Discrete Mathematics Coursework

January 2018

Contents

1 Question 1

In each of the following statements replace '?' by the element such that the statement is true. If there is no solution give a reason, if there are more solutions, give all of them.

i $\{\{?, ?, ?\}\} \subseteq \{1, 2, 3, \{5, 6, 7\}\}$

ii $\{?, ?\} \in \{1, 2, \{3, 4\}\}$

iii $\{?, ?\} \subset \{A, B, C, \{D, E\}\}$

1.1 Answer

i $\{\{5, 6, 7\}\}$

ii $\{3, 4\}$

iii Possible solutions are $\{A, B\}$ or $\{A, C\}$ or $\{B, C\}$

2 Question 2

Give an example of a formula which describes the following infinite set:
$\{8, 11, 14, 17, 20, \ldots\}$

2.1 Answer

$\{x_n \mid x_n \in N \; and \; x_n \geq 8 \; and \; x_{n+1} - x_n = 3\}$

3 Question 3

Rewrite the set by listing its elements in each of the following sets (show your working):

i $\{x + y \mid x \in \{-2, -1, 0, 1, 2\} \; and \; y \in \{0, 1, -1, 2\}\}$

ii $\{n \mid n \in N \; and \; n^3 + n + 5 \; is \; a \; multiple \; of \; 7 \; and \; 1 < n \leq 4\}$

3.1 Answer

i $x \in \{-2,-1,0,1,2\}\ and\, y \in \{0,1,-1,2\}$
$x=-2,\ \ y=0\ \rightarrow\ x+y=-2$
$x=-2,\ \ y=1\ \rightarrow\ x+y=-1$
$x=-2,\ \ y=-1\ \rightarrow\ x+y=-3$
$x=-2,\ \ y=2\ \rightarrow\ x+y=0$
$x=-1,\ \ y=0\ \rightarrow\ x+y=-1$
$x=-1,\ \ y=1\ \rightarrow\ x+y=0$
$x=-1,\ \ y=-1\ \rightarrow\ x+y=-2$
$x=-1,\ \ y=2\ \rightarrow\ x+y=1$
$x=0,\ \ y=0\ \rightarrow\ x+y=0$
$x=0,\ \ y=1\ \rightarrow\ x+y=1$
$x=0,\ \ y=-1\ \rightarrow\ x+y=-1$
$x=0,\ \ y=2\ \rightarrow\ x+y=2$
$x=1,\ \ y=0\ \rightarrow\ x+y=1$
$x=1,\ \ y=1\ \rightarrow\ x+y=2$
$x=1,\ \ y=-1\ \rightarrow\ x+y=0$
$x=1,\ \ y=2\ \rightarrow\ x+y=3$
$x=2,\ \ y=0\ \rightarrow\ x+y=2$
$x=2,\ \ y=1\ \rightarrow\ x+y=3$
$x=2,\ \ y=-1\ \rightarrow\ x+y=1$
$x=2,\ \ y=2\ \rightarrow\ x+y=4$

Collecting all x + y values and excluding duplicates we get
$\{-3,-2,-1,0,1,2,3,4\}$

ii $n=2\ \ \rightarrow\ n^3+n+5=15$ (is not a multiple of 7)
$n=3\ \ \rightarrow\ n^3+n+5=35$ (is a multiple of 7)
$n=4\ \ \rightarrow\ n^3+n+5=73$ (is not a multiple of 7)
$\therefore$ answer is $\{3\}$

4 Question 4

Justify your answers for the following:

i How many subsets of the set $\{a,b,c,d\}$ doesn't contain the element a?

ii How many subsets of the set $\{a,b,c,d\}$ contains the elements a or b (including the subsets which contain both elements a, b)?

4.1 Answer

i Let $set\ A = \{a, b, c, d\}$
Number of subsets of $A = |P(A)|$ where $P(A)$ is the power set of A
$|P(A)| = 2^n$ where $n = |A|$
Number of subsets of A $= 2^4 = 16$

If we remove element 'a' from set A and get the power set of the resulting set, it will be equivalent to the set of subsets of A without 'a'.
Let $B = A - \{a\}$
$|P(B)| = 2^3 = 8$

$\therefore$ Number of subsets of the set $\{a, b, c, d\}$ that doesn't contain the element a is 8.

ii Let $A = \{a, b, c, d\}$

Subsets of A containing element 'a' or 'b' $= \{a\}, \{b\}, \{a, b\}, \{a, c\}, \{a, d\}, \{b, c\}, \{b, d\}, \{a, b, c\}, \{a, b, d\}, \{a, c, d\}, \{b, c, d\}, \{a, b, c, d\}$
$\therefore$ Answer $= 12$

5 Question 5

i How many elements are in the set $\{(a, b)|a, b \in N \times N\ and\ 1 \leq a \leq b \leq 3\}$. Justify your answer.

ii Can you generalise your approach from part (i) to find how many elements are in the set $\{(a, b)|a, b \in N \times N\ and\ 1 \leq a \leq b \leq n\}\ for\ n \geq 1, n \in N$.

