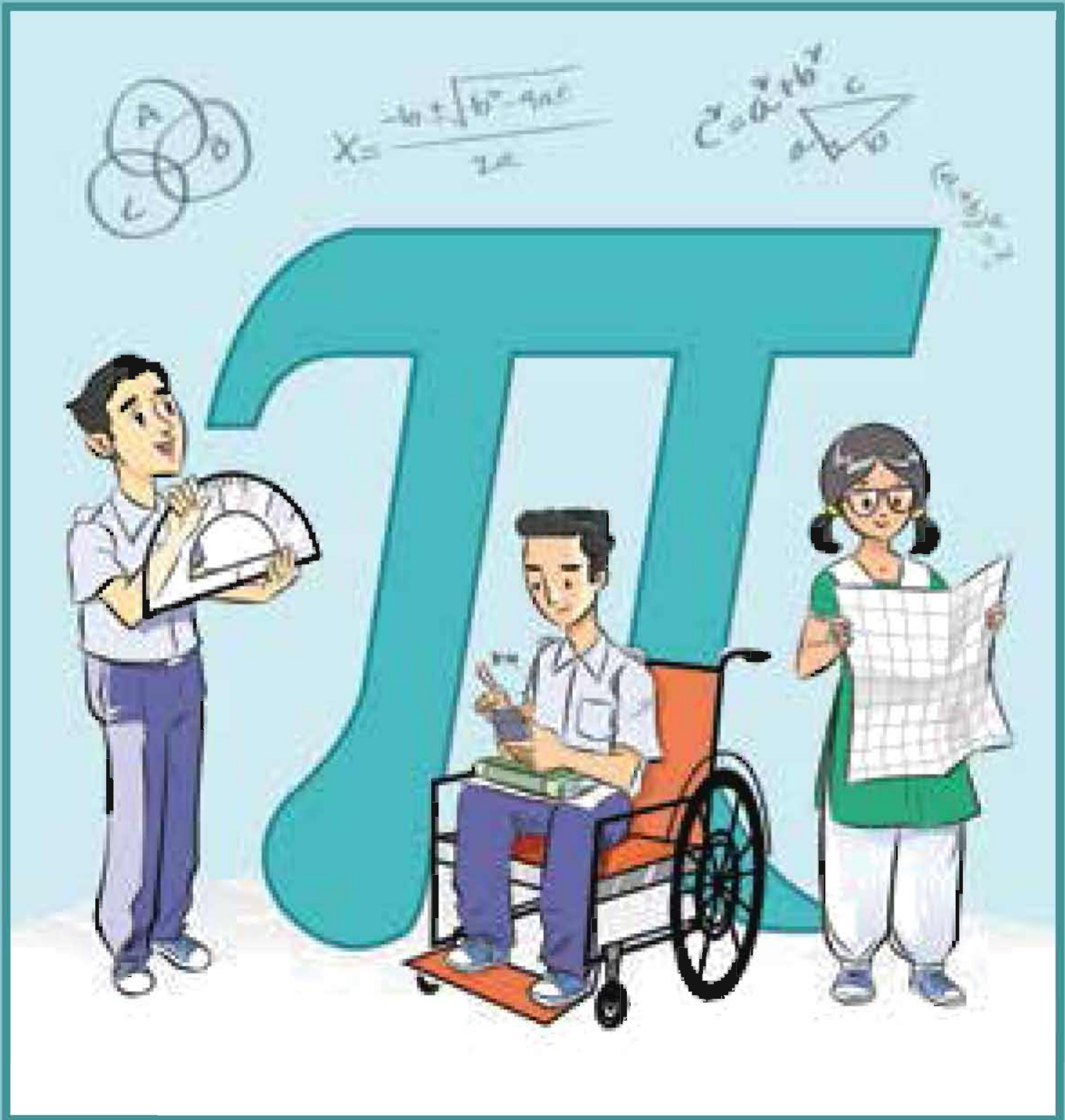
Mathematics

Classes Nine-Ten



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Mathematics

Classes Nine-Ten

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Preface

The aim of secondary education is to make the learners fits for entry into higher education by flourishing their latent talents and prospects with a view to building the nation with the spirit of the Language Movement and the Liberation War. To make the learners skilled and competent citizens of the country based on the economic, social, cultural and environmental settings is also an important issue of secondary education.

The textbooks of secondary level have been written and compiled according to the revised curriculum 2012 in accordance with the aims and objectives of National Education Policy-2010. Contents and presentations of the textbooks have been selected according to the moral and humanistic values of Bengali tradition and culture and the spirit of Liberation War 1971 ensuring equal dignity for all irrespective of caste and creed of different religions and sex.

The present government is committed to ensure the successful implementation of Vision 2021. Honorable Prime Minister, Government of the People's Republic of Bangladesh, Sheikh Hasina expressed her firm determination to make the country free from illiteracy and instructed the concerned authority to give free textbooks to every student of the country. National Curriculum and Textbook Board started to distributed textbooks free of cost since 2010 according to her instruction.

Mathematics plays an important role in developing scientific knowledge at this time of the 21st century. Not only that, the application of Mathematics has increased in family and social life including personal life. With all these things under consideration Mathematics has been presented easily and nicely at the Secondary level to make it useful and delightful to the learners, and a good number of new topics have been included in the textbook.

Considering the challenges and commitments of 21st century and following the revised curriculum the textbook has been written. The textbook has been revised and reedited by a group of prominent educationist to make it learner friendly in 2017.

I thank sincerely all for their intellectual labor who were involved in the process of revision, writing, editing, art and design of the textbook.

Prof. Narayan Chandra Saha
Chairman
National Curriculum and Textbook Board, Bangladesh

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Chapter 1

Real Numbers

History of numbers is as old as that of human civilization. Mathematics was originated from the process of expressing quantity by symbols as numbers. According to Greek Philosopher Aristotle, formal beginning of mathematics started with its practice among ancient Egyptian priests. So it can be said that number based Mathematics was created about two thousand years before the birth of Jesus Christ. After that numbers and operations on them have reached today's universal form through the hands of different nations and civilizations.

Indian mathematicians introduced first the concept of O and decimal number system for satisfying the need of counting natural numbers. This is considered as a milestone in the representation of numbers. Later on Indian and Chinese mathematicians expanded the concepts of 0, negative, real, integer and fractional numbers which the Arabian mathematicians took on the basis with medieval age. Muslim mathematicians of the Middle East are given the credit of expressing numbers by decimal fractions. Again they first introduced irrational numbers in the form of square roots for solution of quadratic algebraic equations in the 11th century. Historians think that around 500 A.D. Greek philosophers felt the necessity of irrational numbers, especially square root of 2, for the purpose of geometrical drawing. In the nineteenth century European mathematicians gave real numbers com lete shape by systematization. In order to satisfying daily needs students are required to have clear understanding about real numbers. In this chapter real numbers have been discussed in detail.

At the end of this chapter students will learn -

- how to classify real numbers.
- how to deduce approximate values of real numbers by expressing it in decimal fractions.
- how to classify decimal fraction.

- how to interpret repeated decimal numbers and convert fractions into repeated decimal numbers.
- how to convert repeated decimal fractions into fractions.
- how to interpret infinite no-nrepeating fractional numbers.
- how to interpret similar and dissimilar decimal fractions.
- how to carry out addition, subtraction, multiplication and division operations on repeated decimal fractions, and will be able to solve different relevant problems.

Classification of Real Numbers

Natural Number: $1, 2, 3, 4, \ldots$ etc are natural numbers or positive whole numbers. $2, 3, 5, 7, \ldots.$ are primes and $4, 6, 8, 9, \ \ldots..$ etc are composite numbers. If GCD (Greatest Common Divisor) of two integers is 1 then they are called mutually primes. For example 6 and 35 are mutually primes.

Integer: All positive and negative fractionless numbers are said to be integers. For example, $\ldots, -3, -2, -1, 0, 1, 2, 3, \ldots$ etc are integers.

Fractional Number: Numbers expressible as $\frac{p}{q}$ are said to be factional numbers where $q \neq 0$, $q \neq 1$ and p are not divisible by q. For example, $\frac{1}{2}, \frac{3}{2}, \frac{-5}{3}, \frac{4}{6}$ etc are simple fractional numbers. In case of a simple fraction $\frac{p}{q}$, if $p < q$, then the fraction is called proper and if $p > q$, then the fraction is termed improper. For example $\frac{1}{2}, \frac{1}{3}, \frac{2}{3}, \frac{1}{4}, \ldots$ etc are proper fractions. $\frac{3}{2}, \frac{4}{3}, \frac{5}{3}, \frac{5}{4}, \ldots$etc are improper fractions.

Rational Number: Numbers of the form $\frac{p}{q}$ are called rational when p and q are integers and $q \neq 0$. For example, $\frac{3}{1} = 3$, $\frac{11}{2} = 5.5$, $\frac{5}{3} = 1.666, \ldots$ etc are rational numbers. Any rational number can be expressed as ratio of two mutually prime numbers. All integers and fractions are rational numbers.

Irrational Number: If a number cannot be expressed in the form $\frac{p}{q}$ where p and q are integers and $q \neq 0$, then it is called an irrational number. Root of a natural number not equal to perfect square or its fraction is an irrational number.

For example, $\sqrt{2} = 1.414213..$, $\sqrt{3} = 1.732..$, $\dfrac{\sqrt{5}}{2} = 1.118\ldots$, etc are irrational numbers. No irrational number can be expressed as ratio of two integers.

Decimal Fractional Number: Rational and irrational numbers when expressed using decimal sign are called decimal fractional number For example, $3 = 3.0$, $\dfrac{5}{2} = 2.5$, $\dfrac{10}{3} = 3.3333\ldots$, $\sqrt{3} = 1.732\ldots$, etc are decimal fractional numbers. If there are finite number of digits after decimal sign, then this is called finite decimal fraction and if there are infinitely many digits, then it is called infinite decimal fractions. For example, 0.52, 3.4152 etc are finite decimal fractions, whereas $\dfrac{4}{3} = 1.333\ldots, \sqrt{5} = 2.123512367\ldots$,etc are infinite decimal fractions. Again, in case of infinite decimal numbers, if some degits after decimal point are repeated, they are called infinite repeated decimal fractions. If digits are not repeated, they are called infinite decimal numbers without repeatation. For example, $\dfrac{122}{99} = 1.2323, .., 5.1\dot{6}5\dot{4}$ etc are infinite repeated decimal fractions and $0.523050056\ldots, 2.12340314\ldots$ etc are infinite decimal numbers without repeatation.

Real Number: All rational and irrational numbers are called real numbers. For example, the following numbers are real numbers.

$0, \pm 1, \pm 2, \pm 3, \cdots$ $\qquad$ $\pm\dfrac{1}{2}, \pm\dfrac{3}{2}, \pm\dfrac{4}{3}, \cdots$

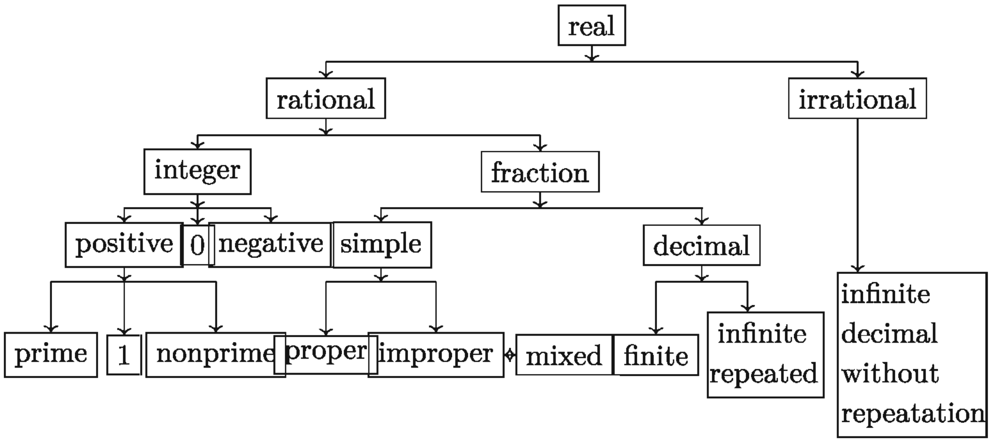
$\sqrt{2}, \sqrt{3}, \sqrt{5}, \sqrt{7}, \cdots$ $\qquad$ $1.23,\ 0.415,\ 1.3333\ldots,\ 0.\dot{6}\dot{2},\ 4.120345061\ldots$

Positive Number: All real numbers bigger than 0 are called positive numbers. For example, $2, \dfrac{1}{2}, \dfrac{3}{2}, \sqrt{2}, 0.415, 0.\dot{6}\dot{2}, 4.120345061\ldots$, etc are positive numbers.

Negative Number:All real numbers smaller than 0 are called negative numbers. For example, $-2, -\dfrac{1}{2}, -\dfrac{3}{2}, -\sqrt{2}, -0.415, -0.\dot{6}\dot{2}, -4.120345061\ldots$, etc are negative numbers.

Non-negative Number:All positive numbers including 0 are called nonnegative numbers. For example, $0,\ 3, \dfrac{1}{2}, 0.612, 1.\dot{3},\ 2.120345\ldots$, etc are no-nnegative numbers.

In the following chart we can see classification of real numbers.



> **Work:** Show the position of the following numbers in the classification of real numbers $\frac{3}{4}$, 5, -7, $\sqrt{13}$, 0, 1, $\frac{9}{7}$, 12, $2\frac{4}{5}$, 1.1234, $0.3\dot{2}\dot{3}$

Example 1. Find an irrational number between $\sqrt{3}$ and 4.

Solution: Here, $\sqrt{3} = 1.7320508\ldots$

Suppose, $a = \dfrac{\sqrt{3}+4}{2} \approx 2.866$ and $b = \dfrac{\sqrt{3}+4+4}{3} \approx 3.244$

Obviously a and b are both real numbers and both are bigger than $\sqrt{3}$ and smaller than 4 because a is average of uneqal numbers $\sqrt{3}$ and 4, and b is average of numbers $\sqrt{3}$ 4 and 4.

That is, $\sqrt{3} < 2.866 < 4$ and $\sqrt{3} < 3.244 < 4$

Again, a and b cannot be expressed as fractions.

$\therefore$ a and b are the tow desired numbers.

In fact, we can construct countless such irrational numbers.

Basic characteristics of addition and multiplication processes of real numbers :

1. If a and b are real numbers, (i) $a+b$ is a real number and (ii) ab is a real number.
2. If a and b are real numbers, (i) $a+b=b+a$ and (ii) $ab=ba$
3. If $a,\ b,\ c$ are real numbers, (i) $(a+b)+c=a+(b+c)$ and (ii) $(ab)c=a(bc)$
4. If a is real number, there are only two real numbers 0 and 1 such that (i) $0 \neq 1$, (ii) $a+0=0+a=a$ and(iii) $a \cdot 1 = 1 \cdot a = a$

5. If a is real number, (i) $a + (-a) = 0$ (ii) If $a \neq 0$,then $a \cdot \frac{1}{a} = 1$
6. If a, b, c are real numbers, $a(b + c) = ab + ac$
7. If a, b are real numbers,$a < b$ or $a = b$ or $a > b$
8. If a, b, c are real numbers and $a < b$,then $a + c < b + c$
9. If a, b c are real numbers and $a < b$, then (i) $ac < bc$ whenever $c > 0$ (ii) $ac > bc$ whenever $c < 0$

Proposition: $\sqrt{2}$ is an irrational number.

Proof: Suppose,$\sqrt{2}$ is an irrational number.

Then there are two mutually prime numbers p, $q > 1$ so that $\sqrt{2} = \frac{p}{q}$.

Or, $2 = \frac{p^2}{q^2}$ [squaring] that is $2q = \frac{p^2}{q}$ [multiplying both sides by q]

Obviously $2q$ is an integer but $\frac{p^2}{q}$ is not an integer, because p and q are natural numbers and they are mutually prime and $q > 1$.

$\therefore$ $2q$ and $\frac{p^2}{q}$ cannot be equal, that is $2q \neq \frac{p^2}{q}$.

$\therefore$ $\sqrt{2}$ cannot be expressed in the form $\frac{p}{q}$, that is $\sqrt{2} \neq \frac{p}{q}$

$\therefore$ $\sqrt{2}$ is an irrational number. □

Remarks : □ is used as the end mark of logical proof

Work: Prove that $\sqrt{3}$ is an irrational number.

Example 2. Prove that when one is added to the product of 4 consecutive natural numbers this becomes a perfect square.

Solution: Let the consecutive integers be x, $x + 1$, $x + 2$, $x + 3$

If we add 1 to the product of these 4 integers we get:

$$x(x + 1)(x + 2)(x + 3) + 1$$
$$= x(x + 3)(x + 1)(x + 2) + 1$$
$$= (x^2 + 3x)(x^2 + 3x + 2) + 1$$
$$= a(a + 2) + 1 \text{ [Assume } x^2 + 3x = a \text{]}$$
$$= a^2 + 2a + 1 = (a + 1)^2 \qquad = (x^2 + 3x + 1)^2$$

which is a perfect square. Therefore, when 1 is added to the product of any four consecutive natural numbers, this becomes a perfect square.

Decimal Fractions

Each real number can be expressed in decimal fractions. For example, $2 = 2.0$, $\frac{2}{5} = 0.4$, $\frac{1}{3} = 0.333\ldots$ etc. Decimal fractions are of three types, namely: finite, repeated and infinite decimal fractions.

Finite decimal fractions: In any finite decimal fraction there is only a finite numbers of digits after the decimal point. For example, 0.12, 1.023, 7.832, 54.67, ... etc are finite decimal fractions.

Repeated decimal fractions: In any repeating decimal fraction all or some digits on the right of decimal point come again and again. For example, 3.333..., 2.454545..., 5.12765765... etc are repeated decimal fractions.

Infinite decimal fractions: In an infinite decimal fraction digits on the right of decimal point never terminate, i.e. the number of digits on the right of the decimal point will not be finite neither will its part appear repeatedly. For example, 1.4142135..., 2.8284271... etc are Infinite decimal fractions.

Remarks: Finite and repeating decimal fractions are rational numbers whereas infinite decimal numbers are irrational numbers. Value of any irrational number can be determined up to any desired decimal place. If numerator and denominator of a fraction can be represented by natural numbers, then the fraction is a rational number.

> **Work:** Classify the following numbers with reasons: 1.723, 5.2333..., 0.0025, 2.1356124..., 0.01050105... and 0.450123...

Repeating decimal fractions

```
6)23(3.833
  18
  ---
   50
   48
   ---
    20
    18
    ---
     20
     18
     ---
      2
```

Express the fraction $\frac{23}{6}$ into decimal fraction. Observe that when numerator was divided by denominator the division process could not be completed. Observe that in the result the digit 3 appears over and over again. So 3.8333... is a repeating decimal number.

The decimal fraction for which the same digit or a set of digits in order appear after decimal point is called repeating decimal fraction. The part of the repeating decimal fraction that appears over and over again is called repeating. The remaining part is nonrepeating.

In a repeating decimal fraction if a single digit repeats, then a point is marked above that digit. Otherwise point is marked on the first and the last repeating digits. For example, $2.555\ldots$ is written as $2.\dot{5}$ and $3.124124124\ldots$ as $3.\dot{1}2\dot{4}$.

In a decimal number if every digit after decimal point is repeating, then it is called a pure repeating fraction. Otherwise the fraction is called mixed. For example, $1.\dot{3}$ is a pure repeating fraction and $4.2351\dot{1}\dot{2}$ is a mixed repeating fraction.

If in the denominator there are digits other than 2, 5 as a factor, then no numerator will be divisible by that denominator. Since there cannot be any digit other than 1, 2, $\ldots$, 9 at some stage as residue, the same numbers will appear. Number of digits in the repeating part is always less than the number in the denominator.

Example 3. Express $\frac{3}{11}$ and $\frac{95}{37}$ into decimal fractions.

Solution: $\frac{3}{11}$ and $\frac{95}{37}$ have been converted into decimal fractions as below.

Actually 3 has been divided. But since 3 is smaller than 11, 0 has been placed in the dividend and a 0 has been placed to the right of 3 and made it 30

```
11)30(0.2727
   22
    80
     77
      30
      22
       80
       77
        3
```

```
37)95(2.56756
   74
   210
   185
    250
    222
     280
     259
      210
      185
       250
       222
        280
        259
         21
```

$\therefore \frac{3}{11} = 0.2727\ldots = 0.\dot{2}\dot{7}$

$\therefore \frac{95}{37} = 2.567567\ldots = 2.\dot{5}6\dot{7}$

Desired decimal fractions are respectively $0.\dot{2}\dot{7}$ and $2.\dot{5}6\dot{7}$

Converting repeating decimal fractions into common fractions

Example 4. Convert $0.\dot{3}$, $0.\dot{2}\dot{4}$, and $42.34\dot{7}\dot{8}$ into common fractions.

Solution: In the following, $0.\dot{3}$, $0.\dot{2}\dot{4}$ and $42.34\dot{7}\dot{8}$ have been cnverted into common fractions.

$$\begin{aligned}
\text{First} \qquad 0.\dot{3} &= 0.3333\ldots \\
0.\dot{3} \times 10 &= 0.333\ldots \times 10 = 3.333\ldots \\
0.\dot{3} \times 1 &= 0.333\ldots \times 1 = 0.333\ldots \\
\hline
\text{subtracting, } 0.\dot{3} \times 10 - 0.\dot{3} \times 1 &= 3 \\
0.\dot{3} \times (10 - 1) &= 3 \\
0.\dot{3} \times 9 &= 3 \\
\therefore 0.\dot{3} &= \frac{3}{9} = \frac{1}{3}
\end{aligned}$$

$$\begin{aligned}
\text{Now} \qquad 0.\dot{2}\dot{4} &= 0.24242424\ldots \\
0.\dot{2}\dot{4} \times 100 &= 0.242424\ldots \times 100 = 24.24242424\ldots \\
0.\dot{2}\dot{4} \times 1 &= 0.242424\ldots \times 1 = 0.24242424\ldots \\
\hline
\text{subtracting, } 0.\dot{2}\dot{4} \times 99 &= 24 \\
\therefore 0.\dot{2}\dot{4} &= \frac{24}{99} = \frac{8}{33}
\end{aligned}$$

$$\begin{aligned}
\text{Finally} \qquad 42.34\dot{7}\dot{8} &= 42.34787878\ldots \\
42.34\dot{7}\dot{8} \times 10000 &= 42.34787878\ldots \times 10000 = 423478.78787878\ldots \\
42.34\dot{7}\dot{8} \times 100 &= 42.34787878\ldots \times 100 = 4234.7878\ldots \\
\hline
\text{subtracting, } 42.34\dot{7}\dot{8} \times 9900 &= 423478 - 4234 = 419244 \\
\therefore 42.34\dot{7}\dot{8} &= \frac{419244}{9900} = \frac{34937}{825} = 42\frac{287}{825}
\end{aligned}$$

$\therefore$ desired common fractions are repsectively $0.\dot{3} = \frac{1}{3}$, $0.\dot{2}\dot{4} = \frac{8}{33}$, $42.34\dot{7}\dot{8} = 42\frac{287}{825}$

Explanation: From the three examples above it appears that,

- Repeating fraction is multiplied with a number equalling 1 followed by as many 0s as there are digits after decimal point of a repeating fraction.

- The repeating decimal fraction has been multiplied with a number equalling 1 followed by as many 0s as many nonrepeating digits are there in the repeating decimal fraction.
- Subtracting the second number from the first results in a whole number. It may be noted that nonrepeating part has been subtracted from the repeating decimal fraction by deleting decimal point and repeating point.
- The result of subtraction has been divided by a number equalling as many 9s as many repeating digits were there in the original repeating decimal fraction, followed by as many 0s as there were nonsepeating digits after decimal point.
- In converting repeating decimal fraction into common fraction we have denominator having as many 9s as many repeating digits followed by as many 0s as many nonrepeating digits after decimal point. Numerator equals a whole number we get after removing decimal and repeating points in the fraction less the number obtained by deleting the decimal point and all digits in the repeating part.

Remarks: Repeating decimal fractions can always be converted into common fractions. All repeating decimal fractions are rational numbers.

Example 5. Convert $5.23\dot{4}5\dot{7}$ into a common fraction.

Solution:

$$\begin{aligned}
5.23\dot{4}5\dot{7} &= \quad 5.23457457457\ldots \\
5.23\dot{4}5\dot{7} \times 100000 &= 523457.457457\ldots \\
5.23\dot{4}5\dot{7} \times 100 &= \quad 523.457457\ldots \\
\hline
\text{subtracting, } 5.23\dot{4}5\dot{7} \times 99900 &= 522934 \\
\therefore 5.23\dot{4}5\dot{7} &= \frac{522934}{99900} = \frac{261467}{49950} = 5\frac{11717}{49950}
\end{aligned}$$

$\therefore$ The required fraction is $5\frac{11717}{49950}$.

Explanation: Since there are 5 digits in the decimal part, the repeating decimal fraction has been multiplied by 100000 (5 0s after 1). Since there are two decimal digits before the repeating part (1 followed by 2 0s), repeating decimal fraction has been multiplied by 100. The second product has been subtracted from the first. On one side of subtraction is a whole number, and in the other the repeating decimal number multiplied by $(100000 - 1000) = 99900$. Both sides are divided by 99900 to and the common fraction is obtained.

Rules for converting repeating decimal fractions into common fractions: = the result by sub tracting

Numerator of the desired fraction the integer obtained by deleting the decimal point of the given decimal fraction and the integer formed by the nonrepeating portion . Denominator of the desired fraction = numbers formed by putting as many 9s as many digits in the repeating part followed by as many 0s in the nonrepeating part after the decimal point.

These rules have been applied in the following examples to convert a repeating decimal fractions into common fractions.

Example 6. Convert $45.2\dot{3}4\dot{6}$ into common fraction.

Solution: $45.2\dot{3}4\dot{6} = \frac{452346 - 452}{9990} = \frac{451894}{9990} = \frac{225947}{4995} = 45\frac{1172}{4995}$

$\therefore$ the desired fraction is $45\frac{1172}{4995}$.

Example 7. Convert $32.\dot{5}6\dot{7}$ into common fraction.

Solution: $32.\dot{5}6\dot{7} = \frac{32567 - 32}{999} = \frac{32535}{999} = \frac{3615}{111} = \frac{1205}{37} = 32\frac{21}{37}$

$\therefore$ the desired fraction is $32\frac{21}{37}$.

Work: Convert $0.\dot{4}\dot{1}$, $3.04\dot{6}2\dot{3}$, $0.0\dot{1}\dot{2}$ and $3.31\dot{2}\dot{4}$ into common fractions.

Similar and dissimilar repeating decimal fractions

If for two or more repeating decimal fractions number of digits both in repeating and nonrepeating parts are the same, then they are called similar repeating decimal fractions. Otherwise they are called dissimilar repeating decimal fractions. For example, $12.\dot{4}\dot{5}$ and $6.\dot{3}\dot{2}$; $9.45\dot{3}$ and $125.89\dot{7}$ are similar repeating decimal fractions. But, $0.3\dot{4}5\dot{6}$ and $7.45\dot{7}8\dot{9}$; $6.43\dot{5}\dot{7}$ and $2.89\dot{3}4\dot{5}$ are dissimilar repeating decimal fractions.

Converting dissimilar repeating decimal fractions into similar ones

If we write repeating part of a repeating decimal fraction, the value does not change. For example, $6.45\dot{3}\dot{7} = 6.45\dot{3}73\dot{7} = 6.453\dot{7}\dot{3} = 6.4537\dot{3}\dot{7}$. Here each one of them is a repeating decimal fraction $6.45373737\ldots$, which is an infinite decimal number. If we convert this repeating decimal number into common fraction, we will see that all of them are equal.

$$6.45\dot{3}\dot{7} = \frac{64537 - 645}{9900} = \frac{63892}{9900}$$
$$6.45\dot{3}73\dot{7} = \frac{6453737 - 645}{999900} = \frac{6453092}{999900} = \frac{63892}{9900}$$
$$6.4537\dot{3}\dot{7} = \frac{6453737 - 64537}{990000} = \frac{6389200}{990000} = \frac{63892}{9900}$$

In order to convert into similar repeating decimal fractions every repeating decimal must have equal number of digits in nonrepeating part. Repeating part must have number of digits equal to lowest common multiple of digits in all repeating parts.

Example 8. Convert $5.\dot{6}$, $7.3\dot{4}\dot{5}$, and $10.78\dot{4}2\dot{3}$ into similar repeating decimal fractions.

Solution: Repeating decimal fractions $5.\dot{6}$, $7.3\dot{4}\dot{5}$, and $10.78\dot{4}2\dot{3}$ have nonrepeating digits equalling 0, 1 and 2 respectively. Here $10.78\dot{4}2\dot{3}$ has the most number of digits in nonrepeating part, so in order to convert into similar repeating decimal fractions we must have 2 digits in nonrepeating part of every number. $5.\dot{6}$, $7.3\dot{4}\dot{5}$ and $10.78\dot{4}2\dot{3}$ in repeating decimal fractions have respectively 1, 2 and 3 digits in the repeating part. LCM of 1, 2 and 3 is 6. So in order to convert into similar repeating decimal fractions we must have 6 digits in the repating part of every number. So, $5.\dot{6} = 5.66\dot{6}6666\dot{6}$, $7.3\dot{4}\dot{5} = 7.34\dot{5}4545\dot{4}$ and $10.78\dot{4}2\dot{3} = 10.78\dot{4}2342\dot{3}$. Desired similar repeating decimal fractions are respectively $5.66\dot{6}6666\dot{6}$, $7.34\dot{5}4545\dot{4}$ and $10.78\dot{4}2342\dot{3}$.

Example 9. Convert 1.7643, $3.\dot{2}\dot{4}$ and $2.78\dot{3}4\dot{6}$ into similar repeating decimal fractions.

Solution: In 1.7643 we have 4 digits in the nonrepeating part and it does not have repeating part. In $3.\dot{2}\dot{4}$ number of digits in nonrepeating part is 0 and that of repeating part is 2; in $2.78\dot{3}4\dot{6}$ number of digits in nonrepeating part is 2 and that in repeating part is 3. Here the largest number of nonrepeating digits is 4 and numbers of digits in the repeating parts are 2 and 3, LCM of which is 6. So each number must have 4 nonrep 2 ating digits and 6 repeating digits.

$\therefore$ $1.7643 = 1.7643\dot{0}0000\dot{0}$, $3.\dot{2}\dot{4} = 3.2424\dot{2}4242\dot{4}$ $2.78\dot{3}4\dot{6} = 2.7834\dot{6}3463\dot{4}$

Desired similar repeating decimal fractions are respectively $1.7643\dot{0}0000\dot{0}$, $3.2424\dot{2}4242\dot{4}$ and $2.7834\dot{6}3463\dot{4}$

Remarks: In order to convert finite decimal fractions into similar decimal fractions we must add sufficient number of 0s at the end of the number to match with number of nonrepeating digits. In case of repeating decimal fractions for each number, numbers of nonrepeating digits have been made equal so should be the case with repating digits.

Work: Convert 3.467, $2.01\dot{2}\dot{4}\dot{3}$ and $7.52\dot{5}\dot{6}$ into similar repeating decimal fractions.

Addition and subtraction of repeating decimal fractions

In order to add or subtract repeating decimal fractions, we need to convert repeating decimal fractions into similar repeating decimals. Then addition or subtraction must be carried out as in case of finite decimal fractions. If addition or subtraction of finite and repeating decimal fractions together are done, in order to make repeating decimal factions similar, the number of digits of non¬repeating part of each repeating fraction should be equal to the number of digits between the numbers of digits after the decimal points of finite decimal fractions and that of the non-repeating parts of repeating decimal fraction. The number of digits of repeating part will be equal to L.C.M. as obtained by applying the rules and in case of finite decimal fractions, necessary numbers of zeros are to be used in its repeating parts. Then the same process of addition and subtraction is to be done following the rules of finite decimal fractions. The sum or the difference thus obtained will not be the actual one. It should be observed that in the process of addition of similar decimal fractions if any number is to be carried over after adding the digits at the extreme left of the repeating part of the decimal fractions, then that number is added to the sum obtained and thus the actual sum is found. In case of subtraction the number to be carried over is to subtract from the difference obtained and thus actual result is found. The sum or difference thus found is the required sum or difference.

Remarks:

1. The sum or difference of repeating decimal is also repeating decimal fractions. In this sum or difference the number of digits in the non-repeating part will be equal to the number of digits in the non- repeating part of that repeating decimal fractions, which have the highest number of digits in its non-repeating part. Similarly, the number of digits in the repeating part of the sum or the result of subtraction will be the equal to L.C.M. of the numbers of digits of repeating parts of repeating decimal fractions. If there are finite decimal fractions, the number of digits in the nonrepeating part of each repeating decimal fraction will be equal to the highest numbers of digits that occurs after the decimal point.

2. Converting the repeating decimal fractions into common fractions, addition and subtraction may be done according to the rules as used in case of fractions and the sum or difference is converted into decimal fractions. But this method of addition or subtraction needs more time.

Example 10. Add $3.\dot{8}\dot{9}$, $2.1\dot{7}\dot{8}$ and $5.89\dot{7}9\dot{8}$.

Solution: We have 2 digits in nonrepeating part, and repeating parts have respectively 2, 2 and 3 for which LCM is 6. At first all these three reepating decimal fractions have been converted into similar repeating decimal fractions.

$$\begin{array}{lr l}
3.\dot{8}\dot{9} & =3.89\dot{8}9898\dot{9} & \\
2.1\dot{7}\dot{8} & =2.17\dot{8}7878\dot{7} & \\
5.89\dot{7}9\dot{8} & =5.89\dot{7}9879\dot{8} & \\
\hline
 & 11.97576574 & [8+8+7+2=25, \text{ here 2 is 2 in hand} \\
 & +2 & \text{here 2 of 25 has been added]} \\
\hline
 & 11.97\dot{5}7657\dot{6} &
\end{array}$$

The desired result is $11.97\dot{5}7657\dot{6}$ or, $11.97\dot{5}7\dot{6}$

Remarks: In the result 576576 is repeating part. But if we assume 576 in the repeating part, then there will be no change of value.

Note: In order to clarify addition at the rightmost digit we have solved the addition in a different way:

$$\begin{array}{lr}
3.\dot{8}\dot{9} & =3.89\dot{8}9898\dot{9}|89 \\
2.1\dot{7}\dot{8} & =2.17\dot{8}7878\dot{7}|87 \\
5.89\dot{7}9\dot{8} & =5.89\dot{7}9879\dot{8}|79 \\
\hline
 & 11.97\dot{5}7657\dot{6}|55
\end{array}$$

Here after the end of repe ating part some more digits have been taken. These extra digits have been separated by a vertical line. Then the addition has been performed. From the addition of digits just after the vertical line 2 in hands has been added to the digits just before the vertical line. It may be noted here the digit on the right side of the vertical line is the same as the first digit of repeatation. Therefore both the sums ore the same.

Example 11. Add $8.9\dot{4}7\dot{8}$, 2.346 and $4.\dot{7}\dot{1}$.

Solution: In order to make repeating decimal fractions similar we must have 3 nonrepeating digits and repeating part must have digits equal to LCM of 2 and 3, that is 6. Now decimal fractions will be added.

$$\begin{array}{lr l}
8.9\dot{4}7\dot{8} & =8.947\dot{8}4784\dot{7} & \\
2.346 & =2.346\dot{0}0000\dot{0} & \\
4.\dot{7}\dot{1} & =4.717\dot{1}7171\dot{7} & \\
\hline
 & 16.011019564 & [8+0+1+1=10, \text{ Here 1 is 1 in hand} \\
 & +1 & \text{Here 1 of 10 has been added]} \\
\hline
 & 16.011\dot{0}1956\dot{5} &
\end{array}$$

The desired sum is $16.011\dot{0}1956\dot{5}$

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Work: Add 1): $2.0\dot{9}\dot{7}$ and $5.12\dot{7}6\dot{8}$ 2) $1.34\dot{5}$, $0.31\dot{5}7\dot{6}$ and $8.05\dot{6}7\dot{8}$

Example 12. Subtract $5.24\dot{6}7\dot{3}$ from $8.2\dot{4}\dot{3}$.

Solution: Here number of digits in nonrepeating part should be 2 and that in repeating part should be 6, LCM of 2 and 3. Now the numbers have been converted into similar numbers, and subtraction has been performed as follows:

$$\begin{array}{lr}
8.2\dot{4}\dot{3} & = 8.24\dot{3}43434\dot{4} \\
5.24\dot{6}7\dot{3} & = 5.24\dot{6}7367\dot{3} \\
\hline
 & 2.99669761 \\
 & -1 \\
\hline
 & 2.99\dot{6}6976\dot{0}
\end{array}$$
[subtracting 6 from 3 results in 1 in hand]

The desired result is $2.99\dot{6}6976\dot{0}$.

Remarks: If the minuend starting at the repeating dot is smaller than the subtrahend, we must subtract 1 from the rightmost digit.

Note: In the following is explained why 1 is subtracted from the rightmost digit in different way.

$$\begin{array}{lr}
8.2\dot{4}\dot{3} & = 8.24\dot{3}43434\dot{4}|34 \\
5.24\dot{6}7\dot{3} & = 5.24\dot{6}7367\dot{3}|67 \\
\hline
 & 2.99\dot{6}6976\dot{0}|67
\end{array}$$

The desired result is $2.99\dot{6}6976\dot{0}|67$. We have obtained the same results in both the cases.

Example 13. Subtract $16.\dot{4}3\dot{7}$ from $24.45\dot{6}4\dot{5}$.

Solution:

$$\begin{array}{lr}
24.45\dot{6}4\dot{5} & = 24.45\dot{6}4\dot{5} \\
16.\dot{4}3\dot{7} & = 16.43\dot{7}4\dot{3} \\
\hline
 & 8.01902 \\
 & -1 \\
\hline
 & 8.01\dot{9}0\dot{1}
\end{array}$$
[if 7 is subtracted from 6, we have 1 in hand]

The desired result is $8.01\dot{9}0\dot{1}$

Note: In the following is explained why 1 is subtracted from the rightmost digit in different way.

$$\begin{array}{lr}
24.45\dot{6}4\dot{5} & = 24.45\dot{6}4\dot{5}|64 \\
16.\dot{4}3\dot{7} & = 16.43\dot{7}4\dot{3}|74 \\
\hline
 & 8.01\dot{9}0\dot{1}|90
\end{array}$$

Work: Subtract: 1) Subtract 10.418 from $13.12\dot{7}8\dot{4}$

2) Subtract $9.12\dot{6}4\dot{5}$ from $23.03\dot{9}\dot{4}$

Multiplication and division of repeating decimal fractions

If repeating decimal fractions are first converted into common fractions and then multiplication or division operations are performed, we can convert the result into repeating decimal fractions. If finite decimal fractions and repeating decimal fractions are multiplied or divided, then this rule should be followed. However, in case of division if both divisor and dividend are repeating decimal fractions, then converting them into similar repeating decimal fractions will ease the operations.

Example 14. Multiply $4.\dot{3}$ with $5.\dot{7}$.

Solution:

$$4.\dot{3} = \frac{43-4}{9} = \frac{39}{9} = \frac{13}{3}$$

$$5.\dot{7} = \frac{57-5}{9} = \frac{52}{9}$$

$$\therefore 4.\dot{3} \times 5.\dot{7} = \frac{13}{3} \times \frac{52}{9} = \frac{676}{27} = 25.\dot{0}3\dot{7}$$

The desired result is $25.\dot{0}3\dot{7}$.

Example 15. Multiply $0.2\dot{8}$ with $42.\dot{1}\dot{8}$.

Solution:

$$0.2\dot{8} = \frac{28-2}{90} = \frac{26}{90} = \frac{13}{45}$$

$$42.\dot{1}\dot{8} = \frac{4218-42}{99} = \frac{4176}{99} = \frac{464}{11}$$

$$\therefore\ 0.2\dot{8} \times 42.\dot{1}\dot{8} = \frac{13}{45} \times \frac{464}{11} = \frac{6032}{495} = 12.1\dot{8}\dot{5}$$

The desired result is $12.1\dot{8}\dot{5}$.

Example 16. How much is $2.5 \times 4.3\dot{5} \times 1.2\dot{3}\dot{4}$?

Solution:

$$2.5 = \frac{25}{10} = \frac{5}{2}$$

$$4.3\dot{5} = \frac{435-43}{90} = \frac{392}{90}$$

$$1.2\dot{3}\dot{4} = \frac{1234-12}{990} = \frac{1222}{990} = \frac{611}{495}$$

$$\therefore 2.5 \times 4.3\dot{5} \times 1.2\dot{3}\dot{4} = \frac{5}{2} \times \frac{392}{90} \times \frac{611}{495} = \frac{119756}{8910} = 13.440628\ldots$$

The desired result is 13.440628 (approx.)

> **Work:** 1) Multiply $1.1\dot{3}$ with 2.6. 2) How much is $0.\dot{2} \times 1.\dot{1}\dot{2} \times 0.0\dot{8}\dot{1}$?

Example 17. Divide $7.\dot{3}\dot{2}$ by $0.2\dot{7}$.

Solution:

$$7.\dot{3}\dot{2} = \frac{732-7}{99} = \frac{725}{99}$$

$$0.2\dot{7} = \frac{27-2}{90} = \frac{25}{90} = \frac{5}{18}$$

$$\therefore 7.\dot{3}\dot{2} \div 0.2\dot{7} = \frac{725}{99} \div \frac{5}{18} = \frac{725}{99} \times \frac{18}{5} = \frac{290}{11} = 26.\dot{3}\dot{6}$$

The desired result is $26.\dot{3}\dot{6}$.

Example 18. Divide $2.\dot{2}71\dot{8}$ by $1.9\dot{1}\dot{2}$.

Solution:

$$2.\dot{2}71\dot{8} = \frac{22718-2}{9999} = \frac{22716}{9999}$$

$$1.9\dot{1}\dot{2} = \frac{1912-19}{990} = \frac{1893}{990}$$

$$\therefore 2.\dot{2}71\dot{8} \div 1.9\dot{1}\dot{2} = \frac{22716}{9999} \div \frac{1893}{990} = \frac{22716}{9999} \times \frac{990}{1893} = \frac{120}{101} = 1.\dot{1}88\dot{1}$$

The desired result is $1.\dot{1}88\dot{1}$.

Example 19. Divide 9.45 by $2.8\dot{6}\dot{3}$.

Solution:

$$9.45 = \frac{945}{100} \qquad\qquad 2.8\dot{6}\dot{3} = \frac{2863 - 28}{990} = \frac{2835}{990}$$

$$\therefore 9.45 \div 2.8\dot{6}\dot{3} = \frac{945}{100} \div \frac{2835}{990} = \frac{945}{100} \times \frac{990}{2835} = \frac{189 \times 99}{2 \times 2835} = \frac{33}{10} = 3.3$$

The desired result is 3.3

Remarks: Multiplication and division of repeating decimal fractions may not be repeating decimal fractions.

Work: 1) Divide $0.\dot{6}$ by $0.\dot{9}$. 2) Divide $0.7\dot{3}\dot{2}$ by $0.0\dot{2}\dot{7}$.

Infinite decimal fractions

There are many decimal numbers that have infinitely many digits after decimal point. Moreover, a digit or a number of digits do not repeat as well. These are called infinite decimal fractions. For example, 5.134248513942301 . . . is an infinite decimal fraction. Square root of 2 is an infinite decimal fraction. Let us calculate square root of 2.

```
1 ) 2 ( 1.4142135...
    1
24 ) 100
      96
281 )  400
       281
2824 ) 11900
       11296
28282 )  60400
         56564
282841 )  383600
          282841
2828423 ) 10075900
           8485269
28284265 ) 159063100
           141421325
            17641775
```

In this way the procedure may continue indefinitely. So $\sqrt{2} = 1.4142135\ldots$ is an infinite decimal fraction.

Value and approximate values up to certain decimal point

It is not the same to calculate an infinite decimal fraction up to decimal places or approximate it to certain decimal places. For example, $5.4325893\ldots$ when calculated up to 4 decimal places will give 5.4325 whereas its approximate value up to 4 decimal places will give us 5.4326. However, these two values are same if we calculate up to 2 decimal places. We can do the same for finite decimal fractions as well.

Remarks: For finding values up to certain decimal places we must write down those digits exactly as they are. However, for finding approximation up to certain decimal places we need to check the next digit. If that digit is 5 or more, 1 should be added to the last position. Otherwise it will be kept in tact.

Example 20. Find square root of 13, and write down its approximate value up to 3 decimal places.

Solution:

```
3 ) 13 ( 3.605551...
     9
66 ) 400
     396
7205 ) 40000
        36025
72105 ) 397500
         360525
721105 ) 3697500
          3605525
7211101 )  9197500
            7211101
            1986399
```

$\therefore$ the desired square root is $3.605551\ldots$, and desired approximation up to 3 decimal places is 3.606

Example 21. Calculate values and approximate values of 4.4623845... up to $1, 2, 3, 4$ and 5 decimal places ?

Solution: For the decimal fraction 4.4623845...

Value up to 1 decimal place is 4.4 but its approximate value up to 1 decimal digit is 4.5

Value up to 2 decimal places is 4.46 and approximate value is 4.46

Value up to 3 decimal places is 4.462 and approximate value is 4.462

Value up to 4 decimal places is 4.4623 and approximate value is 4.4624

Value up to 5 decimal places is 4.462238 and approximate value is 4.46238

> **Work:** Calculate square root of 29 to 2 decimal places, and approximate it to 2 decimal places.

Exercises 1.1

1. Which numbers are irrational?

   1) $0.\dot{3}$ 2) $\sqrt{\dfrac{16}{9}}$ 3) $\sqrt[3]{\dfrac{8}{27}}$ 4) $\dfrac{5}{\sqrt{3}}$

2. If a, b, c, d are four consecutive natural numbers which one of the following will be a whole squared?

   1) $abcd$ 2) $ab + cd$ 3) $abcd + 1$ 4) $abcd - 1$

3. How many primes are there from 1 to 10 ?

   1) 3   2) 4   3) 5   4) 6

4. Which one is the set of all integers?

   1) $\{\ldots, -4,\ -2,\ 0,\ 2,\ 4, \ldots\}$ 2) $\{\ldots, -2,\ -1,\ 0,\ 1,\ 2, \ldots\}$

   3) $\{\ldots, -3,\ -1\ 0,\ 1,\ 3, \ldots\}$ 4) $\{0,\ 1\ 2,\ 3,\ 4\}$

5. In case of real numbers

   (*i*) Square of an odd integer is odd.

   (*ii*) Product of two even numbers is even.

   (*iii*) Square root of a number that is not whole squared is an irrational number.

   Which one is true?

1) *i* and *ii* 2) *i* and *iii* 3) *ii* and *iii* 4) *i*, *ii* and *iii*

6. Product of three consecutive numbers will always be divisible by which of the following numbers?
   1) 5 2) 6 3) 7 4) 11
7. If a and b are two consecutive even numbers, then which of the following numbers is odd?
   1) a^2 2) b^2 3) a^2+1 4) b^2+2
8. If a and b are two integers, then what should be added to a^2+b^2 to obtain a whole squared ?
   1) $-ab$ 2) ab 3) $2ab$ 4) ab
9. Prove that the following numbers are irrational. 1) $\sqrt{5}$ 2) $\sqrt{7}$ 3) $\sqrt{10}$
10. 1) Find two irrational numbers between 0.31 and 0.12.
    2) Find a rational and an irrational number between $\frac{1}{\sqrt{2}}$ and $\sqrt{2}$.
11. 1) Prove that square of an odd integer is an odd integer.
    2) Prove that product of any two consecutive even integers is divisible by 8.
12. Convert the following into repating decimal fractions:
    1) $\frac{1}{6}$ 2) $\frac{7}{11}$ 3) $3\frac{2}{9}$ 4) $3\frac{8}{15}$
13. Convert into simple fractions;
    1) $0.\dot{2}$ 2) $0.\dot{3}\dot{5}$ 3) $0.1\dot{3}$ 4) $3.7\dot{8}$ 5) $6.2\dot{3}0\dot{9}$
14. Express the follwoing into similar repeating decimal fractions:
    1) $2.2\dot{3},\ 5.2\dot{3}\dot{5}$ 2) $7.2\dot{6},\ 4.23\dot{7}$
    3) $5.\dot{7},\ 8.\dot{3}\dot{4},\ 6.\dot{2}4\dot{5}$ 4) $12.32,\ 2.1\dot{9},\ 4.32\dot{5}\dot{6}$
15. Add;
    1) $0.4\dot{5}+0.1\dot{3}\dot{4}$ 2) $2.0\dot{5}+8.0\dot{4}+7.018$ 3) $0.00\dot{6}+0.\dot{9}\dot{2}+0.\dot{1}3\dot{4}$
16. Subtract:
    1) $3.\dot{4}-2.1\dot{3}$ 2) $5.\dot{1}\dot{2}-3.4\dot{5}$
    3) $8.49-5.3\dot{5}\dot{6}$ 4) $19.34\dot{5}-13.2\dot{3}4\dot{9}$
17. Multiply:
    1) $0.\dot{3}\times 0.\dot{6}$ 2) $2.\dot{4}\times 0.\dot{8}\dot{1}$ 3) $0.6\dot{2}\times 0.\dot{3}$ 4) $42.\dot{1}\dot{8}\times 0.2\dot{8}$

18. Divide:

1) $0.\dot{3} \div 0.\dot{6}$ 2) $0.3\dot{5} \div 1.\dot{7}$ 3) $2.3\dot{7} \div 0.4\dot{5}$ 4) $1.\dot{1}8\dot{5} \div 0.\dot{2}\dot{4}$

19. Wrtite down values and approximate values up to 4 decimal places :

1) 12 2) $0.\dot{2}\dot{5}$ 3) $1.\dot{3}\dot{4}$ 4) $5.1\dot{3}0\dot{2}$

20. Write down which of the following numbers are rational and irrational:

1) $0.\dot{4}$ 2) $\sqrt{9}$ 3) $\sqrt{11}$ 4) $\dfrac{\sqrt{6}}{3}$

5) $\dfrac{\sqrt{8}}{\sqrt{7}}$ 6) $\dfrac{\sqrt{27}}{\sqrt{48}}$ 7) $\frac{\frac{2}{3}}{\frac{3}{7}}$ 8) $5.\dot{6}3\dot{9}$

21. Let $n = 2x - 1$, where $x \in N$. Prove that n^2 when divided by 8 gives 1 as remainder.

22. $\sqrt{5}$ and 4 are two real numbers.

1) Specify which one is rational and which one is irrational.

2) Find two irrational numbers between $\sqrt{5}$ and 4.

3) Prove that $\sqrt{5}$ is an irrational number.

23. Simplify:

1) $(0.\dot{3} \times 0.8\dot{3}) \div (0.5 \times 0.\dot{1}) + 0.3\dot{5} \div 0.0\dot{8}$

2) $[(6.27 \times 0.5) \div \{(0.5 \times 0.75) \times 8.36\}] \div$
$\{(0.25 \times 0.1) \times (0.75 \times 21.\dot{3}) \times 0.5\}$

Chapter 2

Sets and Functions

The term 'set' is well known to us. For example, dinner set, set of natural numbers, set of rational numbers etc. As a modern concept set has versatile use. The German mathematician George Cantor (1845-1918) first explained the concept of set.He introduced the notion of infinite set, and created a sensation in mathematical science. His notion is now known as set theory. In this chapter we shall solve different problems by using concepts of set, diagrams and logic and idea of a function will also be introduced.

After completion of this chapter the learners will —

- be able to express the concepts of sets and subsets through symbols.
- be able to describe how sets can be expressed.
- be able to define infinite sets, and distinguish between finite and infinite sets.
- be able to define and verify union and intersection of sets .
- be able to define power sets, and construct power sets of sets with two or three elements.
- be able to define ordered pairs and cartesian products.
- be able to prove simple set theoretic rules by using examples and Venn diagrams, and apply these rules for solving different problems.
- be able to define relations and functions, and construct them.
- be able to determine domain and range.
- be able to draw diagrams for functions.

Sets

A set is simply a well organized object of real or imaginary world. For example, the set of three text books of Bangla, English and Mathematics. Some other examples are the set of first 10 odd natural numbers, set of integers, set of real numbers etc. Sets are usually denoted by the capital letters of English alphabet like $A, B, C, \ldots X, Y, Z$.

For example, the set of three numbers $2, 4, 6$ can be denoted by $A = \{2, 4, 6\}$

Each object or member of the set is called its element. For example, if $B = \{a, b\}$, to express that a and b are its elements the sign $\in$ is used to express the element. a and b are elements of B;
$\therefore a \in B$ and we can read as a, is a member of B (a belongs to B)

$b \in B$ is a member of b, (b belongs to B)

The above set B does not contain c. So it is expressed as $c \notin B$ and is read as c is not a member of B (c does not belong to B)

Methods for expressing sets

Sets can be expressed using two methods, namely Roster method or tabular method and Set Builder Method.

Roster method: In this method all members are listed and put inside {} and if there are more than one element, then elements are separated by commas. For example, $A = \{a, b\}$, $B = \{2, 4, 6\}$, $C = \{$ Niloy, Shuvro, Tisha $\}$ etc.

Set builder method: In this method all elements are not listed rather their common property is described. For example, $A = \{x : x$ is an odd natural number $\}$, $B = \{x : x$ is the first five students of class IX $\}$ etc. Here, by ':' has been expressed such that. Since in this method element of the set is specified by rules, this method is called **Rule Method**.

Example 1. Express $A = \{7, 14, 21, 28\}$ in set building method.

Solution: The elements of A are $7, 14, 21, 28$.

Here, each element is a multiple of 7 and not exceeding 28.

$\therefore A = \{x : x|,$ is a multiple of 7 and $0 < x \leq 28\}$

Example 2. Express $B = \{x : x,$ is a divisor of $28\}$ by using roster method.

Solution: Here, $28 = 1 \times 28 = 2 \times 14 = 4 \times 7$

$\therefore$ factors of 28 are $1, 2, 4, 7, 14, 28$

2020 Hence the desired set is $B = \{1, 2, 4, 7, 14, 28\}$

Example 3. Express $C = \{x : x \text{ positive integer and } x^2 < 18\}$ by using roster method.

Solution: $1, 2, 3, 4, 5, \ldots$ are positive integers.

Here, if

$x = 1$, then $x^2 = 1^2 = 1$; if $x = 2$, then $x^2 = 2^2 = 4$

If $x = 3$, then $x^2 = 3^2 = 9$; if $x = 4$, then $x^2 = 4^2 = 16$

If $x = 5$, then $x^2 = 5^2 = 25$; which is greater than 18.

$\therefore$ as per conditions the acceptable positive integers are $1, 2, 3$ and 4.

$\therefore$ the given set is $C = \{1, 2, 3, 4\}$

Work: Express

1) $C = \{-9, -6, -3, 3, 6, 9\}$ by using set building method.
2) Express $B = \{y : y \text{ integer and } y^3 \leq 18\}$ by using set building method.

Finite Set

The set number of elements of which can be determined by counting is called finite set. For example, $D = \{x, y, z\}$, $E = \{3, 6, 9, \ldots, 60\}$, $F = \{x : x \text{ prime and } 30 < x < 70\}$ etc are finite sets. Here, set D contains 3 elements, set E 20 elements and set F 9 elements.

Infinite Set

The set elements of which cannot be exhausted by counting is called infinite set. For example, $A = \{x : x \text{ odd natural number}\}$, set of natural numbers $N = \{1, 2, 3, 4, \ldots\}$, set of integers $Z = \{\ldots, -3, -2, -1, 0, 1, 2, 3 \ldots\}$, set of rational numbers $Q = \left\{\frac{a}{b} : a \text{ and } b \text{ integer and } b \neq 0\right\}$, set of rational numbers R etc are infinite sets.

Example 4. Prove that the set of all natural numbers is an infinite set.

Solution: The set of natural numbers $N = \{1, 2, 3, 4, 5, 6, 7, 8, \ldots\}$

The set $A = \{1, 3, 5, 7, \ldots\}$ consisting of odd natural numbers from the set N

The set $B = \{2, 4, 6, 8, \ldots\}$ consisting of even natural numbers from the set N

The set of numbers $C = \{3, 6, 9, 12, \ldots\}$ multiple of 3 from N

Here, sets consisting of elements from N cannot be determined by counting. As a result A, B, C are infinite sets.

$\therefore N$ is an infinite set.

Work: Determine which ones are finite and which ones are infinite sets:

1) $\{3, 5, 7\}$
2) $\{1, 2, 2^2, \ldots 2^{10}\}$
3) $\{3, 3^2, 3^3, \ldots\}$
4) $\{x : x \text{ integer and } x < 4\}$
5) $\{\frac{p}{q} : p \text{ and } q \text{ are mutually coprimes and } q > 1\}$
6) $\{y : y \in N \text{ and } y^2 < 100 < y^3\}$

Empty Set

If a set does not have any element, the set is said to be empty set, and is denoted by $\varnothing$. For example, the set of three male students of Holy cross school, $\{x \in N : 10 < x < 11\}$, $\{x \in N : x \text{ prime and } 23 < x < 29\}$ etc.

Venn-Diagram

John Venn (1834-1923) first expressed sets by using pictures. Here every set is represented by a geometric figure like rectangles, cicrles and triangles in a plane. These pictures have been named after John Venn.

Subset

$A = \{a, b\}$ is a set. Sets $\{a, b\}$, $\{a\}$, $\{b\}$ can be formed from the elements of this set. Again, we can form a subset $\varnothing$ with no elements at all. These sets $\{a, b\}$, $\{a\}$, $\{b\}$, $\varnothing$ are subsets of set A. All the sets that can be formed from a set are called subsets of the original set. Subset is denoted by $\subseteq$. If B is a subset of A, then we write $B \subseteq A$. B is a subset of A. Among the subsets above $\{a, b\}$ is equal to set A. Each set is the subset of itself. again $\varnothing$ in formed from any set. $\therefore \varnothing$ is a subset of any set.

Let $P = \{1, 2, 3\}$ and $Q = \{2, 3\}$, $R = \{1, 3\}$,then P, Q and R are subsets of P. That is, $P \subseteq P$, $Q \subseteq P$ and $R \subseteq P$.

Proper Subset

Of the subsets constructed from a set, the ones that have lesser number of elements than the original set is called proper set. For example, $A = \{3, 4, 5, 6\}$ and $B = \{3, 5\}$ are two sets. Here all elements of B are also elements of A, and number of elements in B is less than that of A.

$\therefore$ B is a proper subset of A and is expressed as $B \subset A$.

In the examples of subsets Q and R are each proper subset of P. It may be mentioned here that $\varnothing$ is a proper subset of any set.

Example 5. Write down the subsets of $P = \{x, y, z\}$ and identify those that are proper subsets.

Solution: Given, $P = \{x, y, z\}$

Subsets of P are $\{x, y, z\}, \{x, y\}, \{x, z\}, \{y, z\}, \{x\}, \{y\}, \{z\}, \varnothing$.

Proper subsets of P are $\{x, y\}, \{x, z\}, \{y, z\}, \{x\}, \{y\}, \{z\}, \varnothing$

Observation: If number of elements of a set is n, then it has 2^n subsets, and number of proper subsets is $2^n - 1$.

Equivalent Set

If two sets have the same elements, then they are said to be equivalent sets. For example, $A = \{3, 5, 7\}$ and $B = \{5, 3, 3, 7\}$ are two equivalent sets, then they are denoted by $A = B$. Note that $A = B$ if and only if $A \subseteq B$ and $B \subseteq A$

Again, if $A = \{3, 5, 7\}$, $B = \{5, 3, 3, 7\}$ and $C = \{7, 7, 3, 5, 5\}$, then sets A, B and C are equivalent sets. That is $A = B = C$

Observation: If order of elements in the set are changed or certain element is repeated set is not changed.

Difference of Sets

Let $A = \{1, 2, 3, 4, 5\}$ and $B = \{3, 5\}$. If we delete the elements of set B from that of A, then we get $\{1, 2, 4\}$ and is denoted by $A \setminus B$ or $A - B$ and is read A delete B.

$\therefore A - B = \{1, 2, 3, 4, 5\} - \{3, 5\} = \{1, 2, 4\}$

Example 6. If $P = \{x : x$ is a factor of 12 $\}$ and $Q = \{x : x$ is a multiple of 3 and $x \leq 12\}$, then determine $P - Q$.

Solution: Given , $P = \{x : x \text{ is a factor of } 12\ \}$

Here, factors of 12 are $1, 2, 3, 4, 6, 12$

$\therefore P = \{1, 2, 3, 4, 6, 12\}$

Again, $Q = \{x : 3 \text{ is a factor of } x \text{ and } x \leq 12\}$

Here, multiples of 3 up to 12 are $3, 6, 9, 12$

$\therefore Q = \{3, 6, 9, 12\}$

$\therefore P - Q = \{1, 2, 3, 4, 6, 12\} - \{3, 6, 9, 12\} = \{1, 2, 4\}$

The required set is: $\{1, 2, 4\}$

Universal Set

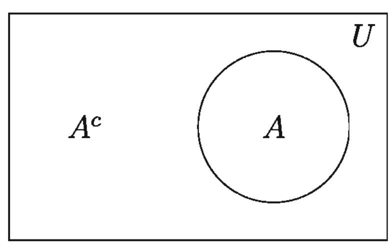
All sets under discussion are subsets of a definite set. For example: the set $A = \{x, y\}$ is a subset of the set $B = \{x, y, z\}$ Here, B set is called as universal set with respect to set A.

So if all sets under discussion are subsets of a given set, then the later set is said to be **universal set** with respect to sets under discussion.

Usually universal set is denoted by U. However, other symbols can also be used to denote universal set. For example, if set of all even natural numbers is $C = \{2, 4, 6, \ldots\}$ and set of all natural numbers is $N = \{1, 2, 3, 4, 5, 6, \ldots\}$, then N will be universal set with respect to the set C.

Complement of a Set

Let U be universal set and A be a subset of U. Then the set of elements outside of A is termed as **complementary set** of A. The complementary set of A is denoted by A^c or A'. Mathematically, $A^c = U \setminus A$.



Let, P and Q be two sets. Then the elements of P that are not elements of Q are said to form the complementary set of P with respect to Q, and is written $Q^c = P \setminus Q$

Example 7. If $U = \{1, 2, 3, 4, 5, 6, 7\}$, $A = \{2, 4, 6, 7\}$ and $B = \{1, 3, 5\}$, then determine A^c and B^c.

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Solution: $A^c = U \setminus A = \{1,2,3,4,5,6,7\} \setminus \{2,4,6,7\} = \{1,3,5\}$

and $B^c = U \setminus B = \{1,2,3,4,5,6,7\} \setminus \{1,3,5\} = \{2,4,6,7\}$

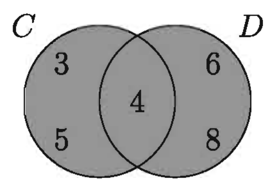
The required sets are $A^c = \{1,3,5\}$ and $B^c = \{2,4,6,7\}$

Union of Sets

The set consisting of elements of two or more sets is said to be **union of sets.** Let, A and B be two sets. The union of sets A and B is denoted by $A \cup B$ and read as A union B. In set building method $A \cup B = \{x : x \in A \text{ or } x \in B\}$

Example 8. If $C = \{3,4,5\}$ and $D = \{4,6,8\}$, determine $C \cup D$.

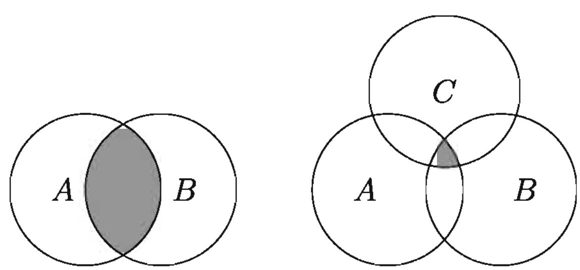
Solution: Given $C = \{3,4,5\}$
and $D = \{4,6,8\}$
$\therefore C \cup D = \{3,4,5\} \cup \{4,6,8\} = \{3,4,5,6,8\}$
The set to be determined is: $\{3,4,5,6,8\}$



$C \cup D$

Intersection of Sets.

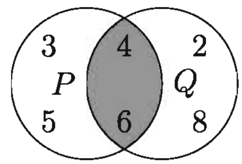
The set consisting of common elements of two or more sets is known as **intersection set** Let A and B be two sets. The intersection of sets A and B is denoted by $A \cap B$ and read as A intersection B. In set building notation $A \cap B = \{x : x \in A \text{ and } x \in B\}$



$A \cap B$ $\qquad$ $A \cap B \cap C$

Example 9. If $P = \{x \in N : 2 < x \leq 6\}$ and $Q = \{x \in N : x \text{ is even and } x \leq 8\}$, determine $P \cap Q$.

Solution: Given,
$P = \{x \in N : 2 < x \leq 6\} = \{3, 4, 5, 6\}$
$Q = \{x \in N : x \text{ is even and } x \leq 8\} = \{2, 4, 6, 8\}$
$\therefore P \cap Q = \{3, 4, 5, 6\} \cap \{2, 4, 6, 8\} = \{4, 6\}$
The set to be determined is $\{4, 6\}$



$P \cap Q$

Disjoint Set

If there is no element common to any two sets, then these two sets are called **disjoint sets.** Let A and B be two sets. If $A \cap B = \varnothing$ then A and B are said to be mutually disjoint sets.

> **Work:** If $U = \{1, 3, 5, 9, 7, 11\}$, $E = \{1, 5, 9\}$ and $F = \{3, 7, 11\}$, then determine $E^c \cup F^c$ and $E^c \cap F^c$.

Power Sets

$A = \{m, n\}$ is a set. Subsets of A are $\{m, n\}, \{m\}, \{n\}, \varnothing$, and set of all sets $\{\{m, n\}, \{m\}, \{n\}, \varnothing\}$ of A is called **power set** of A. Power set of A is denoted by $P(A)$. So the set formed by all subsets of a set is called power set of a given set.

Example 10. Find the number of elements of the power sets $A = \varnothing$, $B = \{a\}$, $C = \{a, b\}$.

Solution: Here $P(A) = \{\varnothing\}$

$\therefore$ number of elements of A is 0 and number of elements of its power set is $= 1 = 2^0$

Again $P(B) = \{\{a\}, \varnothing\}$

$\therefore$ number of elements of B is 1, and number of eleemnts of its power set is $= 2 = 2^1$.

and $P(C) = \{\{a\}, \{b\}, \{a, b\}, \varnothing\}$

$\therefore$ number of elements of C is 2 and number of elements of its power set is $= 4 = 2^2$

So, if number of elements of a set is n, then number of elements of its power set is 2^n.

Work: If $G = \{1, 2, 3\}$,then determine $P(G)$. Show that number of elements of $P(G)$ is 2^3

Ordered Pair

Amena and Sumena of class VIII occupied 1st and 2nd position repectively in the annual examination. According to merit this can be expressed as (Amena, Sumena) pair. Similar definite pairs are called ordered pairs.

So, for a pair of elements if the positions are given and expressed, then the pair is called **Ordered Pair.**

If the first element of an ordered pair is x, and the second element is y, then the ordered pair is denoted by (x, y). For the ordered pairs (x, y) and (a, b) will the equal or $(x, y) = (a, b)$ if and only if $x = a$ and $y = b$.

Example 11. If $(2x + y, 3) = (6, x - y)$ then,determine (x, y).

Solution: Given,$(2x + y, 3) = (6, x - y)$

As per conditions of ordered pairs,

$2x + y = 6 \cdots\cdots (1)$

$x - y = 3 \cdots\cdots (2)$

By adding equations (1) and (2) we get $3x = 9$ or $x = 3$

Putting value of x in equation (1) we get, $6 + y = 6$ or $y = 0$

$\therefore (x, y) = (3, 0)$

(Cartesian Product)

Mr Karim decided to colour interior wall of his rooms using white or blue colour, and outside wall using red, yellow or green. The set of colours of interior wall is $A =$ {white, blue} and that of outside wall is $B =$ {red, yellow and green } Mr Karim can colour walls of his rooms by (white, red), (while, yellow), (white, green), (blue, red), (blue, yellow), (blue, green) ordered pairs. The above ordered pairs are writtena s below:

$A \times B =$ {(white, red), (while, yellow), (white, green), (blue, red), (blue, yellow), (blue, green)}

The above set of ordered pairs is called **Cartesian Product set.**

In set building method, $A \times B = \{(x, y) : x \in A \text{ and } y \in B\}$

$A \times B$ is read as A cross B.

Example 12. If $P = \{1, 2, 3\}$, $Q = \{3, 4\}$, $R = P \cap Q$, then determine $P \times R$ and $R \times Q$.

Solution: Given, $P = \{1, 2, 3\}$, $Q = \{3, 4\}$

and $R = P \cap Q = \{1, 2, 3\} \cap \{3, 4\} = \{3\}$

$\therefore P \times R = \{1, 2, 3\} \times \{3\} = \{(1, 3), (2, 3), (3, 3)\}$

and $R \times Q = \{3\} \times \{3, 4\} = \{(3, 3), (3, 4)\}$

Work:

1) If $\left(\frac{x}{2} + \frac{y}{3}, 1\right) = \left(1, \frac{x}{3} + \frac{y}{2}\right)$, then determine (x, y).
2) If $P = \{1, 2, 3\}$, $Q = \{3, 4\}$ and $R = \{x, y\}$, then determine $(P \cup Q) \times R$ and $(P \cap Q) \times Q$.

Example 13. Determine the set of natural numbers that have the same residue 23 when they divide 311 and 419.

Solution: The natural numbers that leave 23 as residue when they divide 311 and 419 will be bigger than 23 and will be common factors of both $311 - 23 = 288$ and $419 - 23 = 396$.

Let the set of factors of 288 are larger than 23 be A.

Here $288 = 1 \times 288 = 2 \times 144 = 3 \times 96 = 4 \times 72 = 6 \times 48 = 8 \times 36 = 9 \times 32 = 12 \times 24 = 16 \times 18$

$\therefore A = \{24, 32, 36, 48, 72, 96, 144, 288\}$

Let the set of factors of 396 greater than 23 be B.

Here $396 = 1 \times 396 = 2 \times 198 = 3 \times 132 = 4 \times 99 = 6 \times 66 = 9 \times 44 = 11 \times 36 = 12 \times 33 = 18 \times 22$

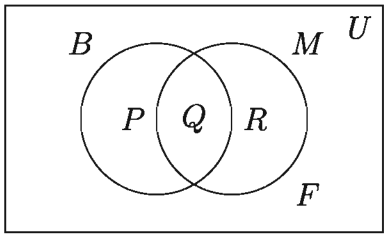
$\therefore B = \{33, 36, 44, 66, 99, 132, 198, 396\}$

$\therefore A \cap B = \{24, 32, 36, 48, 72, 96, 144, 288\} \cap \{33, 36, 44, 66, 99, 132, 198, 396\}$

$\therefore A \cap B = \{36\}$

The required set is $\{36\}$

Example 14. Of 100 students 88 passed in Bangla, 80 in Mathematics and 70 in both subjects. Express the information in Venn diagram, and find how many students failed in both subjects.



Solution: In the Venn diagram rectangular area represents U the set of 100 students. The set of students who passed in Bangla and Mathematics are denoted respectively by B and M. As a result the Venn diagram is divided into 4 disjoint sets denoted by P, Q, R and F.

Here, the set of students passed in both subjects is $Q = B \cap M$, cardinality of which is 70.

P is the set of students passed in Bangla only. Its cardinality is $|P| = 88 - 70 = 18$.

R is the set of students who passed only in Mathmatics. $\therefore$ $|R| = 80 - 70 = 10$

Students who passed in at least one subject is $P \cup Q \cup R = B \cup M$, and corresponding cardinality is $18 + 10 + 70 = 98$

F is the set of students who failed in both the subjects, and $|F| = 100 - 98 = 2$

$\therefore$ number of students failed in both the subjects is 2.

Exercises 2.1

1. Express the following sets in set builder notation:
   1) $\{x \in N : x^2 > 9 \text{ and } x^3 < 130\}$
   2) $\{x \in Z : x^2 > 5 \text{ and } x^3 \leq 36\}$
   3) $\{x \in N : x \text{ is factor of } 36 \text{ and } x \text{ is a multiple of } 6\}$
   4) $\{x \in N : x^3 > 25 \text{ and } x^4 < 264\}$
2. Express the following sets in set builder notation:
   1) $\{3, 5, 7, 9, 11\}$
   2) $\{1, 2, 3, 4, 6, 9, 12, 18, 36\}$
   3) $\{4, 8, 12, 16, 20, 24, 28, 32, 36, 40\}$
   4) $\{\pm 4, \pm 5, \pm 6\}$
3. If $A = \{2, 3, 4\}$, $B = \{1, 2, a\}$ and $C = \{2, a, b\}$,then determine the following sets:
   1) $B \setminus C$ 2) $A \cup B$ 3) $A \cap C$
   4) $A \cup (B \cap C)$ 5) $A \cap (B \cup C)$
4. If $U = \{1, 2, 3, 4, 5, 6, 7\}$, $A = \{1, 3, 5\}$, $B = \{2, 4, 6\}$ and $C = \{3, 4, 5, 6, 7\}$, verify the following statements:
   1) $(A \cup B)' = A' \cap B'$
   2) $(B \cap C)' = B' \cup C'$
   3) $(A \cup B) \cap C = (A \cap C) \cup (B \cap C)$
   4) $(A \cap B) \cup C = (A \cup C) \cap (B \cup C)$
5. If $Q = \{x, y\}$ and $R = \{m, n, l\}$, then determine $P(Q)$ and $P(R)$.
6. If $A = \{a, b\}$, $B = \{a, b, c\}$ and $C = A \cup B$, then show that number of elements of $P(C)$ is 2^n, where n is the number of elements of C.
7. 1) If $(x - 1, y + 2) = (y - 2, 2x + 1)$, then determine x and y.
   2) If $(ax - cy, a^2 - c^2) = (0, ay - cx)$, then determine (x, y).
   3) If $(6x - y, 13) = (1, 3x + 2y)$, then determine (x, y).
8. 1) If $P = \{a\}$, $Q = \{b, c\}$, then determine $P \times Q$ and $Q \times P$.
   2) If $A = \{3, 4, 5\}$, $B = \{4, 5, 6\}$ and $C = \{x, y\}$, determine $(A \cap B) \times C$.
   3) If $P = \{3, 5, 7\}$, $Q = \{5\ 7\}$ and $R = P \setminus Q$, determine $(P \cup Q) \times R$.

9. If A and B are respectively sets of all factors of 35 and 45, then determine $A \cup B$ and $A \cap B$.

10. Determine the natural numbers that divide both 346 and 556 to have 31 as residue.

11. Of the 30 students of a class 20 like football and 15 cricket. Number of students who like both the games is 10. Determine using Venn diagram the number of students who do not like either of the games.

12. Of 100 students 65 passed in Bangla, 48 in both Bangla and English and 15 failed in both the subjects.

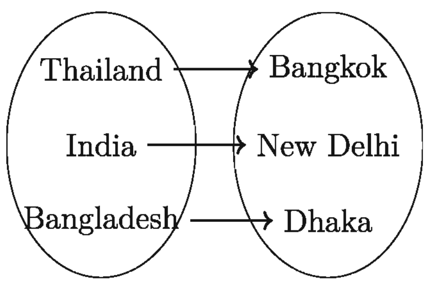
    1) Express the above in Venn diagram along with brief description.

    2) Determine the number of students who passed only in Bangla or passed only in English.

    3) Determine union of the sets of prime factors of number of students passed and failed in both the subjects.

Relations

We know Dhaka is the capital of Bangladesh, New Delhi is of India and Bangkok is of Thailand. There is a relation between capitals and countries. This is country-capital relation. This relation can be expressed as a set in the following way:



That is country-capital relation = {(Bangladesh, Dhaka), (India, New Delhi), (Thailand, Bangkok)}.

If A and B are two sets then, nonnull subset R of ordered pairs of their Cartesian product $A \times B$ is said to be a relation from the set A to the set B. Here set R is a subset of the set $A \times B$. That is, $R \subseteq A \times B$

Example 15. Let us assume that $A = \{3, 5\}$ and $B = \{2, 4\}$

$\therefore A \times B = \{3, 5\} \times \{2, 4\} = \{(3, 2), (3, 4), (5, 2), (5, 4)\}$

$\therefore R \subseteq \{(3, 2), (3, 4), (5, 2), (5, 4)\}$

when for an element x of set A and an element y of set B $(x, y) \in R$, then it is written $x \, R \, y$ and read that x is related to y, that is, x is related to y using relation R.

If $x > y$, then $R = \{(3, 2), (5, 2), (5, 4)\}$

and if $x < y$ then $R = \{(3, 4)\}$

Again, if there is a relation from set A to set A, that is, $R \subseteq A \times A$, then R is a relation of A.

If $x \in A, y \in B$ is related, then the nonempty subset of ordered pairs (x, y) is called **relation**.

Example 16. If $P = \{2, 3, 4\}$, $Q = \{4, 6\}$ and for elements of P and Q a relation $y = 2x$ exists, then find the relations.

Solution: Given, $P = \{2, 3, 4\}$ and $Q = \{4, 6\}$

According to the question $R = \{(x, y) : x \in P, y \in Q \text{ and } y = 2x\}$

Here, $P \times Q = \{2, 3, 4\} \times \{4, 6\} = \{(2, 4), (2, 6), (3, 4), (3, 6), (4, 4), (4, 6)\}$

$\therefore R = \{(2, 4), (3, 6)\}$

The required relation is $\{(2, 4), (3, 6)\}$

Example 17. If $A = \{1, 2, 3\}$ $B = \{0, 2, 4\}$ and elements of A and B have the relation $x = y - 1$, find the relations.

Solution: Given, $A = \{1, 2, 3\}$, $B = \{0, 2, 4\}$

According to the question, relation $R = \{(x, y) : x \in A, y \in B \text{ and } x = y - 1\}$

Here, $A \times B = \{1, 2, 3\} \times \{0, 2, 4\}$

$= \{(1, 0), (1, 2), (1, 4), (2, 0), (2, 2), (2, 4), (3, 0), (3, 2), (3, 4)\}$

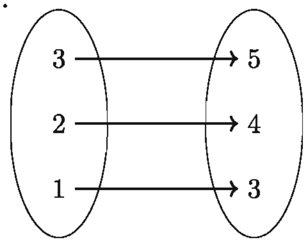
$\therefore R = \{(1, 2), (3, 4)\}$

Work: If $C = \{2, 5, 6\}$, $D = \{4, 5\}$ and the relation $x \leq y$ holds between elements of C and D, find the relations.

Functions

Let us consider the relation between A and B:

For the relation $y = x + 2$
if $x = 1$, then $y = 3$
if $x = 2$, then $y = 4$
if $x = 3$, then $y = 5$



That is, there is exactly one value for y for each value of x and x and y are related by $y = x + 2$. So two variables x and y are related in such a way that for each value of x there is exactly one value of y. Then y is said to be a **function** of x. Usually function of x is denoted by y, $f(x)$, $g(x)$, $F(x)$.

Let $y = x^2 - 2x + 3$ be a function. Here, for each value of x there is exactly one value of y. Here, both x and y are variables. Although both of them are variables value of y depends upon the value of x. That is why x is said to be **independent variable** and y **dependent variable**

Example 18. If $f(x) = x^2 - 4x + 3$, then determine $f(-1)$.

Solution: Given, $f(x) = x^2 - 4x + 3$

$\therefore f(-1) = (-1)^2 - 4(-1) + 3 = 1 + 4 + 3 = 8$

Example 19. If $g(x) = x^3 + ax^2 - 3x - 6$ then for what value of a will bel $g(-2) = 0$?

Solution: Given, $g(x) = x^3 + ax^2 - 3x - 6$

$\therefore g(-2) = (-2)^3 + a(-2)^2 - 3(-2) - 6$

$= -8 + 4a + 6 - 6 = 4a - 8$

As per question $g(-2) = 0$

$\therefore 4a - 8 = 0$ or $4a = 8$ or $a = 2$

$\therefore$ if $a = 2$, then $g(-2) = 0$.

Domain and Range

For ordered pairs of any relation the set of first elements is said to be **domain** and the set of second elements is said to be **range**.

Let R be a relation from set A to the set B, that is, $R \subseteq A \times B$. For the ordered pairs of R the set of first elements will be the domain of R and the set of second elements will be called range of R. The domain of R is denoted by *Dom R* and range by *Range R*.

Example 20. Find domain and range of the relation $S = \{(2,1),(2,2),(3,2),(4,5)\}$.

Solution: Given, $S = \{(2,1),(2,2),(3,2),(4,5)\}$

The first elements of the ordered pairs of S are $2,2,3,4$ and the second elements are $1,2,2,5$

$\therefore$ *Dom* $S = \{2,3,4\}$ and *Range* $S = \{1,2,5\}$

Example 21. If $A = \{0,1,2,3\}$ and $R = \{(x,y) : x \in A, y \in A$ and $y = x+1\}$, then express R in tabular method and determine *Dom* R and *Range* R.

Solution: Given, $R= \{(x,y) : x \in A, y \in A \quad y = x+1\}$

From the conditions of R we get $y = x+1$.

Now for each $x \in A$ determine the value of $y = x+1$.

x	0	1	2	3
y	1	2	3	4

Since $4 \notin A$, $(3,4) \notin R$. $\therefore R = \{(0,1),(1,2),(2,3)\}$

$\therefore$ Dom $R = \{0,1,2\}$ and Range $R = \{1,2,3\}$

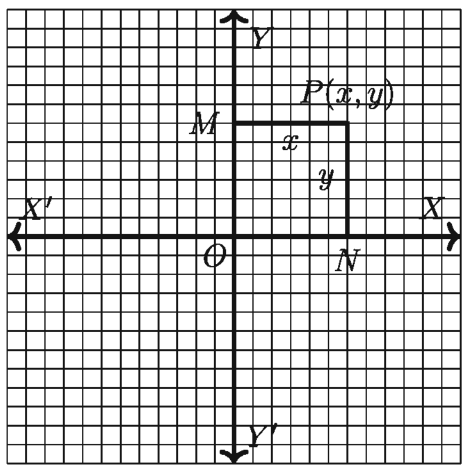
Work:

1) If $S = \{(-3,8),(-2,3),(-1,0),(0,-1),(1,0),(2,3)\}$, then determine domain and range of S.
2) Let $S = \{(x,y) : x,y \in A$ and $y - x = 1\}$, where $A = \{-3,-2,-1,0\}$. Determine Dom S and Range S.

Graph of a Function

Visual display of a function is said to be **graph**. In order to clarify the concept of a function significance of graph is important. French philosopher and mathematician Rene Descartes (1596-1650) was the first to play pioneering role in introducing relations between algebra and geometry. He introduced a modern concept in geometry by defining the position of a point in two dimensional space by drawing two straight lines perpendicularly crossing each other. He termed the perpendicular lines as axes, and their point of intersection as origin. Let two perpendicularly crossing straight lines XOX' and YOY' be drawn on a plane. Any point lying on this plane can be uniquely determined through these two straight lines. Each of these perpendicular straight lines is called **axis**. The line XOX' parallel to horizon is called x-**axis**, and vertical line YOY' is called y-**axis** and point O of intersection of these lines is called **Origin**.

The signed perpendicular distance of a point, located in the plane determined by two axes, from the axes are called **coordinates** of the point. Let P be any point on the plane determined by two axes. Let us draw perpendiculars PN and PM from P to XOX' and YOY'. Then $PM = ON$ which is perpendicular distance to YOY' from P and $PN = OM$ which is perpendicular distance from P to XOX'. If $PM = x$ and $PN = y$, then coordinates of point P are (x, y).



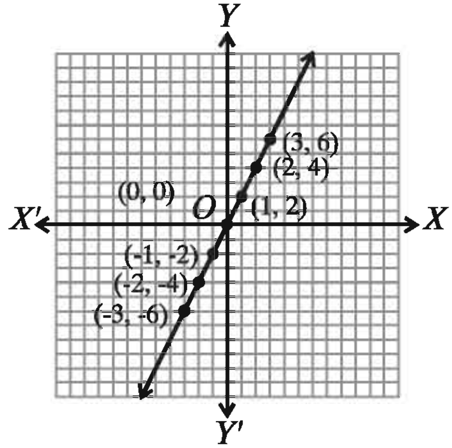
Here x is called **abscissa** and y is called **ordinate**. These coordinates are called **Cartesian coordinates.** It is very easy to depict a function using Cartesian coordinate system. Usually values of independent variable is set along x axis, and those of dependent variable along y axis.

In order to draw a graph for the function $y = f(x)$, some values of independent variable are chosen from domain, and with corresponding values of dependent variables ordered pairs are formed. After that these ordered pairs are placed on the plane. The resultant points are then connected with line which is known as graph of the function $y = f(x)$.

Example 22. Draw the graph for the function $y = 2x$, where $-3 \leq x \leq 3$

Solution: Calculate values of y for some values of x from the domain $-3 \leq x \leq 3$, and draw a list.

x	-3	-2	-1	0	1	2	3
y	-6	-4	-2	0	2	4	6



In the graph paper each considering side of each smallest square as of length points of the list are marked and added. This results in the graph of the function.

Example 23. If $f(y)=\dfrac{y^3-3y^2+1}{y(1-y)}$,then show that $f\left(\dfrac{1}{y}\right)=f(1-y)$

Solution: $f(y)=\dfrac{y^3-3y^2+1}{y(1-y)}$

$$\therefore f\left(\frac{1}{y}\right)=\frac{\left(\frac{1}{y}\right)^3-3\left(\frac{1}{y}\right)^2+1}{\frac{1}{y}\left(1-\frac{1}{y}\right)}=\frac{\frac{1-3y+y^3}{y^3}}{\frac{y-1}{y^2}}$$

$$=\frac{1-3y+y^3}{y^3}\times\frac{y^2}{y-1}=\frac{1-3y+y^3}{y(y-1)}$$

Again, $f(1-y)=\dfrac{(1-y)^3-3(1-y)^2+1}{(1-y)(1-(1-y))}$

$$=\frac{1-3y+3y^2-y^3-3(1-2y+y^2)+1}{(1-y)(1-1+y)}$$

$$=\frac{1-3y+3y^2-y^3-3+6y-3y^2+1}{y(1-y)}$$

$$=\frac{-1+3y-y^3}{y(1-y)}=\frac{-(1-3y+y^3)}{-y(y-1)}$$

$$=\frac{1-3y+y^3}{y(y-1)}$$

$$\therefore f\left(\frac{1}{y}\right)=f(1-y).$$

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Example 24. Let Universal set be $U = \{x : x \in N \text{ and } x \leq 6\}$, $A = \{x : x$ prime and $x \leq 5\}$, $B = \{x : x$ even number and $x \leq 6\}$ and $C = A \setminus B$

1) Determine A^c.
2) Prove that, $A \cup B = (A \setminus B) \cup (B \setminus A) \cup (A \cap B)$
3) Prove that, $(A \cap C) \times B = (A \times B) \cap (C \times B)$

Solution:

1) Given, $U = \{x : x \in N \text{ and } x \leq 6\} = \{1, 2, 3, 4, 5, 6\}$

$A = \{x : x \text{ prime and } x \leq 5\} = \{2, 3, 5\}$

$\therefore\ A^c = U \setminus A = \{1, 2, 3, 4, 5, 6\} - \{2, 3, 5\} = \{1, 4, 6\}$

2) Given,

$B = \{x : x \text{ prime and } x \leq 6\} = \{2, 4, 6\}$

$\therefore\ A \cup B = \{2, 3, 5\} \cup \{2, 4, 6\} = \{2, 3, 4, 5, 6\} \cdots\cdots (1)$

$A \setminus B = \{2, 3, 5\} - \{2, 4, 6\} = \{3, 5\}$

$B \setminus A = \{2, 4, 6\} - \{2, 3, 5\} = \{4, 6\}$

$A \cap B = \{2, 3, 5\} \cap \{2, 4, 6\} = \{2\}$

$\therefore\ (A \setminus B) \cup (B \setminus A) \cup (A \cap B) = \{3, 5\} \cup \{4, 6\} \cup \{2\} = \{2, 3, 4, 5, 6\} \cdots (2)$

Therefore comparing (1) and (2) we get,

$A \cup B = (A \setminus B) \cup (B \setminus A) \cup (A \cap B)$

3) From (2) we get,

$C = A \setminus B = \{3, 5\}$

$A \cap C = \{2, 3, 5\} \cap \{3, 5\} = \{3, 5\}$

$\therefore (A \cap C) \times B = \{3, 5\} \times \{2, 4, 6\}$

$= \{(3, 2), (3, 4), (3, 6), (5, 2), (5, 4), (5, 6)\} \cdots\cdots (3)$

$A \times B = \{2, 3, 5\} \times \{2, 4, 6\}$

$= \{(2, 2), (2, 4), (2, 6), (3, 2), (3, 4), (3, 6), (5, 2), (5, 4), (5, 6)\}$

$C \times B = \{3, 5\} \times \{2, 4, 6\}$

$= \{(3, 2), (3, 4), (3, 6), (5, 2), (5, 4), (5, 6)\}$

$\therefore\ (A \times B) \cap (C \times B)$

$$= \{(2,2),(2,4),(2,6),(3,2),(3,4),(3,6),(5,2),(5,4),(5,6)\}$$
$$\cap \{(3,2),(3,4),(3,6),(5,2),(5,4),(5,6)\}$$
$$= \{(3,2),(3,4),(3,6),(5,2),(5,4),(5,6)\} \cdots\cdots (4)$$

Therefore, comparing (3) and (4) we get,

$(A \cap C) \times B = (A \times B) \cap (C \times B)$

Example 25. Let $A = \{4,5,6,7\}$, $B = \{0,1,2,3\}$ and $R = \{(x,y) : x \in A, y \in A \text{ and } y = x+1\}$

1) Prove that, sets A and B are mutually disjoint.
2) Determine $P(B)$ to show that number of elements of $P(B)$ is 2^n, where n is the number of elements of B.
3) Express the relation R in tabular method and determine its domain.

Solution:

1) Given, $A = \{4,5,6,7\}$ and $B = \{0,1,2,3\}$

   $\therefore\ A \cap B = \{4,5,6,7\} \cap \{0,1,2,3\} = \varnothing$

   Since $A \cap B = \varnothing$

Therefore, A and B are mutually disjoint.

2) Given,

   $B = \{0,1,2,3\}$

   $\therefore\ P(B) = \{\{0\},\{1\},\{2\},\{3\},\{0,1\},\{0,2\},\{0.3\},\{1,2\},\{1,3\},\{2,3\},$
   $\{0,1,2\},\{0,1,3\},\{0,2,3\},\{1,2,3\},\{0,1,2,3\},\varnothing\}$

   Here number of elements of B is 4 and number of elements of the power set is $2^4 = 16$

   $\therefore$ if n is the number of elements of B, then number of elements of power set is 2^n.

   $\therefore$ number of elements of $P(B)$ satisfies 2^n formula.

