_{2,3}^+| \otimes \dots \otimes |\Psi_{2s-2,2s-1}^+\rangle \langle \Psi_{2s-2,2s-1}^+| \otimes \mathbb{1}_{2s}, \quad (32)$$

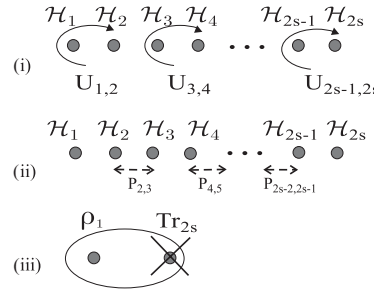


FIG. 7. To obtain a random mixed state with spectral density $\pi^{(s)}$ use a system consisting of $2s$ subsystems, $1, \dots, 2s$ of size N : (i) take the product of s bi-partite random pure states generated by random unitary matrices $U_{1,2}, U_{3,4}, \dots, U_{2s-1,2s} \in U(N^2)$, (ii) measure subsystems $2, \dots, 2s-1$ by a projection onto the product of $s-1$ maximally entangled states, $P_{23} \otimes P_{4,5} \otimes \dots \otimes P_{2s-2,2s-1}$, where $P_{i,j} = |\Psi_{i,j}^+\rangle\langle\Psi_{i,j}^+|$, (iii) perform partial trace average over the subsystem $2s$.

we obtain a pure state $|\phi\rangle$ describing the remaining two subsystems,

$$|\phi\rangle = P|\psi\rangle = N^{1-s} \sum_{i,j} \left(G_1 G_2 \cdots G_s \right)_{ij} |i\rangle_1 \otimes |j\rangle_{2s}. \quad (33)$$

Normalizing this state and performing the partial trace over the last subsystem we obtain an explicit expression for the resulting mixed state on the first subsystem:

$$\rho = \frac{\text{Tr}_{2s} |\phi\rangle\langle\phi|}{\langle\phi|\phi\rangle} = \frac{G_1 G_2 \cdots G_s (G_1 G_2 \cdots G_s)^\dagger}{\text{Tr} [G_1 G_2 \cdots G_s (G_1 G_2 \cdots G_s)^\dagger]}. \quad (34)$$

Alternatively, one may assume that this state is obtained as a result of an orthogonal measurement into the product of $s-1$ maximally entangled bases. The first entangled basis correlates subsystem 2 with subsystem 3, the next one couples subsystem 4 with 5, while due to the last one the subsystem $2s-2$ is correlated with $2s-1$. Thus, eigenvalues of a random state generated in this way coincide with squared singular values of the product of s independent random Ginibre matrices. Their statistical properties will be analyzed in Sec. VI. In general, the Ginibre matrices need not to have the same dimension so the model can be generalized. Assuming that a rectangular matrix G_i has dimensions $N_i \times M_i$ one has to put $M_i = N_{i+1}$ for $i = 1, \dots, s-1$, so that the product (34) is well defined. Setting $N_1 = N$ and defining the ratios $c_i = M_i/N$ for $i = 1, \dots, s$ one obtains ensemble of random states parametrized by the vector of coefficients, $\mathbf{c} := \{c_1, \dots, c_s\}$.

VI. PRODUCT OF GINIBRE MATRICES AND FUSS-CATALAN DISTRIBUTION

For any integer number s , there exists a probability measure $\pi^{(s)}$, called the Fuss-Catalan distribution of order s , whose moments are the generalized Fuss-Catalan numbers^{10,12} given in terms of the binomial symbol,

$$\int_0^{b(s)} x^m \pi^{(s)}(x) dx = \frac{1}{sm+1} \binom{sm+m}{m} =: \text{FC}_m^{(s)}. \quad (35)$$

The measure $\pi^{(s)}$ has no atoms, it is supported on $[0, b(s)]$ where $b(s) = (s+1)^{s+1}/s^s$, its density is analytic on $(0, b(s))$, and bounded at $x = b(s)$, with asymptotic behavior $\sim 1/(\pi x^{s/(s+1)})$ at $x \rightarrow 0$. This distribution arises in random matrix theory as one studies the product of s independent random square Ginibre matrices, $W = \prod_{j=1}^s G_j$. In this case squared singular values of W (i.e., eigenvalues of WW^\dagger) have asymptotic distribution $\pi^{(s)}$. The same Fuss-Catalan distribution describes asymptotically the statistics of singular values of s th power of a single random Ginibre matrix.³⁶ In terms of free probability theory, it is the free multiplicative convolution product of s copies of the Marchenko-Pastur distribution,^{11,37} which is written as $\pi^{(s)} = [\pi^{(1)}]^{\boxtimes s}$.

An explicit expression of the spectral density for $s = 2$ is

$$\pi^{(2)}(x) = \frac{\sqrt[3]{2}\sqrt{3}}{12\pi} \frac{[\sqrt[3]{2}(27 + 3\sqrt{81 - 12x})^{\frac{2}{3}} - 6\sqrt[3]{x}]}{x^{\frac{2}{3}}(27 + 3\sqrt{81 - 12x})^{\frac{1}{3}}}, \quad (36)$$

where $x \in [0, 27/4]$, was derived first in Ref. 38 in context of construction of generalized coherent states from combinatorial sequences. More recently, it was applied in Ref. 5 to describe random quantum states associated with certain graphs.

The spectral distribution of a product of an arbitrary number of s random Ginibre matrices was recently analyzed by Burda *et al.*³⁹ also in the general case of rectangular matrices. The distribution was expressed as a result of a polynomial equation and it was conjectured that the finite size effects can be described by a simple multiplicative correction. Another recent work of Liu *et al.*⁴⁰ provides an integral representation of the distribution $\pi^{(s)}$ derived in the case of s square matrices of size N , which is assumed to be large.

Making use of the inverse Mellin transform and the Meijer G -function one may find a more explicit form of this distribution. It can be represented⁴¹ as a superposition of s hypergeometric functions of the type ${}_sF_{s-1}$,

$$\pi^{(s)}(x) = \sum_{n=1}^s \Lambda_{n,s} x^{\frac{n}{s+1}-1} {}_sF_{s-1}\left(\left[\left\{1 - \frac{1+j}{s} + \frac{n}{s+1}\right\}_{j=1}^s\right]; \left[\left\{1 + \frac{n-j}{s+1}\right\}_{j=1}^{n-1}, \left\{1 + \frac{n-j}{s+1}\right\}_{j=n+1}^s\right]; \frac{s^s}{(s+1)^{s+1}}x\right), \quad (37)$$

where the coefficients $\Lambda_{n,s}$ read for $n = 1, 2, \dots, s$,

$$\Lambda_{n,s} := s^{-3/2} \sqrt{\frac{s+1}{2\pi}} \left(\frac{s^{s/(s+1)}}{s+1}\right)^n \frac{\left[\prod_{j=1}^{n-1} \Gamma\left(\frac{j-n}{s+1}\right)\right] \left[\prod_{j=n+1}^s \Gamma\left(\frac{j-n}{s+1}\right)\right]}{\prod_{j=1}^s \Gamma\left(\frac{j+1}{s} - \frac{n}{s+1}\right)}. \quad (38)$$

Here ${}_pF_q([\{a_j\}_{j=1}^p]; [\{b_j\}_{j=1}^q]; x)$ stands for the hypergeometric function⁴² of the type ${}_pF_q$ with p “upper” parameters a_j and q “lower” parameters b_j , and of the argument x . The symbol $\{a_i\}_{i=1}^r$ represents the list of r elements, a_1, \dots, a_r . The above distribution is exact and it describes the density of squared singular values of s square Ginibre matrices in the limit of large matrix size N .

Observe that in the simplest case $s = 1$ the above form reduces to the Marchenko–Pastur distribution,

$$\pi^{(1)}(x) = \frac{1}{\pi\sqrt{x}} {}_1F_0\left(\left[-\frac{1}{2}\right]; \left[\right]; \frac{1}{4}x\right) = \frac{\sqrt{1-x/4}}{\pi\sqrt{x}}, \quad (39)$$

while the case $s = 2$

$$\pi^{(2)}(x) = \frac{\sqrt{3}}{2\pi x^{2/3}} {}_2F_1\left(\left[-\frac{1}{6}, \frac{1}{3}\right]; \left[\frac{2}{3}\right]; \frac{4x}{27}\right) - \frac{\sqrt{3}}{6\pi x^{1/3}} {}_2F_1\left(\left[\frac{1}{6}, \frac{2}{3}\right]; \left[\frac{4}{3}\right]; \frac{4x}{27}\right) \quad (40)$$

is equivalent to the form (36) obtained in Ref. 38.