3) Given, $R = \{(x,y) : x \in A, y \in A \text{ and } y = x+1\}$ and $A = \{4,5,6,7\}$

   From the conditions of R we get $y = x+1$

   Now, for each $x \in A$ draw a list by finding values of $y = x+1$.

x	4	5	6	7
y	5	6	7	8

Since $8 \notin A$, therefore $(7,8) \notin R$

$\therefore\ R = \{(4,5),(5,6),(6,7)\}$

$Dom\ R = \{4,5,6\}$

Exercises 2.2

1. Which one is the set of factors of 8?
   1) $\{8, 16, 24, \cdots\}$
   2) $\{1, 2, 4, 8\}$
   3) $\{2, 4, 8\}$
   4) $\{1, 2\}$

2. If R is a relation from the set C to the set B, then which one of the following is true ?
   1) $R \subset C$
   2) $R \subset B$
   3) $R \subseteq C \times B$
   4) $C \times B \subseteq R$

3. If $A = \{1, 2\}, B = \{2, 5\}$, which one of the following is the number of elements of $P(A \cap B)$?
   1) 1
   2) 2
   3) 3
   4) 8

4. Which one of the following expresses the set $\{x \in N : 13 < x < 17$ and x is prime $\}$ in tabular method?
   1) $\varnothing$
   2) $\{0\}$
   3) $\{\varnothing\}$
   4) $\{13, 17)\}$

5. If $A \cup B = \{a, b, c\}$, then

   (i) $A = \{a, b\},\ B = \{a, b, c\}$

   (ii) $A = \{a, b, c\},\ B = \{b, c\}$

   (iii) $A = \{a, b\},\ B = \{c\}$

   Based upon the above facts which one of the following is true ?
   1) i
   2) ii
   3) i and ii
   4) i, ii and iii

6. If for two finite sets A and B

   (i) $A \times B = \{(x, y) : x \in A$ and $y \in B\}$

   (ii) If $n(A) = a, n(B) = b$, then $n(A \times B) = ab$

   (iii) Each member of $A \times B$ is an ordered pair.

   Based upon the statements above which one of the following is true?

1) *i* and *ii* 2) *i* and *iii* 3) *ii* and *iii* 4) *i*, *ii* and *iii*

If $A = \{6, 7, 8, 9, 10, 11, 12, 13\}$, then answer the questions 7–9 below:

7. Which one is the correct expression for the set A?
   1) $\{x \in N : 6 < x < 13\}$ 2) $\{x \in N : 6 \leq x < 13\}$
   3) $\{x \in N : 6 \leq x \leq 13\}$ 4) $\{x \in N : 6 < x \leq 13\}$

8. Which one is the set of primes in A?
   1) $\{6, 8, 10, 12\}$ 2) $\{7, 9, 11, 13\}$ 3) $\{7, 11, 13\}$ 4) $\{9, 12\}$

9. Which one of the following sets is a multiple of 3 in set A ?
   1) $\{6, 9\}$ 2) $\{6, 11\}$ 3) $\{9, 12\}$ 4) $\{6, 9, 12\}$

10. If $A = \{3, 4\}, B = \{2, 4\}$, $x \in A$ and $y \in B$, then determine the relation $x > y$ in A and B.

11. If $C = \{2, 5\}, D = \{4, 6, \ 7\}$, $x \in C$ and $y \in D$, then determine the relation $x + 1 < y$ in C and D.

12. If $f(x) = x^4 + 5x - 3$, then determine $f(-1), f(2)$ and $f\left(\frac{1}{2}\right)$.

13. If $f(y) = y^3 + ky^2 - 4y - 8$, then for what value of k will $f(-2) = 0$?

14. If $f(x) = x^3 - 6x^2 + 11x - 6$, then for what value of x is $f(x) = 0$?

15. If $f(x) = \dfrac{2x+1}{2x-1}$,then determine the value of $\dfrac{f\left(\frac{1}{x^2}\right)+1}{f\left(\frac{1}{x^2}\right)-1}$.

16. If $g(x) = \dfrac{1 + x^2 + x^4}{x^2}$,prove that $g\left(\dfrac{1}{x^2}\right) = g(x^2)$

17. Determine domain and range from the following relations.
    1) $R = \{(2, 1), (2, 2), (2, 3)\}$
    2) $S = \{(-2, -4), (-1, 1), (0, 0), (1, 1), (2, 4)\}$
    3) $F = \{(\frac{1}{2}, 0), (1, 1), (1, -1), (\frac{5}{2}, 2), (\frac{5}{2}, -2)\}$

18. Express the following relations in tabular method and determine domain and range for each.
    1) $R = \{(x, y) : x \in A, y \in A \text{ and } x + y = 1\}$ where $A = \{-2, -1, 0, 1, 2\}$
    2) $F = \{(x, y) : x \in C, y \in C \text{ and } y = 2x\}$ where $C = \{-1, 0, 1, 2, 3\}$

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19. Draw the points $(-3, 2), (0, -5), \left(\frac{1}{2}, -\frac{5}{6}\right)$ on graph paper.

20. Draw the points $(1, 2), (-1, 1), (11, 7)$ on the graph paper and show that all the three points are on the same straight line.

21. Universal set $U = \{x : x \in N \text{ and } x \text{ odd number}\}$

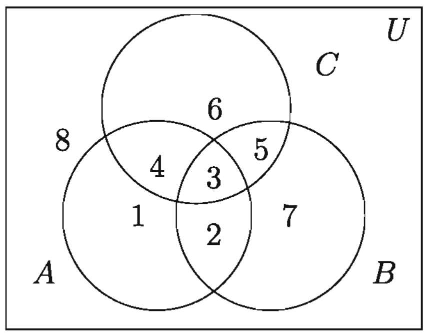
$A = \{x : x \in N \text{ and } 2 \leq x \leq 7\}$

$B = \{x : x \in N \text{ and } 3 < x < 6\}$

$C = \{x : x \in N \text{ and } x^2 > 5 \text{ and } x^3 < 130\}$

1) Express set A in tabular method.
2) Determine A' and $C \setminus B$.
3) Determine $B \times C$ and $P(A \cap C)$.

22. Look at the Venn diagram.



1) Express the set B in set building method.
2) Using the facts mentioned above, verify the relation $A, \cup (B \cap C) = (A \cup B) \cap (A \cup C)$.
3) If $S = (B \cup C)^c \times A$ then determine dom S.

23. $y = f(x) = \dfrac{4x - 7}{2x - 4}$ is a function.

1) Determine the value of $f\left(-\frac{1}{2}\right)$.
2) Determine the value of $\dfrac{f(x) + 2}{f(x) - 1}$.
3) Prove that $f(y) = x$

Chapter 3

Algebraic Expressions

Algebraic formulae are used to solve many algebraic problems. Moreover, many algebraic expressions are presented by resolving them into factors. That is why the problem solved by algebraic formulae and the contents of resolving expressions into factors by making suitable for the students have been presented in this chapter. Moreover, different types of mathematical problems can be solved by resolving into factors with the help of algebraic formulae. In the previous class, algebraic formulae and their related corollaries have been discussed elaborately. In this chapter, those are reiterated and some of their applications are presented through examples. Besides, extension of the formulae of square and cube, resolution into factors using remainder theorem and formation of algebraic formulae and their applications in solving practical problems have been discussed here in detail.

At the end of the chapter, the students will be able to —

- expand the formulae of square and cube by applying algebraic formulae.
- explain the remainder theorem and resolve into factors by applying the theorem.
- form algebraic formulae for solving real life problems and solve the problems by applying the formulae.

Algebraic Expressions

Meaningful organization of operational signs and numerical letter symbols is called **Algebraic Expressions**. Such as, $2a + 3b - 4c$ is an algebraic expression. In algebraic expression, different types of information are expressed through the letters a, b, c, p, q, r, m, n, x, y, z, $\ldots$ etc. These alphabet are used to solve different types of problems related to algebraic expressions. In arithmetic, only positive numbers are used, where as, in algebra, both positive and negative numbers including zero are used. **Algebra is the generalization of arithmetic**.

The numbers used in algebraic expressions are **constants**, their values are fixed.The letter symbols used in algebraic expressions are **variables**, their values are not fixed, they can be of any value.

Algebraic Formulae

Any general rule or resolution expressed by algebraic symbols is called **Algebraic Formula**. In class VII and VIII, algebraic formulae and related corollaries have been discussed. In this chapter, some applications are presented on the basis of that discussion.

Formula 1. $(a+b)^2 = a^2 + 2ab + b^2$

Formula 2. $(a-b)^2 = a^2 - 2ab + b^2$

Remarks: It is seen from Formula 1 and Formula 2 that, adding $2ab$ or $-2ab$ to $a^2 + b^2$, we get a perfect square, i.e. we get $(a+b)^2$ or $(a-b)^2$. Substituting $-b$ instead of b in Formula 1 we get Formula 2 : $\{a + (-b)\}^2 = a^2 + 2a(-b) + (-b)^2$ That is, $(a-b)^2 = a^2 - 2ab + b^2$

Corollary 1. $a^2 + b^2 = (a+b)^2 - 2ab$

Corollary 2. $a^2 + b^2 = (a-b)^2 + 2ab$

Corollary 3. $(a+b)^2 = (a-b)^2 + 4ab$

Proof: $(a+b)^2 = a^2 + 2ab + b^2 = a^2 - 2ab + b^2 + 4ab = (a-b)^2 + 4ab$ □

Corollary 4. $(a-b)^2 = (a+b)^2 - 4ab$

Proof: $(a-b)^2 = a^2 - 2ab + b^2 = a^2 + 2ab + b^2 - 4ab = (a+b)^2 - 4ab$ □

Corollary 5. $a^2 + b^2 = \dfrac{(a+b)^2 + (a-b)^2}{2}$

Proof: From Formula 1 and Formula 2,

$$\begin{array}{r@{\;}l}
a^2 + 2ab + b^2 &= (a+b)^2 \\
a^2 - 2ab + b^2 &= (a-b)^2 \\ \hline
\text{adding, } 2a^2 + 2b^2 &= (a+b)^2 + (a-b)^2 \\
\text{or, } 2(a^2 + b^2) &= (a+b)^2 + (a-b)^2 \\
\text{Hence, } (a^2 + b^2) &= \dfrac{(a+b)^2 + (a-b)^2}{2} \quad \square
\end{array}$$

Corollary 6. $ab = \left(\dfrac{a+b}{2}\right)^2 - \left(\dfrac{a-b}{2}\right)^2$

Proof: From Formula 1 and Formula 2,

$$\begin{array}{r@{\;}l}
a^2 + 2ab + b^2 &= (a+b)^2 \\
a^2 - 2ab + b^2 &= (a-b)^2 \\ \hline
\text{subtracting, } 4ab &= (a+b)^2 - (a-b)^2 \\
\text{or, } ab &= \dfrac{(a+b)^2}{4} - \dfrac{(a-b)^2}{4} \\
\text{hence, } ab &= \left(\dfrac{a+b}{2}\right)^2 - \left(\dfrac{a-b}{2}\right)^2 \quad \square
\end{array}$$

Remark: By applying the Corollary 6 product of any two quantities can be expressed as the difference of two squares.

Formula 3. $a^2 - b^2 = (a+b)(a-b)$

Therefore, the difference of the squares of two expressions = sum of two expressions × difference of two expressions.

Formula 4. $(x+a)(x+b) = x^2 + (a+b)x + ab$

Therefore, $(x+a)(x+b) = x^2 +$ (algebraic sum of a and b) x + (the product of a and b)

Extension of formula for square: There are three terms in the expression $a+b+c$. It can be considered the sum of two terms $(a+b)$ and c. Therefore, by applying Formula 1, the square of the expression is,

$$(a+b+c)^2 = \{(a+b)+c\}^2 = (a+b)^2 + 2(a+b)c + c^2$$

$$= a^2 + 2ab + b^2 + 2ac + 2bc + c^2 = a^2 + b^2 + c^2 + 2ab + 2bc + 2ac$$

Formula 5. $(a+b+c)^2 = a^2 + b^2 + c^2 + 2ab + 2bc + 2ac$

Corollary 7. $a^2 + b^2 + c^2 = (a+b+c)^2 - 2(ab+bc+ac)$

Corollary 8. $2(ab+bc+ac) = (a+b+c)^2 - (a^2+b^2+c^2)$

Note: By applying Formula 5, we get,

1) $(a+b-c)^2 = \{a+b+(-c)\}^2$
$= a^2+b^2+(-c)^2+2ab+2b(-c)+2a(-c)$
$= a^2+b^2+c^2+2ab-2bc-2ac$

2) $(a-b+c)^2 = \{a+(-b)+c\}^2$
$= a^2+(-b)^2+c^2+2a(-b)+2(-b)c+2ac$
$= a^2+b^2+c^2-2ab-2bc+2ac$

3) $(a-b-c)^2 = \{a+(-b)+(-c)\}^2$
$= a^2+(-b)^2+(-c)^2+2a(-b)+2(-b)(-c)+2a(-c)$
$= a^2+b^2+c^2-2ab+2bc-2ac$

Example 1. What is the square of $(4x+5y)$?

Solution: $(4x+5y)^2 = (4x)^2+2\times(4x)\times(5y)+(5y)^2 = 16x^2+40xy+25y^2$

Example 2. What is the square of $(3a-7b)$?

Solution: $(3a-7b)^2 = (3a)^2-2\times(3a)\times(7b)+(7b)^2 = 9a^2-42ab+49b^2$

Example 3. Find the square of 996 by applying the formula of square.

Solution: $(996)^2 = (1000-4)^2 = (1000)^2-2\times1000\times4+4^2$

$= 1000000-8000+16 = 1000016-8000 = 992016$

Example 4. What is the square of $a+b+c+d$?

Solution: $(a+b+c+d)^2 = \{(a+b)+(c+d)\}^2$

$= (a+b)^2+2(a+b)(c+d)+(c+d)^2$

$= a^2+2ab+b^2+2(ac+ad+bc+bd)+c^2+2cd+d^2$

$= a^2+b^2+c^2+d^2+2ab+2ac+2ad+2bc+2bd+2cd$

Work: Find the square with the help of the formulae:

1) $3xy+2ax$ 2) $4x-3y$ 3) $x-5y+2z$

Example 5. Simplify: $(5x+7y+3z)^2+2(7x-7y-3z)(5x+7y+3z)+(7x-7y-3z)^2$

Solution: Let, $5x+7y+3z=a$ and $7x-7y-3z=b$

$\therefore$ Given expression $= a^2+2\cdot b\cdot a+b^2 = a^2+2ab+b^2$

$= (a+b)^2$

$= \{(5x+7y+3z)+(7x-7y-3z)\}^2$ [substituting the values of a and b]

$= (5x+7y+3z+7x-7y-3z)^2$

$= (12x)^2 = 144x^2$

Example 6. If $x-y=2$ and $xy=24$, what is the value of $x+y$?

Solution: $(x+y)^2=(x-y)^2+4xy=(2)^2+4\times 24=4+96=100$

$\therefore x+y=\pm\sqrt{100}=\pm 10$

Example 7. If $a^4+a^2b^2+b^4=3$ and $a^2+ab+b^2=3$, what is the value of a^2+b^2?

Solution: $a^4+a^2b^2+b^4$

$= (a^2)^2+2a^2b^2+(b^2)^2-a^2b^2$

$= (a^2+b^2)^2-(ab)^2$

$= (a^2+b^2+ab)(a^2+b^2-ab)$

$= (a^2+ab+b^2)(a^2-ab+b^2)$

$\therefore 3=3(a^2-ab+b^2)$ [substituting the values]

or, $a^2-ab+b^2=\dfrac{3}{3}=1$

Now adding, $a^2+ab+b^2=3$ and $a^2-ab+b^2=1$

we get, $2(a^2+b^2)=4$

or, $a^2+b^2=\dfrac{4}{2}=2$

$\therefore a^2+b^2=2$

Example 8. Prove that, $(a+b)^4-(a-b)^4=8ab(a^2+b^2)$

Solution: $(a+b)^4-(a-b)^4$

$= \{(a+b)^2\}^2-\{(a-b)^2\}^2$

$= \{(a+b)^2+(a-b)^2\}\{(a+b)^2-(a-b)^2\}$

$= 2(a^2+b^2)\times 4ab$ [applying Corollary 5 and Corollary 6]

$$= 8ab(a^2 + b^2)$$

$$\therefore (a+b)^4 - (a-b)^4 = 8ab(a^2 + b^2)$$

Example 9. If $a + b + c = 15$ and $a^2 + b^2 + c^2 = 83$, what is the value of $ab + bc + ac$?

Solution: First method:

$$2(ab + bc + ac) = (a + b + c)^2 - (a^2 + b^2 + c^2) = (15)^2 - 83 = 225 - 83 = 142$$

$$\therefore ab + bc + ac = \frac{142}{2} = 71$$

Alternative method:

$$(a + b + c)^2 = (a^2 + b^2 + c^2) + 2(ab + bc + ac)$$

or, $(15)^2 = 83 + 2(ab + bc + ac)$

or, $225 - 83 = 2(ab + bc + ac)$

or, $2(ab + bc + ac) = 142$

$$\therefore ab + bc + ac = \frac{142}{2} = 71$$

Example 10. If $a + b + c = 2$ and $ab + bc + ac = 1$, what is the value of $(a + b)^2 + (b + c)^2 + (c + a)^2$?

Solution: $(a + b)^2 + (b + c)^2 + (c + a)^2$

$$= a^2 + 2ab + b^2 + b^2 + 2bc + c^2 + c^2 + 2ca + a^2$$

$$= (a^2 + b^2 + c^2 + 2ab + 2bc + 2ca) + (a^2 + b^2 + c^2)$$

$$= (a + b + c)^2 + (a + b + c)^2 - 2(ab + bc + ca)$$

$$= (2)^2 + (2)^2 - 2 \times 1 = 4 + 4 - 2 = 8 - 2 = 6$$

Example 11. Express $(2x + 3y)(4x - 5y)$ as the difference of two squares.

Solution: Let, $2x + 3y = a$ and $4x - 5y = b$

$$\therefore \text{Given expression } ab = \left(\frac{a+b}{2}\right)^2 - \left(\frac{a-b}{2}\right)^2$$

$$= \left(\frac{2x + 3y + 4x - 5y}{2}\right)^2 - \left(\frac{2x + 3y - 4x + 5y}{2}\right)^2$$ [substituting the values of a and b]

$$= \left(\frac{6x-2y}{2}\right)^2 - \left(\frac{8y-2x}{2}\right)^2$$

$$= \left\{\frac{2(3x-y)}{2}\right\}^2 - \left\{\frac{2(4y-x)}{2}\right\}^2$$

$$= (3x-y)^2 - (4y-x)^2$$

$\therefore (2x+3y)(4x-5y) = (3x-y)^2 - (4y-x)^2$

Work:

1) Simplify: $(4x+3y)^2 + 2(4x+3y)(4x-3y) + (4x-3y)^2$
2) If $x+y+z=12$ and $x^2+y^2+z^2=50$, find the value of $(x-y)^2+(y-z)^2+(z-x)^2$.

Exercise 3.1

1. Find the square with the help of the formulae:

1) $2a+3b$	2) $x^2+\dfrac{2}{y^2}$	3) $4y-5x$
4) $5x^2-y$	5) $3b-5c-2a$	6) $ax-by-cz$
7) $2a+3x-2y-5z$	8) 1007	

2. Simplify:
   1) $(7p+3q-5r)^2 - 2(7p+3q-5r)(8p-4q-5r) + (8p-4q-5r)^2$
   2) $(2m+3n-p)^2 + (2m-3n+p)^2 - 2(2m+3n-p)(2m-3n+p)$
   3) $6.35 \times 6.35 + 2 \times 6.35 \times 3.65 + 3.65 \times 3.65$
   4) $\dfrac{2345 \times 2345 - 759 \times 759}{2345-759}$
3. If $a-b=4$ and $ab=60$, what is the value of $a+b$?
4. If $a+b=9m$ and $ab=18m^2$, what is the value of $a-b$?
5. If $x-\dfrac{1}{x}=4$, prove that, $x^4+\dfrac{1}{x^4}=322$
6. If $2x+\dfrac{2}{x}=3$, what i the value of $x^2+\dfrac{1}{x^2}$?
7. If $a+\dfrac{1}{a}=2$, show that, $a^2+\dfrac{1}{a^2}=a^4+\dfrac{1}{a^4}$

8. If $a+b=\sqrt{7}$ and $a-b=\sqrt{5}$, prove that, $8ab(a^2+b^2)=24$
9. If $a+b+c=9$ and $ab+bc+ca=31$, what is the value of $a^2+b^2+c^2$?
10. If $a^2+b^2+c^2=9$ and $ab+bc+ca=8$, what is the value of $(a+b+c)^2$?
11. If $a+b+c=6$ and $a^2+b^2+c^2=14$, what is the value of $(a-b)^2+(b-c)^2+(c-a)^2$?
12. If $x=3$, $y=4$ and $z=5$, what is the value of $9x^2+16y^2+4z^2-24xy-16yz+12zx$?
13. Express $(a+2b)(3a+2c)$ as the difference of two squares.
14. Express $x^2+10x+24$ as the difference of two squares.
15. If $a^4+a^2b^2+b^4=8$ and $a^2+ab+b^2=4$, find the value of, 1) a^2+b^2, 2) ab

Formulae of Cubes

Formula 6. $(a+b)^3=a^3+3a^2b+3ab^2+b^3=a^3+b^3+3ab(a+b)$

Proof: $(a+b)^3=(a+b)(a+b)^2$

$=(a+b)(a^2+2ab+b^2)$

$=a(a^2+2ab+b^2)+b(a^2+2ab+b^2)$

$=a^3+2a^2b+ab^2+a^2b+2ab^2+b^3$

$=a^3+3a^2b+3ab^2+b^3$

$=a^3+b^3+3ab(a+b)$ □

Corollary 9. $a^3+b^3=(a+b)^3-3ab(a+b)$

Formula 7. $(a-b)^3=a^3-3a^2b+3ab^2-b^3=a^3-b^3-3ab(a-b)$

Proof: $(a-b)^3=(a-b)(a-b)^2$

$=(a-b)(a^2-2ab+b^2)$

$=a(a^2-2ab+b^2)-b(a^2-2ab+b^2)$

$=a^3-2a^2b+ab^2-a^2b+2ab^2-b^3$

$=a^3-3a^2b+3ab^2-b^3$

$=a^3-b^3-3ab(a-b)$ □

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Note: Substituting $-b$ instead of b in Formula 6 we get Formula 7:

$$\{a+(-b)\}^3 = a^3 + (-b)^3 + 3a(-b)\{a+(-b)\}$$

That is, $(a-b)^3 = a^3 - b^3 - 3ab(a-b)$

Corollary 10. $a^3 - b^3 = (a-b)^3 + 3ab(a-b)$

Formula 8. $a^3 + b^3 = (a+b)(a^2 - ab + b^2)$

Proof: $a^3 + b^3 = (a+b)^3 - 3ab(a+b)$

$$= (a+b)\{(a+b)^2 - 3ab\}$$

$$= (a+b)(a^2 + 2ab + b^2 - 3ab)$$

$$= (a+b)(a^2 - ab + b^2) \qquad \square$$

Formula 9. $a^3 - b^3 = (a-b)(a^2 + ab + b^2)$

Proof: $a^3 - b^3 = (a-b)^3 + 3ab(a-b)$

$$= (a-b)\{(a-b)^2 + 3ab\}$$

$$= (a-b)(a^2 - 2ab + b^2 + 3ab)$$

$$= (a-b)(a^2 + ab + b^2) \qquad \square$$

Example 12. Find the cube of $2x + 3y$.

Solution: $(2x+3y)^3$

$$= (2x)^3 + 3(2x)^2 \cdot 3y + 3 \cdot 2x(3y)^2 + (3y)^3$$

$$= 8x^3 + 3 \cdot 4x^2 \cdot 3y + 3 \cdot 2x \cdot 9y^2 + 27y^3$$

$$= 8x^3 + 36x^2y + 54xy^2 + 27y^3$$

Example 13. Find the cube of $2x - y$.

Solution: $(2x-y)^3$

$$= (2x)^3 - 3(2x)^2 \cdot y + 3 \cdot 2x \cdot y^2 - y^3$$

$$= 8x^3 - 3 \cdot 4x^2 \cdot y + 3 \cdot 2x \cdot y^2 - y^3$$

$$= 8x^3 - 12x^2y + 6xy^2 - y^3$$

Work: Find the cube with the help of the formulae.

1) $3x + 2y$ 2) $3x - 4y$ 3) 397

Example 14. If $x = 37$, what is the value of $8x^3 + 72x^2 + 216x + 216$?

Solution: $8x^3 + 72x^2 + 216x + 216$

$= (2x)^3 + 3 \cdot (2x)^2 \cdot 6 + 3 \cdot 2x \cdot (6)^2 + (6)^3$

$= (2x + 6)^3 = (2 \times 37 + 6)^3$ [substituting the values]

$= (74 + 6)^3 = (80)^3 = 512000$

Example 15. If $x - y = 8$ and $xy = 5$, what is the value of $x^3 - y^3 + 8(x + y)^2$?

Solution: $x^3 - y^3 + 8(x + y)^2$

$= (x - y)^3 + 3xy(x - y) + 8\{(x - y)^2 + 4xy\}$

$= (8)^3 + 3 \times 5 \times 8 + 8(8^2 + 4 \times 5)$ [substituting the values]

$= 8^3 + 15 \times 8 + 8(8^2 + 4 \times 5)$

$= 8^3 + 15 \times 8 + 8 \times 84$

$= 8(8^2 + 15 + 84) = 8(64 + 15 + 84)$

$= 8 \times 163 = 1304$

Example 16. If $a = \sqrt{3} + \sqrt{2}$, prove that, $a^3 + \dfrac{1}{a^3} = 18\sqrt{3}$

Solution: Given that, $a = \sqrt{3} + \sqrt{2}$

$\therefore \dfrac{1}{a} = \dfrac{1}{\sqrt{3} + \sqrt{2}}$

$= \dfrac{\sqrt{3} - \sqrt{2}}{(\sqrt{3} + \sqrt{2})(\sqrt{3} - \sqrt{2})}$ [multiplying numerator and denominator by $(\sqrt{3} - \sqrt{2})$]

$= \dfrac{\sqrt{3} - \sqrt{2}}{(\sqrt{3})^2 - (\sqrt{2})^2} = \dfrac{\sqrt{3} - \sqrt{2}}{3 - 2} = \sqrt{3} - \sqrt{2}$

$\therefore a + \dfrac{1}{a} = (\sqrt{3} + \sqrt{2}) + (\sqrt{3} - \sqrt{2}) = \sqrt{3} + \sqrt{2} + \sqrt{3} - \sqrt{2} = 2\sqrt{3}$

Now, $a^3 + \dfrac{1}{a^3} = \left(a + \dfrac{1}{a}\right)^3 - 3 \cdot a \cdot \dfrac{1}{a}\left(a + \dfrac{1}{a}\right)$

$= (2\sqrt{3})^3 - 3(2\sqrt{3}) \left[\because a + \dfrac{1}{a} = 2\sqrt{3}\right]$

$= 2^3 \cdot (\sqrt{3})^3 - 3 \times 2\sqrt{3} = 8 \cdot 3\sqrt{3} - 6\sqrt{3}$

$= 24\sqrt{3} - 6\sqrt{3} = 18\sqrt{3}$ (Proved)

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Example 17. If $x+y=5$, $xy=6$ and $x>y$

1) Find the value of $2(x^2+y^2)$.
2) Find the value of $x^3-y^3-3(x^2+y^2)$.
3) Find the value of x^5+y^5.

Solution:

1) We know, $2(x^2+y^2)=2\{(x+y)^2-2xy\}$

$$=2(5^2-2\cdot 6)=2\times 13=26$$

$$\therefore 2(x^2+y^2)=26$$

2) Given that, $x+y=5$ and $xy=6$, $x>y$ (By the given condition negative value is not acceptable)

$$\therefore x-y=\sqrt{(x+y)^2-4xy}$$

$$=\sqrt{5^2-4\cdot 6}=\sqrt{25-24}=\sqrt{1}=1$$

$$x^3-y^3-3(x^2+y^2)$$

$$=(x-y)^3+3xy(x-y)-\frac{3}{2}\cdot 2(x^2+y^2)$$

$$=1^3+3\cdot 6\cdot 1-\frac{3}{2}\cdot 26 \text{ or, } (-1)^3+3\cdot 6\cdot(-1)-\frac{3}{2}\cdot 26$$

$$=1+18-39$$

$$=-20$$

$$\therefore x^3-y^3-3(x^2+y^2)=-20$$

3) $x+y=5$ and $x-y=1$

adding, $2x=6$ $\quad\therefore x=\frac{6}{2}=3$

subtracting, $2y=4$ $\quad\therefore y=\frac{4}{2}=2$

$$\therefore x^5+y^5=3^5+2^5=243+32=275$$

Work:

1) If $x=-2$, what is the value of $27x^3-54x^2+36x-8$?
2) If $a+b=5$ and $ab=6$, find the value of $a^3+b^3+4(a-b)^2$.
3) If $x=\sqrt{5}+\sqrt{3}$, find the value of $x^3+\frac{8}{x^3}$.

Exercise 3.2

1. Find the cube with the help of the formulae:

    1) $2x^2 + 3y^2$ 2) $7m^2 - 2n$ 3) $2a - b - 3c$

2. Simplify:

    1) $(7x + 3b)^3 - (5x + 3b)^3 - 6x(7x + 3b)(5x + 3b)$
    2) $(a + b + c)^3 - (a - b - c)^3 - 6(b + c)\{a^2 - (b + c)^2\}$
    3) $(m + n)^6 - (m - n)^6 - 12mn(m^2 - n^2)^2$
    4) $(x + y)(x^2 - xy + y^2) + (y + z)(y^2 - yz + z^2) + (z + x)(z^2 - zx + x^2)$
    5) $(2x + 3y - 4z)^3 + (2x - 3y + 4z)^3 + 12x\{4x^2 - (3y - 4z)^2\}$

3. If $a - b = 5$ and $ab = 36$, what is the value of $a^3 - b^3$?

4. If $a^3 - b^3 = 513$ and $a - b = 3$, what is the value of ab?

5. If $x = 19$ and $y = -12$, find the value of $8x^3 + 36x^2y + 54xy^2 + 27y^3$.

6. If $a = 15$, what is the value of $8a^3 + 60a^2 + 150a + 130$?

7. If $a + b = m$, $a^2 + b^2 = n$ and $a^3 + b^3 = p^3$, show that, $m^3 + 2p^3 = 3mn$.

8. If $a + b = 3$ and $ab = 2$, find the value of (a) $a^2 - ab + b^2$ and (b) $a^3 + b^3$.

9. If $a - b = 5$ and $ab = 36$, find the value of (a) $a^2 + ab + b^2$ and (b) $a^3 - b^3$.

10. If $m + \frac{1}{m} = a$, find the value of $m^3 + \frac{1}{m^3}$.

11. If $x - \frac{1}{x} = p$, find the value of $x^3 - \frac{1}{x^3}$.

12. If $a - \frac{1}{a} = 1$, show that, $a^3 - \frac{1}{a^3} = 4$.

13. If $a + b + c = 0$, show that,

    1) $a^3 + b^3 + c^3 = 3abc$
    2) $\frac{(b + c)^2}{3bc} + \frac{(c + a)^2}{3ca} + \frac{(a + b)^2}{3ab} = 1$.

14. If $p - q = r$, show that, $p^3 - q^3 - r^3 = 3pqr$.

15. If $2x - \dfrac{2}{x} = 3$, show that, $8\left(x^3 - \dfrac{1}{x^3}\right) = 63$.

16. If $a = \sqrt{6} + \sqrt{5}$, find the value of $\dfrac{a^6 - 1}{a^3}$.

17. $x - \dfrac{1}{x} = \sqrt{3}$ where $x \neq 0$

    1) Prove that, $x^2 - \sqrt{3}x = 1$.

    2) Prove that, $23\left(x^2 + \dfrac{1}{x^2}\right) = 5\left(x^4 + \dfrac{1}{x^4}\right)$.

    3) Find the value of $x^6 + \dfrac{1}{x^6}$.

Factorization

If an expression is equal to the product of two or more expressions, each of the latter expressions is called a **factor** of the former expression. After finding the possible factors of any algebraic expression and then expressing the expression as the product of these factors are called **factorization** or resolution into factors. The algebraic expressions may consist of one or more terms. So, the factors may also contain one or more terms. Some process of resolving expressions into factors will be discussed here.

Common Factor: If any polynomial expression has common factor in every term, at first they are to be found out. For example,

Example 18. $3a^2b + 6ab^2 + 12a^2b^2 = 3ab(a + 2b + 4ab)$

Example 19. $2ab(x - y) + 2bc(x - y) + 3ca(x - y) = (x - y)(2ab + 2bc + 3ca)$

Perfect Square: An expression can be expressed in the form of a perfect square.

Example 20. Resolve into factors: $4x^2 + 12x + 9$

Solution: $4x^2 + 12x + 9 = (2x)^2 + 2 \times 2x \times 3 + (3)^2$

$= (2x + 3)^2 = (2x + 3)(2x + 3)$

Example 21. Resolve into factors: $9x^2 - 30xy + 25y^2$

Solution: $9x^2 - 30xy + 25y^2$

$= (3x)^2 - 2 \times 3x \times 5y + (5y)^2$

$= (3x - 5y)^2 = (3x - 5y)(3x - 5y)$

Difference of two squares: Expressing an expression as the difference of two squares and then applying the formula $a^2 - b^2 = (a + b)(a - b)$.

Example 22. Resolve into factors: $a^2 - 1 + 2b - b^2$.

Solution: $a^2 - 1 + 2b - b^2 = a^2 - (b^2 - 2b + 1)$

$= a^2 - (b - 1)^2 = \{a + (b - 1)\}\{a - (b - 1)\}$

$= (a + b - 1)(a - b + 1)$

Example 23. Resolve into factors: $a^4 + 64b^4$

Solution: $a^4 + 64b^4 = (a^2)^2 + (8b^2)^2$

$= (a^2)^2 + 2 \times a^2 \times 8b^2 + (8b^2)^2 - 16a^2b^2$

$= (a^2 + 8b^2)^2 - (4ab)^2$

$= (a^2 + 8b^2 + 4ab)(a^2 + 8b^2 - 4ab)$

$= (a^2 + 4ab + 8b^2)(a^2 - 4ab + 8b^2)$

Work: Resolve into factors:

1) $abx^2 + acx^3 + adx^4$ 2) $xa^2 - 144xb^2$ 3) $x^2 - 2xy - 4y - 4$

Middle term factorization: Factors can be determined by applying the formula $x^2 + (a + b)x + ab = (x + a)(x + b)$. In this method, a polynomial of the form $x^2 + px + q$ can be factorized, if two integers a and b can be found so that, $a + b = p$ and $ab = q$. For this, two factors of q with their signs are to be taken whose algebraic sum is p. If $q>0$, a and b will be of same sign and if $q< 0$, a and b will be of opposite sign. To be noted p and q may not be integers.

Example 24. Resolve into factors: $x^2 + 12x + 35$

Solution: $x^2 + 12x + 35 = x^2 + (5 + 7)x + 5 \times 7 = (x + 5)(x + 7)$

Example 25. Resolve into factors: $x^2 + x - 20$

Solution: $x^2 + x - 20 = x^2 + (5 - 4)x + (5)(-4) = (x + 5)(x - 4)$

Middle Term Break-Up: By middle term break-up method of polynomial of the form of ax^2+bx+c would be $ax^2+bx+c=(rx+p)(sx+q)$ if $ax^2+bx+c=rsx^2+(rq+sp)x+pq$. That is, $a=rs$, $b=rq+sp$ and $c=pq$. Hence, $ac=rspq=(rq)(sp)$ and $b=rq+sp$. Therefore, to determine factors of the polynomial ax^2+bx+c, ac, that is, the product of the coefficient of x^2 and the term free from x are to be expressed into two such factors whose algebraic sum is equal to b, the coefficient of x.

Example 26. Resolve into factors: $3x^2-x-14$

Solution: $3x^2-x-14=3x^2-7x+6x-14$

$$=x(3x-7)+2(3x-7)=(3x-7)(x+2)$$

Work: Resolve into factors:

1) x^2+x-56 2) $16x^3-46x^2+15x$ 3) $12x^2+17x+6$

Perfect Cube Form: Factors can be determined by expressing an expression in the form of perfect cube.

Example 27. Resolve into factors: $8x^3+36x^2y+54xy^2+27y^3$

Solution: $8x^3+36x^2y+54xy^2+27y^3$

$$=(2x)^3+3\times(2x)^2\times 3y+3\times 2x\times(3y)^2+(3y)^3$$

$$=(2x+3y)^3=(2x+3y)(2x+3y)(2x+3y)$$

Formulae of addition or subtraction of two cubes: Factors can be determined by applying the formulae $a^3+b^3=(a+b)(a^2-ab+b^2)$ and $a^3-b^3=(a-b)(a^2+ab+b^2)$

Example 28. Resolve into factors: 1) $8a^3+27b^3$ 2) a^6-64

Solution:

1) $8a^3+27b^3=(2a)^3+(3b)^3$

$$=(2a+3b)\{(2a)^2-2a\times 3b+(3b)^2\}$$

$$=(2a+3b)(4a^2-6ab+9b^2)$$

2) $a^6-64=(a^2)^3-(4)^3=(a^2-4)\{(a^2)^2+a^2\times 4+(4)^2\}$

$$= (a^2 - 4)(a^4 + 4a^2 + 16)$$

but $a^2 - 4 = a^2 - 2^2 = (a + 2)(a - 2)$

and $a^4 + 4a^2 + 16 = (a^2)^2 + (4)^2 + 4a^2$

$$= (a^2 + 4)^2 - 2(a^2)(4) + 4a^2$$

$$= (a^2 + 4)^2 - 4a^2$$

$$= (a^2 + 4)^2 - (2a)^2$$

$$= (a^2 + 4 + 2a)(a^2 + 4 - 2a)$$

$$= (a^2 + 2a + 4)(a^2 - 2a + 4)$$

$\therefore\ a^6 - 64 = (a + 2)(a - 2)(a^2 + 2a + 4)(a^2 - 2a + 4)$

Alternative method: $a^6 - 64 = (a^3)^2 - 8^2$

$$= (a^3 + 8)(a^3 - 8)$$

$$= (a^3 + 2^3)(a^3 - 2^3)$$

$$= (a + 2)(a^2 - 2a + 4)(a - 2)(a^2 + 2a + 4)$$

$$= (a + 2)(a - 2)(a^2 + 2a + 4)(a^2 - 2a + 4)$$

Work: Resolve into factors:

1) $2x^4 + 16x$ 2) $8 - a^3 + 3a^2b - 3ab^2 + b^3$ 3) $(a + b)^3 + (a - b)^3$

Factors of the expression with fractional coefficients: Factors of the expression with fractional coefficients can be expressed in several ways. For example, $a^3 + \frac{1}{27} = a^3 + \frac{1}{3^3} = \left(a + \frac{1}{3}\right)\left(a^2 - \frac{a}{3} + \frac{1}{9}\right)$

Again, $a^3 + \frac{1}{27} = \frac{1}{27}(27a^3 + 1) = \frac{1}{27}\{(3a)^3 + (1)^3\} = \frac{1}{27}(3a + 1)(9a^2 - 3a + 1)$

In the second solution, the factors involving the variables are with integral coefficients but the two solutions are same.

$$\frac{1}{27}(3a + 1)(9a^2 - 3a + 1) = \frac{1}{3}(3a + 1) \times \frac{1}{9}(9a^2 - 3a + 1)$$

$$= \left(a + \frac{1}{3}\right)\left(a^2 - \frac{a}{3} + \frac{1}{9}\right)$$

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Example 29. Resolve into factors: $x^3+6x^2y+11xy^2+6y^3$

Solution: $x^3+6x^2y+11xy^2+6y^3$

$$=\{x^3+3\cdot x^2\cdot 2y+3\cdot x\cdot(2y)^2+(2y)^3\}-xy^2-2y^3$$

$$=(x+2y)^3-y^2(x+2y)=(x+2y)\{(x+2y)^2-y^2\}$$

$$=(x+2y)(x+2y+y)(x+2y-y)$$

$$=(x+2y)(x+3y)(x+y)=(x+y)(x+2y)(x+3y)$$

Work: Resolve into factors:

1) $\frac{1}{2}x^2+\frac{7}{6}x+\frac{1}{3}$ 2) $a^3+\frac{1}{8}$ 3) $16x^2-25y^2-8xz+10yz$

Exercises 3.3

Resolve into factors **(1-30)**:

1) $ab(x-y)-bc(x-y)$
2) $9x^2+24x+16$
3) a^4-27a^2+1
4) $x^4-6x^2y^2+y^4$
5) $(a^2-b^2)(x^2-y^2)+4abxy$
6) $4a^2-12ab+9b^2-4c^2$
7) $a^2+6a+8-y^2+2y$
8) $16x^2-25y^2-8xz+10yz$
9) $x^2+13x+36$
10) x^4+x^2-20
11) $a^2-30a+216$
12) a^8-a^4-2
13) $x^2-37x-650$
14) $9x^2y^2-5xy^2-14y^2$
15) $4x^4-27x^2-81$
16) $ax^2+(a^2+1)x+a$
17) $3(a^2+2a)^2-22(a^2+2a)+40$
18) $(a-1)x^2+a^2xy+(a+1)y^2$
19) x^3+3x^2+3x+2
20) $a^3-6a^2+12a-9$
21) $a^3-9b^3+(a+b)^3$
22) $8x^3+12x^2+6x-63$
23) $8a^3+\frac{b^3}{27}$
24) $\frac{a^6}{27}-b^6$
25) $4a^2+\frac{1}{4a^2}-2+4a-\frac{1}{a}$
26) $(3a+1)^3-(2a-3)^3$
27) $(x+2)(x+3)(x+4)(x+5)-48$
28) $(x-1)(x-3)(x-5)(x-7)-65$
29) $2b^2c^2+2c^2a^2+2a^2b^2-a^4-b^4-c^4$
30) $14(x+z)^2-29(x+z)(x+1)-15(x+1)^2$

31) Show that, $(x+1)(x+2)(3x-1)(3x-4)=(3x^2+2x-1)(3x^2+2x-8)$

Remainder Theorem

In the following example, if $6x^2 - 7x + 5$ is divided by $x - 1$,then what is quotient and remainder?

$$\begin{array}{r|rrrl} x-1) & 6x^2 & -7x & +5 & (6x-1 \\ & 6x^2 & -6x & & \\ \hline & & -x & +5 & \\ & & -x & +1 & \\ \hline & & & 4 & \end{array}$$

Here, $x - 1$ is divisor, $6x^2 - 7x + 5$ is dividen , $6x - 1$ is quotient and 4 is remainder.

We know, dividend = divisor × quotient + remainder

Now, if we indicate the dividend by $f(x)$, the quotient by $h(x)$, the remainder by r and the divisor by $(x - a)$, from the above formula, we get,

$f(x) = (x - a) \cdot h(x) + r$, this formula is true to all values of a.

Substituting $x = a$ in both sides we get,

$f(a) = (a - a) \cdot h(a) + r = 0 \cdot h(a) + r = r$

Hence, $r = f(a)$

Therefore, if $f(x)$ is divided by $(x - a)$, the remainder is $f(a)$. This formula is known as remainder theorem. That is, the remainder theorem gives the remainder when a polynomial $f(x)$ of positive degree is divided by $(x-a)$ without performing actual division. In the above example if $a = 1$, then $f(x) = 6x^2 - 7x + 5$.

$\therefore$ $f(1) = 6-7+5 = 4$ which is equal to the remainder. The degree of the divisor polynomial $(x-a)$ is 1, If the divisor is a factor of the dividend, the remainder will be zero and if it is not zero, the remainder will be a number other than zero.

Corollary 11. $(x - a)$ will be a factor of $f(x)$ if and only if $f(a) = 0$.

Proof: Let, $f(a) = 0$. Therefore, according to remainder theorem, if $f(x)$ is divided by $(x - a)$, the remainder will be zero. Hence, $(x - a)$ will be a factor of $f(x)$.

Conversely, let, $(x - a)$ is a factor of $f(x)$.

Therefore, $f(x) = (x - a) \cdot h(x)$, where $h(x)$ is a polynomial.

Putting $x = a$ in both sides we get,

$f(a) = (a - a) \cdot h(a) = 0$

$\therefore f(a) = 0$

Hence, any polynomial $f(x)$ will be divisible by $(x - a)$ if and only if $f(a) = 0$. This formula is known as factor theorem.

Proposition 12. If degree of $f(x)$ is positive and $a \neq 0$, then if $f(x)$ is divided by $(ax + b)$, the remainder will be $f\left(-\frac{b}{a}\right)$.

Proof: Degree of divisor $ax + b$, $(a \neq 0)$ is 1.

Hence, we can write: $f(x) = (ax + b) \cdot h(x) + r = a\left(x + \frac{b}{a}\right) \cdot h(x) + r$

$\therefore f(x) = \left(x + \frac{b}{a}\right) \cdot a \cdot h(x) + r$

It is clear that, if $f(x)$ is divided by $\left(x + \frac{b}{a}\right)$, the quotient will be $a \cdot h(x)$ and the remainder will be r.

Here, divisor $= x - \left(-\frac{b}{a}\right)$

Hence, according to remainder theorem $r = f\left(-\frac{b}{a}\right)$

Therefore, if $f(x)$ is divided by $(ax + b)$, the remainder will be $\left(-\frac{b}{a}\right)$.

Corollary 13. If $a \neq 0$, the expression $ax + b$ will be a factor of any polynomial $f(x)$, if and only if $f\left(-\frac{b}{a}\right) = 0$.

Proof: $a \neq 0$, $ax + b = a\left(x + \frac{b}{a}\right)$ will be a factor of $f(x)$, if and only if $\left(x + \frac{b}{a}\right)$ $= x - \left(-\frac{b}{a}\right)$ is a factor of $f(x)$. Hence, if and only if $f\left(-\frac{b}{a}\right) = 0$. This method of determining the factor by using remainder theorem is called **Vanishing method**.

Example 30. Resolve into factors: $x^3 - x - 6$

Solution: Here, $f(x) = x^3 - x - 6$ is a polynomial. The factors of the constant -6 are $\pm 1, \pm 2, \pm 3, \pm 6$.

Now, putting $x = 1, -1$ we see that the value of $f(x)$ is not zero.

But, putting $x = 2$ we see that the value of $f(x)$ is not zero.

Hence, $f(2) = 2^3 - 2 - 6 = 8 - 2 - 6 = 0$

Therefore, $x - 2$ is a factor of $f(x)$.

$\therefore f(x) = x^3 - x - 6$

$= x^3 - 2x^2 + 2x^2 - 4x + 3x - 6$

$= x^2(x-2) + 2x(x-2) + 3(x-2)$

$= (x-2)(x^2 + 2x + 3)$

Example 31. Resolve into factors: $x^3 - 3xy^2 + 2y^3$ and $x^2 + xy - 2y^2$.

Solution: Here, consider x a variable and y a constant.

We consider the given expression a polynomial of x.

Let, $f(x) = x^3 - 3xy^2 + 2y^3$

Then, $f(y) = y^3 - 3y \cdot y^2 + 2y^3 = 3y^3 - 3y^3 = 0$

$\therefore (x - y)$ is a factor of $f(x)$.

Now, $x^3 - 3xy^2 + 2y^3$

$= x^3 - x^2y + x^2y - xy^2 - 2xy^2 + 2y^3$

$= x^2(x-y) + xy(x-y) - 2y^2(x-y)$

$= (x-y)(x^2 + xy - 2y^2)$

Again let, $g(x) = x^2 + xy - 2y^2$

$\therefore g(y) = y^2 + y^2 - 2y^2 = 0$

$\therefore (x - y)$ is a factor of $g(x)$.

$\therefore g(x) = x^2 + xy - 2y^2$

$= x^2 - xy + 2xy - 2y^2$

$= x(x-y) + 2y(x-y)$

$= (x-y)(x+2y)$

$\therefore x^3 - 3xy^2 + 2y^3 = (x-y)^2(x+2y)$

Example 32. Resolve into factors: $54x^4 + 27x^3a - 16x - 8a$.

Solution: Let, $f(x) = 54x^4 + 27x^3a - 16x - 8a$

$$\text{Then, } f\left(-\frac{1}{2}a\right) = 54\left(-\frac{1}{2}a\right)^4 + 27a\left(-\frac{1}{2}a\right)^3 - 16\left(-\frac{1}{2}a\right) - 8a$$

$$= \frac{27}{8}a^4 - \frac{27}{8}a^4 + 8a - 8a = 0$$

$$\therefore x - \left(-\frac{1}{2}a\right) = x + \frac{a}{2} = \frac{1}{2}(2x + a) \text{ is a factor of } f(x).$$

Hence, $(2x + a)$ is a factor of $f(x)$.

Now, $54x^4 + 27x^3a - 16x - 8a$

$$= 27x^3(2x + a) - 8(2x + a)$$

$$= (2x + a)(27x^3 - 8)$$

$$= (2x + a)\{(3x)^3 - (2)^3\}$$

$$= (2x + a)(3x - 2)(9x^2 + 6x + 4)$$

Example 33. $g(a) = a^3 + a^2 + 10a - 8,\ f(a) = a^3 - 9 + (a + 1)^3$.

1) If $g(a)$ is divided by $(a - 2)$, then determine the remainder.
2) Resolve into factors: $f(a)$.

Solution: 1) Given, $g(a) = a^3 + a^2 + 10a - 8$

According to remainder theorem, if $g(a)$ is divided by $(a - 2)$, then the remainder will be $g(2)$.

$\therefore\ g(2) = 2^3 + 2^2 + 10 \cdot 2 - 8 = 8 + 4 + 20 - 8 = 32 - 8 = 24$

$\therefore\ g(2) = 24$

Calculated remainder is 24.

$f(a) = a^3 - 9 + (a + 1)^3$

$f(a)$ is a polynomial, if we put $a = 1$, then the result of the polynomial will be zero.

Therefore $(a - 1)$ is a factor of the polynomial.

$\therefore\ f(a) = a^3 - 9 + a^3 + 3a^2 + 3a + 1 = 2a^3 + 3a^2 + 3a - 8$

$$= 2a^3 - 2a^2 + 5a^2 - 5a + 8a - 8$$

$$= 2a^2(a - 1) + 5a(a - 1) + 8(a - 1)$$

$= (a-1)(2a^2+5a+8)$

$\therefore\ a^3 - 9 + (a+1)^3 = (a-1)(2a^2+5a+8)$

Work: Resolve into factors:

1) $x^3 - 21x - 20$ 2) $2x^3 - 3x^2 + 3x - 1$ 3) $x^3 + 6x^2 + 11x + 6$

Exercises 3.4

Resolve into factors:

1. $3a^3 + 2a + 5$
2. $x^3 - 7xy^2 - 6y^3$
3. $x^3 + 2x^2 - 5x - 6$
4. $x^3 + 4x^2 + x - 6$
5. $a^3 + 3a + 36$
6. $a^4 - 4a + 3$
7. $a^3 - a^2 - 10a - 8$
8. $x^3 - 3x^2 + 4x - 4$
9. $a^3 - 7a^2b + 7ab^2 - b^3$
10. $x^3 - x - 24$
11. $x^3 + 6x^2 + 11x + 6$
12. $2x^4 - 3x^3 - 3x - 2$
13. $4x^4 + 12x^3 + 7x^2 - 3x - 2$
14. $x^6 - x^5 + x^4 - x^3 + x^2 - x$
15. $4x^3 - 5x^2 + 5x - 1$
16. $18x^3 + 15x^2 - x - 2$

Forming and applying algebraic formulae in solving real life problems

In our daily business we face the realistic problems in different time and in different ways. These problems are described linguistically. In this section, we shall discuss the formation of algebraic formulae and their applications in solving different problems of real surroundings which are described linguistically. As a result of this discussion the students on the one hand, will get the conception about the application of mathematics in real surroundings, on the other hand, they will be eager to learn mathematics for their understanding of the involvement of mathematics with their surroundings.

Methods of solving the problems:

1. At first the problem will have to be observed carefully and to read attentively and then to identify which are unknown and which are to be determined.

2. One of the unknown quantities is to be denoted with any variable (say x). Then realising the problem well, express other unknown quantities in terms of the same variable x.
3. The problem will have to be splitted into small parts and express them by algebraic expressions.
4. Using the given conditions, the small parts together are to be expressed by an equation.
5. The value of the unknown quantity x is to be found by solving the equation.

Different formulae are used in solving the problems based on real life. The formulae are mentioned below :

Related to Payable or Attainable

Suppose, $q =$ amount of money payable or attainable per person

$n =$ number of person

$\therefore$ Amount of payable or attainable money, $A = qn$

Related to Time and Work

Suppose, $q =$ portion of a work performed by every one in unit of time

$n =$ number of performers of work

$x =$ total time of doing work

$W =$ portion of a work done by n persons in time x

$\therefore W = qnx$

Related to Time and Distance

Suppose, $v =$ speed per hour

$t =$ total time

$d =$ total distance

$\therefore d = vt$

Related to pipe and water tank

Suppose, $Q_0 =$ amount of stored water in a tank at the time of opening the pipe

$q =$ amount of water flowing in or flowing out by the pipe in a unit time.

t = time taken.

$Q(t)$ = amount of water in the tank in time t

$\therefore\ Q(t) = Q_0 \pm qt$

'+' sign at the time of flowing of water in and '−' sign, at the time of flowing of water out are to be used.

Related to percentage

Suppose, b = total quantity

r = rate of percentage $= \dfrac{s}{100} = s\%$

p = part of percentage $s\%$ of $= b$

$\therefore\ p = br$

Related to profit and loss

Suppose, C = cost price

r = rate of profit or loss

$\therefore$ selling price $S = C(1 \pm r)$

in case of profit, $S = C(1 + r)$

in case of loss, $S = C(1 - r)$

Related to investment and profit

Suppose, I = profit after time n

n = specific time

P = principal

r = profit of unit principal at unit time

A = principal with profit after time n

In the case of simple profit,

$$I = Pnr$$

$$A = P + I = P + Pnr = P(1 + nr)$$

In the case of compound profit,

$$A = P(1 + r)^n$$

Example 34. For a function of Annual Sports, members of an association made a budget of Tk. 45000 and decided that every member would subscribe equally. But 5 members refused to subscribe. As a result, amount of subscription of each member increased by Tk. 15 per head. How many members were in the association?

Solution: Let the number of members of the association be x and amount of subscription per head be Tk. q.

Then total amount of subscription, Tk $A = qx = 45,000$.

Actually numbers of members were $(x-5)$ and amount of subscription per head became Tk. $(q+15)$.

Then, total amount of subscription $(x-5)(q+15)$

By the question,

$$qx = (x-5)(q+15) \cdots\cdots (1)$$

$$qx = 45000 \cdots\cdots (2)$$

From equation (1) we get,

$$qx = (x-5)(q+15)$$

$$\text{or, } qx = qx - 5q + 15x - 75$$

$$\text{or, } 5q = 15x - 75 = 5(3x-15)$$

$\therefore q = 3x - 15$

Putting the value of q in (2), we get,

$$(3x-15) \times x = 45000$$

$$\text{or, } 3x^2 - 15x = 45000$$

$$\text{or, } x^2 - 5x = 15000 \text{ [dividing both sides by 3]}$$

$$\text{or, } x^2 - 5x - 15000 = 0$$

$$\text{or, } x^2 - 125x + 120x - 15000 = 0$$

$$\text{or, } x(x-125) + 120(x-125) = 0$$

$$\text{or, } (x-125)(x+120) = 0$$

Therefore, $(x-125) = 0$ or $(x+120) = 0$

$$\text{or, } x = 125 \text{ or, } x = -120$$

Since the number of members cannot be negative, hence the value of x as -120 is not acceptable.

Therefore, number of members of the association is 125

Example 35. Rafiq can do a work in 10 days and Shafiq can do that work in 15 days. In how many days do they together finish the work?

Solution: Let, they can finish the work in d days.

Name	Days to complete	Part done in 1 day	Part done in d day
Rafiq	10	$\frac{1}{10}$	$\frac{d}{10}$
Shafiq	15	$\frac{1}{15}$	$\frac{d}{15}$

As per question, $\frac{d}{10}+\frac{d}{15}=1$ or, $d\left(\frac{1}{10}+\frac{1}{15}\right)=1$

or, $d\left(\frac{3+2}{30}\right)=1$ or, $\frac{5d}{30}=1$

or, $d=\frac{30}{5}=6$

Therefore, they together can finish the work in 6 days.

Example 36. A boatman can row x km in time t_1 hour against the current. To cover that distance along the current he takes t_2 hour. How much is the speed of the boat and the current?

Solution: Let the speed of the boat in still water be u km per hour and that of the current be v km per hour.

Then, along the current, the effective speed of boat is $(u+v)$ km per hour and against the current, the effective speed of boat is $(u-v)$ km per hour.

We know, speed $=\frac{\text{distance traversed}}{\text{time}}$

According to the question, $u + v = \dfrac{x}{t_2} \cdots\cdots (1)$

and $u - v = \dfrac{x}{t_1} \cdots\cdots (2)$

Adding equations (1) and (2) we get,

$$2u = \frac{x}{t_2} + \frac{x}{t_1} = x\left(\frac{1}{t_1} + \frac{1}{t_2}\right) \text{ or, } u = \frac{x}{2}\left(\frac{1}{t_1} + \frac{1}{t_2}\right)$$

Subtracting equation (2) from equation (1) we get,

$$2v = \frac{x}{t_2} - \frac{x}{t_1} = x\left(\frac{1}{t_2} - \frac{1}{t_1}\right) \text{ or, } v = \frac{x}{2}\left(\frac{1}{t_2} - \frac{1}{t_1}\right)$$

Hence speed of current is $\dfrac{x}{2}\left(\dfrac{1}{t_2} - \dfrac{1}{t_1}\right)$ km per hour and speed of boat is $\dfrac{x}{2}\left(\dfrac{1}{t_1} + \dfrac{1}{t_2}\right)$ km per hour.

Example 37. A pipe can fill up an empty tank in 12 minutes. Another pipe flows out 14 litre of water per minute. If the two pipes are opened together and the empty tank is filled up in 96 minutes, how much water does the tank contain ?.

Solution: Let x litre of water flows in per minute by the first pipe and the tank can contain y litre of water.

According to the question, the tank is filled up by first pipe in 12 minutes

$\therefore y = 12x \cdots\cdots (1)$

Again, the empty tank is filled up by the two pipes together in 96 minutes.

$\therefore y = 96x - 96 \times 14 \cdots\cdots (2)$

From equation (1) we get, $x = \dfrac{y}{12}$

putting the value of x in equation (2) we get,

$y = 96 \times \dfrac{y}{12} - 96 \times 14$

or, $y = 8y - 96 \times 14$

or, $7y = 96 \times 14$

or, $y = \dfrac{96 \times 14}{7} = 192$

Hence, total 192 litre of water is contained in the tank.

Work:

1) For a picnic, a bus was hired at Tk. 2400 and it was decided that every passenger would have to give equal fare. But due to the absence of 10 passengers, fare per head was increased by Tk. 8. How many passengers did go by the bus and how much money did each of the passengers give as fare?
2) A and B together can do a work in p days. A alone can do that work in q days. In how many days can B alone do the work ?
3) A person rowing against the current can go 2 km per hour. If the speed of the current is 3 km per hour, how much time will he take to cover 32 km, rowing along the current ?

Example 38. Price of a book is Tk. 24. This price is 80% of the actual price. The Government subsidize the due price. How much money does the Government subsidize for each book?

Solution: Market price = 80% of the actual price

We know, $p = br$

Here, $p = \text{Tk.} 24$ and $r = 80\% = \dfrac{80}{100}$

$\therefore 24 = b \times \dfrac{80}{100}$

or, $b = \dfrac{24 \times 100}{80}$

$\therefore b = 30$ Tk.

Hence, the actual price of the book is Tk. 30.

$\therefore$ Amount of subsidized money Tk. $= (30 - 24)$ or Tk. $= 6$

Hence, subsidized money for each book is Tk. 6.

Example 39. The loss is $r\%$ when n oranges are sold per taka. How many oranges are to be sold per taka to make a profit of $s\%$?

Solution: If the cost price is Tk. 100, the selling price at the loss of $r\%$ is Tk $(100 - r)$.

If selling price is Tk. $(100 - r)$, cost price is Tk. 100

$\therefore$ If selling price is Tk. 1, cost price is Tk. $\dfrac{100}{100-r}$.

Again, if cost price is Tk. 100, selling price at the profit of $s\%$ is Tk. $(100+s)$.

$\therefore$ if cost price is Tk. $\dfrac{100}{100-r}$, selling price at the profit of $s\%$ is Tk. $\left(\dfrac{100+s}{100} \times \dfrac{100}{100-r}\right)$

$= \text{Tk. } \dfrac{100+s}{100-r}$

Therefore, in Tk. $\dfrac{100+s}{100-r}$ number of oranges is to be sold is n.

$\therefore$ In tk. 1 number of oranges is to be sold is $n \times \left(\dfrac{100-r}{100+s}\right)$

Hence, $\dfrac{n(100-r)}{100+s}$ oranges are to be sold per taka.

Example 40. What is the profit of Tk. 650 in 6 years at the rate of profit Tk. 7 percent per annum ?

Solution: We know, $I = Pnr$

Here, $P = \text{Tk.}\,650$ $n = 6$ year, percent per annum $s = \text{Tk.}\,7$

$\therefore r = \dfrac{s}{100} = \dfrac{7}{100}$

$\therefore I = 650 \times 6 \times \dfrac{7}{100} = 273$

Hence, profit is Tk. 273.

Example 41. Find the compound principal and compound profit of Tk. 15000 in 3 years at the profit of 6 percent per annum.

Solution: We know, $C = P(1+r)^n$ [where C is the profit principal in the case of compound profit]

Given, $P = 15000$ Tk, $r = 6\% = \dfrac{6}{100}$, $n = 3$ year

$$\therefore C = 15000\left(1+\frac{6}{100}\right)^3 = 15000\left(1+\frac{3}{50}\right)^3 = 15000\left(\frac{53}{50}\right)^3$$

$$= 15000 \times \frac{53}{50} \times \frac{53}{50} \times \frac{53}{50} = \frac{446631}{25} = 17865.24$$

$\therefore$ Compound principal is Tk 17865.24

$\therefore$ Compound profit is Tk. $(17865.24 - 15000)$ Tk. $= 2865.24$.

Work:

1) The loss is 50% when 10 lemons are sold per taka. What will be the profit if 6 lemons are sold per taka?
2) What will be the profit of principal of Tk. 750 in 4 years at simple interest $6\frac{1}{2}$ percent per annum?
3) Find the compound interest of Tk. 2000 in 3 years at compound interest of Tk. 4 percent per annum.

Example 42. The loss is $x\%$ when 10 ice creams are sold per taka. How many ice creams are to be sold per taka to make the profit of $z\%$?

Solution: If the cost price is Tk. 100, the selling price at the loss of $x\%$ is Tk. $= (100 - x)$

If selling price is Tk. $(100 - x)$, cost price is Tk. 100

$\therefore$ If selling price is Tk. 1, cost price is Tk. $\dfrac{100}{100 - x}$

Hence, the cost price of 10 ice creams is Tk. $\dfrac{100}{100 - x}$

$\therefore$ The cost price of 1 ice cream is Tk. $\dfrac{100}{(100 - x) \times 10}$.

Again if the cost price is Tk. 100, selling price at the profit of $z\%$ is Tk. $(100 + z)$

If the cost price is Tk. 100, selling price is Tk. $(100 + z)$.

If the cost price is Tk. 1, selling price is Tk. $\dfrac{100 + z}{100}$

$\therefore$ If the cost price is Tk. $\dfrac{100}{(100 - x) \times 10}$,

$$\text{selling price is Tk. } \frac{100 + z}{100} \times \frac{100}{(100 - x) \times 10} = \frac{(100 + z)}{(100 - x) \times 10}$$

The selling price of 1 ice cream is Tk. $\dfrac{(100 + z)}{(100 - x) \times 10} = \dfrac{100 + z}{1000 - 10x}$

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Hence $\dfrac{1000-10x}{100+z}$ ice creams have to be sold per taka.

Exercises 3.5

1. If $f(x)=x^2-4x+4$, which one of the following is the value of $f(2)$?
   1) 4 2) 2 3) 1 4) 0
2. Which one of the following is the value of $\frac{1}{2}\{(a+b)^2-(a-b)^2\}$?
   1) $2(a^2+b^2)$ 2) a^2+b^2 3) $2ab$ 4) $4ab$
3. If $x+\dfrac{2}{x}=3$, what is the value of $x^3+\dfrac{8}{x^3}$?
   1) 1 2) 8 3) 9 4) 16
4. Which one of the following is the factorized form of p^4+p^2+1 ?
   1) $(p^2-p+1)(p^2+p-1)$ 2) $(p^2-p-1)(p^2+p+1)$
   3) $(p^2+p+1)(p^2+p+1)$ 4) $(p^2+p+1)(p^2-p+1)$
5. If $x=2-\sqrt{3}$, then what is the value of x^2?
   1) 1 2) $7-4\sqrt{3}$
   3) $2+\sqrt{3}$ 4) $\dfrac{1}{2-\sqrt{3}}$
6. If $f(x)=x^2-5x+6$ and $f(x)=0$, what is the value of $x=$?
   1) $2,3$ 2) $-5,1$ 3) $-2,3$ 4) $1,-5$
7. What is to be added to $9x^2+16y^2$, so that their sum will be a perfect square?
   1) $6xy$ 2) $12xy$ 3) $24xy$ 4) $144xy$

   If $x^4-x^2+1=0$, answer the following questions from 8 to 10
8. What is the value of $x^2+\dfrac{1}{x^2}$?
   1) 4 2) 2 3) 1 4) 0
9. What is the value of $(x+\dfrac{1}{x})^2$?
   1) 4 2) 3 3) 2 4) 0
10. What is the value of $x^3+\dfrac{1}{x^3}$?
    1) 3 2) 2 3) 1 4) 0
11. If $a^2+b^2=9$ and $ab=3$

    (i) $(a-b)^2=3$

(*ii*) $(a+b)^2 = 15$

(*iii*) $a^2 + b^2 + a^2b^2 = 18$

Which one of the following is correct?

1) *i*, *ii* 2) *i*, *iii* 3) *ii*, *iii* 4) *i*, *ii* and *iii*

12. If $3a^5 - 6a^4 + 3a + 14$ is a algebraic expression -

(*i*) variable of the expression is a

(*ii*) degree of the expression is 5

(*iii*) constant of a^4 is 6

Which one of the following is correct?

1) *i*, *ii* 2) *i*, *iii* 3) *ii*, *iii* 4) *i*, *ii* and *iii*

13. Factor of $p^3 - \frac{1}{64}$ -

(*i*) $p - \frac{1}{4}$

(*ii*) $p^2 + \frac{p}{4} + \frac{1}{8}$

(*iii*) $p^2 + \frac{p}{4} + \frac{1}{16}$

Which one of the following is correct?

1) *i*, *ii* 2) *i*, *iii*
3) *ii*, *iii* 4) *i*, *ii* and *iii*

14. A can do a work in p days and B can do it in $2p$ days. They started to do the work together and after some days A left the work unfinished. B completed the rest of the work in r days. In how many days was the work finished ?

15. 10 persons can do a work in 7 days by working 6 hours a day. Working how many hours per day can 14 persons finish the work in 6 days?

16. Mita can do a work in 10 days. Rita can do that work in 15 days.In how many days will they together complete the work ?

17. A bus was hired at Tk. 5700 to go for a picnic under the condition that every passenger would bare equal fare. But due to the absence of 5 passengers, the fare was increased by Tk. 3 per head. How many passengers availed the bus ?

18. A boatman can go d km in p hours against the current. He takes q hours to cover that distance along the current. What is the speed of the current and the boat ?

19. A boatman goes 15 km and returns from there in 4 hours plying by oar. He goes 5 km at a period of time along the current and goes 3 km at the same period of time against the current. Find the speed of the oar and current.

20. Two pipes are connected with a tank. The empty tank is filled up in t_1 minutes by the first pipe and it becomes empty in $t_2$minutes by the second pipe. If the two pipes are opened together, in how much time will the tank be filled up ?(Here $t_2 > t_1$)

21. A tank is filled up in 12 minutes by a pipe. Another pipe flows out 15 litre of water in 1 minute. When the tank remains empty, the two pipes are opened togethr and the tank is filled up in 48 minutes. How much water does the tank contain ?

22. Divide Tk. 260 among A, B and C in such a way that 2 times the share of A, 3 times the share of B and 4 times the share of C are equal to one another.

23. Due to the selling of a commodity at the loss $x\%$ such price is obtained that due to the selling at the profit of $3x\%$ Tk. $18x$ more is obtained. What was the cost price of the commodity ?

24. If a pen is sold at Tk. 11, there is a profit of 10%. What was the cost price of the pen ?

25. Due to the sale of a notebook at Tk. 36, there was a loss. If the notebook would be sold at Tk. 72, there would be profit amounting twice the loss. What was the cost price of the notebook ?

26. If the simple profit of Tk. 300 in 4 years and the simple profit of Tk. 400 in 5 years together are Tk. 148, what is the percentage of profit ?

27. If the difference of simple profit and compound profit of some principal in 2 years is Tk. 1 at the rate of profit 4%, what is the principal ?

28. Some principal beomes Tk. 460 with simple profit in 3 years and Tk. 600 with simple profit in 5 years. What is the rate of profit ?

29. How much money will become Tk. 985 as profit principal in 13 years at the rate of simple profit 5% per annum ?

30. How much money will become Tk. 1280 as profit principal in 12 years at the rate of profit 5% per annum ?

31. Find the difference of simple profit and compound profit of Tk. 8000 in 3 years at the rate of profit 5%.

32. The Value Added Tax (VAT) of sweets is $x\%$. If a trader sells sweets at Tk. P including VAT, how much VAT is he to pay ? If $x = 15$, $P = 2300$, what is the amount of VAT ?

33. Sum of a number and its multiplicative inverse is 3.

    1) Taking the number as the variable x, express the above information by an equation.

    2) Find the value of $x^3 - \dfrac{1}{x^3}$.

    3) Prove that, $x^5 + \dfrac{1}{x^5} = 123$

34. Each of the members of an association decided to subscribe 100 times the number of members. But 7 members did not subscribe. As a result, amount of subscription for each member was increased by Tk. 500 than the previous.

    1) If the number of members is x and total amount of subscription is Tk. A, find the relation between them.

    2) Find the number of members of the association and total amount of subscription.

    3) $\dfrac{1}{4}$ of total amount of subscription at the rate of simple profit 5% and rest of the money at the rate of simple profit 4% were invested for 2 years. Find the total profit.

35. A bus was hired at Tk. 2400 to go for a picnic under the condition that every passenger would bare equal fare. But due to the absence of 10 passengers, the fare was increased by Tk. 8 per head.

    1) Determine the increased fare per head and percentage of absent passengers.

    2) Determine the fare per head of the passengers.

    3) Find the difference of simple profit and compound profit of the amount equivalent to the bus fare in 13 years at the rate of profit 5%.

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36. Going from point A to point B in a canal, it has to be returned. If the speed of oar is constant, then which time will be greater if there is current or not?

37. The grass in a field increases at a constant rate. 17 cows can eat up all grass in 30 days whereas it takes 24 days for 19 cows. 4 cows were sold from a herd after the herd had eaten grass for 6 days. It took 2 more days for the herd to eat up the grass. How many cows were in that herd?

38. Two brothers had a trained horse which can follow any order. Getting out from the house at the same time two brothers wanted to go to a Baishakhi fair which is 20 miles away from their house. The hourse can carry only one brother at any moment. If the speed of each brother is 4 miles per hour and speed of the horse (with or without person) is 10 miles per hour, then what is the minimum time they need to reach the fair? How much distance each brother has to walk?

Chapter 4

Exponents and Logarithms

Very large or very small numbers or expressions can easily be expressed by using exponents. As a result, calculations and solution of mathematical problems become easier. Scientific or standard form of a number is expressed by using exponents. Therefore, every student should have the knowledge of exponents and its applications.

Exponents be get logarithms. Multiplication and division of numbers or expressions and exponent related calculations have become easier with the help of logarithms. Use of logarithm in scientific calculations was the only way before calculators and computers became available. Still the use of logarithm is important as the alternative of calculator and computer.

In this chapter, exponents and logarithms have been discussed in detail.

At the end of the chapter, the students will be able to —

- explain the rational exponent.
- explain and apply the positive integral exponents, zero and negative integral exponents.
- solve problems by describing and applying the rules of exponents.
- explain the n th root and rational fractional exponents and express the n th root in terms of exponents.
- explain logarithms.
- prove and apply the formulae of logarithms.
- explain natural logarithm and common logarithm.
- explain the scientific form of numbers.
- explain the characteristic and mantissa of common logarithm.
- calculate common and natural logarithm by calculator.

Exponents or Indices

In class VI, we have got the idea of exponents and in class VII, we have learnt the exponential rules for multiplication and division. Expressions associated with exponent and base are called **exponential expression**.

Work: Fill in the blanks

Successive multiplication of the same expression	Exponential expression	base	power or exponent
$2 \times 2 \times 2$	2^3	2	3
$3 \times 3 \times 3 \times 3$		3	
$a \times a \times a$	a^3		
$b \times b \times b \times b \times b$			5

If a is any real number, successive multiplication of n as is a^n; that is, $a \times a \times a \times \ldots \times a$ (n times a) $= a^n$, where n is a positive integer. Here, n is **index or power** and a is **base**. Conversely $a^n = a \times a \times a \times \ldots \times a$ (n times a).

Exponents may not only be positive integers, they may be negative integers or positive fractions or negative fractions. That is, for base $a \in R$ (set of real numbers) and power $n \in Q$ (set of rational numbers), a^n is defined. The case of $n \in N$ (set of natural numbers) is especially considered. Besides, exponents may also be irrational. But as it is out of curriculum, it has not been discussed in this chapter.

Index Laws

Let, $a \in R$ (set of real numbers) and $m, n \in N$ (set of natural numbers).

Formula 1 (Multiplication). $a^m \times a^n = a^{m+n}$

Formula 2 (Division). $\dfrac{a^m}{a^n} = \begin{cases} a^{m-n} \text{ when } m \geq n \\ \dfrac{1}{a^{n-m}} \text{ when } n > m \end{cases}$

Fill in the blanks of the following table:

$a \neq 0, m > n$ $\quad m = 5, n = 3$	$a \neq 0, n > m$ $\quad m = 3, n = 5$
$a^5 \times a^3 = (a\times a\times a\times a\times a)\times(a\times a\times a)$ $= a\times a\times a\times a\times a\times a\times a\times a = a^8 = a^{5+3}$	$a^3 \times a^5 =$
$\frac{a^5}{a^3} =$	$\frac{a^3}{a^5} = \frac{a\times a\times a}{a\times a\times a\times a\times a} = \frac{1}{a^2} = \frac{1}{a^{5-3}}$

$\therefore$ In general $a^m \times a^n = a^{m+n}$ and $\frac{a^m}{a^n} = \begin{cases} a^{m-n} & \text{when } m \geq n \\ \frac{1}{a^{n-m}} & \text{when } n > m \end{cases}$

Formula 3 (power of product). $\quad (ab)^n = a^n \times b^n$

We observe, $(5\times 2)^3 = (5\times 2)\times(5\times 2)\times(5\times 2)$ $[\because a^3 = a\times a\times a,\ a = 5\times 2]$
$= (5\times 5\times 5)\times(2\times 2\times 2)$
$= 5^3 \times 2^3$

In general, $(ab)^n = ab\times ab\times ab\times\ldots\times ab$ [successive multiplication of n ab's]
$= (a\times a\times a\times\ldots\times a)\times(b\times b\times b\times\ldots\times b)$
$= a^n \times b^n$

Formula 4 (power of quotient). $\quad \left(\frac{a}{b}\right)^n = \frac{a^n}{b^n},\ (b \neq 0)$

We observe, $\left(\frac{5}{2}\right)^3 = \frac{5}{2}\times\frac{5}{2}\times\frac{5}{2} = \frac{5^3}{2^3}$

In general, $\left(\frac{a}{b}\right)^n = \frac{a}{b}\times\frac{a}{b}\times\frac{a}{b}\ldots\times\frac{a}{b}$ [successive multiplication of n times $\frac{a}{b}$]
$= \frac{a\times a\times a\times\ldots\times a}{b\times b\times b\times\ldots\times b} = \frac{a^n}{b^n}$

Formula 5 (power of power). $\quad (a^m)^n = a^{mn}$

$(a^m)^n = a^m\times a^m\times a^m\times\ldots\times a^m$ [successive multiplication of n times a^m]
$= a^{m+m+m\ldots+m}$ [in the power, multiplication of n times of exponent]
$= a^{m\times n} = a^{mn}$

$\therefore (a^m)^n = a^{mn}$

Zero and Negative Indices

For 0 and negative indices a^0 and a^{-n} are defined as follows (where n is natural number).

Definition 1 (zero index). $\quad a^0 = 1,\ (a \neq 0)$

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Definition 2 (negative index). $a^{-n} = \dfrac{1}{a^n}, \ (a \neq 0, n \in N)$

For all integer indices m and n the definition $\dfrac{a^m}{a^n} = a^{m-n}$ holds true.

Observe, $\dfrac{a^n}{a^n} = a^{n-n} = a^0$

But $\dfrac{a^n}{a^n} = \dfrac{a \times a \times a \times \ldots \times a \quad (n \text{ times})}{a \times a \times a \times \ldots \times a \quad (n \text{ times})} = 1$

$\therefore a^0 = 1$

and $\dfrac{1}{a^n} = \dfrac{a^0}{a^n} = a^{0-n} = a^{-n}$

Example 1. Find the values: 1) $\dfrac{5^2}{5^3}$ 2) $\left(\dfrac{2}{3}\right)^5 \times \left(\dfrac{2}{3}\right)^{-5}$

Solution:

1) $\dfrac{5^2}{5^3} = 5^{2-3} = 5^{-1} = \dfrac{1}{5^1} = \dfrac{1}{5}$

2) $\left(\dfrac{2}{3}\right)^5 \times \left(\dfrac{2}{3}\right)^{-5} = \left(\dfrac{2}{3}\right)^{5-5} = \left(\dfrac{2}{3}\right)^0 = 1$

Example 2. Simplify: 1) $\dfrac{5^4 \times 8 \times 16}{2^5 \times 125}$ 2) $\dfrac{3 \cdot 2^n - 4 \cdot 2^{n-2}}{2^n - 2^{n-1}}$

Solution:

1) $\dfrac{5^4 \times 8 \times 16}{2^5 \times 125} = \dfrac{5^4 \times 2^3 \times 2^4}{2^5 \times 5^3} = \dfrac{5^4 \times 2^{3+4}}{5^3 \times 2^5} = \dfrac{5^4}{5^3} \times \dfrac{2^7}{2^5}$

$= 5^{4-3} \times 2^{7-5} = 5^1 \times 2^2 = 5 \times 4 = 20$

2) $\dfrac{3 \cdot 2^n - 4 \cdot 2^{n-2}}{2^n - 2^{n-1}} = \dfrac{3 \cdot 2^n - 2^2 \cdot 2^{n-2}}{2^n - 2^n \cdot 2^{-1}} = \dfrac{3 \cdot 2^n - 2^{2+n-2}}{2^n - 2^n \cdot \dfrac{1}{2}}$

$$= \frac{3 \cdot 2^n - 2^n}{\left(1 - \dfrac{1}{2}\right) \cdot 2^n} = \frac{(3-1) \cdot 2^n}{\dfrac{1}{2} \cdot 2^n} = \frac{2 \cdot 2^n}{\dfrac{1}{2} \cdot 2^n} = 2 \cdot 2 = 4$$

Example 3. Prove that, $(a^p)^{q-r} \cdot (a^q)^{r-p} \cdot (a^r)^{p-q} = 1$

Solution: $(a^p)^{q-r} \cdot (a^q)^{r-p} \cdot (a^r)^{p-q} = a^{p(q-r)} \cdot a^{q(r-p)} \cdot a^{r(p-q)}$ $[\because (a^m)^n = a^{mn}]$

$= a^{pq-pr} \cdot a^{qr-pq} \cdot a^{pr-qr} = a^{pq-pr+qr-pq+pr-qr} = a^0 = 1$

Work: Fill in the blanks:

1) $3 \times 3 \times 3 \times 3 = 3^{\square}$ 2) $5^{\square} \times 5^3 = 5^5$ 3) $a^2 \times a^{\square} = a^{-3}$

4) $(-5)^0 = \square$ 5) $\dfrac{4}{4^{\square}} = 1$

n th Root of a

We observe, $5^{\frac{1}{2}} \times 5^{\frac{1}{2}} = \left(5^{\frac{1}{2}}\right)^2$

Again, $5^{\frac{1}{2}} \times 5^{\frac{1}{2}} = 5^{\frac{1}{2}+\frac{1}{2}} = 5^{2\times\frac{1}{2}} = 5$

$\therefore \left(5^{\frac{1}{2}}\right)^2 = 5$

Square (power 2) of $(5^{\frac{1}{2}}) = 5$ and square root (second root) of $5 = 5^{\frac{1}{2}}$

$5^{\frac{1}{2}}$ is written as $\sqrt{5}$.

Again we observe, $5^{\frac{1}{3}} \times 5^{\frac{1}{3}} \times 5^{\frac{1}{3}} = \left(5^{\frac{1}{3}}\right)^3$

Again, $5^{\frac{1}{3}} \times 5^{\frac{1}{3}} \times 5^{\frac{1}{3}} = 5^{\frac{1}{3}+\frac{1}{3}+\frac{1}{3}} = 5^{3\times\frac{1}{3}} = 5$

Cube (power 3) of $5^{\frac{1}{3}} = 5$ and cube root (third root) of $5 = 5^{\frac{1}{3}}$

$5^{\frac{1}{3}}$ is written as $\sqrt[3]{5}$.

In the case of n th root,

$a^{\frac{1}{n}} \times a^{\frac{1}{n}} \times a^{\frac{1}{n}} \times \ldots \times a^{\frac{1}{n}}$ [successive multiplication of n $a^{\frac{1}{n}}$'s] $= \left(a^{\frac{1}{n}}\right)^n$

Again, $a^{\frac{1}{n}} \times a^{\frac{1}{n}} \times a^{\frac{1}{n}} \times \ldots \times a^{\frac{1}{n}}$

$= a^{\frac{1}{n}+\frac{1}{n}+\frac{1}{n}+\ldots+\frac{1}{n}}$ [in the exponent, sum of n $\frac{1}{n}$'s]

$= a^{n\times\frac{1}{n}} = a$

$\therefore \left(a^{\frac{1}{n}}\right)^n = a$

The n th power of $a^{\frac{1}{n}}$ is a and the n th root of a is $a^{\frac{1}{n}}$

i.e, n th power of $a^{\frac{1}{n}}$ is $\left(a^{\frac{1}{n}}\right)^n = a$ and n th root of a is $(a)^{\frac{1}{n}} = a^{\frac{1}{n}} = \sqrt[n]{a}$

The n th root of a is written as $\sqrt[n]{a}$.

Example 4. Simplify: 1) $(12)^{-\frac{1}{2}} \times \sqrt[3]{54}$ 2) $(-3)^3 \times \left(-\frac{1}{2}\right)^2$

Solution:

1) $(12)^{-\frac{1}{2}} \times \sqrt[3]{54} = \frac{1}{(12)^{\frac{1}{2}}} \times (54)^{\frac{1}{3}} = \frac{1}{(2^2 \times 3)^{\frac{1}{2}}} \times (3^3 \times 2)^{\frac{1}{3}}$

$= \frac{1}{(2^2)^{\frac{1}{2}} \times 3^{\frac{1}{2}}} \times (3^3)^{\frac{1}{3}} \cdot 2^{\frac{1}{3}} = \frac{1}{2 \cdot 3^{\frac{1}{2}}} \times (3^3 \times 2)^{\frac{1}{3}}$

$= \frac{2^{\frac{1}{3}}}{2^1} \times \frac{3^1}{3^{\frac{1}{2}}} = \frac{3^{1-\frac{1}{2}}}{2^{1-\frac{1}{3}}} = \frac{3^{\frac{1}{2}}}{2^{\frac{2}{3}}} = \frac{3^{\frac{1}{2}}}{4^{\frac{1}{3}}} = \frac{\sqrt{3}}{\sqrt[3]{4}}$

2) $(-3)^3 \times \left(-\frac{1}{2}\right)^2 = (-3)(-3)(-3) \times \left(-\frac{1}{2}\right)\left(-\frac{1}{2}\right) = -27 \times \frac{1}{4} = -\frac{27}{4}$

Work: Simplify: 1) $\frac{2^4 \cdot 2^2}{32}$ 2) $\left(\frac{2}{3}\right)^{\frac{5}{2}} \times \left(\frac{2}{3}\right)^{-\frac{5}{2}}$ 3) $8^{\frac{3}{4}} \div 8^{\frac{1}{2}}$

To be noticed:

1) Under the condition $a > 0,\ a \neq 1$ if $a^x = a^y$, then $x = y$
2) Under the condtion $a > 0,\ b > 0,\ x \neq 0$ if $a^x = b^x$, then $a = b$

Example 5. Solve: $4^{x+1} = 32$

Solution: $4^{x+1} = 32$ or, $(2^2)^{x+1} = 32$ or, $2^{2x+2} = 2^5$

$\therefore 2x + 2 = 5$ [if $a^x = a^y$, $x = y$]

or, $2x = 5 - 2$ or, $2x = 3$

$\therefore x = \frac{3}{2}$

Exercises 4.1

Simplify (1 - 8):

1. $\frac{7^3 \times 7^{-3}}{3 \times 3^{-4}}$
2. $\frac{\sqrt[3]{7^2} \cdot \sqrt[3]{7}}{\sqrt{7}}$
3. $(2^{-1} + 5^{-1})^{-1}$
4. $(2a^{-1} + 3b^{-1})^{-1}$
5. $\left(\frac{a^2b^{-1}}{a^{-2}b}\right)^2$
6. $\sqrt{x^{-1}y} \cdot \sqrt{y^{-1}z} \cdot \sqrt{z^{-1}x}$ $(x > 0,\ y > 0,\ z > 0)$
7. $\frac{2^{n+4} - 4 \cdot 2^{n+1}}{2^{n+2} \div 2}$
8. $\frac{3^{m+1}}{(3^m)^{m-1}} \div \frac{9^{m+1}}{(3^{m-1})^{m+1}}$

Prove (9 - 15):

9. $\dfrac{4^n-1}{2^n-1}=2^n+1$

10. $\dfrac{2^{2p+1}\cdot 3^{2p+q}\cdot 5^{p+q}\cdot 6^p}{3^{p-2}\cdot 6^{2p+2}\cdot 10^p\cdot 15^q}=\dfrac{1}{2}$

11. $\left(\dfrac{a^l}{a^m}\right)^n\cdot\left(\dfrac{a^m}{a^n}\right)^l\cdot\left(\dfrac{a^n}{a^l}\right)^m=1$

12. $\dfrac{a^{p+q}}{a^{2r}}\times\dfrac{a^{q+r}}{a^{2p}}\times\dfrac{a^{r+p}}{a^{2q}}=1$

13. $\left(\dfrac{x^a}{x^b}\right)^{\frac{1}{ab}}\cdot\left(\dfrac{x^b}{x^c}\right)^{\frac{1}{bc}}\cdot\left(\dfrac{x^c}{x^a}\right)^{\frac{1}{ca}}=1$

14. $\left(\dfrac{x^a}{x^b}\right)^{a+b}\cdot\left(\dfrac{x^b}{x^c}\right)^{b+c}\cdot\left(\dfrac{x^c}{x^a}\right)^{c+a}=1$

15. $\left(\dfrac{x^p}{x^q}\right)^{p+q-r}\cdot\left(\dfrac{x^q}{x^r}\right)^{q+r-p}\cdot\left(\dfrac{x^r}{x^p}\right)^{r+p-q}=1$

16. If $a^x=b$, $b^y=c$ and $c^z=a$, then show that, $xyz=1$

Solve (17 - 20):

17. $4^x=8$

18. $2^{2x+1}=128$

19. $(\sqrt{3})^{x+1}=(\sqrt[3]{3})^{2x-1}$

20. $2^x+2^{1-x}=3$

21. $P=x^a$, $Q=x^b$ and $R=x^c$

1) Find the values of $P^{bc}\cdot Q^{-ca}$.

2) Find the values of $\left(\dfrac{P}{Q}\right)^{a+b}\times\left(\dfrac{Q}{R}\right)^{b+c}\div 2(RP)^{a-c}$

3) Show that, $\left(\dfrac{P}{Q}\right)^{a^2+ab+b^2}\times\left(\dfrac{Q}{R}\right)^{b^2+bc+c^2}\times\left(\dfrac{R}{P}\right)^{c^2+ca+a^2}=1$

22. $X=(2a^{-1}+3b^{-1})^{-1}$, $Y=\sqrt[pq]{\dfrac{x^p}{x^q}}\times\sqrt[qr]{\dfrac{x^q}{x^r}}\times\sqrt[rp]{\dfrac{x^r}{x^p}}$

and $Z=\dfrac{5^{m+1}}{(5^m)^{m-1}}\div\dfrac{25^{m+1}}{(5^{m-1})^{m+1}}$, where $x,p,q,r>0$

1) Find the value of X.

2) Show that, $Y+\sqrt[3]{81}=4$

3) Show that, $Y\div Z=25$

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Logarithms

Logarithms are used to find the values of exponential expressions. Logarithm is written in brief as **Log**. Product, quotient, etc. of large numbers or quantities can easily be determined with the help of log.

We know, $2^3 = 8$; this mathematical statement is written in terms of log as $\log_2 8 = 3$. Again, conversely, if $\log_2 8 = 3$, it can be written in terms of exponents as $2^3 = 8$. That is, if $2^3 = 8$, then $\log_2 8 = 3$ and conversely, if $\log_2 8 = 3$, then $2^3 = 8$. Similarly, $2^{-3} = \frac{1}{2^3} = \frac{1}{8}$ can be written in terms of log as, $\log_2 \frac{1}{8} = -3$.

If $a^x = N, (a > 0,\ a \neq 1)$, then $x = \log_a N$ is defined as a **based** $log_a N$

Note: If $a > 0$, then a^x is always positive whatever may be the values of x, positive or negative. So, only the log of positive numbers has values that are real; log of zero or negative numbers have no real value.