The distributions (37), shown in Fig. 8, are thus directly applicable to describe the level density of random mixed states obtained from a $2s$ -partite pure states by projection onto maximally entangled states and partial trace as described in Sec. V. This result becomes exact in the asymptotic limit, in which the dimension N of a single subsystem tends to infinity. However, based on recent results of Burda *et al.*³⁹ one can conjecture that the finite N effects can be described by a multiplicative correction.

Note that the upper edge $b(s) = (s+1)^{s+1}/s^s$ of the FC distribution $\pi^{(s)}(x)$ for large matrices determines the size of the largest eigenvalue λ_{\max} of the reduced density matrix ρ of size N . In the case of the structureless ensemble of random pure states on $N \times N$ system, corresponding to $s = 1$, one has $b(1) = 4$ so that $\lambda_{\max} \approx 4/N$. For states obtained by the projection of a $2s$ partite system on maximally entangled states, as described in Sec. V, the largest component behaves as $\lambda_{\max} \approx b(s)/N$.

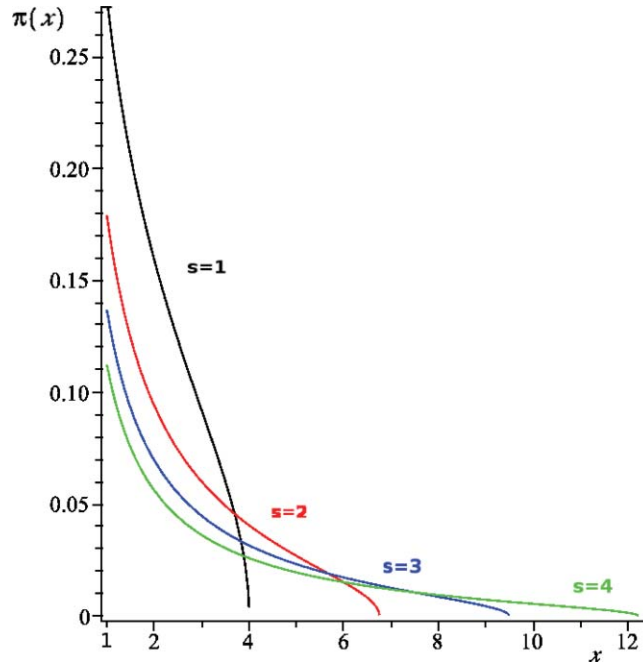


FIG. 8. (Color online) Fuss–Catalan distributions $\pi^{(s)}(x)$ plotted for $s = 1, 2, 3, 4$ supported on the interval $[0, (s+1)^{s+1}/s^s]$. To demonstrate the behavior at the right edge of the support the figure is depicted for $x \geq 1$.

The number λ_{\max} , equal to the largest component of the Schmidt vector of a random pure state on the bi-partite system, can be used to measure the degree of quantum entanglement. For instance, for a bi-partite system, the “geometric measure” of entanglement, related to the distance to the closest separable state (in the sense of the natural Fubini-Study distance) reads,⁴³ $E_g(|\phi\rangle) = -\ln \lambda_{\max}$. This quantity can also be considered as the Chebyshev entropy S_∞ – the generalized Renyi entropy $S_q = \frac{1}{1-q} \ln \text{Tr} \rho^q$ in the limit $q \rightarrow \infty$.⁹

Thus, the right edge $b(s)$ of the support of the spectral density for the reduced state $\rho = \text{Tr}_B |\phi\rangle\langle\phi|$ determines the geometric measure of entanglement of the corresponding random pure states. In the case of structureless random pure states, related to the Marchenko–Pastur distribution one becomes an asymptotic expression for the average $\langle E_g(|\phi\rangle) \rangle = -\ln(4/N)$. In the general case of random state corresponding the FC distribution $\pi^{(s)}(x)$ the typical value of the entropy reads

$$\langle S_\infty \rangle_s = \ln N - \ln b(s) = \ln N + s \ln s - (s+1) \ln(s+1). \quad (41)$$

The larger the value of s , the smaller the Chebyshev entropy S_∞ , and the less entangled a typical random state obtained by the projection of the $2s$ partite system.