Work: Express the following values in exponents in terms of log:

In terms of exponent	in terms of log
$10^2 = 100$	
$3^{-2} = \frac{1}{9}$	
$2^{\frac{1}{2}} = \sqrt{2}$	
$2^{-\frac{1}{2}} = \frac{1}{\sqrt{2}}$	
$\sqrt[4]{2^4} = 2$	

In terms of exponent	in terms of log
$10^0 = 1$	$\log_{10} 1 = 0$
$e^0 = \ldots$ $a^0 = 1$	$\log_e 1 = \ldots$ $\ldots = \ldots$
$10^1 = 10$	$\log_{10} 10 = 1$
$e^1 = \ldots$	$\ldots = \ldots$
$\ldots = \ldots$	$\log_a a = 1$

Laws of Logarithms

Let, $a > 0,\ a \neq 1;\ b > 0,\ b \neq 1$ and $M > 0,\ N > 0$

Formula 6 (zero and one log). If $a > 0,\ a \neq 1$ 1) $\log_a 1 = 0$ 2) $\log_a a = 1$

Proof: We know from the formula of exponents, $a^0 = 1$

$\therefore$ from the definition of log, we get, $\log_a 1 = 0$ (proved)

Again, we know, from the formula of exponents, $a^1 = a$

$\therefore$ from the definition of log, we get, $\log_a a = 1$ (proved)

Formula 7 (Log of products). $\log_a (MN) = \log_a M + \log_a N$

Proof: Let, $\log_a M = x,\ \log_a N = y$

$\therefore M = a^x,\ N = a^y$

Now, $MN = a^x \cdot a^y = a^{x+y}$

$\therefore \log_a(MN) = x + y$

or, $\log_a(MN) = \log_a M + \log_a N$ [putting the values of x, y]

$\therefore \log_a(MN) = \log_a M + \log_a N$ (proved)

Note: $\log_a(MNP\ldots) = \log_a M + \log_a N + \log_a P + \ldots$

Note: $\log_a(M \pm N) \neq \log_a M \pm \log_a N$

Formula 8 (log of quotient). $\log_a \dfrac{M}{N} = \log_a M - \log_a N$

Proof: Let, $\log_a M = x,\ \log_a N = y$

$\therefore M = a^x,\ N = a^y$

Now, $\dfrac{M}{N} = \dfrac{a^x}{a^y} = a^{x-y}$

$\therefore \log_a \dfrac{M}{N} = x - y$

$\therefore \log_a \dfrac{M}{N} = \log_a M - \log_a N$ (proved)

Formula 9 (Log of power). $\log_a M^r = r\log_a M$

Proof: Let, $\log_a M = x,\ \therefore M = a^x$

or, $(M)^r = (a^x)^r$ or, $M^r = a^{rx}$

$\therefore \log_a M^r = rx$ or, $\log_a M^r = r \log_a M$

$\therefore \log_a M^r = r\log_a M$ (proved)

Note: $(\log_a M)^r$ and $r\log_a M$ may not be equal.

For example, $(\log_2 4)^5 = (\log_2 2^2)^5 = 2^5 = 32,\ 5\log_2 4 = 5 \cdot 2 = 10 \neq 32$

Formula 10 (change of base). $\log_a M = \log_b M \times \log_a b$

Proof: Let, $\log_a M = x,\ \log_b M = y$

$\therefore a^x = M,\ b^y = M$

$\therefore a^x = b^y$ or, $(a^x)^{\frac{1}{y}} = (b^y)^{\frac{1}{y}}$ or, $b = a^{\frac{x}{y}}$

$\therefore \dfrac{x}{y} = \log_a b$ or, $x = y\log_a b$

or, $\log_a M = \log_b M \times \log_a b$ (proved)

Corollary 1. $\log_a b = \dfrac{1}{\log_b a}$ or, $\log_b a = \dfrac{1}{\log_a b}$

Proof: We know, $\log_a M = \log_b M \times \log_a b$

Putting $M = a$ we get, $\log_a a = \log_b a \times \log_a b$

or, $1 = \log_b a \times \log_a b$

$\therefore \log_a b = \dfrac{1}{\log_b a}$ or $\log_b a = \dfrac{1}{\log_a b}$ (proved)

Example 6. Find the values: 1) $\log_{10}100$ 2) $\log_3\dfrac{1}{9}$ 3) $\log_{\sqrt{3}}81$

Solution:

1) $\log_{10}100 = \log_{10}10^2 = 2\log_{10}10$ $[\because \log_{10}M^r = r\log_{10}M]$

$= 2 \times 1 = 2$ $[\because \log_a a = 1]$

2) $\log_3\left(\dfrac{1}{9}\right) = \log_3\left(\dfrac{1}{3^2}\right) = \log_3 3^{-2} = -2\log_3 3$ $[\because \log_a M^r = r\log_a M]$

$= -2 \times 1 = -2$ $[\because \log_a a = 1]$

3) $\log_{\sqrt{3}}81 = \log_{\sqrt{3}}3^4 = \log_{\sqrt{3}}\{(\sqrt{3})^2\}^4 = \log_{\sqrt{3}}(\sqrt{3})^8$

$= 8\log_{\sqrt{3}}\sqrt{3} = 8 \times 1 = 8$ $[\because \log_a a = 1]$

Example 7. 1) What is the log of $5\sqrt{5}$ to the base of 5? 2) If the log of 400 is 4, then what is the base of log?

Solution:

1) Log of $5\sqrt{5}$ to the base of 5

$= \log_5 5\sqrt{5} = \log_5(5 \times 5^{\frac{1}{2}}) = \log_5 5^{\frac{3}{2}}$

$= \dfrac{3}{2}\log_5 5$ $[\because \log_a M^r = r\log_a M]$

$= \dfrac{3}{2} \times 1 = \dfrac{3}{2}$ $[\because \log_a a = 1]$

2) Let the base be a

$\therefore$ By the question, $\log_a 400 = 4$

$\therefore a^4 = 400$ or, $a^4 = (20)^2 = \{(2\sqrt{5})^2\}^2 = (2\sqrt{5})^4$

$\therefore a = 2\sqrt{5} \qquad [\because a^x = b^x,\ a^x \neq 0,\ a = b]$

$\therefore$ Base $2\sqrt{5}$

Example 8. Find the value of x: 1) $\log_{10}x = -2$ 2) $\log_x 324 = 4$

Solution:

1) $\log_{10}x = -2$

or, $x = 10^{-2} = \dfrac{1}{10^2} = \dfrac{1}{100} = 0.01$

$\therefore x = 0.01$

2) $\log_x 324 = 4$

or, $x^4 = 324 = 3 \times 3 \times 3 \times 3 \times 2 \times 2 = 3^4 \times 2^2$

or, $x^4 = 3^4 \times (\sqrt{2})^4$

or, $x^4 = (3\sqrt{2})^4$

$\therefore x = 3\sqrt{2}$

Example 9. Prove that, $3\log_{10}2 + \log_{10}5 = \log_{10}40$

Solution: Left hand side $= 3\log_{10}2 + \log_{10}5$

$= \log_{10}2^3 + \log_{10}5 \qquad [\because \log_a M^r = r\log_a M]$

$= \log_{10}8 + \log_{10}5$

$= \log_{10}(8 \times 5) \qquad [\because \log_a(MN) = \log_a M + \log_a N]$

$= \log_{10}40 =$ right hand side (proved)

Example 10. Simplify: $\dfrac{\log_{10}\sqrt{27} + \log_{10}8 - \log_{10}\sqrt{1000}}{\log_{10}1.2}$

Solution: $\dfrac{\log_{10}\sqrt{27} + \log_{10}8 - \log_{10}\sqrt{1000}}{\log_{10}1.2}$

$$= \frac{\log_{10}(3^3)^{\frac{1}{2}} + \log_{10}8 - \log_{10}(10^3)^{\frac{1}{2}}}{\log_{10}\dfrac{12}{10}}$$

$$= \frac{\log_{10}3^{\frac{3}{2}} + \log_{10}2^3 - \log_{10}(10)^{\frac{3}{2}}}{\log_{10}12 - \log_{10}10}$$

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$$= \frac{\frac{3}{2}\log_{10}3 + 3\log_{10}2 - \frac{3}{2}\log_{10}10}{\log_{10}(3 \times 2^2) - \log_{10}10}$$

$$= \frac{\frac{3}{2}(\log_{10}3 + 2\log_{10}2 - 1)}{\log_{10}3 + 2log_{10}2 - 1} \qquad [\because \log_{10}10 = 1]$$

$$= \frac{3}{2}$$

Exercises 4.2

1. Find the values:

   1) $\log_3 81$ 2) $\log_5 \sqrt[3]{5}$ 3) $\log_4 2$

   4) $\log_{2\sqrt{5}} 400$ 5) $\log_5(\sqrt[3]{5} \cdot \sqrt{5})$

2. Find the value of x :

   1) $\log_5 x = 3$ 2) $\log_x 25 = 2$ 3) $\log_x \frac{1}{16} = -2$

3. Show that,

   1) $5\log_{10}5 - \log_{10}25 = \log_{10}125$

   2) $\log_{10}\frac{50}{147} = \log_{10}2 + 2\log_{10}5 - \log_{10}3 - 2\log_{10}7$

   3) $3\log_{10}2 + 2\log_{10}3 + \log_{10}5 = \log_{10}360$

4. Simplify:

   1) $7\log_{10}\frac{10}{9} - 2\log_{10}\frac{25}{24} + 3\log_{10}\frac{81}{80}$

   2) $\log_7(\sqrt[5]{7} \cdot \sqrt{7}) - \log_3\sqrt[3]{3} + \log_4 2$

   3) $\log_e\frac{a^3b^3}{c^3} + \log_e\frac{b^3c^3}{d^3} + \log_e\frac{c^3d^3}{a^3} - 3\log_e b^2c$

5. $x = 2, y = 3, z = 5, w = 7$

   1) What is the log of $\sqrt{y^3}$ to the base 3.

   2) Find the value of $w\log\frac{xz}{y^2} - x\log\frac{z^2}{x^2y} + y\log\frac{y^4}{x^4z}$

   3) Show that, $\dfrac{\log\sqrt{y^3} + y\log x - \frac{y}{x}\log(xz)}{\log(xy) - \log z} = \log_y\sqrt{y^3}$

Scientific or Standard Form of Numbers

We can express very large numbers or very small numbers easily by using exponents.

Such as, the velocity of light = 300000 km/sec = 300000000 m/sec

$= 3 \times 100000000$ m/sec $= 3 \times 10^8$ m/sec

Again, the radius of a hydrogen atom

$$= 0.0000000037 \text{ cm}$$
$$= \frac{37}{10000000000} \text{ cm} = 37 \times 10^{-10} \text{ cm}$$
$$= 3.7 \times 10 \times 10^{-10} \text{ cm} = 3.7 \times 10^{-9} \text{ cm}$$

For convenience, very large or very small numbers are expressed in the form $a \times 10^n$ where, $1 \leq a < 10$ and $n \in Z$. The number of the form $a \times 10^n$ is called the **scientific or standard form** of number.

Work: Express the following numbers in scientific form :

1) 15000 2) 0.000512 3) 123.000512

Logarithmic Methods

Logarithmic systems are of two kinds :

1) **Natural Logarithm:** Mathematician John Napier (1550-1617) of Scotland published the first book on logarithms in 1614 by taking e as its base. e is an irrational number, $e = 2.71828\ldots$. Such logarithms are called **Napierian logarithm** or e **based logarithm** or **natural logarithm**. $\log_e x$ is also written in the form $\ln x$.

2) **Common Logarithm:** Mathematician Henry Briggs (1561-1630) of England prepared a **table of logarithm** (**log table**) in 1624 with 10 as the base. This logarithm is called **Briggs logarithm** or 10 **based logarithm** or **practical logarithm**. This logarithm is written as $\log_{10} x$.

Note: If there is no mention of base, in case of algebraic expressions e and in case of numbers 10, are considered the base. In log table 10 is taken as the base.

Characteristics of Common Log

Let a number N be expressed in the scientific form as

$N = a \times 10^n$, where $N > 0, 1 \leq a < 10$ and $n \in Z$

Taking base 10 log of both sides we get.

$\log_{10}N = \log_{10}(a \times 10^n) = \log_{10}a + \log_{10}10^n = \log_{10}a + n\log_{10}10$

$\therefore \log_{10}N = n + \log_{10}a$ [$\because \log_{10}10 = 1$]

Suppressing the base 10, we have, $\log N = n + \log a$

n is called the characteristic of $\log N$.

Note: It is evident from the table that characteristic of a number having m digits to the left of decimal point is 1 less than the number of digits to the left of decimal sign and that will be positive. That is if the mentioned number of digits is m, the characteristic of logarithm will be m-1.

N	Scientific form of N	**Exponent**	Number of digits on the left of the decimal point	Characteristic
6237	6.237×10^3	3	4	$4 - 1 = 3$
623.7	6.237×10^2	2	3	$3 - 1 = 2$
62.37	6.237×10^1	1	2	$2 - 1 = 1$
6.237	6.237×10^0	0	1	$1 - 0 = 0$
0.6237	6.237×10^{-1}	-1	0	$0 - 1 = -1$

Note: Now observe from the following table: If there is no integral part of a number and the number has k 0's immediately after decimal point, then the characteristic of the logarithm of the number is $\{-(k+1)\}$.

If any characteristic is negative, '$-$' sign is not placed on the left of the characteristic, rather '$-$' (bar sign) over the characteristic. Such as, characteristic -3 will be written as $\bar{3}$. Otherwise, whole part of the log including mantissa will be negative.

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N	Scientific form of N	Exponent	Number of zeroes between decimal point and its next first significant digit	Characteristic
0.6237	6.237×10^{-1}	-1	0	$-(0+1) = -1 = \bar{1}$
0.06237	6.237×10^{-2}	-2	1	$-(1+1) = -2 = \bar{2}$
0.006237	6.237×10^{-3}	-3	2	$-(2+1) = -3 = \bar{3}$

Note: Characteristic may be either positive or negative, but mantissa will always be positive.

Example 11. Find the characteristics of log of the following numbers :

1) 5570 2) 45.70 3) 0.4305 4) 0.000435

Solution:

1) $5570 = 5.570 \times 1000 = 5.570 \times 10^3$

$\therefore$ Characteristic of log of the number is 3

Otherwise, the number of digits in the number 5570 is 4.

$\therefore$ Characteristic of log of the number is $= 4 - 1 = 3$

2) $45.70 = 4.570 \times 10^1$

$\therefore$ Characteristic of log of the number is 1

Otherwise, there are 2 digits in the integral part (i.e. on the left of the decimal point) of the number.

$\therefore$ Characteristic of log of the number is $= 2 - 1 = 1$

3) $0.4305 = 4.305 \times 10^{-1}$ $\therefore$ Characteristic of log of the number is -1

$\therefore$ Characteristic of log of the number is $= 0 - 1 = -1 = \bar{1}$

Otherwise, there is no zero in between decimal point and its next first significant digit of the number 0.4305, i.e. there is 0 zeroes.

$\therefore$ Characteristic of log of the number is $= -(0+1) = -1 = \bar{1}$

$\therefore$ Characteristic of log of the number 0.4305 is $\bar{1}$

4) $0.000435 = 4.35 \times 10^{-4}$

$\therefore$ Characteristic of log of the number is -4 or $\bar{4}$
Otherwise, there are 3 zeroes in between decimal point and its next first significant digit of the number 4.

$\therefore$ Characteristic of log of the number is $= -(3+1) = -4 = \bar{4}$

$\therefore$ Characteristic of log of the number 0.000435 is $\bar{4}$

Mantissa of Common Log

Mantissa of Common Logarithm of any number is a non-negative number less than 1. It is mainly an irrational number. But the value of mantissa is determined upto a certain decimal place. Mantissa of the log of a number can be found from the log table. Again, it can also be found by calculator. We shall find the mantissa of the log of any number by using 2nd method, that is by calculator.

Example 12. Find the characteristic and the mantissa of log2717 :

Solution: Let us use the calculator: $\boxed{AC}\ \boxed{\log}\ \boxed{2717}\ \boxed{=}$ 3.43409

$\therefore$ Characteristic of log2717 is 3 and the mantissa is .43409

Example 13. Find the characteristic and the mantissa of log43.517.

Solution: Let us use the calculator: $\boxed{AC}\ \boxed{\log}\ \boxed{43.517}\ \boxed{=}$ 1.63866

$\therefore$ Characteristic of log43.517 is 1 and mantissa is .63866

Example 14. What are the characteristics and mantissa of log 0.00836?

Solution: Let us use the calculator: $\boxed{AC}\ \boxed{\log}\ \boxed{0.00836}\ \boxed{=}$ -2.07779

$-2.07779 = -3 + 0.92221 = \bar{3}.92221$

$\therefore$ Characteristic of log0.00836 is -3 and the mantissa is .92221.

Example 15. Find $\log_e 10$:

Solution: $\log_e 10 = \dfrac{1}{\log_{10} e} = \dfrac{1}{\log_{10} 2.71828} = \dfrac{1}{0.43429}$ [using the calculator]

$= 2.30259$ (approx)

Alternative : We use the calculator : $\boxed{AC}\ \boxed{\ln}\ \boxed{10}\ \boxed{=}$ 2.30259

Work: Compute base 10 and base e logarithm of the following numbers by using calculator:

1) 2550 2) 52.143 3) 0.4145 4) 0.0742

Exercises 4.3

1. On what condition $a^0 = 1$?

   1) $a = 0$ 2) $a \neq 0$ 3) $a > 0$ 4) $a \neq 1$

2. Which one of the following is the value of $\sqrt[3]{5} \cdot \sqrt[3]{5}$?

   1) $\sqrt[6]{5}$ 2) $(\sqrt[3]{5})^3$ 3) $(\sqrt{5})^3$ 4) $\sqrt[3]{25}$

3. On what exact condition $\log_a a = 1$?

   1) $a > 0$ 2) $a \neq 1$ 3) $a > 0, a \neq 1$ 4) $a \neq 0, a > 1$

4. If $\log_x 4 = 2$, what is the value of x?

   1) 2 2) ± 2 3) 4 4) 10

5. Under what condition can a number be written in the form $a \times 10^n$?

   1) $1 < a < 10$ 2) $1 \leq a \leq 10$ 3) $1 \leq a < 10$ 4) $1 < a \leq 10$

6. If $a > 0,\ b > 0$ and $a \neq 1,\ b \neq 1$, then

   (i) $\log_a b \times \log_b a = 1$

   (ii) $\log_a M^r = M \log_a r$

   (iii) $\log_a(\sqrt[3]{a} \cdot \sqrt{a}) = \dfrac{5}{6}$

   Which of the above information are correct?

   1) i 2) ii 3) i and iii 4) ii and iii

7. What is the characteristic of the common log of 0.0035?

   1) 3 2) 1 3) $\bar{2}$ 4) $\bar{3}$

   Considering the number 0.0225 answer the following questions (8 – 10):

8. Which one of the following is in the form a^n?

1) $(2.5)^2$ 2) $(.015)^2$ 3) $(1.5)^2$ 4) $(.15)^2$

9. Which one of the following is the scientific form of the number ?
   1) 225×10^{-4} 2) 22.5×10^{-3} 3) 2.25×10^{-2} 4) $.225 \times 10^{-1}$

10. What is the characteristic of the common log of the number ?
    1) $\bar{2}$ 2) $\bar{1}$ 3) 0 4) 2

11. Express into scientific form:
    1) 6530 2) 60.831 3) 0.000245 4) 37500000
    5) 0.00000014

12. Express in ordinary decimals:
    1) 10^5 2) 10^{-5} 3) 2.53×10^4 4) 9.813×10^{-3}
    5) 3.12×10^{-5}

13. Find the characteristics of common logarithm of the following numbers (without using calculator) :
    1) 4820 2) 72.245 3) 1.734 4) 0.045
    5) 0.000036

14. Find the characteristics and mantissa of the common logarithm of the following numbers by using calculator:
    1) 27 2) 63.147 3) 1.405 4) 0.0456
    5) 0.000673

15. Find the common logarithm of the product/quotient (approximate value upto five decimal places) :
    1) 5.34×8.7 2) 0.79×0.56 3) $22.2642 \div 3.42$
    4) $0.19926 \div 32.4$

16. If $\log 2 = 0.30103$, $\log 3 = 0.47712$ and $\log 7 = 0.85410$, find the value of the following expressions :
    1) $\log 9$ 2) $\log 28$ 3) $\log 42$

17. Given, $x = 1000$ and $y = 0.0625$
    1) Express x in the form $a^n b^n$, where a and b are prime numbers.
    2) Express the product of x and y in scientific form.
    3) Find the characteristic and mantissa of the common logarithm of xy.

Chapter 5

Equations in One Variable

We have known in the previous class what equation is and learnt its usage. We have also learnt the solution of simple equations with one variable and acquired knowledge thoroughly about the solution of simple equations by forming equations from real life problems. In this chapter, linear and quadratic equations and identities have been discussed and their usages have been shown to solve the real life problems.

At the end of the chapter, the students will be able to —

- ▶ explain the conception of variable.
- ▶ explain the difference between equation and identity.
- ▶ solve the linear equations.
- ▶ solve by forming linear equations based on real life problems.
- ▶ solve the quadratic equations and find the solution sets.
- ▶ form the quadratic equations based on real life problems and solve.

Variable

We know, $x+3=5$ is an equation. To solve it, we find the value of the unknown quantity x. Here the unknown quantity x is a variable. Again, to solve the equation $x+a=5$, we find the value of x, not the value of a. Here, x is assumed as variable and a as constant. In this case, we shall get the values of x in terms of a. But if we determine the value of a, we shall write $a=5-x$; that is, the value of a will be obtained in terms of x. Here a is considered a variable and x a constant. But if no direction is given, conventionally x is considered a variable. Generally, the small letter x, y, z, the ending part of English alphabet are taken as variables and a, b, c, the starting part of the alphabet are taken as constants.

The equation, which contains only one variable, is called a linear equation with one variable. Such as, $x+3=5,\ x^2-5x+b=0,\ 2y^2+5y-3=0$ etc.

We know what the set is. If a set, $S = \{x : x \in R, 1 \leq x \leq 10\}$, x-may be any real number from 1 to 10. Here, x is a variable. So, we can say that when a letter symbol means the element of a set, it is called variable.

Degree of an equation : The highest degree of a variable in any equation is called the degree of the equation. Degree of each of the equations $x + 1 = 5$, $2x - 1 = x + 5$, $y + 7 = 2y - 3$ is 1; these are linear equations with one variable.

Again, the degree of each of the equations $x^2 + 5x + 6 = 0$, $y^2 - y = 12$, $4x^2 - 2x = 3 - 6x$ is 2; these are linear equations with two variable. The equation $2x^3 - x^2 - 4x + 4 = 0$ is the equation of degree 3 with one variable.

Equation and Identity

Equation: There are two polynomials on two sides of the equal sign of an equation, or there may be zero on one side (mainly on right hand side). Degree of the variable of the polynomials on two sides may not be equal. Solving an equation, we get the number of values of the variable equal to the highest degree of that variable. This value or these values are called the roots of the equation. The equation will be satisfied by the root or roots. In the case of more than one root, these may be equal or unequal. Such as, roots of $x^2 - 5x + 6 = 0$ are $2, 3$. Again, though the value of x in equation $(x - 3)^2 = 0$ is 3, the roots of the equations are $3, 3$.

Identity: There are two polynomials of same (equal) degree on two sides of equal sign. Identity will be satisfied by more values than the number of highest degree of the variable. There is no difference between the two sides of equal sign; that is why, it is called identity. Such as, $(x+1)^2 - (x-1)^2 = 4x$ is an identity; it will be satisfied for all values of x. So this equation is an identity. Each algebraic formula is an identity. Such as, $(a+b)^2 = a^2 + 2ab + b^2$, $(a-b)^2 = a^2 - 2ab + b^2$, $a^2 - b^2 = (a+b)(a-b)$, $(a+b)^3 = a^3 + 3a^2b + 3ab^2 + b^3$ etc. are identities.

All equations are not identities, In identity $\equiv$ sign is used instead of equal ($=$) sign. But as all identities are equations, in the case of identity also, generally the equal sign is used.

Distinctions between equation and identity are given below :

Equation	Identity
1. Two polynomials may exist on both sides of equal sign, or there may be zero on one side.	1. Two polynomials exist on two sides.
2. Degree of the polynomials on both sides may be unequal.	2. Degree of the polynomials on both sides is equal.
3. The equality is true for one or more values of the variable.	3. Generally, the equality is true for all values of the original set of the variable.
4. The number of values of the variable may be equal to the highest degree of the equation	4. Equality is true for infinite number of values of the variable.
5. All equations are not identities.	5. All algebraic identities are equations.

Work:

1) What is the degree of and how many roots has each of the following equations?

   (1) $3x + 1 = 5$ (2) $\frac{2y}{5} - \frac{y-1}{3} = \frac{3y}{2}$

2) Write down three identities.

Solving Linear Equations

In case of solving equations, some rules are to be applied. If the rules are known, solution of equations becomes easier. The rules are as follows:

1. If the same number or quantity is added to both sides of an equation, two sides remain equal.
2. If the same number or quantity is subtracted from both sides of an equation, two sides remain equal.
3. If both sides of an equation are multiplied by the same number or quantity, the two sides remain equal.
4. If both sides of an equation are divided by same non- zero number or quantity, the two sides remain equal.

The rules stated above may be expressed in terms of algebraic expressions as follows:

If $x = a$ and $c \neq 0$,

(i) $x + c = a + c$ $\quad$ (ii) $x - c = a - c$ $\quad$ (iii) $xc = ac$ $\quad$ (iv) $\dfrac{x}{c} = \dfrac{a}{c}$

Besides, if a, b and c are three quantities, if $a = b + c$, $a - b = c$ and if $a + c = b$, $a = b - c$.

This law is known as transposition law and different equations can be solved by applying this law. If the terms of an equation are in fractional form and if the degree of the variables in each numerator is 1 and the denominator in each term is constant, such equations are linear equations.

Example 1. Solve: $\dfrac{5x}{7} - \dfrac{4}{5} = \dfrac{x}{5} - \dfrac{2}{7}$

Solution: $\dfrac{5x}{7} - \dfrac{4}{5} = \dfrac{x}{5} - \dfrac{2}{7}$ or, $\dfrac{5x}{7} - \dfrac{x}{5} = \dfrac{4}{5} - \dfrac{2}{7}$ [by transposition]

or, $\dfrac{25x - 7x}{35} = \dfrac{28 - 10}{35}$ or, $\dfrac{18x}{35} = \dfrac{18}{35}$

or, $18x = 18$ or, $x = 1$

$\therefore$ Solution is $x = 1$

Now, we shall solve such equations which are in quadratic form. These equations are transformed into their equivalent equations by simplifications and lastly the equations is transformed into linear equation of the form $ax = b$. Again, even if there are variables in the denominator, they are also transformed into linear equation by simplification.

Example 2. Solve: $(y - 1)(y + 2) = (y + 4)(y - 2)$

Solution: $(y - 1)(y + 2) = (y + 4)(y - 2)$

or, $y^2 - y + 2y - 2 = y^2 + 4y - 2y - 8$

or, $y - 2 = 2y - 8$

or, $y - 2y = -8 + 2$ [by transposition]

or, $-y = -6$

or, $y = 6$

$\therefore$ Solution is $y = 6$

Example 3. Solve and write the solution set: $\dfrac{6x+1}{15} - \dfrac{2x-4}{7x-1} = \dfrac{2x-1}{5}$

Solution: $\dfrac{6x+1}{15} - \dfrac{2x-4}{7x-1} = \dfrac{2x-1}{5}$

or, $\dfrac{6x+1}{15} - \dfrac{2x-1}{5} = \dfrac{2x-4}{7x-1}$ [by transposition]

or, $\dfrac{6x+1-6x+3}{15} = \dfrac{2x-4}{7x-1}$

or, $\dfrac{4}{15} = \dfrac{2x-4}{7x-1}$

or, $15(2x-4) = 4(7x-1)$ [by cross-multiplication]

or, $30x - 60 = 28x - 4$

or, $30x - 28x = 60 - 4$ [by transposition]

or, $2x = 56$ or, $x = 28$

$\therefore$ Solution is $x = 28$

and solution set is $S = \{28\}$

Example 4. Solve: $\dfrac{1}{x-3} + \dfrac{1}{x-4} = \dfrac{1}{x-2} + \dfrac{1}{x-5}$

Solution: $\dfrac{1}{x-3} + \dfrac{1}{x-4} = \dfrac{1}{x-2} + \dfrac{1}{x-5}$

or, $\dfrac{x-4+x-3}{(x-3)(x-4)} = \dfrac{x-5+x-2}{(x-2)(x-5)}$

or, $\dfrac{2x-7}{x^2-7x+12} = \dfrac{2x-7}{x^2-7x+10}$

Values of the fractions of two sides are equal. Again, numerators of two sides are equal, but denominators are unequal. In this case, if only the value of the numerators is zero, two sides will be equal.

$\therefore 2x - 7 = 0$ or, $2x = 7$ or, $x = \dfrac{7}{2}$

$\therefore$ Solution is $x = \dfrac{7}{2}$

Work: If $(\sqrt{5}+1)x + 4 = 4\sqrt{5}$ then show that , $x = 6 - 2\sqrt{5}$

Usage of linear equations

In real life we have to solve different types of problems. In most cases of solving these problems mathematical knowledge, skill and logic are necessary. In real cases, in the application of mathematical knowledge and skill, as on one side the problems are solved smoothly, on the other side in daily life, solutions of the problems are obtained by mathematics. As a result, the students are interested in mathematics. Here different types of problems based on real life will be expressed by equations and they will be solved.

For determining the unknown quantity in solving the problems based on real life, variable is assumed instead of the unknown quantity and then equation is formed by the given conditions. Then by solving the equation, value of the variable, that is the unknown quantity is found.

Example 5. The digit of the units place of a number consisting of two digits is 2 more than the digit of its tens place. If the places of the digits are interchanged, the number thus formed will be less by 6 than twice the given number. Find the number.

Solution: Let the digit of tens place be x.Then the digit of units place will be $x+2$.

$\therefore$ the number is $10x+(x+2)$ or, $11x+2$

Now, if the places of the digits are interchanged, the changed number will be $10(x+2)+x$ or, $11x+20$

By the question, $11x+20=2(11x+2)-6$

or, $11x+20=22x+4-6$

or, $22x-11x=20+6-4$ [by transposition]

or, $11x=22$

or, $x=2$

$\therefore$ the number is $11x+2=11\times 2+2=24$

$\therefore$ given number is 24

Example 6. In a class if 4 students are seated in each bench, 3 benches remain vacant. But if 3 students are seated on each bench, 6 students are to remain standing. What is the number of students in that class ?

Solution: Let the number of students in the class be x

Since, if 4 students are seated in a bench, 3 benches remain vacant, the number of benches of that class $= \frac{x}{4} + 3$

Again, since, if 3 students are seated in each bench, 6 students are to remain standing, the number of benches of that class $= \frac{x-6}{3}$

Since the number of benches is fixed,

therefore, $\frac{x}{4} + 3 = \frac{x-6}{3}$ or, $\frac{x+12}{4} = \frac{x-6}{3}$

or, $4x - 24 = 3x + 36$ or, $4x - 3x = 36 + 24$

or, $x = 60$

$\therefore$ number of students of the class is 60

Example 7. Mr. Kabir, from his Tk. 56000, invested some money at the rate of profit 12% per annum and the rest of the money at the rate of profit 10% per annum. After one year he got the total profit of Tk. 6400. How much money did he invest at the rate of profit 12%?

Solution: Let Mr. Kabir invest Tk. x at the rate of profit 12%.

$\therefore$ he invested Tk. $(56000 - x)$ at the rate of profit 10%.

Now, profit of Tk. x in 1 year is Tk. $x \times \frac{12}{100}$ or Tk. $\frac{12x}{100}$.

Again, profit of Tk. $(56000 - x)$ in 1 year is Tk. $(56000 - x) \times \frac{10}{100}$ or, Tk. $\frac{10(56000-x)}{100}$.

By the question, $\frac{12x}{100} + \frac{10(56000-x)}{100} = 6400$

or, $12x + 560000 - 10x = 640000$

or, $2x = 640000 - 560000$

or, $2x = 80000$

or, $x = 40000$

$\therefore$ Mr. Kabir invested Tk. 40000 at the rate of profit 12%.

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Work: Solve by forming equations:

1) What is the number, if any same number is added to both numerator and denominator of the fraction $\frac{3}{5}$, the fraction will be $\frac{4}{5}$?
2) If the difference of the squares of two consecutive natural numbers is 151, find the two numbers.
3) If 120 coins of Tk.1 and Tk.2 together are Tk. 180, what is the number of coins of each kind ?

Exercises 5.1

Solve (1 - 8):

1. $\frac{ay}{b} - \frac{by}{a} = a^2 - b^2$
2. $(z+1)(z-2) = (z-4)(z+2)$
3. $\frac{4}{2x+1} + \frac{9}{3x+2} = \frac{25}{5x+4}$
4. $\frac{1}{x+1} + \frac{1}{x+4} = \frac{1}{x+2} + \frac{1}{x+3}$
5. $\frac{a}{x-a} + \frac{b}{x-b} = \frac{a+b}{x-a-b}$
6. $\frac{x-a}{b} + \frac{x-b}{a} + \frac{x-3a-3b}{a+b} = 0$
7. $\frac{x-a}{a^2-b^2} = \frac{x-b}{b^2-a^2}$
8. $(3+\sqrt{3})z + 2 = 5 + 3\sqrt{3}$

Find the solution set (9 - 14):

9. $2x + \sqrt{2} = 3x - 4 - 3\sqrt{2}$
10. $\frac{z-2}{z-1} = 2 - \frac{1}{z-1}$
11. $\frac{1}{x} + \frac{1}{x+1} = \frac{2}{x-1}$
12. $\frac{m}{m-x} + \frac{n}{n-x} = \frac{m+n}{m+n-x}$

13. $\dfrac{1}{x+2}+\dfrac{1}{x+5}=\dfrac{1}{x+3}+\dfrac{1}{x+4}$

14. $\dfrac{2t-6}{9}+\dfrac{15-2t}{12-5t}=\dfrac{4t-15}{18}$

Solve by forming equations (15 - 25):

15. A number is $\frac{2}{5}$ times of another number. If the sum of the numbers is 98, find the two numbers.

16. Difference of numerator and denominator of a proper fraction is 1; if 2 is subtracted from numerator and 2 is added to denominator of the fraction, it will be equal to $\frac{1}{6}$. Find the fraction.

17. Sum of the digits of a number consisting of two digits is 9; if the number obtained by interchanging the places of the digits is less by 45 than the given number, what is the number ?

18. The digit of the tens place of a number consisting of two digits is twice the digit of the units place. Show that, the number is seven times the sum of the digits.

19. A petty merchant by investing Tk. 5600 got the profit 5% on some of the money and profit of 4% the rest of the money. If the total profit is 256, then on how much money did he get the profit of 5%?

20. In a girls, school if 6 students sit in each bench, 2 benches remain empty. But if 5 students sit in each bench, 6 students have to remain standing. What is the number of benches in the class?

21. Number of passengers in a launch is 47. The fare per head for the cabin is twice that for the deck. The fare per head for the deck is Tk. 30. If the total fare collected is Tk. 1680 , what is the number of passengers in the cabin?

22. 120 coins of twenty five paisa and fifty paisa together is Tk. 35. What is the number of coins of each kind ?

23. A car passed over some distance at the speed of 60 km per hour and passed over the rest of the distance at the speed of 40 km per hour. The car passed

over the total distance of 240 km in 5 hours. How far did the car pass over at the speed of 60 km per hour ?

24. Distance between Dhaka New Market and Gabtoli is 12 km. From New Market Sajal departed for Gabtoli by rickshaw at the speed of 6 km per hour and Kajal from the same place departed for Gabtoli on foot at the speed of 4 km per hour. After reaching Gabtoli Sajal took rest for 30 minutes and then departed for New Market at the same speed. At which distance from New Market will they meet?

25. Number of passengers in a steamer is 376. The fare per head for the cabin is thrice that for the deck. The fare per head for the deck is Tk. 60 and the total fare collected is Tk. 33840.

    1) Letting the number of passengers on the deck as x, form an equation.

    2) What is the number of passengers on the deck and in the cabin?

    3) What is the fare per head for the cabin?

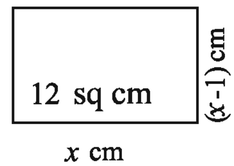
Quadratic Equations in One Variable

Equations of the form $ax^2 + bx + c = 0$ [where, a, b, c are constants and $a \neq 0$] are called the quadratic equation in one variable. Left hand side of a quadratic equation is a polynomial of second degree, right hand side is generally taken to be zero.

Length and breadth of a rectangular region of area 12 square cm. are respectively x cm. and $(x - 1)$ cm.

$\therefore$ area of the rectangular region is $= x(x-1)$ square cm.

By the question, $x(x-1) = 12$ or, $x^2 - x - 12 = 0$



x is the variable in the equation and highest power of x is 2. Such equation is a quadratic equation. The equation, which has the highest degree 2 of the variable, is called the quadratic equation.

In class VIII, we have factorized the quadratic expressions in one variable of the forms $x^2 + px + q$ and $ax^2 + bx + c$. Here, we shall solve the equations of the forms $x^2 + px + q = 0$ and $ax^2 + bx + c = 0$ by factorizing the left hand side and by finding the value of the variable.

An important law of real numbers is applied to the method of factorization. The law is as follows :

If the product of two quantities is equal to zero, either only one of the quantities or both quantities will be zero.That is, if the product of two quantities a and b i.e., $ab = 0$, $a = 0$ or, $b = 0$, or, both $a = 0$ and $b = 0$.

Example 8. Solve: $(x+2)(x-3) = 0$

Solution: $(x+2)(x-3) = 0$

$\therefore x + 2 = 0$ or, $x - 3 = 0$

If $x + 2 = 0$, $x = -2$

Again, if $x - 3 = 0$, $x = 3$

$\therefore$ solution is $x = -2$ or, $x = 3$

Example 9. Find the solution set: $y^2 = \sqrt{3}y$

Solution: $y^2 = \sqrt{3}y$

or, $y^2 - \sqrt{3}y = 0$ [by transposition, right hand side has been done zero]

or, $y(y - \sqrt{3}) = 0$

$\therefore y = 0$ or $y - \sqrt{3} = 0$

Again, if $y - \sqrt{3} = 0$, $y = \sqrt{3}$

$\therefore$ Solution set is $\{0, \sqrt{3}\}$

Example 10. Solve and write the solution set: $x - 4 = \dfrac{x-4}{x}$

Solution: $x - 4 = \dfrac{x-4}{x}$

or, $x(x-4) = x - 4$ [by cross-multiplication]

or, $x(x-4) - (x-4) = 0$ [by transposition]

or, $(x-4)(x-1) = 0$

$\therefore x - 4 = 0$ or $x - 1 = 0$

If $x - 4 = 0$, $x = 4$

Again, if $x - 1 = 0$, $x = 1$

$\therefore$ Solution set is $\{1, 4\}$

Example 11. Solve: $\left(\dfrac{x+a}{x-a}\right)^2 - 5\left(\dfrac{x+a}{x-a}\right) + 6 = 0$

Solution: $\left(\dfrac{x+a}{x-a}\right)^2 - 5\left(\dfrac{x+a}{x-a}\right) + 6 = 0 \ldots (1)$

Let, $\dfrac{x+a}{x-a} = y$

$\therefore$ from(1) we get, $y^2 - 5y + 6 = 0$

or, $y^2 - 2y - 3y + 6 = 0$

or, $y(y-2) - 3(y-2) = 0$

or, $(y-2)(y-3) = 0$

$\therefore$ when, y– $2 = 0$, $y = 2$

or, if $y - 3 = 0$, $y = 3$

Now, if $y = 2$,

$\dfrac{x+a}{x-a} = \dfrac{2}{1}$ [putting the value of y]

or, $x + a = 2(x - a)$ [by cross-multiplication]

or, $x + a = 2x - 2a$

or, $2x - x = a + 2a$

or, $x = 3a$

Again, when $y = 3$,

$\frac{x+a}{x-a} = \frac{3}{1}$

or, $x + a = 3(x - a)$ [by cross-multiplication]

or, $x + a = 3x - 3a$

or, $3x - x = a + 3a$

or, $x = 2a$

$\therefore$ Solution is $x = 2a$ or, $x = 3a$

Work:

1) Comparing the equation $x^2 - 1 = 0$ with $ax^2 + bx + c = 0$, write down the values of a, b, c.
2) What is the degree of the equation $(x-1)^2$? How many roots does it have and what are these?

Usage of quadratic equations

Many problems of our daily life can be solved easily by forming linear and quadratic equations. Here, the formation of quadratic equations from the given conditions based on real life problems and techniques for solving them are discussed.

Example 12. Denominator of a proper fraction is 4 more than the numerator. If the fraction is squared, its denominator will be 40 more than the numerator. Find the fraction.

Solution: Let the numerator of the fraction be x and denominator be $x + 4$

Hence the fraction is $\frac{x}{x+4}$

Square of the fraction $= \left(\frac{x}{x+4}\right)^2 = \frac{x^2}{(x+4)^2} = \frac{x^2}{x^2+8x+16}$

Here, numerator $= x^2$ and denominator $= x^2 + 8x + 16$

by the question, $x^2 + 8x + 16 = x^2 + 40$

2020

or, $8x + 16 = 40$

or, $8x = 40 - 16$

or, $8x = 24$

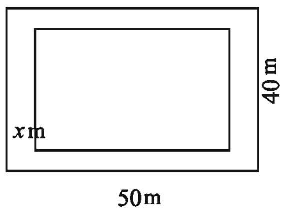
or, $x = 3$

$\therefore x + 4 = 3 + 4 = 7$

$\therefore \dfrac{x}{x+4} = \dfrac{3}{7}$

$\therefore$ the fraction is $\dfrac{3}{7}$

Example 13. A rectangular garden with length 50 metre and breadth 40 metre has of equal width all around the inside of the garden. If the area of the garden except the path is 1200 square metre, how much is the path wide in metre ?



Solution: Let the path be x metre wide.

Without the path length of the garden is $(50 - 2x)$ metre and breadth is $(40 - 2x)$ metre

$\therefore$ Without the path area of the garden $= (50 - 2x) \times (40 - 2x)$ square metre

By the question, $(50 - 2x) \times (40 - 2x) = 1200$

or, $2000 - 80x - 100x + 4x^2 = 1200$

or, $4x^2 - 180x + 800 = 0$

or, $x^2 - 45x + 200 = 0$ [dividing by 4]

or, $x^2 - 5x - 40x + 200 = 0$

or, $x(x - 5) - 40(x - 5) = 0$

or, $(x - 5)(x - 40) = 0$

$\therefore x - 5 = 0$ or $x - 40 = 0$

if $x - 5 = 0$, $x = 5$

if $x - 40 = 0$, $x = 40$

But the breadth of the path will be less than 40 metre from the breadth of the garden.

$\therefore x \neq 40;\ \therefore x = 5$

$\therefore$ the path is 5 metres wide

Example 14. Shahik bought some pens for Tk. 240. If he would get one more pen in that money, average cost of each pen would be less by Tk. 1. How many pens did he buy?

Solution: Let, shahik bought x pens in total for Tk. 240. Then each pen costs Tk. $\frac{240}{x}$.

If he would get $(x+1)$ pens by Tk. 240, then the cost of each pen would be Tk. $\frac{240}{x+1}$.

By the question, $\frac{240}{x+1} = \frac{240}{x} - 1$

or, $\frac{240}{x+1} = \frac{240-x}{x}$

or, $240x = (x+1)(240-x)$ [by cross-multiplication]

or, $240x = 240x + 240 - x^2 - x$

or, $x^2 + x - 240 = 0$ [by transposition]

or, $x^2 + 16x - 15x - 240 = 0$

or, $x(x+16) - 15(x+16) = 0$

or, $(x+16)(x-15) = 0$

$\therefore x + 16 = 0$, or $x - 15 = 0$

If $x + 16 = 0$, $x = -16$

If $x - 15 = 0$, $x = 15$

But the number of pen x cannot be negative.

$\therefore x \neq -16;\ \therefore x = 15$

$\therefore$ Shahik bought 15 pens.

Work: Solve by forming equations:

1) If a natural number is added to its square, the sum will be equal to nine times of exactly its next natural number. What is the number?
2) Length of a perpendicular drawn from the centre of a circle of radius 10 cm. to a chord is less by 2 cm, than the semi-chord. Find the length of the chord by drawing a probable picture.

Example 15. In an examination of class IX of a school, total marks of x students obtained in mathematics is 1950. If at the same examination, marks of a new student in mathematics is 34 and it is added to the former total marks, the average of the marks become less by 1.

1) Write down the average of the obtained marks of all students including the new student and separately x students in terms of x.
2) By forming equation from the given conditions, show that, $x^2+35x-1950=0$
3) By finding the value of x, find the average of the marks in the two cases

Solution:

1) Average of the marks obtained by x students $=\dfrac{1950}{x}$

   Average of the marks obtained by $(x+1)$ students including the new student $=\dfrac{1950+34}{x+1}=\dfrac{1984}{x+1}$

2) By the question, $\dfrac{1950}{x}=\dfrac{1984}{x+1}+1$

   or, $\dfrac{1950}{x}-\dfrac{1984}{x+1}=1$ [by transposition]

   or, $\dfrac{1950x+1950-1984x}{x(x+1)}=1$

   or, $x^2+x=1950x-1984x+1950$ [by cross-multiplication]

   or, $x^2+x=1950-34x$

   $\therefore x^2+35x-1950=0$ [showed]

3) $x^2+35x-1950=0$

   or, $x^2+65x-30x-1950=0$

   or, $x(x+65)-30(x+65)=0$

   or, $(x+65)(x-30)=0$

$\therefore x + 65 = 0$ or $x - 30 = 0$

If $x + 65 = 0$, $x = -65$

Again, if $x - 30 = 0$, $x = 30$

Since the number of students, i.e. x cannot be negative,

Hence, $x \neq -65$

$\therefore x = 30$

$\therefore$ in the first case, average $= \dfrac{1950}{30} = 65$ and in the second case,

average $= \dfrac{1984}{31} = 64$

Exercises 5.2

1. Assuming x as the variable in the equation $a^2x + b = 0$ which one of the following is the degree of the equation?
   1) 3 2) 2 3) 1 4) 0
2. Which one of the following is an identity ?
   1) $(x+1)^2 + (x-1)^2 = 4x$ 2) $(x+1)^2 + (x-1)^2 = 2(x^2+1)$
   3) $(a+b)^2 + (a-b)^2 = 2ab$ 4) $(a-b)^2 = a^2 + 2ab + b^2$
3. How many roots are there in the equation $(x-4)^2 = 0$?
   1) 1 2) 2 3) 3 4) 4
4. Which one of the following are the two roots of the equation $x^2 - x - 12 = 0$?
   1) $3, 4$ 2) $3, -4$ 3) $-3, 4$ 4) $-3, -4$
5. What is the coefficient of x in the equation $3x^2 - x + 5 = 0$?
   1) 3 2) 2 3) 1 4) -1
6. If the product of the two algebraic expressions x and y is $xy = 0$,then
   (i) $x = 0$ or $y = 0$
   (ii) $x = 0$ and $y \neq 0$
   (iii) $x \neq 0$ and $y = 0$
   Which one of the following is correct?

1) i and ii 2) ii and iii 3) i and iii 4) i, ii and iii

7. Which one of the following is the solution set of the equation $x^2-(a+b)x+ab=0$?

   1) $\{a,b\}$ 2) $\{a,-b\}$ 3) $\{-a,b\}$ 4) $\{-a,-b\}$

The digit of the tens place of a number consisting of two digits is twice the digit of the units place and digit of the units place is x. In respect of the information, answer the following questions (8-10):

8. What is the number?

   1) $2x$ 2) $3x$ 3) $12x$ 4) $21x$

9. If the places of the digits are interchanged, what will be the number?

   1) $3x$ 2) $4x$ 3) $12x$ 4) $21x$

10. If $x=2$, what will be the difference between the original number and the number by interchanging their places?

    1) 18 2) 20 3) 34 4) 36

Solve (11 - 17):

11. $(y+5)(y-5)=24$
12. $(\sqrt{2}x+3)(\sqrt{3}x-2)=0$
13. $2(z^2-9)+9z=0$
14. $\dfrac{3}{2z+1}+\dfrac{4}{5z-1}=2$
15. $\dfrac{x-2}{x+2}+\dfrac{6(x-2)}{x-6}=1$
16. $\dfrac{x}{a}+\dfrac{a}{x}=\dfrac{x}{b}+\dfrac{b}{x}$
17. $\dfrac{x-a}{x-b}+\dfrac{x-b}{x-a}=\dfrac{a}{b}+\dfrac{b}{a}$

Find the solution set (18 - 22):

18. $\dfrac{3}{x}+\dfrac{4}{x+1}=2$
19. $\dfrac{x+7}{x+1}+\dfrac{2x+6}{2x+1}=5$
20. $\dfrac{1}{x}+\dfrac{1}{a}+\dfrac{1}{b}=\dfrac{1}{x+a+b}$

21. $x + \dfrac{1}{x} = 2$

22. $\dfrac{(x+1)^3 - (x-1)^3}{(x+1)^2 - (x-1)^2} = 2$

Solve by forming equations (23 - 34):

23. Sum of the two digits of a number consisting of two digits is 15 and their product is 56; find the number.

24. Area of the floor of a rectangular room is 192 square metre. If the length of the floor is decreased by 4 metre and the breadth is increased by 4 metre, the area remains unchanged. Find the length and breadth of the floor.

25. Length of the hypotenuse of a right angled triangle is 15 cm. and the difference of the lengths of other two sides is 3 cm. Find the lengths of those two sides.

26. The base of a triangle is 6 cm. more than twice its height. If the area of the triangle is 810 square cm., what is its height ?

27. As many students are there in a class, each of them contribute equal to the number of class-mates of the class and thus total Tk. 420 was collected. What is the number of students in the class and how much did each student contribute ?

28. As many students are there in a class, each of them contributed 30 paisa more than the number of paisa equal to the number of students and thus total Tk. 70 was collected. What is the number of students in that class ?

29. Sum of the digits of a number consisting of two digits is 7. If the places of the digits are interchanged, the number so formed is 9 less than the given number.

    1) Write down the given number and the number obtained by interchanging their places in terms of variable x.

    2) Find the number.

    3) If the digits of the original number indicate the length and breadth of a rectangular region in centimetre, find the length of its diagonal. Assuming the diagonal as the side of a square, find the length of the diagonal of the square.

30. The base and height of a right angle triangle are respectively $(x-1)$ cm. and x cm. and the length of the side of a square is equal to the height of the triangle. Again, the length of a rectangular region is $(x+3)$ cm. and its breadth is x cm.

    1) Show the information in only one picture.
    2) If the area of the triangular region is 10 square centimetre, what is its height?
    3) Find the successive ratio of the areas of the triangular, square and rectangular regions.

31. The area of a land is 192 square metre. If the length of the land is decreased by 4 metre and the breadth is increased by 4 metre,then the area remains unchanged. Again a circle of 20 diametre was drawn in the center of the land. A line drawn from the center of the circle perpendicular to one of the chords is 2 cm less than the length of that chord.

    1) Letting the length as x and the breadth as y, express the information by an equation.
    2) Find the perimeter of the land.
    3) Find the length of the chord of the circle.

32. When Nabil's age was same as Shuva's present age, at that time Nabil's age was twice as Shuva's age. When Shuva's age will be same as Nabil's present age,sum of their ages will be 63. What is the present age of each one?

33. In the queue of bus, two more passenger are standing in front of Sohag than the number passengers standing behind Sohag. Total number of passengers in the queue is thrice as the number of passengers standing behind him. How many passengers are standing in the queue?

34. Sabuj went to drawing class at 3 : 30 from home. While he was returning home from school, minute hand of the clock was still down the steep; but the distance between two hands was 15 degrees less than that was at 3 : 30. When did Sabuj return home?

Chapter 6

Lines, Angles and Triangles

Geometry is an old branch of mathematics. The word 'geometry' comes from the Greek words 'geo', meaning the 'earth', and 'metrein', meaning 'to measure'. So, the word 'geometry' means 'the measurement of land.' Geometry appears to have originated from the need for measuring land in the age of agricultural based civilization. However, now a days geometry is not only used for measuring lands, rather knowledge of geometry is now indispensable for solving many complicated mathematical problems. The practice of geometry is evident in relics of ancient civilization. According to the historians, concepts and ideas of geometry were applied to the survey of lands about four thousand years ago in ancient Egypt. Signs of application of geometry are visible in different practical works of ancient Egypt, Babylon, India, China and the Incas civilisation. In the Indian subcontinent there were extensive usages of geometry in the Indus Valley civilisation. The excavations at Harappa and Mohenjo-Daro show the evidence of that there was a well planned city. For example, the roads were parallel to each other and there was a developed underground drainage system. Besides the shape of houses shows that the town dwellers were skilled in mensuration. In Vedic period in the construction of altars (or vedis) definite geometrical shapes and areas were maintained. Usually these were constituted with triangles, quadrilaterals and trapeziums.

But geometry as a systematic discipline evolved in the age of Greek civilization. A Greek mathematician, Thales is credited with the first geometrical proof. He proved logically that a circle is bisected by its diameter. Thales' pupil Pythagoras developed the theory of geometry to a great extent. About 300 BC Euclid, a Greek scholar, collected all the work and placed them in an orderly manner in his famous treatise, 'Elements'. 'Elements' completed in thirteen chapters is the foundation of modern geometry for generations to come. In this chapter, we shall discuss logical geometry in accordance with Euclid.

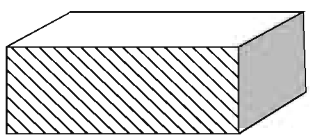
At the end of this chapter, the students will be able to -

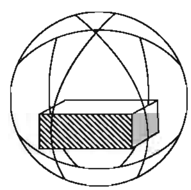
- describe the basic postulates of plane geometry.
- prove the theorems related to triangles.
- apply the theorems and corollaries related to triangles to solve problems.

Concepts of space, plane, line and point

The space around us is limitless. It is occupied by different solids, small and large. By solids we mean the grains of sand, pin, pencil, paper, book, chair, table, brick, rock, house, mountain, the earth, planets and stars. The concept of geometry springs from the study of space occupied by solids and the shape, size, location and properties of the space.

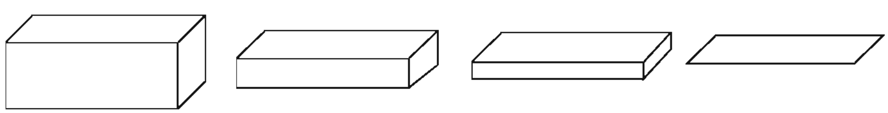
A solid occupies space which is spread in three directions. This spread in three directions denotes the three dimensions (length, breadth and height) of the solid. Hence every solid is three dimensional. For example, a brick or a box has three dimensions (length, breadth and height). A sphere also has three dimensions. Although the dimensions are not distinctly visible, it can be divided distinctly into length-breadth-height.





The boundary of a solid denotes a surface, that is, every solid is bounded by one or more surfaces. For example, the six faces of a box represent six surfaces. The upper face of a sphere is also a surface. But the surfaces of a box and of a sphere are different. The first one is plane while the second one is curved.

Two-dimensional surface: A surface is two dimensional; it has only length and breadth and is said to have no thickness. Keeping the two dimension of a box unchanged, if the third dimension is gradually reduced to zero, we are left with a face or boundary of the box. In this way, we can get the idea of surface from a solid.



When two surfaces intersect, a line is formed. For example, two faces of a box meet at one side in a line. This line is a straight line. Again, if a lemon is cut by a knife, a curved line is formed on the plane of intersection of curved surface of the lemon.

Line: A line is one-dimensional; it has only length and no breadth or thickness. If the width of a face of the box is gradually receded to zero, we are left with only line of the boundary. In this way, we can get the idea of line from the idea of surface.

The intersection of two lines produces a point. That is, the place of intersection of two lines is denoted by a point. For example, the two edges of a box meet at a point. A point has no length, breadth and thickness. If the length of a line is gradually reduced to zero, at last it ends in a point. Thus, a point is considered an entity of zero dimension.

Euclid's Axioms and Postulates

The discussion above about surface, line and point do not lead to any definition – they are merely description. This description refers to height, breadth and length, neither of which has been defined. We only can represent them intuitively. The definitions of point, line and surface which Euclid mentioned in the beginning of the first volume of his 'Elements' are incomplete from modern point of view. A few of Euclid's axioms are given below:

1. A point is that which has no part.
2. A line has no end point.
3. A line has only length, but no breath and height
4. A straight line is a line which lies evenly with the points on itself.
5. A surface is that which has length and breadth only.
6. The edges of a surface are lines.
7. A plane surface is a surface which lies evenly with the straight lines on itself.

It is observed that in this description, part, length, width, evenly etc have been accepted without any definition. It is assumed that we have primary ideas about them. The ideas of point, straight line and plane surface have been imparted on this assumption. As a matter of fact, in any mathematical discussion one or more elementary ideas have to be taken granted. Euclid called them axioms. Some of the axioms given by Euclid are:

1. Things which are equal to the same thing, are equal to one another.
2. If equals are added to equals, the wholes are equal.
3. If equals are subtracted from equals, the remainders are equal.
4. Things which coincide with one another, are equal to one another.
5. The whole is greater than the part.

In modern geometry, we take a point, a line and a plane as undefined terms and some of their properties are also admitted to be true. These admitted properties are called geometric postulates. These postulates are chosen in such a way that they are consistent with real conception. The five postulates of Euclid are:

Postulate 1: A straight line may be drawn from any one point to any other point.

Postulate 2: A terminated line can be produced indefinitely.

Postulate 3: A circle can be drawn with any centre and any radius.

Postulate 4: All right angles are equal to one another.

Postulate 5: If a straight line falling on two straight lines makes the interior angles on the same side of it taken together less than two right angles, then the two straight lines, if produced indefinitely, meet on that side on which the sum of angles is less than two right angles.

After Euclid stated his postulates and axioms, he used them to prove other results. Then using these results, he proved some more results by applying deductive reasoning. The statements that were proved are called propositions or theorems. Euclid is his 'elements' proved a total of 465 propositions in a logical chain. This is the foundation of modern geometry.

Note that there are some incompleteness in Euclid's first postulate. The drawing of a unique straight line passing through two distinct points has been ignored. Postulate 5 is far more complex than any other postulate. On the other hand, Postulates 1 through 4 are so simple and obvious that these are taken as 'self-evident truths'. However, it is not possible to prove them. So, these statements are accepted without any proof. Since the fifth postulate is related to parallel lines, it will be discussed later.

Plane Geometry

It has been mentioned earlier that point, straight line and plane are three fundamental concepts of geometry. Although it is not possible to define them properly, based on our real life experience we have ideas about them. As a concrete geometrical conception space is regarded as a set of points and straight lines and planes are considered the subsets of this universal set. That is,

Postulate 1: Space is a set of all points and plane and straight lines are the sub-sets of this set.

From this postulate we observe that each of plane and straight line is a set and points are its elements. However, in geometrical description the notation of sets is usually avoided. For example, a point included in a straight line or plane is expressed by ‘ the point lies on the straight line or plane’ or ‘the straight line or plane passes through the point’. Similarly if a straight line is the subset of a plane, it is expressed by such sentences as ‘the straight line lies on the plane, or the plane passes through the straight line’.

It is accepted as properties of straight line and plane that,

Postulate 2: For two different points there exists one and only one straight line, on which both the points lie.

Postulate 3: For three points which are not colinear, there exists one and only one plane, on which all the three points lie.

Postulate 4: A straight line passing through two different points on a plane lie completely in the plane.

Postulate 5:

1) Space contains more than one plane.
2) In each plane more than one straight lines lie.
3) The points on a straight line and the real numbers can be related in such a way that every point on the line corresponds to a unique real number and conversely every real number corresponds to a unique point of the line.

Remarks: The postulates from 1 to 5 are called incidence postulates.

The concept of distance is also an elementary concept. It is assumed that,

Postulate 6:

1) Each pair of points P and Q determines a unique real number which is known as the distance between point P and Q and is denoted by PQ.
2) If P and Q are different points, the number PQ is positive. Otherwise, $PQ = 0$.
3) The distance between P and Q and that between Q and P are the same, i.e. $PQ = QP$.

Since $PQ = QP$, this distance is called the distance between point P and point Q. In practical, this distance is measured by previously determined unit.

According to postulate $5(c)$ one to one correspondence can be established between the set of points in every straight line and the set of real numbers. In this connection, it is admitted that,

Postulate 7: One-to-one correspondence can be established between the set of points in a straight line and the set of real numbers such that, for any points P, Q, $PQ = |a - b|$ where, the one-to-one correspondence associates points P and Q to real numbers a and b respectively.

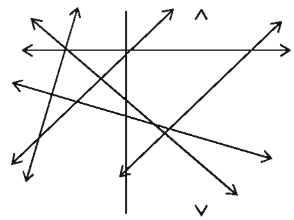
If the correspondence stated in this postulate is made, the line is said to have been reduced to a number line. If P corresponds to a in the number line, P is called the graph point of P and a the coordinates of P. To convert a straight line into a number line the co-ordinates of two points are taken as 0 and 1 respectively. Thus a unit distance and the positive direction are fixed in the straight line. For this, it is also admitted that,

Postulate 8: Any straight line AB can be converted into a number line such that the coordinate of A is 0 and that of B is positive.

Remarks: Postulate 6 is known as **distance postulate** and Postulate 7 as **ruler postulate** and Postulate 8 as **ruler placement postulate**.

Geometrical figures are drawn to make geometrical description clear. The model of a point is drawn by a thin dot by tip of a pencil or pen on a paper. The model of a straight line is constructed by drawing a line along a ruler. The arrows at ends of a line indicate that the line is extended both ways indefinitely. By postulate 2, two different points A and B define a unique straight line on which the two points lie. This line is called AB or BA line. By postulate $5(c)$ every such straight line contains infinite number of points.

According to postulate $5(a)$ more than one plane exist. There is infinite number of straight lines in every such plane. The branch of geometry that deals with points, lines lying in same plane and different geometrical entities related to them, is known as plane Geometry. In this textbook, plane geometry is the matter of our discussion. Hence, whenever something is not mentioned in particular, we will assume that all discussed points, lines etc lie in a plane.



Proof of Mathematical statements

In any mathematical theory different statements related to the theory are logically established on the basis of some elementary concepts, definitions and postulates. Such statements are generally known as propositions. In order to prove correctness of statements some methods of logic are applied. The methods are:

1. Mathematical Induction
2. Mathematical Deduction
3. Proof by contradiction etc.

Proof by contradiction

Philosopher Aristotle first introduced this method of logical proof. The basis of this method is:

1. A property cannot be accepted and rejected at the same time.
2. The same object cannot possess opposite properties.
3. One cannot think of anything which is contradictory to itself.
4. If an object attains some property, that object cannot unattain that property at the same time.

Geometric proof

In geometry, special importance is attached to some propositions which are taken as theorems and used successively in establishing other propositions. In geometric proof different statements are explained with the help of figures. But the proof must be logical.

In describing geometric propositions general or particular enunciation is used. The general enunciation is the description independent of the figure and the particular enunciation is the description based on the figure. If the general enunciation of a proposition is given, subject matter of the proposition is specified through particular enunciation. For this, necessary figure is to be drawn.

Generally, in proving the geometric theorem the following steps should be followed :

1. General enunciation.
2. Figure and particular enunciation.
3. Description of the necessary constructions and
4. Description of the logical steps of the proof.

If a proposition is proved directly from the conclusion of a theorem, it is called a corollary of that theorem. Besides, proof of various propositions, proposals for construction of different figures are considered. These are known as constructions. By drawing figures related to problems, it is necessary to narrate the description of construction and its logical truth.

Exercises 6.1

1. Give a concept of space, surface, line and point.
2. State Euclid's five postulates.
3. State five postulates of incidence.
4. State the distance postulate
5. State the ruler postulate.

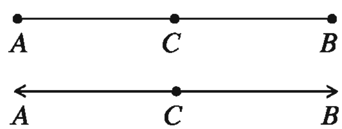
6. Explain the number line.

7. State the postulate of ruler placement.

8. Define intersecting straight line and parallel straight line.

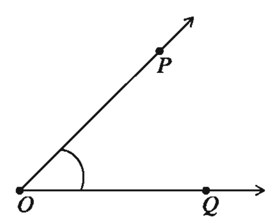
Line, Ray, Line Segment

By postulates of plane geometry, every point of a straight line lies in a plane. Let AB be a line in a plane and C be a point on it. The point C is called internal to A and B if the points A, C and B are different points on a line and $AC+CB = AB$. The points A, C and B are also called collinear points. The set of points including A and B and all the internal points is known as the line segment AB. The points between A and B are called internal points.



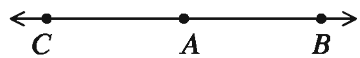
Angle

When two rays in a plane meet at a point, an angle is formed. The rays are known as the sides of the angle and the common point as vertex. In the figure, two rays OP and OQ make an angle $\angle POQ$ at their common point O. O is the vertex of the angle $\angle POQ$. The set of all points lying in the plane on the Q side of OP and P side of OQ is known as the interior region of the $\angle POQ$. The set of all points not lying in the interior region or on any side of the angle is called exterior region of the angle.



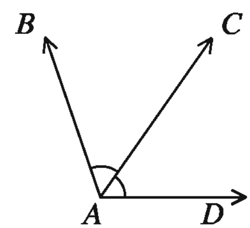
Straight angle

The angle made by two opposite rays at their common end point is a straight angle. In the adjacent figure, a ray AC is drawn from the end point A of the ray AB. Thus the rays AB and AC have formed an angle $\angle BAC$ at their common point A. $\angle BAC$ is a straight angle. The measurement of a right angle is 2 right angles or 180°.



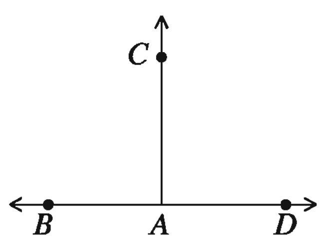
Adjacent angle

If two angles in a plane have the same vertex, a common side and the angles lie on opposite sides of the common side, each of the two angles is said to be an adjacent angle of the other.
In the adjacent figure, the angles $\angle BAC$ and $\angle CAD$ have the same vertex A, a common side AC and are on opposite sides of AC. $\angle BAC$ and $\angle CAD$ are adjacent angles.



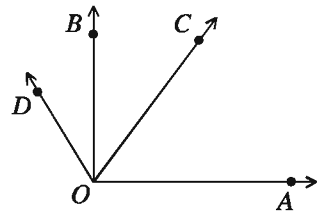
Perpendicular and Right angle

If two adjacent angles are on the same line and equal, then each of the related adjacent angle is a right angle or 90° : The two sides of a right angle are mutually perpendicular. In the adjacent figure two angles $\angle BAC$ and $\angle DAC$ are produced at the point A of BD by the ray AC. A is the vertex of these two angles.
AC is the common side of the angles $\angle BAC$ and $\angle DAC$. The tow angles lie on the tow sides of the common side AC. If $\angle BAC$ and $\angle DAC$ are equal, then each of the two angles is right angle.The line segments AC and BD are mutually perpendicular.



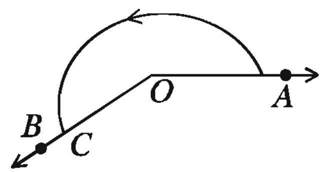
Acute angle and obtuse angle

An angle which is less than a right angle is called an acute angle and an angle greater than one right angle but less than two right angles is an obtuse angle. In the figure, $\angle AOC$ is an acute angle and $\angle AOD$ is an obtuse angle. Here $\angle AOB$ is a right angle.



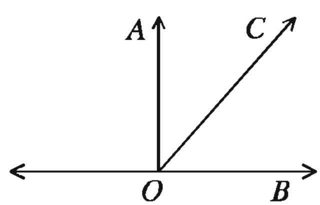
Reflex angle

An angle which is greater than two right angles and less than four right angles is called a reflex angle. In the figure, $\angle AOC$ is a reflex angle.



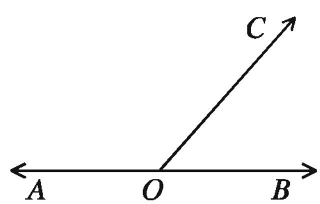
Complementary angle

If the sum of two angles is one right angle, the two angles are called complementary angles.
In the adjacent figure, $\angle AOB$ is a right angle. The ray OC is in the inner side of the angle and makes two angles $\angle AOC$ and $\angle COB$.So taking together the measurement of these two angles is one right angle. The angles $\angle AOC$ and $\angle COB$ are complementary angles.



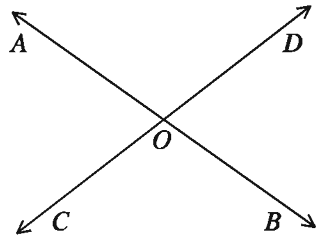
Supplementary angle

If the sum of two angles is 2 right angles, two angles are called supplementary angles.
The point O is an internal point of the line AB. OC is a ray which is different from the ray OA and ray OB. As a result two angles $\angle AOC$ and $\angle COB$ are formed. The measurement of this two angles is equal to the measurement of the straight angle $\angle AOB$ i.e., two right angles. The angles $\angle AOC$ and $\angle COB$ are supplementary angles.



Vertical angle

Two angles are said to be the opposite angles if the sides of one are the opposite rays of the other.

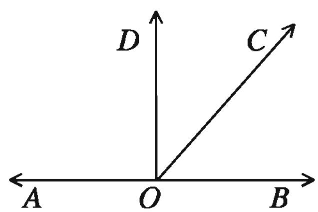


In the adjoining figure OA and OB are mutually opposite rays. So are the rays OC and OD. The angles $\angle BOD$ and $\angle AOC$ are a pair of opposite angles.

Similarly, $\angle BOC$ and $\angle DOA$ are another pair of opposite angles. Therefore, two intersecting lines produce two pairs of opposite angles.

Theorem 1. The sum of the two adjacent angles which a ray makes with a straight line on its meeting point is equal to two right angles.

Proof:
Let, the ray OC meets the straight line AB at O. As a result two adjacent angles $\angle AOC$ and $\angle COB$ are formed. Draw a perpendicular DO on AB.



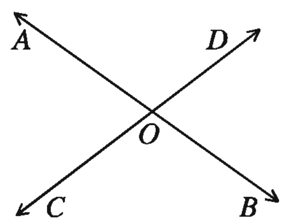
Sum of the adjacent two angles

$= \angle AOC + \angle COB = \angle AOD + \angle DOC + \angle COB$

$= \angle AOD + \angle DOB = 2$ right angles.

Theorem 2. When two straight lines intersect, the vertically opposite angles are equal.

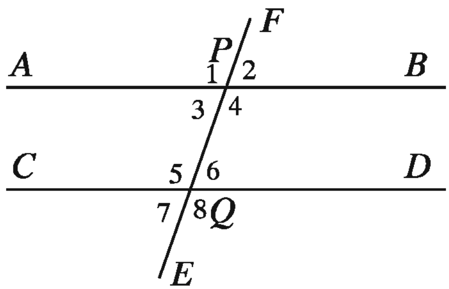
Let AB and CD be two straight lines, which intersect at O. As a result the angles $\angle AOC$, $\angle COB$, $\angle BOD$, $\angle AOD$ are formed at O.



$\angle AOC$ = opposite $\angle BOD$ and $\angle COB$ = opposite $\angle AOD$.

Parallel lines

Alternate angles, corresponding angles and interior angles of the traversal



In the figure, two straight lines AB and CD are cut by a straight line EF at P and Q. The straight line EF is a traversal of AB and CD. The traversal has made eight angles $\angle 1$, $\angle 2$, $\angle 3$, $\angle 4$, $\angle 5$, $\angle 6$, $\angle 7$, $\angle 8$ with the lines AB and CD. Among the angles

1) $\angle 1$ and $\angle 5$, $\angle 2$ and $\angle 6$, $\angle 3$ and $\angle 7$, $\angle 4$ and $\angle 8$ are corresponding angles.
2) $\angle 3$ and $\angle 6$, $\angle 4$ and $\angle 5$ are alternate angles.
3) $\angle 4$, $\angle 6$ are interior angles on the right.
4) $\angle 3$, $\angle 5$ are interior angles on the left.

In a plane two straight lines may intersect or they are parallel. The lines intersect if there exists a point which is common to both lines. Otherwise, the lines are parallel. Note that two different straight lines may at most have only one point in common.

The parallelism of two straight lines in a plane may be defined in three different ways:

1) The two straight lines never intersect each other (even if extended to infinity).
2) Every point on one line lies at equal smallest distance from the other.
3) The corresponding angles made by a transversal of the pair of lines are equal.

According to definition (a) in a plane two straight lines are parallel, if they do not intersect each other. Two line segments taken as parts of the parallel lines are also parallel.

According to definition (b) the perpendicular distance of any point of one of the parallel lines from the other is always equal. Perpendicular distance is the length of the perpendicular from any point on one of the lines to the other. Conversely, if the perpendicular distances of two points on any of the lines to the other are equal, the lines are parallel. This perpendicular distance is known as the distance of the parallel lines.

The definition (c) is equivalent to the fifth postulate of Euclid. This definition is more useful in geometric proof and constructions.

Observe that, through a point not on a line, a unique line parallel to it can be drawn.

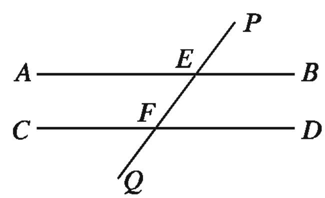
Theorem 3. When a transversal cuts two parallel straight lines,

1) the pair of corresponding angles are equal.

2) the pair of alternate angles are equal.

3) that pair of interior angles on the same side of the transversal are supplementary.

In the figure, $AB \parallel CD$ and the traversal PQ intersects them at E and F respectively.
Therefore,
1) $\angle PEB =$ corresponding $\angle EFD$ [by definition]
2) $\angle AEF =$ alternate $\angle EFD$
3) $\angle BEF + \angle EFD = 2$ right angles



Work:

Using alternate definitions of parallel lines prove the theorems related to parallel straight lines.

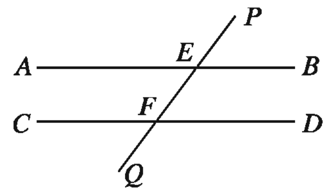
Theorem 4. When a transversal cuts two straight lines, such that

1) pairs of corresponding angles are equal, or

2) pairs of alternate angles are equal, or

3) pairs of interior angles on the same side of the transversal are equal to the sum of two right angles, then those two straight lines will be parallel.

In the figure the line PQ intersects the straight lines AB and CD at E and F respectively and



1) $\angle PEB =$ corresponding $\angle EFD$ or,
2) $\angle AEF =$ alternate $\angle EFD$ or,
3) $\angle BEF + \angle EFD = 2$ right angles

Therefore, the straight lines AB and CD are parallel.

Corollary 1. The lines which are parallel to a given line are parallel to each other.

Exercises 6.2

1. Define interior and exterior of an angle.
2. If there are three different points in a line, identify the angles in the figure.
3. Define adjacent angles and locate its sides.
4. Define with a figure of each: opposite angles, complementary angle, supplementary angle, right angle, acute and obtuse angle.

Triangle

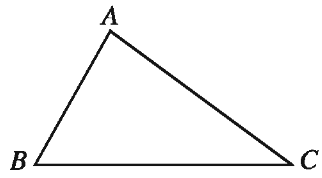
A triangle is a figure closed by three line segments. The line segments are known as sides of the triangle. The point common to any pair of sides is the vertex. The sides form angles at the vertices.

A triangle has three sides and three angles. Triangles are classified by sides into three types: equilateral, isosceles and scalene.

By angles triangles are also classified into three types: acute angled, right angled and obtuse angled.

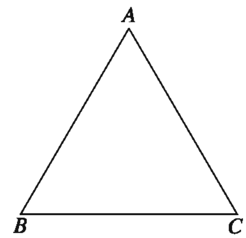
The sum of the lengths of three sides of the triangle is the perimeter. By triangle we also denote the region closed by the sides. The line segment drawn from a vertex to the mid-point of opposite side is known as the median. Again, the perpendicular distance from any vertex to the opposite side is the height of the triangle.

In the adjacent figure ABC is a triangle. A, B,C are three vertices. AB , BC ,CA are three sides and $\angle ABC$, $\angle BCA$, $\angle CAB$ are three angles of the triangle. The sum of the measurement of AB, BC and CA is the perimeter of the triangle.



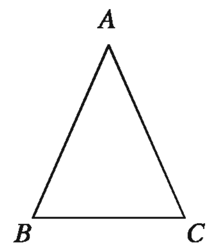
Equilateral triangle

An equilateral triangle is a triangle of three equal sides. In the adjacent figure, triangle ABC is an equilateral triangle; because, $AB = BC = CA$ i.e., the lengths of three sides are equal.



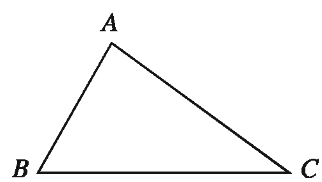
Isosceles triangle

An isosceles triangle is triangle with two equal sides. In the adjacent figure triangle ABC is an isosceles triangle; because $AB = AC \neq BC$ i.e. the lengths of only two sides are equal.



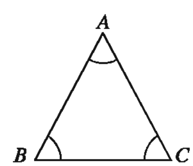
Scalene triangle

Sides of scalene triangle are unequal. Triangle ABC is a scalene triangle, since the lengths of its sides AB, BC , CA are unequal



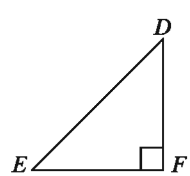
Acute triangle

A triangle having all the three angles acute is acute angled triangle. In the triangle ABC each of the angles $\angle BAC$, $\angle ABC$, $\angle BCA$ is acute i.e., the measurement of any angle is less than 90°. So $\triangle ABC$ is acute angled.

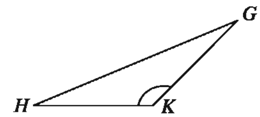


Right triangle

A triangle with one of the angles right is a right angled triangle. In the figure, the $\angle DFE$ is a right angle; each of the two other angles $\angle DEF$ and $\angle EDF$ is acute. The triangle $\triangle DEF$ is a right angled triangle.



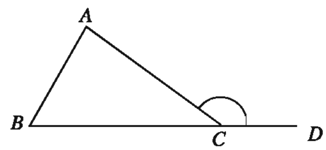
Obtuse triangle



A triangle having an angle obtuse is an obtuse angled triangle. In the figure, the $\angle GKH$ is an obtuse angle; the two other angles $\angle GHK$ and $\angle HGK$ are acute. $\triangle GHK$ is an obtuse angled triangle.

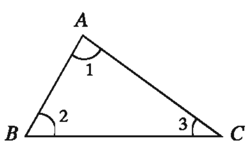
Interior and Exterior Angles

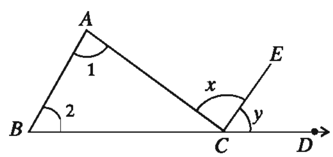
If a side of a triangle is produced, a new angle is formed. This angle is known as exterior angle. Except the angle adjacent to the exterior angle, the two others angles of the triangle are known as opposite interior angles.



In the adjacent figure, the side BC of $\triangle ABC$ is extended to D. The angle $\angle ACD$ is an exterior angle of the triangle. $\angle ABC$, $\angle BAC$ and $\angle ACB$ are three interior angles. $\angle ACB$ is the adjacent interior angle of the exterior angle $\angle ACD$. Each of $\angle ABC$ and $\angle BAC$ is an opposite interior angle with respect to $\angle ACD$.

Theorem 5. The sum of the three angles of a triangle is equal to two right angles.





Let ABC be a triangle. In the triangle $\angle BAC + \angle ABC + \angle ACB = 2$ right angles.

Draw CE from point C so that $AB \parallel CE$. Now $\angle ABC = \angle ECD$ [corresponding angles] and $\angle BAC = \angle ACE$ [alternate angles]

$\therefore \angle ABC + \angle BAC = \angle ECD + \angle ACE = \angle ACD$

$\angle ABC + \angle BAC + \angle ACB = \angle ACD + \angle ACB = 2$
right angles.

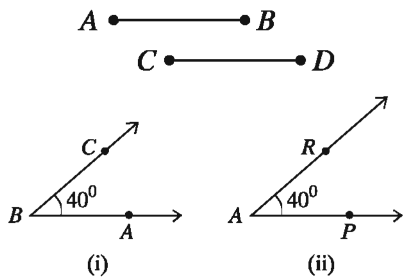
Corollary 2. If a side of a triangle is produced, then exterior angle so formed is equal to the sum of the two opposite interior angles.

Corollary 3. If a side of a triangle is produced, the exterior angle so formed is greater than each of the two interior opposite angles.

Corollary 4. The acute angles of a right angled triangle are complementary to each other.

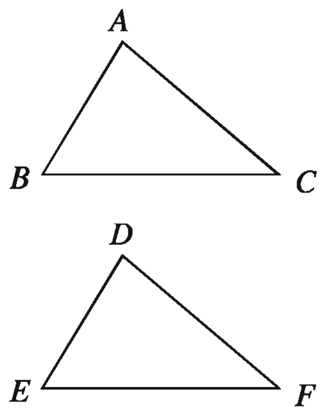
> **Work:**
>
> Prove that if a side of a triangle is extended, the exterior angle so formed is greater than each of the two interior opposite angles.

Congruence of Sides and Angles

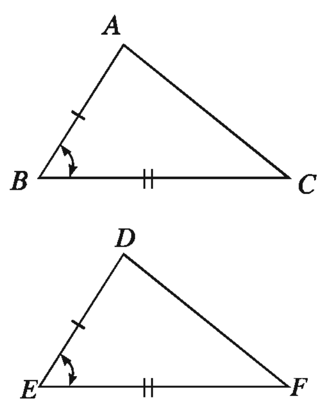


If two line segments have the same length, they are congruent. Conversely, if two line segments are congruent, they have the same length.
If the measurement of two angles is equal, the angles are congruent. Conversely, if two angles are congruent, their measurement is the same.

Congruence of Triangles



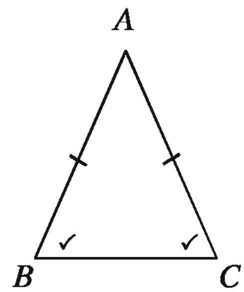
If a triangle when placed on another exactly covers the other, the triangles are congruent. The corresponding sides and angles of two congruent triangles are equal.
In the adjacent figure, $\triangle ABC$ and $\triangle DEF$ are congruent. If two triangles $\triangle ABC$ and $\triangle DEF$ are congruent and vertices A, B, C superpose on vertices D, E, F respectively, then $AB = DE$, $AC = DF$, $BC = EF$ and $\angle A = \angle D, \angle B = \angle E, \angle C = \angle F$. To express $\triangle ABC$ and $\triangle DEF$ congruent it is written as $\triangle ABC \cong \triangle DEF$.



Theorem 6. (Side-Angle-Side criterion)
If two sides and the angle included between them of a triangle are equal to two corresponding sides and the angle included between them of another triangle, the triangles are congruent.

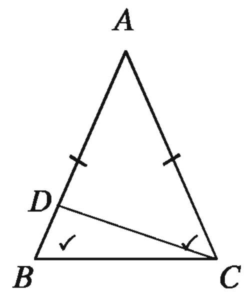
Let $\triangle ABC$ and $\triangle DEF$ be two triangles in which $AB = DE$, $BC = EF$ and the included $\angle ABC =$ the included $\angle DEF$. Then, $\triangle ABC \cong \triangle DEF$.

Theorem 7. If two sides of a triangle are equal, the angles opposite the equal sides are also equal.



Suppose in the triangle ABC, $AB = AC$. Then, $\angle ABC = \angle ACB$

Theorem 8. If two angles of a triangle are equal, the sides opposite the equal angles are also equal.



Special Nomination: Let, in the triangle ABC, $\angle ABC = \angle ACB$. It is to be proved that, $AB = AC$

Proof:

Step 1. If $AB \neq AC$, then (i) $AB > AC$ or, (i) $AB < AC$.

Suppose, (i) $AB > AC$. Cut from AB a part AD equal to AC. Now, the triangle ADC is an isosceles triangle. So,

$\angle ADC = \angle ACD$ [$\because$ The base angles of an isosceles triangles are equal]

In $\triangle DBC$ exterior angle $\angle ADC > \angle ABC$ [$\because$ [Exterior angle is greater than each of the interior opposite angles]

$\therefore$ $\angle ACD > \angle ABC$. Therefore, $\angle ACB > \angle ABC$, but this is against the given condition.

Step 2. Similarly, (ii) if $AB < AC$, it can be proved that

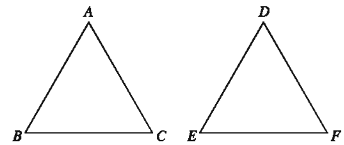
$\angle ABC > \angle ACB$, but this is also against the condition.

Step 3. So neither $AB > AC$ nor $AB < AC$.

$\therefore$ $AB = AC$ (Proved)

Theorem 9. (SSS criterion)
If the three sides of one triangle are equal to the three corresponding sides of another triangle, the triangles are congruent.

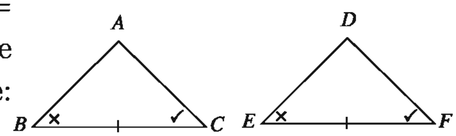
Let, in $\triangle ABC$ and $\triangle DEF$, $AB = DE$, $AC = DF$ and $BC = EF$. Therefore, $\triangle ABC \cong \triangle DEF$



Theorem 10. (ASA criterion)

If two angles and the included side of a triangle are equal to two corresponding angles and the included side of another triangle, the triangles are congruent.

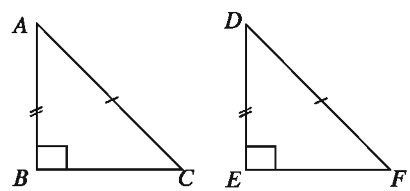
Let, in $\triangle ABC$ and $\triangle DEF$, $\angle B = \angle E$, $\angle C = \angle F$ and the side BC = the corresponding side EF. Then the triangles are congruent, i,e: $\triangle ABC \cong \triangle DEF$



Theorem 11. (Hypotenuse-side criterion)

If the hypotenuse and one side of a right-angled triangle are respectively equal to the hypotenuse and one side of another right-angled triangle, the triangles are congruent.

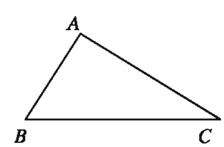
Let $\triangle ABC$ and $\triangle DEF$ be two right angled triangles, in which the hypotenuse AC = hypotenuse DF and $AB = DE$. Then, $\triangle ABC \cong \triangle DEF$



There is a relation between the sides and angles of a triangle. This relation is described in the following criteria 12 and 13.

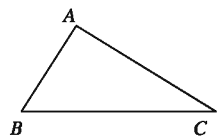
Theorem 12. If one side of a triangle is greater than another, the angle opposite the greater side is greater than the angle opposite the lesser sides.

Let, in triangle $\triangle ABC$, $AC > AB$. Therefore $\angle ABC > \angle ACB$



Theorem 13. If one angle of a triangle is greater than another, the side opposite the greater angle is greater than the side opposite the lesser.

Special Nomination: Let, in triangle $\triangle ABC$, $\angle ABC > \angle ACB$ It is to be proved that, $AC > AB$



Proof:

Step 1. If the side AC is not greater than AB, then (i) $AC = AB$ or, (ii) $AC < AB$.

(i) If $AC = AB$, then $\angle ABC = \angle ACB$ [$\because$ The base angles of isosceles triangle are equal]

Which is against the supposition, since by supposition $\angle ABC > \angle ACB$.

(ii) Again, if $AC < AB$, then $\angle ABC < \angle ACB$. [$\because$ The angle opposite to smaller side is smaller]

But this is also against the supposition.

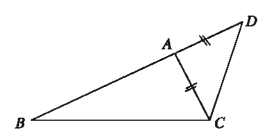
Step 2. Therefore, the side AC is neither equal to nor less than AB.

$\therefore AC > AB$ (Proved).

There is a relation between the sum or the difference of the lengths of two sides and the length of the third side of a triangle.

Theorem 14. The sum of the lengths of any two sides of a triangle is greater than the third side.

Let ABC be a triangle. Let BC be the greatest side of the triangle. Then, $AB + AC > BC$.



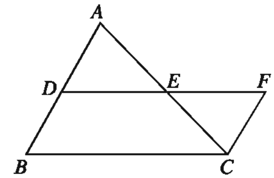
Corollary 5. The difference of the lengths of any two sides of a triangle is smaller than the third side.

Let ABC be a triangle. Then the difference of the lengths of any two of its sides is smaller than the length of third side, i.e: $AB - AC < BC$.

Theorem 15. The line segment joining the mid-points of any two sides of a triangle is parallel to the third side and in length it is half.

Special Nomination: Let ABC be a triangle. D and E are respectively mid-points of the AB and AC. It is required to prove $DE \parallel BC$ and $DE = \frac{1}{2}BC$.

Drawing: Join D and E and extend to F so that $EF = DE$. Draw a straight line segment from C to F.



Proof:

Step 1. Between $\triangle ADE$ and $\triangle CEF$, $AE = EC$ [given]

$DE = EF$ [by construction]

$\angle AED = \angle CEF$ [opposite angles]

$\therefore \triangle ADE \cong \triangle CEF$ [SAS theorem]

$\therefore \angle ADE = \angle EFC$ and $\angle DAE = \angle ECF$ [alternate angle]

$\therefore AD \parallel CF$ or, $AB \parallel CF$

Again, $BD = AD = CF$ and $BD \parallel CF$

Therefore $BDFC$ is a parallelogram.

$\therefore DF \parallel BC$ or, $DE \parallel BC$.

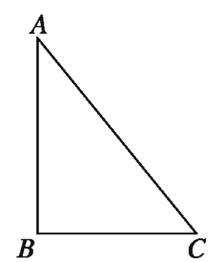
Step 2. Again, $DF = BC$ or, $DE + EF = BC$

or, $DE + DE = BC$ or, $2DE = BC$ or, $DE = \frac{1}{2}BC$

$\therefore DE \parallel BC$ and $DE = \frac{1}{2}BC$ (Proved).

Theorem 16. (Pythagoras theorem)

In a right-angled triangle the square on the hypotenuse is equal to the sum of the squares of regions on the two other sides.



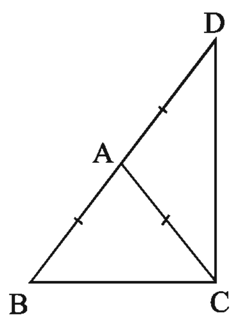
Let in the triangle ABC, $\angle ABC = 1$ right angle and AC is the hypotenuse.Then, $AC^2 = AB^2 + BC^2$

Example 1. In triangle $\triangle ABC$, $AB = AC$, BA is produced to D such that $AD = AC$. C, D are added.

1) Draw the figure based on the stem.
2) Prove that, $BC + CD > 2AC$
3) Prove that, $\angle BCD = 1$ right angle.

Solution:

1)



2) Given, $AB = AC$ and by construction $AC = AD$

   For $\triangle BCD$

   $BC + CD > BD$ [The sum of the lengths of any two sides of a triangle is greater than the third side.]

   or, $BC + CD > AB + AD$

   or, $BC + CD > AD + AD$

   or, $BC + CD > 2AD$

   $\therefore\ BC + CD > 2AC \quad [\because AB = AC = AD]$

3) Given $AB = AC$ Therefore $\angle ABC = \angle ACB$

   Hence $\angle DBC = \angle ACB$

   By construction $AC = AD$. Therefore $\angle ADC = \angle ACD$

   Hence $\angle BDC = \angle ACD$

   In triangle $\triangle BCD$,

   $\angle BDC + \angle DBC + \angle BCD = 2$ right angle [Sum of three angles of a triangle is equal to 2 right angles] or, $\angle ACD + \angle ACB + \angle BCD = 2$ right angles

or, $\angle BCD + \angle BCD = 2$ right angles

or, $2\angle BCD = 2$ right angles.

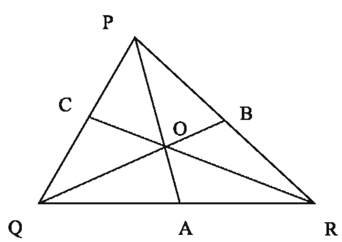
$\therefore \angle BCD = 1$ right angle

Example 2. PQR is a triangle. PA, QB and RC are three medians which intersect at point O.

1) Draw a figure based on the information.
2) Prove that, $PQ + PR > QO + RO$
3) Prove that, $PA + QB + RC < PQ + QR + PR$

Solution:

1)



2) From figure 1) it is to be proved that, $PQ + PR > QO + RO$

Proof: Any two sides of a triangle are greater than the third one.

In triangle $\triangle PQB$, $PQ + PB > QB$

Again in triangle $\triangle BOR$, $BR + BO > RO$

$\therefore PQ + PB + BR + BO > QB + RO$

or, $PQ + PR + BO > QO + OB + RO$

$\therefore PQ + PR > QO + RO$

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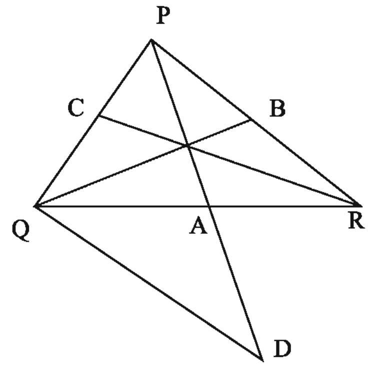
3) **Drawing:** Produce PA to D such that $PA = AD$. Add Q, D.

Proof:



In triangle $\triangle QAD$ and $\triangle PAR$,
$QA = AR$, $AD = PA$
and the included $\angle QAD$ = the included $\angle PAR$
$\therefore \triangle QAD \cong \triangle PAR$ and $QD = PR$
Now, in $\triangle PQD$, $PQ + QD > PD$
or, $PQ + PR > 2PA$ [$\because$ A is the mid point of PD]

Similarly, $PQ + QR > 2QB$ $\quad PR + QR > 2RC$

$\therefore PQ + PR + PQ + QR + PR + QR > 2PA + 2QB + 2RC$

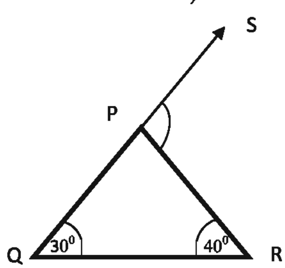
or, $2PQ + 2QR + 2PR > 2PA + 2QB + 2RC$

or, $PQ + QR + PR > PA + QB + RC$

$\therefore PA + QB + RC < PQ + QR + PR$

Exercises 6.3

1. The lengths of three sides of a triangle are given below. In which case it is possible to draw a triangle (the numbers are in unit of length)?
   1) 5, 6, 7 2) 5, 7, 14
   3) 3, 4, 7 4) 2, 4, 8
2. If one side of a equilateral triangle is produced to both, sides then what is difference between the generated exterior angles??
   1) 0° 2) 120° 3) 180° 4) 240°

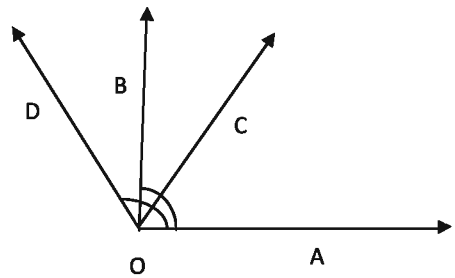


3. In figure what is the value of $\angle RPS$?

   1) 40° 2) 70° 3) 90° 4) 110°

4. In the adjacent figure

   (i) $\angle AOC$ is an acute angle
   (ii) $\angle AOB$ is a right angle
   (iii) $\angle AOD$ is a reflex angle



   Which one of the following is correct?

   1) i 2) ii 3) i and ii 4) ii and iii

5. If a triangle when placed on another exactly covers the other

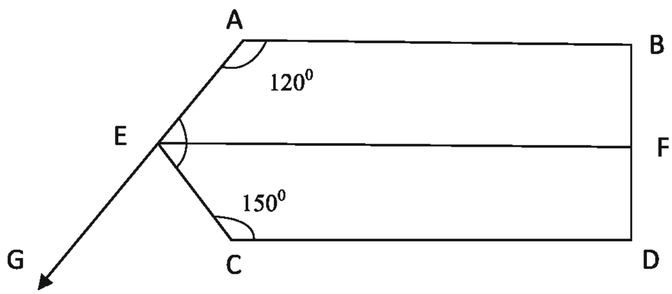
   (i) the two triangles are congruent

   (ii) the corresponding sides of the two triangles are equal

   (iii) corresponding angles are equal

   Which one of the following is correct?

   1) i, ii 2) i, iii 3) ii, iii 4) i, ii and iii



In the above figure $AB \parallel EF \parallel CD$ and $BD \perp CD$. Answer the questions (6 − 8) based on the given figure:

6. What is the value of $\angle AEF$?

   1) 30° 2) 60° 3) 240° 4) 270°

7. Which one of the following is the value of $\angle BFE$?

   1) 30° 2) 60° 3) 90° 4) 120°

8. $\angle CEF + \angle CEG =$ what?

1) 60° 2) 120° 3) 180° 4) 210°

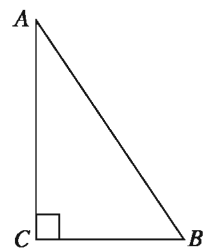
9. Prove that, the triangle formed by joining the mid points of the sides of an equilateral triangle is also equilateral.

10. Prove that, the three medians of an equilateral triangle are equal.

11. Prove that, the sum of any two exterior angles of a triangle is greater than two right angles.

12. In $\triangle ABC$, D is the mid point of side BC. Prove that, $AB + AC > 2AD$

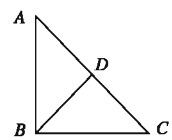
13. In the figure given, $\angle C = 1$ right angle and $\angle B = 2\angle A$. Prove that, $AB = 2BC$



14. Prove that, the exterior angle so formed by producing any side of a triangle is equal to the sum of the interior opposite angles.

15. Prove that, the difference between any two sides of a triangle is less than the third.

16. In the figure, in triangle ABC, $\angle B = 1$ right angle and D is the mid point of hypotenuse AC. Prove that, $BD = \frac{1}{2}AC$



17. In $\triangle ABC$, $AB > AC$ and bisector of $\angle A$ is AD which intersects side BC at point D. Prove that, $\angle ADB$ is obtuse angle.

18. Prove that, any point on the perpendicular bisector of a line segment is equidistant from the terminal points of that line segment

19. In triangle ABC, $\angle A = 1$ right angle. Mid point of BC is D
   1) Draw a triangle ABC with given information.
   2) Show that, $AB + AC > 2AD$
   3) Prove that, $AD = \frac{1}{2}BC$

20. In $\triangle ABC$, D and E are the mid points of AB and AC respectively and bisectors of $\angle B$ and $\angle C$ intersect at point O.

    1) Express the information of the stem through figure.
    2) Prove that, $DE \parallel BC$ and $DE = \dfrac{1}{2}BC$
    3) Prove that, $\angle BOC = 90° + \dfrac{1}{2}\angle A$

21. Prove that, bisector of the vertical angle of an isosceles triangle bisects the base and perpendicular to base.

22. Prove that, sum of the three medians of a triangle is less than the perimeter of the triangle.

23. An industrious father called upon his only son and said that he hid gold in a nearby forest which he bought with his earned money. When the son asked about the position of the gold, he informed him that there were two similar trees A and B and a stone S. After reaching from S to A, he would have to go perpendicularly the same distance to find the point C. Again coming from S to B, he would have to travel the same distance perpendicularly to find the point D. Then he would find the gold at the midpoint of the line CD. The son found the trees A and B but unfortunately did not find S. Will the son able to find the gold? If so, then how?

Chapter 7

Practical Geometry

In the previous classes geometrical figures were drawn in proving different propositions and in the exercises. There was no need for precision in drawing these figures. But sometimes precision is necessary in geometrical constructions. For example, when an architect makes a design of a house or an engineer draws different parts of a machine, high precision of drawing is required. In such geometrical constructions, one makes use of ruler and compasses only. We have already learnt how to construct triangles and quadrilaterals with the help of ruler and compasses. In this chapter we will discuss the construction of some special triangles and quadrilaterals.

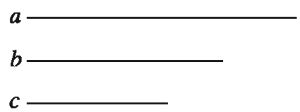
At the end of the chapter, the students will be able to –

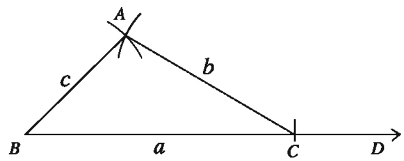
- explain triangles and quadrilaterals with the help of figures
- construct triangle by using given data
- construct quadrilateral, parallelogram, trapezium by using given data.

Construction of Triangles

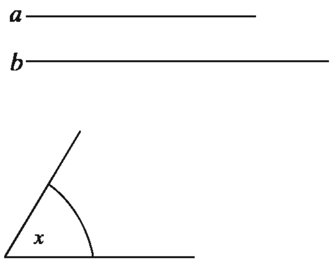
Every triangle has three sides and three angles. But, to specify the shape and size of a triangle, all sides and angles need not to be specified. For example, as sum of the three angles of a triangle is two right angles, one can easily find the measurement of the third angle when the measurement of the two angles of the triangle is given. Again, from the theorems on congruence of triangles it is found that the following combination of three sides and angles are enough to be congruent. That is, a combination of these three parts of a triangle is enough to construct a unique triangle. In class seven we have learnt how to construct triangles from the following data:

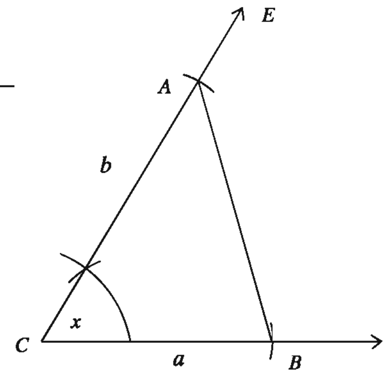
1. Three sides



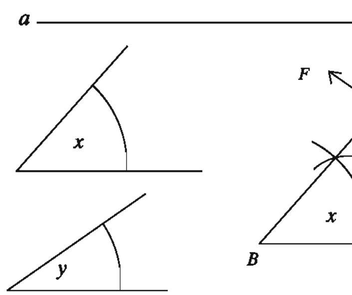


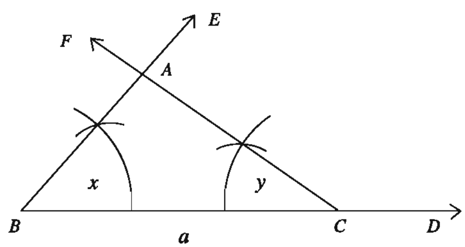
2. Two sides and their included angle.



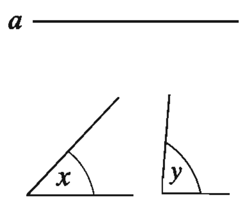


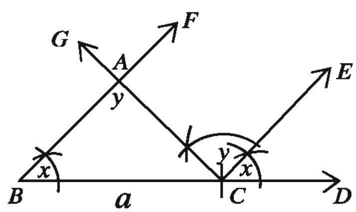
3. Two angles and their adjacent sides



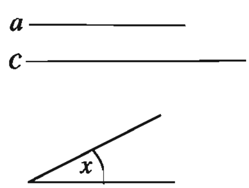


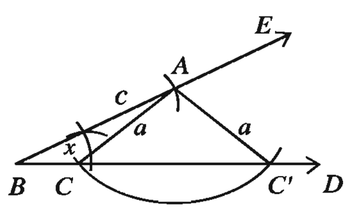
4. Two angles and an opposite side



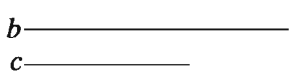


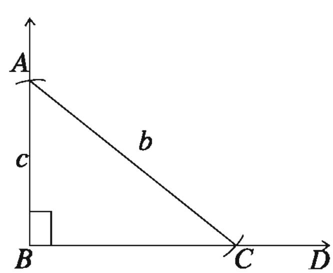
5. Two sides and an opposite angle



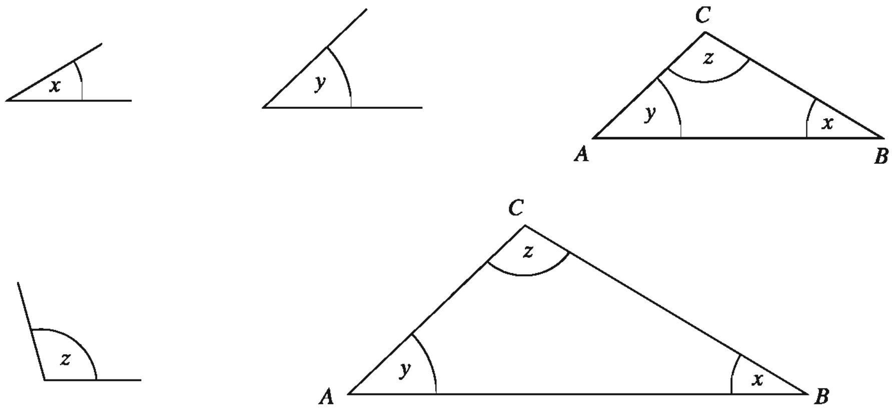


6. Hypotenuse and a side of a right-angled triangle





Observe that in each of the cases above, three parts of a triangle have been specified. But any three parts do not necessarily specify a unique triangle. As for example, if three angles are specified, infinite numbers of triangles of different sizes can be drawn with the specified angles (which are known as similar triangles).



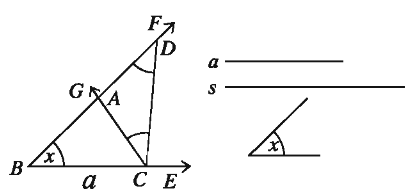
Sometimes for construction of a triangle three such data are provided by which we can specify the triangle through various drawing. Construction in a few such cases is stated below.

Construction 1. The base, the base adjacent angle and the sum of other two sides of a triangle are given. Construct the triangle.

Let the base a, a base adjacent angle $\angle x$ and the sum s of the other two sides of a triangle ABC be given. It is required to construct it.

Drawing:

1. From any ray BE cut the line segment BC equal to a. At B of the line segment BC, draw an angle $\angle CBF = \angle x$.
2. Cut a line segment BD equal to s from the ray BF.
3. Join C, D and at C make an angle $\angle DCG$ equal to $\angle BDC$ on the side of DC in which B lies.
4. Let the ray CG intersect BD at A.



Then, $\triangle ABC$ is the required triangle.

Proof: In $\triangle ACD$, $\angle ADC = \angle ACD$ [by construction]

$\therefore AC = AD$

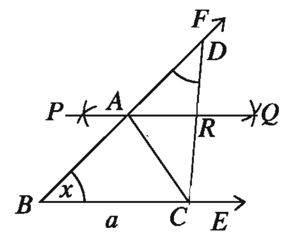
Now, in $\triangle ABC$, $\angle ABC = \angle x, BC = a$ [by construction]

and $BA+AC = BA+AD = BD = s$. Therefore, $\triangle ABC$ is the required triangle.

Alternate Method Let the base a, a base adjacent $\angle x$ and the sum s of the other two sides of a triangle be given. It is required to construct the triangle.

Drawing:

1. From any ray BE cut the line segment BC equal to a. At B of the line segment BC draw an angle $\angle CBF = \angle x$.
2. Cut a line segment BD equal to s from the ray BF.
3. Join C, D and construct the perpendicular bisector PQ of CD.
4. Let the ray PQ intersect BD at A and CD at R. Join A, C.



Then, $\triangle ABC$ is the required triangle.

Proof: In $\triangle ACR$ and $\triangle ADR$, $CR = DR$, $AR = AR$ and the included $\angle ARC =$ included $\angle ARD$ [right angle]

$\triangle ACR \cong \triangle ADR$.

$\therefore AC = AD$

Now, In $\triangle ABC$, $\angle ABC = \angle x, BC = a$ [by construction]

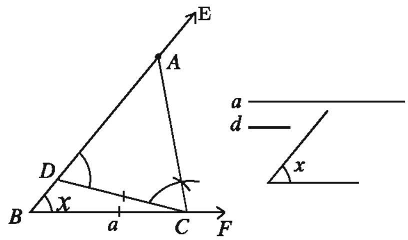
and $BA+AC = BA+AD = BD = s$. Therefore, $\triangle ABC$ is the required triangle.

Construction 2. The base of a triangle, an acute angle adjacent the base and the difference between the other two sides are given. Construct the triangle.

Let the base a, a base adjacent acute angle $\angle x$ and the difference d of the other two sides of a triangle be given. It is required to construct the triangle.

Drawing:

1. From any ray BF cut the line segment BC equal to a. At B of the line segment BC draw an angle $\angle CBF = \angle x$.
2. Cut a line segment BD equal to d from the ray BE.
3. Join C, D and at C make an angle $\angle DCA$ equal to $\angle EDC$ on the side of DC in which E lies.



Let the ray CA intersect BE at A. Then, $\triangle ABC$ is the required triangle.

Proof: By construction, in $\triangle ACD$, $\angle ACD = \angle ADC$

$\therefore AD = AC$

So, the difference of two sides, $AB - AC = AB - AD = BD = d$

Now, in $\triangle ABC$, $BC = a$, $AB - AC = d$ and $\angle ABC = \angle x$

Therefore, $\triangle ABC$ s the required triangle.

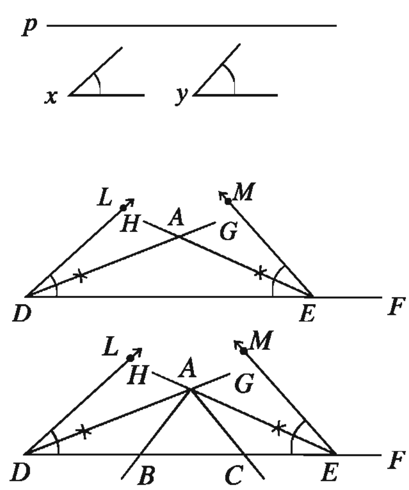
Work:

1) If the given angle is not acute, the above construction is not possible. Why ? Explore any way for the construction of the triangle under such circumstances.
2) The base, the base adjacent angle and the difference of the other two sides of a triangle are given. Construct the triangle in an alternate method.

Construction 3. Two angles adjacent to the base and the perimeter of a triangle are given. Construct the triangle.

Let the perimeter p and base adjacent angles $\angle x$ and $\angle y$ be given. It is required to construct the triangle.

Drawing:



1. From any ray DF, cut the part DE equal to the perimeter p. Make angles $\angle EDL$ equal to $\angle x$ and $\angle DEM$ equal to $\angle y$ on the same side of the line segment DE at D and E.
2. Draw the bisectors DG and EH of the two angles.
3. Let these bisectors DG and EH intersect at a point A. At the point A, draw $\angle DAB$ equal to $\angle ADE$ and $\angle EAC$ equal to $\angle AED$.
4. Let the rays AB and AC intersects DE at the point B and C respectively.

Then, $\triangle ABC$ is the required triangle.

Proof: In $\triangle ABD$, $\angle ADB = \angle DAB$ [by construction]

$\therefore AB = DB$

Again, $\triangle ACE$, $\angle AEC = \angle EAC$

$\therefore CA = CE$

Therefore, $\triangle ABC$, $AB + BC + CA = DB + BC + CE = DE = p$

$\angle ABC = \angle ADB + \angle DAB = \frac{1}{2}\angle x + \frac{1}{2}\angle x = \angle x$

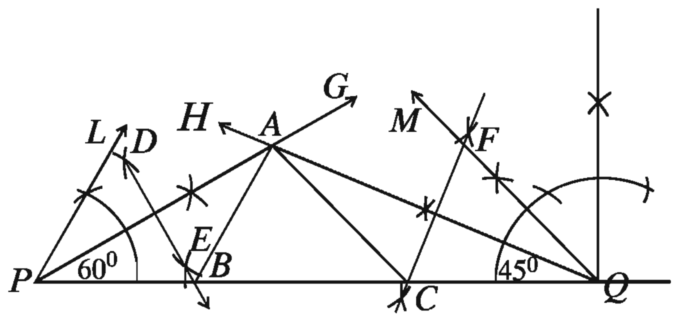
and $\angle ACB = \angle AEC + \angle EAC = \frac{1}{2}\angle y + \frac{1}{2}\angle y = \angle y$

Therefore, $\triangle ABC$ is the required triangle.

Work:

Two acute base adjacent angles and the perimeter of a triangle are given. Construct the triangle in an alternative way.

Example 1. Construct a triangle ABC, in which $\angle B = 60°$, $\angle C = 45°$ and perimeter $AB + BC + CA = 11$ cm.



Drawing: Follow the steps below:

1. Draw a line segment $PQ = 11$ cm.
2. At P, construct an angle of $\angle QPL = 60°$ and at Q, an angle of $\angle PQM = 45°$ on the same side of PQ.
3. Draw the bisectors PG and QH of the two angles. Let the bisectors PG and QH of these angles intersect at A.
4. Draw perpendicular bisector of the segments PA, QA to intersect PQ at B and C.
5. Join A, B and A, C.

Then, $\triangle ABC$ is the required triangle.

Work:

An adjacent side with the right angle and the difference of hypotenuse and the other side of a right-angled triangle are given. Construct the triangle.

Example 2. Base of a triangle $a = 3$ cm., base adjacent acute angle 45° and the sum of other two sides $s = 6$ cm.

1) Express the information of the stem in the picture.
2) Construct the triangle.(Construction and description of construction are must)
3) If the perimeter of a square is $2s$, construct the square. (Construction and description of construction are must)

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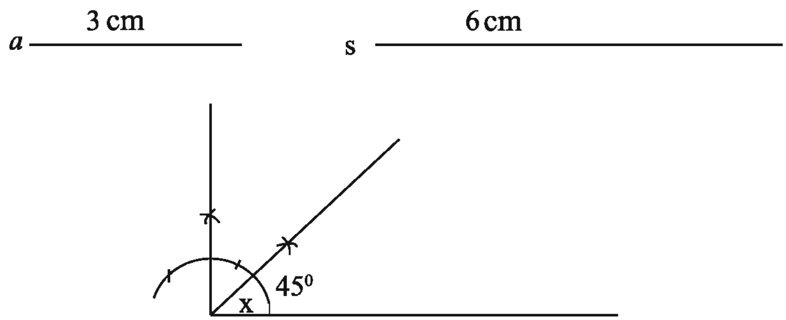
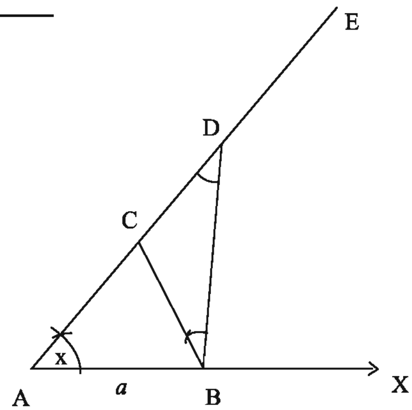
Solution:

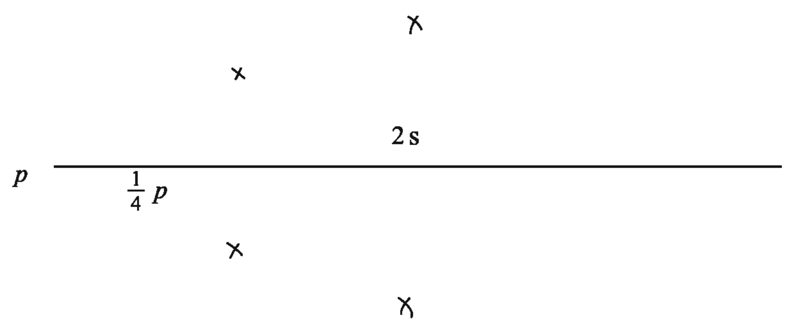
1)



2) Cut $AB = a$ from any ray AX. Draw $\angle XAE = x$ at the point A. Take $AD = s$ from AE. Join B, D. Now draw $\angle DBC$ equal to $\angle ADB$ at the point B. The line segment BC intersects AD at the point C. $\therefore$ ABC is the required triangle.

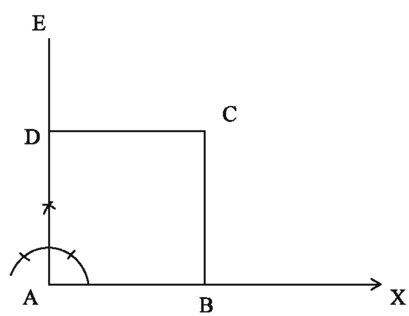


3) Let the perimeter of the square $p = 2s$. We are to draw the square.



Cut $AB = \dfrac{1}{4}p$ from any ray AX. Draw $AE \perp AB$ at the point A. Cut $AD = AB$ from AE. centring the points B and D draw two arcs equal radius of $\dfrac{1}{4}p$ in the interior of $\angle BAD$. The arcs intersects each other at the point C. Join B, C and C, D.

$\therefore$ $ABCD$ is the required square.



Exercise 7.1

1. Construct a triangle with the following data:

    1) The lengths of three sides are respectively 3 cm, 3.5 cm, 2.8 cm.

    2) The lengths of two sides are 4 cm, 3 cm and the included angle is 60°.

    3) Two angles are 60° and 45° and their adjacent side is 5 cm.

    4) Two angles are 60° and 45° and the side opposite the angle 45° is 5 cm.

    5) The lengths of two sides are 4.5 cm and 3.5 cm respectively and the angle opposite to the second side is 30°.

    6) The lengths of the hypotenuse and a side are 6 cm. and 4 cm. respectively.

2. Construct a triangle with the following data:

    1) Base 3.5 cm, base adjacent angle 60° and the sum of the two other sides 8 cm.

    2) Base 5 cm, base adjacent angle 45° and the difference of the two other sides 1 cm.

    3) Base adjacent angles 60° and 45° and the perimeter 12 cm.

3. Construct a triangle when the two base adjacent angles and the length of the perpendicular from the vertex to the base are given.

4. Construct a right-angled triangle when the hypotenuse and the sum of the other two sides are given.

5. Construct a triangle when a base adjacent angle, the height and the sum of the other two sides are given.

6. Construct an equilateral triangle whose perimeter is given.

7. The base, an obtuse base adjacent angle and the difference of the other two sides of a triangle are given. Construct the triangle.

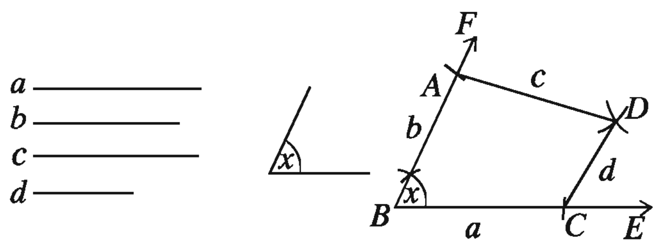
Construction of Quadrilaterals

We have seen if three independent data are given, in many cases it is possible to construct a definite triangle. But with four given sides the construction of a definite quadrilateral is not possible. Five independent data are required for construction of a definite quadrilateral. A definite quadrilateral can be constructed if any one of the following combinations of data is known :

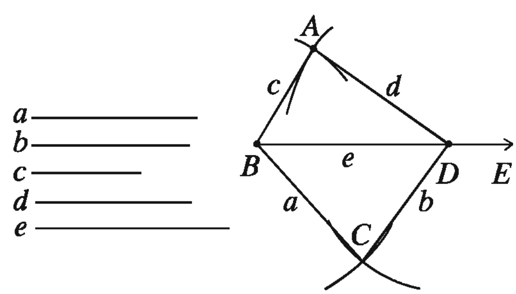
1. Four sides and an angle
2. Four sides and a diagonal
3. Three sides and two diagonals
4. Three sides and two included angles
5. Two sides and three angles.

In class VIII, the construction of quadrilaterals with the above specified data has been discussed. If we closely look at the steps of construction, we see that in some cases it is possible to construct the quadrilaterals directly. In some cases, the construction is done by constructions of triangles. Since a diagonal divides the quadrilateral into two triangles, when one or two diagonals are included in data, construction of quadrilaterals is possible through construction of triangle.

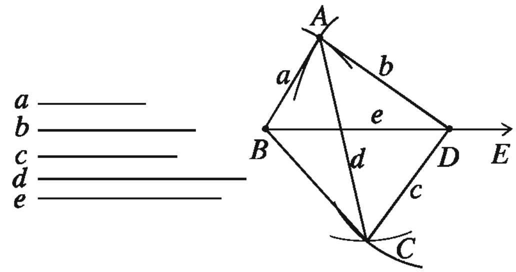
1. Four sides and an angle



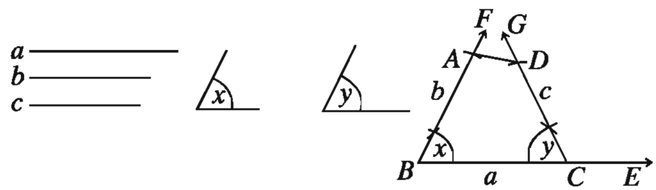
2. Four sides and a diagonal



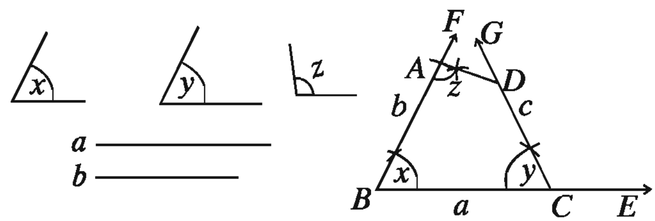
3. Three sides and two diagonals



4. Three sides and two included angles



5. Two sides and three angles

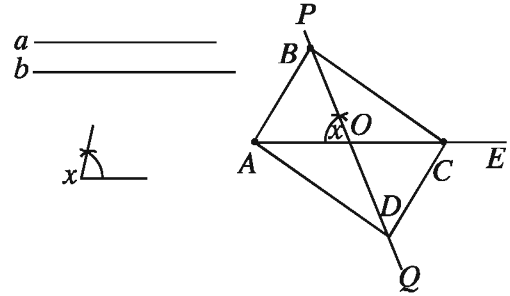


Sometimes special quadrilaterals can be constructed with fewer data. In such a case, from the properties of quadrilaterals, we can retrieve five necessary data. For example, a parallelogram can be constructed if only the two adjacent sides and the included angle are given. In this case, only three data are given. Again, a square can be constructed when only one side of the square is given. The four sides of a square are equal and an angle is a right angle; so five data are easily specified.

Construction 4. Two diagonals and an included angle between them of a parallelogram are given. Construct the parallelogram.

Let a and b be the diagonals of a parallelogram and $\angle x$ be an angle included between them. The parallelogram is to be constructed.

Drawing: From any ray AE, cut the line segment AC equal to a. Bisect the line segment AC to find the mid-point O. At O construct the angle $\angle AOP$ equal to $\angle x$ and extend the ray OP to the opposite ray OQ. From the rays OP and OQ cut two line segments OB and OD equal to $\frac{1}{2}b$. Join A, B; A, D; C, B and C, D.



Then, $ABCD$ is the required parallelogram.

Proof: In $\triangle AOB$ and $\triangle COD$,

$OA = OC = \frac{1}{2}a$, $OB = OD = \frac{1}{2}b$ [by construction]

and included $\angle AOB$ = included $\angle COD$ [opposite angle]

Therefore, $\triangle AOB \cong \triangle COD$

So, $AB = CD$

mp and $\angle ABO = \angle CDO$; but the two angles are alternate angles.

$\therefore$ AB and CD are parallel and equal.

Similarly, AD and BC are parallel and equal.

Therefore, $ABCD$ is a parallelogram with diagonals

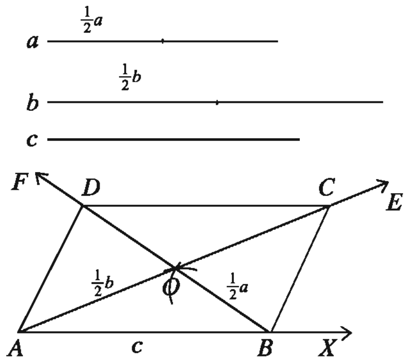
$AC = AO+OC = \frac{1}{2}a+\frac{1}{2}a = a$

and $BD = BO+OD = \frac{1}{2}b+\frac{1}{2}b = b$ and the angle included between the diagonals is $\angle AOB = \angle x$.

Therefore, $ABCD$ is the required parallelogram.

Construction 5. Two diagonals and a side of a parallelogram are given. Construct the parallelogram.

Let a and b be the diagonals and c be a side of the parallelogram. The parallelogram is to be constructed.



Drawing: Bisect the diagonals a and b into two equal parts. From any ray AX, cut the line segment AB equal to c. With centre at A and B draw two arcs with radius $\frac{a}{2}$ and $\frac{b}{2}$ respectively on the same side of AB . Let the arcs intersect at O. Join A, O and B, O. Extend AO to AE and BO to BF. Now cut $\frac{a}{2} = OC$ from OE and $\frac{b}{2} = OD$ from OF. Join A, D; D, C and B, C.

Therefore, $ABCD$ is the required parallelogram.

Proof: In $\triangle AOB$ and $\triangle COD$,

$OA = OC = \frac{a}{2}$; $OB = OD = \frac{b}{2}$ [by construction]

and included $\angle AOB$ = included $\angle COD$ [opposite angle]

$\therefore \triangle AOB \cong \triangle COD$

$\therefore AB = CD$ and $\angle ABO = \angle ODC$; but the angles are alternate angles.

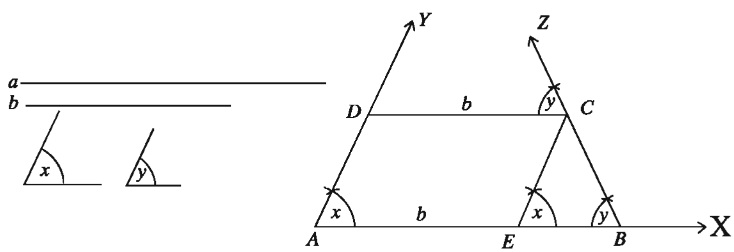
AB and CD are parallel and equal.

Similarly, AD and BC are parallel and equal.

Therefore, $ABCD$ is the required parallelogram.

Example 3. The parallel sides and two angles included with the larger side of a trapezium are given. Construct the trapezium.

Let a and b be the parallel sides of a trapezium where $a > b$ and $\angle x$ and $\angle y$ be two angles included with the side a. The trapezium is to be constructed.



Drawing: From any ray AX, cut the line segment $AB = a$. At A of the line segment AB, construct the angle $\angle BAY$ equal to $\angle x$ at B, construct the angle

$\angle ABZ$ equal to $\angle y$.

Now from the line segment AB, cut a line segment $AE = b$. At E, construct $EC \parallel AY$ which cuts BZ at C. Now construct $CD \parallel BA$. The line segment CD intersects the ray AY at D. Then, $ABCD$ is the required trapezium.

Proof: By construction, $AE \parallel CD$ and $AD \parallel EC$. Therefore $AECD$ is a parallelogram and $CD = AE = b$.

Now in the quadrilateral $ABCD$, $AB = a$, $CD = b$, $AB \parallel CD$ and $\angle BAD = \angle x$, $\angle ABC = \angle y$ [by construction]

Therefore, $ABCD$ is the required trapezium.

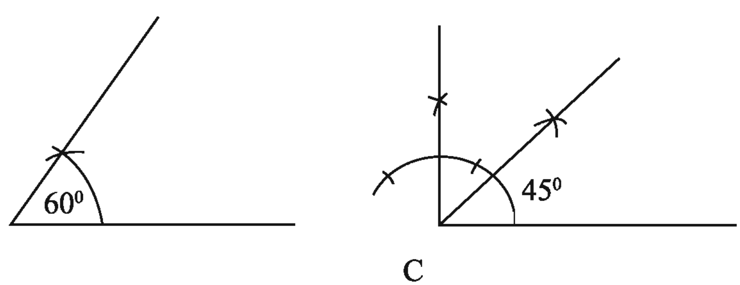
Work: The perimeter and an angle of a rhombus are given. Construct the rhombus.

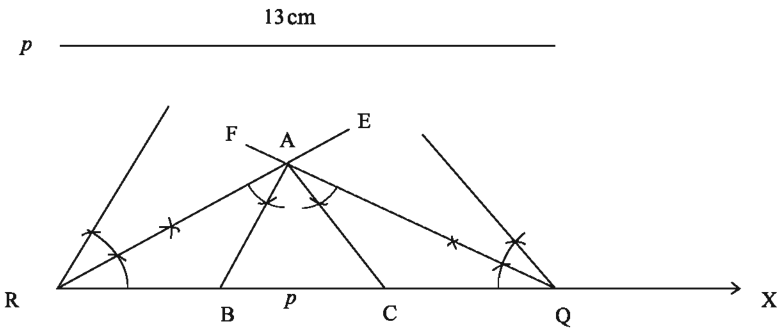
Example 4. In the triangle ABC, $\angle B = 60°$, $\angle C = 45°$ and perimeter $p = 13$ c m.

1) Construct $\angle B$ and $\angle C$ with scale and compass.
2) Construct the triangle. (Construction and description of construction are must)
3) Construct a rhombus whose length is equal to $\frac{p}{3}$ and an angle equal to $\angle B$. (Construction and description of construction is are must)

Solution:

1)

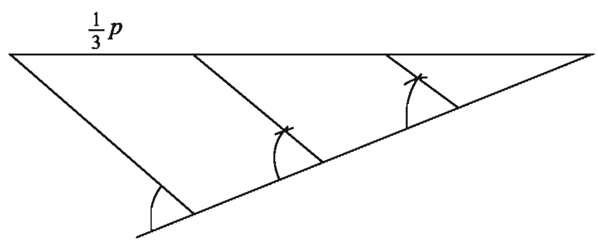


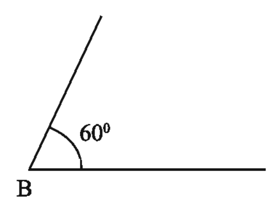
2) 

Cut $RQ = p$ from any ray RX. Draw $\frac{1}{2}\angle B = \angle ERX$ at the point R and draw $\frac{1}{2}\angle C = \angle FQR$ at the point Q. ER and FQ intersect each other at the point A . Now draw $\angle RAB = \frac{1}{2}\angle B$ and $\angle QAC = \frac{1}{2}\angle C$ at the point A where $\angle ERX$ of the side ER and $\angle FQR$ of the side FQ lie, respectively. The line segment AB and AC intersect at the point B and C of the line RQ.

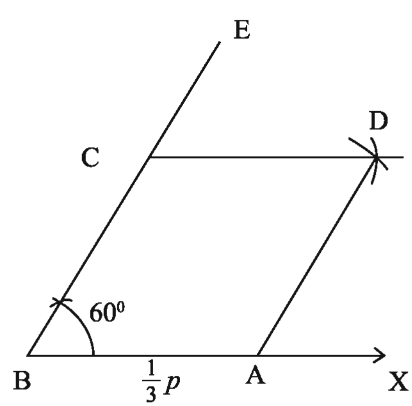
$\therefore$ ABC is the required triangle.

3) Given sides of rhombus is $\frac{1}{3}p$ and an angle is $\angle B = 60°$. Construct the rhombus.





Cut $BA = \frac{1}{3}p$ from any ray BX. Draw $\angle ABE = 60°$ at the point B. Take $BC = AB$ from BE. centring the point A and C, draw two arcs of equal radius with $\frac{1}{3}p$ in the interior of $\angle ABC$. The arcs intersects each other at the point D. Join A, D; C, D.
$\therefore$ $ABCD$ is the required rhombus.



Exercise 7.2

1. The two angles of a right angled triangle are given. Which one of the following combinations allows constructing the triangle?

   1) 60° and 36° 2) 40° and 50°
   3) 30° and 70° 4) 80° and 20°

2. If the lengths of two sides of a triangle are respectively 4 c m and 9 c m, then what of the following will be the length of the third side?

   1) 4 2) 5 3) 6 4) 13

3. If the lengths of two equal sides of a isosceles right angled triangle are 18 cm, then what of the following is the measurement of the triangle?

   1) 36 2) 81 3) 162 4) 324

4. The construction of a particular quadrilateral is possible if there is given-

   (*i*) four sides and an angle

   (*ii*) three sides and two included angles

   (*iii*) two sides and three angles

   Which one of the following is correct?

   1) *i* 2) *ii* 3) *i*, *ii* 4) *i*, *ii* and *iii*

5. In a rhombus-

   (*i*) four sides are mutually equal

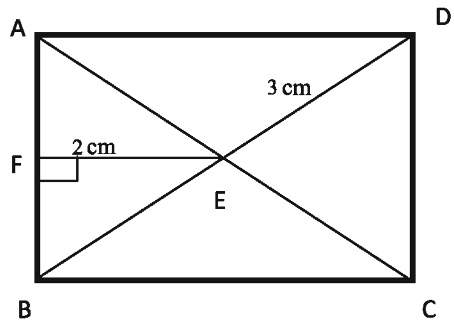
   (*ii*) opposite angles are equal

   (*iii*) two diagonals bisect each other in right angle

   Which one of the following is correct?

   1) *i*, *ii* 2) *i*, *iii* 3) *ii*, *iii* 4) *i*, *ii* and *iii*

In the picture, $ABCD$ is a rectangle where $EF = 2$ c m and $DE = 3$ c m. Use the above information to answer the question (6-8):



6. How many c m is the length of BF?
   1) 1 2) $\sqrt{5}$ 3) $\sqrt{13}$ 4) 5

7. How many c m is the length of AB?
   1) 2 2) $2\sqrt{5}$ 3) $5\sqrt{2}$ 4) 10

8. How many square c m is the measurement of $ABCD$?
   1) $8\sqrt{5}$ 2) 20 3) $12\sqrt{5}$ 4) $32\sqrt{5}$

9. Construct a quadrilateral with the following data :
   1) The lengths of four sides are 3 c m, 3.5 c m, 2.5 c m and 3 c m and an angle is 45°.
   2) The lengths of four sides are 3.5 c m, 4 c m, 2.5 c m and 3.5 c m and a diagonal is 5 c m.
   3) The lengths of three sides are 3.2 c m, 3 c m, 3.5 c m and two diagonals are 2.8 c m. and 4.5 c m.
   4) The lengths of three sides are 3 c m, 3.5 c m, 4 c m and two angles are 60° and 45°.

10. Construct a parallelogram with the following data:
    1) The lengths of two diagonals are 4 c m, 6.5 c m and the included angle is 45°.
    2) The length of a side is 4 c.m. and the lengths of two diagonals are 5 c.m., 6.5 c.m.

11. The sides AB and BC and the angles $\angle B$, $\angle C$ and $\angle D$ of the quadrilateral $ABCD$ are given. Construct the quadrilateral.

12. The four segments made by the intersecting points of the diagonals of a quadrilateral $ABCD$ and an included angle between them are $OA = 4$ c m,

$OB = 5$ c m, $OC = 3.5$ c m, $OD = 4.5$ c m and $\angle AOB = 80°$ respectively. Construct the quadrilateral.

13. The length of a side of a rhombus and an angle are 3.5 c m and 45° respectively; construct the rhombus.

14. The length of a side and a diagonal of a rhombus are given; construct the rhombus.

15. The length of two diagonals of a rhombus are given. Construct the rhombus.

16. The perimeter of a square is given. Construct the square.

17. The lengths of the hypotenuse and a side of right angled triangle are 5 c m and 4 c m Use the information to answer the following questions:

    1) Find the length of the other sides of the triangle.

    2) Construct the triangle.(Construction is must)

    3) Construct a square whose perimeter is equal to the perimeter of the triangle. (Construction is must)

18. $AB = 4$ c m, $BC = 5$ c m, $\angle A = 85°$, $\angle B = 80°$ and $\angle C = 95°$ of the quadrilateral $ABCD$. Use the information to answer the following questions:

    1) Find the value of $\angle D$.

    2) Use the above information to construct the quadrilateral $ABCD$.(Constructon is must)

    3) Construct the parallelogram considering the given sides as two sides of the parallelogram and $\angle B = 80°$. (Constructon is must)

19. The length of two parallel sides of a trapezium 4 c m and 6 c m and two angles adjacent greater side are $\angle x = 60°$ and $\angle y = 50°$.

    1) Express the given information in a picture.

    2) Construct the trapezium. (Construction and description of construction are must)

    3) Supposing two sides of the stem as two diagonals of a parallelogram and $\angle y$ as included angle, construct the parallelogram. (Construction and description of construction are must)

Chapter 8

Circle

We have already known that a circle is a geometrical figure in a plane consisting of points equidistant from a fixed point. Different concepts related to circles like centre, diameter, radius, chord etc have been discussed in previous class. In this chapter, the propositions related to arcs and tangents of a circle in the plane will be discussed.

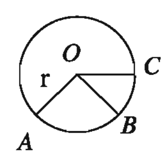
At the end of the chapter, the students will be able to

- explain arcs, angle at the centre, angle in the circle, quadrilaterals inscribed in the cirlce
- prove theorems related to circle
- apply of the theorems to solve many problems related to circle
- state constructions related to circle.

Circle

A circle is a geometrical figure in a plane whose points are equidistant from a fixed point. The fixed point is the centre of the circle. The closed path traced by a point that keeps it distance from the fixed centre is a circle. The distance from the centre is the radius of the circle.

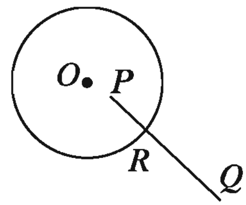
Let, O be a fixed point in a plane and r be a fixed measurement. The set of points which are at a distance r from O is the circle with centre O and radius r. In the figure, O is the centre of the circle and A, B and C are three points on the circle. Each of OA, OB and OC is a radius of the circle.



Some coplanar points are called concylcic if a circle passes through these points, i.e. there is a circle on which all these points lie. In the above figure, the points A, B and C are concyclic.

Interior and Exterior of a Circle

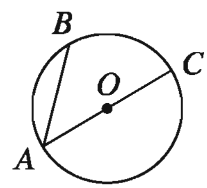
If O is the centre of a circle and r is its radius, the set of all points on the plane whose distances from O are less than r is called the interior region of the circle and the set of all points on the plane whose distances from O are greater than r is called the exterior region of the circle. The line segment joining two points of a circle lies inside the circle.



The line segment drawn from an interior point to an exterior point of a circle intersects a circle at one and only one point. In the figure, P is the interior point and Q is the exterior point of the circle. The line segment PQ intersects the circle at R only.

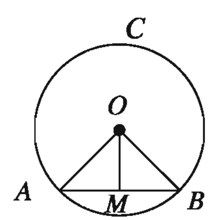
Chord and diameter of a circle

The line segment connecting two different points of a circle is a chord of the circle. If the chord passes through the centre, it is known as diameter. That is, any chord forwarding to the centre of the circle is diameter. In the figure, AB and AC are two chords and O is the centre of the circle. The chord AC is a diameter, since it passes through the centre. OA and OC are two radii of the circle. Therefore, the centre of a circle is the midpoint of any diameter. The length of a diameter is $2r$, where r is the radius of the circle.



Theorem 17. The line segment drawn from the centre of a circle to bisect a chord other than diameter is perpendicular to the chord.

Let AB be a chord (other than diameter) of a circle ABC with center O and M be the midpoint of the chord. Join O, M. It is to be proved that the line segment OM is perpendicular to the chord AB.



Drawing: Join O, A and O, B.

Proof:

Step 1. In $\triangle OAM$ and $\triangle OBM$,

$AM = BM$ [$\because M$ is the mid point of AB]

$OA = OB$ [$\because$ radius of same circle]

and $OM = OM$ [common side]

Therefore, $\triangle OAM \cong \triangle OBM$ [SSS theorem]

$\therefore \angle OMA = \angle OMB$

Step 2. Since the two angles together make a straight angle and are equal.

Therefore, $\angle OMA = \angle OMB = 1$ right angle.

Therefore, $OM \perp AB$. (Proved)

Corollary 1. The perpendicular bisector of any chord passes through the centre of the circle.

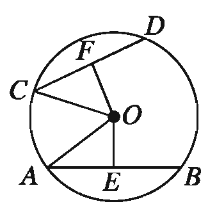
Corollary 2. A straight line can not intersect a circle in more than two points.

Work:

The theorem 17 opposite of the theorem states as follows:
the perpendicular from the centre of a circle to a chord bisects the chord. Prove the theorem.

Theorem 18. All equal chords of a circle are equidistant from the centre.

Let AB and CD be two equal chords of a circle with centre O. It is to be proved that the chords AB and CD are equidistant from the centre O.



Drawing: Draw from O the perpendiculars OE and OF to the chords AB and CD respectively. Join O, A and O, C.

Proof:

Step 1. $OE \perp AB$ and $OF \perp CD$

Therefore, $AE = BE$ and $CF = DF$ [$\because$ The perpendicular from the centre bisects the chord]

$\therefore AE = \frac{1}{2}AB$ and $CF = \frac{1}{2}CD$

Step 2. But $AB = CD$ [supposition]

$\therefore AE = CF$

Step 3. Now in the rig t-angled triangles $\triangle OAE$ and $\triangle OCF$

hypotenuse $OA =$ hypotenuse OC [radius of same circle]

and

$AE = CF$ [Step 2]

$\therefore \triangle OAE \cong \triangle OCF$ [RHS theorem]

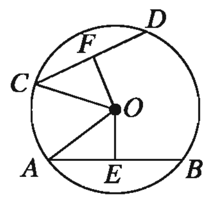
$\therefore OE = OF$

Step 4. But OE and OF are the distances from O to the chords AB and CD respectively.

Therefore, the chords AB and CD are equidistant from the centre of the circle. (Proved)

Theorem 19. Chords equidistant from the centre of a circle are equal.

Let AB and CD be two chords of a circle with centre O. OE and OF are the perpendiculars from O to the chords AB and CD respectively. Then OE and OF represent the distance from centre to the chords AB and CD respectively. It is to be proved that if $OE = OF$, $AB = CD$



Drawing: Join O, A and O, C.

Proof:

Step 1. Since $OE \perp AB$ and $OF \perp CD$

Therefore, $\angle OEA = \angle OFC = 1$ right angle.

Step 2. Now in the right angled triangles $\triangle OAE$ and $\triangle OCF$

hypotenuse $OA =$ hypotenuse OC [radius of same circle]

and

$OE = OF$ [supposition]

$\therefore \triangle OAE \cong \triangle OCF$ [RHS theorem]

$\therefore AE = CF$

Step 3. $AE = \frac{1}{2}AB$ and $CF = \frac{1}{2}CD$ [$\because$ The perpendicular from the centre bisects the chord]

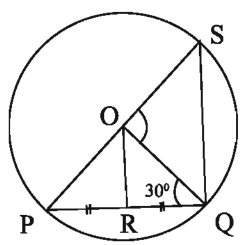
Step 4. Therefore, $\frac{1}{2}AB = \frac{1}{2}CD$

i.e., $AB = CD$ (Proved)

Corollary 3. The diameter is the greatest chord of a circle.

Exercise 8.1

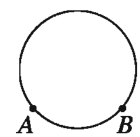
1. Prove that the straight line joining the middle points of two parallel chords of a circle passes through the centre and is perpendicular to the chords.
2. Two chords AB and AC of a circle subtend equal angles with the radius passing through A. Prove that, $AB = AC$.
3. A circle passes through the vertices of a right angled triangle. Show that, the centre of the circle is the middle point of the hypotenuse.
4. A chord AB of one of the two concentric circles intersects the other circle at points C and D. Prove that, $AC = BD$.
5. If two equal chords of a circle intersect each other, show that two segments of one are equal to two segments of the other.
6. Show that, the two equal chords drawn from two ends of the diameter on its opposite sides are parallel.
7. Show that, of the two chords of a circle the bigger chord is nearer to the centre than the smaller.
8. A chord $PQ = x$ c m of a circle with the centre O and $OR \perp PQ$.
   1) What is the measurement of $\angle QOS$?
   2) Prove that, the chord PS is the largest chord of the circle.
   3) If $OR = \left(\frac{x}{2} - 2\right)$ c m, find out the measurement of x.

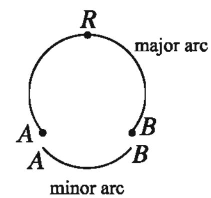


9. Prove that if the straight line joining two points make equal angles at two different points on the same side of the straight line, then all these four points are concentric.
10. Prove that, the middle points of equal chords of a circle are concyclic.
11. Show that, the two parallel chords of a circle drawn from two ends of a diameter on its opposite sides are equal.
12. Prove that, if two chords of a circle bisect each other, their point of intersection is the centre of the circle.

The arc of a circle

An arc is the piece of the circle between any two points of the circle. Look at the pieces of the circle between two points A and B in the figure. We find that there are two pieces, one comparatively larger and the other smaller. The larger one is called the major arc and the smaller one is called the minor arc. A and B are the terminal points of this arc and all other points are its internal points. With an internal fixed point R the arc is called arc ARB and is expressed by the symbol ARB. Again, sometimes minor arc is expressed by the symbol AB. The two points A and B of the circle divide the circle into two arcs. The terminal points of both arcs are A and B and there is no other common point of the two arcs other than the terminal points.

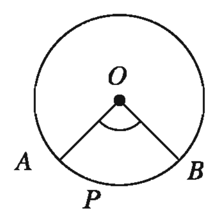




Arc cut by an Angle

An angle is said to cut an arc of a circle if

1. each terminal point of the arc lies on the sides of the angle,
2. each side of the angle contains at least one terminal point and
3. Every interior point of the arc lies inside the angle. The angle shown in the figure cuts the APB arc of the circle with centre O.

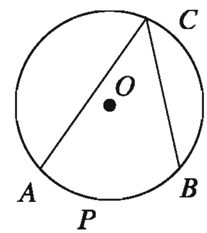


Angle in a Circle (Inscribed angle)

If two chords of a circle meet at a point on the circle then the angle formed between these chords is called circular angle or angle inscribed in a circle.

In the figure, $\angle ACB$ is an angle in a circle. Every angle in a circle cuts an arc of the circle. This arc may be a major or minor arc or a semi-circle.

The angle in a circle cuts an arc of the circle and the angle is said to be standing on the cut off arc. The angle is also known as the angle inscribed in the conjugate arc.



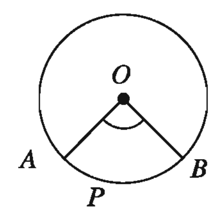
In the adjacent figure, the angle stands on the arc APB and is inscribed in the conjugate arc ACB.

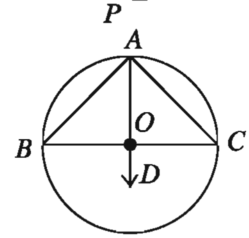
It is to be noted that, APB and ACB are mutually conjugate.

Remarks: The angle inscribed in an arc of a circle is the angle with vertex in the arc and the sides passing through the terminal points of the arc. An angle standing on an arc is the angle inscribed in the conjugate arc.

Angle at the Centre (Central angle)

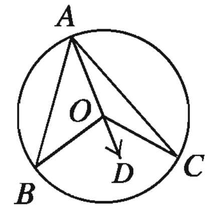
The angle with vertex at the centre of the circle is called an angle at the centre. An angle at the centre cuts an arc of the circle and is said to stand on the arc. In the adjacent figure, $\angle AOB$ is an angle at the centre and it stands on the arc APB. Every angle at the centre stands on a minor arc of the circle. In the figure APB is the minor arc. So the vertex of an angle at the centre always lies at the centre and the sides pass through the two terminal points of the arc. To consider an angle at the centre standing on a semi-circle the above description is not meaningful. In the case of semicircle, the angle at the centre $\angle BOC$ is a straight angle and the angle on the arc $\angle BAC$ is a right angle.





Theorem 20. The angle subtended by the same arc at the centre is double of the angle subtended by it at any point on the remaining part of the circle.

Given an arc BC of a circle subtending angles $\angle BOC$ at the centre O and $\angle BAC$ at a point A of the circle ABC.



We need to prove that $\angle BOC = 2\angle BAC$

Drawing: Suppose, the line segment AC does not pass through the centre. In this case, draw a line segment AD at A passing through the centre.

Proof:

Step 1. In $\triangle AOB$ the external angle $\angle BOD = \angle BAO + \angle ABO$ [$\because$ An exterior angle of a triangle is equal to the sum of the two interior opposite angles]

Step 2. In $\triangle AOB$, $OA = OB$ [$\because$ Radius of same circle]

Therefore, $\angle BAO = \angle ABO$ [$\because$ Base angles of an isosceles triangle are equal]

Step 3. From steps (1) and (2), $\angle BOD = 2\angle BAO$

Step 4. Similarly, from $\triangle AOC$, $\angle COD = 2\angle CAO$

Step 5. From steps (3) and (4),

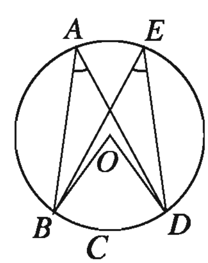
$\angle BOD + \angle COD = 2\angle BAO + 2\angle CAO$ [by adding]

This is the same as $\angle BOC = 2\angle BAC$ (Proved)

We can state the theorem in a different way. The angle standing on an arc of the circle is half the angle subtended by the arc at the centre.

> **Work:** Prove the theorem 20 when AC passes through the centre O of the circle ABC.

Theorem 21. Angles in a circle standing on the same arc are equal.



Let O be the centre of a circle and standing on the arc BCD, $\angle BAD$ and $\angle BED$ be the two angles in the circle. We need to prove that $\angle BAD = \angle BED$.

Drawing: Join O, B and O, D.

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Proof:

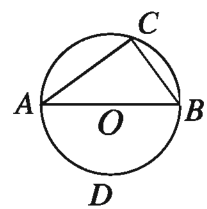
Step 1. The arc BCD subtends an angle $\angle BOD$ at the centre O.

Therefore, $\angle BOD = 2\angle BAD$ and $\angle BOD = 2\angle BED$ [$\because$ The angle subtended by an arc at the centre is double of the angle subtended on the circle]

$\therefore 2\angle BAD = 2\angle BED$

or, $\angle BAD = \angle BED$ (Proved)

Theorem 22. The angle inscribed in the semi- circle is a right angle.



Let AB be a diameter of circle with centre at O and $\angle ACB$ is the angle subtended by a semi-circle. It is to be proved that $\angle ACB$ is a right angle.

Drawing: Take a point D on the circle on the opposite side of AB of the circle where C is located.

Proof:

Step 1. The angle standing on the arc ADB

$\angle ACB = \frac{1}{2}$ (straight angle in the centre $\angle AOB$) [$\because$ The angle standing on an arc at any point of the circle is half the angle at the centre]

Step 2. But the straight angle $\angle AOB$ is equal to 2 right angles.

$\therefore \angle ACB = \frac{1}{2}$ (2 right angles) $=$ 1 right angle (Proved).

Corollary 4. The circle drawn with hypotenuse of a right-angled triangle as diameter passes through the vertices of the triangle.

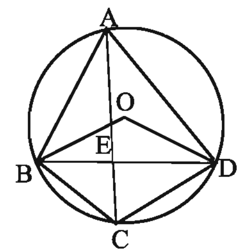
Corollary 5. The angle inscribed in the major arc of a circle is an acute angle.

Work:

Prove that any angle inscribed in a minor arc is obtuse.

Exercise 8.2

1. $ABCD$ is a quadrilateral inscribed in a circle with centre O. If the diagonals AC and BD intersect at the point E, prove that $\angle AOB + \angle COD = 2\angle AEB$.
2. $ABCD$ is a quadrilateral inscribed in a circle with centre O where $\angle ADB + \angle BDC = 1$ right angle. Prove that, A, O and C lie in the same straight line.
3. Show that, the oblique sides of a cyclic trapezium are equal.
4. In the figure, the centre of the circle is O and $OB = 2.5$ c m.
   1) Evaluate the parameter of the circle $ABCD$.
   2) Prove that $\angle BAD = \frac{1}{2}\angle BOD$
   3) If AC and BD intersect at the point E, prove that, $\angle AOB + \angle COD = 2\angle AEB$.



5. Two chords AB and CD of the circle $ABCD$ intersect at the point E. Show that, $\triangle AED$ and $\triangle BEC$ are equiangular.

Quadrilateral inscribed in a circle (Inscribed Quadrilaterals)

An inscribed quadrilateral or a quadrilateral inscribed in a circle is a quadrilateral having all four vertices on the circle. Such quadrilaterals possess a special property. The following work helps us understand this property.

Work: Draw a few inscribed quadrilaterals. This can easily be accomplished by drawing circles with different radius and then by taking four arbitrary points on each of the circles. Measure the angles of the quadrilaterals and fill in the following table.

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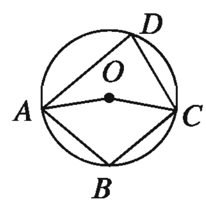
Serial No.	$\angle A$	$\angle B$	$\angle C$	$\angle D$	$\angle A + \angle C$	$\angle B + \angle D$
1						
2						
3						
4						
5						

What do you infer from the table?

Theorem 23. The sum of the two opposite angles of a quadrilateral inscribed in a circle is two right angles.

Let $ABCD$ be a quadrilateral inscribed in a circle with centre O. It is required to prove that, $\angle ABC + \angle ADC = 2$ right angles and $\angle BAD + \angle BCD = 2$ right angles.



Drawing: Join O, A and O, C.

Proof:

Step 1. Standing on the same arc ADC, the angle at centre reflex $\angle AOC = 2$ ($\angle ABC$ at the circumference)

that is, reflex $\angle AOC = 2\angle ABC$ [The angle subtended by an arc at the centre is double of the angle subtended by it at the circle]

Step 2. Again, standing on the same arc ABC, the angle at the centre $\angle AOC = 2$ ($\angle ADC$ at the circumference)

that is, $\angle AOC = 2\angle ADC$ [The angle subtended by an arc at the centre is double of the angle subtended by it at the circle]

$\therefore$ reflex $\angle AOC + \angle AOC = 2(\angle ABC + \angle ADC)$

But reflex $\angle AOC + \angle AOC = 4$ right angles

$\therefore 2(\angle ABC + \angle ADC) = 4$ right angles

$\therefore \angle ABC + \angle ADC = 2$ right angles

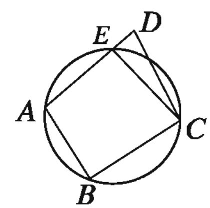
In the same way, it can be proved that $\angle BAD + \angle BCD = 2$ right angles. (Proved)

Corollary 6. If one side of a cyclic quadrilateral is extended, the exterior angle formed is equal to the opposite interior angle.

Corollary 7. A parallelogram inscribed in a circle is a rectangle.

Theorem 24. If two opposite angles of a quadrilateral are supplementary, the four vertices of the quadrilateral are concyclic.

Let $ABCD$ be the quadrilateral with $\angle ABC + \angle ADC = 2$ right angles. It is required to prove that the four points A, B, C, D are concyclic.



Drawing: Since the points A, B, C are not collinear, there exists a unique circle which passes through these three points. Let the circle intersect AD at E. Join C, E.

Proof: $ABCE$ is a quadrilateral inscribed in the circle.

Therefore, $\angle ABC + \angle AEC = 2$ right angles. [The sum of the two opposite angles of an inscribed quadrilateral is two right angles.]

But $\angle ABC + \angle ADC = 2$ right angles. [given]

$\therefore \angle AEC = \angle ADC$

But this is impossible, since in $\triangle CED$, exterior $\angle AEC >$ opposite interior $\angle ADC$

Therefore, E and D points can not be different points. So, E must coincide with the point D.

Therefore, the points A, B, C, D are concyclic.(Proved)

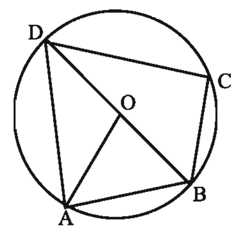
Exercise 8.3

1. If the internal and external bisectors of the angles $\angle B$ and $\angle C$ of $\triangle ABC$ meet at P and Q respectively, prove that B, P, C, Q are concyclcic.

2. $ABCD$ is a circle. If the bisectors of $\angle CAB$ and $\angle CBA$ meet at the point P and the bisectors of $\angle DBA$ and $\angle DAB$ meet at Q, prove that, the four points A, Q, P, B are concyclic.

3. The chords AB and CD of a circle with centre O meet at right angles at some point within the circle, prove that, $\angle AOD + \angle BOC = 2$ right angles.

4. The opposite angles of the quadrilateral $ABCD$ are supplementary to each other. If the line AC is the bisector of $\angle BAD$, prove that, $BC = CD$.
5. O is the centre of the circle with radius 2.5 c.m., $AB = 3$ c m and BD is the bisector of $\angle ADC$.

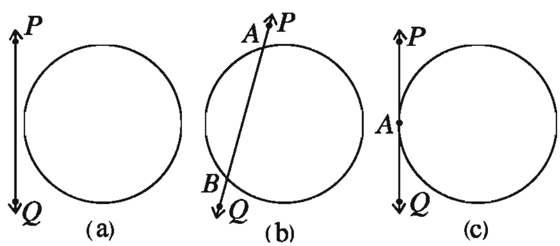


   1) Findthe length of AD.
   2) Show that, $\angle ADC + \angle ABC = 180°$.
   3) Prove that, $AB = BC$.
6. If the vertical angles of two triangles standing on equal bases are supplementary, prove that their circum circles are equal.
7. Prove that, the bisector of any angle of a cyclic quadrilateral and the exterior bisector of its opposite angle meet on the circumference of the circle.

Secant and Tangent of the circle

Consider the relative position of a circle and a straight line in the plane. Three possible situations of the following given figures may arise in such a case:

1) The circle and the straight line have no common points
2) The straight line has cut the circle at two points
3) The straight line has touched the circle at a point.



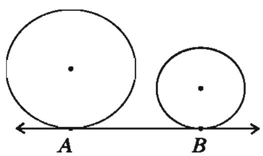
A circle and a straight line in a plane may at best have two points of intersection. If a circle and a straight line in a plane have two points of intersection, the straight line is called a secant to the circle and if the point of intersection is one and only one, the straight line is called a tangent to the circle. In the latter case, the common point is called the point of contact of the tangent. In the above figure, the relative position of a circle and a straight line is shown.

In figure (i) the circle and the straight line PQ have no common point; in figure (ii) the line PQ is a secant, since it intersects the circle at two points A and B and in figure (iii) the line PQ has touched the circle at A. PQ is a tangent to the circle and A is the point of contact of the tangent.

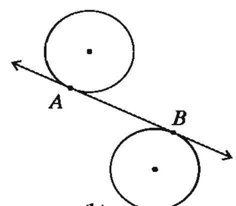
Remarks: All the points between two points of intersection of every secant of the circle lie interior of the circle.

Common tangent

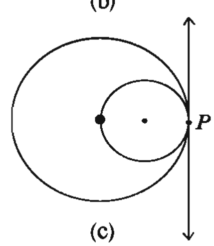
If a straight line is a tangent to two circles, it is called a common tangent to the two circles. In the adjoining figures, AB is a common tangent to both the circles. In figure (a) and (b), the points of contact are different. In figure (c) and (d), the points of contact are the same.



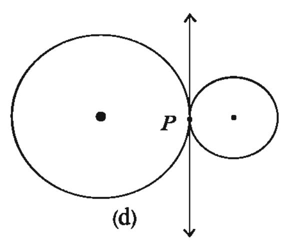
(a)



(b)



(c)



(d)

If the two points of contact of the common tangent to two circles are different, the tangent is said to be

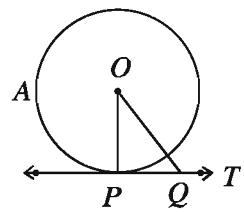
1) direct common tangent if the two centres of the circles lie on the same side of the tangent and
2) transverse common tangent, if the two centres lie on opposite sides of the tangent.

The tangent in figure (a) is a direct common one and in figure (c) it is a transverse common tangent.

If a common tangent to a circle touches both the circles at the same point, the two circles are said to touch each other at that point. In such a case, the two circles are said to have touched internally or externally according to their centres lie on the same side or opposite side of the tangent. In figure (c) the two circles have touched each other internally and in figure (d) externally.

Theorem 25. The tangent drawn at any point of a circle is perpendicular to the radius through the point of contact of the tangent.

Let PT be a tangent at the point P to the circle with centre O and OP is the radius through the point of contact. It is required to prove that, $PT \perp OP$.



Drawing: Take any point Q on PT and join O, Q.

Proof: Since PT is a tangent to the circle at the point P, hence every point on PT except P lies outside the circle. Therefore, the point Q is outside of the circle.

$\therefore$ OQ is greater than OP that is, $OQ > OP$ and it is true for every point Q on the tangent PT except P.

$\therefore$ So, OP is the shortest distance from the centre O to PT.

Therefore, $PT \perp OP$ (Proved)

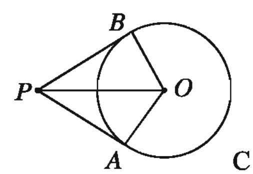
Corollary 8. At any point on a circle, only one tangent can be drawn.

Corollary 9. The perpendicular to a tangent at its point of contact passes through the centre of the circle.

Corollary 10. At any point of the circle the perpendicular to the radius is a tangent to the circle.

Theorem 26. If two tangents are drawn to a circle from an external point, the distances from that point to the points of contact are equal.

Let P be a point outside a circle ABC with centre O and two line segments PA and PB be two tangents to the circle at points A and B. It is required to prove that, $PA = PB$



Drawing: Join O, A; O, B and O, P.

Proof:

Step 1. Since PA is a tangent and OA is the radius through the point of tangent, $PA \perp OA$

$\therefore$ $\angle PAO = 1$ right angle. [$\because$ The tangent is perpendicular to the radius through the point of contact of the tangent]

Similarly, $\angle PBO = 1$ right angle.

$\therefore$ $\triangle PAO$ and $\triangle PBO$ are right-angled triangles.

Step 2. Now in the right angled triangles $\triangle PAO$ and $\triangle PBO$, hypotenuse $PO =$ hypotenuse PO and $OA = OB$ [$\because$ Radius of the same circle]

$\therefore\ \triangle PAO \cong \triangle PBO$ [Hypotenuse-side unanimity of right angled triangle]

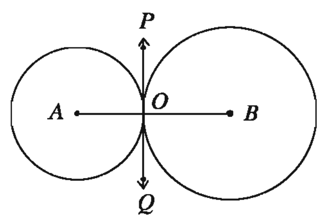
$\therefore\ PA = PB$ (Proved)

Remarks:

1. If two circles touch each other externally, all the points of one excepting the point of contact will lie outside the other circle.
2. If two circles touch each other internally, all the points of the smaller circle excepting the point of contact lie inside the greater circle.

Theorem 27. If two circles touch each other externally, the point of contact of the tangent and the centres are collinear.

Let the two circles with centres at A and B touch each other externally at O. It is required to prove that the points A, O, B are collinear.



Drawing: Since the given circles touch each other at O,they have a common tangent at the point O. Now draw the common tangent POQ at O and join O, A and O, B.

Proof:

In the circles with the centre A, OA is the radius through the point of contact of the tangent and POQ is the tangent.

Therefore $\angle POA = 1$ right angle. Similarly, $\angle POB = 1$ right angle.

$\angle POA + \angle POB = 1$ right angle $+ 1$ right angle $= 2$ right angles

Or, $\angle AOB = 2$ right angles

i.e. $\angle AOB$ is a straight angle.

$\therefore A,\ O,\ B$ are collinear. (Proved)

Corollary 11. If two circles touch each other externally, the distance between their centres is equal to the sum of their radii

Corollary 12. If two circles touch each other internally, the distance between their centres is equal to the difference of their radii.

Work: Prove that, if two circles touch each other internally, the point of contact of the tangent and the centres are collinear.

Exercise 8.4

1. Two tangents are drawn from an external point P to the circle with centre O. Prove that OP is the perpendicular bisector of the chord through the touch points.

2. Prove that, if two circles are concentric and if a chord of the greater circle touches the smaller, the chord is bisected at the point of contact.

3. AB is a diameter of a circle and BC is a chord equal to its radius. If the tangents drawn at A and C meet each other at the point D, prove that, ACD is an equilateral triangle.

4. Prove that a circumscribed quadrilateral of a circle having the angles subtended by opposite sides at the centre are supplementary.

5. PA and PB are two tangent from the point P external to circle with centre O.

   1) Construct the figure according to the stimulus.

   2) Prove that, $PA = PB$

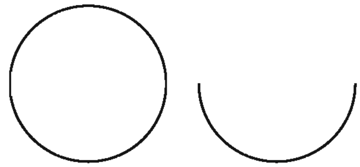
   3) Prove that, the segment OP is perpendicular bisector of the tangent chord.

6. Given, O is the centre of the circle and two tangents PA and PB touch the circle at the point A and B respectively. Prove that, PO is the bisector of $\angle APB$.

Constructions related to circles

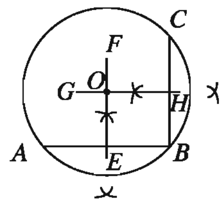
Construction 6. To determine the centre of a circle or an arc of a circle.

Given a circle as in figure (a) or an arc of a circle as in figure (b). It is required to determine the centre of the circle or the arc.



(a): Circle (b): Arc

Drawing: In the given circle or the arc of the circle, three different points A, B and C are taken.
Join A, B and B, C. The perpendicular bisectors EF and GH of the chords AB and BC are drawn respectively. Let the bisectors intersect at O. Threfore, the O is the required centre of the circle or of the arc of the circle.



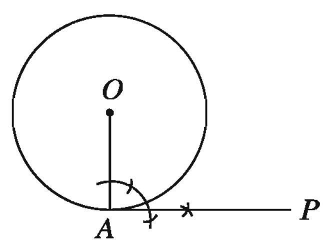
Proof: By construction, the line segment EF is the perpendicular bisector of chord AB and GH is the perpendicular bisector of chord BC. But both EF and GH pass through the centre and their common point is O. Therefore, the point O is the centre of the circle or of the arc of the circle.

Tangents to a Circle

We have known that a tangent can not be drawn to a circle from a point internal to it. If the point is on the circle, a single tangent can be drawn at that point. The tangent is perpendicular to the radius drawn from the specified point. Therefore, in order to construct a tangent to a circle at a point on it, it is required to construct the radius from the point and then construct a perpendicular to it. Again, if the point is located outside the circle, two tangents to the circle can be constructed.

Construction 7. To draw a tangent at any point of a circle.

Let A be a point of a circle whose centre is O. It is required to draw a tangent to the circle at the point A.



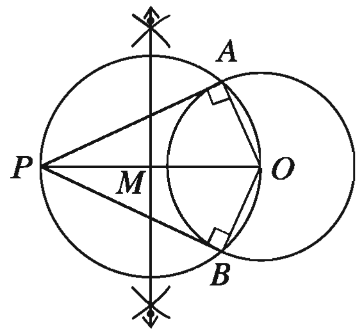
Drawing: O, A are joined. At the point A, a perpendicular AP is drawn to OA. Then AP is the required tangent.

Proof: The line segment OA is the radius passing through A and AP is perpendicular to it. Hence AP is the required tangent.

Note : At any point of a circle only one tangent can be drawn.

Construction 8. To draw a tangent to a circle from a point outside.

Let P be a point outside of a circle whose centre is O. A tangent is to be drawn to the circle from the point P.



Drawing:

1. Join P, O. The middle point M of the line segment PO is determined.
2. Now with M as centre and MO as radius, a circle is drawn. Let the new circle intersect the given circle at the points A and B.
3. Join A, P and B, P.

Then both AP, BP are the required tangents.

Proof: A, O and B, O are joined. PO is the diameter of the circle APB.

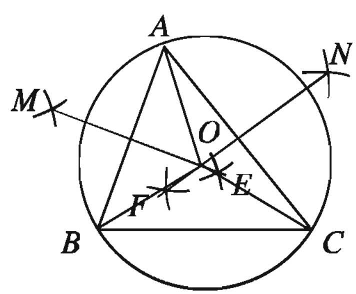
$\therefore \angle PAO = 1$ right angle [$\because$ the angle inscribed in the semi-circle is a right angle]

So the line segment OA is perpendicular to AP. Therefore, the line segment AP is a tangent at A to the circle with centre at O. Similarly, the line segment BP is also a tangent to the circle.

Nota Bene: Two and only two tangents can be drawn to a circle from an external point.

Construction 9. To draw a circle circumscribing a given triangle.

Let ABC be a triangle. It is required to draw a circle circumscribing it. That is, a circle which passes through the three vertices A, B and C of the triangle is to be drawn.



Drawing:

1. EM and FN, the perpendicular bisectors of AB and AC respectively are drawn. Let the line segments intersect each other at O.
2. A, O are joined. With O as centre and radius equal to OA, a circle is drawn.

Then the circle will pass through the points A, B and C and this circle is the required circumcircle of $\triangle ABC$.

Proof: B, O and C, O are joined. The point O stands on EM, the perpendicular bisector of AB.

$\therefore OA = OB$, Similarly, $OA = OC$

$\therefore OA = OB = OC$

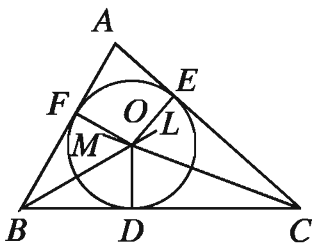
Hence, the circle drawn with O as the centre and OA as the radius passes through the three points A, B and C. This circle is the required circumcircle of $\triangle ABC$.

> **Work:** In the above figure, the circumcircle of an acute angled-triangle is constructed. Construct the circumcircle of an obtuse and right-angled triangles.

Notice that for in obtuse-angled triangle, the circumcentre lies outside the triangle, in acute-angled triangle, the circumcentre lies within the triangle and in right-angled triangle, the circumcentre lies on the hypotenuse of the triangle.

Construction 10. To draw a circle inscribed in a triangle.

Let $\triangle ABC$ be a triangle. It is required to draw am inscribed circle. that is,to inscribe a circle in it or to draw a circle in it such that it touches each of the three sides BC, CA and AB of $\triangle ABC$.



Drawing: BL and CM, the bisectors respectively of the angles $\angle ABC$ and $\angle ACB$ are drawn. Let the line segments intersect at O. OD is drawn perpendicular to BC from O and let it intersect BC at D. With O as centre and OD as radius, a circle is drawn. Then, this circle is the required inscribed circle.

Proof: From O, OE and OF are drawn perpendiculars respectively to AC and AB. Let these two perpendiculars intersect the respective sides at E and F.

The point O lies on the bisector of $\angle ABC$.

$\therefore OF = OD$

Similarly, as O lies on bisector of $\angle ACB$, $OE = OD$

$\therefore OD = OE = OF$

Hence, the circle drawn with centre as O and OD as radius passes through D, E and F.

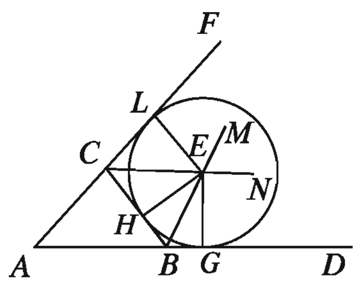
Again, OD, OE and OF respectively are perpendiculars to BC, AC and AB at their extremities.

Hence, the circle lying inside $\triangle ABC$ touches its sides at the points D, E and F respectively.

Hence, the circle DEF is the required inscribed circle of $\triangle ABC$.

Construction 11. To draw an ex-circle of a given triangle.

Let ABC be a triangle. It is required to draw its ex-circle. That is, to draw a circle which touches one side of triangle and the other two sides produced.



Drawing: Let AB and AC be produced to D and F respectively. BM and CN, the bisectors of $\angle DBC$ and $\angle FCB$ respectively are drawn. Let E be their point of intersection. From E, perpendicular EH is drawn on BC and let EH intersects BC at H. With E as centre and radius equal to EH, a circle is drawn. The circle HGL is the ex-circle of the triangle ABC.

Proof: From E, perpendiculars EG and EL respectively are drawn to line segments BD and CF.Let the perpendicular intersect line segments BD and CF at the point G and L respectively.

Since E lies on the bisector of $\angle DBC$ $\therefore EH = EG$

Similarly, the point E lies on the bisector of $\angle FCB$, so, $EH = EL$

$\therefore EH = EG = EL$

Hence, the circle drawn with E as centre and radius equal to EL passes through H, G and L.

Again, the line segments BC, BD and CF respectively are perpendiculars at the extremities of EH, EG and EL.

Hence, the circle touches the three line segments at the three points H, G and L.

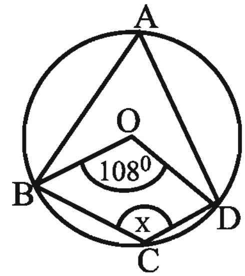
Therefore, the circle HGL is the ex-circle of $\triangle ABC$.

Remarks: Three ex-circles can be drawn with any triangle.

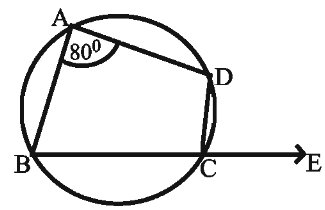
Work: Construct the two other ex-circles of a triangle.

Exercise 8.5

1. The angle inscribed in a major arc is-
   1) acute angle
   2) obtuse angle
   3) right angle
   4) complementary angle



2. What is the value of the angle x in the circle with centre O?
   1) 126°
   2) 108°
   3) 72°
   4) 54°



3. In the adjacent figure $\frac{1}{2}\angle ECD =$ how much degrees?
   1) 40°
   2) 50°
   3) 80°
   4) 100°

4. Two circles intersects each other externally. If one of their diametes is 8 c m and the radius of the another is 4 c m, what is the distance between their two centres?
   1) 0
   2) 4
   3) 8
   4) 12

5. If two tangents PQ and PR are drawn from the external point P in a circle with centre O, then $\triangle PQR$ will be-

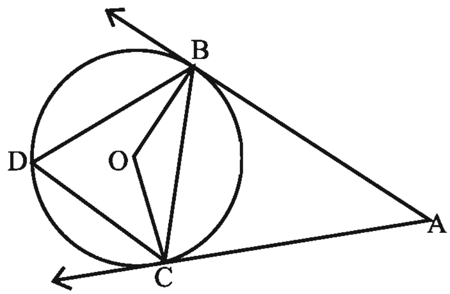
   (i) equilateral

   (ii) isosceles

   (iii) right angled

   Which one the following is correct?
   1) i
   2) i and ii
   3) ii and iii
   4) i, ii and iii

6. If O is the circumcentre of the equilateral triangle ABC, then $\angle BOC =$ how much degrees?
   1) 30°
   2) 60°
   3) 90°
   4) 120°



AB and AC are the tangents of the circle BCD. Centre of the circle in O and $\angle BAC = 60°$. According to this information answer the questions (7 - 8)

7. What is the value of $\angle BOC$?

 1) 300° 2) 270° 3) 120° 4) 90°

8. If D be the midpoint of the arc BDC ,then

 (i) $\angle BDC = \angle BAC$

 (ii) $\angle BAC = \frac{1}{2}\angle BOC$

 (iii) $\angle BOC = \angle DBC + \angle BCD$

 Which one of the following is correct?

 1) i and ii 2) i and iii 3) ii and iii 4) i, ii and iii

9. Draw a tangent to a circle which is parallel to a given straight line.

10. Draw a tangent to a circle which is perpendicular to a given straight line

11. Draw two tangents to a circle such that the angle between them is 60°.

12. Draw the circumcircle of the triangle whose sides are 3 c m, 4 c.m. and 4.5 c.m. and find the radius of this circle.

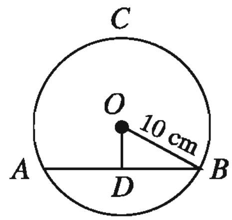
13. Draw an ex-circle to an equilateral triangle ABC touching the side AC of the triangle, the length of each side being 5 c m.

14. Draw the inscribed and the circumscribed circles of a square.

15. If the chords AB and CD of a circles with centre O intersect at an internal point E, prove that, $\angle AEC = \frac{1}{2}(\angle BOD + \angle AOC)$

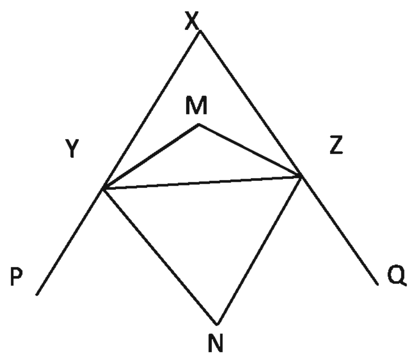
16. AB is the common chord of two circles of equal radius. If a line segment drawn from the point B meet through the circles at P and Q, prove that, $\triangle OAQ$ is an isosceles triangle.

17. The chord $AB = x$ c m and $OD \perp AB$, are in the circle ABC with centre O. use the adjoint figure to answer the following questions:



1) Find the area of the circle.
2) Show that, D is the mid point of AB.
3) If $OD = \left(\frac{x}{2} - 2\right)$ c m, determine the value of x.

18. In the figure, YM and ZM are internal bisectors of $\angle Y$ and $\angle Z$ respectively and YN and ZN are external bisectors of $\angle Y$ $\angle Z$.



1) Show that, $\angle MYZ + \angle NYZ = 90°$
2) Prove that, $\angle YNZ = 90° - \frac{1}{2}\angle X$
3) Prove that, Y, M, Z and N are concyclic.

19. The lengths of three sides of a triangle are 4 c m, 5 c m and 6 c m. Use this information to answer the following questions:

1) Construct the triangle.
2) Draw the circumcircle of the triangle.
3) From an exterior point of the circumcircle, draw two tangents to it and show that their lengths are equal.

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Chapter 9

Trigonometric Ratio

In our day to day life we make use of triangles, and in particular, right-angled triangles. Many different examples from our surroundings can be drawn where right triangles can be imagined to be formed. In ancient times, with the help of geometry men learnt the technique of determining the width of a river by standing on its bank. Without climbing the tree they knew how to measure the height of the tree accurately by comparing its shadow with that of a stick. In all the situations given above, the distances or heights can be found by using some mathematical technique which come under a special branch of mathematics called Trigonometry. The word 'Trigonometry' is derived from Greek words 'tri' (means three), 'gon' (means edge) and 'metron' (means measure). In fact, trigonometry is the study of relationship between the sides and angles of a triangle. There are evidence of using the Trigonometry in Egyptian and Babilonian civilization. It is believed that the Egyptians made its extensive use in land survey and engineering works. Early astrologers used it to determine the distances from the Earth to the farthest planets and stars. At present trigonometry is in use in all branches of mathematics. There are wide used of trigonometry for the solution of triangle related problems and in navigation etc. Now a days trigonometry is in wide use in Astronomy and Calculus.

At the end of the chapter, the students will be able to –

- describe the trigonometric ratios of acute angles.
- determine the mutual relations among the trigonometric ratios of acute angle.
- solve and prove the mathematical problems justifying the trigonometric ratios of acute angle.
- determine and apply trigonometric ratios of acute angles $30°, 45°, 60°$ geometrically.
- determine and apply the value of meaningful trigonometric ratios of the

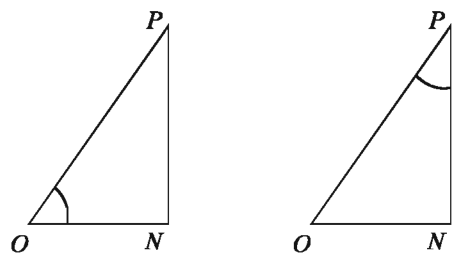
angles 0° and 90°.

- prove the trigonometric identities.
- apply the trigonometric identities.

Naming of sides of a right-angled triangle

We know that, the sides of right-angle triangle are known as hypotenuse, base and height. This is successful for the horizontal position of triangle. Again, the naming of sides is based on the position of one of the two acute angles of right-angled triangle. As for example :

1. '(hypotenuse)',the side of a right-angled triangle, which is the opposite side of the right angle.
2. '(opposite side)', which is the direct opposite side of a given angle.
3. '(adjacent side)', which is a line segment constituting the given angle.



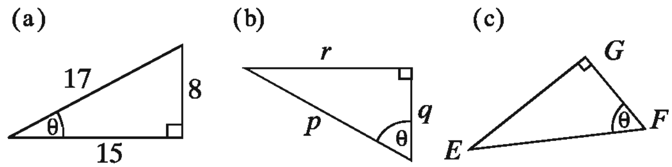
For the angle $\angle PON$, OP is the hypotenuse, ON is the adjacent side and PN is the opposite side. For the angle $\angle OPN$, OP is the hypotenuse, PN is the adjacent side and ON is the opposite side.

In the geometric figure, the capital letters are used to indicate the vertices and small letters are used to indicate the sides of a triangle. We often use the Greek letters to indicate angle. Widely used six letters of Greek alphabet are :

alpha α	beta β	gamma γ	theta θ	phi ϕ	omega ω
alpha	beta	gamma	theta	phi	omega

Greek letter are used in geometry and trigonometry through all the great mathematician of ancient Greek.

Example 1. Indicate the hypotenuse, the adjacent side and the opposite side for the angleθ.



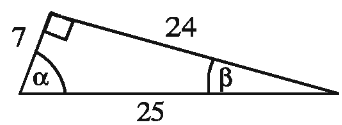
Solution:

1) Hypotenuse 17 units
opposite side 8 units
adjacent side 15 units

2) Hypotenuse p
opposite side r
adjacent side q

3) Hypotenuse EF
opposite side EG
adjacent side FG

Example 2. Find the lengths of hypotenuse, the adjacent side and the opposite side for the angles α and β.



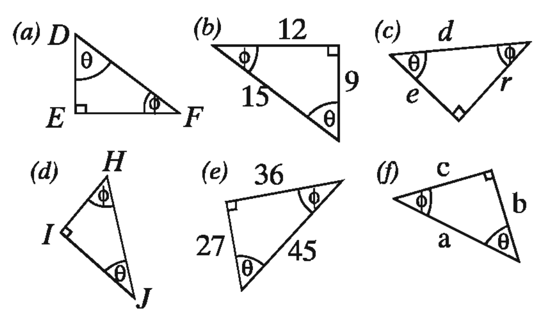
Solution:

1) For α angle
hypotenuse 25 units
adjacent side 7 units
opposite side 24 units

2) For β angle
hypotenuse 25 units
adjacent side 24 units
opposite side 7 units

Work:

Indicate the hypotenuse, adjacent side and opposite for the angle θ and ϕ.