The average von Neumann entropy, $S_1 = -\text{Tr} \rho \ln \rho$, of mixed states of size N generated according to the FC distribution reads $\langle S(\rho) \rangle_s = \ln N - \sum_{j=2}^{s+1} \frac{1}{j}$.⁵ The second moment of the FC distribution given in (35) implies the asymptotic average purity $\langle \text{Tr} \rho^2 \rangle_s \approx (s+1)/N$ – the larger the number s , the more pure the typical mixed state generated by a projection onto $s-1$ maximally entangled states.

VII. CONCLUDING REMARKS

In this work we analyzed structured ensembles of random pure states on composite systems. They are defined with respect to a given decomposition of the entire system into its subsystems, what induces a concrete tensor product structure in the Hilbert space. The structured ensembles are thus invariant with respect to the group of local unitary transformations.

TABLE I. Reduction of pure states from structured ensembles leads to random mixed states of the form $\rho = WW^\dagger / \text{Tr}WW^\dagger$. Random matrix W is constructed out of random unitary matrices U_i distributed according to the Haar measure and/or (independent) random Ginibre matrices G_j of a given size N . Asymptotic distribution $P(x)$ of the density of a rescaled eigenvalue $x = N\lambda$ of ρ for $N \rightarrow \infty$ is characterized by the singularity at 0, its support $[a, b]$, the second moment M_2 determining the average purity $\langle \text{Tr}\rho^2 \rangle = M_2/N$, and the mean entropy, $\int_a^b [-x \ln x P(x) dx]$, according to which the table is ordered.

k	s	Matrix W	Distribution $P(x)$	Singularity at $x \rightarrow 0$	Support $[a, b]$	M_2	Mean entropy
1	0	U_1	$\delta(1) = \pi^{(0)}$	—	$\{1\}$	1	0
2	0	$U_1 + U_2$	arcsine	$x^{-1/2}$	$[0, 2]$	$3/2$	$\ln 2 - 1 \approx -0.307$
3	0	$U_1 + U_2 + U_3$	3 entangled states	$x^{-1/2}$	$[0, 2\frac{2}{3}]$	$5/3$	≈ -0.378
4	0	$U_1 + U_2 + U_3 + U_4$	4 entangled states	$x^{-1/2}$	$[0, 3]$	$7/8$	≈ -0.411
1	1	$G \sim UG$	Marchenko–Pastur $\pi^{(1)}$	$x^{-1/2}$	$[0, 4]$	2	$-1/2 \approx -0.5$
2	1	$(U_1 + U_2)G$	Bures	$x^{-2/3}$	$[0, 3\sqrt{3}]$	$5/2$	$-\ln 2 \approx -0.693$
1	2	$G_1 G_2$	Fuss–Catalan $\pi^{(2)}$	$x^{-2/3}$	$[0, 6\frac{3}{4}]$	3	$-5/6 \approx -0.833$
1
1	s	$G_1 \cdots G_s$	Fuss–Catalan $\pi^{(s)}$	$x^{-s/(s+1)}$	$[0, (s+1)^{s+1}/s^s]$	$s+1$	$-\sum_{j=2}^{s+1} \frac{1}{j}$

Performing a partial trace over selected subsystems one obtains an ensemble of random mixed states defined on the remaining subsystems. The particular ensemble depends thus on the number of systems traced out and on the way the initial random pure states are prepared.

Quantum states obtained by the partial trace of a superposition of k maximally entangled pure states of the bi-partite system involve the sum of k random unitary matrices. To generate states which involve a product of an arbitrary number of s matrices one needs to consider a system consisting of $2s$ subsystems, in which an orthogonal measurement is performed in the product of $s-1$ maximally entangled bases. Selected ensembles of random states, recipe to generate numerically the corresponding density matrices, and some properties of the distribution of the Schmidt coefficients are collected in Table I.

We are going to conclude this work by writing down a more general class of structured random states, which contains all particular cases discussed in the paper and listed in Table I. Consider the following ensemble of non-hermitian random matrices parametrized by an arbitrary k -dimensional probability vector $p = \{p_1, \dots, p_k\}$ and a non-negative integer s :

$$W_{k,s} := [p_1 U_1 + p_2 U_2 + \cdots + p_k U_k] G_1 \cdots G_s. \quad (42)$$

Here U_1, \dots, U_k denote k independent random unitary matrices distributed according to the Haar measure on $U(N)$, while G_1, \dots, G_s are independent square random matrices of size N from the complex Ginibre ensemble. Random density matrix is obtained as a normalized Wishart-like matrix:

$$\rho_{k,s} := \frac{W_{k,s} W_{k,s}^\dagger}{\text{Tr}(W_{k,s} W_{k,s}^\dagger)}. \quad (43)$$

Note that any particular ensemble from the above class can be physically realized by taking a superposition of k random pure states weighted by the vector p . Each pure state is defined on the system containing $2s$ subsystems. Performing a measurement in the product of $(s-1)$ maximally entangled bases one gets a random pure state of the desired structured ensemble. Eventually, averaging over the last subsystem one arrives at the mixed state (43).

Consider first the case of a uniform probability vector, $p_i = 1/k$ for $i = 1, \dots, k$. For $k = 1$ one obtains ensembles leading to the Fuss–Catalan distributions $\pi^{(s)}$, which in the case $s = 1$ reduces to the Marchenko–Pastur distribution. Taking $k = 2$ and $s = 0$ one obtains the arcsine ensemble (7), while for larger k one obtains the distribution (10), which converges to $\pi^{(1)}$ in the limit $k \rightarrow \infty$. Moreover, the case $k = 2$ and $s = 1$ corresponds to the Bures ensemble (23). Thus, the case $k = 2$ and arbitrary s can be called *higher order Bures ensemble*.

In a more general case, taking an arbitrary probability vector p and varying the weights in a continuous manner one can study the transition between given structured ensembles. For instance, by

fixing the parameter s , setting $k = 2$, and varying the weight $p_2 = 1 - p_1$, one defines a continuous interpolation between the higher order Bures ensemble and the Fuss–Catalan ensemble. We have shown therefore that having at our disposal simple algorithms to generate random unitary and random Ginibre matrices we can construct a wide class of ensembles of random quantum states. Furthermore, we provide constructive physical recipe to generate such states by means of generic two-particle interaction, superposition of states, selective measurements in maximally entangled basis, and performing averages over certain subsystems.

As discussed in Appendix B it is also possible to introduce analogous ensembles of real random density matrices. Physically this corresponds to imposing restrictions on the class of the interactions used to generate random pure states. In contrast with the complex case, the ensemble based on square real Ginibre matrices does not lead to the Bures measure in the space of real states. To achieve such a measure one needs to generalize the ensemble even further to allow also rectangular Ginibre matrices.¹⁵ In physical terms this implies that the dimension of the principal system and the auxiliary system have to be different in this case.

The notion of random quantum states is closely related with the concept of random quantum maps. Due to the Jamiołkowski isomorphism any quantum operation Φ acting on density matrices of size N can be represented by a state on the extended Hilbert space,⁹

$$\sigma = (\Phi \otimes \mathbb{1})|\psi^+\rangle\langle\psi^+|. \quad (44)$$

Here $|\psi^+\rangle = \frac{1}{\sqrt{N}} \sum_{j=1}^N |j, j\rangle$ denotes the maximally entangled state from the bi-partite Hilbert space $\mathcal{H} = \mathcal{H}_N \otimes \mathcal{H}_N$. Any state σ on the composed Hilbert space $\mathcal{H} = \mathcal{H}_A \otimes \mathcal{H}_B$ defines a completely positive, trace preserving map provided the following partial trace condition is satisfied, $\text{Tr}_A \sigma = \mathbb{1}/N$.

It is possible to impose this partial trace condition on an arbitrary state ω acting on \mathcal{H} . To this end one finds the reduced state $Y := \text{Tr}_A \omega$ which is positive and allows one to take its square root \sqrt{Y} , and write the normalized cognate state:⁴⁴

$$\sigma = \frac{1}{N} \left(1 \otimes \frac{1}{\sqrt{Y}} \right) \omega \left(1 \otimes \frac{1}{\sqrt{Y}} \right). \quad (45)$$

The required property, $\text{Tr}_A \sigma = \mathbb{1}/N$, is satisfied by construction, so the state σ represents a quantum operation. As the matrix elements of the corresponding superoperator Φ can readily be obtained by reshuffling⁹ the entries of the density matrix σ , any random state ω determines by (45) and (44) a quantum operation. Therefore, any ensemble of random states introduced in this paper, applied for bi-partite, $N \times N$ systems determines the corresponding ensemble of random operations. For instance, the induced measure with $K = N^2$ corresponds to the flat measure in the space of quantum operations,^{44,45} but other ensembles of random states can be also applied to generate random quantum operations.⁹