Constancy of ratios of the sides of similar right-angled triangles

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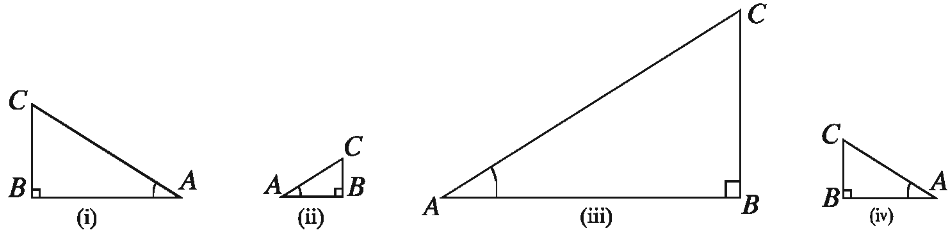
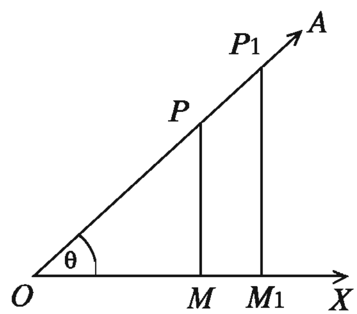
Work: Measure the lengths of the sides of the following four similar triangles and complete the table below. What do you observe about the ratios of the triangles?



length of sides			ratio(related to angle)		
BC	AB	AC	BC/AC	AB/AC	BC/AB

Let $\angle XOA$ is an acute angle. We take a point P on the side OA. We draw a perpendicular PM from P to OX. As a result, a right-angled triangle POM is formed. The three ratios of the sides PM, OM and OP of $\triangle POM$ do not depend on the position of the point P on the side OA.



If we draw the perpendiculars PM and P_1M_1 from two points P and P_1 to the side OX, two similar triangles $\triangle POM$ and $\triangle P_1OM_1$ are formed.

Now, $\triangle POM$ and $\triangle P_1OM_1$ are being similar,

$$\frac{PM}{P_1M_1} = \frac{OP}{OP_1} \text{ or, } \frac{PM}{OP} = \frac{P_1M_1}{OP_1}$$

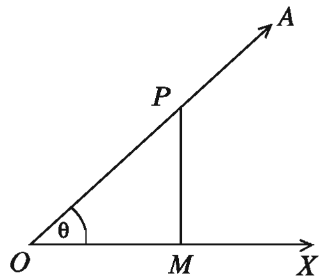
$$\frac{OM}{OM_1} = \frac{OP}{OP_1} \text{ or, } \frac{OM}{OP} = \frac{OM_1}{OP_1}$$

$$\frac{PM}{P_1M_1} = \frac{OM}{OM_1} \text{ or, } \frac{PM}{OM} = \frac{P_1M_1}{OM_1}$$

That is, each of these ratios is constant. These ratios are called trigonometric ratios.

Trigonometric ratios of an acute angle

Let $\angle XOA$ be an acute angle. We take any point P on OA. We draw a perpendicular PM from the point P to OA. A right angled triangle POM is formed. The six ratios are obtained from the sides PM, OM and OP of $\triangle POM$ which are called trigonometric ratios of the angle $\angle XOA$ and each of them are named particularly. With respect to the $\angle XOA$ of right-angled triangle POM, PM is the opposite side, OM is the adjacent side and OP is the hypotenuse. Denoting $\angle XOA = \theta$, the obtained six ratios are described below for the angle θ.



From the figure :

$$\sin\theta = \frac{PM}{OP} = \frac{\text{opposite side}}{\text{hypotenuse}} \text{ [sine of angle } \theta]$$

$$\cos\theta = \frac{OM}{OP} = \frac{\text{adjacent side}}{\text{hypotenuse}} \text{ [cosine of angle } \theta]$$

$$\tan\theta = \frac{PM}{OM} = \frac{\text{opposite side}}{\text{adjacent side}} \text{ [tangent of angle } \theta]$$

And opposite ratios of them are

$$\text{cosec}\theta = \frac{1}{\sin\theta} \text{ [cosecant of angle } \theta]$$

$$\sec\theta = \frac{1}{\cos\theta} \text{ [secant of angle } \theta]$$

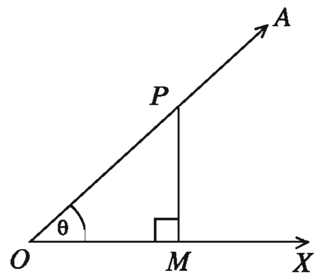
$$\cot\theta = \frac{1}{\tan\theta} \text{ [cotangent of angle } \theta]$$

We observe, the symbol $\sin\theta$ means the ratio of sine of the angle θ, not the multiplication of sin and θ. sin is meaningless without θ. It is applicable for the other trigonometric ratios as well

Relation among the trigonometric ratios

FLet $\angle XOA = \theta$ is an acute angle.

rom the adjacent figure, according to the definition



$\sin\theta = \dfrac{PM}{OP}$, $\text{cosec}\theta = \dfrac{1}{\sin\theta} = \dfrac{OP}{PM}$

$\cos\theta = \dfrac{OM}{OP}$, $\sec\theta = \dfrac{1}{\cos\theta} = \dfrac{OP}{OM}$

$\tan\theta = \dfrac{PM}{OM}$, $\cot\theta = \dfrac{1}{\tan\theta} = \dfrac{OM}{PM}$

Again, $\tan\theta = \dfrac{PM}{OM} = \dfrac{\frac{PM}{OP}}{\frac{OM}{OP}}$ [dividing the numerator and the denominator by OP]

Or, $\tan\theta = \dfrac{\sin\theta}{\cos\theta}$

$\therefore \boxed{\tan\theta = \dfrac{\sin\theta}{\cos\theta}}$

and similarly,

$\boxed{\cot\theta = \dfrac{\cos\theta}{\sin\theta}}$

Trigonometric identity

(i) $(\sin\theta)^2 + (\cos\theta)^2 = \left(\dfrac{PM}{OP}\right)^2 + \left(\dfrac{OM}{OP}\right)^2$

$= \dfrac{PM^2}{OP^2} + \dfrac{OM^2}{OP^2} = \dfrac{PM^2 + OM^2}{OP^2} = \dfrac{OP^2}{OP^2}$ [by the formula of Pythagoras]

$= 1$

Or, $(\sin\theta)^2 + (\cos\theta)^2 = 1$

$\therefore \boxed{(\sin\theta)^2 + (\cos\theta)^2 = 1}$

Remarks: For integer indices n we can write $\sin^n\theta$ for $(\sin\theta)^n$ and $\cos^n\theta$ for $(\cos\theta)^n$.

(ii) $\sec^2\theta = (\sec\theta)^2 = \left(\dfrac{OP}{OM}\right)^2$

$= \dfrac{OP^2}{OM^2} = \dfrac{OM^2 + PM^2}{OM^2}$ [OP is the hypotenuse of right-angled $\triangle POM$]

$= \dfrac{OM^2}{OM^2} + \dfrac{PM^2}{OM^2}$

$= 1 + \left(\dfrac{PM}{OM}\right)^2 = 1 + (\tan\theta)^2 = 1 + \tan^2\theta$

$\therefore \boxed{\sec^2\theta - \tan^2\theta = 1}$ and $\boxed{\tan^2\theta = \sec^2\theta - 1}$

(iii) $\text{cosec}^2\theta = (\text{cosec}^2\theta)^2 = \left(\dfrac{OP}{PM}\right)^2$

$= \dfrac{OP^2}{PM^2} = \dfrac{PM^2 + OM^2}{PM^2}$ [OP is the hypotenuse of right-angled $\triangle POM$]

$= \dfrac{PM^2}{PM^2} + \dfrac{OM^2}{PM^2} = 1 + \left(\dfrac{OM}{PM}\right)^2$

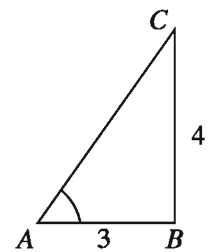
$= 1 + (\cot\theta)^2 = 1 + \cot^2\theta$

$\therefore \boxed{\text{cosec}^2\theta - \cot^2\theta = 1}$ and $\boxed{\cot^2\theta = \text{cosec}^2\theta - 1}$

Example 3. If $\tan A = \dfrac{4}{3}$, find the other trigonometric ratios of the angle A.

Solution: Given that, $\tan A = \dfrac{4}{3}$

So, opposite side of the angle A= 4, adjacent side= 3
hypotenuse $=\sqrt{4^2+3^2}=\sqrt{25}=5$
Therefore, $\sin A=\frac{4}{5}$, $\cos A=\frac{3}{5}$, $\cot A=\frac{3}{4}$
$\operatorname{cosec} A=\frac{5}{4}$, $\sec A=\frac{5}{3}$

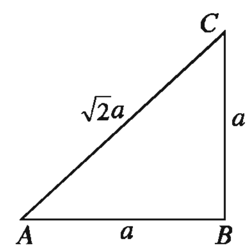


Work: Construct a table of the following trigonometric formulae for memorizing easily.

$\operatorname{cosec}\theta=\frac{1}{\sin\theta}$ $\sec\theta=\frac{1}{\cos\theta}$ $\cot\theta=\frac{1}{\tan\theta}$	$\tan\theta=\frac{\sin\theta}{\cos\theta}$ $\cot\theta=\frac{\cos\theta}{\sin\theta}$	$\sin^2\theta+\cos^2\theta=1$ $\sec^2\theta=1+\tan^2\theta$ $\operatorname{cosec}^2\theta=1+\cot^2\theta$

Example 4. $\angle B$ is the right-angle of a right angled triangle ABC. If $\tan A=1$ then verify the truth of $2\sin A\cos A=1$.

Solution: Given that, $\tan A=1$
So, opposite side of the angle=adjacent side=a
hypotenuse $=\sqrt{a^2+a^2}=\sqrt{2}a$
Therefore, $\sin A=\frac{a}{\sqrt{2}a}=\frac{1}{\sqrt{2}}$, $\cos A=\frac{a}{\sqrt{2}a}=\frac{1}{\sqrt{2}}$
Now, left hand side $=2\sin A\cos A=2\cdot\frac{1}{\sqrt{2}}\cdot\frac{1}{\sqrt{2}}=$
$2\cdot\frac{1}{2}=1=$ right hand side.
$\therefore 2\sin A\cos A=1$ is true.



Work:

$\angle C$ is the right angle of a right angled triangle ABC, if $AB=29$ c m, $BC=21$ c m and $\angle ABC=\theta$, find the value of $\cos^2\theta-\sin^2\theta$.

Example 5. Prove that, $\tan\theta+\cot\theta=\sec\theta\cdot\operatorname{cosec}\theta$

Solution:

Left hand side $=\tan\theta+\cot\theta$
$=\frac{\sin\theta}{\cos\theta}+\frac{\cos\theta}{\sin\theta}$

$$= \frac{\sin^2\theta + \cos^2\theta}{\sin\theta \cdot \cos\theta}$$

$$= \frac{1}{\sin\theta \cdot \cos\theta} \; [\because \sin^2\theta + \cos^2\theta = 1]$$

$$= \frac{1}{\sin\theta} \cdot \frac{1}{\cos\theta}$$

$$= \operatorname{cosec}\theta \cdot \sec\theta$$

$= \sec\theta \cdot \operatorname{cosec}\theta =$ Right hand side (Proved)

Example 6. Prove that, $\sec^2\theta + \operatorname{cosec}^2\theta = \sec^2\theta \cdot \operatorname{cosec}^2\theta$

Solution:

L.H.S. $= \sec^2\theta + \operatorname{cosec}^2\theta$

$$= \frac{1}{\cos^2\theta} + \frac{1}{\sin^2\theta}$$

$$= \frac{\sin^2\theta + \cos^2\theta}{\cos^2\theta \cdot \sin^2\theta}$$

$$= \frac{1}{\cos^2\theta \cdot \sin^2\theta}$$

$$= \frac{1}{\cos^2\theta} \cdot \frac{1}{\sin^2\theta}$$

$$= \sec^2\theta \cdot \operatorname{cosec}^2\theta$$

= R.H.S (Proved)

Example 7. Prove that, $\dfrac{1}{1+\sin^2\theta} + \dfrac{1}{1+\operatorname{cosec}^2\theta} = 1$

Solution:

L.H.S. $= \dfrac{1}{1+\sin^2\theta} + \dfrac{1}{1+\operatorname{cosec}^2\theta}$

$$= \frac{1}{1+\sin^2\theta} + \frac{1}{1+\dfrac{1}{\sin^2\theta}}$$

$$= \frac{1}{1+\sin^2\theta} + \frac{\sin^2\theta}{1+\sin^2\theta}$$

$$= \frac{1+\sin^2\theta}{1+\sin^2\theta}$$

$= 1 =$ R.H.S (Proved)

Example 8. Prove that: $\dfrac{1}{2-\sin^2\theta}+\dfrac{1}{2+\tan^2\theta}=1$

Solution:

$$\text{L.H.S.} = \frac{1}{2-\sin^2\theta}+\frac{1}{2+\tan^2\theta}$$

$$=\frac{1}{2-\sin^2\theta}+\frac{1}{2+\dfrac{\sin^2\theta}{\cos^2\theta}}$$

$$=\frac{1}{2-\sin^2\theta}+\frac{\cos^2\theta}{2\cos^2\theta+\sin^2\theta}$$

$$=\frac{1}{2-\sin^2\theta}+\frac{\cos^2\theta}{2(1-\sin^2\theta)+\sin^2\theta}$$

$$=\frac{1}{2-\sin^2\theta}+\frac{\cos^2\theta}{2-2\sin^2\theta+\sin^2\theta}$$

$$=\frac{1}{2-\sin^2\theta}+\frac{1-\sin^2\theta}{2-\sin^2\theta}$$

$$=\frac{2-\sin^2\theta}{2-\sin^2\theta}$$

$= 1 =$ R.H.S (Proved)

Example 9. Prove that: $\dfrac{\tan A}{\sec A+1}-\dfrac{\sec A-1}{\tan A}=0$

Solution:

$$\text{L.H.S.} = \frac{\tan A}{\sec A+1}-\frac{\sec A-1}{\tan A}$$

$$=\frac{\tan^2 A-(\sec^2 A-1)}{(\sec A+1)\tan A}$$

$$=\frac{\tan^2 A-\tan^2 A}{(\sec A+1)\tan A}\ [\because \sec^2 A-1=\tan^2 A]$$

$$=\frac{0}{(\sec A+1)\tan A}$$

$= 0 =$ R.H.S (Proved)

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Example 10. Prove that: $\sqrt{\dfrac{1-\sin A}{1+\sin A}} = \sec A - \tan A$

Solution:

L.H.S. $= \sqrt{\dfrac{1-\sin A}{1+\sin A}}$

$= \sqrt{\dfrac{(1-\sin A)(1-\sin A)}{(1+\sin A)(1-\sin A)}}$ [Multiplying the numerator and denominator by $\sqrt{1-\sin A}$]

$= \sqrt{\dfrac{(1-\sin A)^2}{1-\sin^2 A}}$

$= \sqrt{\dfrac{(1-\sin A)^2}{\cos^2 A}}$

$= \dfrac{1-\sin A}{\cos A}$

$= \dfrac{1}{\cos A} - \dfrac{\sin A}{\cos A}$

$= \sec A - \tan A$

$=$ R.H.S (Proved)

Example 11. If $\tan A + \sin A = a$ and $\tan A - \sin A = b$, prove that, $a^2 - b^2 = 4\sqrt{ab}$

Solution: Here given that, $\tan A + \sin A = a$ and $\tan A - \sin A = b$

L.H.S. $= a^2 - b^2$

$= (\tan A + \sin A)^2 - (\tan A - \sin A)^2$

$= 4\tan A \sin A$ [$\because (a+b)^2 - (a-b)^2 = 4ab$]

$= 4\sqrt{\tan^2 A \sin^2 A}$

$= 4\sqrt{\tan^2 A(1-\cos^2 A)}$

$= 4\sqrt{\tan^2 A - \tan^2 A . \cos^2 A}$

$= 4\sqrt{\tan^2 A - \sin^2 A}$ [$\because \tan A = \dfrac{\sin A}{\cos A}$]

$= 4\sqrt{(\tan A + \sin A)(\tan A - \sin A)}$

$= 4\sqrt{ab}$

$=$ R.H.S (Proved)

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Work:

1) If $\cot^4 A - \cot^2 A = 1$, prove that, $\cos^4 A + \cos^2 A = 1$
2) If $\sin^4 A + \sin^2 A = 1$, prove that, $\tan^4 A - \tan^2 A = 1$

Example 12. If $\sec A + \tan A = \frac{5}{2}$, find the value of $\sec A - \tan A$.

Solution: Here given that, $\sec A + \tan A = \frac{5}{2} \ldots (1)$

We know that, $\sec^2 A = 1 + \tan^2 A$

Or, $\sec^2 A - \tan^2 A = 1$

Or, $(\sec A + \tan A)(\sec A - \tan A) = 1$

Or, $\frac{5}{2}(\sec A - \tan A) = 1$ [From (1)]

$\therefore \sec A - \tan A = \frac{2}{5}$

Exercise 9.1

1. Verify whether each of the following mathematical statements is true or false. Give argument in favour of your answer.
   1) The value of $\tan A$ is always less than 1.
   2) $\cot A$ is the multiplication of cot and A
   3) For any value of A, $\sec A = \frac{12}{5}$
   4) cos is the smallest form of cotangent
2. If $\sin A = \frac{3}{4}$, find the other trigonometric ratios of the angle A.
3. Given that, $15\cot A = 8$, find the values of $\sin A$ and $\sec A$.
4. If $\angle C$ is the right angle of the right-angled triangle ABC, $AB = 13$ c m, $BC = 12$ c m and $\angle ABC = \theta$, find the values of $\sin\theta$, $\cos\theta$ and $\tan\theta$.
5. $\angle B$ is the right-angle of the right angled triangle ABC. If $\tan A = \sqrt{3}$, verify the truth of $\sqrt{3}\sin A\cos A = \frac{3}{4}$

   Prove (6-20):
6. 1) $\frac{1}{\sec^2 A} + \frac{1}{\mathrm{cosec}^2 A} = 1$
   2) $\frac{1}{\cos^2 A} - \frac{1}{\cot^2 A} = 1$

3) $\dfrac{1}{\sin^2 A} - \dfrac{1}{\tan^2 A} = 1$

7. 1) $\dfrac{\sin A}{\operatorname{cosec} A} + \dfrac{\cos A}{\sec A} = 1$

2) $\dfrac{\sec A}{\cos A} - \dfrac{\tan A}{\cot A} = 1$

3) $\dfrac{1}{1+\sin^2 A} + \dfrac{1}{1+\operatorname{cosec}^2 A} = 1$

8. 1) $\dfrac{\tan A}{1-\cot A} + \dfrac{\cot A}{1-\tan A} = \sec A \operatorname{cosec} A + 1$

2) $\dfrac{1}{1+\tan^2 A} + \dfrac{1}{1+\cot^2 A} = 1$

9. $\dfrac{\cos A}{1-\tan A} + \dfrac{\sin A}{1-\cot A} = \sin A + \cos A$

10. $\tan A\sqrt{1-\sin^2 A} = \sin A$

11. $\dfrac{\sec A + \tan A}{\operatorname{cosec} A + \cot A} = \dfrac{\operatorname{cosec} A - \cot A}{\sec A - \tan A}$

12. $\dfrac{\operatorname{cosec} A}{\operatorname{cosec} A - 1} + \dfrac{\operatorname{cosec} A}{\operatorname{cosec} A + 1} = 2\sec^2 A$

13. $\dfrac{1}{1+\sin A} + \dfrac{1}{1-\sin A} = 2\sec^2 A$

14. $\dfrac{1}{\operatorname{cosec} A - 1} - \dfrac{1}{\operatorname{cosec} A + 1} = 2\tan^2 A$

15. $\dfrac{\sin A}{1-\cos A} + \dfrac{1-\cos A}{\sin A} = 2\operatorname{cosec} A$

16. $\dfrac{\tan A}{\sec A + 1} - \dfrac{\sec A - 1}{\tan A} = 0$

17. $(\tan\theta + \sec\theta)^2 = \dfrac{1+\sin\theta}{1-\sin\theta}$

18. $\dfrac{\cot A + \tan B}{\cot B + \tan A} = \cot A \cdot \tan B$

19. $\sqrt{\dfrac{1-\sin A}{1+\sin A}} = \sec A - \tan A$

20. $\sqrt{\dfrac{\sec A + 1}{\sec A - 1}} = \cot A + \operatorname{cosec} A$

21. If $\cos A + \sin A = \sqrt{2}\cos A$, prove that, $\cos A - \sin A = \sqrt{2}\sin A$

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22. If $\tan A = \frac{1}{\sqrt{3}}$, find the value of $\frac{\operatorname{cosec}^2 A - \sec^2 A}{\operatorname{cosec}^2 A + \sec^2 A}$.

23. If $\operatorname{cosec} A - \cot A = \frac{4}{3}$, what is the value of $\operatorname{cosec} A + \cot A$?

24. If $\cot A = \frac{b}{a}$, find the value of $\frac{a\sin A - b\cos A}{a\sin A + b\cos A}$.

25. If $\operatorname{cosec} A - \cot A = \frac{1}{x}$,

    1) find the value of $\operatorname{cosec} A + \cot A$.

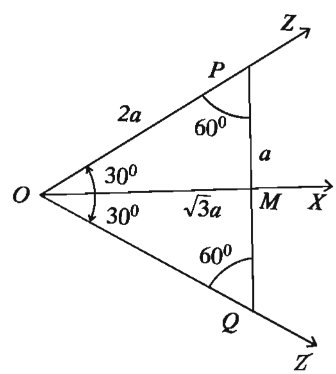
    2) Show that, $\sec A = \frac{x^2+1}{x^2-1}$

    3) According to the stimulus prove that, $\tan A + \cot A = \sec A \cdot \operatorname{cosec} A$

Trigonometric ratios for the angles 30°, 45°, and 60°.

We have learnt how to draw angles 30°, 45° **and** 60° in geometric way. Exact values of these trigonometric ratios for all these angles can also be determined geometrically.

Trigonometric ratios of the angles 30° **and** 60° :



Let, $\angle XOZ = 30°$ and P is a point on OZ. Let us draw $PM \perp OX$ and extend PM to Q so that $MQ = PM$. Add O, Q and extend to Z.

Now for $\triangle POM$ and $\triangle QOM$ $PM = QM$

OM common side and included $\angle PMO$ = included $\angle QMO = 90°$

$\therefore \triangle POM \cong \triangle QOM$

$\therefore$, $\angle QOM = \angle POM = 30°$

and $\angle OQM = \angle OPM = 60°$

Again, $\angle POQ = \angle POM + \angle QOM = 30° + 30° = 60°$

$\therefore \triangle OPQ$ is an equilateral triangle.

If $OP = 2a$ then $PM = \frac{1}{2}PQ = \frac{1}{2}OP = a$ [since $\triangle OPQ$ is an equilateral triangle.]

Form the right-angled $\triangle OPM$ we get,

$OM = \sqrt{OP^2 - PM^2} = \sqrt{4a^2 - a^2} = \sqrt{3}a$

Let us find the trigonometric ratios:

$$\sin 30^\circ = \frac{PM}{OP} = \frac{a}{2a} = \frac{1}{2}, \cos 30^\circ = \frac{OM}{OP} = \frac{\sqrt{3}a}{2a} = \frac{\sqrt{3}}{2}$$

$$\tan 30^\circ = \frac{PM}{OM} = \frac{a}{\sqrt{3}a} = \frac{1}{\sqrt{3}}$$

$$\operatorname{cosec} 30^\circ = \frac{OP}{PM} = \frac{2a}{a} = 2, \sec 30^\circ = \frac{OP}{OM} = \frac{2a}{\sqrt{3}a} = \frac{2}{\sqrt{3}}$$

$$\cot 30^\circ = \frac{OM}{PM} = \frac{\sqrt{3}a}{a} = \sqrt{3}$$

In the same way,

$$\sin 60^\circ = \frac{OM}{OP} = \frac{\sqrt{3}a}{2a} = \frac{\sqrt{3}}{2}, \cos 60^\circ = \frac{PM}{OP} = \frac{a}{2a} = \frac{1}{2},$$

$$\tan 60^\circ = \frac{OM}{PM} = \frac{\sqrt{3}a}{a} = \sqrt{3}$$

$$\operatorname{cosec} 60^\circ = \frac{OP}{OM} = \frac{2a}{\sqrt{3}a} = \frac{2}{\sqrt{3}}, \sec 60^\circ = \frac{OP}{PM} = \frac{2a}{a} = 2,$$

$$\cot 60^\circ = \frac{PM}{OM} = \frac{a}{\sqrt{3}a} = \frac{1}{\sqrt{3}}$$

Trigonometric ratios for 45° :

Let us assume $\angle XOZ = 45^\circ$ and P is a point on OZ.

Draw $PM \perp OX$

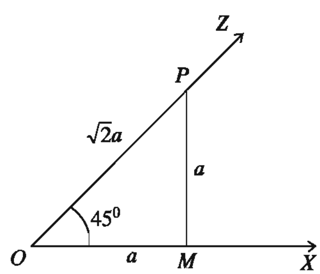
In the right-angled $\triangle OPM$, $\angle POM = 45^0$

$\therefore$, $\angle OPM = 45^0$

So $PM = OM = a$ (denote)

Now, $OP^2 = OM^2 + PM^2 = a^2 + a^2 = 2a^2$

or, $OP = \sqrt{2}a$



From the definitions of trigonometric ratios we get,

$$\sin 45^\circ = \frac{PM}{OP} = \frac{a}{\sqrt{2}a} = \frac{1}{\sqrt{2}}, \cos 45^\circ = \frac{OM}{OP} = \frac{a}{\sqrt{2}a} = \frac{1}{\sqrt{2}},$$

$$\tan 45^\circ = \frac{PM}{OM} = \frac{a}{a} = 1$$

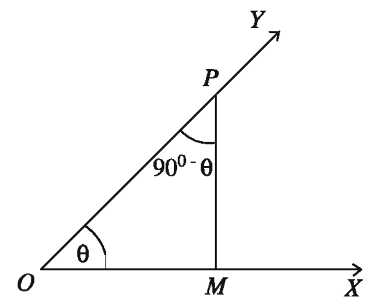
$$\operatorname{cosec} 45^\circ = \frac{1}{\sin 45^\circ} = \sqrt{2}, \sec 45^\circ = \frac{1}{\cos 45^\circ} = \sqrt{2},$$

$$\cot 45^\circ = \frac{1}{\tan 45^\circ} = 1$$

Trigonometric ratios of complementary angles

We know, if the sum of two acute angles is 90°, one of them is called complementary angle to the other. For example, 30° and 60°, and 15° and 75° are complementary angles to each other.

In general, the angles θ and $(90° - \theta)$ are complementary angles to each other.



Let $\angle XOY = \theta$ and P is the point on the side OY of the angle. We draw $PM \perp OX$.
Since the sum of the three angles of a triangle is two right angles, therefore, in the right angled triangle POM, $\angle PMO = 90°$
and $\angle OPM + \angle POM =$ one right angle $= 90°$
$\angle OPM = 90° - \angle POM = 90° - \theta$
[Since $\angle POM = \angle XOY = \theta$]

$$\therefore \sin(90° - \theta) = \frac{OM}{OP} = \cos\angle POM = \cos\theta$$

$$\cos(90° - \theta) = \frac{PM}{OP} = \sin\angle POM = \sin\theta$$

$$\tan(90^0 - \theta) = \frac{OM}{PM} = \cot\angle POM = \cot\theta$$

$$\cot(90^0 - \theta) = \frac{PM}{OM} = \tan\angle POM = \tan\theta$$

$$\sec(90^0 - \theta) = \frac{OP}{PM} = \mathrm{cosec}\angle POM = \mathrm{cosec}\theta$$

$$\mathrm{cosec}(90^0 - \theta) = \frac{OP}{OM} = \sec\angle POM = \sec\theta$$

We can express the above formulae in words below :

sine of complementary angle = cosine of angle

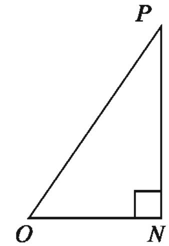
cosine of complementary angle = sine of angle

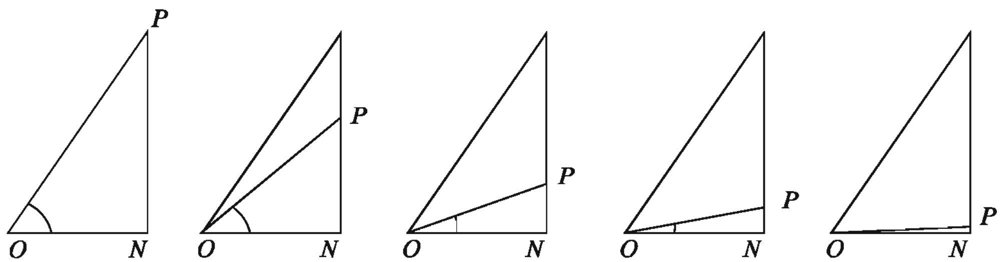
tangent of complementary angle = cotangent of angle etc.

Work: If $\sec(90° - \theta) = \frac{5}{3}$, find the value of $\mathrm{cosec}\theta - \cot\theta$.

Trigonometric ratios of the angles 0^0 and 90^0

We have learnt how to determine the trigonometric ratios for the acute angle θ of a right-angled triangle. Now, we see, if the angle is made gradually smaller, how the trigonometric ratios change. As θ get smaller the length of the side, PN also gets smaller. The point P closes to the point N and finally the angle θ comes closer to the angle $0°$, OP is reconciled with ON approximately.





When the angle θ comes closer to $0°$, the length of the line segment PN reduces to zero and in this case the value of $\sin\theta = \frac{PN}{OP}$ is approximately zero. At the same time, the length of OP is equal to the length of ON and the value of $\cos\theta = \frac{ON}{OP}$ is 1 approximately.

The angle, $0°$ is introduced for the convenience of discussion in trigonometry, and the edge line and the original line of the angle $0°$ are supposed the same ray. Therefore, in line with the prior discussion, it is said that, $\cos 0° = 1$, $\sin 0° = 0$

If θ is the acute angle, we see $\tan\theta = \frac{\sin\theta}{\cos\theta}$, $\cot\theta = \frac{\cos\theta}{\sin\theta}$,

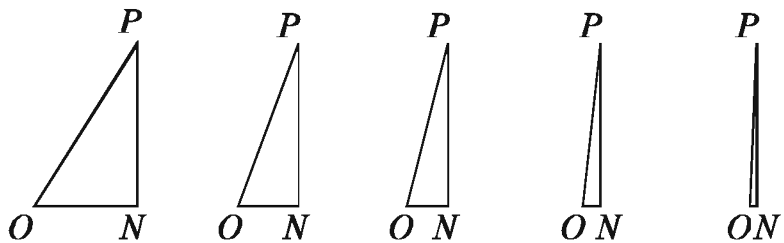
$\sec\theta = \frac{1}{\cos\theta}$, $\operatorname{cosec}\theta = \frac{1}{\sin\theta}$

We define the angle $0°$ in probable cases so that, those relations exists.

$$\tan 0° = \frac{\sin 0°}{\cos 0°} = \frac{0}{1} = 0$$

$$\sec 0° = \frac{1}{\cos 0°} = \frac{1}{1} = 1$$

Since division by 0 is not allowed, $\operatorname{cosec} 0°$ $\cot 0°$ can not be defined.



Again, when the angle θ is very closed to $90°$, hypotenuse OP is approximately equal to PN. So the value of $\sin\theta$ is approximately 1. on the other hand, if the angle θ is equal to 90^0, ON is nearly zero; the value of $\cos\theta$ is approximately 0.

So, in agreement of formulae that are described above, we can say, $\cos 90° = 0$, $\sin 90° = 1$

$$\cot 90° = \frac{\cos 90°}{\sin 90°} = \frac{0}{1} = 0$$

$$\text{cosec}90° = \frac{1}{\sin 90°} = \frac{1}{1} = 1$$

Since one cannot divided by 0 as before, $\tan 90°$ and $\sec 90°$ are not defined.

Note: For convenience of using the values of trigonometric ratios of the angles $0°$, $30°$, $45°$, $60°$ and $90°$ are shown in the following table :

Ratio/Angle	$0°$	$30°$	$45°$	$60°$	$90°$
sine	0	$\frac{1}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{\sqrt{3}}{2}$	1
cosine	1	$\frac{\sqrt{3}}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{1}{2}$	0
tangent	0	$\frac{1}{\sqrt{3}}$	1	$\sqrt{3}$	undefined
cotangent	undefined	$\sqrt{3}$	1	$\frac{1}{\sqrt{3}}$	0
secant	1	$\frac{2}{\sqrt{3}}$	$\sqrt{2}$	2	undefined
cosecant	undefined	2	$\sqrt{2}$	$\frac{2}{\sqrt{3}}$	1

Observe: Easy method for remembering of the values of trigonometric ratios of some fixed angles.

(i) If we divide the numbers 0, 1, 2, 3 and 4 by 4 and take square root of the quotients, we get the values of $\sin 0°$, $\sin 30°$, $\sin 45°$, $\sin 60°$ and $\sin 90°$ respectively.

(*ii*) If we divide the numbers 4, 3, 2, 1 and 0 by 4 and take square root of quotients, we get the values of $\cos 0°$, $\cos 30°$, $\cos 45°$, $\cos 60°$ and $\cos 90°$ respectively.

(*iii*) If we divide the numbers 0, 1, 3 and 9 by 3 and take square root of the quotients, we get the values of $\tan 0°$, $\tan 30°$, $\tan 45°$ and $\tan 60°$ respectively. (It is noted that $\tan 90°$ is undefined).

(*iv*) If we divide the numbers 9, 3, 1 and 0 by 3 and take square root of the quotients, we get the values of $\cot 30°$, $\cot 45°$, $\cot 60°$ and $\cot 90°$ respectively. (It is noted that $\cot 0°$ is undefined).

Example 13. Find the values :

1) $\dfrac{1-\sin^2 45°}{1+\sin^2 45°}+\tan^2 45°$

2) $\cot 90° \cdot \tan 0° \cdot \sec 30° \cdot \operatorname{cosec} 60°$

3) $\sin 60° \cdot \cos 30° + \cos 60° \cdot \sin 30°$

4) $\dfrac{1-\tan^2 60°}{1+\tan^2 60°}+\sin^2 60°$

Solution:

1) Given expression $= \dfrac{1-\sin^2 45°}{1+\sin^2 45°}+\tan^2 45°$

$$= \frac{1-\left(\frac{1}{\sqrt{2}}\right)^2}{1+\left(\frac{1}{\sqrt{2}}\right)^2}+(1)^2 \ [\because \sin 45° = \frac{1}{\sqrt{2}} \text{ and } \tan 45° = 1]$$

$$= \frac{1-\frac{1}{2}}{1+\frac{1}{2}}+1 = \frac{\frac{1}{2}}{\frac{3}{2}}+1 = \frac{1}{3}+1 = \frac{4}{3}$$

2) Given expression $= \cot 90° \cdot \tan 0° \cdot \sec 30° \cdot \operatorname{cosec} 60°$

$$= 0 \cdot 0 \cdot \frac{2}{\sqrt{3}} \cdot \frac{2}{\sqrt{3}} = 0$$

$$[\because \cot 90° = 0, \tan 0° = 0, \sec 30° = \frac{2}{\sqrt{3}}, \operatorname{cosec} 60° = \frac{2}{\sqrt{3}}\ ]$$

3) Given expression $= \sin 60° \cdot \cos 30° + \cos 60° \cdot \sin 30°$

$$= \frac{\sqrt{3}}{2} \cdot \frac{\sqrt{3}}{2} + \frac{1}{2} \cdot \frac{1}{2}$$

$$[\because \sin 60° = \cos 30° = \frac{\sqrt{3}}{2}, \cos 60° = \sin 30° = \frac{1}{2}]$$

$$= \frac{3}{4} + \frac{1}{4} = \frac{4}{4} = 1$$

4) Given expression $= \dfrac{1 - \tan^2 60°}{1 + \tan^2 60°} + \sin^2 60°$

$$= \frac{1 - (\sqrt{3})^2}{1 + (\sqrt{3})^2} + \left(\frac{\sqrt{3}}{2}\right)^2 \quad [\because \tan 60° = \sqrt{3}, \sin 60° = \frac{\sqrt{3}}{2}]$$

$$= \frac{1-3}{1+3} + \frac{3}{4} = \frac{-2}{4} + \frac{3}{4}$$

$$= \frac{-2+3}{4} = \frac{1}{4}$$

Example 14. 1) If $\sqrt{2}\cos(A-B) = 1$, $2\sin(A+B) = \sqrt{3}$ and A, B are acute angles, find the values of A and B.

2) If $\dfrac{\cos A - \sin A}{\cos A + \sin A} = \dfrac{1-\sqrt{3}}{1+\sqrt{3}}$, find the value of A.

3) If $A = 45°$. Prove that, $\cos 2A = \dfrac{1 - \tan^2 A}{1 + \tan^2 A}$.

4) Solve: $2\cos^2\theta + 3\sin\theta - 3 = 0$, where θ is an acute angle.

Solution:

1) $\sqrt{2}\cos(A-B) = 1$

or, $\cos(A-B) = \dfrac{1}{\sqrt{2}}$

or, $\cos(A-B) = \cos 45°$ $[\because \cos 45° = \dfrac{1}{\sqrt{2}}]$

$\therefore A - B = 45^0 \ldots (1)$

and $2\sin(A+B) = \sqrt{3}$

or, $\sin(A+B) = \dfrac{\sqrt{3}}{2}$

or, $\sin(A+B) = \sin 60°$ $[\because \sin 60° = \dfrac{\sqrt{3}}{2}]$

$\therefore A + B = 60^0 \ldots (2)$

Adding (1) and (2), we get,

$2A = 105^0$

$\therefore A = \frac{105^0}{2} = 52\frac{1^0}{2}$

Again subtracting (1) from (2), we get,

$2B = 15^0$

$\therefore B = \frac{15^0}{2} = 7\frac{1^0}{2}$

Required $A = 52\frac{1^0}{2}$ and $B = 7\frac{1^0}{2}$

2) $\frac{\cos A - \sin A}{\cos A + \sin A} = \frac{1-\sqrt{3}}{1+\sqrt{3}}$

or, $\frac{\cos A - \sin A + \cos A + \sin A}{\cos A - \sin A - \cos A - \sin A} = \frac{1-\sqrt{3}+1+\sqrt{3}}{1-\sqrt{3}-1-\sqrt{3}}$ [By addition-subtraction]

or, $\frac{2\cos A}{-2\sin A} = \frac{2}{-2\sqrt{3}}$

or, $\frac{\cos A}{\sin A} = \frac{1}{\sqrt{3}}$

or, $\cot A = \cot 60°$

$\therefore A = 60°$

3) Given that, $A = 45°$

we have to prove that, $\cos 2A = \frac{1-\tan^2 A}{1+\tan^2 A}$

L.H.S. $= \cos 2A$

$= \cos(2 \times 45°) = \cos 90° = 0$

R.H.S. $= \frac{1-\tan^2 A}{1+\tan^2 A}$

$= \frac{1-\tan^2 45°}{1+\tan^2 45°} = \frac{1-(1)^2}{1+(1)^2}$

$= \frac{0}{2} = 0$

$\therefore$ L.H.S. = R.H.S. (Proved)

4) Given equation $2\cos^2\theta + 3\sin\theta - 3 = 0$

or, $2(1 - \sin^2\theta) + 3\sin\theta - 3 = 0$

or, $2(1 + \sin\theta)(1 - \sin\theta) - 3(1 - \sin\theta) = 0$

or, $(1 - \sin\theta)\{2(1 + \sin\theta) - 3\} = 0$

or, $(1 - \sin\theta)\{2\sin\theta - 1\} = 0$

$\therefore 1 - \sin\theta = 0$

or, $\sin\theta = 1$

or, $\sin\theta = \sin 90^0$

or, $\theta = 90°$

or, $2\sin\theta - 1 = 0$

or, $2\sin\theta = 1$

or, $\sin\theta = \frac{1}{2}$

or, $\sin\theta = \sin 30°$

or, $\theta = 30°$

θ is an acute angle, so $\theta = 30°$.

Exercises 9.2

1. If $\cos\theta = \frac{1}{2}$, which one is the value of $\cot\theta$?

   1) $\frac{1}{\sqrt{3}}$ 2) 1 3) $\sqrt{3}$ 4) 2

2. If $\cos^2\theta - \sin^2\theta = \frac{1}{3}$, what is the value of $\cos^4\theta - \sin^4\theta$?

   1) 3 2) 2 3) 1 4) $\frac{1}{3}$

3. If $\cot(\theta - 30°) = \frac{1}{\sqrt{3}}$, $\sin\theta =$ what ?

   1) $\frac{1}{2}$ 2) 0 3) 1 4) $\frac{\sqrt{3}}{2}$

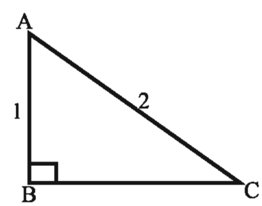
4. If $\tan(3A) = \sqrt{3}$, $A =$ what ?

   1) 45° 2) 30° 3) 20° 4) 15°

5. For $0° \leq \theta \leq 90°$, what is the maximum value of $\sin\theta =$?

   1) -1 2) 0 3) $\frac{1}{2}$ 4) 1

6. ABC is a right-angled triangle whose hypotenuse $AC = 2$, $AB = 1$

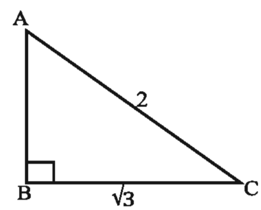
   

   (*i*) $\angle ACB = 30°$
   (*ii*) $\tan A = \sqrt{3}$
   (*iii*) $\sin(A + C) = 0$

   Which one of the following is correct?

   1) *i* 2) *ii* 3) *i* and *ii* 4) *ii* and *iii*

7. ABC is a right-angled triangle whose hypotenuse $AC = 2$, $AB = 1$

   (*i*) $\cos A = \sin C$
   (*ii*) $\cos A + \sec A = \frac{5}{2}$
   (*iii*) $\tan C = \frac{2}{\sqrt{3}}$

   Which one of the following is correct?

   1) *i* and *ii* 2) *ii* and *iii* 3) *i* and *iii* 4) *i*, *ii* and *iii*

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Determine the value (8- 11)

8. $\dfrac{1-\cot^2 60°}{1+\cot^2 60°}$

9. $\tan 45° \cdot \sin^2 60° \cdot \tan 30° \cdot \tan 60°$

10. $\dfrac{1-\cos^2 60°}{1+\cos^2 60°} + \sec^2 60°$

11. $\cos 45° \cdot \cot^2 60° \cdot \operatorname{cosec}^2 30°$

Show that, (12- 17)

12. $\cos^2 30° - \sin^2 30° = \cos 60°$

13. $\sin 60° \cdot \cos 30° + \cos 60° \cdot \sin 30° = \sin 90°$

14. $\cos 60° \cdot \cos 30° + \sin 60° \cdot \sin 30° = \cos 30°$

15. $\sin 3A = \cos 3A$, when $A = 15°$.

16. $\sin 2A = \dfrac{2\tan A}{1+\tan^2 A}$ when $A = 45°$.

17. $\tan 2A = \dfrac{2\tan A}{1-\tan^2 A}$ when $A = 30°$.

18. If $2\cos(A+B) = 1 = 2\sin(A-B)$ and A, B are acute angles, show that $A = 45°$, $B = 15°$.

19. If $\cos(A-B) = 1$, $2\sin(A+B) = \sqrt{3}$ and A, B are acute angles, determine the values of A and B.

20. Solve: $\dfrac{\cos A - \sin A}{\cos A + \sin A} = \dfrac{\sqrt{3}-1}{\sqrt{3}+1}$

21. If A and B are acute angles and $\cot(A+B) = 1$, $\cot(A-B) = \sqrt{3}$, determine the values of A and B.

22. Show that, $\cos 3A = 4\cos^3 A - 3\cos A$, when $A = 30°$.

23. Solve: $\sin\theta + \cos\theta = 1$, when $0° \le \theta \le 90°$

24. Solve: $\cos^2\theta - \sin^2\theta = 2 - 5\cos\theta$, when θ is an acute angle.

25. Solve: $2\sin^2\theta + 3\cos\theta - 3 = 0$, θ is an acute angle.

26. Solve: $\tan^2\theta - (1+\sqrt{3})\tan\theta + \sqrt{3} = 0$

27. Solve: $3\cot^2 60^0 + \frac{1}{4}\text{cosec}^2 30^0 + 5\sin^2 45^0 - 4\cos^2 60^0$

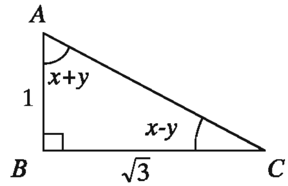
28. If $\angle B = 90^0$, $AB = 5$ c m, $BC = 12$c m of $\triangle ABC$

1) Find the length of AC.

2) If $\angle C = \theta$, find the value of $\sin\theta + \cos\theta$.

3) If $\angle A = 30^0$ Show that, $\sec^2 A + \text{cosec}^2 A = \sec^2 A \cdot \text{cosec}^2 A$

29. In accordance of the given figure



1) what is the measurement of AC?
2) Find the value of $\tan A + \tan C$.
3) Find the values of x and y.

30. $\sin\theta = p$, $\cos\theta = q$, $\tan\theta = r$, where θ is an acute angle.

1) If $r = \sqrt{(3)^{-1}}$, find the value of θ.

2) If $p + q = \sqrt{2}$, prove that $\theta = 45^0$.

3) If $7p^2 + 3q^2 = 4$, show that, $\tan\theta = \frac{1}{\sqrt{3}}$.

31. Prove that, for a triangle ABC, $\frac{AB + BC}{AC} = \cot\left(\frac{B}{2}\right)$

32. Prove that, for a triangle ABC, if $AC \neq BC$,

$$\frac{BC\cos C - AC\cos B}{BC\cos B - AC\cos A} + \cos C = 0$$

33. Prove that, for a triangle ABC,

if $\cot A + \cot B = 2\cot C$, then $AC^2 + BC^2 = 2AB^2$.

Chapter 10

Distance and Elevation

From very ancient times trigonometric ratios are applied to find the distance and height of any distant object. At present trigonometric ratios are of boundless importance because of its increasing usage. The heights of the hills, mountains towers, trees and the widths of those rivers which cannot be measured in ordinary method are measured with the help of trigonometry. It is necessary to know the trigonometrical ratios values of acute angles in this regard.

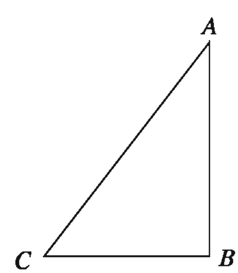
At the end of this chapter, the students will be able to —

- explain the geoline, vertical plane and angles of elevation and declination.
- solve mathematical problem related to distance and height with the help of trigonometry.
- measure practically different types of distances and heights with the help of trigono etry.

Horizontal Line, Vertical Line and Vertical Plane

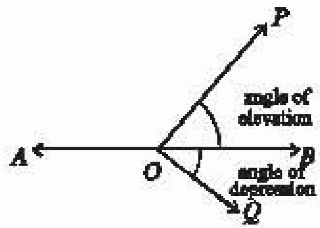
A horizontal line is a straight line on the horizontal plane. A straight line parallel to horizon is also called a horizontal line. A vertical line is a line perpendicular to the horizontal plane. It is also called a normal line. A horizontal line and a vertical line intersected at right angles on the plane define a plane. It is known as vertical plane.

In the figure, a tree with height of AB is standing vertically at a distance of CB from a point C on the horizontal plane. Here, CB is the horizontal line. BA is the vertical line and the plane ABC is perpendicular to the horizontal plane which is a vertical plane.

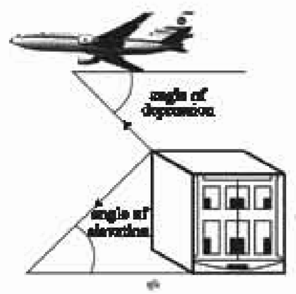


Angle of Elevation and Angle of Depression

Observe the figure, AB is a straight line parallel to the horizon. The points A, O, B, P and Q lie on the same vertical plane. The point P on the straight line AB makes angle $\angle POB$ with the line AB. Here at O, the angle of elevation of P is $\angle POB$.



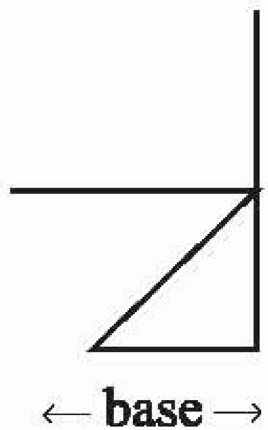
So, the angle at any point above the plane with the straight line parallel to horizon is called the angle of elevation.



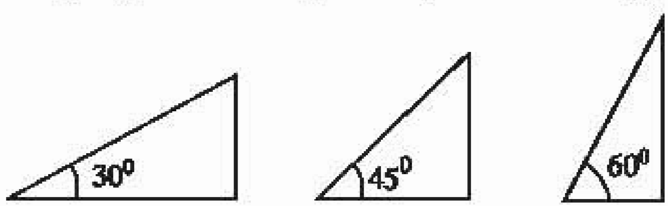
The points Q, lie at lower side of the straight line AB parallel to horizon. Here, the angle of depression at O of Q is $\angle QOB$. So, the angle at any point below the straight line parallel to the plane is called the angle of depression.

Work:

Point the figure and indicate the horizontal line, vertical line, vertical plane, angle of elevation and angle of depression.



Note : For solving the problems in this chapter approximately right figure is needed. While drawing the figure, the following techniques are to be applied.



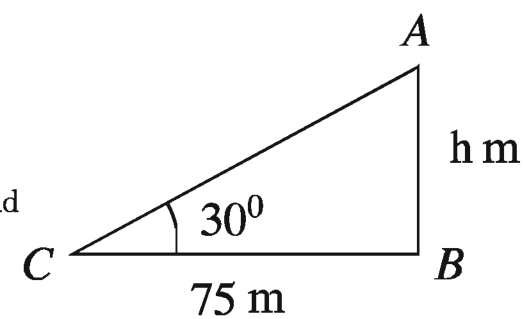
1. While drawing 30° angle, it is needed base > perpendicular.

2. While drawing 45° angle, it is needed base = perpendicular.

3. While drawing 60° angle, it is needed base < perpendicular.

Example 1. The angle of elevation at the top of a tower at a point on the ground is 30° at a distance of 75 metre from the foot. Find the height of the tower.

Solution: Let, the height of the tower is $AB = h$ metre. The angle of elevation at C from the foot of the tower $BC = 75$ metre of A on the ground is $\angle ACB = 30°$



From $\triangle ABC$ we get, $\tan\angle ACB = \dfrac{AB}{BC}$

or, $\tan 30° = \dfrac{h}{75}$ or, $\dfrac{1}{\sqrt{3}} = \dfrac{h}{75}$ or, $\sqrt{3}h = 75$

or, $h = \dfrac{75}{\sqrt{3}}$

or, $h = \dfrac{75\sqrt{3}}{3}$ [multiplying denominator and numerator by $\sqrt{3}$]

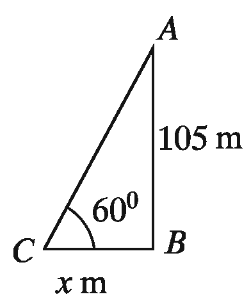
or, $h = 25\sqrt{3}$

$\therefore h = 43.301$ (approx.).

$\therefore$ Height of the tower is 43.30 metre (approx.)

Example 2. The height of a tree is 105 metre. If the angle of elevation of the tree at a point from its foot on the ground is 60°, find the distance of the point on the ground from the foot of the tree.

Solution:
Let, the distance of the point on the ground from the foot of tree is $BC = x$ metre. Height of the tree $AB = 105$ metre and at C the angle of elevation of the vertex of tree is $\angle ACB = 60°$. From right-angle $\triangle ABC$ we. get,



$\tan\angle ACB = \dfrac{AB}{BC}$

or, $\tan 60° = \dfrac{105}{x}$

or, $\sqrt{3} = \dfrac{105}{x}$ [$\because \tan 60^0 = \sqrt{3}$]

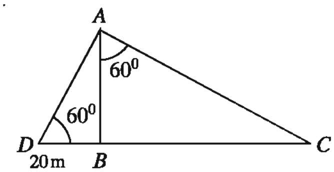
or, $\sqrt{3}x = 105$ or, $x = \dfrac{105}{\sqrt{3}}$ or, $x = \dfrac{105\sqrt{3}}{3}$ or, $x = 35\sqrt{3}$

$\therefore x = 60.622$ (approx.)

$\therefore$ The required distance of the point on the ground from the foot of the tree is 60.62 metre (approx.).

Work: In the picture, AB is a tree. From the information given in the picture

1) Find the height of the tree.
2) Find the distance of the point C on the ground from the foot of the tree.



Example 3. A ladder of 18 metre long touches the roof of a wall and makes an angle of 45° with the horizon. Find the height of the wall.

Solution: Let, the height of the wall $AB = h$ metre, length of ladder AC =18m and makes angle $\angle ACB = 45°$ with the ground.

From $\triangle ABC$ we get, $\sin\angle ACB = \dfrac{AB}{AC}$

or, $\sin 45° = \dfrac{h}{18}$

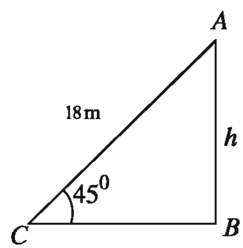
or, $\dfrac{1}{\sqrt{2}} = \dfrac{h}{18}$ $\left[\because \sin 45° = \dfrac{1}{\sqrt{2}}\right]$

or, $\sqrt{2}h = 18$ or, $h = \dfrac{18}{\sqrt{2}}$

or, $h = \dfrac{18\sqrt{2}}{2}$ [multiplying the numerator and denominator by $\sqrt{2}$]

or, $h = 12.728$ (approx.)

Therefore, The required height of the wall is 12.73 (approx.)



Example 4. A tree leaned due to storm. The stick with height of 7 metre from its foot was leaned against the tree to make it straight. If the angle of depression at the point of contacting with the stick on the ground is 30°, find the length of the stick.

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Solution: Let, the height of the stick from the foot leaned against the tree of $AB = 7$ metre and angle of depression $\angle DBC = 30°$.

$\therefore \angle ACB = \angle DBC = 30°$ [alternate angle]

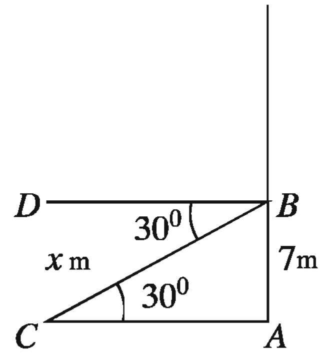
From right-angled $\triangle ABC$ we get,

$\sin\angle ACB = \dfrac{AB}{BC}$ or, $\sin 30° = \dfrac{7}{BC}$

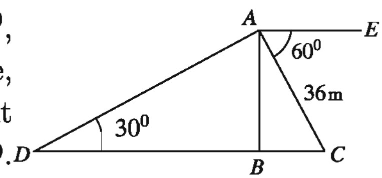
or, $\dfrac{1}{2} = \dfrac{7}{BC}$ $\left[\because \sin 30° = \dfrac{1}{2}\right]$

$\therefore BC = 14$

$\therefore$ The required height of the stick is 14 metre.



Work:
In the figure, if depression angle $\angle CAE = 60^0$, elevation angle $\angle ADB = 30^0$, $AC = 36$ metre, $AB \perp DC$ and D, C, B lie on the same straight line, find the lengths of the sides AD, AB and CD.



Example 5. The angle of elevation at a point of the roof of a building is 60^0 in any point on the ground. Moving back 42 metre from the angle of elevation of the point of the place of the building becomes 45^0. Find the height of the building.

Solution: Let, the height of the building is $AB = h$ metres. The angle of elevation at the top $\angle ACB = 60^0$. The angle of elevation becomes $\angle ADB = 45^0$ moving back from C by $CD = 42$ metre.

Let, $BC = x$ metre.

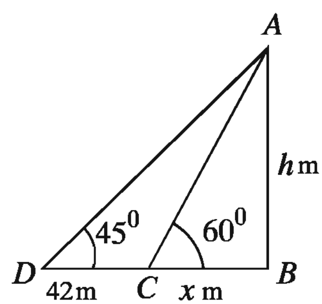
$\therefore BD = BC + CD = (x + 42)$ metre.

From $\triangle ABC$ we get,

$\tan\angle ACB = \dfrac{AB}{BC}$ or, $\tan 60° = \dfrac{h}{x}$

or, $\sqrt{3} = \dfrac{h}{x}$ $[\because \tan 60° = \sqrt{3}]$

$\therefore x = \dfrac{h}{\sqrt{3}} \cdots (1)$



Again, from $\triangle ABD$ we get, $\tan\angle ADB = \tan 45° = \dfrac{AB}{BD}$

or, $\tan 45° = \dfrac{h}{x+42}$ or, $1 = \dfrac{h}{x+42}$ $[\because \tan 45^0 = 1]$

or, $h = x + 42$ or, $h = \dfrac{h}{\sqrt{3}} + 42$ [from (1)]

or, $\sqrt{3}h = h + 42\sqrt{3}$ or, $\sqrt{3}h - h = 42\sqrt{3}$ or, $(\sqrt{3}-1)h = 42\sqrt{3}$ or, $h = \dfrac{42\sqrt{3}}{\sqrt{3}-1}$

$\therefore h = 99.373$ (approx.)

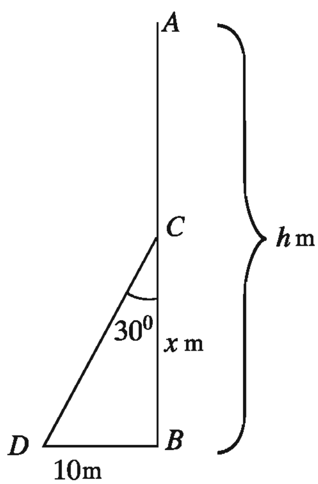
$\therefore$ Height of the building is 99.37 metre (approx.).

Example 6. A pole is broken such that the undetached broken part makes an angle of 30^0 with the other and touches the ground at a distance of 10 metre form its foot. Find the length of the pole.

Solution:
Let, the total height of the pole is $AB = h$ metre. It breaks at the height of $BC = x$ metre without separation and makes an angle with the other, $\angle BCD = 30^0$ and touches the ground at a distance $BD = 10$ metre from the foot.
Here, $CD = AC = AB - BC = (h - x)$ metre
From $\triangle BCD$ we get,

$\tan\angle BCD = \dfrac{BD}{BC}$ or, $\tan 30^\circ = \dfrac{10}{x}$

or, $\dfrac{1}{\sqrt{3}} = \dfrac{10}{x}$ $\therefore x = 10\sqrt{3}$



Again, $\sin\angle BCD = \dfrac{BD}{CD}$ or, $\sin 30^0 = \dfrac{BD}{CD}$ or, $\dfrac{1}{2} = \dfrac{10}{h-x}$

or, $h - x = 20$ or, $h = 20 + x$ or, $h = 20 + 10\sqrt{3}$ [putting the value of x]

$\therefore h = 37.321$ (approx.)

$\therefore$ Height of the pole is 37.32 metre (approx.).

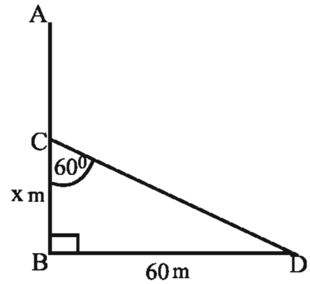
> **Work:** A balloon is flying above any point between two mile posts. At the point of the balloon the angle of depression of the two posts are 30° and 60° respectively. Find the height of the balloon.

Exercises 10

1. The square of the length of a stick is one third of its shadow length. What is the angle of elevation of sun at the edge of the shadow.

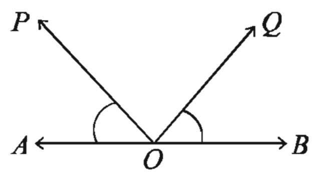
   1) 15° 2) 30° 3) 45° 4) 60°

2. What is the value of x in the picture?



   1) $\dfrac{\sqrt{3}}{60}$
   2) $\dfrac{60}{\sqrt{3}}$
   3) $60\sqrt{2}$
   4) $60\sqrt{3}$

3. What is the elevation angle of point P from the point O?

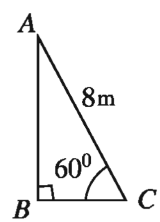


   1) $\angle QOB$ 2) $\angle POA$
   3) $\angle QOA$ 4) $\angle POB$

4. For which value of the angle of depression length of a stick and length of its' shadow is equal?

   1) 30° 2) 45° 3) 60° 4) 90°

According to the figure given here, answer 5 and 6.



5. What is the length of BC?

   1) $\dfrac{4}{\sqrt{3}}$ metre 2) 4 metre
   3) $4\sqrt{2}$ metre 4) $4\sqrt{3}$ metre

6. What is the length of AB?

   1) $\dfrac{4}{\sqrt{3}}$ metre 2) 4 metre
   3) $4\sqrt{2}$ metre 4) $4\sqrt{3}$ metre

7. Angle of elevation –

   (i) if 30°, then base $>$ perpendicular.

   (ii) if 45°, then base $=$ perpendicular.

   (iii) if 60°, then base $<$ perpendicular.

   Which one of the following is correct?

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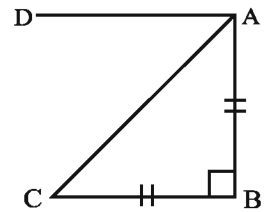
1) *i* and *ii* 2) *ii* and *iii* 3) *i* and *iii* 4) *i*, *ii* and *iii*

8. In the adjacent picture -



(*i*) $\angle DAC$ is an angle of depression
(*ii*) $\angle ACB$ is an angle of elevation
(*iii*) $\angle DAC = \angle ACB$

Which one of the following is correct?

1) *i* and *ii* 2) *ii* and *iii* 3) *i* and *iii* 4) *i*, *ii* and *iii*

9. What is the other name is of geoline?

1) perpendicular line
2) Parallel line
3) Lying line
4) Vertical line

10. If the angle of the elevation of the top of the miner is 30° at a point on the ground and the height is 26 metre, then find the distance of the plane from the Miner.

11. If the top of a tree is 20 metre distance from the foot on the ground at any point on the ground andthe angle of elevation is 60°, find the height of the tree.

12. Forming 45° angle with ground an 18 metre long ladder touches the top of the wall, find the height of the wall.

13. If the angle of depression at a point on the ground 20 metre from the top of the house is 30°, then find the height of the house.

14. The angle of elevation of a tower at any point on the ground is 60°. If moved back 25 metre, the angle of elevation becomes 30°, find the height of the tower.

15. The angle of elevation of a tower becomes 60° from 45° by moving 60 metre towards a minar. Find the height of the minar.

16. A man standing at a place on the bank of a river observesd that the angle of elevation of a tower exactly opposite to him on the other bank was 60°. Moving 32 metre back he observed that the angle of elevation of the tower was 30°. Find the height of the tower and the width of the river.

17. A pole of 64 metre long breaks into two parts without complete separation and makes an angle 60° with the ground. Find the length of the broken part of the pole.

18. A tree is broken by a storm such that the broken part makes an angle of 30° with the other and touches the ground at a distance of 12 metre from it. Find the length of the whole tree.

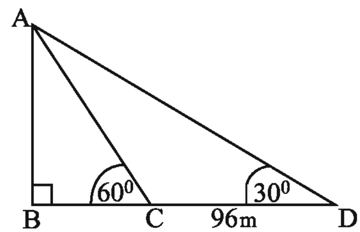
19. Standing any where on the bank of a river, a man observed that the angle of elevation of a 150 metre long tree exactly straight to him on the other bank is 30°. The man started for the tree by a boat. But he reached at 10 metre away from the tree due to current.

    1) According to the stem show the above description by a figure.
    2) Find the width of the river.
    3) Find the distance from the starting point to the destination.

20. Forming 60° angle with ground an 16 metre long ladder touches the top of the vertical wall.

    1) According to the stem show the above description by a figure.
    2) Find the height of the wall.
    3) How far does the foot of the ladder need to be moved along horizontal line so that it still touches the wall and makes an angle of 30° with ground?

21. In the figure, $CD = 96$ metre.
    1) Find the measurement of $\angle CAD$ in degree.
    2) Find the length of BC.
    3) Find the perimeter of $\triangle ACD$.



Chapter 11

Algebraic Ratio and Proportion

It is important for us to have a clear conception of ratio and proportion in mathematics. Arithmetical ratio and proportion have been elaborately discussed in class VII. In this chapter, algebraic ratio and proportion will be discussed. We regularly use the concept of ratio and proportion in many applications: the production of construction materials, food staff and goods , applying fertilizer in land, making beautiful shapes and design to make things attractive and good looking, and so on. Many problems of daily lives can also be solved by using the concept of ratio and proportion.

After studying this chapter, the students will be able to —

- exp ain algebraic ratio and proportion.
- use different types of rules of transformation related to proportion.
- learn successive proportion.
- use ratio, proportion, successive proportion in solving real life problems.

Ratio and Proportion

Ratio

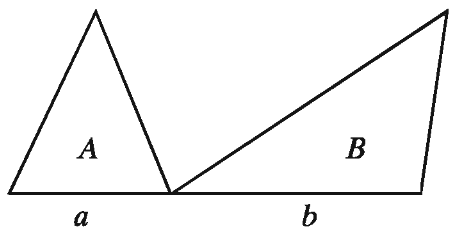
If we have two quantities (or numbers) of the same unit, we can express one quantity as a multiple (or a fraction) of the other quantity. This fraction is called the ratio of two quantities. Thus, a ratio says how much of one thing there is compared to another thing.

The ratio of two quantities p and q is written as $p : q = \frac{p}{q}$. These two quantities, p and q are of same kind and same unit. Here, p is called antecedent and q is called subsequent of the ratio.

Sometimes we use the ratio as an approximate measure. For example, the number of cars on the road at 10 A.M. is two times higher than the number of cars at 8 A.M. In this case, it is not necessary to know the exact number of cars to determine the ratio. Again, in many occasions, we say that the area of your house is three times to the area of mine. Also in this case, it is not necessary to know the exact area of the house. We use the concept of ratio in many other cases of practical life.

Proportion

If four quantities (or numbers), are such that the ratio of the first and the second quantities is equal to the ratio of the third and the fourth quantities. those four quantities form a proportion. If a, b, c, d are four such quantities, we write $a : b = c : d$. The four quantities need not to be of same kind. The two quantities of the ratio need to be of the same kind.



In the above figure, let the bases of two triangles be a and b, respectively and each of their height is h unit. If the areas of these triangle are A and B square units, respectively, we can write.

$$\frac{A}{B} = \frac{\frac{1}{2}ah}{\frac{1}{2}bh} = \frac{a}{b} \qquad \text{or, } A : B = a : b$$

i.e. the ratio of two areas is equal to the ratio of two bases.

Continued proportion

By continued proportion of a, b, c, it is meant that a : b=b: c. a, b, c will be continued proportion, If and only if b2=ac. In case of continued proportional, all the quantities are to be of same kind. In this case, c is called the third proportional of a and b, and b is called the mid-proportional of a and c.

Example 1. A and B traverse the fixed distance in t_1 and t_2 minutes. Find the ratio of the average velocity of A and B

Solution: Let the average velocities of A and B be v_1 metre and v_2 metre per minute respectively. So, in time t_1 minutes A traverses v_1t_1 metre and in t_2 minutes B traverses v_2t_2 metra. According to the problem, $v_1t_1 = v_2t_2$ $\therefore \frac{v_1}{v_2} = \frac{t_2}{t_1}$ Here, the ratio of the velocities is inversely proportional to the ratio of time.

> **Work:**
>
> 1) Express 3.5 : 5.6 into 1 : a and b;.1.
> 2) If $x : y = 5 : 6$, then what is the value of $3x : 5y =$ what?

Transformation of ratio

Here, the quantities of ratios are positive numbers.

1. if $a : b = c : d$, then $b : a = d : c$ [Invertendo]

   Proof: Given that,
   $\frac{a}{b} = \frac{c}{d}$
   or, $ad = bc$ [multiplying both the sides by bd]
   or, $\frac{ad}{ac} = \frac{bc}{ac}$ [dividing both the sides by ac where a and c are non-zero]
   or, $\frac{d}{c} = \frac{b}{a}$
   i.e, $b : a = d : c$

2. If $a : b = c : d$, then, $a : c = b : d$ [Alternendo]

   Proof: Given that,
   $\frac{a}{b} = \frac{c}{d}$
   or, $ad = bc$ [multiplying both the sides by bd]
   or, $\frac{ad}{cd} = \frac{bc}{cd}$ [dividing both the sides by cd where c, d are non-zero]
   or, $\frac{a}{c} = \frac{b}{d}$
   i.e., $a : c = b : d$

3. if $a : b = c : d$, then, $\dfrac{a+b}{b} = \dfrac{c+d}{d}$ [Componendo]

Proof: Given that,

$\dfrac{a}{b} = \dfrac{c}{d}$

or, $\dfrac{a}{b} + 1 = \dfrac{c}{d} + 1$ [adding 1 to both the sides]

i.e, $\dfrac{a+b}{b} = \dfrac{c+d}{d}$

4. If $a : b = c : d$, then $\dfrac{a-b}{b} = \dfrac{c-d}{d}$ [dividendo]

Proof: Given that,

$\dfrac{a}{b} = \dfrac{c}{d}$

or, $\dfrac{a}{b} - 1 = \dfrac{c}{d} - 1$ [subtracting 1 from both the sides]

i.e., $\dfrac{a-b}{b} = \dfrac{c-d}{d}$

5. If $a : b = c : d$, then $\dfrac{a+b}{a-b} = \dfrac{c+d}{c-d}$ [componendo-Dividendo]

Proof: Given that, $a : b = c : d$

or, $\dfrac{a}{b} = \dfrac{c}{d}$

By componendo, $\dfrac{a+b}{b} = \dfrac{c+d}{d} \dots (1)$

Again, by dividendo, $\dfrac{a-b}{b} = \dfrac{c-d}{d}$

or, $\dfrac{b}{a-b} = \dfrac{d}{c-d}$ [[by invertendo]] $\dots (2)$

Therefore, $\dfrac{a+b}{b} \times \dfrac{b}{a-b} = \dfrac{c+d}{d} \times \dfrac{d}{c-d}$ [multiplying(1) & (2)]

i.e., $\dfrac{a+b}{a-b} = \dfrac{c+d}{c-d}$ [here $a \neq b, c \neq d$]

6. If $\dfrac{a}{b} = \dfrac{c}{d} = \dfrac{e}{f} = \dfrac{g}{h}$, then each of the ratio $= \dfrac{a+c+e+g}{b+d+f+h}$

Proof: let,

$\dfrac{a}{b} = \dfrac{c}{d} = \dfrac{e}{f} = \dfrac{g}{h} = k$

$\therefore a = bk,\ c = dk,\ e = fk,\ g = hk$

$$\therefore \frac{a+c+e+g}{b+d+f+h} = \frac{bk+dk+fk+hk}{b+d+f+h} = \frac{k(b+d+f+h)}{b+d+f+h} = k$$

But, k is equal to each of the ratio.

$$\therefore \frac{a}{b} = \frac{c}{d} = \frac{e}{f} = \frac{g}{h} = \frac{a+c+e+g}{b+d+f+h}$$

Work:

1) The sum of ages of mother and sister is s years. The ratio of their ages, before t years was $r : p$. After x years, what will be the ratio of their ages?
2) Let a man be standing at p metre distance from a light-post, r be the height of the man, s be the shadow length. Determine the height of the light-post in terms of p, r and s.

Example 2. The ratio of present ages of father and son is 7 : 2, and after 5 years, the ratio will be 8 : 3. What are their present ages?

Solution: Let the present age of father be a and that of the son be b.

So, by the conditions of first and second of the problems, we have,

$$\frac{a}{b} = \frac{7}{2} \dots (1)$$

$$\frac{a+5}{b+5} = \frac{8}{3} \dots (2)$$

From equation(1),we get,

$$a = \frac{7b}{2} \dots (3)$$

From equation(2),we get,

$3(a+5) = 8(b+5)$

or, $3a + 15 = 8b + 40$

or, $3a - 8b = 40 - 15$

or, $3 \times \frac{7b}{2} - 8b = 25$ [by using(3)]

or, $\frac{21b - 16b}{2} = 25$

or, $5b = 50$

$\therefore b = 10$

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In equation (3), by putting $b = 10$, we get $a = \dfrac{7 \times 10}{2} = 35.$

$\therefore$ The present age of father is 35 years, and that of the son is 10 years.

Example 3. If $a : b = b : c$, prove that, $\left(\dfrac{a+b}{b+c}\right)^2 = \dfrac{a^2+b^2}{b^2+c^2}$

Solution: Given that, $a : b = b : c$

$\therefore b^2 = ac$

Now, $\left(\dfrac{a+b}{b+c}\right)^2 = \dfrac{(a+b)^2}{(b+c)^2}$

$= \dfrac{a^2+2ab+b^2}{b^2+2bc+c^2}$

$= \dfrac{a^2+2ab+ac}{ac+2bc+c^2}$

$= \dfrac{a(a+2b+c)}{c(a+2b+c)}$

$= \dfrac{a}{c}$

And, $\dfrac{a^2+b^2}{b^2+c^2} = \dfrac{a^2+ac}{ac+c^2}$

$= \dfrac{a(a+c)}{c(a+c)}$

$= \dfrac{a}{c}$

$\therefore \left(\dfrac{a+b}{b+c}\right)^2 = \dfrac{a^2+b^2}{b^2+c^2}$

Example 4. If $\dfrac{a}{b} = \dfrac{c}{d}$, show that, $\dfrac{a^2+b^2}{a^2-b^2} = \dfrac{ac+bd}{ac-bd}$

Solution: Let, $\dfrac{a}{b} = \dfrac{c}{d} = k$

$\therefore a = bk$ and $c = dk$

Now, $\dfrac{a^2+b^2}{a^2-b^2} = \dfrac{(bk)^2+b^2}{(bk)^2-b^2} = \dfrac{b^2(k^2+1)}{b^2(k^2-1)} = \dfrac{k^2+1}{k^2-1}$

And, $\dfrac{ac+bd}{ac-bd} = \dfrac{bk \cdot dk + bd}{bk \cdot dk - bd} = \dfrac{bd(k^2+1)}{bd(k^2-1)} = \dfrac{k^2+1}{k^2-1}$

$$\therefore \frac{a^2+b^2}{a^2-b^2}=\frac{ac+bd}{ac-bd}$$

Example 5. Solve: $\dfrac{1-ax}{1+ax}\sqrt{\dfrac{1+bx}{1-bx}}=1$ where $0<b<2a<2b$

Solution: Given that, $\dfrac{1-ax}{1+ax}\sqrt{\dfrac{1+bx}{1-bx}}=1$

or, $\sqrt{\dfrac{1+bx}{1-bx}}=\dfrac{1+ax}{1-ax}$

or, $\dfrac{1+bx}{1-bx}=\dfrac{(1+ax)^2}{(1-ax)^2}$ [squaring both the sides]

or, $\dfrac{1+bx}{1-bx}=\dfrac{1+2ax+a^2x^2}{1-2ax+a^2x^2}$

or, $\dfrac{1+bx+1-bx}{1+bx-1+bx}=\dfrac{1+2ax+a^2x^2+1-2ax+a^2x^2}{1+2ax+a^2x^2-1+2ax-a^2x^2}$ [by componendo and dividendo]

or, $\dfrac{2}{2bx}=\dfrac{2(1+a^2x^2)}{4ax}$

or, $2ax=bx(1+a^2x^2)$

or, $x\{2a-b(1+a^2x^2)\}=0$

$\therefore\ x=0$

Again, $2a-b(1+a^2x^2)=0$

or, $b(1+a^2x^2)=2a$

or, $1+a^2x^2=\dfrac{2a}{b}$

or, $a^2x^2=\dfrac{2a}{b}-1$

or, $x^2=\dfrac{1}{a^2}\left(\dfrac{2a}{b}-1\right)$

$\therefore x=\pm\dfrac{1}{a}\sqrt{\dfrac{2a}{b}-1}$

Thus, the required solution is $x=0,\pm\dfrac{1}{a}\sqrt{\dfrac{2a}{b}-1}$

Example 6. If $\dfrac{\sqrt{1+x}+\sqrt{1-x}}{\sqrt{1+x}-\sqrt{1-x}}=p$, prove that, $p^2-\dfrac{2p}{x}+1=0$

Solution: Given that, $\dfrac{\sqrt{1+x}+\sqrt{1-x}}{\sqrt{1+x}-\sqrt{1-x}}=p$

or, $\dfrac{\sqrt{1+x}+\sqrt{1-x}+\sqrt{1+x}-\sqrt{1-x}}{\sqrt{1+x}+\sqrt{1-x}-\sqrt{1+x}+\sqrt{1-x}}=\dfrac{p+1}{p-1}$ [by componendo - dividendo]

or, $\dfrac{2\sqrt{1+x}}{2\sqrt{1-x}}=\dfrac{p+1}{p-1}$

or, $\dfrac{\sqrt{1+x}}{\sqrt{1-x}}=\dfrac{p+1}{p-1}$

or, $\dfrac{1+x}{1-x}=\dfrac{(p+1)^2}{(p-1)^2}=\dfrac{p^2+2p+1}{p^2-2p+1}$ [squaring both the sides]

or, $\dfrac{1+x+1-x}{1+x-1+x}=\dfrac{p^2+2p+1+p^2-2p+1}{p^2+2p+1-p^2+2p-1}$ [by componendo - dividendo]

or, $\dfrac{2}{2x}=\dfrac{2(p^2+1)}{4p}$

or, $\dfrac{1}{x}=\dfrac{p^2+1}{2p}$

or, $p^2+1=\dfrac{2p}{x}$

$\therefore p^2-\dfrac{2p}{x}+1=0$

Example 7. If $\dfrac{a^3+b^3}{a-b+c}=a(a+b)$, prove that, a, b, c are continued proportion.

Solution: Given that, $\dfrac{a^3+b^3}{a-b+c}=a(a+b)$

or, $\dfrac{(a+b)(a^2-ab+b^2)}{a-b+c}=a(a+b)$

or, $\dfrac{a^2-ab+b^2}{a-b+c}=a$ [dividing both the sides by $(a+b)$]

or, $a^2-ab+b^2=a^2-ab+ac$

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or, $b^2=ac$

$\therefore a, b, c$ are continued proportion.

Example 8. If $\frac{a+b}{b+c}=\frac{c+d}{d+a}$, prove that, $c=a$ or $a+b+c+d=0$

Solution: Given that, $\frac{a+b}{b+c}=\frac{c+d}{d+a}$

or, $\frac{a+b}{b+c}-1=\frac{c+d}{d+a}-1$ [subtracting 1 from both the sides]

or, $\frac{a+b-b-c}{b+c}=\frac{c+d-d-a}{d+a}$

or, $\frac{a-c}{b+c}=-\frac{a-c}{d+a}$

or, $\frac{a-c}{b+c}+\frac{a-c}{d+a}=0$

or, $(a-c)\left(\frac{1}{b+c}+\frac{1}{d+a}\right)=0$

or, $(a-c)\frac{(d+a+b+c)}{(b+c)(d+a)}=0$

or, $(a-c)\ (d+a+b+c)=0$

$\therefore\ a-c=0$ or, $d+a+b+c=0$

$\therefore\ c=a$ or, $a+b+c+d=0$

Example 9. If $\frac{x}{y+z}=\frac{y}{z+x}=\frac{z}{x+y}$ and x, y, z are not mutually equal, prove that the value of each ratio is either equal to -1 or equal to $\frac{1}{2}$

Solution: Let, $\frac{x}{y+z}=\frac{y}{z+x}=\frac{z}{x+y}=k$

$\therefore x=k(y+z)\ldots(1)$

$y=k(z+x)\ldots(2)$

$z=k(x+y)\ldots(3)$

By subtracting equation (2) from (1),

$x-y=k(y-x)$ or, $k(y-x)=-(y-x)$

$\therefore\ k=-1$

Again, adding equations (1), (2) and (3), we get,

$x+y+z=k(y+z+z+x+x+y)=2k(x+y+z)$

or, $k = \dfrac{(x+y+z)}{2(x+y+z)}$

$\therefore\ k = \dfrac{1}{2}$

$\therefore$ The value of each of the ratio is -1 or $\dfrac{1}{2}$.

Example 10. If $ax = by = cz$, show that, $\dfrac{x^2}{yz} + \dfrac{y^2}{zx} + \dfrac{z^2}{xy} = \dfrac{bc}{a^2} + \dfrac{ca}{b^2} + \dfrac{ab}{c^2}$

Solution: Let, $ax = by = cz = k$

$\therefore x = \dfrac{k}{a},\ y = \dfrac{k}{b},\ z = \dfrac{k}{c}$

Now, $\dfrac{x^2}{yz} + \dfrac{y^2}{zx} + \dfrac{z^2}{xy} = \dfrac{k^2}{a^2} \times \dfrac{bc}{k^2} + \dfrac{k^2}{b^2} \times \dfrac{ca}{k^2} + \dfrac{k^2}{c^2} \times \dfrac{ab}{k^2} = \dfrac{bc}{a^2} + \dfrac{ca}{b^2} + \dfrac{ab}{c^2}$

That is, $\dfrac{x^2}{yz} + \dfrac{y^2}{zx} + \dfrac{z^2}{xy} = \dfrac{bc}{a^2} + \dfrac{ca}{b^2} + \dfrac{ab}{c^2}$

Example 11. If a, b, c and d are continued proportional, and $x = \dfrac{10pq}{p+q}$

1) Show that, $\dfrac{a}{c} = \dfrac{a^2+b^2}{b^2+c^2}$
2) Prove that, $(a^2+b^2+c^2)(b^2+c^2+d^2) = (ab+bc+cd)^2$
3) Determine $\dfrac{x+5p}{x-5p} + \dfrac{x+5q}{x-5q}$,where $p \neq q$

Solution:

1) Given that, $a : b = b : c$ or, $\dfrac{a}{b} = \dfrac{b}{c}$ or, $ac = b^2$

   $$\text{RHS} = \frac{a^2+b^2}{b^2+c^2} = \frac{a^2+ac}{ac+c^2} = \frac{a(a+c)}{c(a+c)} = \frac{a}{c} = \text{LHS}$$

   $\therefore\ \dfrac{a}{c} = \dfrac{a^2+b^2}{b^2+c^2}$

2) Given that, a, b, c and d are continued proportional.

   $\therefore \dfrac{a}{b} = \dfrac{b}{c} = \dfrac{c}{d}$

   Let, $\dfrac{a}{b} = \dfrac{b}{c} = \dfrac{c}{d} = k$, where k is a constant proportional.

   $\therefore\ \dfrac{c}{d} = k$ or, $c = dk$

   $\dfrac{b}{c} = k$ or, $b = ck = dk \cdot k = dk^2$

$\frac{a}{b} = k$ or, $a = bk = dk^2 \cdot k = dk^3$

LHS $= (a^2 + b^2 + c^2)(b^2 + c^2 + d^2)$

$= \{(dk^3)^2 + (dk^2)^2 + (dk)^2\}\{(dk^2)^2 + (dk)^2 + d^2\}$

$= (d^2k^6 + d^2k^4 + d^2k^2)(d^2k^4 + d^2k^2 + d^2)$

$= d^2k^2(k^4 + k^2 + 1)d^2(k^4 + k^2 + 1)$

$= d^4k^2(k^4 + k^2 + 1)^2$

RHS $= (ab + bc + cd)^2$

$= (dk^3 \cdot dk^2 + dk^2 \cdot dk + dk \cdot d)^2$

$= (d^2k^5 + d^2k^3 + d^2k)^2$

$= \{d^2k(k^4 + k^2 + 1)\}^2$

$= d^4k^2(k^4 + k^2 + 1)^2 =$ LHS

$\therefore\ (a^2 + b^2 + c^2)(b^2 + c^2 + d^2) = (ab + bc + cd)^2$

3) Given that, $x = \frac{10pq}{p+q}$

or, $\frac{x}{5p} = \frac{2q}{p+q}$

or, $\frac{x+5p}{x-5p} = \frac{2q+p+q}{2q-p-q}$ [by componendo-dividendo]

or, $\frac{x+5p}{x-5p} = \frac{p+3q}{q-p} \ldots (1)$

Again, $x = \frac{10pq}{p+q}$

or, $\frac{x}{5q} = \frac{2p}{p+q}$

or, $\frac{x+5q}{x-5q} = \frac{2p+p+q}{2p-p-q}$ [by componendo-dividendo]

or, $\frac{x+5q}{x-5q} = \frac{3p+q}{p-q} \ldots (2)$

Here, by adding equations (1) and (2) we get,

$$\frac{x+5p}{x-5p} + \frac{x+5q}{x-5q} = \frac{p+3q}{q-p} + \frac{3p+q}{p-q} = \frac{p+3q}{q-p} - \frac{3p+q}{q-p}$$

$$= \frac{p+3q-3p-q}{q-p} = \frac{2q-2p}{q-p} = \frac{2(q-p)}{q-p} = 2$$

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Exercise 11.1

1. If the sides of two squares are a and b metre respectively, what will be the ratio of their areas?
2. If the area of a circle is equal to the area of a square, find the ratio of their perimeter.
3. If the ratio of two numbers is $3:4$ and their L.C.M. is 180, find out the two numbers.
4. The ratio of absent and present students of a day in your class is found to be $1:4$. Express the number of absent students in percentage with respect to the total number of students.
5. A thing is first bought and then sold with 28% loss. Find the ratio of buying and selling cost.
6. Sum of the ages of father and son is 70 years. 7 years ago the ratio of their ages were $5:2$. What will the ratio of their ages after 5 years?
7. If $a:b=b:c$, prove that,

   1) $\dfrac{a}{c}=\dfrac{a^2+b^2}{b^2+c^2}$

   2) $a^2b^2c^2\left(\dfrac{1}{a^3}+\dfrac{1}{b^3}+\dfrac{1}{c^3}\right)=a^3+b^3+c^3$

   3) $\dfrac{abc(a+b+c)^3}{(ab+bc+ca)^3}=1$

8. Solve:

   1) $\dfrac{1-\sqrt{1-x}}{1+\sqrt{1-x}}=\dfrac{1}{3}$

   2) $\dfrac{a+x-\sqrt{a^2-x^2}}{a+x+\sqrt{a^2-x^2}}=\dfrac{b}{x}$, $2a>b>0$ and $x\neq 0$

   3) $81\left(\dfrac{1-x}{1+x}\right)^3=\dfrac{1+x}{1-x}$

9. if $\frac{a}{b} = \frac{b}{c} = \frac{c}{d}$, show that,

   1) $\dfrac{a^3+b^3}{b^3+c^3} = \dfrac{b^3+c^3}{c^3+d^3}$

   2) $(a^2+b^2+c^2)(b^2+c^2+d^2) = (ab+bc+cd)^2$

10. If $x = \dfrac{4ab}{a+b}$, show that, $\dfrac{x+2a}{x-2a} + \dfrac{x+2b}{x-2b} = 2,\ a \neq b$

11. If $x = \dfrac{\sqrt[3]{m+1}+\sqrt[3]{m-1}}{\sqrt[3]{m+1}-\sqrt[3]{m-1}}$, prove that, $x^3 - 3mx^2 + 3x - m = 0$

12. If $x = \dfrac{\sqrt{2a+3b}+\sqrt{2a-3b}}{\sqrt{2a+3b}-\sqrt{2a-3b}}$, show that, $3bx^2 - 4ax + 3b = 0$

13. If $\dfrac{a^2+b^2}{b^2+c^2} = \dfrac{(a+b)^2}{(b+c)^2}$,show that, a, b, c are continued proportion.

14. If $\dfrac{x}{b+c} = \dfrac{y}{c+a} = \dfrac{z}{a+b}$, prove that, $\dfrac{a}{y+z-x} = \dfrac{b}{z+x-y} = \dfrac{c}{x+y-z}$.

15. If $\dfrac{bz-cy}{a} = \dfrac{cx-az}{b} = \dfrac{ay-bx}{c}$, prove that, $\dfrac{x}{a} = \dfrac{y}{b} = \dfrac{z}{c}$.

16. If $\dfrac{a+b-c}{a+b} = \dfrac{b+c-a}{b+c} = \dfrac{c+a-b}{c+a}$ and $a+b+c \neq 0$, prove that, $a = b = c$

17. If $\dfrac{x}{xa+yb+zc} = \dfrac{y}{ya+zb+xc} = \dfrac{z}{za+xb+yc}$ and $x+y+z \neq 0$, show that each of the ratio is $= \dfrac{1}{a+b+c}$.

18. If $(a+b+c)p = (b+c-a)q = (c+a-b)r = (a+b-c)s$, then prove that, $\dfrac{1}{q} + \dfrac{1}{r} + \dfrac{1}{s} = \dfrac{1}{p}$.

19. If $lx = my = nz$, then show that $\dfrac{x^2}{yz} + \dfrac{y^2}{zx} + \dfrac{z^2}{xy} = \dfrac{mn}{l^2} + \dfrac{nl}{m^2} + \dfrac{lm}{n^2}$.

20. If $\dfrac{p}{q} = \dfrac{a^2}{b^2}$ and $\dfrac{a}{b} = \dfrac{\sqrt{a+q}}{\sqrt{a-q}}$,show that, $\dfrac{p+q}{a} = \dfrac{p-q}{q}$.

Continued Ratio

Let us assume that Roni's earning is Tk 1000, Soni's earning is Tk 1500 and Samir's earning is Tk 2500. Here, Roni's earning : Soni's earning = 1000 : 1500 =

2 : 3; Soni's earning : Samir's earning = 1500 : 2500 = 3 : 5. Hence, Roni's : Soni's : Sami's earning = 2 : 3 : 5.

If two ratios are of the form $a : b$ and $b : c$, they can be re-written in the form $a : b : c$. This is called continued ratio. Any two or more ratios can be expressed in this form. It is noted that if two ratios are to be expressed in the form $a : b : c$, the antecedent of the first ratio and the subsequent of the second ratio need to be equal. For example, if two ratios 2 : 3 and 4 : 3 are to be put in the form $a : b : c$, the subsequent quantity of the first ratio is to be made equal to the antecedent quantity of the second ratio. That is, those quantities are to be made equal to their L.C.M.

Here, L.C.M of 3 and 4 is 12.

Here, $2 : 3 = \frac{2}{3} = \frac{2 \times 4}{3 \times 4} = \frac{8}{12} = 8 : 12$

Again, $4 : 3 = \frac{4}{3} = \frac{4 \times 3}{3 \times 3} = \frac{12}{9} = 12 : 9$

Therefore, if the ratios 2 : 3 and 4 : 3 are expressed in the form, $a : b : c$ will be 8 : 12 : 9

It is to be noted that if the earning of Sami in the above example is 1125, the ratio of their earnings will be 8 : 12 : 9.

Example 12. If a, b, c are quantities of same kind and $a : b = 3 : 4$, $b : c = 6 : 7$, what will be $a : b : c$?

Solution: $a : b = \frac{3}{4} = \frac{3 \times 3}{4 \times 3} = \frac{9}{12}$ and $b : c = \frac{6}{7} = \frac{6 \times 2}{7 \times 2} = \frac{12}{14}$ [LC.M. of 4 and 6 is 12]]

$\therefore a : b : c = 9 : 12 : 14$

Example 13. The ratio of angles of a triangle is 3 : 4 : 5. Express the angles in degree.

Solution: Let the angles, according to given ratio, be $3x$, $4x$ and $5x$. Sum of three angles = 108°. According to the problem, $3x + 4x + 5x = 180°$ or, $12x = 180°$ or, $x = 15°$, Therefore, the angles are $3x = 3 \times 15° = 45°$

$4x = 4 \times 15° = 60°$

and $5x = 5 \times 15° = 75°$

Example 14. If the sides of a square increase by 10%, how much will the area be increased in percentage?

Solution: Let, each side of the square be a metre. Therefore, the area of the square be a^2 square metre. If the side of the square increases by 10%, each side will be $(a + a$ of $10\%)$ metre, or $1.10a$ metre. In this case, the area of the square will be $(1.10a)^2$ square metre, or $1.21a^2$ square metre. Thus the the area increases by $(1.21a^2 - a^2) = 0.21a^2$ square metre.

$\therefore$ The percentage of increment of the area will be $\dfrac{0.21a^2}{a^2} \times 100\% = 21\%$

Work:

1) There are 35 male and 25 female students in your class. The ratio of rice and pulse given by each of the male and female students for making Khicuri in a picnic 3 : 1 and 5 : 2, respectively. Find the ratio of total rice and total pulse.
2) The amount of pulse, mustard and paddy produced in a farmer's farm are 75 kg, 100 kg and 525 kg, respectively. The grains are sold at price of 100, 120 and 30 taka respectively. After selling all the grains, calculate the ratio of income from the individual grain.

Proportional Division

Division of a quantity into fixed ratio is called proportional division. If S is to be divided in a given ratio $a : b : c : d$, dividing S by $a+b+c+d$, the parts a, b, c and d need to be taken. Therefore, 1st part $= S$ of $\dfrac{a}{a+b+c+d} = \dfrac{Sa}{a+b+c+d}$

2nd part $= S$ of $\dfrac{b}{a+b+c+d} = \dfrac{Sb}{a+b+c+d}$

3rd part $= S$ of $\dfrac{c}{a+b+c+d} = \dfrac{Sc}{a+b+c+d}$

4th part $= S$ of $\dfrac{d}{a+b+c+d} = \dfrac{Sd}{a+b+c+d}$

In this way, any quantity may be divided into any fixed ratio.

Example 15. The area of a rectangular land is 12 hectors and length of its diagonal is 500 metres. The ratio of length and breadth of this land to that of other land are 3 : 4 and 2 : 3, respectively.

1) What is the area of the given land in square metre?
2) What is the area of the other land?
3) What is the breadth of the given land?

Solution:

1) We know, 1 hector $= 10,000$ square metres.

   $\therefore$ 12 hectors $= 12 \times 10,000 = 120000$ square metres

2) Given that, the ratio of length and breadth of the given land to that of other land are 3 : 4 and 2 : 3, respectively.

   Let length of the given land be $3x$ metres and breadth be $2y$ metres.

   Therefore, the length of the other land is $4x$ metres and breadth is $3y$ metres.

   $\therefore$ the area of the given land is $= 3x \cdot 2y = 6xy$ square metres and the area of the other land is $= 4x \cdot 3y = 12xy$ square metres.

   According to the given question, $6xy = 120000$ or, $xy = 20000$ $\therefore$ the area of the other land is $= 12xy = 12 \times 20000 = 240000$ square metres.

3) Let the length of the given land be $3x$ metres and the breadth be $2y$ metres.

   Therefore, the length of its one diagonal is $\sqrt{(3x)^2 + (2y)^2}$ metres.

   We get from the previous (b) problem, $xy = 20000$

   According to the question, $\sqrt{(3x)^2 + (2y)^2} = 500$

   or, $9x^2 + 4y^2 = 250000$

   or, $(3x + 2y)^2 - 2 \cdot 3x \cdot 2y = 250000$

   or, $(3x + 2y)^2 - 12xy = 250000$

   or, $(3x + 2y)^2 - 12 \times 20000 = 250000$

   or, $(3x + 2y)^2 = 250000 + 240000$

   or, $(3x + 2y)^2 = 490000$

   or, $3x + 2y = 700 \cdots (1)$

Again, $(3x-2y)^2 = (3x+2y)^2 - 4 \cdot 3x \cdot 2y$

or, $(3x-2y)^2 = (3x+2y)^2 - 24xy$

or, $(3x-2y)^2 = (700)^2 - 24 \times 20000$

or, $(3x-2y)^2 = 490000 - 480000$

or, $(3x-2y)^2 = 10000$

or, $3x - 2y = 100 \ldots (2)$

By subtracting (2) from (1) we get,

$4y = 600$ or, $y = 150$

$\therefore$ the breadth of the given land is 150 metres.

Exercise 11.2

1. If a, b, c are continued proportional, which one of the followings is correct?

   1) $a^2 = bc$ 2) $b^2 = ac$
   3) $ab = bc$ 4) $a = b = c$

2. The ratio of ages of Arif and Akib is 5 : 3. If Arif is of 20 years old, how many years later the ratio of their ages will be 7 : 5?

   1) 5 years 2) 6 years
   3) 8 years 4) 10 years

3. If the sides of a square double, how much will the area of a square be increased?

   1) 2 times 2) 4 times
   3) 8 times 4) 6 times

4. If $x : y = 7 : 5$, $y : z = 5 : 7$, then $x : z =$ how much?

   1) 35 : 49 2) 35 : 35
   3) 25 : 49 4) 49 : 25

5. If b, a, c are continued proportional, then-

   (i) $a^2 = bc$

   (ii) $\frac{b}{a} = \frac{c}{a}$

   (iii) $\frac{a+b}{a-b} = \frac{c+a}{c-a}$

   Which one of the following is correct?

   1) i 2) i and ii 3) i and iii 4) i, ii and iii

6. If $x : y = 2 : 1$ and $y : z = 2 : 1$, then-

   (i) x, y, z are continued proportional

   (ii) $z : x = 1 : 4$

   (iii) $y^2 + zx = 4yz$

   Which one of the following is correct?

   1) i and ii 2) i and iii 3) ii and iii 4) i, ii and iii

7. If $\frac{a}{x} = \frac{m^2+n^2}{2mn}$, then $\frac{\sqrt{a+x}}{\sqrt{a-x}} =$ what?

   1) $\frac{m}{n}$ 2) $\frac{m+n}{m-n}$ 3) $\frac{m-n}{m+n}$ 4) $\frac{n}{m}$

If the perimeter of a triangle is 36 cm and the ratio of length of its side is $3:4:5$, then answer question 8 and 9:

8. What is the length of its greatest side in c m?

   1) 5   2) 9   3) 12   4) 15

9. What is the area of the triangle in square cm?

   1) 6   2) 54   3) 67   4) 90

10. Weight of 1 cubic c.m wood is 7 decigram. What is the percentage of weight of the wood to the weight of equal volume of water?

11. Distribute Tk. 300 among a, b, c, d in such a way that the ratios are a's part : b's part $= 2:3$, b's part : c's part $= 1:2$ and c's part : d's part $3:2$.

12. Three fishermen have caught 690 pieces of fishes. The ratios of their parts are $\frac{2}{3}, \frac{4}{5}$ and $\frac{5}{6}$. How many fishes will each of them get?

13. The perimeter of a triangle is 45 cm. The ratio of the lengths of the sides is $3:5:7$. Find the length of each sides.

14. If the ratio of two numbers is $5:7$ and their H.C.F. is 4, what is L.C.M. of the numbers ?

15. In a cricket game, the total runs scored by Sakib, Mushfique and Mashrafi were 171. The ratio of runs scored by Sakib and Mushfique, and Mushfique and Mashrafi was $3:2$. What were the runs scored by them individually?

16. In an office, there were 2 officers, 7 clarks and 3 bearers. If a bearer gets Tk. 1, a clerk gets Tk. 2 and an officer gets Tk. 4. Their total salary is Tk. $150,000$. What is their individual salary?

17. If the sides of a square are increased by 20%, what is percentage of increment of the area of the square?

18. If the length of a rectangle is increased by 10% and the breadth is decreased by 10%, what is the percentage of increase or decrease of the area of the rectangle?

19. In a field, the ratio of production is $4:7$ before and after irrigation. In that field, the production of paddy in a land previously was 304 quintal. What would be the production of paddy after irrigation?

20. If the ratio of paddy and rice produced from paddy is $3:2$ and the ratio of wheat and suzi produced from wheat is $4:3$, find the ratio of rice and suzi produced from equal quantity of rice and wheat.

21. The area of a land is 432 square metre. If the ratios of lengths and breadths of that land and that of another land are 3 : 4 and 2 : 5 respectively, then what is the area of another land?

22. Zami and Simi take loans of different amounts at the rate of 10% simple profit on the same day from the same Bank. The amount on capital and profit which Zami refunds after 2 years is the the same amount that Simi refunds after 3 years on capital and profit. Find the ratio of their loan.

23. The ratio of sides of a triangle is 5 : 12 : 13 and perimeter is 30 c m.

    1) Draw the triangle and write down the type of triangle with respect to the angles.

    2) Determine the area of a square drawn with the diagonal of a rectangle, where the the length of the rectangle is the largest side of the triangle and the breadth of the triangle is the smallest side of the triangle.

    3) If the length is increased by 10% and the breadth is increased by 20%, what will be percentage of increase of the area?

24. The ratio of present and absent students of a day in a class is 1 : 4.

    1) Express the percentage of absent students against total students.

    2) The ratio of present and absent students would be 1 : 9, if 5 more students were present. What was the total number of students?

    3) Of the total number of students, the number of male students is 10 less than the twice of the number of female students. Find the ratio of male and female students.

25. Ashik, Mizan, Anika and Ahona started a business with a total capital of Tk. 195000 and had a profit of Tk. 26500 after the end of a year. In the capital of that business, Ashik's part : Mizan's part = 2 : 3, Mizan's part : Anika's part = 4 : 5 and Anika's part : Ahona's part = 5 : 6.

    1) Determine the simple ratio of the capital.

    2) Determine the amount of capital of each person in that business.

    3) 60% of the profit is invested in the business at the end of year. If the remaining profit is divided according to the capital ratio, then between Ahona and Ashik who will get the greater amount of profit money?

Chapter 12

Simple Simultaneous Equations in Two Variables

For solving the mathematical problems, the most important topic of Algebra is equation. In classes VI and VII, we have got the idea of simple equation and have known how to solve the simple equation in one variable. In class VIII, we have solved the simple simultaneous equations by the methods of substitution and elimination and by graphs. We have also learnt how to form and solve simple simultaneous equations related to real life problems. In this chapter, the idea of simple simultaneous equations have been expanded and new methods of solution have been discussed. Besides, in this chapter, solution by graphs and formation of simultaneous equations related to real life problems and their solutions have been discussed in detail.

At the end of the chapter, the students will be able to —

- verify the consistency of simple simultaneous equations in two variables.
- verify the mutual dependence of two simple simultaneous equations in two variables
- explain the method of cross-multiplication.
- form and solve simultaneous equations related to real life mathematical problems.
- solve the simultaneous equations with two variables by graphs.

Simple simultaneous equations

Simple simultaneous equations means two simple equations in two variables when they are presented together and the two variables are of same characteristics. Such two equations together are also called system of simple equations. In class VIII, we have solved such system of equations and learnt to form and solve simultaneous equations related to real life problems. In this chapter, these have been discussed in more details.

First, we consider the equation $2x + y = 12$. This is a simple equation in two variables.

In the equation, can we get such values of x and y on the left hand side for which the sum of twice the first with the second will be equal to 12 of the right hand side; that is, the equation will be satisfied by those two values?

Now, we fill in the following chart from the equation 2x+y=12:

Value of x	Value of y	Value of L.H.S $(2x + y)$	R.H.S
-2	16	$-4 + 16 = 12$	12
0	12	$0 + 12 = 12$	12
3	6	$6 + 6 = 12$	12
5	2	$10 + 2 = 12$	12
...	...	$\ldots = 12$	12

The equation has infinite number of solutions. Among those, four solutions are:

$(-2, 16)$, $(0, 12)$, $(3, 6)$, $(5, 2)$.

Again, we fill in the following chart from another equation $x - y = 3$:

Value of x	Value of y	Value of L.H.S $(x - y)$	R.H.S
-2	-5	$-2 + 5 = 3$	3
0	-3	$0 + 3 = 3$	3
3	0	$3 - 0 = 3$	3
5	2	$5 - 2 = 3$	3
...	...	$\ldots = 3$	3

The equation has infinite number of solutions. Among those, four solutions are:

$(-2, -5)$, $(0, -3)$, $(3, 0)$, $(5, 2)$.

If the two equations discussed above are considered together a system, both the equations will be satisfied simultaneously only by $(5, 2)$. Both the equations will not be satisfied simultaneously by any other values.

Therefore, the solution of the system of equations $2x + y = 12$ and $x - y = 3$ is $(x, y) = (5, 2)$.

Work: Write down five solutions for each of the two equations $x-2y+1=0$ and $2x+y-3=0$ so that among the solutions, the common solution also exists.

Conformability for the solution of simple simultaneous equations in two variables

1) As discussed earlier, the system of equations $\left.\begin{aligned} 2x+y&=12 \\ x-y&=3 \end{aligned}\right\}$ has unique (only one) solution. Such systems of equations are called consistent. Comparing the coefficient of x and y (taking the ratio of the coefficients) of the two equations, we get, $\frac{2}{1} \neq \frac{1}{-1}$; any equation of the system of equations cannot be expressed in terms of the other. That is why, such systems of equations are called mutually independent. In the case of consistent and mutually independent system of equations, the ratios are not equal. In this case, the constant terms need not to be compared.

2) Now we shall consider the system of equations $\left.\begin{aligned} 2x-y&=6 \\ 4x-2y&=12 \end{aligned}\right\}$. Can these two equations be solved?

   Here, if both sides of first equation are multiplied by 2, we shall get the second equation. Again, if both sides of second equation are divided by 2, we shall get the first equation. That is, the two equations are mutually dependent.

   We know, the first equation has infinite number of solutions. So, the second equation has also the same infinite number of solutions. Such systems of equations are called consistent and mutually dependent. Such systems of equations have infinite number of solutions. Here, comparing the coefficients of x and y, and the constant terms of the two equations, we get, $\frac{2}{4} = \frac{-1}{-2}$ $= \frac{6}{12}\left(=\frac{1}{2}\right)$

   That is, in the case of the system of such simultaneous equations, the ratios become equal.

3) Now, we shall try to solve the system of equations $\left.\begin{aligned} 2x+y&=12 \\ 4x+2y&=5 \end{aligned}\right\}$.

Here, multiplying both sides of first equation by 2, we get, $4x + 2y = 24$

second equation is, $4x + 2y = 5$

subtracting, $0 = 19$ which is impossible.

So, we can say, such systems of equations cannot be solved. Such systems of equations are inconsistent and mutually independent. Such systems of equations have no solution.

Here, comparing the coefficients of x and y and constant terms from the two equations, we get, $\frac{2}{4} = \frac{1}{2} \neq \frac{12}{5}$

That is, in case of the system of inconsistent and mutually independent equations ratios of the coefficients of the variables are not equal to the ratio of the constant terms.

Generally, conditions for conformability of two simple simultaneous equations, such as, $\left.\begin{matrix} a_1x + b_1y = c_1 \\ a_2x + b_2y = c_2 \end{matrix}\right\}$ are given in the chart below:

	system of equations	comparison of oeff. and const. terms	consistent/ inconsistent	mutually dependent/ independent	has solutions(how many)/no
(i)	$a_1x + b_1y = c_1$ $a_2x + b_2y = c_2$	$\frac{a_1}{a_2} \neq \frac{b_1}{b_2}$	consistent	independent	yes (only one)
(ii)	$a_1x + b_1y = c_1$ $a_2x + b_2y = c_2$	$\frac{a_1}{a_2} = \frac{b_1}{b_2} = \frac{c_1}{c_2}$	consistent	dependent	yes (infinite)
(iii)	$a_1x + b_1y = c_1$ $a_2x + b_2y = c_2$	$\frac{a_1}{a_2} = \frac{b_1}{b_2} \neq \frac{c_1}{c_2}$	inconsistent	independent	no

Now, if there is no constant terms in both the equations of a system of equations i.e $c_1 = c_2 = 0$, then from the table:

According to (i) if $\frac{a_1}{a_2} \neq \frac{b_1}{b_2}$, the system of equations is always consistent and independent of each other. In that case, there will be only one (unique) solution.

According to (ii) if $\frac{a_1}{a_2} = \frac{b_1}{b_2}$, the system of equations is consistent and dependent of each other. In that case, there will be infinite number of solutions.

Example 1. Explain whether the following systems of equations are consistent/ inconsistent, dependent/ independent of each other and indicate the number of solutions in each case.

1) $x + 3y = 1$
$2x + 6y = 2$

2) $2x - 5y = 3$
$x + 3y = 1$

3) $3x - 5y = 7$
$6x - 10y = 15$

Solution:

1) Given system of equations is: $\left.\begin{aligned} x + 3y &= 1 \\ 2x + 6y &= 2 \end{aligned}\right\}$

Ratio of the coefficients of x is $\frac{1}{2}$

Ratio of the coefficients of y is $\frac{3}{6}$ or, $\frac{1}{2}$

Ratio of constant terms is $\frac{1}{2}$

$\therefore \frac{1}{2} = \frac{3}{6} = \frac{1}{2}$

Therefore, the system of equations is consistent and mutually dependent. The system of equations has infinite number of solutions.

2) Given system of equations is: $\left.\begin{aligned} 2x - 5y &= 3 \\ x + 3y &= 1 \end{aligned}\right\}$

Ratio of the coefficients of x is $\frac{2}{1}$

Ratio of the coefficients of x is $\frac{-5}{3}$

We get, $\frac{2}{1} \neq \frac{-5}{3}$

$\therefore$ Therefore, the system of equations is consistent and mutually independent. The system of equations has only one (unique) solution.

3) Given system of equations is: $\left.\begin{aligned} 3x - 5y &= 7 \\ 6x - 10y &= 15 \end{aligned}\right\}$

Ratio of the coefficients of x is $\frac{3}{6}$ or, $\frac{1}{2}$

Ratio of the coefficients of y is $\frac{-5}{-10}$ or, $\frac{1}{2}$

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Ratio of constant terms is $\frac{7}{15}$

We get, $\frac{3}{6} = \frac{-5}{-10} \neq \frac{7}{15}$

$\therefore$ Therefore, the system of equations is inconsistent and mutually independent. The system of equations has no solution.

> **Work:** Verify whether the system of equations $x-2y+1=0$, $2x+y-3=0$ is consistent and dependent and indicate how many solutions the system of equations may has.

Exercises 12.1

Mention with arguments whether the following simple simultaneous equations are consistent/inconsistent, mutually dependent/ independent and indicate the number of solutions.

1. $x - y = 4$
   $x + y = 10$
2. $2x + y = 3$
   $4x + 2y = 6$
3. $x - y - 4 = 0$
   $3x - 3y - 10 = 0$
4. $3x + 2y = 0$
   $6x + 4y = 0$
5. $3x + 2y = 0$
   $9x - 6y = 0$
6. $5x - 2y - 16 = 0$
   $3x - \frac{6}{5}y = 2$
7. $-\frac{1}{2}x + y = -1$
   $x - 2y = 2$
8. $-\frac{1}{2}x - y = 0$
   $x - 2y = 0$
9. $-\frac{1}{2}x + y = -1$
   $x + y = 5$
10. $ax - cy = 0$
    $cx - ay = c^2 - a^2$

Solution of simple simultaneous equations

We shall discuss the solutions of only the consistent and independent simple simultaneous equations. Such system of equation has only one (unique) solution.

Here, four methods of solutions are discussed :

1. Method of substitution 2. Method of elimination 3. Method of cross-multiplication 4. Graphical method.

In class VIII, we have known how to solve by the methods of substitution and elimination. Here, examples of one for each of these two methods are given.

Example 2. Solve by the method of substitution:

$$2x + y = 8$$

$$3x - 2y = 5$$

Solution: Given equations are;

$$2x + y = 8 \ldots (1)$$

$$3x - 2y = 5 \ldots (2)$$

From equation (1), $y = 8 - 2x \ldots (3)$

Putting the value of y from equation (3) in equation (2) we get,

$3x - 2(8 - 2x) = 5$

or, $3x - 16 + 4x = 5$

or, $7x = 5 + 16$

or, $7x = 21$

or, $x = 3$

Putting the value of x in equation (3) we get,

$y = 8 - 2 \times 3$

or, $y = 8 - 6$

or, $y = 2$

$\therefore$ solution $(x, y) = (3, 2)$

Substitution method: Conveniently from any of the two equations, value of one variable is expressed in terms of the other variable and putting the obtained value in the other equation, we get an equation in one variable. Solving this equation, value of the variable can be found. This value can be put in any of the equations. But, if it is put in the equation in which one variable has been expressed in terms of the other variable, the solution will be easier. From this equation, value of the other variable will be found.

Example 3. Solve by the method of elimination:

$$2x + y = 8$$
$$3x - 2y = 5$$

Note: To show the difference between the methods of substitution and elimination, same equations of example 2 have been taken in this example 3.

Solution: Given equations are:

$$2x + y = 8 \ldots (1)$$

$$3x - 2y = 5 \ldots (2)$$

Multiplying both sides of equation (1) by 2, $4x + 2y = 16 \ldots (3)$

Adding (2) and (3),

$7x = 21$

or, $x = 3$

Putting the value of x in (1) we get,

$2 \times 3 + y = 8$

or, $y = 8 - 6$

or, $y = 2$

$\therefore$ solution $(x, y) = (3, 2)$

Elimination method: Conveniently one equation or both equations are multiplied by such a number so that after multiplication, absolute values of the coefficients of the same variable become equal. Then as per need, if the equations are added or subtracted, the variable with equal coefficient will be eliminated. Then, solving the obtained equation, the value of the existing variable will be found. If that value is put conveniently in any of the given equations, value of the other variable will be found.

Cross multiplication method:

We consider the following two equations:

$$a_1x + b_1y + c_1 = 0 \ldots (1)$$

$$a_2x + b_2y + c_2 = 0 \ldots (2)$$

Multiplying equation (1) by b_2 and equation (2) by b_1 we get,

$$a_1b_2x + b_1b_2y + b_2c_1 = 0 \ldots (3)$$

$$a_2b_1x + b_1b_2y + b_1c_2 = 0 \ldots (4)$$

Subtracting equation (4) from equation (3) we get,

$$(a_1b_2 - a_2b_1)x + b_2c_1 - b_1c_2 = 0$$

$$\text{or, } (a_1b_2 - a_2b_1)x = b_1c_2 - b_2c_1$$

$$\text{or, } \frac{x}{b_1c_2 - b_2c_1} = \frac{1}{a_1b_2 - a_2b_1} \ldots (5)$$

Again, multiplying equation (1) by a_2 and equation (2) by a_1 we get,

$$a_1a_2x + a_2b_1y + c_1a_2 = 0 \ldots (6)$$

$$a_1a_2x + a_1b_2y + c_2a_1 = 0 \ldots (7)$$

Subtracting equation (7) from equation (6) we get,

$$(a_2b_1 - a_1b_2)y + c_1a_2 - c_2a_1 = 0$$

$$\text{or, } -(a_1b_2 - a_2b_1)y = -(c_1a_2 - c_2a_1)$$

$$\text{or, } \frac{y}{c_1a_2 - c_2a_1} = \frac{1}{a_1b_2 - a_2b_1} \ldots (8)$$

From (5) and (8) we get,

$$\boxed{\frac{x}{b_1c_2 - b_2c_1} = \frac{y}{c_1a_2 - c_2a_1} = \frac{1}{a_1b_2 - a_2b_1}}$$

From such relation between x and y, the technique of finding their values is called the method of cross- multiplication.

From the above relation between x and y we get,

$$\frac{x}{b_1c_2 - b_2c_1} = \frac{1}{a_1b_2 - a_2b_1}, \text{ or, } x = \frac{b_1c_2 - b_2c_1}{a_1b_2 - a_2b_1}$$

$$\text{Again, } \frac{y}{c_1a_2 - c_2a_1} = \frac{1}{a_1b_2 - a_2b_1}, \text{ or, } y = \frac{c_1a_2 - c_2a_1}{a_1b_2 - a_2b_1}$$

$\therefore$ The solution of the given equations: $(x, y) = \left(\dfrac{b_1c_2 - b_2c_1}{a_1b_2 - a_2b_1}, \dfrac{c_1a_2 - c_2a_1}{a_1b_2 - a_2b_1}\right)$

We observe:

Equation	Relation between x and y	Illustration
$a_1x+b_1y+c_1=0$ $a_2x+b_2y+c_2=0$	$\dfrac{x}{b_1c_2-b_2c_1}=\dfrac{y}{c_1a_2-c_2a_1}=\dfrac{1}{a_1b_2-a_2b_1}$	$\begin{array}{c\|ccccc} & & x & y & 1 & \\ a_1 & b_1 & c_1 & a_1 & b_1 \\ a_2 & b_2 & c_2 & a_2 & b_2 \end{array}$

Note: The method of cross-multiplication can also be applied by keeping the constant terms of both equations on the right hand side. In that case, changes of sign will occur but the solution will remain the same.

Work:

If the system of equations $\left.\begin{array}{l} 4x-y-7=0 \\ 3x+y=0 \end{array}\right\}$ are expressed as the system of equations $\left.\begin{array}{l} a_1x+b_1y+c_1=0 \\ a_2x+b_2y+c_2=0 \end{array}\right\}$, find the values of a_1, b_1, c_1, a_2, b_2, c_2.

Example 4. Solve by the method of cross-multiplication:

$6x - y = 1$

$3x + 2y = 13$

Solution:

Making the right hand side of the equations 0 (zero) by transposition, we get,

$6x - y - 1 = 0$
$3x + 2y - 13 = 0$

comparing the equations with
$a_1x + b_1y + c_1 = 0$ and
$a_2x + b_2y + c_2 = 0$
respectively we get,
$a_1 = 6$, $b_1 = -1$, $c_1 = -1$
$a_2 = 3$, $b_2 = 2$, $c_2 = -13$

By the method of cross-multiplication we get,

$$\frac{x}{b_1c_2 - b_2c_1} = \frac{y}{c_1a_2 - c_2a_1} = \frac{1}{a_1b_2 - a_2b_1}$$

or, $$\frac{x}{(-1)\times(-13) - 2\times(-1)} = \frac{y}{(-1)\times 3 - (-13)\times 6} = \frac{1}{6\times 2 - 3\times(-1)}$$

or, $$\frac{x}{13+2} = \frac{y}{-3+78} = \frac{1}{12+3}$$

or, $$\frac{x}{15} = \frac{y}{75} = \frac{1}{15}$$

Therefore, $\frac{x}{15} = \frac{1}{15}$ or, $x = \frac{15}{15} = 1$

Again, $\frac{y}{75} = \frac{1}{15}$ or, $y = \frac{75}{15} = 5$

$\therefore$ solution $(x, y) = (1, 5)$

$$\begin{array}{c|ccccc} & & x & y & 1 & \\ a_1 & b_1 & c_1 & a_1 & b_1 \\ a_2 & b_2 & c_2 & a_2 & b_2 \end{array}$$

$$\downarrow$$

$$\begin{array}{c|ccccc} & & x & y & 1 & \\ 6 & -1 & -1 & 6 & -1 \\ 3 & 2 & -13 & 3 & 2 \end{array}$$

Example 5. Solve by the method of cross-multiplication:

$3x - 4y = 0$

$2x - 3y = -1$

Solution: Given equations are:

$3x - 4y = 0$
$2x - 3y = -1$

or,

$3x - 4y + 0 = 0$
$2x - 3y + 1 = 0$

By the method of cross-multiplication, we get:

$$\frac{x}{-4\times 1 - (-3)\times 0} = \frac{y}{0\times 2 - 1\times 3} = \frac{1}{3\times(-3) - 2\times(-4)}$$

or, $$\frac{x}{-4+0} = \frac{y}{0-3} = \frac{1}{-9+8}$$

or, $$\frac{x}{-4} = \frac{y}{-3} = \frac{1}{-1}$$

$$\begin{array}{c|cccc} & & x & y & 1 \\ 3 & -4 & 0 & 3 & -4 \\ 2 & -3 & 1 & 2 & -3 \end{array}$$

or, $\dfrac{x}{4}=\dfrac{y}{3}=\dfrac{1}{1}$

$\therefore \dfrac{x}{4}=\dfrac{1}{1}$ or, $x=4$

Again, $\dfrac{y}{3}=\dfrac{1}{1}$ or, $y=3$

$\therefore$ solution $(x,y)=(4,3)$

Example 6. Solve by the method of cross-multiplication:

$$\frac{x}{2}+\frac{y}{3}=8$$

$$\frac{5x}{4}-3y=-3$$

Solution: Arranging the given equations in the form $ax+by+c=0$ we get,

$\dfrac{x}{2}+\dfrac{y}{3}=8$

or, $\dfrac{3x+2y}{6}=8$

or, $3x+2y-48=0$

Again, $\dfrac{5x}{4}-3y=-3$

or, $\dfrac{5x-12y}{4}=-3$

or, $5x-12y+12=0$

$\therefore$ the given equations are:

$3x+2y-48=0$

$5x-12y+12=0$

By the method of cross-multiplication we get,

$$\frac{x}{2\times 12-(-12)\times(-48)}=\frac{y}{(-48)\times 5-12\times 3}=\frac{1}{3\times(-12)-5\times 2}$$

or, $\dfrac{x}{24-576}=\dfrac{y}{-240-36}=\dfrac{1}{-36-10}$

$$\begin{array}{c|cccccc} & & x & & y & 1 & \\ 3 & 2 & & -48 & 3 & & 2 \\ 5 & -12 & & 12 & 5 & & -12 \end{array}$$

or, $\dfrac{x}{-552}=\dfrac{y}{-276}=\dfrac{1}{-46}$

or, $\dfrac{x}{552}=\dfrac{y}{276}=\dfrac{1}{46}$

$\therefore \dfrac{x}{552}=\dfrac{1}{46}$ or, $x=\dfrac{552}{46}=12$

Again, $\dfrac{y}{276}=\dfrac{1}{46}$ or, $y=\dfrac{276}{46}=6$

$\therefore$ solution: $(x, y) = (12, 6)$

Verification of the correctness of the solution:
Putting the values of x y in given equations,

In first equation, L.H.S $= \dfrac{x}{2} + \dfrac{y}{3} = \dfrac{12}{2} + \dfrac{6}{3} = 6 + 2 = 8 =$ R.H.S

In second equation, L.H.S $= \dfrac{5x}{4} - 3y = \dfrac{5 \times 12}{4} - 3 \times 6 = 15 - 18 = -3 =$ R.H.S.

$\therefore$ the solution is correct.

Example 7. Solve by the method of cross-multiplication: $ax - by = ab = bx - ay$

Solution: Given equations are:

$ax - by = ab$ or, $ax - by - ab = 0$
$bx - ay = ab$ $bx - ay - ab = 0$ By the

By the method of cross-multiplication, we get,

$$\frac{x}{(-b)\times(-ab) - (-a)(-ab)} = \frac{y}{(-ab)\times b - (-ab)\times a}$$
$$= \frac{1}{a \times (-a) - b \times (-b)}$$

$$\text{or, } \frac{x}{ab^2 - a^2b} = \frac{y}{-ab^2 + a^2b} = \frac{1}{-a^2 + b^2}$$

$$\text{or, } \frac{x}{-ab(a-b)} = \frac{y}{ab(a-b)} = \frac{1}{-(a+b)(a-b)}$$

$$\begin{array}{c|ccccc} & & x & & y & 1 \\ a & -b & -ab & a & -b & \\ b & -a & -ab & b & -a & \end{array}$$

$$\text{or, } \frac{x}{ab(a-b)} = \frac{y}{-ab(a-b)} = \frac{1}{(a+b)(a-b)}$$

$$\therefore \frac{x}{ab(a-b)} = \frac{1}{(a+b)(a-b)}, \text{ or, } x = \frac{ab(a-b)}{(a+b)(a-b)} = \frac{ab}{a+b}$$

$$\text{Again, } \frac{y}{-ab(a-b)} = \frac{1}{(a+b)(a-b)}, \text{ or, } y = \frac{-ab(a-b)}{(a+b)(a-b)} = \frac{-ab}{a+b}$$

$$\therefore (x, y) = \left(\frac{ab}{a+b}, \frac{-ab}{a+b}\right)$$

Exercise 12.2

Solve by the method of substitution $(1-3)$:

1. $7x - 3y = 31$
   $9x - 5y = 41$
2. $\dfrac{x}{2} + \dfrac{y}{3} = 1$
   $\dfrac{x}{3} + \dfrac{y}{2} = 1$
3. $\dfrac{x}{a} + \dfrac{y}{b} = 2$
   $ax + by = a^2 + b^2$

Solve by the method of elimination$(4-6)$:

4. $7x - 3y = 31$
   $9x - 5y = 41$
5. $7x - 8y = -9$
   $5x - 4y = -3$
6. $ax + by = c$
   $a^2x + b^2y = c^2$

Solve by the method of cross-multiplication $(7-15)$:

7. $2x + 3y + 5 = 0$
   $4x + 7y + 6 = 0$
8. $3x - 5y + 9 = 0$
   $5x - 3y - 1 = 0$
9. $x + 2y = 7$
   $2x - 3y = 0$
10. $4x + 3y = -12$
    $2x = 5$
11. $-7x + 8y = 9$
    $5x - 4y = -3$
12. $3x - y - 7 = 0$
    $2x + y - 3 = 0$
13. $ax + by = a^2 + b^2$
    $2bx - ay = ab$
14. $y(3 + x) = x(6 + y)$
    $3(3 + x) = 5(y - 1)$
15. $(x + 2)(y - 3) = y(x - 1)$
    $5x - 11y - 8 = 0$

Solving by using Graphical Method

In a simple equation in two variables, the relation of existing variables x and y and can be expressed by picture. This picture is called the graph of that relation. In the graph of such equation, there exist infinite number of points. Plotting a few such points and joining them with each other, we get the graph.

Each of a simple simultaneous equations has infinite number of solutions. Graph of each equation is a straight line. Coordinates of each point of the straight line satisfies the equation. To indicate a graph, two or more than two points are necessary.

Now we shall try to solve graphically the following system of equations:

$$2x + y = 3 \ldots (1)$$

$$4x + 2y = 6 \ldots (2)$$

From equation (1) we get, $y = 3 - 2x$.

Taking some values of x in the equation, we find the corresponding values of y and make the adjoining table:

x	-1	0	3
y	5	3	-3

$\therefore$ three points on the graph of the equation are: $(-1, 5), (0, 3)\quad(3, -3)$.

Again, from equation (2) we get, $2y = 6 - 4x$ or, $y = \dfrac{6 - 4x}{2}$

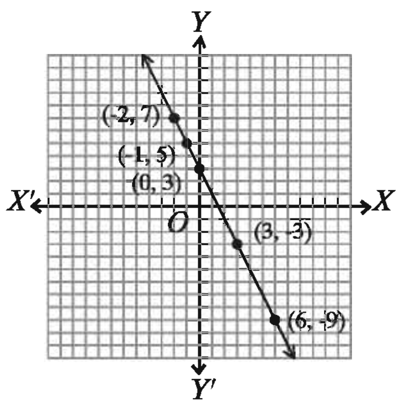
Taking some values of x in the equation, we find the corresponding values of y and make the adjoining table:

x	-2	0	6
y	7	3	-9

$\therefore$ three points on the graph of the equation are: $(-2, 7), (0, 3)\quad(6, -9)$.

In a graph paper let XOX' and YOY' be respectively the X- axis and Y- axis and O is the origin.

We take each side of the smallest squares of the graph paper as unit along with both axes. Now, we plot the points $(-1, 5), (0, 3)$ and $(3, -3)$ obtained from equation (1) and join them each other. The graph is a straight line.



Again, we plot the points $(-2, 7), (0, 3)\quad(6, -9)$ obtained from equation (2) and join them each other. In this case also the graph is a straight line.

But we observe that the two straight lines coincide and they have turned into the one straight line. Again, if both sides of equation (2) are divided by 2, we get he equation (1). That is why the graphs of the two equations coincide.

Here, the system of equations, $\left.\begin{aligned} 2x + y &= 3 \ldots (1) \\ 4x + 2y &= 6 \ldots (2) \end{aligned}\right\}$ are consistent and mutually dependent. Such system of equations have infinite number of solutions and its graph is a straight line.

$$2x - y = 4 \ldots (1)$$

$$4x - 2y = 12 \ldots (2)$$

From equation (1) we get, $y = 2x - 4$.

Taking some values of x in the equation, we find the corresponding values of y and make the adjoining table:

x	-1	0	4
y	-6	-4	4

$\therefore$ three points on the graph of the equation are: $(-1,-6),(0,-4),(4,4)$.

Again, from equation (2) we get,

$4x-2y=12$, or, $2x-y=6$ [dividing both sides by 2]

or, $y=2x-6$

Taking some values of x in the equation, we find the corresponding values of y and make the adjoining table:

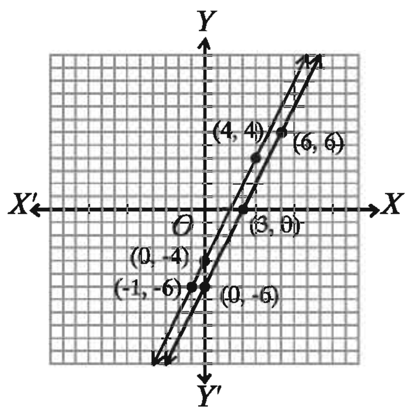
x	0	3	6
y	-6	0	6

$\therefore$ three points on the graph of the equation are: $(0,-6),(3,0),(6,6)$.

In a graph paper let XOX' and YOY' be respectively the X- axis and Y- axis and O is the origin.

We take each side of smallest squares of the graph paper as unit along with both axes. Now, we plot the points $(-1,-6),(0,-4)$ $(4,4)$ obtained from equation (1) and join them each other. The graph is a straight line.

Again, we plot the points $(0,-6),(3,0),(6,6)$ obtained from equation (2) and join them each other. In this case also the graph is a straight line.



We observe in the graph, though each of the given equations has separately infinite number of solutions, they have no common solution as system of simultaneous equations. Further, we observe that the graphs of the two equations are straight lines parallel to each other. That is, the lines will never intersect each other. Therefore, there will be no common point of the lines. In this case we say, such system of equations has no solution. We know, such system of equations is inconsistent and independent of each other.

Now, we shall solve the system of two consistent and independent equations by graphs. Graphs of two such equations in two variables intersect at a point.

Both the equations will be satisfied by the coordinates of that point. The very coordinates of the point of intersection will be the solution of the two equations.

Example 8. Solve and show the solution in graph:

$$2x + y = 8$$

$$3x - 2y = 5$$

Solution: Given equations are:

$$2x + y - 8 = 0 \ldots (1)$$

$$3x - 2y - 5 = 0 \ldots (2)$$

By the method of cross-multiplication we get,

$$\frac{x}{1 \times (-5) - (-2) \times (-8)} = \frac{y}{(-8) \times 3 - (-5) \times 2} = \frac{1}{2(-2) - 3 \times 1}$$

or, $\frac{x}{-5 - 16} = \frac{y}{-24 + 10} = \frac{1}{-4 - 3}$

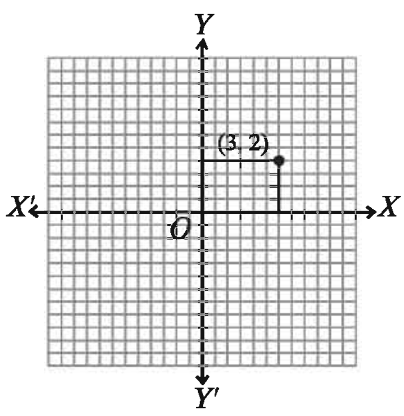
or, $\frac{x}{-21} = \frac{y}{-14} = \frac{1}{-7}$

or, $\frac{x}{21} = \frac{y}{14} = \frac{1}{7}$

$\therefore \frac{x}{21} = \frac{1}{7}$, or, $x = \frac{21}{7} = 3$

Again, $\frac{y}{14} = \frac{1}{7}$, or, $y = \frac{14}{7} = 2$

$\therefore$ solution: $(x, y) = (3, 2)$



Let XOX' and YOY' be X- axis and $Y-$ axis respectively and O be the origin. Taking each two sides of the smallest squares along with both axes of the graph paper as one unit, we plot the point $(3, 2)$.

Example 9. Solve with the help of graphs:

$$3x - y = 3$$

$$5x + y = 21$$

Solution: Given equations are:

$3x - y = 3 \ldots (1)$

$5x + y = 21 \ldots (2)$

From equation (1) we get, $3x - y = 3$, or, $y = 3x - 3$

Taking some values of x in the equation, we find the corresponding values of y and make the adjoining table:

x	-1	0	3
y	-6	-3	6

$\therefore$ three points on the graph of the equation are: $(-1, -6), (0, -3), (3, 6)$

Again, from equation (2) we get, $5x + y = 21$, or, $y = 21 - 5x$

Taking some values of x in the equation, we find the corresponding values of y and make the adjoining table:

x	3	4	5
y	6	1	-4

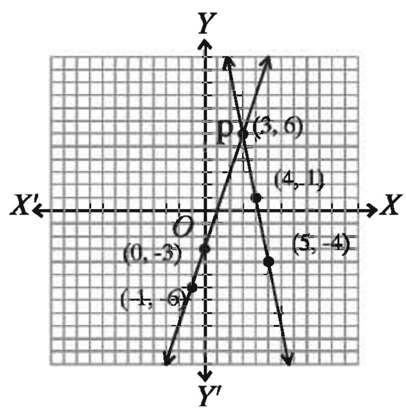
$\therefore$ three points on the graph of the equation are: $(3, 6), (4, 1), (5, -4)$.

In a graph paper let XOX' and YOY' be respectively the X- axis and Y- axis and O is the origin.

We take each side of the smallest squares of the graph paper as unit along with both axes.

Now, we plot the points $(-1, -6), (0, -3), (3, 6)$ obtained from equation (1) and join them each other. The graph is a straight line.

Again, we plot the points $(3, 6), (4, 1), (5, -4)$ obtained from equation (2) and join them each other. In this case also the graph is a straight line.



Let the two straight lines intersect each other at P. It is seen from the picture that the coordinates of P are $(3, 6)$.

$\therefore$ solution: $(x, y) = (3, 6)$

Example 10. Solve by graphical method:

$2x + 5y = -14$

$4x - 5y = 17$

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Solution: Given equations are:

$2x + 5y = -14 \ldots (1)$

$4x - 5y = 17 \ldots (2)$

From equation (1) we get, $5y = -14 - 2x$, or, $y = \dfrac{-2x - 14}{5}$

Taking some convenient values of x in the equation, we find the corresponding values of y and make the adjoining table:

x	3	$\frac{1}{2}$	-2
y	-4	-3	-2

$\therefore$ three points on the graph of the equation are: $(3, -4)$, $\left(\frac{1}{2}, -3\right)$, $(-2, -2)$.

Again, from equation (2) we get, $5y = 4x - 17$, or, $y = \dfrac{4x - 17}{5}$

Taking some convenient values of x in the equation, we find the corresponding values of y and make the adjoining table:

x	3	$\frac{1}{2}$	-2
y	-1	-3	-5

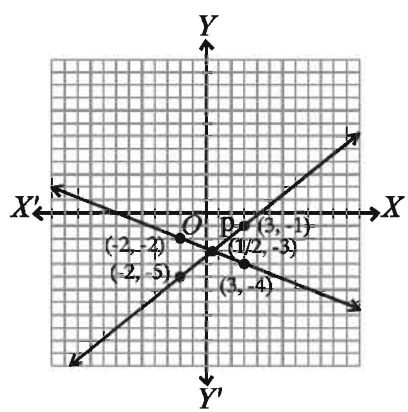
$\therefore$ three points on the graph of the equation are: $(3, -1)$, $\left(\frac{1}{2}, -3\right)$, $(-2, -5)$

In a graph paper let XOX' and YOY' be respectively the X- axis and Y- axis and O is the origin.

We take two sides of the smallest squares of the graph paper as unit along with both axes.

Now, we plot the points $(3, -4)$, $\left(\frac{1}{2}, -3\right)$, $(-2, -2)$ obtained from equation (1) and join them each other. The graph is a straight line.

Again, we plot the points $(3, -1)$, $\left(\frac{1}{2}, -3\right)$, $(-2, -5)$ obtained from equation (2) and join them each other. In this case also the graph is a straight line.



Let the two straight lines intersect each other at P. It is seen from the picture that the coordinates of P are $\left(\frac{1}{2}, -3\right)$.

$\therefore$ solution: $(x, y) = \left(\frac{1}{2}, -3\right)$

Example 11. Solve with the help of graphs: $3 - \frac{3}{2}x = 8 - 4x$

Solution: Given equation is $3 - \frac{3}{2}x = 8 - 4x$

Let, $y = 3 - \frac{3}{2}x = 8 - 4x$

$\therefore y = 3 - \frac{3}{2}x \ldots (1)$

and $y = 8 - 4x \ldots (2)$

Taking some values of x in equation (1), we find the corresponding values of y and make the adjoining table:

x	-2	0	2
y	6	3	0

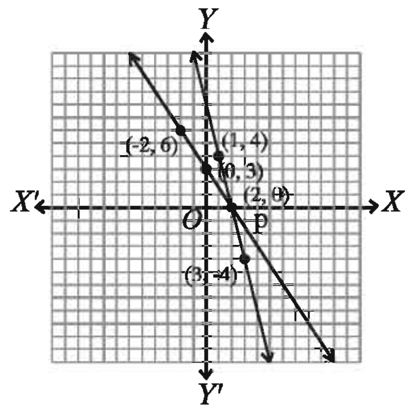
$\therefore$ three points on the graph of the equation are: $(-2, 6), (0, 3), (2, 0)$.

Again, taking some values of x in equation (2), we find the corresponding values of y and make the adjoining table:

x	1	2	3
y	4	0	-4

$\therefore$ three points on the graph of the equation are: $(1, 4), (2, 0), (3, -4)$.

In a graph paper let XOX' and YOY' be respectively the X- axis and Y- axis and O is the origin.
We take each side of the smallest squares of the graph paper as unit along with both axes.
Now, we plot the points $(-2, 6), (0, 3), (2, 0)$ obtained from equation (1) and join them each other. The graph is a straight line.
Again, we plot the points $(1, 4), (2, 0), (3, -4)$ obtained from equation (2) and join them each other. In this case also the graph is a straight line.



Let the two straight lines intersect each other at P. It is seen from the picture that the coordinates of P are $(2, 0)$.

$\therefore$ solution: $x = 2$

Work: Find four points on the graph of the equation $2x - y - 3 = 0$ in terms of a table. Then, taking unit of a fixed length on the graph paper, plot the points and join them each other. Is the graph a straight line?

Exercise 12.3

Solve by graphs:

1. $3x + 4y = 14$
   $4x - 3y = 2$
2. $2x - y = 1$
   $5x + y = 13$
3. $2x + 5y = 1$
   $x + 3y = 2$
4. $3x - 2y = 2$
   $5x - 3y = 5$
5. $\dfrac{x}{2} + \dfrac{y}{3} = 2$
   $2x + 3y = 13$
6. $3x + y = 6$
   $5x + 3y = 12$
7. $3x + 2y = 4$
   $3x - 4y = 1$
8. $\dfrac{x}{2} + \dfrac{y}{3} = 3$
   $x + \dfrac{y}{6} = 3$
9. $3x + 2 = x - 2$
10. $3x - 7 = 3 - 2x$

Formation of simultaneous equations from real life problems and solution

In everyday life, there occur some such mathematical problems which are easier to solve by forming equations. For this, from the condition or conditions of the problem, two mathematical symbols, mostly the variables x, y are assumed for two unknown expressions. Two equations are to be formed for determining the values of those unknown expressions. If the two equations thus formed are solved, values of the unknown quantities will be found.

Example 12. If 5 is added to the sum of the two digits of a number consisting of two digits, the sum will be three times the digits of the tens place. Moreover, if the places of the digits are interchanged, the number thus found will be 9 less than the original number. Find the number.

Solution: Let the digit of the tens place of the required number be x and its digits of the units place is y. Therefore, the number is $10x + y$.

$\therefore$ by the 1st condition, $x + y + 5 = 3x \ldots (1)$

and by the 2nd condition, $10y + x = (10x + y) - 9 \ldots (2)$

From equation (1) we get, $y = 3x - x - 5$, or, $y = 2x - 5 \ldots (3)$

Again, from equation (2) we get,

$10y - y + x - 10x + 9 = 0$

or, $9y - 9x + 9 = 0$

or, $y - x + 1 = 0$

or, $2x - 5 - x + 1 = 0$ [putting the value of y from equation (3) we get]

or, $x = 4$

putting the value of x in equation (3) we get, $y = 2 \times 4 - 5 = 8 - 5 = 3$

$\therefore$ the number is $10x + y = 10 \times 4 + 3 = 40 + 3 = 43$

Example 13. 8 years ago, father's age was eight times the age of his son. After 10 years, father's age will be twice the age of the son. What are their present ages?

Solution: Let the present age of father be x years and age of son is y years.

$\therefore$ by 1st condition, $x - 8 = 8(y - 8) \ldots (1)$

and by 2nd condition, $x + 10 = 2(y + 10) \ldots (2)$

From (1) we get, $x - 8 = 8y - 64$

or, $x = 8y - 64 + 8$

or, $x = 8y - 56 \ldots (3)$

From (2) we get, $x + 10 = 2y + 20$

or, $8y - 56 + 10 = 2y + 20$ [Putting the value of x from (3)]

or, $8y - 2y = 20 + 56 - 10$

or, $6y = 66$

or, $y = 11$

(3) we get, $x = 8 \times 11 - 56 = 88 - 56 = 32$

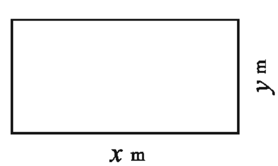
$\therefore$ at present, Father's age is 32 years and son's age is 11 years.

Example 14. Twice the breadth of a rectangular garden is 10 metre more than its length and perimeter of the garden is 100 metre. There is a path of width 2 metre around the outside boundary of the garden. To make the path by bricks, it costs TK. 110 per square metre.

1) Assuming the length of the garden to be x metre and its breadth to be y metre, form system of simultaneous equations.
2) Find the length and breadth of the garden.
3) What will be the total cost to make the path by bricks?

Solution:

1) Length of the rectangular garden is x metre and breadth is y metre.



$\therefore$ by first condition, $2y = x + 10 \ldots (1)$

and by second condition, $2(x + y) = 100 \ldots (2)$

2) from equation (2) we get, $2x + 2y = 100$

or, $2x + x + 10 = 100$ [putting the value of $2y$ from (1)]

or, $3x = 90$

or, $x = 30$

$\therefore$ from (1) we get, $2y = 30 + 10$ [putting the value of x]

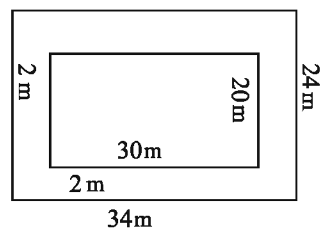
or, $2y = 40$

or, $y = 20$

$\therefore$ length of the garden is 30 metre and breadth is 20 metre.

3) Length of the garden with the path $= (30 + 4)$ m. $= 34$ m.

and breadth of the garden with the path $= (20 + 4)$ m. $= 24$ m.



$\therefore$ Area of the path= Area of the garden with the path - Area of the garden

$= (34 \times 24 - 30 \times 20)$ square metre.

$= (816 - 600)$ square metre.

$= 216$ square metre.

$\therefore$ cost for making the path by bricks = Tk. (216×110) = Tk. 23760

Example 15. How many times will minute hand and hour hand coincide? Find the times.

Solution: Let minute hand and hour hand coincide at the time $x : y$. We need to remember that x ($x = 0, 1, \cdots 11$ where 0 means 12) is always an integer but y may not be. We know, minute hand runs 12 times faster than the hour hand. At time x the hour hand is exactly on the x and minute hand is on the 12. Within y minutes the hour hand passes $\frac{y}{12}$ and the minute hand passes y ticks. So,

$5x + \frac{y}{12} = y$

or, $y - \frac{y}{12} = 5x$

or, $\frac{11}{12}y = 5x$

$\therefore y = \frac{60}{11}x$

Now we put the possible values of x:

If $x = 0$, $y = 0$ minute i.e. $12 : 00$.

If $x = 1$, $1 : 5\frac{5}{11}$ minute.

If $x = 2$, $2 : 10\frac{10}{11}$ minute.

$\cdots \quad \cdots \quad \cdots$

If $x = 11$, $11 : 60$ minute or, $12 : 00$.

As the first and last times are same, these two hands coincides 11 times and the times are: $x : \frac{60}{11}x$ minute.

Work: If in triangle ABC $\angle B = 2x°$, $\angle C = x°$, $\angle A = y°$ and $\angle A = \angle B + \angle C$, find the value of x and y.

Exercise 12.4

1. For which of the following conditions is the system of equations $ax+by+c=0$ and $px+qy+r=0$ consistent and mutually independent?
   1) $\frac{a}{p} \neq \frac{b}{q}$ 2) $\frac{a}{p} = \frac{b}{q} = \frac{c}{r}$ 3) $\frac{a}{p} = \frac{b}{q} \neq \frac{c}{r}$ 4) $\frac{a}{p} = \frac{b}{q}$
2. If $x+y=4$, $x-y=2$, which one of the following is the value of (x, y)?
   1) $(2, 4)$ 2) $(4, 2)$ 3) $(3, 1)$ 4) $(1, 3)$
3. If $x+y=6$ and $2x=4$, what is the value of y?
   1) 2 2) 4 3) 6 4) 8
4. For which one of the following equation is the adjoining chart correct?

x	0	2	4
y	-4	0	4

   1) $y=x-4$ 2) $y=8-x$ 3) $y=4-2x$ 4) $y=2x-4$
5. If $2x-y=8$ and $x-2y=4$, then $x+y=$ what?
   1) 0 2) 4 3) 8 4) 12
6. The equations $x-y-4=0$ and $3x-3y-10=0$:

   (i) are mutually dependent.

   (ii) are mutually consistent.

   (iii) do not have any solution.

   On the basis of information above, which one of the following is correct?
   1) ii 2) iii 3) i and iii 4) ii and iii

On the basis of information given below answer questions 7 – 9.

Length of the floor of a rectangular room is 2 metres more than its breadth and perimeter of the floor is 20 metre. For decorating the floor with mosaic it costs Tk. 900 per square metre.

7. What is the length of the floor of the room in metre?
   1) 10 2) 8 3) 6 4) 4
8. What is the area of the floor of the room in square metre?
   1) 24 2) 32 3) 48 4) 80

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9. How much taka will be the total cost for decorating the floor with mosaic?
   1) 72000 2) 43200 3) 28800 4) 21600

Solve by forming simultaneous equations (10 – 17):

10. If 1 is added to each of numerator and denominator of a fraction, the fraction will be $\frac{4}{5}$. Again, if 5 is subtracted from each of numerator and denominator, the fraction will be $\frac{1}{2}$. Find the fraction.
11. If 1 is subtracted from numerator and 2 is added to denominator of a fraction, the fraction will be $\frac{1}{2}$. Again, if 7 is subtracted from numerator and 2 is subtracted from denominator, the fraction will be $\frac{1}{3}$. Find the fraction.
12. The digit of the units place of a number consisting of two digits is 1 more than three times the digit of tens place. But if the places of the digits are interchanged, the number thus found will be equal to eight times the sum of the digits. What is the number?
13. Difference of the digits of a number consisting of two digits is 4. If the places of the digits are interchanged, sum of the numbers so found and the original number will be 110 ; find the number.
14. Present age of mother is four times the sum of the ages of her two daughters. After 5 years, mother's age will be twice the sum of the ages of the two daughters. What is the present age of the mother ?
15. If the length of a rectangular region is decreased by 5 metre and breadth is increased by 3 metre, the area will be less by 9 square metre. Again, if the length is increased by 3 metre and breadth is increased by 2 metre, the area will be increased by 67 square metre. Find the length and breadth of the rectangle.
16. A boat, rowing in favour of current, goes 15 km per hour and rowing against the current goes 5 km per hour. Find the speed of the boat and current.
17. A labourer of a garments serves on the basis of monthly salary. At the end of every year she gets a fixed increment. Her monthly salary becomes Tk. 4500 after 4 years and Tk. 5000 after 8 years. Find the salary at the beginning of her service and amount of annual increment of salary.

18. A system of simple equations is $x + y = 10$, $3x - 2y = 0$

    1) Show that the equations are consistent. How many solutions do they have?

    2) Solving the system of equations, find (x, y) .

    3) Find the area of the triangle formed by the straight lines indicated by the equations with the X- axis.

19. If 7 is added to the numerator of a fraction, the value of the fraction is the integer 2. Again, if 2 is subtracted from the denominator, value of the fraction is the integer 1.

    1) Form a system of equations by the fraction $\frac{x}{y}$.

    2) Find (x, y) by solving the system of equations by the method of cross-multiplication. What is the fraction?

    3) Draw the graph of the system of equations and verify the correctness of the obtained values of (x, y).

20. If the total number of sides of two polygons is 17 and total number of diagonals of those polygons is 53, then what is the number of sides of each polygon?

21. An assignment was given to some students where they can work alone or in a male-female group of two members. $\frac{2}{3}$ of male students and $\frac{3}{5}$ of female students worked in groups. What fraction of students did that assignment alone?

22. It takes 5 seconds for two trains of 100 and 200 metre lengths to cross each other when they pass in the opposite direction and 15 seconds in the same direction. Find the velocity of the two trains.

23. How many con ecutive integers are needed so that their product is divisible by 5040?

24. How many times the hour hand and minute hand of a clock make an angle of 30 degree with each other? Find the times.

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Chapter 13

Finite Series

The term 'order' is widely used in our day to day life. For example, the concept of order is used to arrange the commodities in the shops, to arrange the events of drama and ceremony, to keep the commodities in attractive way in the godown. Again, to make many tasks easier and attractive, we use large to small, child to old, light to heavy etc. types of order. Mathematical series have been originated from these concepts of order. In this chapter, the relation between sequence and series and contents related to them have been presented.

At the end of this chapter, the students will be able to —

- describe the sequence and series and determine the differences between them.
- explain finite series.
- form formulae for determining the fixed term of the series and the sum of fixed numbers of terms and solve mathematical problems by applying the formulae.
- determine the sum of squares and cubes of natural numbers.
- solve mathematical problems by applying different formulae of series.
- construct formulae to find the fixed term of a geometric progression and sum of fixed numbers of terms and solve mathematical problems by applying the formulae.

Sequence

Let us note the following relation:

$$\begin{array}{ccccccc} 1 & 2 & 3 & 4 & \cdots & n & \cdots \\ \downarrow & \downarrow & \downarrow & \downarrow & & \downarrow & \\ 2 & 4 & 6 & 8 & \cdots & 2n & \cdots \end{array}$$

Here, every natural number n is related to twice the number $2n$. That means, the set of positive even numbers $\{2, 4, 6, \cdots\}$ is obtained by a method from the set of natural numbers $\{1, 2, 3, \cdots\}$. This arranged set of even number is a sequence.

Hence, some quantities are arranged in a particular way such that the antecedent and subsequent terms becomes related. The set of quantities arranged in such a way is called a sequence.

The aforesaid relation is called a function and written as $f(n) = 2n$. The general term of this sequence is $2n$. The number of terms of any sequence is infinite. The way of writing the sequence with the help of general term is:

$\{2n\}$, $n = 1, 2, 3, \cdots$ or, $\{2n\}_{n=1}^{+\infty}$ or, $\{2n\}$

The first quantity of the sequence is called the first term, the second quantity is called second term, the third quantity is called the third term etc. The first term of the sequence $1, 3, 5, 7, \cdots$ is 1, the second term is 2, etc. Followings are the four examples of sequence :

$1, 2, 3, \cdots, n, \cdots$

$1, 3, 5, \cdots, 2n - 1, \cdots$

$1, 4, 9, \cdots, n^2, \cdots$

$\frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \cdots, \frac{n}{n+1}, \cdots$

Work:

1) The general terms of six sequences are given below. Write down the sequences:
   (1) $\frac{1}{n}$ (2) $\frac{n-1}{n+1}$ (3) $\frac{1}{2^n}$
   (4) $\frac{1}{2^{n-1}}$ (5) $(-1)^{n+1}\frac{n}{n+1}$ (6) $(-1)^{n-1}\frac{n}{2n+1}$
2) Each of you write a general term and then write the sequence.

Series

If the terms of a sequence are connected successively by + sign, a series is obtained. Such as, $1 + 3 + 5 + 7 + \cdots$ is a series. The differences between any two successive terms of the series are equal. Again, $2 + 4 + 8 + 16 + \cdots$ is a series. The ratio of two successive terms is equal. Hence, the characterstic of any series depends upon the relation between its two successive terms. Among the series, two important series are arithmetic series and geometric series.

Arithmetic Series

If the difference between any term and its antecedent term is always equal, the series is called arithmetic series.

Example 1. $1+3+5+7+9+11$ is a series. The first term of the series is 1, the second term is 3, the third term is 5 etc.

Here, second term – first term $= 3-1 = 2$,

third term –second term $= 5-3 = 2$, fourth term – third term $= 7-5 = 2$,

fifth term – fourth term $= 9-7 = 5$, sixth term – fifth term $= 11-9 = 2$

Hence, the series is an arithmetic series.

In this series, the difference between two terms is called common difference. The common difference of the mentioned series is 2. The numbers of terms of the series are fixed. That is why the series is finite series. It is to be noted that if the terms of the series are not fixed, the series is called infinite series, such as, $1+4+7+10+\cdots$ is an infinite series. In an arithmetic series, the first term and the common difference are generally denoted by a and d respectively. Then by definition, if the first term is a, the second term is $a+d$, the third term is $a+2d$ etc. Hence, the series will be $a+(a+d)+(a+2d)+\cdots$.

Determining common term of an arithmetic series

Let the first term of an arithmetic series be a and the common difference be d. Then the terms of the series are:

First term $= a = a+(1-1)d$

Second term $= a+d = a+(2-1)d$

Third term $= a+2d = a+(3-1)d$

Fourth term $= a+3d = a+(4-1)d$

$\cdots\ \cdots\ \cdots$

$\cdots\ \cdots\ \cdots$

$\therefore n$ th term $= a+(n-1)d$

This nth term is called common term of arithmetic series. If the first term of an arithmetic series in a and common difference is d are known, all the terms of the series can be determined successively by putting $n = 1, 2, 3, 4, \cdots$ in the nth term.

Let the first term of an arithmetic series be 3 and the common difference be 2. Then nth term of the series is $= 3 + (n-1) \times 2 = 2n + 1$.

> **Work:** If the first term of an arithmetic series is 5 and common difference is 7, determine the first six terms, 22nd term, rth term and $(2p+1)$th term.

Example 2. Of the series $5 + 8 + 11 + 14 + \cdots$ which term is 383?

Solution: The first term of the series $a = 5$, common difference $d = 8 - 5 = 11 - 8 = 14 - 11 = 3$

$\therefore$ It is an arithmetic series.

Let, nth term of the series $= 383$

We know that, nth term $= a + (n-1)d$

$\therefore a + (n-1)d = 383$

or, $5 + (n-1)3 = 383$

or, $5 + 3n - 3 = 383$

or, $3n = 383 - 5 + 3$

or, $3n = 381$

or, $n = \dfrac{381}{3}$

or, $n = 127$

$\therefore$ 127 th term of the given series is 383

Sum of n terms of an arithmetic series

Let the first term of any arithmetic series be a, last term be p, common difference be d, number of terms be n and sum of n terms be S_n.

Writing from the first term to the last and conversely from the last term to the first of the series we get,

$$S_n = a + (a+d) + (a+2d) + \cdots + (p-2d) + (p-d) + p \ldots (1)$$
$$\text{and } S_n = p + (p-d) + (p-2d) + \cdots + (a+2d) + (a+d) + a \ldots (2)$$
$$\text{Adding, } 2S_n = (a+p) + (a+p) + (a+p) + \cdots + (a+p) + (a+p) + (a+p)$$

or, $2S_n = n(a+p)$ [$\because$ number of terms of the series is n]

$\therefore S_n = \frac{n}{2}(a+p) \ldots (3)$

Again, nth term $= p = a + (n-1)d$. Putting this value of p in (3) we get,

$S_n = \frac{n}{2}[a + \{a + (n-1)d\}]$

i.e., $S_n = \frac{n}{2}\{2a + (n-1)d\} \ldots (4)$

If the first term of an arithmetic series a, last term p and number of terms n are known, the sum of the series can be determined by the formula (3). But if the first term a, common difference d and number of terms n are known, the sum of the series are determined by the formula (4).

Determination of the sum of first n terms of natural numbers

Let S_n be the sum of n-numbers of natural numbers i.e.

$S_n = 1 + 2 + 3 + \cdots + (n-1) + n$

Writing from the first term and conversely from the last term of the series we get,

$$S_n = 1 + 2 + 3 + \cdots + (n-2) + (n-1) + n \ldots (1)$$
$$\text{and } S_n = n + (n-1) + (n-2) + \cdots + 3 + 2 + 1 \ldots (2)$$
$$\text{Adding, } 2S_n = (n+1) + (n+1) + (n+1) + \cdots + (n+1) \quad [n\text{-number of terms}]$$

or, $2S_n = n(n+1)$

$\therefore S_n = \frac{n(n+1)}{2} \ldots (3)$

Example 3. Find the sum total of first 50 natural numbers.

Solution: Using formula (3) we get,

$S_{50} = \frac{50(50+1)}{2} = 25 \times 51 = 1275$

$\therefore$ The sum total of first 50 natural numbers is 1275.

Example 4. $1+2+3+4+\cdots+99=$ what?

Solution:

The first term of the series $a=1$, common difference $d=2-1=1$ and the last term $p=99$.

$\therefore$ It is an arithmetic series.

Let the nth term of the series $=9$

We know, nth term of an arithmetic progression $=a+(n-1)d$

$\therefore a+(n-1)d=99$

or, $1+(n-1)1=99$

or, $1+n-1=99$

$\therefore n=99$

From (4) formula, the sum of first n-terms of an arithmetic series,

$S_n=\frac{n}{2}\{2a+(n-1)d\}$

Hence, the sum of first 99 terms of the series $S_{99}=\frac{99}{2}\{2\times 1+(99-1)\times 1\}=\frac{99}{2}(2+98)$

$=\dfrac{99\times 100}{2}=99\times 50=4950$

Alternative method: From formula (3), $S_n=\frac{n}{2}(a+p)$

$\therefore S_{99}=\frac{99}{2}(1+99)=\dfrac{99\times 100}{2}=4950$

Example 5. What is the sum of 30 terms of the series $7+12+17+\cdots$?

Solution: First term of the series $a=7$, common difference $d=12-7=5$

$\therefore$ It is an arithmetic series. Here, number of terms $n=30$

We know that the sum of n-terms of an arithmetic series ,

$S_n=\frac{n}{2}\{2a+(n-1)d\}$

So, the sum of 30 terms $S_{30}=\frac{30}{2}\{2\cdot 7+(30-1)5\}=15(14+29\times 5)$

$=15(14+145)=15\times 159=2385$

Example 6. Rashid deposits Tk. 1200 from his salary in the first month and every subsequent month, he deposits Tk. 100 more than the previous month.

1) Express the aforesaid problem as a series upto n terms.
2) How much does he deposit in the 18 th month and how much does he deposit in first 18 months?
3) In how many years does he deposit a total of Tk. 106200?

Solution:

1) As per the question, first term of the arithmetic series, $a = 1200$, common difference $d = 100$

   $\therefore$ second term $= 1200 + 100 = 1300$

   Third term $= 1300 + 100 = 1400$

   n-th term $= a + (n-1)d = 1200 + (n-1)\,100 = 1100 + 100\text{n}$

   $\therefore$ The series upto n is $1200 + 1300 + 1400 + \cdots$ upto n-terms

2) We know, n-th term $= a + (n-1)d$

   $\therefore$ deposit in the 18 th month $= a + (18-1)d = 1200 + 17 \times 100 = 2900$ Tk.

   Again, summation of first n-terms $= \dfrac{n}{2}\{2a + (n-1)d\}$

   $\therefore$ deposit in first 18 months $= \dfrac{18}{2}\{2 \times 1200 + (18-1) \times 100\}$ Tk.

   $= 9(2400 + 1700) = 36900$ Tk.

3) Let, he deposits 106200 Tk. in n months.

   According to the question, $\dfrac{n}{2}\{2a + (n-1)d\} = 106200$

   or, $\dfrac{n}{2}\{2 \times 1200 + (n-1) \times 100\} = 106200$

   or, $n(2400 + 100n - 100) = 212400$

   or, $100n^2 + 2300n - 212400 = 0$

   or, $n^2 + 23n - 2124 = 0$

   or, $n^2 + 59n - 36n - 2124 = 0$

   or, $(n+59)(n-36) = 0$

   So, $n = -59$ or, $n = 36$

   Number of months cannot be negative.

   $\therefore$ The required time: 36 months or 3 years.

Exercise 13.1

1. What is the number of terms of the series $13 + 20 + 27 + 34 + \cdots + 111$?
   1) 10 2) 13 3) 15 4) 20

2. The series $5 + 8 + 11 + 14 + \cdots + 62$ –
   (*i*) is a finite series (*ii*) is a geometric series
   (*iii*) 19th term of the series is 59

   which one of the following is correct?
   1) *i* and *ii* 2) *i* and *iii* 3) *ii* and *iii* 4) *i*, *ii* and *iii*

   Based on the following information answer questions 3 – 4.

$7 + 13 + 19 + 25 + \cdots$ is a series.

3. Which one is the 15th term of the series?
   1) 85 2) 91 3) 97 4) 104

4. What is the summation of first 20 terms of the series?
   1) 141 2) 1210 3) 1280 4) 2560

5. Find common difference and the 12th term of the series $2 - 5 - 12 - 19 - \cdots$.

6. Which term of the series $8 + 11 + 14 + 17 + \cdots$ is 392?

7. Which term of the series $4 + 7 + 10 + 13 + \cdots$ is 301?

8. If the mth term of an arithmetic series is n and nth term is m, what is the $(m + n)$th term of that series?

9. What is the sum of first n terms of the series $1 + 3 + 5 + 7 + \cdots$?

10. What is the sum of first 9 terms of the series $8 + 16 + 24 + \cdots$?

11. $5 + 11 + 17 + 23 + \cdots + 59 =$ what?

12. $29 + 25 + 21 + \cdots - 23 =$ what?

13. If the 12th term of an arithmetic series is 77, what is the summation of the first 23 terms of that series?

14. If the 16th term of an arithmetic series is -20, what will be the sum of the first 31 terms?

15. The sum of the first n terms of the series $9+7+5+\cdots$ is -144. Find the value of n.

16. The sum of the first n terms of the series $2+4+6+8+\cdots$ is 2550. Find the value of n.

17. If the sum of the first n terms of a series is $n(n+1)$, find the series.

18. If the sum of the first n terms of a series is $n(n+1)$, what is the sum of the first 10 terms?

19. If the sum of 12 terms of an arithmetic series is 144 and the first 20 terms is 560, find the sum of the first 6 terms.

20. The sum of the first m terms of an arithmetic series is n and the first n terms is m. Find the sum of the first $(m+n)$ terms.

21. If the pth, qth and rth term of an arithmetic series are a, b, c respectively, show that $a(q-r)+b(r-p)+c(p-q)=0$.

22. Show that, $1+3+5+7+\cdots+125=169+171+173+\cdots+209$.

23. A man agrees to refund the loan of Tk. 2500 in some installments. Each installment is Tk.2 more than the previous installment. If the first installment is Tk.1, in how many installments will the man be able to refund that amount?

24. The lth term of an arithmetic series is l^2 and the kth term is k^2.
    1) Construct two equations according to the information of the stem considering a as the first term of the series and d as common difference.
    2) Find the $(l+k)$ th term.
    3) Prove that summation of the first $(l+k)$ terms of the series is $\frac{l+k}{2}(l^2+k^2+l+k)$.

Formulas of series

Determination of the sum of squares of the first n natural numbers

Let S_n be the sum of squares of the first n natural numbers.

i.e, $S_n = 1^2 + 2^2 + 3^2 + \cdots + n^2$

we know,

$r^3 - 3r^2 + 3r - 1 = (r-1)^3$

or, $r^3 - (r-1)^3 = 3r^2 - 3r + 1$

in the above identity, putting $r = 1, 2, 3, \cdots, n$ we get,

$1^3 - 0^3 = 3 \cdot 1^2 - 3 \cdot 1 + 1$

$2^3 - 1^3 = 3 \cdot 2^2 - 3 \cdot 2 + 1$

$3^3 - 2^3 = 3 \cdot 3^2 - 3 \cdot 3 + 1$

$\cdots \quad \cdots \quad \cdots$

$\cdots \quad \cdots \quad \cdots$

$n^3 - (n-1)^3 = 3 \cdot n^2 - 3 \cdot n + 1$

adding we get,

$n^3 - 0^3 = 3(1^2 + 2^2 + 3^2 + \cdots + n^2) - 3(1 + 2 + 3 + \cdots + n) + (1 + 1 + 1 + \cdots + 1)$

or, $n^3 = 3S_n - \dfrac{3n(n+1)}{2} + n \left[\because 1 + 2 + 3 + \cdots + n = \dfrac{n(n+1)}{2}\right]$

or, $3S_n = n^3 + \dfrac{3n(n+1)}{2} - n$

$= \dfrac{2n^3 + 3n^2 + 3n - 2n}{2} = \dfrac{2n^3 + 3n^2 + n}{2} = \dfrac{n(2n^2 + 3n + 1)}{2}$

$= \dfrac{n(2n^2 + 2n + n + 1)}{2} = \dfrac{n\{2n(n+1) + 1(n+1)\}}{2}$

or, $3S_n = \dfrac{n(n+1)(2n+1)}{2}$

$\therefore S_n = \dfrac{n(n+1)(2n+1)}{6}$

The sum of cubes of the first n natural numbers

Let S_n be the sum of cubes of the first n natural numbers.

That is, $S_n = 1^3 + 2^3 + 3^3 + \cdots + n^3$

We know that, $(r+1)^2 - (r-1)^2 = (r^2 + 2r + 1) - (r^2 - 2r + 1) = 4r$

or, $(r+1)^2 r^2 - r^2(r-1)^2 = 4r \cdot r^2 = 4r^3$ [Multiplying both the sides by r^2]

We get, in the above identity, putting, $r = 1, 2, 3, \cdots, n$

$2^2 \cdot 1^2 - 1^2 \cdot 0^2 = 4 \cdot 1^3$

$3^2 \cdot 2^2 - 2^2 \cdot 1^2 = 4 \cdot 2^3$

$4^2 \cdot 3^2 - 3^2 \cdot 2^2 = 4 \cdot 3^3$

$\cdots \; \cdots \; \cdots$

$\cdots \; \cdots \; \cdots$

$(n+1)^2 \cdot n^2 - n^2 \cdot (n-1)^2 = 4n^3$

Adding we get,

$(n+1)^2 \cdot n^2 - 1^2 \cdot 0^2 = 4(1^3 + 2^3 + 3^3 + \cdots + n^3)$

or, $(n+1)^2 \cdot n^2 = 4S_n$

or, $S_n = \dfrac{n^2(n+1)^2}{4}$

$\therefore S_n = \left\{\dfrac{n(n+1)}{2}\right\}^2$

Necessary formulae

1. $1 + 2 + 3 + \cdots + n = \dfrac{n(n+1)}{2}$
2. $1^2 + 2^2 + 3^2 + \cdots + n^2 = \dfrac{n(n+1)(2n+1)}{6}$
3. $1^3 + 2^3 + 3^3 + \cdots + n^3 = \left\{\dfrac{n(n+1)}{2}\right\}^2$

N.B: $1^3 + 2^3 + 3^3 + \cdots + n^3 = (1 + 2 + 3 + \cdots + n)^2$

Work:

1) Find the sum of the first n natural even numbers.
2) Find the sum of the squares of the first n natural odd numbers.

Geometric Series

If the ratio of any term and its antecedent term of any series is always equal i.e., if any term divided by its antecedent term, the quotient is always equal, the series is called a geometric series and the quotient is called common ratio. Such as, of the series $2 + 4 + 8 + 16 + 32$ the first term is 2, the second term is 4, the third term is 8, the fourth term is 16 and the fifth term is 32. Here,

the ratio of the second term to the first term $= \dfrac{4}{2} = 2$

the ratio of the third term to the second term $= \dfrac{8}{4} = 2$

the ratio of the fourth term to the third term $= \dfrac{16}{8} = 2$

the ratio of the fifth term to the fourth term $= \dfrac{32}{16} = 2$.

So, this series is a geometric series. In this series, the ratio of any term to its antecedent term is always equal. The common ratio of the mentioned series is 2. The numbers of terms of the series are finite. That is why the series is finite geometric series.

The geometric series is widely used in different areas of physical and biological science, in organizations like Banks and Life Insurance etc, and in different branches of technology.

If the numbers of terms are not fixed in a geometric series, it is called an infinite geometric series. The first term of a geometric series is generally expressed by a and common ratios by r. So by definition, if the first term is a, the second term is ar, the third term is ar^2 etc. Hence the series will be $a + ar + ar^2 + ar^3 + \cdots$.

Work: Write down the geometric series in the following cases:

1) The first term 4, common ratio 10
2) The first term 4, common ratio $\frac{1}{3}$
3) The first term 7, common ratio $\frac{1}{10}$
4) The first term 3, common ratio 1
5) The first term 1, common ratio $-\frac{1}{2}$
6) The first term 3, common ratio -1

General term of a Geometric series

Let the first term of a geometric series be a and common ratio be r. Then, of the series

First term $= a = ar^{1-1}$ Second term $= ar = ar^{2-1}$

Third term $= ar^2 = ar^{3-1}$ Fourth term $= ar^3 = ar^{4-1}$

$\cdots \quad \cdots \quad \cdots$

$\cdots \quad \cdots \quad \cdots$

nth term $= ar^{n-1}$

This nth term is called the general term of the geometric series. If the first term of a geometric series a and the common ratio r are known, any term of the series can be determined by putting $n = 1, 2, 3, \cdots$ etc. successively in the equation of the nth term.

Example 7. What is the 10th term of the series $2 + 4 + 8 + 16 + \cdots$?

Solution: The first term of the series $a = 2$, common ratio $r = \frac{4}{2} = 2$

$\therefore$ The given series is a geometric series.

We know, the nth term of a geometric series $= ar^{n-1}$

$\therefore$ 10th term of the series$= 2 \times 2^{10-1} = 2 \times 2^9 = 1024$

Example 8. What is the general term of the series $128 + 64 + 32 + \cdots$?

Solution: The first term of the series $a = 128$, common ratio $r = \frac{64}{128} = \frac{1}{2}$

$\therefore$ It is a geometric series.

We know, general term of geometric series $= ar^{n-1}$

So, the general term of the series $= 128 \times \left(\frac{1}{2}\right)^{n-1} = \frac{2^7}{2^{n-1}} = \frac{1}{2^{n-1-7}} = \frac{1}{2^{n-8}}$

Example 9. first and the second terms of a geometric series are 27 and 9. Find the 5th and the 10th terms of the series.

Solution: The first term of the given series $a = 27$, the second term $= 9$

$\therefore$ The common ratio $r = \frac{9}{27} = \frac{1}{3}$

$\therefore$ The fifth term $= ar^{5-1} = 27 \times \left(\frac{1}{3}\right)^4 = \frac{27 \times 1}{27 \times 3} = \frac{1}{3}$

and the tenth term $= ar^{10-1} = 27 \times \left(\frac{1}{3}\right)^9 = \frac{3^3}{3^3 \times 3^6} = \frac{1}{3^6} = \frac{1}{729}$

Determination of the sum of a Geometric series

Let the first term of the geometric series be a, common ratio r and number of terms n. If S_n is the sum of n terms,

$$S_n = a + ar + ar^2 + \cdots + ar^{n-2} + ar^{n-1} \ldots (1)$$

$$\text{and } r \cdot S_n = ar + ar^2 + ar^3 + \cdots + ar^{n-1} + ar^n \text{ [multiplying (1) by } r \text{]} \ldots (2)$$

Subtracting, $S_n - rS_n = a - ar^n$

or, $S_n(1 - r) = a(1 - r^n)$

$\therefore S_n = \frac{a(1 - r^n)}{1 - r}$, when $r < 1$

Again, subtracting (1) from (2),

$rS_n - S_n = ar^n - a$

or, $S_n(r - 1) = a(r^n - 1)$

$\therefore S_n = \frac{a(r^n - 1)}{r - 1}$, when $r > 1$

Observe: If common ratio is $r = 1$, each term $= a$
Here, in this case $S_n = a + a + a + \cdots$ upto $n = an$

Work: Mr Rahim employed a man from the first of April for taking his son to school and taking back home for a month. His wages were fixed to be – one paisa in first day, twice of the first day in second day i.e. two paisa, twice of the second day in the third day i.e. four paisa. If the wages were paid in this way, how much would he get after one month including holidays of the week ?

Example 10. What is the sum of the series $12 + 24 + 48 + \cdots + 768$?

Solution: The first term of the series is $a = 12$, common ratio $r = \frac{24}{12} = 2 > 1$

$\therefore$ it is a geometric series.

Let, the nth term of the series $= 768$

We know, the nth term $= ar^{n-1}$

$\therefore ar^{n-1} = 768$

or, $12 \times 2^{n-1} = 768$

or, $2^{n-1} = \frac{768}{12} = 64$

or, $2^{n-1} = 2^6$

or, $n - 1 = 6$

$\therefore n = 7$

Therefore, the sum of the series $= \frac{a(r^n - 1)}{(r - 1)}$, when $r > 1$

$= \frac{12(2^7 - 1)}{2 - 1} = 12 \times (128 - 1) = 12 \times 127 = 1524$

Example 11. Find the sum of the first eight terms of the series $1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots$.

Solution: The first term of the series $a = 1$, common ratio $r = \frac{\frac{1}{2}}{1} = \frac{1}{2} < 1$.

$\therefore$ It is a geometric series. Here the number of terms $n = 8$.

We know, sum of n terms of a geometric series

$S_n = \frac{a(1 - r^n)}{1 - r}$, when $r < 1$

Hence, sum of eight terms of the series is $S_8 = \frac{1 \times \left\{1 - \left(\frac{1}{2}\right)^8\right\}}{1 - \frac{1}{2}} = \frac{1 - \frac{1}{256}}{\frac{1}{2}}$

$$= 2\left(\frac{256-1}{256}\right) = \frac{255}{128} = 1\frac{127}{128}$$

Example 12. Palash Sarker joined the job in January 2005 at a yearly salary of Tk. 120000. His yearly increment is Tk. 5000. 10% of his salary is deducted every year for provident fund. At the end of year he deposits 12000 Tk. in a bank at a compound interest of 12%. He will retire from the job on December 31, 2030.

1) Which series done the basic salary of Palash Sarker follow? Write down that series.
2) How much salary (in Tk.) would he receive in his entire job life excluding the provident fund money?
3) What is the amount of total deposited money with interest in the bank by December 31, 2031?

Solution:

1) Basic salary of Palash Sarker follows arithmetic series.

   The first term of the series $a = 120000$ and common difference $= 5000$

   $\therefore$ The second term $= 120000 + 5000 = 125000$

   The third term $= 125000 + 5000 = 130000$

   $\therefore$ The series is, $120000 + 125000 + 130000 + \cdots$

2) The total amount of salary excluding the provident fund from January 2005 to December 31, 2030 i.e. $(2030 - 2005 + 1)$ or, 26 years is:

   $(120000 - 10\%$ of $120000) + (125000 - 10\%$ of $125000) + (130000 - 10\%$ of $130000) + \cdots$

   $= (120000 - 12000) + (125000 - 12500) + (130000 - 13000) + \cdots$

   $= 108000 + 112500 + 117000 + \cdots$

   In this cane it is an arithmetic series whose first term, $a = 108000$, common difference $d = 112500 - 108000 = 4500$ and number of terms $n = 26$

   $\therefore$ Total salary he receives in 26 years $= \frac{26}{2}\{2 \times 108000 + (26-1) \times 4500\}$ Tk.

   $= 13(216000 + 112500) = 13 \times 328500 = 4270500$ Tk.

3) Total time from 2005 to 2031 is $(2031 - 2005)$ or 26 years

   Deposit of 12000 Tk. after 1 year $12000\left(1 + \frac{12}{100}\right) = 12000 \times 1.12$ Tk.

Deposit of 12000 Tk. after 2 years $12000 \times (1.12)^2$ Tk.

Deposit of 12000 Tk. after 3 years $12000 \times (1.12)^3$ Tk.

$\therefore$ Total deposited amount after 26 years $= 12000 \times 1.12 + 12000 \times (1.12)^2 + \cdots$ upto 26 th term.

$= 12000\{1.12 + (1.12)^2 + \cdots + (1.12)^{26}\}$

$= 12000 \times 1.12 \times \dfrac{(1.12)^{26} - 1}{1.12 - 1} = 12000 \times 1.12 \times \dfrac{18.04}{0.12}$

$= 2020488$ Tk. (approx.)

Exercise 13.2

1. a, b, c and d are four consecutive terms of an arithmetic series. Which one of the following is true?

   1) $b = \dfrac{c+d}{2}$ 2) $a = \dfrac{b+c}{2}$ 3) $c = \dfrac{b+d}{2}$ 4) $d = \dfrac{a+c}{2}$

2. For $n \in N$

   (i) $\sum n = \dfrac{n^2+n}{2}$

   (ii) $\sum n^2 = \dfrac{1}{6}n(n+1)(n+2)$

   (iii) $\sum n = \dfrac{n^2(n^2+2n+1)}{4}$

   Which one of the following is true?

   1) i and ii 2) i and iii 3) ii and iii 4) i, ii and iii

   On the basis of the following series, answer questions 3 and 4.

   $\log 2 + \log 4 + \log 8 + \cdots$

3. What is the common difference of the series?

   1) 2 2) 4 3) log2 4) 2log2

4. Which one is the 7th term of the series?

   1) log32 2) log64 3) log128 4) log256

5. Find the eighth term of the series $64 + 32 + 16 + 8 + \cdots$.

6. Find the sum of the first 14 terms of the series $3 + 9 + 27 + \cdots$.

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7. Which term of the series $128 + 64 + 32 + \cdots$ is $\dfrac{1}{2}$?

8. If $\dfrac{2\sqrt{3}}{9}$ is the fifth and $\dfrac{8\sqrt{2}}{81}$ is the tenth term of a geometric series, find its third term.

9. Which term of the series $\dfrac{1}{\sqrt{2}} - 1 + \sqrt{2} - \cdots$ is $8\sqrt{2}$?

10. If $5 + x + y + 135$ is a geometric series, find the value of x and y.

11. If $3 + x + y + z + 243$ is a geometric series, find the value of x, y and z.

12. What is the sum of the first 7 terms of the series $2 - 4 + 8 - 16 + \cdots$?

13. Find the sum of $(2n + 1)$ terms of the series $1 - 1 + 1 - 1 + \cdots$.

14. What is the sum of the first 7 terms of the series $\log 2 + \log 4 + \log 8 + \cdots$?

15. What is the sum of the first 12 terms of the series $\log 2 + \log 16 + \log 512 + \cdots$?

16. If the sum of n terms of the series $2 + 4 + 8 + 16 + \cdots$ is 254, find the value of n.

17. What is the sum of $(2n + 2)$ terms of the series $2 - 2 + 2 - 2 + \cdots$?

18. If the sum of cubes of n natural numbers is 441, find the value of n and find the sum of those n terms.

19. If the sum of cubes of n natural numbers is 225, find the value of n and find the sum of squares of those n terms.

20. Show that, $1^3 + 2^3 + 3^3 + \cdots + 10^3 = (1 + 2 + 3 + \cdots + 10)^2$

21. If $\dfrac{1^3 + 2^3 + 3^3 + \cdots + n^3}{1 + 2 + 3 + \cdots + n} = 210$, what is the value of n?

22. An iron-bar with length of one metre is divided into ten-pieces such that the lengths of the pieces form a geometric progression. If the largest piece is ten times than that of the smallest one, find the length in approximate millimetre of the smallest piece.

23. The first term of a geometric series is a, common ratio r, the fourth term of the series is -2 and the ninth term is $8\sqrt{2}$.

    1) Express the above information by two equations.

    2) Find the 12th term of the series.

3) Find the series and then determine the sum of the first seven terms of the series.

24. The nth term of a series is $2n - 4$.

1) Find the series.

2) Find the 10 th term of the series and determine the sum of the first 20 terms.

3) Considering the first term of the obtained series as the 1st term and the common difference as common ratio, construct a new series and find the sum of the first 8 terms of the series by applying the formula.

25. An S.S.C. examinee gets his result at 1 : 15 p.m. At 1 : 20 p.m. 8 students get their results, at 1 : 25 p.m 27 students get theirs.

1) As per the stem write down the two patterns.

2) How many students will know their results at exactly 2 : 10 p.m.? How many students would be knowing their results by 2 : 10 p.m.?

3) When will 6175225 students get their results?

Chapter 14

Ratio, Similarity and Symmetry

For comparing two quantities, their ratios are to be considered. Again, for determining ratios, the two quantities are to be measured in the same units. In algebra we have discussed this in detail.

At the end of this chapter, the students will be able to —

- explain geometric ratios.
- explain the internal division of a line segment.
- verify and prove theorems related to ratios.
- verify and prove theorems related to similarity.
- explain the concepts of symmetry.
- verify line and rotational symmetry of real objects practically.

Properties of Ratio and Proportion

(*i*) If $a : b = x : y$ and $c : d = x : y,$ it follows that, $a : b = c : d$.

(*ii*) If $a : b = b : a,$ it follows that, $a = b$

(*iii*) If $a : b = x : y,$ it follows that, $b : a = y : x$ (inversendo).

(*iv*) If $a : b = x : y,$ it follows that, $a : x = b : y$ (alternendo).