The present study on ensembles of random states should be concluded with a remark that apart of the methods developed in this paper several other approaches are advocated in the literature. In very recent papers^{46,47} the authors follow a statistical approach introducing a partition function which leads to a generalization of the Hilbert-Schmidt measure. Varying the parameter of the model, which corresponds to the inverse temperature, they demonstrate a phase transition during an interpolation between Marchenko-Pastur and semicircle distribution of spectral density. In another recent approach Garnerone *et al.* studied statistical properties of random matrix product states,⁴⁸ which are obtained out of products of truncated random unitary matrices. Although these models of random states do differ from the one presented in this work, possible links and relations between results obtained in these approaches are currently under investigation.

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APPENDIX A: SUM OF k RANDOM UNITARIES AND THE DISTRIBUTION ν_k

We compute, in the limit of large matrix size $N \rightarrow \infty$, the asymptotic eigenvalue distribution of the random matrix,

$$V_k = \frac{1}{k}(U_1 + U_2 + \cdots + U_k)(U_1^* + U_2^* + \cdots + U_k^*), \quad (\text{A1})$$

where U_1, \dots, U_k are $N \times N$ random independent Haar unitary matrices. Obviously, this is equivalent to computing the singular value distribution of $k^{-1/2} \sum_{i=1}^k U_i$.

For now, we forget about the normalization pre-factor and we put $W_k = kV_k$. It is a well known result in free probability theory that independent large unitary matrices are free from each other (and, very importantly, but not of interest here, they are also free from deterministic diagonal matrices).

Theorem A.1 (Refs. 49 and 50): *Let $U_{1,N}, \dots, U_{k,N} \in \mathcal{U}(N)$ be k independent Haar unitary random matrices. Then, as $N \rightarrow \infty$,*

$$U_{1,N}, \dots, U_{k,N} \xrightarrow{*-\text{distr}} u_1, \dots, u_k, \quad (\text{A2})$$

where u_1, \dots, u_k are free Haar unitary elements in a non-commutative W^* -probability space (\mathcal{M}, τ) .

Hence, computing the limit distribution of W_k amounts to understanding the distribution of a sum of k free Haar unitary elements in a von Neumann algebra $w_k = (u_1 + \cdots + u_k)(u_1^* + \cdots + u_k^*)$. This problem has been related to random walks on k -regular trees by Kesten.⁵¹ Indeed, the number of alternating words of length $2p$ in the letters u_i, u_i^* which reduce to the unit is bijectively the same as the number of walks of length $2p$ on the k -regular tree beginning and ending at some fixed vertex. Using standard formulas for the number of such walks, we can deduce moment information for $k = 2$:

$$\lim_{N \rightarrow \infty} \mathbb{E} \left[\frac{1}{N} \text{Tr } W_2^p \right] = \tau(w_2^p) = \binom{2p}{p}, \quad (\text{A3})$$

and a moment generating function in the general case:

$$F_k(z) = \sum_{p=0}^{\infty} \tau(w_k^p) z^p = \frac{2(k-1)}{k-2+k\sqrt{1-4(k-1)z}}. \quad (\text{A4})$$

From the last formula, using Cauchy transform techniques, one can easily deduce the probability density function of the distribution of w :

$$d\mu_k(x) = \frac{k}{2\pi} \frac{\sqrt{4(k-1)x - x^2}}{k^2x - x^2} \mathbf{1}_{[0, 4(k-1)]}(x) dx. \quad (\text{A5})$$

This result has been obtained by Haagerup and Larsen in Example 5.3 of Ref. 52. The authors were interested in the Brown measure of the non-normal element $\tilde{w} = u_1 + \cdots + u_k$. They found that the distribution of $|\tilde{w}|$ is given by

$$d\tilde{\mu}_k(x) = \frac{k}{\pi} \frac{\sqrt{4(k-1) - x^2}}{k^2 - x^2} \mathbf{1}_{[0, 2\sqrt{k-1}]}(x) dx. \quad (\text{A6})$$

One can easily recover Eq. (A5) by using $w = \tilde{w}\tilde{w}^*$ and by noticing that $\mu_k = \tilde{\mu}_k \circ \text{sq}$, where sq is the square function $\text{sq}(x) = x^2$.

If ν_k is the distribution of the rescaled element $v = w/k$, then one arrives at the desired distribution function,

$$d\nu_k(x) = \frac{1}{2\pi} \frac{\sqrt{4k(k-1)x - k^2x^2}}{kx - x^2} \mathbf{1}_{[0, 4\frac{k-1}{k}]}(x) dx. \quad (\text{A7})$$

In Ref. 52 it is shown that the Brown measure of \tilde{w} is a rotationally invariant measure, supported on the centered disk of radius \sqrt{k} with radial density

$$f_{\tilde{w}}(r) = \frac{k^2(k-1)}{\pi(k^2 - r^2)^2}, \quad 0 < r < \sqrt{k}. \quad (\text{A8})$$

With the proper $k^{-1/2}$ rescaling, it is easy to see that the above Brown measure converges to the uniform measure on the unit disk. Hence, we recover the Ginibre behavior in the limit $k \rightarrow \infty$ (one has to take first the limit $N \rightarrow \infty$). Let us add that a more general study on statistical properties of a sum of random unitary matrices was recently presented by Jarosz.⁵³

APPENDIX B: REAL RANDOM STATES, REAL GINIBRE, AND RANDOM ORTHOGONAL MATRICES

Although most often one considers complex density matrices, it is also interesting to study quantum states described by real density matrices. The dimensionality of the set of real states on \mathcal{H}_N is $N(N+1)/2 - 1$, so its geometry is easier to study than that of the $N^2 - 1$ dimensional set of complex states.⁹ For instance, the set of real states of a qubit forms a two-dimensional disk, which can be considered as a cross-section of the three-dimensional Bloch ball of complex states. Euclidean volume of the set of real density matrices of size N was derived in Ref. 54, while the corresponding measure can be derived from the real Ginibre ensemble.

In this appendix we define ensembles of real states based on random orthogonal matrices and real Ginibre ensemble, and show that the level density does not differ from the complex case. To this end we formulate two lemmas.

Lemma B.1: Consider k independent orthogonal matrices O_1, \dots, O_k , distributed according to the Haar measure on $O(N)$ and define a normalized real density matrix as

$$\rho_{\text{ort}} = \frac{(O_1 + \dots + O_k)(O_1^T + \dots + O_k^T)}{\text{Tr}(O_1 + \dots + O_k)(O_1^T + \dots + O_k^T)}. \quad (\text{B1})$$

Then for large N its spectral density is described by the distribution ν_k given in (10).

This lemma follows directly from the fact that the moments of orthogonal and unitary random matrices have the same behavior for large matrix size N , since random orthogonal matrices are asymptotically free.⁵⁵

Lemma B.2: Consider s independent random matrices R_1, \dots, R_s taken from the real Ginibre ensemble of square matrices of size N . Define a normalized real density matrix as

$$\rho_R = \frac{R_1 R_2 \dots R_s (R_1 R_2 \dots R_s)^T}{\text{Tr}[R_1 R_2 \dots R_s (R_1 R_2 \dots R_s)^T]}. \quad (\text{B2})$$

Then for large N its spectral density is described by the Fuss-Catalan distribution $\pi^{(s)}$ given in (37).

To prove this, one needs to show that in the case of large matrix size N the moments of this distribution are indeed given by the Fuss–Catalan numbers (35), exactly as for the product of complex Ginibre matrices. This follows from the interpretation of the Wick formula for random matrices in terms of maps gluing and from the fact that leading terms have to be non-crossing, therefore orientable as for complex Gaussian matrices – see, e.g., Refs. 56 and 57.

It is natural to combine both definitions and propose a more general ensemble of real density matrices, each obtained out of k random orthogonal matrices and s square real Ginibre matrices in a direct analogy to Eq. (42).

Let us close this work with a remark that the differences between the real and complex case can be significant in some cases. For instance, to get a real state distributed according to the Bures measure one needs to use a symmetric random unitary matrix and a rectangular, $N \times (N + 1)$, real Ginibre matrix, while the complex Bures state is obtained from a random unitary and a square matrix of the complex Ginibre ensemble.¹⁵

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