(*v*) If $a : b = c : d,$ it follows that, $ad = bc$ (cross-multiplication)

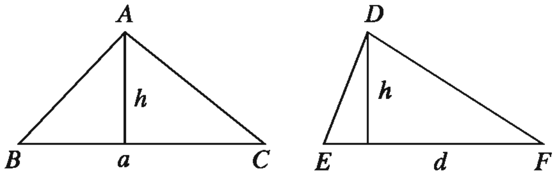
(*vi*) If $a : b = x : y,$ it follows that, $a + b : b = x + y : y$ (componendo) and $a - b : b = x - y : y$ (dividendo)

(*vii*) If $\frac{a}{b} = \frac{c}{d}$, it follows that $\frac{a+b}{a-b} = \frac{c+d}{c-d}$ (componendo and dividendo).

Geometric proportions

We have learnt to find the area of a triangular region. Two necessary concepts of ratio are to be formed from this.

1. If the heights of two triangles are equal, their bases and areas are proportional.



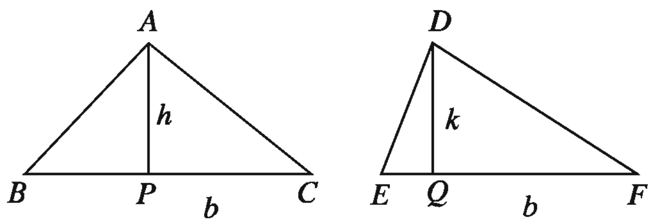
Let the bases of the triangles ABC and DEF be $BC = a$, $EF = d$ respectively and the height in both cases be h.

Hence, the area of the triangle $ABC = \frac{1}{2} \times a \times h$, the area of the triangle DEF $= \frac{1}{2} \times d \times h$

Therefore, the area of the triangle ABC: area of the triangle DEF

$= \frac{1}{2} \times a \times h : \frac{1}{2} \times d \times h = a : d = BC : EF$

2. If the bases of two triangles are equal, their heights and areas are proportional.



Let the heights of the triangles ABC and DEF be $AP = h$, $DQ = k$ respectively and the base in both cases be b.

Hence, the area of the triangle $ABC = \frac{1}{2} \times b \times h$, and the area of the triangle $DEF = \frac{1}{2} \times b \times k$

Therefore, the area of the triangle ABC: area of the triangle DEF

$= \frac{1}{2} \times b \times h : \frac{1}{2} \times b \times k = h : k = AP : DQ$

Theorem 28. A straight line drawn parallel to one side of a triangle intersects the other two sides or those sides produced proportionally.

Special Nomination: In the figure, the straight line DE is parallel to the side BC of the triangle ABC. DE intersects AB and AC (figure-1) or their produced sections (figure-2) at D and E respectively. It is required to prove that $AD : DB = AE : EC$

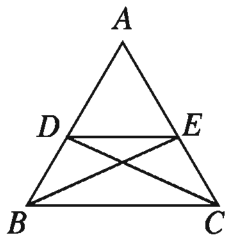


Figure 1

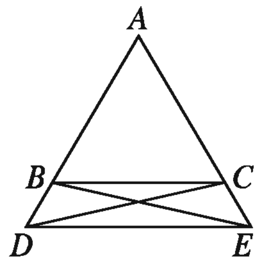


Figure 2

Drawing: Join B, E and C, D.

Proof:

Step 1. The heights of $\triangle ADE$ and $\triangle BDE$ are equal.

$\therefore \dfrac{\triangle ADE}{\triangle BDE} = \dfrac{AD}{DB}$ [The bases of the triangles of equal height are proportional]

Step 2. The heights of $\triangle ADE$ and $\triangle DEC$ are equal

$\therefore \dfrac{\triangle ADE}{\triangle DEC} = \dfrac{AE}{EC}$ [The bases of the triangles of equal height are proportional]

Step 3. But $\triangle BDE = \triangle DEC$ [On the same base DE and between same pair of lines]

$\therefore \dfrac{\triangle ADE}{\triangle BDE} = \dfrac{\triangle ADE}{\triangle DEC}$

Step 4. Therefore, $\dfrac{AD}{DB} = \dfrac{AE}{EC}$

i.e, $AD : DB = AE : EC$

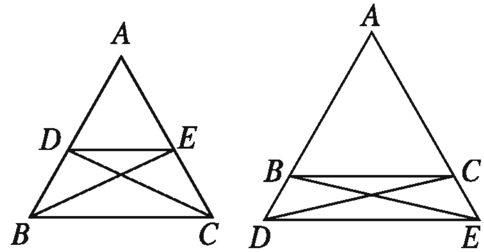
Corollary 1. If the line parallel to BC of the triangle ABC intersects the sides AB and AC at D and E respectively, then $\dfrac{AB}{AD} = \dfrac{AC}{AE}$ and $\dfrac{AB}{BD} = \dfrac{AC}{CE}$.

Corollary 2. The line through the mid point of a side of a triangle parallel to another side bisects the third line.

The proposition opposite of theorem 28 is also true. That is if a line segment divides the two sides of a triangle or the line produced proportionally, it is parallel to the third side. Here follows the proof of the theorem.

Theorem 29. If a line segment divides the two sides or their produced sections of a triangle proportionally, it is parallel to the third side.

Special Nomination: In the triangle ABC the line segment DE divides the two sides AB and AC or their produced sections proportionally. That is, $AD : DB = AE : EC$. It is required to prove that DE and BC are parallel.



Drawing: Join B, E and C, D.

Proof:

Step 1. $\dfrac{\triangle ADE}{\triangle BDE} = \dfrac{AD}{DB}$ [Triangles with equal height]

and $\dfrac{\triangle ADE}{\triangle DEC} = \dfrac{AE}{EC}$ [Triangles with equal height]

Step 2. But $\dfrac{AD}{DB} = \dfrac{AE}{EC}$ [Given]

Step 3. Therefore, $\dfrac{\triangle ADE}{\triangle BDE} = \dfrac{\triangle ADE}{\triangle DEC}$ [from (i) and (ii)]

$\therefore \triangle BDE = \triangle DEC$

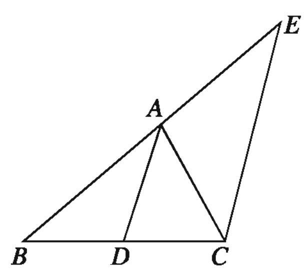
Step 4. But $\triangle BDE$ and $\triangle DEC$ are on the same side of the common base DE. So they lie between a pair of parallel lines.

$\therefore$ BC and DE are parallel.

Theorem 30. The internal bisector of an angle of a triangle divides its opposite side in the ratio of the sides constituting to the angle.

Special Nomination: In $\triangle ABC$ the line segment AD bisects the internal angle $\angle A$ and intersects the side BC at D. It is required to prove that $BD : DC = BA : AC$.

Drawing: Draw the line segment CE parallel to DA, so that it intersects the side BA produced at E.



Proof:

Step 1. Since, $DA \parallel CE$ and BE is their transversal [by construction]

$\therefore \angle AEC = \angle BAD$ [corresponding angles]

Again, $DA \parallel CE$ and AC is their transversal $\therefore \angle ACE = \angle CAD$ [corresponding angles]

Step 2. But $\angle BAD = \angle CAD$ [supposition]

$\therefore \angle AEC = \angle ACE$ Hence, $AC = AE$ [Chapter 6 Theorem 8]

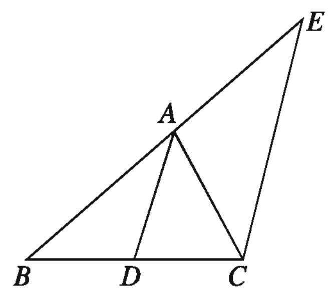
Step 3. Again, since $DA \parallel CE$ Therefore, $\dfrac{BD}{DC} = \dfrac{BA}{AE}$ [step 2]

Step 4. But $AE = AC$

$\therefore \dfrac{BD}{DC} = \dfrac{BA}{AC}$

Theorem 31. If any side of a triangle is divided internally, the line segment from the point of division to the opposite vertex bisects the angle at the vertex.

Special Nomination: Let ABC be a triangle and the line segment AD from vertex A divides the side BC at D such that $BD : DC = BA : AC$. It is required to prove that AD bisects $\angle BAC$ i.e. $\angle BAD = \angle CAD$



Drawing: Draw at C the line segment CE parallel to DA, so that it intersects the side BA produced at E.

Proof:

Step 1. For $\triangle BCE$ $DA \parallel CE$ [by construction]

$\therefore BA : AE = BD : DC$ [theorem 28]

Step 2. But $BD : DC = BA : AC$ [supposition]

$\therefore BA : AE = BA : AC$ [from step 1 and step 2]

$\therefore AE = AC$

Therefore, $\angle ACE = \angle AEC$ [base angles of isosceles triangle are equal]

Step 3. But $\angle AEC = \angle BAD$ [corresponding angles]

and $\angle ACE = \angle CAD$ [alternate angles]

Therefore, $\angle BAD = \angle CAD$ [from step 2]

$\therefore$ the line segment AD bisects $\angle BAC$.

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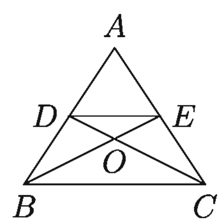
Exercise 14.1

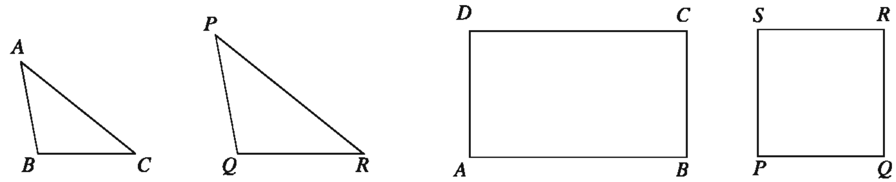
1. The bisectors of two base angles of a triangle intersect the opposite sides at X and Y respectively. If XY is parallel to the base, prove that the triangle is an isosceles triangle.
2. Prove that if two lines intersect a few parallel lines, the matching sides are proportional.
3. Prove that the diagonals of a trapezium are divided in the same ratio at their point of intersection.
4. Prove that the line segment joining the mid points of oblique sides of a trapezium and two parallel sides are parallel.
5. The medians AD and BE of the triangle ABC intersect each other at G. A line segment is drawn through G parallel to DE which intersects AC at F. Prove that $AC = 6EF$.
6. In the triangle ABC, X is any point on BC and O is a point on AX. Prove that $\triangle AOB : \triangle AOC = BX : XC$
7. In the triangle ABC, the bisector of A intersects BC at D. A line segment drawn parallel to BC intersects AB and AC at E and F respectively. Prove that $BD : DC = BE : CF$.
8. If the heights of the equiangular triangles ABC and DEF are AM and DN respectively, prove that $AM : DN = AB : DE$.
9. In the adjacent figure $BC \parallel DE$
   1) Prove that, $\triangle BOC$ and $\triangle DOE$ are similar.
   2) Prove that, $AD : BD = AE : CE$.
   3) Prove that, $BO : OE = CO : OD$



Similarity

The congruence and similarity of triangles have been discussed earlier in class VII. In general, congruence is a special case of similarity. If two figures are congruent, they are similar; but two similar triangles are not always congruent.

Equiangular Polygons: If the angles of two polygons with equal number of sides are sequentially equal, the polygons are known as equiangular polygons.



Similar Polygons: If the vertices of two polygons with equal number of sides can be matched in such a sequential way that

(*i*) The matching angles are equal and (*ii*) The ratios of matching sides are equal, then the two polygons are called similar polygons.

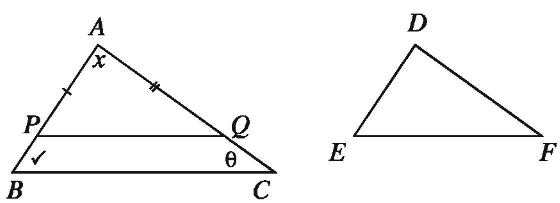
In the above figures, the rectangle $ABCD$ and the square $PQRS$ are equiangular since the number of sides in both the figures is 4 and the angles of the rectangle are sequentially equal to the angles of the square (all right angles). Though the similar angles of the figure are equal, the ratios of the matching sides are not the same. Hence the figures are not similar. In case of triangles, situation like this does not arise. As a result of matching the vertices of triangles, if one of the conditions of similarity is true, the other condition automatically becomes true and the triangles are similar. That is, similar triangles are always equiangular and equiangular triangles are always similar.

If two triangles are equiangular and one of their matching pairs is equal, the triangles are congruent. The ratio of the matching sides of two equiangular triangles is a constant. Proofs of the related theorems are given below.

Theorem 32. If two triangles are equiangular, their matching sides are proportional.

Special Nomination: Let ABC and DEF be triangles with $\angle A = \angle D$, $\angle B = \angle E$ and $\angle C = \angle F$

We need to prove that, $\dfrac{AB}{DE} = \dfrac{AC}{DF} = \dfrac{BC}{EF}$



Drawing: Consider the matching sides of the triangles ABC and DEF unequal. Take two points P and Q on AB and AC respectively so that $AP = DE$ and $AQ = DF$. Join P and Q and complete the drawing.

Proof:

Step 1. In $\triangle APQ$ and $\triangle DEF$

$AP{=}DE, AQ{=}DF, \angle A = \angle D$ Therefore, $\triangle APQ \cong \triangle DEF$ [SAS theorem]

Therefore, $\angle APQ = \angle DEF = \angle ABC$ and $\angle AQP = \angle DFE = \angle ACB$.

That is, the corresponding angles produced as a result of intersections of AB and AC by the line segment PQ are equal.

Therefore $PQ \parallel BC$ $\quad \therefore \dfrac{AB}{AP} = \dfrac{AC}{AQ}$ or, $\dfrac{AB}{DE} = \dfrac{AC}{DF}$ [corollary 1]

Step 2. Similarly, cutting line segments ED and EF from BA and BC respectively, it can be shown that,

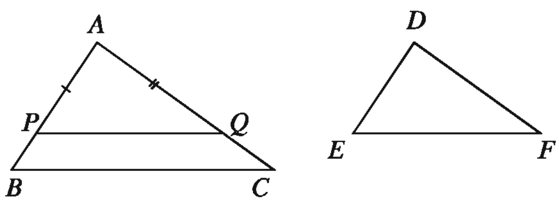
$\dfrac{BA}{ED} = \dfrac{BC}{EF}$ [theorem 28]

i.e. $\dfrac{AB}{DE} = \dfrac{BC}{EF}$ $\quad \therefore \dfrac{AB}{DE} = \dfrac{AC}{DF} = \dfrac{BC}{EF}$

The proposition opposite of theorem 32 is also true.

Theorem 33. If the sides of two triangles are proportional, the opposite angles of their matching sides are equal.

Special Nomination: Let, in $\triangle ABC$ and $\triangle DEF$, $\dfrac{AB}{DE} = \dfrac{AC}{DF} = \dfrac{BC}{EF}$. It is to prove that, $\angle A = \angle D$, $\angle B = \angle E$, $\angle C = \angle F$.



Drawing: Consider the matching sides of the triangles ABC and DEF unequal. Take two points P and Q on AB and AC respectively, so that $AP = DE$ and $AQ = DF$. Join P and Q.

Proof:

Since $\frac{AB}{DE} = \frac{AC}{DF}$, Therefore $\frac{AB}{AP} = \frac{AC}{AQ}$

Therefore $PQ \parallel BC$ [theorem 29]

$\therefore \angle ABC = \angle APQ$ [corresponding angles made by the transversal AB]

and $\angle ACB = \angle AQP$ [corresponding angles made by the transversal AC]

$\therefore \triangle ABC$ and $\triangle APQ$ are equiangular.

Therefore, $\frac{AB}{AP} = \frac{BC}{PQ}$ or, $\frac{AB}{DE} = \frac{BC}{PQ}$ [theorem 32]

But $\frac{AB}{DE} = \frac{BC}{EF}$ [supposition]

$\therefore \frac{BC}{EF} = \frac{BC}{PQ}$

$\therefore EF = PQ$

Therefore $\triangle APQ$ and $\triangle DEF$ are congruent. [SSS theorem]

$\therefore \angle PAQ = \angle EDF,\ \angle APQ = \angle DEF,\ \angle AQP = \angle DFE$

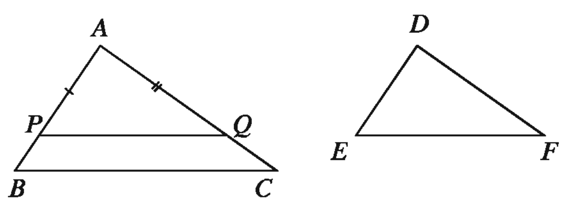
$\therefore \angle APQ = \angle ABC$ and $\angle AQP = \angle ACB$

$\angle A = \angle D,\ \angle B = \angle E, \angle C = \angle F$

Theorem 34. If one angle of a triangle is equal to an angle of the other and the sides adjacent to the equal angles are proportional, the triangles are similar.

Special Nomination: Let in $\triangle ABC$ and $\triangle DEF$, $\angle A = \angle D$ and $\frac{AB}{DE} = \frac{AC}{DF}$.

It is to be proved that, $\triangle ABC$ and $\triangle DEF$ are similar.



Drawing: Consider the matching sides of ABC and DEF unequal. Take two points P and Q on AB and AC respectively so that $AP = DE$ and $AQ = DF$. Join P and Q.

Proof:

For $\triangle APQ$ and $\triangle DEF$, $AP = DE,\ AQ = DF$ and internal $\angle A =$ internal $\angle D$

$\therefore \triangle APQ \cong \triangle DEF$ [SAS theorem]

$\therefore \angle A = \angle D,\ \angle APQ = \angle E,\ \angle AQP = \angle F$

Again, since $\dfrac{AB}{DE} = \dfrac{AC}{DF}$, therefore $\dfrac{AB}{AP} = \dfrac{AC}{AQ}$ [theorem 29]

$\therefore PQ \parallel BC$

Therefore $\angle ABC = \angle APQ$ and $\angle ACB = \angle AQP$

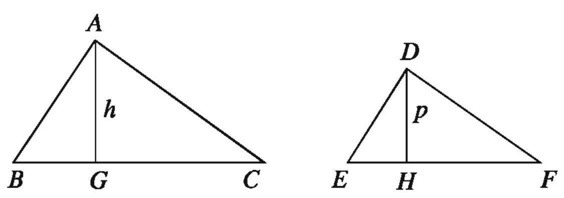
$\therefore\ \angle A = \angle D,\ \angle B = \angle E$, and $\angle C = \angle F$

i.e. $\triangle ABC$ and $\triangle DEF$ are equiangular.

Therefore $\triangle ABC$ and $\triangle DEF$ are similar.

Theorem 35. The ratio of the areas of two similar triangles is equal to the ratio of squares on any two matching sides.

Special Nomination: Let the triangles ABC and DEF be similar and BC and EF be their matching sides respectively. It is required to prove that, $\triangle ABC : \triangle DEF = BC^2 : EF^2$



Drawing: Draw perpendiculars AG and DH on BC and EF respectively. Let $AG = h,\ DH = p$.

Proof:

Step 1. $\triangle ABC = \frac{1}{2} \times BC \times h$ and $\triangle DEF = \frac{1}{2} \times EF \times p$

$$\frac{\triangle ABC}{\triangle DEF} = \frac{\frac{1}{2} \times BC \times h}{\frac{1}{2} \times EF \times p} = \frac{h}{p} \times \frac{BC}{EF}$$

Step 2. But in the triangles ABG and DEH, $\angle B = \angle E$, $\angle AGB = \angle DHE$

[1 right angle]

$\therefore\ \angle BAG = \angle EDH$

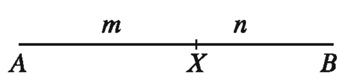
$\therefore \triangle ABC$ and $\triangle DEF$ are equiangular, so similar.

$\therefore \frac{h}{p} = \frac{AB}{DE} = \frac{BC}{EF}$ [as $\triangle ABC$ and $\triangle DEF$ are similar]

Step 3. $\frac{\triangle ABC}{\triangle DEF} = \frac{h}{p} \times \frac{BC}{EF} = \frac{BC}{EF} \times \frac{BC}{EF} = \frac{BC^2}{EF^2}$

Internal Division of a Line Segment in definite ratio

If A and B are two different points in a plane and m and n are two natural numbers, we acknowledge that there exists a unique point X lying between A and B and $AX : XB = m : n$.

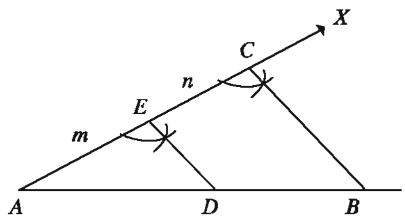


In the above figure, the line segment AB is divided at X internally in the ratio $m : n$ i.e. $AX : XB = m : n$.

Construction 12. To divide a given line segment internally in a given ratio.

Special Nomination: Let the line segment AB be divided internally in the ratio $m : n$.

Drawing: Let an angle BAX be drawn at A. From AX cut the lengths $AE = m$ and $EC = n$ sequentially. Join B, C. At E, draw line segment ED parallel to CB which intersects AB at D. Then the line segment AB is divided at D internally in the ratio $m : n$.



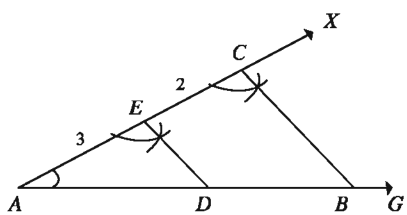
Proof: Since the line segment DE is parallel to a side BC of the triangle ABC.

$\therefore AD : DB = AE : EC = m : n$

Work: Divide a given line segment in definite ratio internally by an alternative method.

Example 1. Divide a line segment of length 7 c m internally in the ratio 3 : 2.

Solution: Draw any ray AG. From AG, cut a line segment $AB = 7$ c m. Draw an angle $\angle BAX$ at A. From AX, cut the lengths $AE = 3$ c.m. and $EC = 2$ c m from EX. Join B, C. At E, draw an $\angle AED$ equal to $\angle ACB$ whose side intersects AB at D. Then the line segment AB is divided at D internally in the ratio 3 : 2.



Work: Draw a triangle similar to a particular triangle whose sides are $\frac{3}{5}\times$ the sides of the given triangle.

Exercise 14.2

1. In $\triangle ABC$, if the line DE parallel to BC intersects AB and AC at D and E respectively, then-

   (i) $\triangle ABC$ and $\triangle ADE$ are similar.

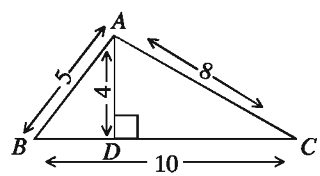
   (ii) $\dfrac{AD}{BD} = \dfrac{CE}{AE}$

   (iii) $\dfrac{\triangle ABC}{\triangle ADE} = \dfrac{BC^2}{DE^2}$

   Which one of the following is true?

   1) i and ii 2) i and iii 3) ii and iii 4) i, ii and iii

   Use the information from the adjacent figure to answer the questions 2 and 3:



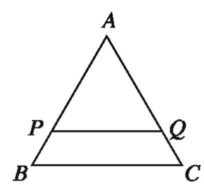
2. What is the ratio of the height and base of $\triangle ABC$?

   1) $\dfrac{1}{2}$ 2) $\dfrac{4}{5}$ 3) $\dfrac{2}{5}$ 4) $\dfrac{5}{4}$

3. What is the area of $\triangle ABD$ in square units?

   1) 6 2) 20 3) 40 4) 50

4. If in $\triangle ABC$, $PQ \parallel BC$, then which one of the following is true?



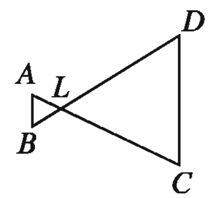
   1) $AP : PB = AQ : QC$
   2) $AB : PQ = AC : PQ$
   3) $AB : AC = PQ : BC$
   4) $PQ : BC = BP : BQ$

5. Prove that if each of the two triangles is similar to a third triangle, they are congruent to each other.

6. Prove that, if one acute angle of a right-angled triangle is equal to an acute angle of another right-angled triangle, the triangles are similar.

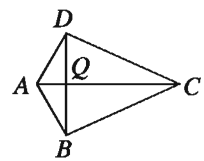
7. Prove that the two right-angled triangles formed by the perpendicular from the vertex containing the right angle are similar to each other and also to the original triangle.

8. In the adjacent figure, $\angle B = \angle D$ and $CD = 4AB$. Prove that, $BD = 5BL$.



9. A line segment drawn through the vertex A of the parallelogram $ABCD$ intersects the BC and DC at M and N respectively. Prove that $BM \times DN$ is a constant.

10. In the adjacent figure $BD \perp AC$ and $DQ = BQ = 2AQ = \frac{1}{2}QC$. Prove that, $DA \perp DC$.

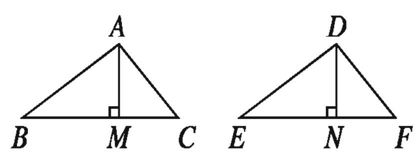


11. In the triangles $\triangle ABC$ and $\triangle DEF$, $\angle A = \angle D$. Prove that, $\triangle ABC : \triangle DEF = AB \cdot AC : DE \cdot DF$

12. The bisector AD of $\angle A$ of the triangle ABC intersects BC at D. The line segment CE parallel to DA intersects the line segment BA extended at E.

    1) Draw the figure according to the information.
    2) Prove that, $BD : DC = BA : AC$
    3) If a line segment parallel to BC intersect AB and AC at P and Q respectively, prove that $BD : DC = BP : CQ$.

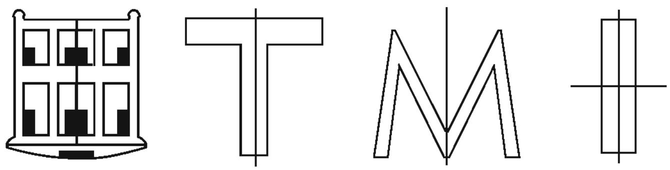
13. In the figure, ABC and DEF are two similar triangles.



    1) Name the matching sides and matching angles of the triangles.
    2) Prove that, $\dfrac{\triangle ABC}{\triangle DEF} = \dfrac{AB^2}{DE^2} = \dfrac{AC^2}{DF^2} = \dfrac{BC^2}{EF^2}$
    3) If $BC = 3$ c m, $EF = 8$ c m, $\angle B = 60°$, $\dfrac{BC}{AB} = \dfrac{3}{2}$ and area of $\triangle ABC$ is 3 square c m, then draw the triangle $\triangle DEF$ and find its area.

Symmetry

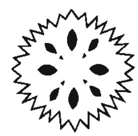
Symmetry is an important geometrical concept, commonly exhibited in nature and is used almost in every field of our activity. Artists, designers, architects, carpenters always make use of the idea of symmetry. Tree-leaves, flowers, beehives, houses, tables, chairs -everywhere we find symmetrical designs. A figure has line symmetry, if there is a line about which the figure may be folded so that the two parts of the figure will coincide.



Each of the above figures has the line of symmetry.

Work:

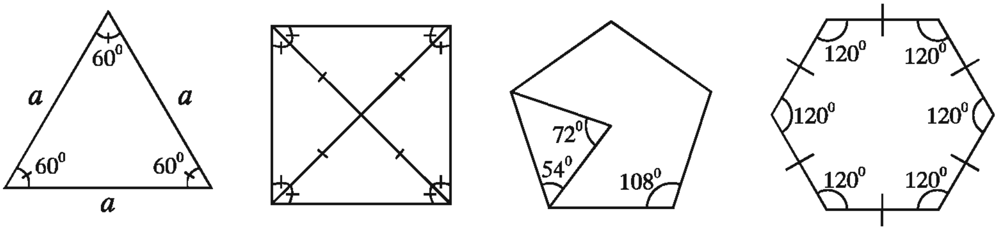
1) Sumi has made some paper-cut design as shown in the adjacent figure. In the figure, mark the lines of symmetry. How many lines of symmetry does the figure have?
2) Write and identify the letters in English alphabet having line symmetry. Also mark their line of symmetry.



Line of symmetry of a regular polygon

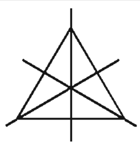
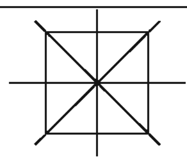
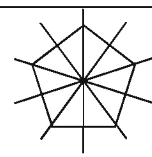
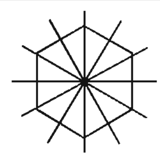
A polygon is a closed figure made of several line segments. A polygon is said to be regular if all its sides are of equal length and all its angles are equal. The triangle is a polygon made up of the least number of line segments. An equilateral triangle is a regular polygon of three sides. An equilateral triangle is regular because its sides as well as angles are equal. A square is the regular polygon of four sides.

The sides of a square are equal and each of the angles is equal to one right angle. Similarly, in regular pentagons and hexagons, the sides are equal and the angles are equal as well.

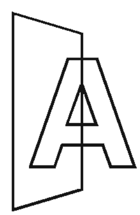


Equilateral triangle | Square | Regular pentagon | Regular hexagon

Each regular polygons is a figure of symmetry. Therefore, it is necessary to know their lines of symmetry. Each regular polygon has many lines of symmetry as it has many sides.

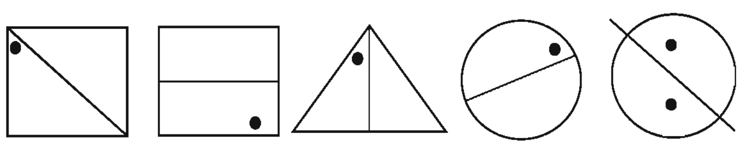
Three lines of symmetry	Four lines of symmetry	Five lines of symmetry	Six lines of symmetry
			
Equilateral triangle	Square	Regular pentagon	Regular hexagon

The concept of line symmetry is closely related to mirror reflection. A geometrical figure has line symmetry when one half of it is the mirror image of the other half. So, the line of symmetry is also called the reflection symmetry.



Work:

1) The line of symmetry is given, find the other hole.



2) Identify the lines of symmetry in the following geometrical figures.
   (1) An isosceles triangle (2) A scalene triangle
   (3) A square (4) A rhombus (5) A regular hexagon
   (6) A pentagon (7) A circle

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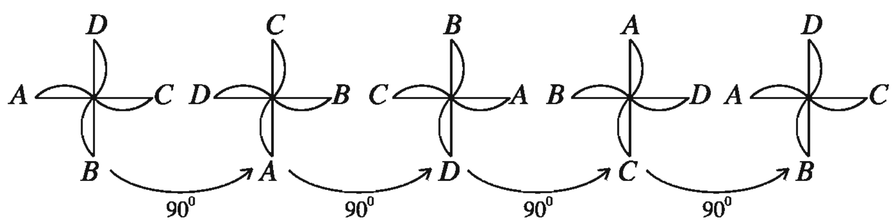
Rotational symmetry

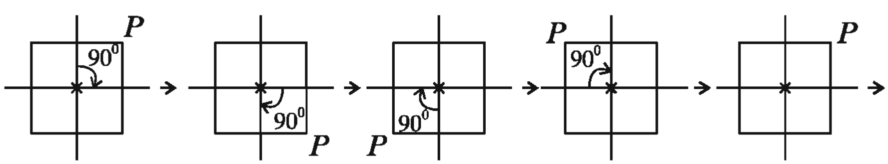
When an object rotates around any fixed point, its shape and size do not change. But the different parts of the object change their position. If the new position of the object after rotation becomes identical to the original position, we say the object has a rotational symmetry. The wheels of a bicycle, ceiling fan, square etc. are examples of objects having rotational symmetry. As a result of rotation the blades of the fan looks exactly the same as the original position more than once. The blades of a fan may rotate in the clockwise direction or in the anticlockwise direction. The wheels of a bicycle may rotate in the clockwise direction or in the anticlockwise direction. The rotation in the anticlockwise direction is considered the positive direction of rotation.

This fixed point around which the object rotates is the **centre of rotation**.The angle of turning during rotation is called the angle of rotation. A full-turn means rotation by 360°; a half-turn is rotation by 180°.

In the figure below, a fan with four blades rotating by 90° is shown in different positions. It is noted during a complete revolution in four positions (rotating about the angle by 90°, 180°, 270° and 360°), the fan looks exactly the same. For this reason, it is said that the rotational symmetry of the fan is order 4.



Here is one more example for rotational symmetry. Consider the intersection of two diagonals of a square the centre of rotation. In the quarter turn about the centre of the square, any diagonal position will be as like as the second figure. In this way, when you complete four quarter-turns, the square reaches its original position. It is said that a square has a rotational symmetry of order 4.

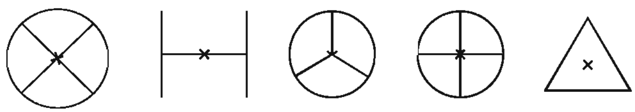


Observe also that every object occupies same position after one complete revolution. So every geometrical object has a rotational symmetry of order 1. For finding the rotational symmetry of an object, one needs to consider the following matter.

1) The centre of rotation
2) The angle of rotation
3) The direction of rotation
4) The order of rotational symmetry

Work:

1) Give examples of 5 plane objects from your surroundings which have rotational symmetry.
2) Find the order of rotational symmetry of the following figures.



(a) (b) (c) (d) (e)

Line symmetry and rotational symmetry

We have seen that some geometrical shapes have only line symmetry, some have only rotational symmetry and some have both line symmetry and rotational symmetry. For example, the square has four lines of symmetry as well rotational symmetry of order 4.

The circle is the most symmetrical figure, because it can be rotated around its centre through any angle. Therefore, it has unlimited order of rotational of symmetry. At the same time, every line through the centre forms a line of reflection symmetry and so it has unlimited number of lines of symmetry.

Work: Determine the line of symmetry and the rotational symmetry of the given alphabet and complete the table below:

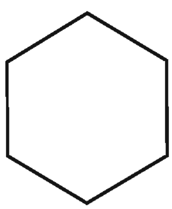
Letter	Line of symmetry	Number of lines of symmetry	Rotational symmetry	Order of rotational symmetry
Z	NO	0	YES	2
H				
O				
E				
C				

Exercise 14.3

1. In plane geometry:
   (*i*) The triangle is a polygon made up of the least number of line segments.
   (*ii*) A rhombus is the regular polygon of four sides
   (*iii*) Sides of a regular polygon are equal, but angles are not
   Which of the following is true?
   1) *i* 2) *i* and *ii* 3) *i* and *iii* 4) *i*, *ii* and *iii*

2. How many lines of symmetry does a scalene triangle have?
   1) 0 2) 1 3) 3 4) Countless

From the following figure, answer question 3 and 4.



The length of each side of the polygon is 6 c.m.

3. How many lines of symmetry does the polygon have?
   1) 3 2) 6 3) 7 4) Countless

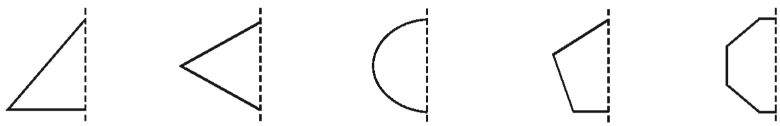
4. For the polygon-
   (*i*) Order of rotational symmetry is 4
   (*ii*) The angle of rotation is 60°
   (*iii*) Each angles in equal

   Which of the following is true?

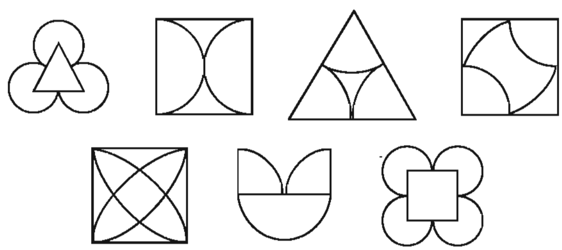
   1) *i* 2) *ii* 3) *ii* and *iii* 4) *i*, *ii* and *iii*

5. Which one of the followings has line of symmetry?
   1) Figure of a house 2) Figure of a mosque 3) Figure of a temple
   4) Figure of a church 5) Figure of a pagoda 6) Figure of a parliament
   7) Figure of a mask 8) Figure of the building of the Tajmahal.

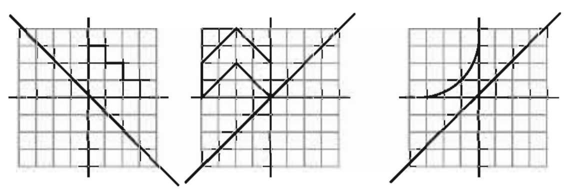
6. Lines of symmetry are given (dashed lines), complete the geometrical figures and identify:



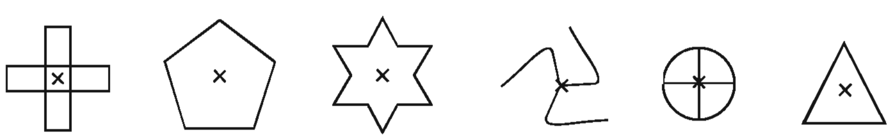
7. Find the lines of symmetry of the following figures:



8. Complete the following figures such that reflection symmetry is achieved:

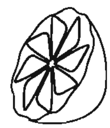


9. Find the rotational symmetry of the following figure:



(a) (b) (c) (d) (e) (f)

10. Draw those English letters that have symmetry with respect to–
    1) Horizontal mirror
    2) Vertical mirror
    3) Both horizontal and vertical mirrors
11. Draw three figures which do not have symmetry.
12. When you slice a lemon, the cross-section looks as shown in the figure. Determine the rotational symmetry of the figure.



13. Fill in the blanks:

Shape	Centre of Rotation	Order of Rotation	Angle of rotation
Square			
Rectangle			
Rhombus			
Equilateral triangle			
Semi-circle			
Regular pentagon			

14. Name the quadrilaterals which have line of symmetry and rotational symmetry of order more than 1.
15. Can we have a rotational symmetry of a body of order more than 1 whose angle of rotation is 18°? Justify your answer.

Chapter 15

Area Related Theorems and Constructions

We know that bounded plane figures may have different shapes. If the region is bounded by four sides, it is known as quadrilateral.Quadrilaterals have classification and they are also named based on their shapes and properties. Apart from these, there are many regions bounded by more than four sides. These are polygonal regions or simply polygons. Every closed region has a certain measurement which is called the area of the region. For measurement of areas usually the area of a square with sides of 1 unit of length is used as the unit area and their areas are expressed in square units. For example, the area of Bangladesh is 1.47 lacs square kilometers (approximately). In our day to day life we need to know and measure areas of polygons for meeting the necessity of life. So, it is important for the learners to have a comprehensive knowledge about areas of polygons. Areas of polygons and related theorems and constructions are presented here.

At the end of the chapter, the students will be able to —

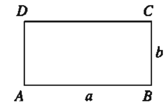
- explain the area of polygons
- verify and prove theorems related to areas.
- construct polygons and justify construction by using given data.
- construct a quadrilateral with area equal to the area of a triangle.
- construct a triangle with area equal to the area of a quadrilateral.

Area of a Plane Region

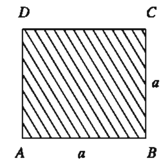
Every closed plane region has definite area. In order to measure such area, usually the area of a square having sides of unit length is taken as the unit. For example, the area of a square with a side of length 1 cm is 1 square centimere.

We know that,

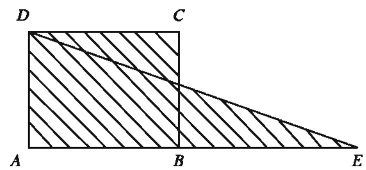
1) If the length of the rectangular region $ABCD$ $AB = a$ units (say, metre), breadth $BC = b$ units (say, metre), the area of the region $ABCD = ab$ square units (say, square metres).



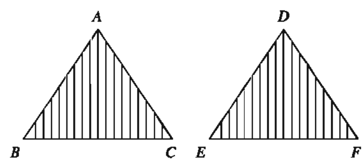
2) If the length of a side of the square region $ABCD$, $AB = a$ units (say, metre), the area of the region $ABCD = a^2$ square units (say, square metres).



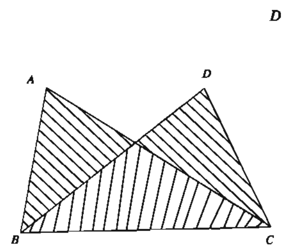
When the area of two regions are equal, the sign '=' is used between them. For example, in the figure the area of the rectangular region $ABCD$ = area of the triangular region AED, where $AB = BE$.



It should be mentioned that, if $\triangle ABC$ and $\triangle DEF$ are congruent, we write $\triangle ABC \cong \triangle DEF$. In this case, the area of the triangular region $\triangle ABC$ = area of the triangular region $\triangle DEF$.

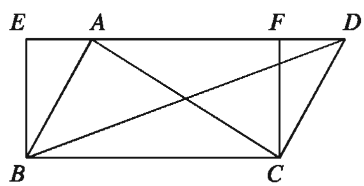


But, two triangles are not necessarily congruent when they have equal areas. For example, in the figure, area of $\triangle ABC$ = area of $\triangle DBC$ but $\triangle ABC$ and $\triangle DBC$ are not congruent



Theorem 36. Areas of all the triangular regions having same base and lying between the same pair of parallel lines are equal to one another.

Let, the triangular regions ABC and DBC stand on the same base BC and lie between the pair of parallel lines BC and AD. It is required to prove that, $\triangle$region $ABC = \triangle$ region DBC.



Drawing: At the points B and C of the line segment BC, draw perpendiculars BE and CF respectively. They intersect the line AD or extended AD at the points E and F respectively. As a result a rectangular region $EBCF$ is formed.

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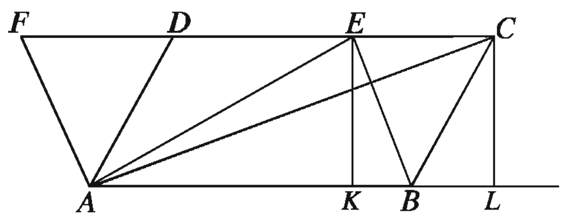
Proof: According to the construction, $EBCF$ is a rectangular region. The triangular region $\triangle ABC$ and rectangular region $EBCF$ stand on the same base BC and lie between the two parallel line segments BC and ED.

Hence, $\triangle$ region $ABC = \frac{1}{2} \times$ rectangular region $EBCF$

Similarly, $\triangle$ region $DBC = \frac{1}{2} \times$ rectangular region $EBCF$

$\therefore$ $\triangle$ region $ABC = \triangle$ region DBC (proved)

Theorem 37. Parallelograms lying on the same base and between the same pair of parallel lines are of equal area.



Let the parallelograms regions $ABCD$ and $ABEF$ stand on the same base AB and lie between the pair of parallel lines AB and FC. It is required to prove that, area of the parallelogram $ABCD$ = area of the parallelogram $ABEF$.

Drawing: Join A, C and A, E. From the points C and E, draw perpendiculars EK and CL to the base AB and extended AB respectively.

Proof: The area of $\triangle ABC = \frac{1}{2} \times AB \times CL$ and

The area of $\triangle ABE = \frac{1}{2} \times AB \times EK$

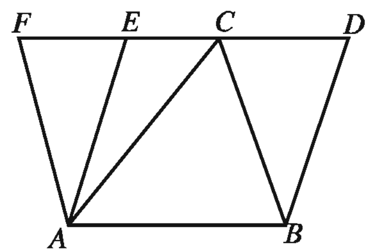
$\because CL = EK$, [by construction $AL \parallel FC$]

$\therefore$ The area of $\triangle ABC$ = The area of $\triangle ABE$

$\implies \frac{1}{2}$ area of the parallelogram $ABCD = \frac{1}{2}$ area of the parallelogram $ABEF$.

$\therefore$ Area of the parallelogram $ABCD$ = area of the parallelogram $ABEF$. (proved)

Theorem 38. Area of a triangle is exactly half of the area of a parallelogram lying on the same base and between the same pair of parallel lines of the triangle.



Let, $\triangle ABC$ and parallelogram $ABDE$ stand on the same base AB and lie between the pair of parallel lines AB and ED. It is required to prove that, $\triangle ABC = \frac{1}{2}$ parallelogram $ABDE$

Drawing: Through A draw the straight line AF parallel to BC which intersects the line DC extended at the point F.

Proof:

1. $AF \parallel BC$ [by construction] and $AB \parallel FC$ [by supposition]

   $\therefore ABCF$ is a parallelogram.

2. Parallelograms $ABDE$ and $ABCF$ stand on the same base AB and lie between the two parallel line segments AB and FD

   $\therefore$ parallelogram $ABDE$ = parallelogram $ABCF$ [theorem 37]

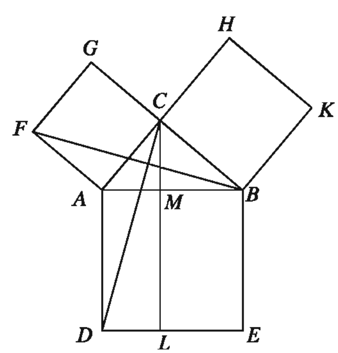
3. AC is a diagonal of parallelogram $ABCF$

   $\therefore \triangle ABC = \frac{1}{2}$ parallelogram $ABCF = \frac{1}{2}$ parallelogram $ABDE$ (proved)

Corollary 1. If a triangle and a parallelogram lie on bases with equal length and between same pair of parallel lines, the area of the triangle is equal to exactly half of the area of the parallelogram.

Theorem 39. Pythagoras Theorem

In a right-angled triangle, the area of the square drawn on the hypotenuse is equal to the sum of the areas of the squares drawn on the other two sides.



Special Nomination: Let ABC be a right-angled triangle in which $\angle ACB$ is a right angle and hypotenuse is AB. It is to be proved that, $AB^2 = BC^2 + AC^2$.

Drawing: Draw three squares $ABED$, $ACGF$ and $BCHK$ on the external sides of AB, AC and BC respectively. Through C, draw the line segment CL parallel to AD or BE. Let CL intersects AB at M and DE at L respectively. Join C, D and B, F.

Proof:

Step 1. In $\triangle CAD$ and $\triangle BAF$, $CA = AF$, $AD = AB$ and included $\angle CAD = \angle CAB + \angle BAD = \angle CAB + \angle CAF =$ included $\angle BAF$ [$\angle BAD = \angle CAF =$ 1 right angle]

Therefore, $\triangle CAD \cong \triangle BAF$

Step 2. Triangle $\triangle CAD$ and rectangular region $ADLM$ lie on the same base AD and between the parallel lines AD and CL. Therefore, Rectangular region $ADLM = 2\triangle CAD$ [Theorem 37]

Step 3. $\triangle BAF$ and the square $ACGF$ lie on the same base AF and between the parallel lines AF and BG. Hence, square region $ACGF = 2\triangle FAB = 2\triangle CAD$ [Theorem 37]

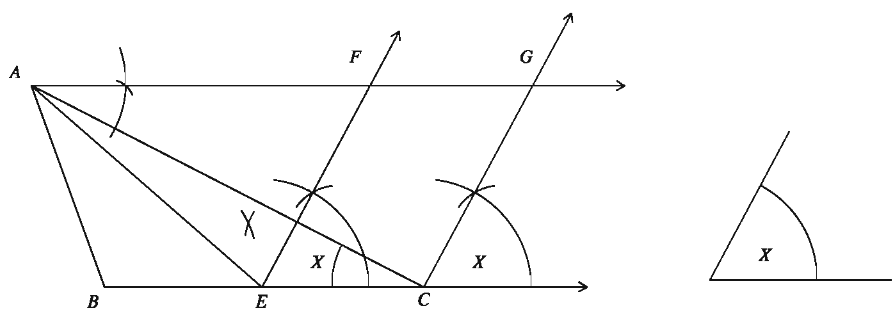
Step 4. Rectangular region $ADLM =$ square region $ACGF$.

Step 5. Similarly joining C, E and A, K it can be proved that rectangular region $BELM =$ square region $BCHK$.

Step 6. Rectangular region $(ADLM + BELM) =$ square region $ACGF+$ square region $BCHK$. or, square region $ABED =$ square region $ACGF+$ square region $BCHK$.

That is, $AB^2 = BC^2 + AC^2$ (Proved)

Construction 13. Construct a parallelogram with an angle equal to a definite angle and area equal to that of a triangular region.



Let ABC be a triangular region and $\angle x$ be a definite angle. It is required to construct a parallelogram with angle equal to $\angle x$ and area equal to the area of the triangular region ABC.

Drawing: Bisect the line segment BC at E. At the point E of the line segment EC, draw CEF equal to $\angle x$. Through A, draw AG parallel to BC which intersects the ray EF at F. Again, through C, draw the ray CG parallel to EF which intersects the ray AG at G. Hence, $ECGF$ is the required parallelogram.

Proof: Join A, E .

Now, area of $\triangle ABE$ = area of $\triangle AEC$[since base BE = base EC and heights of both the triangles are equal].

$\therefore$ area of $\triangle ABC = 2$ (area of $\triangle AEC$).

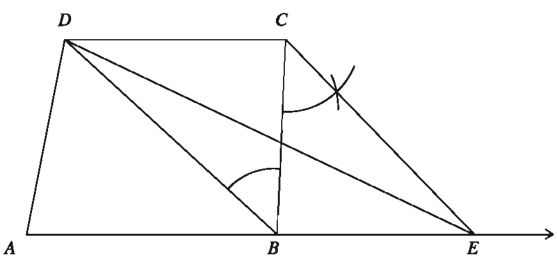
Again, area of the parallelogram $ECGF = 2$ (area of $\triangle AEC$) [since both lie on the same base EC and $EC \parallel AG$).

$\therefore$ area of the parallelogram region $ECGF$ = area of $\triangle ABC$.

Again $\angle CEF = \angle x$[since $EF || CB$ by construction].

So the parallelogram $ECGF$ is the required parallelogram.

Construction 14. Construct a triangle with area of the triangular region equal to that of a quadrilateral region.



Let $ABCD$ be a quadrilateral region. It is required to construct a triangle such that area of the triangular region is equal to that of a rectangular region $ABCD$.

Drawing: Join D, B. Through C, draw CE parallel to DB which intersects the side AB extended at E. Join D, E. Then, $\triangle DAE$ is the required triangle.

Proof: $\triangle BDC$ and $\triangle BDE$ lie on the same base BD and $CE \parallel DB$ (by construction).

$\therefore$ area of $\triangle BDC$ = area of $\triangle BDE$.

$\therefore$ area of $\triangle BDC+$ area of $\triangle ABD$ = area of $\triangle BDE+$ area of $\triangle ABD$

$\therefore$ area of the quadrilateral region $ABCD$ = area of $\triangle ADE$. Therefore, $\triangle ADE$ is the required triangle.

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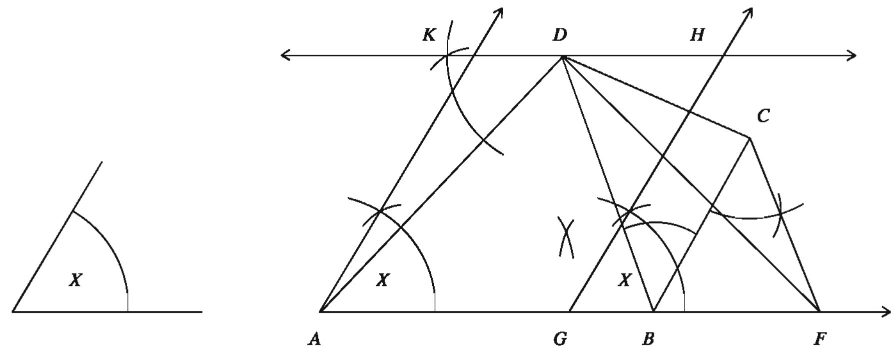
Nota Bene: Applying the above mentioned method innumerable numbers of triangles can be drawn whose area is equal to the area of a given quadrilateral region.

Construction 15. Construct a parallelogram, with a given angle and the area of the bounded region equal to that of a quadrilateral region.



Let $ABCD$ be a quadrilateral region and $\angle x$ be a definite angle. It is required to construct a parallelogram with angle $\angle x$ and the area equal to area of the quadrilateral region $ABCD$.

Drawing: Join B, D. Through C, draw CF parallel to DB which intersects the side AB extended at F. Find the midpoint G of the line segment AF. At A of the line segment AG, draw GAK equal to $\angle x$ and draw $AK \parallel GH$ through G. Again, draw $KDH \parallel AG$ through D and let, KDH intersects AK and GH at K and H respectively. Hence $AGHK$ is the required parallelogram.

Proof: Join D, F. By construction $AGHK$ is a parallelogram, where $\angle GAK = \angle x$. Again, area of $\triangle DAF =$ area of the rectangular region $ABCD$ and area of the parallelogram $AGHK =$ area of the triangular region DAF.Therefore, $AGHK$ is the required parallelogram.

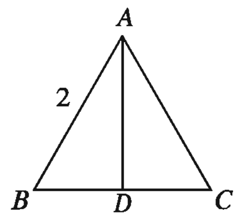
Exercise 15

1. The lengths of three sides of a triangle are given, in which case below the construction of a right-angled triangle is not possible ?
   1) 3 cm, 4 cm, 5 cm 2) 6 cm, 8 cm, 10 cm
   3) 5 cm, 7 cm, 9 cm 4) 5 cm, 12 cm, 13 cm

2. In plane geometry :
   (*i*) Each of the bounded plane has definite area.
   (*ii*) If the area of two triangles is equal, the two triangles are congruent.
   (*iii*) If the two triangles are congruent, their area is equal.

   Which one of the following is correct ?
   1) *i* and *ii* 2) *i* and *iii* 3) *ii* and *iii* 4) *i*, *ii* and *iii*

In the adjacent figure, $\triangle ABC$ is equilateral, $AD \perp BC$ and $AB = 2$.
Based on the information mentioned above, answer question no. 3 and 4:



3. $BD =$ what?
   1) 1 2) $\sqrt{2}$ 3) 2 4) 4

4. What is the height of the triangle?
   1) $\dfrac{4}{\sqrt{3}}$ 2) $\sqrt{3}$ 3) $\dfrac{2}{\sqrt{3}}$ 4) $2\sqrt{3}$

5. Prove that, the diagonals of a parallelogram divide the parallelogram into four equal triangular regions.

6. Prove that, the area of a square is half the area of the square drawn on its diagonal.

7. Prove that, any median of a triangle divides the triangular region into two regions of equal area.

8. A parallelogram and a rectangular region of equal area lie on the same side of the bases. Show that, the perimeter of the parallelogram is greater than that of the rectangle.

9. X and Y are the mid points of the sides AB and AC of $\triangle ABC$. Prove that the area of $\triangle AXY = \frac{1}{4}$(area of the triangular region $\triangle ABC$).

10. $ABCD$ is a trapezium. The sides AB and CD are parallel to each other. Find the area of the region bounded by the trapezium $ABCD$.

11. P is any point interior to the parallelogram $ABCD$. Prove that the area of $\triangle PAB$ + the area of $\triangle PCD = \frac{1}{2}$(area of the parallelogram $ABCD$).

12. A line parallel to base BC of the triangle ABC intersects AB and AC at D and E respectively. Prove that, $\triangle DBC = \triangle EBC$ and $\triangle DBF = \triangle CDE$.

13. $\angle A = 1$ right angle of the triangle ABC. D is a point on AC. Prove that $BC^2 + AD^2 = BD^2 + AC^2$.

14. ABC is an isosceles right triangle. BC is its hypotenuse and P is any point on BC. Prove that $PB^2 + PC^2 = 2PA^2$.

15. $\angle C$ is an obtuse angle of $\triangle ABC$; AD is perpendicular to BC. Show that $AB^2 = AC^2 + BC^2 + 2BC \cdot CD$.

16. $\angle C$ is an acute angle of $\triangle ABC$; AD is perpendicular to BC. Show that $AB^2 = AC^2 + BC^2 - 2BC \cdot CD$.

17. QD is a median of the $\triangle PQR$.
    1) Draw a proportional figure according to the stem.
    2) Prove that, $PQ^2 + QR^2 = 2(PD^2 + QD^2)$
    3) If $PQ = QR = PR$, then prove that, $4PD^2 = 3PQ^2$

18. $ABCD$ is a parallelogram where $AB = 5$ cm, $AD = 4$ cm and $\angle BAD = 75°$ $APML$ is another parallelogram where $\angle LAP = 60°$. Area of $\triangle AED$ and area of parallelogram $APML$ are equal to the area of parallelogram $ABCD$.
    1) Draw $\angle BAD$ by using pencil, compass and scale.
    2) Draw $\triangle AED$ [drawing and description are must].
    3) Draw parallelogram $APML$ [drawing and description are must].

Chapter 16

Mensuration

The length of a line, the area of a place, the volume of a solid etc. are determined for practical purposes. In the case of measuring any such quantity, another quantity of the same kind having some definite magnitude is taken as unit. The ratio of the quantity measured and the unit defined in the above process is the amount of the quantity.

i.e. magnitude $= = \dfrac{\text{Quantity measured}}{\text{Unit quantity}}$

In the case of a fixed unit, every measure is a number which denotes how many times the magnitude of the unit is the magnitude of the quantity measured. For example, the bench is 5 meter long. Here metre is a definite length which is taken as a unit and in comparison to that the bench is 5 times in length.

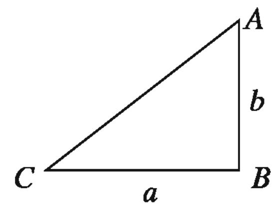
At the end of the Chapter, the students will be able to —

- determine the area of polygonal region by applying the laws of area of triangle and quadrilateral and solve allied problems.
- determine the circumference of the circle and a length of the chord of a circle.
- determine the area of circle.
- determining the area of a circle and its segment, solve the alied problems.
- determine the area of solid rectangles, cubes and cylinder and solve the allied problems.
- determine the area of uniform and non-uniform solids

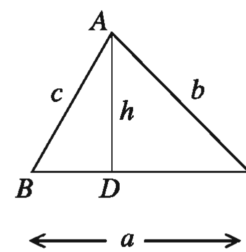
Area of Triangular region

In the previous class, we learned that area of triagular region $= \dfrac{1}{2} \times$ base $\times$ height.

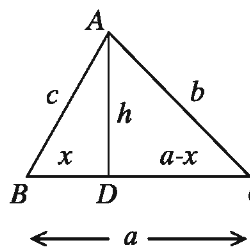
1. **Right-angled triangle**: Let in the right-angled triangle ABC, $BC = a$ and $AB = b$ are the adjacent sides of the right angle. Here if we consider BC as the base and AB as the height,
   Area of $\triangle ABC = \frac{1}{2}\times$ base $\times$ height $= \frac{1}{2}ab$



2. **Two sides of a triangular region and the angle included between them are given**:
   Let in $\triangle ABC$, the sides are $BC = a$, $CA = b$, $AB = c$. AD is drawn perpendicular from A to BC. Let altitude (height) $AD = h$.
   Considering the angle C we get, $\frac{AD}{CA} = \sin C$
   or, $\frac{h}{b} = \sin C$ or, $h = b\sin C$
   Area of $\triangle ABC = \frac{1}{2}BC \times AD$
   $= \frac{1}{2}a \times b\sin C = \frac{1}{2}ab\sin C$
   Similarly, area of $\triangle ABC$
   $= \frac{1}{2}bc\sin A = \frac{1}{2}ca\sin B$



3. **Three sides of a triangle are given**:
   Let in $\triangle ABC$, $BC = a$, $CA = b$ and $AB = c$.
   $\therefore$ Perimeter of the triangle $2s = a + b + c$.
   We draw $AD \perp BC$.
   Let, $BD = x$, then $CD = a - x$
   In right-angled $\triangle ABD$ and $\triangle ACD$



$\therefore AD^2 = AB^2 - BD^2$ and $AD^2 = AC^2 - CD^2$

$\therefore AB^2 - BD^2 = AC^2 - CD^2$

or, $c^2 - x^2 = b^2 - (a - x)^2$

or, $c^2 - x^2 = b^2 - a^2 + 2ax - x^2$

or, $2ax = c^2 + a^2 - b^2$

$\therefore x = \frac{c^2 + a^2 - b^2}{2a}$

Again,

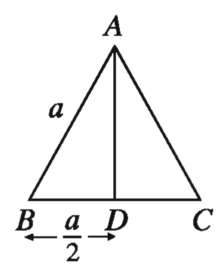
$$\begin{aligned}
AD^2 &= c^2 - x^2 \\
&= c^2 - \left(\frac{c^2+a^2-b^2}{2a}\right)^2 \\
&= \left(c + \frac{c^2+a^2-b^2}{2a}\right)\left(c - \frac{c^2+a^2-b^2}{2a}\right) \\
&= \frac{2ac+c^2+a^2-b^2}{2a} \cdot \frac{2ac-c^2-a^2+b^2}{2a} \\
&= \frac{\{(c+a)^2-b^2\}\{b^2-(c-a)^2\}}{4a^2} \\
&= \frac{(c+a+b)(c+a-b)(b+c-a)(b-c+a)}{4a^2} \\
&= \frac{(a+b+c)(a+b+c-2b)(a+b+c-2a)(a+b+c-2c)}{4a^2} \\
&= \frac{2s(2s-2b)(2s-2a)(2s-2c)}{4a^2} \\
&= \frac{4s(s-a)(s-b)(s-c)}{a^2}
\end{aligned}$$

$\therefore AD = \dfrac{2}{a}\sqrt{s(s-a)(s-b)(s-c)}$

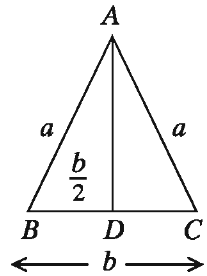
$\therefore$ Area of $\triangle ABC$

$= \dfrac{1}{2}BC \cdot AD = \dfrac{1}{2} \cdot a \cdot \dfrac{2}{a}\sqrt{s(s-a)(s-b)(s-c)} = \sqrt{s(s-a)(s-b)(s-c)}$

4. **Equilateral triangle**: Let the length of each side of the equilateral triangular region ABC be a.
Draw $AD \perp BC$.
$\therefore BD = CD = \dfrac{a}{2}$
In right-angled $\triangle ABD$
$BD^2 + AD^2 = AB^2$
or, $AD^2 = AB^2 - BD^2 = a^2 - \left(\dfrac{a}{2}\right)^2 = a^2 - \dfrac{a^2}{4} = \dfrac{3a^2}{4}$
$\therefore AD = \dfrac{\sqrt{3}a}{2}$
Area of $\triangle ABC = \dfrac{1}{2} \cdot BC \cdot AD = \dfrac{1}{2} \cdot a \cdot \dfrac{\sqrt{3}a}{2} = \dfrac{\sqrt{3}}{4}a^2$

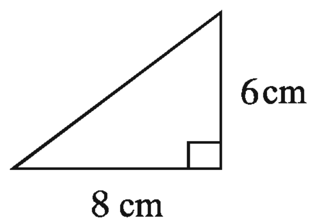


5. **Isosceles triangle**: Let ABC be an isosceles triangle in which
$AB = AC = a$ and $BC = b$
Draw $AD \perp BC$. $\therefore BD = CD = \frac{b}{2}$
$\triangle ABD$ is right angled.
$\therefore AD^2 = AB^2 - BD^2$
$= a^2 - \left(\frac{b}{2}\right)^2 = a^2 - \frac{b^2}{4} = \frac{4a^2 - b^2}{4}$
$\therefore AD = \frac{\sqrt{4a^2 - b^2}}{2}$
Area of isosceles $\triangle ABC = \frac{1}{2} \cdot BC \cdot AD$
$= \frac{1}{2} \cdot b \cdot \frac{\sqrt{4a^2 - b^2}}{2} = \frac{b}{4}\sqrt{4a^2 - b^2}$



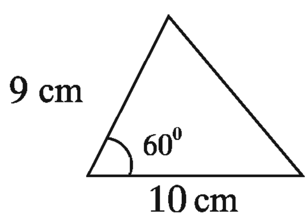
Example 1. The lengths of the two sides of a right-angled triangle, adjacent to right angle are 6 cm and 8 cm respectively. Find the area of the triangle.

Solution: Let, the sides adjacent to right angle are $a = 8$ cm and $b = 6$ cm respectively. $\therefore$ Its area $= \frac{1}{2}ab = \frac{1}{2} \times 6 \times 8$ square c m $= 24$ square cm



Example 2. The lengths of the two sides of a triangle are 9 c m and 10 c m respectively and the angle included between them is $60°$. Find the area.

Solution: Let, the sides of triangle are $a = 9$ cm. and $b = 10$ cm. respectively. Their included angle $\theta = 60°$.
$\therefore$ Area of the tringle $= \frac{1}{2}ab\sin 60°$
$= \frac{1}{2} \times 9 \times 10 \times \frac{\sqrt{3}}{2}$ sq cm $= 38.97$ sq cm (approx.)
Required area 38.97 sq cm (approx.)



Example 3. The lengths of the three sides of a triangle are 7 cm, 8 cm and 9 cm. respectively. Find its area.

Solution: et, the lengths of the sides of the triangle are $a = 7$ cm., $b = 8$ cm and $c = 9$ cm.

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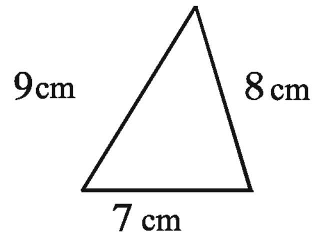
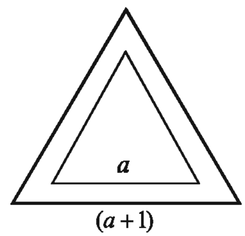
Semi perimeter $s = \dfrac{a+b+c}{2} = \dfrac{7+8+9}{2}$ cm $= 12$ cm



$\therefore$ Its area $= \sqrt{s(s-a)(s-b)(s-c)}$
$= \sqrt{12(12-7)(12-8)(12-9)}$ sq. c m
$= \sqrt{12 \times 5 \times 4 \times 3}$ sq. c m
$= \sqrt{720} = 26.83$ sq. c m (approx.)
$\therefore$ The area of the triangle 26.83 sq. c m (approx.)

Example 4. The area of an equilateral triangle increases by $3\sqrt{3}$ sq. metre when the length of each side increases by 1 metre. Find the length of the side of the triangle.

Solution:
Let, the length of each side of the equilateral triangle is a metre. $\therefore$ Its area $= \dfrac{\sqrt{3}}{4}a^2$ sq. metre.
The area of the triangle when the length of each side increases by 1 metre $= \dfrac{\sqrt{3}}{4}(a+1)^2$ sq. metre.



According to the question, $\dfrac{\sqrt{3}}{4}(a+1)^2 - \dfrac{\sqrt{3}}{4}a^2 = 3\sqrt{3}$

or, $(a+1)^2 - a^2 = 12$ $\quad \left[\text{dividing by } \dfrac{\sqrt{3}}{4}\right]$

or, $a^2 + 2a + 1 - a^2 = 12$ or, $2a = 11$ or, $a = 5.5$

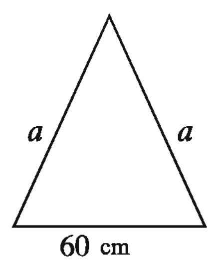
The required length is 5.5 metre.

Example 5. The length of the base of an isosceles triangle is 60 cm. If its area is 1200 sq. metre, find the length of equal sides.

Solution: Let the base of the isosceles triangle be $b = 60$ cm and the length of equal sides b e a.

Area of the triangle $= \dfrac{b}{4}\sqrt{4a^2 - b^2}$

According to the question, $\dfrac{b}{4}\sqrt{4a^2 - b^2} = 1200$



or, $\dfrac{60}{4}\sqrt{4a^2 - (60)^2} = 1200$

or, $15\sqrt{4a^2 - 3600} = 1200$

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or, $\sqrt{4a^2 - 3600} = 80$

or, $4a^2 - 3600 = 6400$ [by squaring]

or, $4a^2 = 10000$

or, $a^2 = 2500$

$\therefore a = 50$

The length of equal sides of the triangle is 50 cm.

Example 6. From a certain place two roads run in two directions making an angle of 120°. From that place, persons move in the two directions with speed of 10 km per hour and 8 km per hour respectively. What will be the direct distance between them after 5 hours ?

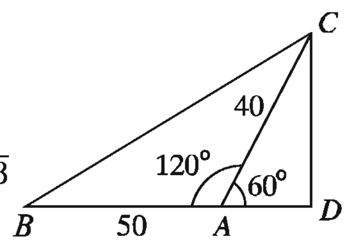
Solution: Let two persons start from A with velocities 10 km/hour and 8 km/hour respectively and reach B and C after 5 hours. Then after 5 hours, the direct distance between them is BC. From C perpendicular CD is drawn on BA produced.

$\therefore AB = 5 \times 10$ km $= 50$ km, $AC = 5 \times 8$ km $= 40$ km

and $\angle BAC = 120°$

$\therefore \angle DAC = 180° - 120° = 60°$

$\triangle ACD$ is right-angled.



$\therefore \dfrac{CD}{AC} = \sin 60°$ or, $CD = AC \sin 60° = 40 \times \dfrac{\sqrt{3}}{2} = 20\sqrt{3}$

and $\dfrac{AD}{AC} = \cos 60°$ or, $AD = AC \cos 60° = 40 \times \dfrac{1}{2} = 20$

Again, from right-angled triangle BCD we get,

$BC^2 = BD^2 + CD^2 = (BA + AD)^2 + CD^2$

$= (50 + 20)^2 + (20\sqrt{3})^2 = 4900 + 1200 = 6100$

$\therefore BC = 78.1$ (approx.)

The required distance is 78.1 km (approx.)

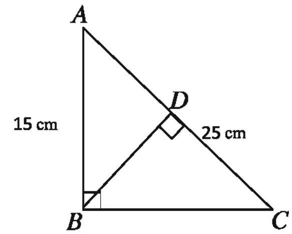
Example 7. Consider the diagram given and –

1) Find the length of the side BC.

2) Find the value of BD?

3) Find the ratio of areas of $\triangle ABD$ and $\triangle BCD$.

Solution:



1) $AB = 15,\ AC = 25$
$\therefore BC = \sqrt{AC^2 - AB^2} = \sqrt{(25)^2 - (15)^2} = \sqrt{400} = 20$

2) Area of $\triangle ABC = \frac{1}{2}BC \cdot AB = \frac{1}{2}AC \cdot BD$
$\frac{1}{2}AC \cdot BD = \frac{1}{2}BC \cdot AB$
$\therefore 25 \times BD = 20 \times 15$
$\therefore BD = 12$

3) From right-angled triangle $\triangle ABD$ we get,

$AD^2 + BD^2 = AB^2$

or, $AD^2 + 12^2 = 15^2$

or, $AD^2 = 225 - 144 = 81$

$\therefore AD = 9$ and $CD = AC - AD = 25 - 9 = 16$

So, ratio of areas of $\triangle ABD$ and $\triangle BCD$ is:

$$\frac{\triangle ABD}{\triangle BCD} = \frac{\frac{1}{2}BD \cdot AD}{\frac{1}{2}BD \cdot CD} = \frac{9}{16}$$

$\triangle ABD : \triangle BCD = 9 : 16$

Exercise 16.1

1. The hypotenuse of a right-angled triangle is 25 m. If one of its sides is $\frac{3}{4}$ of the other, find the length of the two sides.

2. A ladder with length 20 m. stands vertically against a wall. How much further should the lower end of the ladder be moved so that its upper end descends 4 metre?

3. The perimeter of an isosceles triangle is 16 m. If the length of equal sides is $\frac{5}{6}$ of base, find the area of the triangle.

4. The lengths of the two sides of a triangle are 25 cm, 27 cm and perimeter is 84 cm. Find the area of the triangle.

5. When the length of each side of an equilateral triangle is increased by 2 metre, its area is increased by $6\sqrt{3}$ square metre. Find the length of side of

the triangle.

6. The lengths of the two sides of a triangle are 26 m., 28 m. respectively and its area is 182 square metre. Find the angle between the two sides.

7. The length of equal sides of an isosceles triangle is 10 m and area 48 square metre. Find the length of the base.

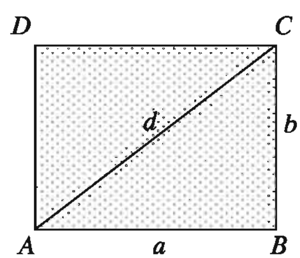
8. Two roads run from a certain place with an angle of 135° in two directions. Two persons move from that place in two directions with the speed of 7 km per hour and 5 km per hour respectively. What will be the direct distance between them after 4 hours?

9. If the lengths of the perpendiculars from a point interior of an equilateral triangle to three sides are 6 cm, 7 cm, 8 cm respectively; find the length of sides of the triangle and the area of the triangular region.

10. The perpendicular of a right-angled triangle is 6 cm less than $\frac{11}{12}$ times of the base, and the hypotenuse is 3 cm less than $\frac{4}{3}$ times of the base.
    1) Let the base be x. Express the area of the triangle in terms of x.
    2) Find the length of the base.
    3) If the length of the base of the triangle is 12 cm., find the area of the equilateral triangle having the same perimeter as its perimeters.

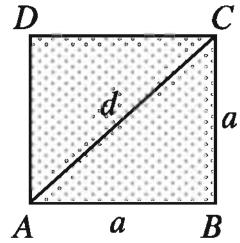
Area of Quadrilateral Region

1. **Area of Rectangular Region**: Let, the length of $AB = a$, breadth $BC = b$ and diagonal $AC = d$ of rectangle $ABCD$. We know, the diagonal of a rectangle divides the rectangle into two equal triangular regions.



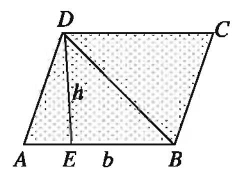
$\therefore$ Area of the rectangle $ABCD = 2\times$ area of $\triangle ABC$
$= 2 \times \frac{1}{2} a \cdot b = ab$
perimeter of the rectangular region, $s = 2(a + b)$ and observe that triangle ABC is right-angled.
$\therefore AC^2 = AB^2 + BC^2$ or, $d^2 = a^2 + b^2$
$\therefore d = \sqrt{a^2 + b^2}$

2. **Area of Square Region**: Let the length of each side of a square $ABCD$ be a and diagonal d. The diagonal AC divides the square region into two equal triangular regions.



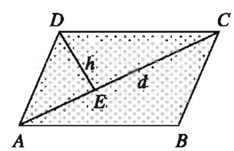
$\therefore$ Area of the square region $ABCD = 2\times$ area of $\triangle ABC = 2 \times \frac{1}{2} a \cdot a = a^2 =$ (length of a side)^2
Observe that, the perimeter of the square region $s = 4a$ and diagonal $d = \sqrt{a^2 + a^2} = \sqrt{2a^2} = \sqrt{2}a$

3. **Area of a parallelogram region**:
a) Base and height are given: Let, the base $AB = b$ and height $DE = h$ of parallelogram $ABCD$. The diagonal BD divides the parallelogram into two equal triangular regions.



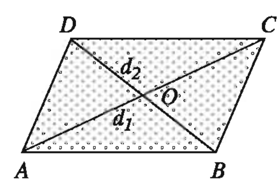
$\therefore$ The area of the parallelogram $ABCD$
$= 2\times$ area of $\triangle ABD = 2 \times \frac{1}{2} b \cdot h = bh$

b) The length of a diagonal and the length of a perpendicular drawn from the opposite angular point on that diagonal are given:
Let, in a parallelogram $ABCD$, the diagonal be $AC = d$ and the perpendicular from opposite angular point D on AC be $DE = h$. Diagonal AC divides the parallelogram into two equal triangular regions.



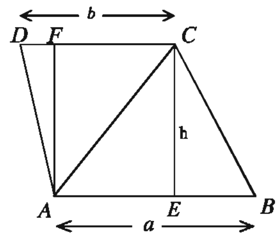
$\therefore$The area of the parallelogram $ABCD$
$= 2\times$ area of $\triangle ACD = 2 \times \frac{1}{2} d \cdot h = dh$

4. **Area of Rhombus Region**: Two diagonals of a rhombus region are given. Let the diagonals be $AC = d_1$, $BD = d_2$ of the rhombus $ABCD$ and the diagonals intersect each other at O.
Diagonal AC divides the rhombus region into two equal triangular regions. We know that the diagonals of a rhombus bisect each other at right angles.



$\therefore$ Height of $\triangle ACD = \frac{d_2}{2}$
$\therefore$ The area of the rhombus $ABCD$
$= 2\times$ area of $\triangle ACD = 2 \times \frac{1}{2} d_1 \cdot \frac{d_2}{2} = \frac{1}{2} d_1 d_2$

5. **Area of trapezium region**: Two parallel sides of trapezium region and the distance of perpendicular between them are given. Let $ABCD$ be a trapezium whose lengths of parallel sides are $AB = a$ unit, $CD = b$ unit and distance between them be $CE = AF = h$. Diagonal AC divides the trapezium region $ABCD$ into $\triangle ABC$ and $\triangle ACD$.
Area of trapezium region $ABCD$
$= \text{area of } \triangle ABC + \text{area of } \triangle ACD$
$= \frac{1}{2}AB \times CE + \frac{1}{2}CD \times AF$
$= \frac{1}{2}ah + \frac{1}{2}bh = \frac{h(a+b)}{2}$



Example 8. Length of a rectangular room is $\frac{3}{2}$ times of breadth. If the area is 384 square metre, find the perimeter and length of the diagonal.

Solution: Let breadth of the rectangular room is x metre.

$\therefore$ Length of the room is $\frac{3}{2}x$ and area $\frac{3}{2}x \times x = \frac{3}{2}x^2$

According to the question, $\frac{3}{2}x^2 = 384$ or, $3x^2 = 768$ or, $x^2 = 256$

$\therefore x = 16$ metre.

Length of the rectangular room $= \frac{3}{2} \times 16 = 24$ metre and breadth $= 16$ metre.

$\therefore$ Its perimeter $= 2(24 + 16)$ metre $= 80$ metre and length of the diagonal $= \sqrt{24^2 + 16^2}$ metre $= \sqrt{832}$ metre $= 28.84$ metre (approx.)

The required perimiter is 80 metre and length of the diagonal is 28.84 meter (approx.)

Example 9. The area of a rectangular region is 2000 square meter. If the length is reduced by 10 metre, it becomes a square region. Find the length and breadth of the rectangular region.

Solution: Let length of the rectangular region be x metre and breadth y metre.

$\therefore$ area of the rectangular region is $= xy$ sq. metre

According to the question, $xy = 2000 \ldots (1)$ and $x - 10 = y \ldots (2)$

Putting y= x-10 in equation (1) we get,

$x(x-10) = 2000$ or, $x^2 - 10x - 2000 = 0$

or, $x^2 - 50x + 40x - 2000 = 0$ or, $(x - 50)(x + 40) = 0$

$\therefore x = 50$ or, $x = -40$

But length can never be negative. $\therefore x = 50$

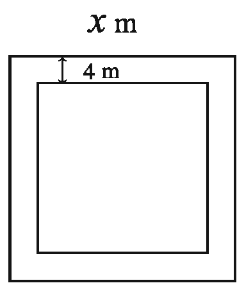
Now putting the value of x in equation (2) we get, $y = 50 - 10 = 40$

$\therefore$ The length of the rectangular is 50 and breadth 40 metre.

Example 10. There is a road of 4 metre width inside around a square field. If the area of the road is 1 hector, determine the area of the field excluding the road.

Solution: Let, the length of the square field is x metre.

$\therefore$ Its area is x^2 sq. metre
There is a road around the field with width 4 metre.
Length of the square field excluding the road $= (x - 2 \times 4)$ or, $(x - 8)$ metre.
Area of the square field excluding the road is $= (x - 8)^2$ sq. metre
$\therefore$ Area of the road $= x^2 - (x - 8)^2$ sq. metre
We know, 1 hector $= 10000$ sq. metre



According to the question, $x^2 - (x - 8)^2 = 10000$

or, $x^2 - x^2 + 16x - 64 = 10000$

or, $16x = 10064$

$\therefore x = 629$

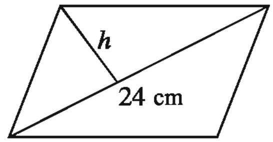
Area of the square field excluding the road

$= (629 - 8)^2$ sq. metre $= 385641$ sq. metre $= 38.56$ hector (approx.)

The required area $= 38.56$ hector (approx.).

Example 11. The area of a parallelogram is 120 sq. cm and length of one of its diagonal is 24 cm. Determine the length of the perpendicular drawn on that diagonal from the opposite vertex.

Solution: Let a diagonal of a parallelogram be $d = 24$cm and the length of the perpendicular drawn on the diagonal from the opposite vertex be h cm.



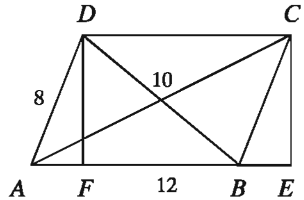
$\therefore$ Area of the parallelogram $= dh$ square cm.

As per question, $dh = 120$ or, $h = \dfrac{120}{d} = \dfrac{120}{24} = 5$

The required length of the perpendicular is 5 cm.

Example 12. The length of the sides of a parallelogram are wetre and 8 metre If the length of the smaller diagonal is 10 m, determine the length of the other diagonal.

Solution:



Let, in the parallelogram $ABCD$; $AB = a = 12$ metre and $AD = c = 8$ metre and diagonal $BD = b = 10$ metre Let us draw the perpendiculars DF and CE from D and C on AB and the extended part of AB respectively. Join C, A and D, B.

$\therefore$ Semi perrimeter of $\triangle ABD$, $s = \dfrac{12 + 10 + 8}{2}$ metre $= 15$ metre.

$\therefore$ area of $\triangle ABD$ $\sqrt{s(s-a)(s-b)(s-c)} = \sqrt{15(15-12)(15-10)(15-8)}$ sq. metre $= \sqrt{15 \times 3 \times 5 \times 7}$ sq. metre $= \sqrt{1575}$ sq. metre $= 39.68$ sq. metre (approx.)

Again, area of $\triangle ABC = \dfrac{1}{2} AB \times DF$

or, $39.68 = \dfrac{1}{2} \times 12 \times DF$ or, $6DF = 39.68$ $\therefore DF = 6.61$ (approx.)

Now, $\triangle BCE$ is a right-angled triangle.

$\therefore BE^2 = BC^2 - CE^2 = AD^2 - DF^2 = 8^2 - (6.61)^2 = 20.31$

$\therefore BE = 4.5$ (approx.)

So, $AE = AB + BE = 12 + 4.5 = 16.5$ (approx.)

In the right-angled $\triangle ACE$

$AC^2 = AE^2 + CE^2 = (16.5)^2 + (6.61)^2 = 315.94$

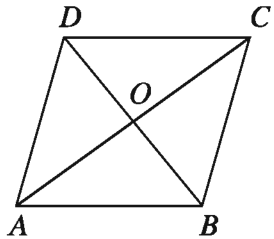
$\therefore AC = 17.77$ (approx.)

The required length of the diagonal is 17.77 metre (approx.)

Example 13. The length of a diagonal of a rhombus is 10 metre and its area is 120 sq. metre. Determine the length of the other diagonal and its perimeter.

Solution:

Let, the length of a diagonal of rhombus $ABCD$ is $BD = d_1 = 10$ metre and another diagonal d_2 metre.



Area of the rhombus $= \frac{1}{2}d_1d_2$ sq. meter

As per question, $\frac{1}{2}d_1d_2 = 120$ or, $d_2 = \frac{120 \times 2}{10} = 24$ metre.

We know, the diagonals of rhombus bisect each other at right angles. Let the diagonals interset at the point O.

$\therefore OD = OB = \frac{10}{2}$ metre $= 5$ metre and $OA = OC = \frac{24}{2}$ metre $= 12$ metre.

In the right-angled $\triangle AOD$

$AD^2 = OA^2 + OD^2 = 12^2 + 5^2$

$\therefore AD = 13$

$\therefore$ The length of each sides of the rhombus is 13 metre.

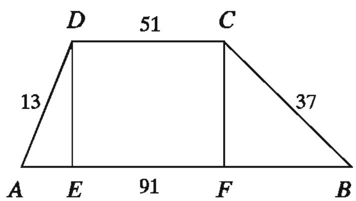
The perimeter of the rhombus $= 4 \times 13$ metre $= 52$ metre

The required length of the diagonal is 24 metre and perimeter 52 metre.

Example 14. The lengths of two parallel sides of a trapezium are 91 cm and 51 cm and the lengths of two other sides are 37 cm and 13 cm respectively. Determine the area of the trapezium.

Solution:

Let, in trapezium $ABCD$; $AB = 91$cm. $CD = 51$cm. Let us draw the perpendiculars DF and CF on AB from D and C respectively.



$\therefore$ $CDEF$ is a rectangle.

$\therefore$ $EF = CD = 51$ c m

Let, $AE = x$ and $DE = CF = h$

$\therefore$ $BF = AB - AF = 91 - (AE + EF) = 91 - (x + 51) = 40 - x$

From the right-angled $\triangle ADE$ we get,

$AE^2 + DE^2 = AD^2$ or, $x^2 + h^2 = 13^2$ or, $x^2 + h^2 = 169 \ldots (1)$

Again in the-right angled triangle BCF

$BF^2 + CF^2 = BC^2$ or, $(40 - x)^2 + h^2 = 37^2$

or, $1600 - 80x + x^2 + h^2 = 1369$

or, $1600 - 80x + 169 = 1369$ [From (1)]

or, $1600 + 169 - 1369 = 80x$

or, $80x = 400 \therefore x = 5$

Now putting the value of x in equation(1) we get,

$5^2 + h^2 = 169$ or, $h^2 = 169 - 25 = 144 \therefore h = 12$

Area of the trapezium $ABCD = \frac{1}{2}(AB + CD) \cdot h$

$= \frac{1}{2}(91 + 51) \times 12$ square cm $= 71 \times 12$ square cm $= 852$ square cm

The required area is 852 square cm.

Area of regular polygon

The lengths of all sides of a regular polygon are equal. Again, the angles are also equal. Regular polygon with n sides produces n isosceles triangles by adding centre to the vertices.

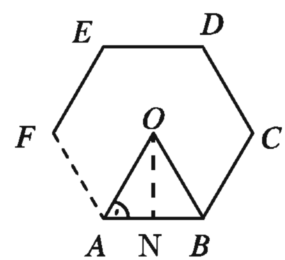
So, area of the regular polygon $= n\times$ area of one triangular region

$ABCDEF\cdots$ is a regular polygon whose centre is O. $\therefore$ It has n sides and the length of each side is a. We join O, A; *and* O, B.

Let height of $\triangle AOB$, ON $= h$ and $\angle OAB = \theta$

The angle produced at each of the vertices of regular polygon $= 2\theta$

$\therefore$ Total angle produced by n number of vertices in the polygon $= 2\theta n$



Angle produced in the polygon at the centre = 4 right angles.

$\therefore$ The sum of angles of number of triangles $(2\theta n + 4)$ right angles.

$\therefore$ Sum of 3 angles of $\triangle OAB = 2$ right angles.

$\therefore$ Summation of the angles of n numbers of triangles $= 2n$ right angles.

$\therefore 2\theta \cdot n + 4$ right angles $= 2n$ right angles.

or, $2\theta \cdot n = (2n - 4)$ right angles.

or, $\theta = \frac{2n-4}{2n}$ right angles.

or, $\theta = \left(1 - \dfrac{2}{n}\right) \times 90^\circ$

$\therefore \theta = 90^\circ - \dfrac{180^\circ}{n}$

Here, $\tan\theta = \dfrac{\text{ON}}{\text{AN}} = \dfrac{h}{\frac{a}{2}} = \dfrac{2h}{a}$

$\therefore h = \dfrac{a}{2}\tan\theta$

$$\begin{aligned}\text{Area of } \triangle OAB &= \frac{1}{2}ah \\ &= \frac{1}{2}a \times \frac{a}{2}\tan\theta \\ &= \frac{a^2}{4}\tan\left(90^\circ - \frac{180^\circ}{n}\right) \\ &= \frac{a^2}{4}\cot\frac{180^\circ}{n}\ [\because \tan(90^\circ - A) = \cot A]\end{aligned}$$

$\therefore$ The require area of a regular polygon having n sides $= \dfrac{na^2}{4}\cot\dfrac{180^\circ}{n}$

Example 15. If the length of each side of a regular pentagon is 4 cm, determine its area.

Solution: Let, length of each side of a regular pentagon is $a = 4$ cm and number of sides $n = 5$

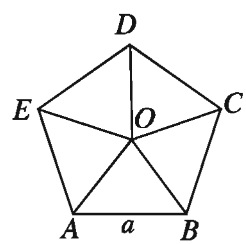
We know, area of a regular polygon $= \dfrac{na^2}{4}\cot\dfrac{180^\circ}{n}$

$\therefore$ Area of the pentagon $= \dfrac{5 \times 4^2}{4}\cot\dfrac{180^\circ}{5}$ square cm

$= 20 \times \cot 36^\circ$ square cm

$= 20 \times 1.376$ square cm (using calculator)

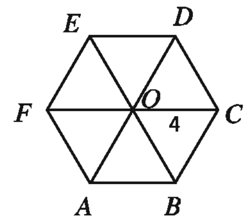
$= 27.528$ square cm (approx.)



Required area 27.528 square cm (approx.)

Example 16. The distance of the centre to the vertex of a regular hexagon is 4 metre. Determine its area.

Solution: Let, $ABCDEF$ is a regular hexagon whose centre is O, O is joined to each of the vertex and thus 6 triangles of equal area are formed.



$\therefore \angle COD = \dfrac{360°}{6} = 60°$

Let the distance of centre O to its vertex is a metre.

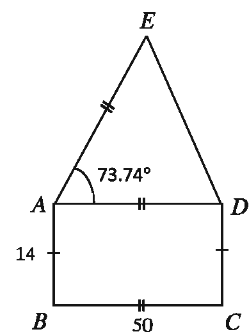
$\therefore$ area of $\triangle COD = \dfrac{1}{2} \cdot a \cdot a \sin 60°$

$= \dfrac{\sqrt{3}}{4} \times 4^2$ square metre$= 4\sqrt{3}$ square metre

Area of the regular hexagon $= 6\times$ area of $\triangle COD$

$= 6 \times 4\sqrt{3}$ square metre $= 24\sqrt{3}$ square metre

The required area in $24\sqrt{3}$ square metre.



Example 17. According to the given figure–

1) Find length of the diagonal of the rectangle.
2) Find the integer value of its area.
3) Find the perimeter of the isosceles triangle.

Solution:

1) As per the given figure, the area is divided into rectangle $ABCD$ and isosceles triangle ADE.

   Length of diagonal of rectangle $ABCD = \sqrt{50^2 + 14^2}$ cm $= 51.92$ cm (approx.)

2) Area of rectangle $ABCD = 50 \times 14$ square cm $= 700$ square cm

   Area of triangle $ADE = \dfrac{1}{2} AD \cdot AE \cdot \sin\angle DAE = \dfrac{1}{2} \times 50 \times 50 \times \sin 73.74°$ square cm $= 24 \times 50 \times 0.960001$ square cm $= 1200$ square cm (approx.)

   Total area $= (700 + 1200)$ square cm $= 1900$ square cm

3) Let, in $\triangle ADE$, $AD = AE = 50$ cm $= a$, $DE = b$

   $\therefore$ Area of the isosceles triangle $ADE = \dfrac{b}{4}\sqrt{4a^2 - b^2}$

   As per the question, $\dfrac{b}{4}\sqrt{4a^2 - b^2} = 1200$

   $b\sqrt{4(50)^2 - b^2} = 4800$

   or, $b^2(10000 - b^2) = 23040000$ [by squaring] or, $10000b^2 - b^4 = 23040000$

or, $b^4 - 10000b^2 + 23040000 = 0$

or, $b^4 - 6400b^2 - 3600b^2 + 2304000 = 0$

or, $(b^2 - 6400)(b^2 - 3600) = 0$

$\therefore b^2 - 6400 = 0$ or $b^2 - 3600 = 0$

or, $b^2 = 6400$ or $b^2 = 3600$

$\therefore b = 80$ or $b = 60$

If $b = 80$, then $\frac{1}{2} \cdot AD \cdot DE \cdot \sin\angle ADE = 1200$

or, $\frac{1}{2} \times 50 \times 80 \times \sin\angle ADE = 1200$

or, $\sin\angle ADE = 0.6$

$\therefore \angle ADE = 36.87°$ (approx.)

Sum of three angles of $\triangle ADE = 73.74° + 36.87° + 36.87° = 147.48°$

But sum of three angles of any triangle $= 180°$, So $b \neq 80$

If $b =$ then 60, $\frac{1}{2} \cdot AD \cdot DE \cdot \sin\angle ADE = 1200$

or, $\frac{1}{2} \times 50 \times 60 \times \sin\angle ADE = 1200$

or, $\sin\angle ADE = 0.8$

$\therefore \angle ADE = 53.13°$ (approx.)

Sum of three angles of $\triangle ADE = 73.74° + 53.13° + 53.13° = 180°$, $\therefore b = 60$

$\therefore$ The perimeter of the triangle $(50 + 50 + 60)$ cm $= 160$ cm

Exercises 16.2

1. The length of a rectangular region is twice its width. If its area is 512 sq. metre, determine its perimeter.

2. The length of a plot is 80 metre and the breadth is 60 metre A pond was excavated in the plot. If the width of each side of the border around the pond is 4 metre, determine the area of the border of the pond.

3. The length of a garden is 40 metre and its breadth is 30 metre. There is a pond inside the garden with border of equal width surrounding it. If the area of the pond is $\frac{1}{2}$ of that of the garden, find the length and breadth of the pond.

4. Outside a square there is a path of width 5 metre around it. If the area of the path is 500 square meter, find the area of the field.

5. The perimeter of a square region is equal to the perimeter of a rectangular region. The length of the rectangular region is thrice its breadth and the area is 768 sq. metre. How many stones will be required to cover the square region with square stones of 40 cm each?

6. Area of a rectangular region is 160 sq. metre. If the length is reduced by 6 metre, it becomes a square region. Determine the length and the breadth of the rectangle.

7. The base of a parallelogram is $\frac{3}{4}$ th of the height and area is 363 square inches. Determine the base and the height of the parallelogram.

8. The area of a parallelogram is equal to the area of a square region. If the base of the parallelogram is 125 metre and the height is 5 metre, find the length of the diagonal of the square.

9. The length of two sides of a parallelogram are 30 cm and 26 cm. If its smaller diagonal is 28 cm, find the length of the other diagonal.

10. The perimeter of a rhombus is 180 cm and the smaller diagonal is 54 cm. Find its other diagonal and the area.

11. Difference of the length of two parallel sides of a trapezium is 8 cm and their perpendicular distance is 24 cm. If the area of the trapezium is 312 sq. cm, find the lengths of the two parallel sides of the trapezium.

12. The lengths of two parallel sides of a trapezium are 31 cm and 11 cm respectively and two other sides are 10 and 12 cm respectively. Find the area of the trapezium.

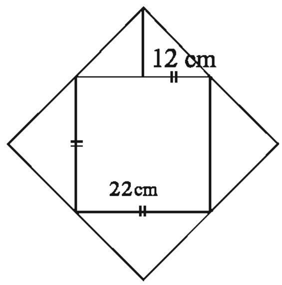
13. The distance from the centre to the vertex of a regular octagon is 1.5 metre Find the area of the regular octagon.

14. The length of a rectangular flower garden is 150 metre and breadth is 100 metre. For nursing the garden, there is a path of width 3 metre along its length and

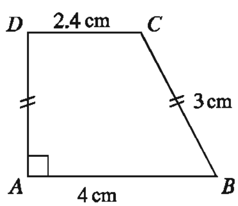
breadth right at the middle of the garden.

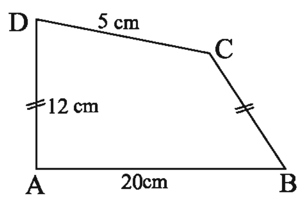
1) Describe the above information with figure.
2) Determine the area of the path.
3) How many bricks of 25 cm length and 12.5 cm width will be required to make the path metalled?

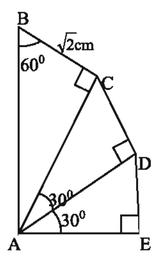
15. From the information of the figure below, determine the area of the polygon.

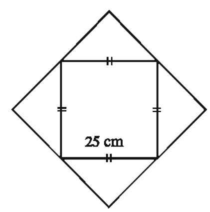


16. From the information of the figures below, determine the area of the polygons.





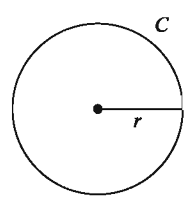




Measurement regarding circle

1. **Circumference of a circle**

The length of a circle is called its circumference. Let r be the radius of a circle, its circumference $c = 2\pi r$, where $\pi = 3.14159265\cdots$ which is an irrational number. Value of $\pi = 3.1416$ is used as the approximate value. Therefore, if the radius of a circle is known, we can find the approximate value of the circumference of the circle by using the value of π.



Example 18. The diameter of a circle is 26 cm. Find its circumference.

Solution: Let, the radius of the circle is r.

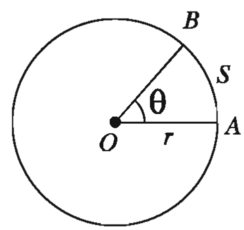
$\therefore$ diameter of the circle $= 2r$ and circumference $= 2\pi r$

As per question, $2r = 26$ or, $r = \frac{26}{2}$ or, $r = 13$ cm

$\therefore$ circumference of the circle $= 2\pi r = 2 \times 3.1416 \times 13$ cm $= 81.68$ cm (approx.)

2. **Length of arc of a circle**

Let O be the centre of a circle whose radius is r and arc $AB = s$, which produces θ° angle at the centre.
$\therefore$ circumference of the circle $= 2\pi r$
Total angle produced at the centre of the circle $= 360^\circ$ and arc s produces angle θ° at the centre. We know, any interior angle at the centre of a circle produced by any arc is proportional to the arc.



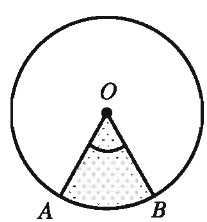
$$\therefore \frac{\theta}{360^\circ} = \frac{s}{2\pi r} \text{ or, } s = \frac{\pi r \theta}{180^\circ}$$

3. **Area of circular region and circular segment**

The area, surrounded by any circle, is called a circular region and the circle is called the boundary of the circular region.

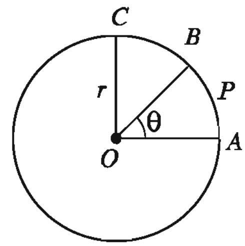
Circular segment: The area formed by an arc and the radius that joins the end points of that arc to the centre of the circle is called circular segment.

If A and B are two points on a circle with centre O, interior to $\angle AOB$, radius OA and OB, and the arc AB, form a circular segment. In previous class, we have learnt that if the radius of a circle is r, the area $= \pi r^2$.
We know, any angle produced by an arc at the centre of a circle is proportional to the arc.



So, at this stage we can accept that the area of two circular segments of the same circle are proportional to the two arcs on which they stand.

Let us draw a radius r with centre O. The circular segment AOB stands on the arc APB whose measurement is θ. Draw a perpendicular OC on OA



$$\therefore \frac{\text{Area of circular segment } AOB}{\text{Area of circular segment } AOC} = \frac{\text{Measurement of} \angle AOB}{\text{Measurement of} \angle AOC}$$

or, $$\frac{\text{Area of circular segment } AOB}{\text{Area of circular segment } AOC} = \frac{\theta}{90^\circ}$$

$[\because \angle AOC = 90^\circ]$

or, Area of circular segment $AOB = \dfrac{\theta}{90^\circ} \times$ area of circular segment AOC

$$= \frac{\theta}{90^\circ} \times \frac{1}{4} \times \text{ area of the circle}$$

$$= \frac{\theta}{90^\circ} \times \frac{1}{4} \times \pi r^2$$

$$= \frac{\theta}{360^\circ} \times \pi r^2$$

So, area of circular segment $= \dfrac{\theta}{360^\circ} \times \pi r^2$

Example 19. The radius of a circle is 8 cm and a circular segment subtends an angle 56° at the centre. Find the length of the arc and area of the circular segment.

Solution: Let, radius of the circle $r = 8$ c m, length of arc is s and the angle subtended by the circular segment is $\theta = 56^\circ$.

We know, $s = \dfrac{\pi r \theta}{180^\circ} = \dfrac{3.1416 \times 8 \times 56^\circ}{180^\circ}$ cm $= 7.82$ cm (approx.) and

Area of circular segment $= \dfrac{\theta}{360^\circ} \times \pi r^2 = \dfrac{56}{360} \times 3.1416 \times 8^2$ square cm $= 31.28$ square cm (approx.)

Example 20. If the difference between the radius and circumference of a circle is 90 cm, find the radius of the circle.

Solution: Let the radius of the circle be r.

$\therefore$ Diameter of the circler is $2r$ and circumference $= 2\pi r$

As per question, $2\pi r - 2r = 90$

or, $2r(\pi - 1) = 90$

or, $r = \dfrac{90}{2(\pi - 1)} = \dfrac{45}{3.1416 - 1} = 21.01\ \text{cm}$ (approx.)

The required radius of the circle is 21.01 cm (approx.)

Example 21. The diameter of a circular field is 124 metre There is a path with 6 metre. width around the field. Find the area of the path.

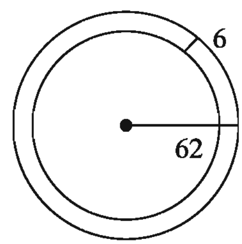
Solution:

Let the radius of the circular field be r and radius of the field with the path be R

$\therefore\ r = \dfrac{124}{2}$ metre $= 62$ metre and $R = (62 + 6)$ metre $= 68$ metre

Area of the circular field $= \pi r^2$ and area of the circular field with the path $= \pi R^2$

$\therefore$ Area of the path = Area of field with path − Area of the field



$= (\pi R^2 - \pi r^2) = \pi(R^2 - r^2)$

$= 3.1416(68^2 - 62^2) = 3.1416(4624 - 3844)$

$= 3.1416 \times 780 = 2450.44$ square metre (approx.)

The required area of the path is 2450.44 square metre (approx.)

> **Work:** Circumference of a circle is 440 metre. Determine the length of the sides of the inscribed square in it.

Example 22. The radius of a circle is 12 cm and the length of an arc is 14 cm. Determine the angle subtended by the circular segment at its centre.

Solution: Let, radius of the circle is $r = 12$ cm, the length of the arc is $s = 14$ cm and the angle subtended at the centre is θ.

We know, $s = \dfrac{\pi r \theta}{180}$

or, $\pi r \theta = 180 \times s$

or, $\theta = \dfrac{180 \times s}{\pi r} = \dfrac{180 \times 14}{3.1416 \times 12} = 66.84°$ (approx.)

The required angle is 66.84° (approx.)

Example 23. Diameter of a wheel is 4.5 metre. To traverse a distance of 360 metre, how many times the wheel will revolve?

Solution: Given that, the diameter of the wheel is 4.5 metre.

$\therefore$ The radius of the wheel, $r = \dfrac{4.5}{2} = 2.25$ metre and circumference $= 2\pi r$.

Let, for traversing 360 metre , the wheel will revolve n times.

As per question, $n \times 2\pi r = 360$

or, $n = \dfrac{360}{2\pi r} = \dfrac{360}{2 \times 3.1416 \times 2.25} = 25.46$ (approx.)

$\therefore$ The wheel will revolve 25 times. (approx.)

Example 24. Two wheels revolve 32 and 48 times respectively to cover a distance of 211 metre 20cm. Determine the difference of their radii.

Solution: 211 metre 20 cm $= 21120$ cm

Let, the radii of two wheels are R and r respectively ; where $R > r$.

$\therefore$ Circumferences of two wheels are $2\pi R$ and $2\pi r$ respectively and the difference of radii is $(R - r)$.

As per the question, $32 \times 2\pi R = 21120$.

or, $R = \dfrac{21120}{32 \times 2\pi} = \dfrac{21120}{32 \times 2 \times 3.1416} = 105.04$ cm (approx.)

and $48 \times 2\pi r = 21120$

or, $r = \dfrac{21120}{48 \times 2\pi} = \dfrac{21120}{48 \times 2 \times 3.1416} = 70.03$ cm (approx.)

$\therefore R - r = (105.04 - 70.03) = 35.01$ cm $= 0.35$ m (approx.)

The difference of radii of the two wheels is 0.35 metre (approx.)

Example 25. The radius of a circle is 14 cm. The area of a square is equal to the area of the circle. Determine the length of the square.

Solution: Let the radius of the circle, $r = 14$ cm and the length of the square is a.

$\therefore$ Area of the square region $= a^2$ and the area of the circle $= \pi r^2$.

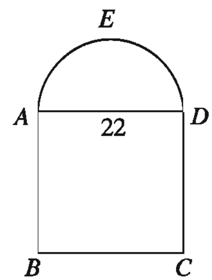
According to the question, $a^2 = \pi r^2$

or, $a = \sqrt{\pi} r = \sqrt{3.1416} \times 14 = 24.81$ (approx.)

The required length is 24.81 cm (approx.)

Example 26. In the figure, $ABCD$ is a square whose length of each side is 22 metre and region AED is a half circle. Determine the area of the whole region.

Solution: Let the length of each side of the square $ABCD$ be a.



$\therefore$ Area of square region $= a^2$
Again, AED is a half circle.

$\therefore$ Radius of the half circle $r = \dfrac{22}{2}$ metre $= 11$ metre.

Therefore, area of the half circle $= \dfrac{1}{2}\pi r^2$

$\therefore$ Area of the whole region $=$ Area of the square $ABCD$ + area of the half circle AED

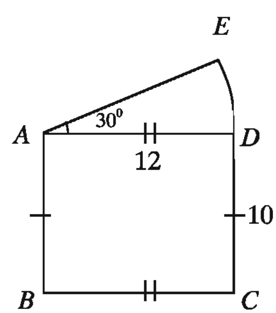
$= (a^2 + \dfrac{1}{2}\pi r^2)$

$= (22^2 + \dfrac{1}{2} \times 3.1416 \times 11^2) = 674.07$ square metre (approx.)

The required area is 674.07 square metre (approx.)

Example 27. In the figure, $ABCD$ is a rectangle whose length is 12 metre the breadth is 10 metre. and DAE is a circular segment. Determine the length of the arc DE and the area of the whole region.

Solution: The radius of the circular segment, $r = AD = 12$ metre and the angle subtended at centre $\theta = 30°$.



$\therefore$ length of the arc $DE = \dfrac{\pi r\theta}{180}$

$= \dfrac{3.1416 \times 12 \times 30}{180} = 6.28$ metre (approx.)

Area of the circular segment $ADE = \dfrac{\theta}{360} \times \pi r^2$.

$= \dfrac{30}{360} \times 3.1416 \times 12^2 = 37.7$ square metre (approx.)

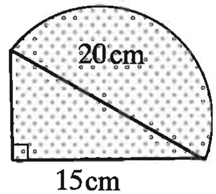
The length of the rectangle $ABCD$ is 12 metre and breadth is 10 metre.

$\therefore$ Area of the rectangle $=$ length $\times$ breadth $= 12 \times 10 = 120$ square metre

$\therefore$ Area of the whole region $= (37.7 + 120)$ square metre $= 157.7$ square metre (approx.)

The required area is 157.7 square metre (approx.).

Work: Determine the area of the dark marked region in the figure.



Exercises 16.3

1. Angle subtended by a circular segment at the centre is 30°. If the diameter of the circle is 126 cm, determine the length of the arc.

2. A horse circled around a field with a speed of 66 metre per minute in $1\frac{1}{2}$ minute. Determine the diameter of the field.

3. Area of a circular segment is 77 square metre and the radius is 21 metre. Determine the angle subtended at the centre by the circular arc.

4. The radius of a circle is 14 cm and an arc subtends an angle 75° at its centre. Determine the area of the circular segment.

5. There is a road around a circular field. The outer circumference of the road is greater than the inner circumference by 44 meters. Find the width of the road.

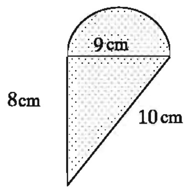
6. The diameter of a circular park is 26 metre. There is a road of 2 metre width surrounding the park. Determine the area of the road.

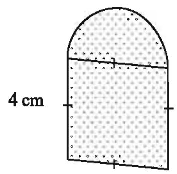
7. The diameter of the front wheel of a car is 28 cm and the back wheel is 35 cm. To cover a distance of 88 metre, how many integer number of times more the front wheel will revolve than the back one ?

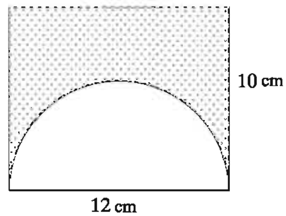
8. The circumference of a circle is 220 metre. Determine the length of the side of the square, inscribed in the circle.

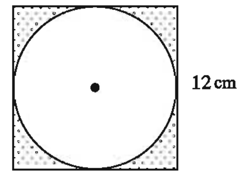
9. The circumference of a circle is equal to the perimeter of an equilateral triangle. Determine the ratio of their areas.

10. Determine the area of the dark marked region with the help of the information given below :









Solids

Rectangular solid

The region surrounded by three pairs of parallel rectangular planes or surfaces is known as rectangular solid.

Let, $ABCDEFGH$ is a rectangular solid, whose length $AB = a$, and breadth $BC = b$ and height $AH = c$.

1. Determining the diagonal: AF is the diagonal of the rectangular solid $ABCDEFGH$.

   In $\triangle ABC$, $BC \perp AB$ and AC is the hypotenuse.

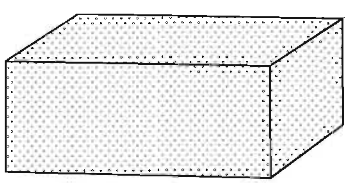
   $\therefore AC^2 = AB^2 + BC^2 = a^2 + b^2$

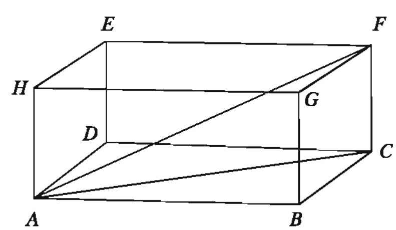
   Again, in $\triangle ABC, FC \perp AC$ and AF is hypotenuse.

   $\therefore AF^2 = AC^2 + CF^2 = a^2 + b^2 + c^2$

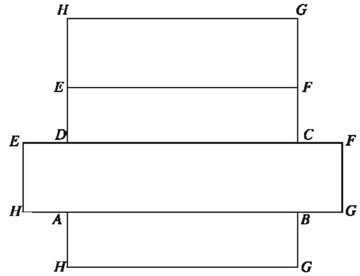
   $\therefore AF = \sqrt{a^2 + b^2 + c^2}$

   $\therefore$ the diagonal of the rectangular solid $= \sqrt{a^2 + b^2 + c^2}$


Rectangular solid



2. Determination of area of the whole surface: There are 6 surfaces of the rectangular solid where the opposite surfaces are of equal dimensions.



Area of the whole surface of the rectangular solid
$= 2($ area of the surface of $ABCD$ + area of the surface of $ABGH$ + area of the surface of $BCFG)$
$= 2(AB \times AD + AB \times AH + BC \times BG)$
$= 2(ab + ac + bc) = 2(ab + bc + ca)$

3. Volume of the rectangular solid = length × width × height = abc

Example 28. The length, width and height of a rectangular solid are 25 cm, 20 cm and 15 cm respectively. Determine its area of the whole surface, volume and the length of the diagonal.

Solution: Let, the length of the rectangular solid is $a = 25$ cm, width $b = 20$ cm and height $c = 15$ cm

$\therefore$ Area of the whole surface of the rectangular solid $= 2(ab + bc + ca)$

$= 2(25 \times 20 + 20 \times 15 + 15 \times 25) = 2350$ square cm

and Volume $= abc = 25 \times 20 \times 15 = 7500$ cube cm

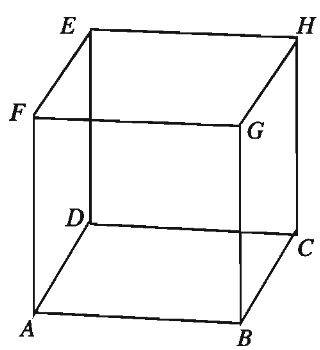
and length of its diagonal $= \sqrt{a^2 + b^2 + c^2}$

$= \sqrt{25^2 + 20^2 + 15^2} = \sqrt{624 + 400 + 225} = \sqrt{1250} = 35.363$ cm (approx.)

The required area of the whole surface is 2350 square cm, volume 7500 cubic cm and length of its diagonal 35.363 cm (approx.).

Work: Determine the volume, area of the whole surface and the length of the diagonal of your mathematics book after measuring its length, width and height.

Cube



Cube

If the length, width and height of a rectangular solid are equal, it is called a cube.
Let, $ABCDEFGH$ is a cube.
Its length = width = height = a units

1. The length of diagonal of the cube
$= \sqrt{a^2 + a^2 + a^2} = \sqrt{3a^2} = \sqrt{3}a$
2. The area of the whole surface of the cube
$= 2(a \cdot a + a \cdot a + a \cdot a) = 2(a^2 + a^2 + a^2) = 6a^2$
3. The volume of the cube $= a \cdot a \cdot a = a^3$

Example 29. The area of the whole surface of a cube is 96 square metre. Determine the length of its diagonal.

Solution: Let, the sides of the cube is a.

$\therefore$ The area of its whole surface $= 6a^2$ and length of its diagonal $= \sqrt{3}a$

As per question, $6a^2 = 96$ or, $a^2 = 16 \therefore a = 4$

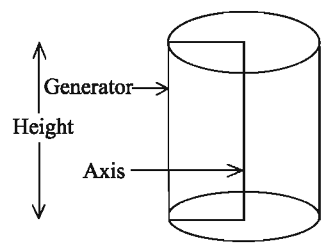
$\therefore$ The length of the diagonal $= \sqrt{3} \cdot 4 = 6.928$ metre (approx.).

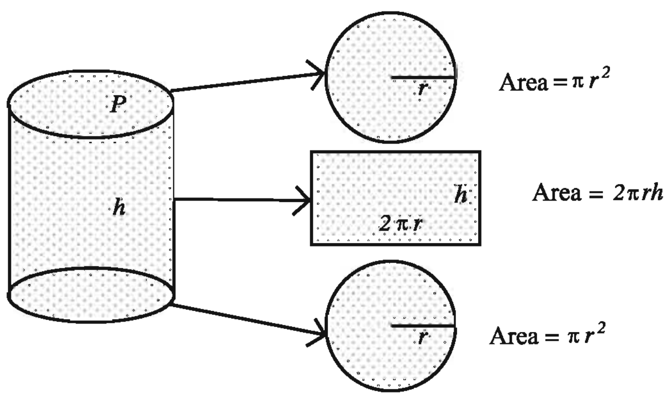
The required length of the diagonal 6.928 metre (approx.).

Work: The sides of 3 metal cubes are 3 cm, 4 cm and 5 cm respectively. A new cube is formed by melting these 3 cubes. Determine the area of the whole surface and the length of the diagonal of the new cube.

Cylinder

The solid formed by a complete revolution of any rectangle about one of its sides as axis is called a cylinder or a right circular cylinder. The two ends of a right circular cylinder are circular surfaces. The curved face is called curved surface and the total plane is called whole surface. The side of the rectangle which is parallel to the axis and revolves about the axis is called the generator line of the cylinder.





The figure above is a right circular cylinder, whose radius is r and height h.

1. Area of the base $= \pi r^2$
2. Area of the curved surface $=$ perimeter of the base $\times$ height$= 2\pi rh$
3. Area of the whole surface

   $= (\pi r^2 + 2\pi rh + \pi r^2) = 2\pi r(r+h)$
4. Volume $=$ Area of the base $\times$ height$= \pi r^2 h$

Example 30. If the height of a right circular cylinder is 10 cm and radius of the base is 7 cm, determine its volume and the area of the whole surface.

Solution: Let, the height of the right circular cylinder is $h = 10$ cm and radius of the base is r.

$\therefore$ Its volume $= \pi r^2 h$

$= 3.1416 \times 7^2 \times 10 = 1539.38$ cube cm (approx.)

and the area of the whole surface $= 2\pi r(r+h)$

$= 2 \times 3.1416 \times 7(7+10) = 747.7$ square metre (approx.)

> **Work:** Make a right circular cylinder using a rectangular paper. Determine the area of its whole surface and the volume.

Example 31. The outer measurements of a box with its top are 10 cm, 9 cm and 7 cm respectively and the area of the whole inner surface is 262 square metre and the thickness of its wall is uniform on all sides.

1) Find the volume of the box.
2) Find the thickness of its wall.
3) If the length of a diagonal of a rhombus having sides equal to the largest length of the box is 16 cm, then find its area.

Solution:

1) The outer measurements of the box with top are 10 cm, 9 cm and 7 cm.

   $\therefore$ The outer volume of the box $= 10 \times 9 \times 7 = 630$ cube cm
2) Let, the thickness of the box is x. The outer measurements of the box with top are 10 cm, 9 cm 7 cm.

   $\therefore$ The inside measurements of the box are respectively $a = (10 - 2x)$, $b = (9 - 2x)$ and $c = (7 - 2x)$ cm.

   The area of the whole surface of the inner side of the box $= 2(ab + bc + ca)$

As per question, $2(ab+bc+ca)=262$

or, $(10-2x)(9-2x)+(9-2x)(7-2x)+(7-2x)(10-2x)=131$

or, $90-38x+4x^2+63-32x+4x^2+70-34x+4x^2-131=0$

or, $12x^2-104x+92=0$

or, $3x^2-26x+23=0$

or, $3x^2-3x-23x+23=0$

or, $3x(x-1)-23(x-1)=0$

or, $(x-1)(3x-23)=0$

or, $x-1=0$ or $3x-23=0$

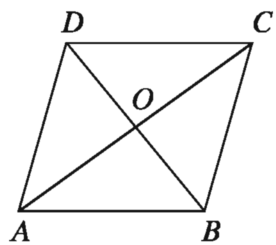
or, $x=1$ or, $x=\dfrac{23}{3}=7.67$ (approx.)

But the thickness of a box cannot be greater than or equal to any of the sides.

The required thickness of the box is 1 cm

3) Let, length of each side of $ABCD$ is 10 cm and the diagonals intersect each other at the point O.

We know, diagonals of rhombus bisect each other at right angle.



$\therefore OA=OC,\ OB=OD$

In the right-angled triangle $\triangle AOB$, hypotenuse $AB=10$

Here, $AB^2=10^2=100=36+64$

$=6^2+8^2=OB^2+OA^2$ [according to the figure]

$\therefore\ OB=6,\ OA=8$

$\therefore$ diagonal $AC=2\times 8=16$ cm and diagonal $BD=2\times 6=12$ cm

$\therefore$ area of the rhombus $ABCD=\dfrac{1}{2}\times AC\times BD=\dfrac{1}{2}\times 16\times 12=96$ square cm

Example 32. If the length of diagonal of the surface of a cube is $8\sqrt{2}$ cm, determine the length of its diagonal and volume.

Solution:

Let, the side of the cube is a.

$\therefore$ The length of diagonal of the surface $= \sqrt{2}a$. Length of diagonal $= \sqrt{3}a$ and volume $= a^3$

As per question, $\sqrt{2}a = 8\sqrt{2}$ or, $a = 8$

$\therefore$ The length of the cube's diagonal $= \sqrt{3} \times 8 = 13.856$ cm (approx.)

and volume $= 8^3 = 512$ cube cm.

The required length of the diagonal is 13.856 cm (approx.) and volume 512 cubic cm.

Example 33. The length of a rectangle is 12 cm and width 5 cm. Determine the area of its whole surface and the volume of the solid that is formed by revolving the rectangle around its greater side.

Solution: Given that, the length of a rectangle is 12 cm. and width 5 cm. If it is revolved around the greater side, a right circular cylinder is formed with height $h = 12$ cm and radius of the base $r = 5$ cm

The whole surface of the produced solid $= 2\pi r(r + h)$

$= 2 \times 3.1416 \times 5(5 + 12) = 534.071$ square cm (approx.)

and volume $= \pi r^2 h$

$= 3.1416 \times 5^2 \times 12 = 942.48$ cube cm (approx.)

The required area of whole surface is 534.071 square cm (approx.) and volume 942.48 cube cm (approx.)

Exercises 16.4

1. The length and witdth of two adjacent sides of a parallelogram are 7 cm, and 5 c.m. respectively. What is the half of its perimeter in cm?

   1) 12 2) 20 3) 24 4) 28

2. The length of the side of an equilateral triangle is 6 cm. What is its area ($c.m^2$)?

   1) $3\sqrt{3}$ 2) $4\sqrt{3}$ 3) $6\sqrt{3}$ 4) $9\sqrt{3}$

3. In plane geometry:

   (*i*) Each angle of equilateral triangle is less than one right angle.

   (*ii*) Sum of acute angles of a right-angled triangle is one right angle.

   (*iii*) An exterior angle of a triangle formed by extending one side of triangle is greater than each of the opposite interior angle .

   Which of the following is true?

   1) *i* and *ii* 2) *i* and *iii* 3) *ii* and *iii* 4) *i*, *ii* and *iii*

4. If the length of each side of a square is a and diagonal is d, then:

   (*i*) Its area is a^2 square units

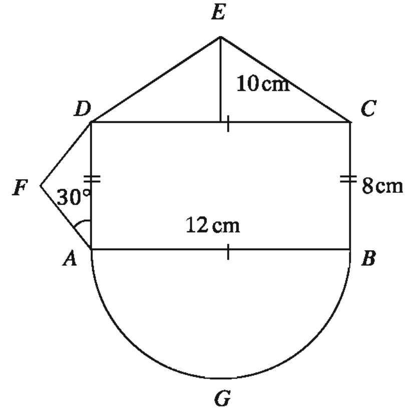
   (*ii*) Perimeter is $2ad$ units

   (*iii*) $d = \sqrt{2}a$

   Which of the following is true?

   1) *i* and *ii* 2) *i* and *iii* 3) *ii* and *iii* 4) *i*, *ii* and *iii*

   Answer the following questions (5–7) as per information from the picture below:



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5. What is the length of the diagonal of the rectangle $ABCD$ in cm?
   1) 13   2) 14   3) 14.4   4) 15

6. What is the area of the triangle ADF in square cm?
   1) 16   2) 32   3) 64   4) 128

7. What is the circumference of the half circle AGB in cm?
   1) 18   2) 18.85 (approx.)   3) 37.7 (approx.)   4) 96

8. The length, width and height of a rectangular solid are 16 metre, 12 metre and 4.5 metre respectively. Determine the area of its whole surface, length of the diagonal and the volume.

9. The ratios of the length, width and height of a rectangular solid are 21 : 16 : 12 and the length of diagonal is 87 cm. Determine the area of the whole surface of the solid.

10. A rectangular solid is standing on a base of area 48 square metre. Its height is 3 metre and diagonal is 13 metre. Determine the length and width of the rectangular solid.

11. The outer measurements of a rectangular wooden box are 8 cm., 6 cm. and 4 cm., respectively and the area of the whole inner surface is $88cm^2$. Find the thickness of the wood of the box.

12. The length of a wall is 25 metre, height is 6 metre. and breadth is 30 cm. The length, breadth and height of a brick is 10 cm 5 cm and 3 cm. respectively. Determine the number of bricks required to build the wall with the bricks.

13. The area of the surface of a cube is 2400 square cm. Determine the diagonal of the cube.

14. The radius of the base of a cylinder with height 12 cm, is 5 cm. Find the area of the whole surface and the volume of the cylinder.

15. The area of a curved surface of a right circular cylinder is 100 square cm. and its volume is 150 cubic cm. Find the height and the radius of the cylinder.

16. The area of the curved surface of a right circular cylinder is 4400 square cm. If its height is 30 cm., find the area of its whole surface.

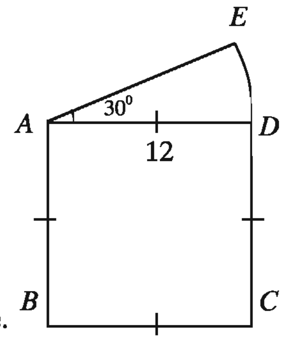
17. The inner and outer diameter of a iron pipe is 12 cm and 14 cm respectively and the height of the pipe is 5 metre Find the weight of the iron pipe where weight of 1 cm^3 iron is 7.2 gm.

18. The length and the breadth of a rectangular region are 12 metre and 5 metre respectively. There is a circular region just around the rectangle. The places which are not occupied by the rectangle, are planted with grass.

    1) Describe the information above with a figure.
    2) Find the diameter of the circular region.
    3) If the cost of planting grass per sq. metre is Tk. 50, find the total cost.

19. The figure is composed of a square and a circular segment.

    1) Determine the length of the diagonal and perimeter of the square.
    2) Find the total area of the whole region.
    3) If a regular hexagon having sides equal to the length of the square is inscribed in a circle, then determine the area of the unoccupied region of the circle.



20. $ABCD$ is a parallelogram and $BCEF$ is a rectangle and BC is the base of both of them.

    1) Draw a figure of the rectangle and the parallelogram assuming the same height.
    2) Show that the perimeter of $ABCD$ is greater than the perimeter of $BCEF$.
    3) Ratio of length and width of the rectangle is 5 : 3 and its perimeter is 48 metre. Determine the area of the parallelogram

21. The perimeters of a square region and a rectangular region are same. The length of the rectangle is 3 times of its width and its area is 1200 square metre.

    1) Determine the perimeter of the rectangle using the variable x.
    2) Determine the area of the square region.
    3) Consider a 1.5 metre wide path just around the rectangular region. How many 25×12.5 square cm bricks will be required to make the path metalled ?

Chapter 17

Statistics

Owing to the contribution of information and data, the world has become a global village for the rapid advancement of science and information. Globalization has been made possible due to rapid transformation and expansion of information and data. So, to keep the continuity of development and for participating and contribute in globalization, it is essential for the students at this stage to have clear knowledge about information and data. In the context, to meet the demands of students in acquiring knowledge, information and data have been discussed from class VI and class-wise contents have been arranged step by step. In continuation of this, the students of this class will know and learn cumulative frequency, frequency polygon, ogive curve in measuring of central tendency mean, median, mode etc. in short-cut method.

At the end of this chapter, the students will be able to —

- explain cumulative frequency, frequency polygon and ogive curve.
- explain data by the frequency polygon, and ogive curve.
- explain the method of measuring of central tendency.
- explain the necessity of short-cut method in the measurement of central tendency.
- find the mean, median and mode by the short-cut method.
- explain the diagram of frequency polygon and ogive curve.

Presentation of Data: We know that numerical information which are not qualitative are the data of statistics. The data under investigation are the raw materials of statistics. They are in unorganized form and it is not possible to take necessary decision directly from the unorganized data. It is necessary to organize and tabulate the data. And the tabulation of data is the presentation of the data. In previous class we have learnt how to organize the data in tabulation. We know that it is required to determine the range of data for tabulation. Then determining the class interval and the number of classes by using tally marks, the frequency distribution table is made. Here, the methods of making frequency distribution table are to be re-discussed through example for convenient understanding.

Example 1. In a winter season, the temperature (in celsius) of the month of January in the district of Srimangal is placed below. Find the frequency distribution table of the temperature.

14°, 14°, 14°, 13°, 12°, 13°, 10°, 10°, 11°, 12°, 11°, 10°, 9°, 8°, 9°, 11°, 10°, 10°, 8°, 9°, 7°, 6°, 6°, 6°, 6°, 7°, 8°, 9°, 9°, 8°, 7°

Solution: Here the minimum and maximum numerical values of the data of temperature are 6 and 14 respectively.
Hence the range $= (14 - 6) + 1 = 9$
If the class interval is considered to be 3, the numbers of class will be $\frac{9}{3}$ or 3
Considering 3 to be the class interval, if the data are arranged in 3 classes, the frequency table will be:

Temperature (in celcius)	Tally	Frequency
$6° - 8°$	$\cancel{\|\|\|\|}$ $\cancel{\|\|\|\|}$ $\|$	11
$9° - 11°$	$\cancel{\|\|\|\|}$ $\cancel{\|\|\|\|}$ $\|\|\|$	13
$12° - 14°$	$\cancel{\|\|\|\|}$ $\|\|$	7
	Total	31

Work: Form two groups of all the students studying in your class. Find the frequency distribution table of the weights (in Kgs) of all the members of the groups.

Cumulative Frequency: In Example 1 considering 3 the class interval and determining the number of classes, the frequency distribution table has been made. The number of classes of the mentioned data is 3. The limit of the first class is $6° - 8°$. The lowest range of the class is 6° and the highest range is 8°. The frequency of this class is 11. Similarly, The limit of the second class is $9° - 11°$ and the frequency of this class is 13. Now if the frequency 11 of first class is added to the frequency 13 of the second class, we get 24. This 24 will be the cumulative frequency of the second class and the cumulative frequency of first class as begins with the class will be 11. Again, if the cumulative frequency 24 of the second class is added to the frequency of the third class, we get $24 + 7 = 31$ which is the cumulative frequency of the third class. Thus cumulative frequency distribution table is made. In the context of the above discussion, the cumulative frequency distribution of temperature in Example 1 is as follows:

Temperature (in celcius)	Frequency	Cumulative Frequency
$6^\circ - 8^\circ$	11	11
$9^\circ - 11^\circ$	13	$(11 + 13) = 24$
$12^\circ - 14^\circ$	7	$(24 + 7) = 31$

Example 2. The marks obtained in English by 40 students in an annual examination are given below. Make a cumulative frequency table of the marks obtained.

70, 40, 35, 60, 55, 58, 45, 60, 65, 80, 70, 46, 50, 60, 65, 70, 58, 60, 48, 70, 36, 85, 60, 50, 46, 65, 55, 61, 72, 85, 90, 68, 65, 50, 40, 56, 60, 65, 46, 76

Solution:
Range of the data = (highest numerical value − lowest numerical value) + 1 = $(90 - 35) + 1 = 55 + 1 = 56$

Let the class interval be 5 then the number of classes $= \frac{56}{5} = 11.2$ or 12 [taking the immediate next integer when class interval is fractional]

Hence the cumulative frequency distribution table at a class interval of 5 will be as follows:

Obtained marks	Tally	Frequency	Cumulative frequency
$35 - 39$	||	2	2
$40 - 44$	||	2	$2 + 2 = 4$
$45 - 49$	卌	5	$5 + 4 = 9$
$50 - 54$	|||	3	$3 + 9 = 12$
$55 - 59$	卌	5	$5 + 12 = 17$
$60 - 64$	卌 ||	7	$7 + 17 = 24$
$65 - 69$	卌 |	6	$6 + 24 = 30$
$70 - 74$	卌	5	$5 + 30 = 35$
$75 - 79$	|	1	$1 + 35 = 36$
$80 - 84$	|	1	$1 + 36 = 37$
$85 - 89$	||	2	$2 + 37 = 39$
$90 - 94$	|	1	$1 + 39 = 40$

Variable: We know that the numerical information is the data of statistics. The numbers used in data are variable. Such as, in example 1 the numbers indicating temperatures are variable. Similarly, in example 2, the secured marks used in the data are the variables.

Discrete and Indiscrete variable: The variables used in statistics are of two types. Such as, discrete and indiscrete vari ables. The variables whose values are only integers, are discrete variables. The marks obtained in example 2 are discrete variables. Similarly, only integers are used in population indicated data. That is why, the variables of data used for population are discrete variables. A nd the variables whose numerical values can be any real number are indiscrete variables. Such as, in example 1, the temperature indicated data which can be any real number. Besides, any real number can be used for the data related to age, height, weight etc. That is why, the variables used for those are indiscrete variables. The number between two indiscrete variables can be the value of those variables. Some times it becomes necessary to make class interval indiscrete. To make the class interval indiscrete, the actual higher limit of a class and the lower limit of the next class are determined by fixing mid-point of a higher limit of any class and the lower limit of the next class. Such as, in example 1 the actual higher-lower limits of the first class are 8.5° and 5.5° respectively and that of the second class are 11.5° 8.5° etc.

Work: Form a group of maximum 40 students of your class. Form frequency distribution table and cumulative frequency table of the group with the weights/heights of the members.

Diagram of Data: We have seen that the collected data under investigation are the raw materials of the statistics. If the frequency distribution and cumulative frequency distribution table are made with them, it becomes clear to comprehend and to draw a conclusion. If that tabulated data are presented through diagram, they become easier to understand as well as attractive. That is why, presentation of statistical data in tabulation and diagram is widely and frequently used method. In class VIII, different types of diagram in the form of line graph and histogram have been discussed elaborately and the students have been taught how to draw them. Here, how frequency polygon, pie-chart, ogive curve are drawn from frequency distribution and cumulative frequency table will be discussed.

Frequency Polygon: In class VIII, we have learnt how to draw the histogram of discrete data. Here how to draw frequency polygon from histogram of indiscrete data will be put for discussion through example.

Example 3. The frequency distribution table of the weights (in Kg) of 60 students of class X of a school is given below:

Weight (Kg)	46 – 50	51 – 55	56 – 60	61 – 65	66 – 70
Frequency (No. of students)	5	10	20	15	10

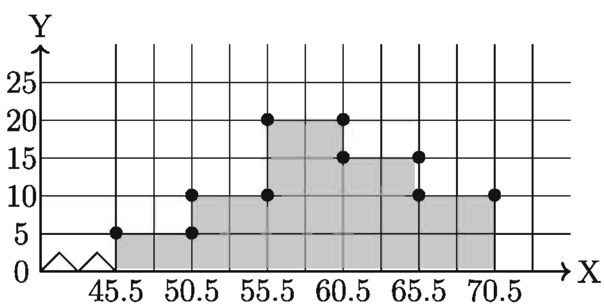
1) Draw the histogram of frequency distribution.

2) Draw frequency polygon of the histogram.

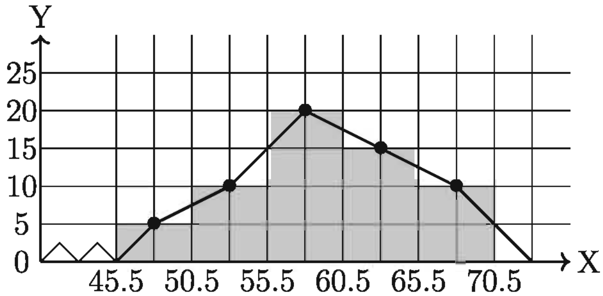
Solution: The class interval of the data in the table is discrete. If the class interval are made indiscrete, the table will be :

Class interval of the weight (in Kg)	In Discrete class interval	Mid point of class	Frequency
46 – 50	45.5 – 50.5	48	5
51 – 55	50.5 – 55.5	53	10
56 – 60	55.5 – 60.5	58	20
61 – 65	60.5 – 65.5	63	15
66 – 70	65.5 – 70.5	68	10

1) Histogram has been drawn taking each square of graph paper as 5 unit of class interval along with x-axis and frequency along with y-axis. The class interval along with x-axis has started from 45.5. The broken segments $\triangle\triangle$ have been used to show the presence of previous squares starting from from origin to 45.5.



2) The mid-points of the opposite sides parallel to the base of rectangle of the histogram have been fixed for drawing frequency polygon from histogram. The mid-points have been joined by line segments to draw the frequency polygon (shown in the adjacent figure). The mid-points of the first and the last rectangles have been joined with x-axis representing the class interval by the end points of line segments to show the frequency polygon attractive

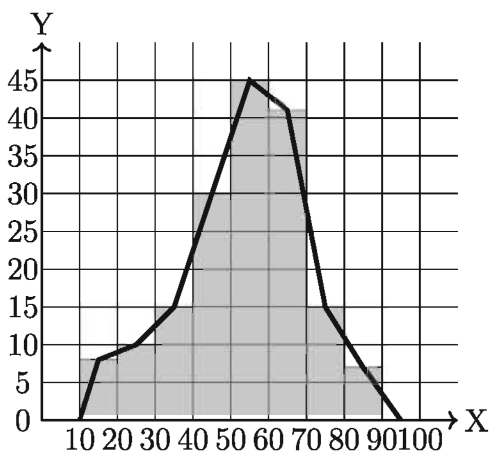


Frequency Polygon: The diagram drawn by joining frequency indicated points opposite to the class interval of indiscrete data by line segments successively is frequency polygon.

Example 4. Draw polygon of the following frequency distribution table.

Class interval	10 – 20	20 – 30	30 – 40	40 – 50	50 – 60	60 – 70	70 – 80	80 – 90
Mid-point	15	25	35	45	55	65	75	85
Frequency	8	10	15	30	45	41	15	7

Solution: Histogram of frequency distribution is drawn taking each square of graph paper as 10 units of class interval along with x-axis and each square of graph paper as 5 units of frequency along with y-axis. The mid-points of the sides opposite to the base of rectangle of histogram are identified which are the mid-points of the class. Now the fixed mid-points are joined. The end-points of the first and the last classes are joined to x-axis representing the class interval to draw frequency polygon.



Work: Draw frequency polygon from the marks obtained in Bangla by the students of your class in first terminal examination.

Example 5. The frequency distribution table of the marks obtained by 50 students of class X in science are given. Draw the frequency polygon of the data (without using histogram):

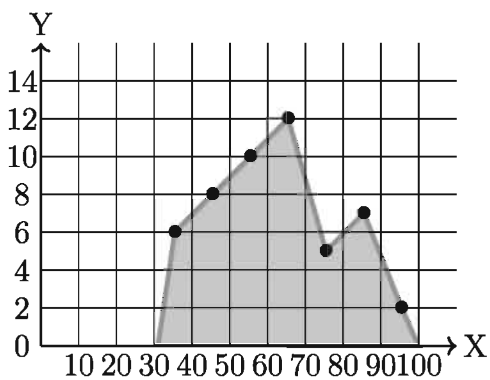
Class interval	31 – 40	41 – 50	51 – 60	61 – 70	71 – 80	81 – 90	91 – 100
Frequency	6	8	10	12	5	7	2

Solution: Here the given data are discrete. In this case, it is convenient to draw frequency polygon directly by finding the mid-point of class interval.

The Mid-point of the first class interval (31 – 40) is $\dfrac{31+40}{2} = 35.5$

Class interval	31 – 40	41 – 50	51 – 60	61 – 70	71 – 80	81 – 90	91 – 100
Mid-point	35.5	45.5	55.5	65.5	75.5	85.5	95.5
Frequency	6	8	10	12	5	7	2

The polygon is drawn by taking each squares of graph paper as 1 units of mid-points of class interval along with x-axis and taking each square of graph paper as 2 units of frequency along with y-axis.



Work: Draw frequency polygon from the frequency distribution table of heights of 100 students of a college.

Heights (in cm.)	141 – 150	151 – 160	161 – 170	171 – 180	181 – 190
Frequency	5	16	56	11	12

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Cumulative Frequency Diagram or Ogive curve: Cumulative frequency diagram or Ogive curve is drawn by taking the upper limit of class interval along with x-axis and cumulative frequency along with y-axis after classification of a data.

Example 6. The frequency distribution table of the marks obtained by 50 students out of 60 students is as follows:

Class interval of marks obtained	1 – 10	11 – 20	21 – 30	31 – 40	41 – 50
Frequency	8	12	15	18	7

Draw the Ogive curve of this frequency distribution.

Solution: The cumulative frequency table of frequency distribution of the given data is :

Class interval of marks obtained	1 – 10	11 – 20	21 – 30	31 – 40	41 – 50
Frequency	8	12	15	18	7
Cumulative Frequency	8	$8 + 12 = 20$	$15 + 20 = 35$	$18 + 35 = 53$	$7 + 53 = 60$

Ogive curve of cumulative frequency of data is drawn taking each square of graph paper as two units of upper limit of class interval along with x-axis and cumulative frequency along with y-axis.

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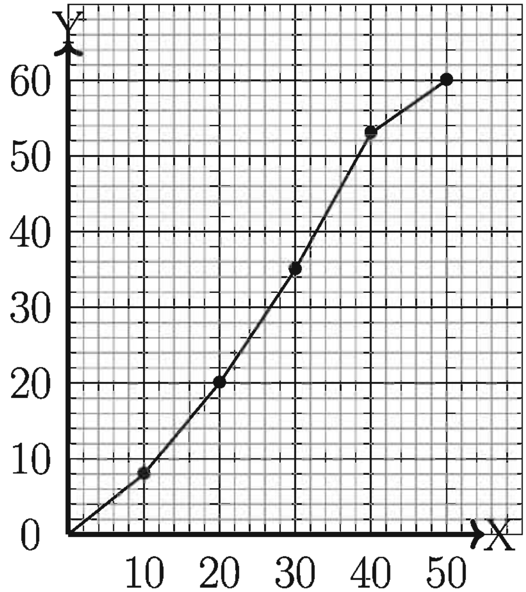

> **Work:** Make cumulative frequency table of the marks obtained 50 and above in Mathematics by the students of your class in an examination and draw an Ogive curve.

Central Tendency: Central tendency and its measurement have been discussed in class VII and VIII. We have seen if the data under investigation are arranged in order of values, the data cluster round near any central value. A gain if the disorganized data are placed in frequency distribution table, the frequency is found to be abundant in a middle class i.e. frequency is maximum in middle class. In fact, the tendency of data to be clustered around the central value is number and it represents the data. The central tendency is measured by this number. Generally, the measurement of central tendency is of three types (1) Arithmetic means (2) Median (3) Mode :

Arithmetic Mean: We know if the sum of data is divided by the numbers of the data, we get the arithmetic mean. But this method is complex, time consuming and there is every possibility of committing mistake for large numbers of data. In such cases, the data are tabulated through classification and the arithmetic mean is determined by short-cut method.

Example 7. The frequency distribution table of the marks obtained by the students of a class is as follows. Find the arithmetic mean of the marks.

Class interval	25 – 34	35 – 44	45 – 54	55 – 64	65 – 74	75 – 84	85 – 94
Frequency	5	10	15	20	30	16	4

Solution: Here class interval is given and that is why it is not possible to know the individual marks of the students. In such case, it becomes necessary to know the mid-value of the class.

$$\text{Mid-value of the class} = \frac{\text{Class upper value} + \text{class lower value}}{2}$$

If the class mid-value is $x_i(i = 1 \dots k)$ the mid-value related table will be as follows:

Class interval	Class mid-value (x_i)	Frequency (f_i)	$(f_i x_i)$
25 – 34	29.5	5	147.5
35 – 44	39.5	10	395.5
45 – 54	49.5	15	742.5
55 – 64	59.5	20	1190.5
65 – 74	69.5	30	2085.5
75 – 84	79.5	16	1272.5
85 – 94	89.5	4	358.5
	Total	$n = 100$	6190.0

The required arithmetic mean

$$= \frac{1}{n} \sum_{i=1}^{k} f_i x_i = \frac{1}{100} \times 6190 = 61.9$$

Arithmetic mean of classified data (short-cut method): The short-cut method is easy for determining arithmetic mean of classified data. The steps to determine mean by short-cut method are :

1. To find the mid-value of classes.
2. To take convenient approximated mean (a) from the mid-values.
3. To determine steps deviation, the difference between class mid-values and approximate mean are divided by the class interval i.e.

   $$\text{steps deviation } u = \frac{\text{mid value} - \text{approximate mean}}{\text{class interval}}$$
4. To multiply the steps deviation by the corresponding class frequency.
5. To determine the mean of the deviation and to add this mean with approximate mean to find the required mean.

Short-cut method: The formula used for determining the mean of the data by this method is

$$\bar{x} = a + \frac{\sum f_i u_i}{n} \times h$$

where, $\bar{x}$ = required mean, a = approximate mean, f_i =class frequency of ith class, $u_i f_i$ =-the product of step deviation with class intervals of ith class and h = class interval.

Example 8. The production cost (in hundred taka) of a commodity at different stages is shown in the following table. Find the mean of the expenditure by short-cut method.

Production cost(in hundred taka)	$2-6$	$6-10$	$10-14$	$14-18$	$18-22$	$22-26$	$26-30$	$30-34$
Frequency	1	9	21	47	52	36	19	3

Solution: To determine mean in the light of followed steps in short-cut method, the table will be :

Class interval	Mid-value x_i	Frequency f_i	Step deviation $u_i = \frac{x_i - a}{h}$	Frequency and step deviation $f_i u_i$
$2-6$	4	1	-4	-4
$6-10$	8	9	-3	-27
$10-14$	12	21	-2	-42
$14-18$	16	47	-1	-47
$18-22$	$20 \leftarrow a$	52	0	0
$22-26$	24	36	1	36
$26-30$	28	19	2	38
$30-34$	32	3	3	9
Total		188		-37

Mean $\bar{x} = a + \frac{\sum f_i u_i}{n} \times h = 20 + \frac{-37}{188} \times 4 = 20 - 0.79 = 19.21$

$\therefore$ Mean production cost is Tk. 19 hundred.

Weighted mean: In many cases the numerical values $x_1, x_2, \ldots, x_n$ of statistical data under investigation may be influenced by different reasons/ importance/ weight. In such case, the values of the data $x_1, x_2, \ldots, x_n$ along with their reasons/ importance/ weight $w_1, w_2, \ldots, w_n$ are considered to find the arithmetic mean. If the values of n numbers of data are $x_1, x_2, \ldots, x_n$ and their weights are $w_1, w_2, \ldots, w_n$, the weighted mean will be

$$\overline{x}_w = \frac{\sum_{i=1}^{n} x_i w_i}{\sum_{i=1}^{n} w_i}$$

Example 9. The rate of passing in degree Honours class and the number of students of some department of a University are presented in the table below. Find the mean rate of passing in degree Honours class of those departments of the university.

Name of the department	Math	Statistics	English	Bangla	Zoology	Political Science
Rate of passing (%)	70	80	50	90	60	85
Number of students	80	120	100	225	135	300

Solution: Here, the rate of passing and the number of students are given. The weight of rate of passing is the number of students. If the variables of rate of passing are x and numerical variable of students is w, the table for determining the arithmetic mean of given weight will be as follows :

Department	Rate of passing x_i	Number of students w_i	$x_i w_i$
Math	70	80	5600
Satistics	80	120	9600
English	50	100	5000
Bangla	90	225	20250
Zoology	60	135	8100
Political Science	85	300	25500
Total		960	74050

$$\overline{x_w} = \frac{\sum_{i=1}^{6} x_i w_i}{\sum_{i=1}^{6} w_i} = \frac{74050}{960} = 77.14$$

$\therefore$ Mean rate of passing 77.14

Work: Collect the rate of passing students and their numbers in S.S.C. examination of some schools in your Upazilla and find mean rate of passing.

Median:

We have already learnt in class VIII the value of the data which divide the data when arranged in ascending order into two equal parts are median of the data.

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We have also learnt if the numbers of data are n and n is an odd number, the median will be the value of $\frac{n+1}{2}$ th term. But if n is an even number, the median will be numerical mean of the value of $\frac{n}{2}$ and $\left(\frac{n}{2}+1\right)$ th terms. Here we present through example how mean is determined with or without the help formulae.

Example 10. The frequency distribution table of 51 students is placed below. Find the median.

Height(cm)	150	155	160	165	170	175
Frequency	4	6	12	16	8	5

Solution: Cumulative frequency distribution table for finding mean is as follows :

Height(cm)	150	155	160	165	170	175
Frquency	4	6	12	16	8	5
Cumulative Frequency	4	10	22	38	46	51

Here, $n = 51$ is an odd number.

$\therefore$ Median = the value of $\frac{51+1}{2}$ th term = the value of26th term = 165

Required median 165 cm.

Note: The value of the terms from 23th to 38th is 165.

Example 11. The frequency distribution table of marks obtained in mathematics of 60 students is as follows. Find the median :

Marks obtained	40	45	50	55	60	70	80	85	90	95	100
Frequency	2	4	4	3	7	10	16	6	4	3	1

Solution: Cumulative frequency distribution table for determining median is :

Marks obtained	40	45	50	55	60	70	80	85	90	95	100
Frequency	2	4	4	3	7	10	16	6	4	3	1
Cumulative Frequency	2	6	10	13	20	30	46	52	56	59	60

Here, $n = 60$, which is an even number.

$$\therefore \text{ Median} = \frac{\frac{60}{2}\text{th term} + (\frac{60}{2}+1)\text{th term}}{2} = \frac{30\text{th term} + 31\text{th term}}{2}$$

$$= \frac{70+80}{2} = 75$$

$\therefore$ Required median 75

Work:

1) Make frequency distribution table of the heights (in cm.) of 49 students of your class and find the mean without using any formula.
2) From the above problem, deduct the heights of 9 students and then find the median of heights (in cm.) of 40 students.

Determining Median of Classified Data: If the number of classified data is n, the value of $\frac{n}{2}$th term of classified data is median. And the formula used to determine the median or the value of $\frac{n}{2}$th term is : Median $= L + \left(\frac{n}{2} - F_c\right) \times \frac{h}{f_m}$, where L is the lower limit of the median class, n is the frequency, F_c is the cumulative frequency of previous class to median class, f_m is the frequency of median class and h is the class interval.

Example 12. Determine median from the following frequency distribution table :

Time(sec.)	30 – 35	36 – 41	42 – 47	48 – 53	54 – 59	60 – 65
Frequency	3	10	18	25	8	6

1) What is a frequency distribution table?
2) Determine median from the frequency distribution table given above.
3) Draw the frequency polygon of the given data.

Solution:

1) Frequency distribution table refers to organize and tabulate a dataset by determining specific class interval and number of classes.
2) The frequency distribution table to determine median is given below:

Class interval	Frequency	Cumulative frequency
30 – 35	3	3
36 – 41	10	13
42 – 47	18	31
48 – 53	25	56
54 – 59	8	64
60 – 65	6	70
	$n = 70$	

Here, $n = 70$ and $\frac{n}{2} = \frac{70}{2}$ or 35

Therefore, median is the value of 35th term. 35th term lies in the class (48–53). Hence the median class is (48–53).

$\therefore L = 48, F_c = 31, f_m = 25$ and $h = 6$

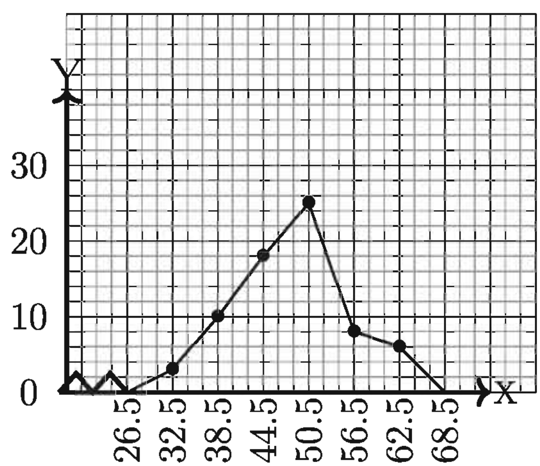
So median $= 48 + (35 - 31) \times \frac{6}{25} = 48 + 4 \times \frac{6}{25} = 48 + 0.96 = 48.96$

Required median 48.96.

3) The table useful to draw the frequency polygon is given below: The median of the class prior to the first class would be 26.5 and the median of the class following the last class would be 68.5.

Now, taking convenient units of median values per square unit along the X axis where, $\triangle\triangle$ sign indicates $0 - 26.5$ and taking each sqaure unit as two units of frequency along Y axis the frequency polygon has been drawn.

Class interval	Median	Frequency
30 – 35	32.5	3
36 – 41	38.5	10
42 – 47	44.5	18
48 – 53	50.5	25
54 – 59	56.5	8
60 – 65	62.5	6



Work: Make two groups with all the students of your class. (a) Make a frequency distribution table of the time taken by each of you to solve a problem, (b) find the median from the table.

Mode:

In class VIII, we have learned that the number which appears maximum times in a data is the mode of the data. In a data, there may be one or more than one mode. If there is no repetition of a member in a data, data will have no mode. Now we shall discuss how to determine the mode of classified data using formula.

Determining Mode of Classified Data: Mode $= L + \dfrac{f_1}{f_1 + f_2} \times h$, where

L is the lower limit of mode-class i.e. the class where the mode lies,

f_1 = frequency of mode-class – frequency of the class previous

f_2 = frequency of mode class – frequency of next class of mode class and h =class interval.

Example 13. Following is a table.

Class interval	31 – 40	41 – 50	51 – 60	61 – 70	71 – 80	81 – 90	91 – 100
Frequency	4	6	8	12	9	7	4

1) What is central tendency?
2) Find mode from the above table.
3) Draw ogive curve of the given data.

Solution:

1) If the disorganized data of statistics are arranged according to the value, the data cluster round near any central value. Moreover, abundance of data is observed in a single class when these data are presented in some frequency distribution table. This tendency of data to cluster around central value is known as central tendency.

2) The table to determine mode is given below:

Class	31 – 40	41 – 50	51 – 60	61 – 70	71 – 80	81 – 90	91 – 100
Frequency	4	6	8	12	9	7	4

$$\text{Mode} = L = \frac{f_1}{f_1 + f_2} \times h$$

Here, the maximum numbers of repetition of frequency is 12 which lies in the class (61–70).

$$\therefore L = 61, f_1 = 12 - 8 = 4, f_2 = 12 - 9 = 3, h = 10$$

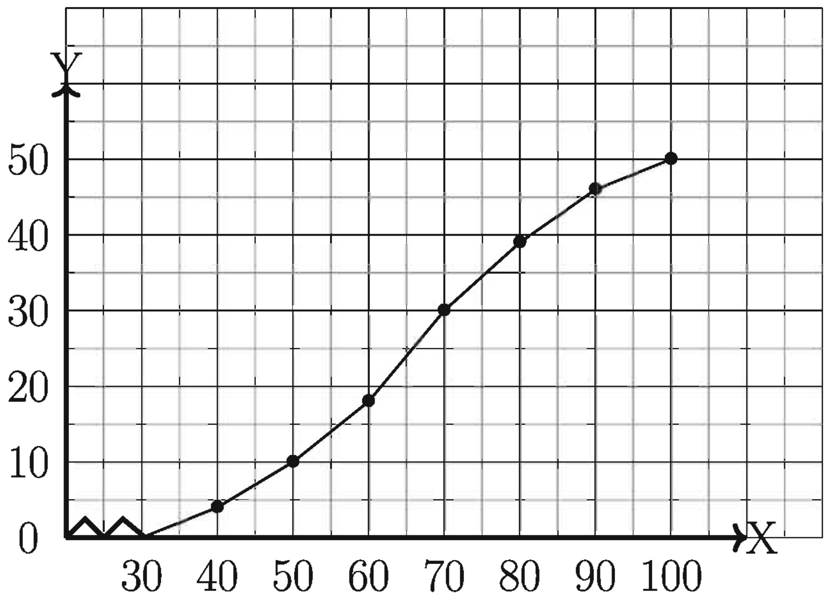
$$\therefore \text{Mode} = 61 + \frac{4}{4+3} \times 10 = 61 + \frac{4}{7} \times 10 = 61 + \frac{40}{7} = 61 + 5.7 = 66.7$$

Required mode 66.7

3) The table to draw the ogive curve is given below:

Class interval	Discontinuous class interval	Frequency	Cumulative frequency
31 – 40	30 – 40	4	4
41 – 50	40 – 50	6	10
51 – 60	50 – 60	8	18
61 – 70	60 – 70	12	30
71 – 80	70 – 80	9	39
81 – 90	80 – 90	7	46
91 – 100	90 – 100	4	50

Using convenient scaling along X axis where △△ (sign indicates 0 – 30 and taking per square unit as 5 units of cumulative frequency along Y axis points along higher limits of classes have been marked. Then these marked points have been joined and thus the Ogive curve has been drawn.



Example 14. Find mode from the frequency distribution table given below.

Class	41 – 50	51 – 60	61 – 70	71 – 80
Frequency	25	20	15	8

Solution: Here, maximum numbers of frequency are 25 which lie in the class (41–50). So it is evident that mode is in this class. We know that

Mode$= L + \dfrac{f_1}{f_1 + f_2} \times h$. Here, $L = 41,\ f_1 = 25 - 0,\ f_2 = 25 - 20$ [If the frequency is maximum in the first class, the frequency of previous class is zero]

$$\therefore \text{Mode} = 41 + \frac{25}{25+5} \times 10 = 41 + \frac{25}{30} \times 10 = 41 + 8.33 = 49.33$$

Required mode 49.33.

Example 15. Find mode from the frequency distribution table given below.

Class	11 – 20	21 – 30	31 – 40	41 – 50
Frequency	4	16	20	25

Solution: The maximum numbers of Class Frequency frequency are 25 which lie in the class (41 – 50). So it is obvious that this class is the class of mode. We know that, Mode $= L + \dfrac{f_1}{f_1 + f_2} \times h$

Here, $L = 41, f_1 = 25 - 20 = 5, f_2 = 25 - 0, h = 10$ [If the frequency is maximum in the last class, the frequency of following class is zero]

$\therefore$ Mode $= 41 + \dfrac{5}{25+5} \times 10 = 41 + \dfrac{5}{30} \times 10 = 41 + \dfrac{5}{3} = 41 + 1.67 = 42.67$

Required mode 42.67 (approximately)

> If the frequency is maximum in the first class, the frequency of previous class is zero. If the frequency is maximum in the last class, the frequency of following class is zero.

Exercise 17

1. Which one indicates the data included in each class when the data are classified?

   1) Class interval 2) Mid-point of the class 3) Number of classes 4) Class frequency

2. If the disorganized data of statistics are arranged according to the value, the data cluster round near any central value. This tendency of data is called

   1) Mode 2) Central tendency 3) Mean 4) Median

3. Consider the table below:

Temperature	$6^\circ - 8^\circ$	$8^\circ - 10^\circ$	$10^\circ - 12^\circ$
Frequency	5	9	4

   (i) Class interval is 3

   (ii) Median class is $8^\circ - 10^\circ$

   (iii) Temperature is a discontinuous variable

   Which of the following is true?

   1) i and ii 2) i and iii 3) ii and iii 4) i, ii, and iii

4. To draw histograms we need -

   (i) Discontinuous class interval along x axis

   (ii) Frequency along y axis

   (iii) Class mid-point

   Which of the following is true?

1) i and ii 2) i and iii 3) ii and iii 4) i, ii, and iii

5. In case of data, Mode is -

(*i*) Measures of central tendency

(*ii*) Represented value which is mostly occurred

(*iii*) May not unique in all respect

Which is correct on the basis of above information?

1) i and ii 2) i and iii 3) ii and iii 4) i, ii and iii

In winter, the statistics of temperatures (in Celsius) of a region in Bangladesh is $10°, 9°, 8°, 6°, 11°, 12°, 7°, 13°, 14°, 5°$. In the context of this statistics, answer questions(6- 8).

6. Which is the mode of the above numerical data?

1) 12° 2) 5° 3) 14° 4) no mode

7. Which one is the mean of temperature of the above numerical data?

1) 8° 2) 8.5° 3) 9.5° 4) 9°

8. Which one is the median of the data?

1) 9.5° 2) 9° 3) 8.5° 4) 8°

9. The number of classified data included in the table is n, the lower limit of median class is L, the cumulative data of previous class to median class is F_c, the frequency of median class is f_m and class interval is h. In the light of these information, which one is the formula for determining the median

1) $L + \left(\frac{n}{2} - F_c\right) \times \frac{h}{F_m}$ 2) $L + \left(\frac{n}{2} - F_m\right) \times \frac{h}{F_m}$

3) $L - \left(\frac{n}{2} - F_c\right) \times \frac{h}{F_m}$ 4) $L - \left(\frac{n}{2} - F_m\right) \times \frac{h}{F_m}$

10. The frequency distribution table of marks obtained in mathematics of 60 students of class X is given below. Draw frequency diagram and ogive curve.

Class interval	31 – 40	41 – 50	51 – 60	61 – 70	71 – 80	81 – 90	91 – 100
Frequency	6	8	10	12	5	7	2

11. Frequency distribution table of the marks obtained in mathematics of 50 students of class X are provided. Draw the frequency polygon of the provided data

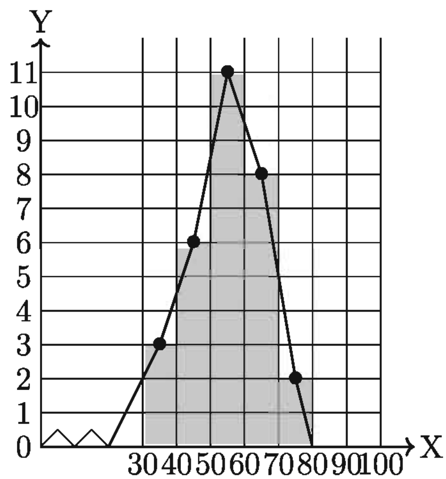
Weight (Kg)	45	50	55	60	65	70
Frequency	2	6	8	16	12	6

12. The following are the marks obtained in Mathematics of fifty students of class IX in a school:

    76, 65, 98, 79, 64, 68, 56, 73, 83, 57, 55, 92, 45, 77, 87, 46, 32, 75, 89, 48
    97, 88, 65, 73, 93, 58, 41, 69, 63, 39, 84, 56, 45, 73, 93, 62, 67, 69, 65, 53
    78, 64, 85, 53, 73, 34, 75, 82, 67, 62

    1) What is the type of the given information? What indicate frequency in a class of distribution?
    2) Make frequency table taking appropriate class interval.
    3) Determine the mean of the given number by short-cut method.

13.



    1) In the above figure, what is the mid-point value of the first class and what is the frequency of the last class?
    2) Express by data of information demonstrated in the above figure.
    3) Find the median of frequency obtained from 142.

14. The frequency distribution table of weights (in kg) of 60 students of a class are:

Class interval	45 – 49	50 – 54	55 – 59	60 – 64	65 – 69	70 – 74
Frequency	4	8	10	20	12	6

1) Write the formula to determine median.
2) Find the mode of the data.
3) Draw histogram of the data.

15. Temperature is always changing. Usually in Bangladesh temperature is comparatively lower during first week of January and comparatively higher during fourth week of June. Following is the list of temperatures of 52 weeks is celsius.

    35, 30, 27, 42, 20, 19, 27, 36, 39, 14, 15, 38, 37, 40, 40, 12, 10, 9, 7, 20, 21, 24, 33, 30, 29, 21, 19, 31, 28, 26, 32, 30, 22, 23, 24, 41, 26, 23, 25, 22, 17, 19, 21, 23, 8, 13, 23, 24, 20, 32, 11, 17

    1) Calculate the number of classes considering a class interval of 5.
    2) Express the given data in a tabular form and find the mean value of lowest and highest temperatures.
    3) Draw histogram using the table and find the mode.

Answers to exercises

Exercises 1

12. 1) $0.1\dot{6}$ 2) $0.\dot{6}\dot{3}$ 3) $3.\dot{2}$ 4) $3.5\dot{3}$

13. 1) $\dfrac{2}{9}$ 2) $\dfrac{35}{99}$ 3) $\dfrac{2}{15}$ 4) $3\dfrac{71}{90}$ 5) $6\dfrac{769}{3330}$

14. 1) $2.3\dot{3}\dot{3}$, $5.2\dot{3}\dot{5}$ 2) $7.2\dot{6}\dot{6}$, $4.2\dot{3}\dot{7}$
3) $5.\dot{7}77777\dot{7}$, $8.\dot{3}43434\dot{4}$, $6.\dot{2}45245\dot{5}$ 4) $12.32\dot{0}\dot{0}$, $2.19\dot{9}\dot{9}$, $4.32\dot{5}\dot{6}$

15. 1) $0.58\dot{9}$ 2) $17.117\dot{9}$ 3) $1.0700\dot{9}37\dot{2}$

16. 1) $1.3\dot{1}$ 2) $1.6\dot{6}\dot{5}$ 3) $3.13\dot{3}\dot{4}$ 4) $6.11\dot{6}0\dot{2}$

17. 1) $0.\dot{2}$ 2) 2 3) $0.2\dot{0}7\dot{4}$ 4) $12.1\dot{8}\dot{5}$

18. 1) 0.5 2) 0.2 3) $5.\dot{2}195\dot{1}$ 4) $4.\dot{8}$

19. 1) 3.4641, 3.464 2) 0.5025, 0.503
3) 1.1590, 1.160 4) 2.2650, 2.265

20. 1) rational 2) rational 3) irrational 4) irrational
5) irrational 6) rational 7) rational 8) rational

23. 1) 9 2) 5

Exercises 2.1

1. 1) $\{4, 5\}$ 2) $\{\ \ldots, -5, -4, -3,\ 3\ \}$ 3) $\{6, 12, 18, 36\}$ 4) $\{3, 4\}$

2. 1) $\{x \in N : x \text{ odd and } 1 < x < 13\}$
2) $\{x \in N : x \text{ is a factor of } 36\}$
3) $\{x \in N : x \text{ is a multiplier of } 4 \text{ and } x \leq 40\}$
4) $\{x \in Z : x^2 \geq 16 \text{ and } x^3 \leq 216\}$

3. 1) $\{1\}$ 2) $\{1, 2, 3, 4, a\}$ 3) $\{2\}$
4) $\{2, 3, 4, a\}$ 5) $\{2\}$

5. $P(Q) = \{\varnothing, \{x\}, \{y\}, \{x, y\}\}$

$P(R) = \{\varnothing, \{m\}, \{n\}, \{l\}, \{m, n\}, \{m, l\}, \{n, l\}, \{m, n, l\}\}$

7. 1) $2, 3$ 2) (c, a) 3) $(1, 5)$

8. 1) $\{(a, b), (a, c)\}, \{(b, a), (c, a)\}$ 2) $\{(4, x), (4, y), (5, x), (5, y)\}$
3) $\{(3, 3), (5, 3), (7, 3)\}$

9. $\{1, 3, 5, 7, 9, 15, 35, 45\}$ and $\{1, 5\}$

10. $\{35, 105\}$

11. 5 persons

Exercises 2.2

10. $\{(3, 2),\ (4, 2)\}$

11. $\{(2, 4),\ (2, 6)\}$

12. $-7,\ 23,\ -\dfrac{7}{16}$

13. 2

14. 1 or 2 or 3

15. $\dfrac{2}{x^2}$

17. 1) $\{2\},\ \{1, 2, 3\}$
2) $\{-2, -1, 0, 1, 2\},\ \{0, 1, 4\}$
3) $\left\{\dfrac{1}{2}, 1, \dfrac{5}{2}\right\},\ \{-2, -1, 0, 1, 2\}$

18. 1) $\{(-1, 2), (0, 1), (1, 0), (2, -1)\}, \{-1, 0, 1, 2\}, \{-1, 0, 1, 2\}$
2) $\{(0, 0), (1, 2)\},\ \{0, 1\},\ \{0, 2\}$

Exercises 3.1

1. 1) $4a^2 + 12ab + 9b^2$ 2) $x^4 + \dfrac{4x^2}{y^2} + \dfrac{4}{y^4}$
3) $16y^2 - 40xy + 25x^2$ 4) $25x^4 - 10x^2y + y^2$
5) $9b^2 + 25c^2 + 4a^2 - 30bc + 20ca - 12ab$
6) $a^2x^2 + b^2y^2 + c^2z^2 - 2abxy + 2bcyz - 2cazx$
7) $4a^2 + 9x^2 + 4y^2 + 25z^2 + 12ax - 8ay - 20az - 12xy - 30xz + 20yz$
8) 1014049

2. 1) $p^2 + 49q^2 - 14pq$ 2) $36n^2 - 24pn + 4p^2$
   3) 100 4) 3104

3. ± 16
4. $\pm 3m$
6. $\frac{1}{4}$
9. 19
10. 25
11. 6
12. 9
13. $(2a + b + c)^2 - (b - a - c)^2$
14. $(x + 5)^2 - 1^2$
15. 1) 3 2) 1

Exercises 3.2

1. 1) $8x^6 + 36x^4y^2 + 54x^2y^4 + 27y^6$ 2) $343m^6 - 294m^4n + 84m^2n^2 - 8n^3$
   3) $8a^3 - b^3 - 27c^3 - 12a^2b - 36a^2c + 6ab^2 + 54ac^2 - 9b^2c - 27bc^2 + 36abc$
2. 1) $8x^3$ 2) $8(b + c)^3$ 3) $64m^3n^3$
   4) $2(x^3 + y^3 + z^3)$ 5) $64x^3$

3. 665
4. 54
5. 8
6. 42880
8. 1) 3 2) 9
9. 1) 133 2) 665
10. $a^3 - 3a$
11. $p^3 + 3p$
16. $46\sqrt{5}$

Exercises 3.3

1. $b(x - y)(a - c)$
2. $(3x + 4)^2$
3. $(a^2 + 5a - 1)(a^2 - 5a - 1)$
4. $(x^2 + 2xy - y^2)(x^2 - 2xy - y^2)$
5. $(ax + by + ay - bx)(ax + by - ay + bx)$
6. $(2a - 3b + 2c)(2a - 3b - 2c)$
7. $(a + y + 2)(a - y + 4)$
8. $(4x - 5y)(4x + 5y - 2z)$
9. $(x + 4)(x + 9)$
10. $(x + 2)(x - 2)(x^2 + 5)$
11. $(a - 18)(a - 12)$
12. $(a^4 - 2)(a^4 + 1)$
13. $(x + 13)(x - 50)$
14. $y^2(x + 1)(9x - 14)$
15. $(x + 3)(x - 3)(4x^2 + 9)$
16. $(x + a)(ax + 1)$
17. $(a^2 + 2a - 4)(3a^2 + 6a - 10)$

18. $(x+ay+y)(ax-x+y)$
19. $(x+2)(x^2+x+1)$
20. $(a-3)(a^2-3a+3)$
21. $(a-b)(2a^2+5ab+8b^2)$
22. $(2x-3)(4x^2+12x+21)$
23. $\frac{1}{27}(6a+b)(36a^2-6ab+b^2)$
24. $\left(\frac{a^2}{3}-b^2\right)\left(\frac{a^4}{9}+\frac{a^2b^2}{3}+b^4)\right)$
25. $\left(2a-\frac{1}{2a}\right)\left(2a-\frac{1}{2a}+2)\right)$
26. $(a+4)(19a^2-13a+7)$
27. $(x^2+7x+4)(x^2+7x-18)$
28. $(x^2-8x+20)(x^2-8x+2)$
29. $(a+b+c)(b+c-a)(c+a-b)(a+b-c)$
30. $(2z-3x-5)(10x+7z+3)$

Exercises 3.4

1. $(a+1)(3a^2-3a+5)$
2. $(x+y)(x-3y)(x+2y)$
3. $(x-2)(x+1)(x+3)$
4. $(x-1)(x+2)(x+3)$
5. $(a+3)(a^2-3a+12)$
6. $(a-1)(a-1)(a^2+2a+3)$
7. $(a+1)(a-4)(a+2)$
8. $(x-2)(x^2-x+2)$
9. $(a-b)(a^2-6ab+b^2)$
10. $(x-3)(x^2+3x+8)$
11. $(x+1)(x+2)(x+3)$
12. $(x-2)(2x+1)(x^2+1)$
13. $(2x-1)(2x+1)(x+1)(x+2)$
14. $x(x-1)(x^2+x+1)(x^2-x+1)$
15. $(4x-1)(x^2-x+1)$
16. $(2x+1)(3x+2)(3x-1)$

Exercises 3.5

14. $\frac{2}{3}(p+r)$ days
15. 5 hours
16. 6 days
17. 100 persons
18. Speed of current $\frac{d}{2}\left(\frac{1}{q}-\frac{1}{p}\right)$ km/hr and speed of boat $\frac{d}{2}\left(\frac{1}{p}+\frac{1}{q}\right)$ km/hr
19. Speed of boat 8 km/hr and speed of current 2 km/hr
20. $\frac{t_1t_2}{t_2-t_1}$ minute
21. 240 litre
22. 1) Tk. 120 2) Tk. 80 3) Tk. 60
23. Tk. 450
24. Tk. 10
25. Tk. 48
26. 4%
27. Tk. 625
28. 28%
29. Tk. 600
30. Tk. 800
31. Tk. 61
32. Tk. $\frac{px}{100+x}$; VAT amount Tk. 300

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36. Time required will be more in presence of current

37. 40   38. $3\frac{1}{11}$ hour

Exercises 4.1

1. 27 2. $\sqrt{7}$ 3. $\frac{10}{7}$ 4. $\frac{ab}{3a+2b}$

5. $\frac{a^8}{b^4}$ 6. 1 7. 4 8. $\frac{1}{9}$

17. $\frac{3}{2}$ 18. 3 19. 5 20. $0, 1$

Exercises 4.2

1. 1) 4 2) $\frac{1}{3}$ 3) $\frac{1}{2}$ 4) 4 5) $\frac{5}{6}$

2. 1) 125 2) 5 3) 4

4. 1) $\log_{10}2$ 2) $\frac{13}{15}$ 3) 0

Exercises 4.3

11. 1) 6.530×10^3 2) 6.0831×10^1 3) 2.45×10^{-4}
 4) 3.75×10^7 5) 1.4×10^{-7}

12. 1) 100000   2) 0.00001   3) 25300
 4) 0.009813   5) 0.0000312

13. 1) 3 2) 1 3) 0
 4) $\bar{2}$ 5) $\bar{5}$

14. 1) Char 1, Mant .43136   2) Char 1, Mant .80035
 3) Char 0, Mant .14768   4) Char $\bar{2}$, Mant .65896
 5) Char $\bar{4}$, Mant .82802

15. 1) 1.66706   2) $\bar{1}.64562$ 3) 0.81358   4) $\bar{3}.78888$

16. 1) 0.95424   2) 1.44710   3) 1.62325

Exercises 5.1

1. ab 2. -6 3. $-\frac{3}{5}$ 4. $-\frac{5}{2}$
5. $\frac{a+b}{2}$ 6. $a+b$ 7. $\frac{a+b}{2}$ 8. $\sqrt{3}$
9. $\{4(1+\sqrt{2})\}$ 10. $\varnothing$ 11. $\{-\frac{1}{3}\}$ 12. $\{\frac{m+n}{2}\}$
13. $\{-\frac{7}{2}\}$ 14. $\{6\}$ 15. $28, 70$ 16. $\frac{3}{4}$
17. 72 19. 3200 20. 18 21. 9
22. 100, 20 23. 120 km 24. $10\frac{4}{5}$ km

Exercises 5.2

11. ± 7 12. $-\frac{3\sqrt{2}}{2}, \frac{2\sqrt{3}}{3}$ 13. $-6, \frac{3}{2}$
14. $1, -\frac{3}{20}$ 15. $0, \frac{2}{3}$ 16. $\sqrt{ab}$
17. $0, a+b$ 18. $3, -\frac{1}{2}$ 19. $2, -\frac{2}{3}$
20. $-a, -b$ 21. 1, 1 22. $1, \frac{1}{3}$
23. 78 or 87 24. 16 m., 12 m. 25. 9 cm., 12 cm.
26. 27 cm. 27. 21 persons, Tk. 20 28. 70 persons
32. Nabila's age 28 years, Shuvo's age 21 years 33. 9 persons
34. 4 : 30

Exercises 9.1

2. $\cos A = \frac{\sqrt{7}}{4}, \tan A = \frac{3}{\sqrt{7}}, \cot A = \frac{\sqrt{7}}{3}, \sec A = \frac{4}{\sqrt{7}}, \operatorname{cosec} A = \frac{4}{3}$
3. $\sin A = \frac{15}{17}, \sec A = \frac{17}{8}$
4. $\sin\theta = \frac{5}{13}, \cos\theta = \frac{12}{13}, \tan\theta = \frac{5}{12}$
22. $\frac{1}{2}$
23. $\frac{3}{4}$

24. $\dfrac{a^2 - b^2}{a^2 + b^2}$

Exercises 9.2

8. $\dfrac{1}{2}$
9. $\dfrac{3}{4}$
10. $\dfrac{23}{5}$
11. $\dfrac{2\sqrt{2}}{3}$
19. $A = 30°, \ B = 30°$
20. $A = 30°$
21. $A = 37\dfrac{1}{2}°, \ B = 7\dfrac{1}{2}°$
23. $\theta = 90°$
24. $\theta = 60°$
25. $\theta = 60°$
26. $\theta = 45°, 60°$
27. $\dfrac{7}{2}$

Exercises 10

10. 45.033 m. (approx)
11. 34.641 m. (approx)
12. 12.728 m. (approx)
13. 10 m.
14. 21.651 m. (approx)
15. 141.962 m. (approx)
16. 27.713 m. (approx) and 16 m.
17. 34.298 m. (approx)
18. 44.785 m. (approx)

Exercises 11.1

1. $a^2 : b^2$
2. $\pi : 2\sqrt{\pi}$
3. 45, 60
4. 20%
5. 18 : 25
6. 13 : 7
8. 1) $\dfrac{3}{4}$ 2) $\pm\sqrt{2ab - b^2}$ 3) $\dfrac{1}{2}$, 2

Exercises 11.2

10. 70%
11. A Tk. 40, B Tk. 60, C Tk. 120, D Tk. 80
12. 200, 240, 250
13. 9, 15, 21
14. 140

15. 81 runs, 54 runs, 36 runs
16. Officers Tk. 24000, Clerks Tk. 12000 and Assistants Tk. 6000
17. 44%
18. 1% reduction
19. 532 Quintal
20. 8 : 9
21. 1440 sq.m.
22. 13 : 12

Exercises 12.1

1. consistent, independent, unique solution
2. consistent, dependent, infinitely many solutions
3. inconsistent, independent, no solution
4. consistent, dependent, infinitely many solutions
5. consistent, independent, unique solution
6. inconsistent, independent, no solution
7. consistent, dependent, infinitely many solutions
8. consistent, independent, unique solution
9. consistent, independent, unique solution
10. consistent, independent, unique solution

Exercises 12.2

1. $(4, -1)$
2. $(\frac{6}{5}, \frac{6}{5})$
3. (a, b)
4. $(4, -1)$
5. $(1, 2)$
6. $\left(\frac{c(b-c)}{a(b-a)}, \frac{c(c-a)}{b(b-a)}\right)$
7. $(-\frac{17}{2}, 4)$
8. $(2, 3)$
9. $(3, 2)$
10. $(\frac{5}{2}, -\frac{22}{3})$
11. $(1, 2)$
12. $(2, -1)$
13. (a, b)
14. $(2, 4)$
15. $(-5, -3)$

Exercises 12.3

1. $(2, 2)$
2. $(2, 3)$
3. $(-7, 3)$
4. $(4, 5)$
5. $(2, 3)$
6. $(\frac{3}{2}, \frac{3}{2})$
7. $(1, \frac{1}{2})$
8. $(2, 6)$
9. -2
10. 2

Exercises 12.4

10. $\frac{7}{9}$
11. $\frac{15}{26}$
12. 27
13. 37 or 73
14. 30 years
15. length 17 m., width9 m.
16. speed of boat 10 km/hr
17. Tk. 4000, Tk. 125
20. 11 and 6
21. $\frac{29}{57}$
22. 40 and 20 m./sec
23. 7
24. 22 times

Exercises 13.1

5. -7 and -75
6. 129th
7. 100th
8. 0
9. n^2
10. 360
11. 320
12. 42
13. 1771
14. -620
15. 18
16. 50
17. $2 + 4 + 6 + \ldots$
18. 110
19. 0
20. $-(m + n)$
23. 50

Exercises 13.2

5. $\frac{1}{2}$
6. $\frac{3}{2}(3^{14} - 1)$
7. 9th element
8. $\frac{1}{\sqrt{3}}$
9. 9th element
10. $x = 15$ and $y = 45$
11. $x = 9, y = 27, z = 81$
12. 86
13. 1
14. 55log2
15. 650log2
16. $n = 7$
17. 0
18. $n = 6, S = 21$
19. $n = 5, S = 55$
21. 20
22. 24.47 mm. (approx)

Exercises 16.1

1. 20 m., 15 m. 2. 12 m. 3. 12 sq.m.
4. 327.26 sq.cm. (approx) 5. 5 m.
6. 30° 7. 12 or 16 m. 8. 44.44 km. (approx)
9. 24.249 cm. (approx), 254.611 sq.cm. (approx)

Exercises 16.2

1. 96 m. 2. 1056 sq.m. 3. 30 m. and 20 m.
4. 400 sq.m. 5. 6400 6. 16 m. and 10 m.
7. 16.5 m. and 22 m. 8. 35.35 m. (approx) 9. 48.66 cm. (approx)
10. 72 cm., 1944 sq.cm. 11. 17 cm. and 9 cm. 12. 95.75 sq.m. (approx)
13. 6.363 sq.m. (appox)

Exercises 16.3

1. 32.987 cm. (approx) 2. 31.513 m. (approx) 3. 20.008° (approx)
4. 128.282 sq.cm.(approx) 5. 7.003 m. (approx)
6. 175.93 sq.m. (approx) 7. 20 times 8. 49.517 m.(approx)
9. $3\sqrt{3} : \pi$

Exercises 16.4

8. 636 sq.m., 20.5 m., 864 cubic m. 9. 14040 sq.cm.
10. 12 m., 4 m. 11. 1 cm. 12. 300000
13. 34.641 cm. (appox) 14. 534.071 sq.cm. (appox), 942.48 cubic cm. (approx) 15. 5.305 cm., 3 cm. 16. 7823.591 sq.cm.
17. 147.027 kg. (approx)

Exercises 17

10. Do it yourself 11. 60 kg.

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9-10 Math

‘একজন ঘুমন্ত মানুষ আরেকজন ঘুমন্ত মানুষকে জাগিয়ে তুলতে পারে না।’
–শেখ সাদি

সমৃদ্ধ বাংলাদেশ গড়ে তোলার জন্য যোগ্যতা অর্জন কর
– মাননীয় প্রধানমন্ত্রী শেখ হাসিনা

তথ্য, সেবা ও সামাজিক সমস্যা প্রতিকারের জন্য ‘৩৩৩’ কলসেন্টারে ফোন করুন
নারী ও শিশু নির্যাতনের ঘটনা ঘটলে প্রতিকার ও প্রতিরোধের জন্য ন্যাশনাল হেল্পলাইন সেন্টারে ১০৯ নম্বর-এ (টোল ফ্রি, ২৪ ঘণ্টা সার্ভিস) ফোন করুন

Ministry of Education

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