5.1 Answer

i Let A $= \{(a, b)|a, b \in N \times N\ and\ 1 \leq a \leq b \leq 3\}$
$A = \{(1, 1), (1, 2), (1, 3), (2, 2), (2, 3), (3, 3)\}$
$|A| = 3 + 2 + 1 = 6$
$\therefore$ Answer $= 6$

ii For the set $\{(a, b)|a, b \in N \times N\ and\ 1 \leq a \leq b \leq n\}\ for\ n \geq 1, n \in N$
there will be:
n pairs for a=1,
n-1 pairs for a=2,
n-2 pairs for a=3 ...

and so on until for a=n where there will only be one pair.
Hence the total number of elements is $n + (n-1) + (n-2) + \cdots + 2 + 1$

$\therefore$ Answer $= n + (n-1) + (n-2) + \cdots + 2 + 1 = n(n+1)/2$

6 Question 6

Let A, B and C be sets. Is it true that always $A \cup (B \cap C) = (A \cup B) \cap C$? Either prove the statement of find a counter example.

6.1 Answer

Plotting the venn diagram of $A \cup (B \cap C)$

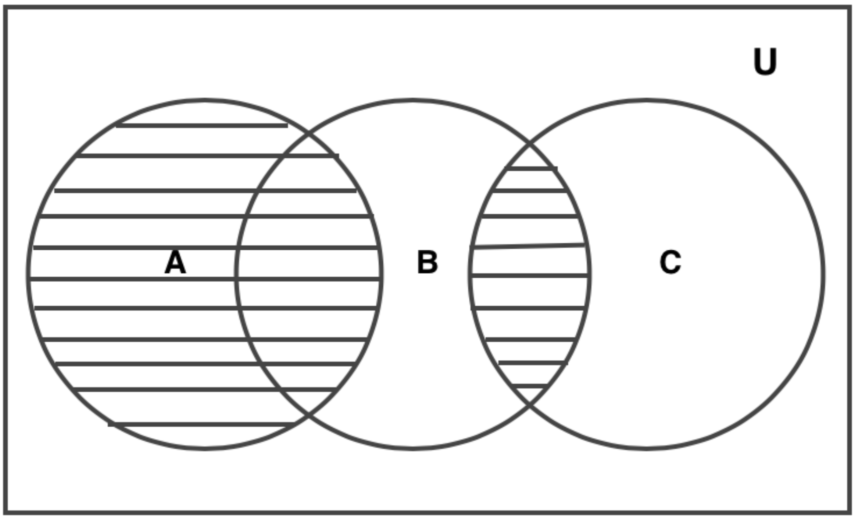

Figure 1: $A \cup (B \cap C)$

Plotting the venn diagram of $(A \cup B) \cap C$

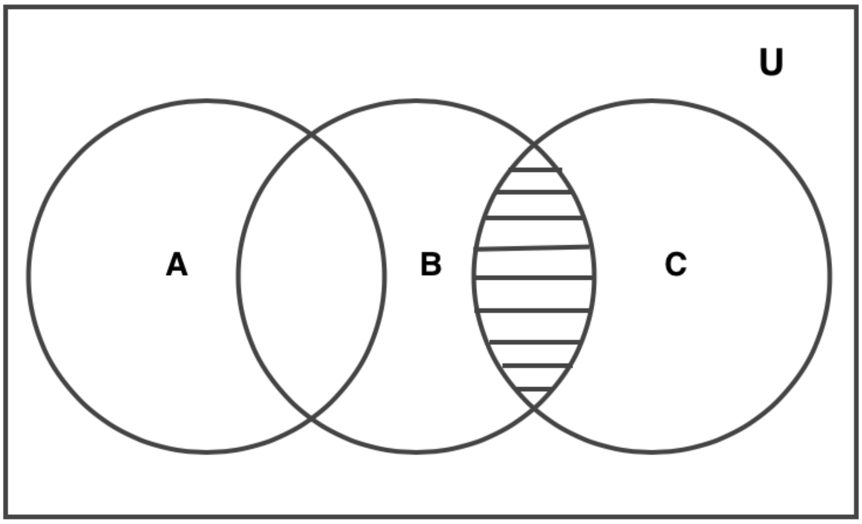

Figure 2: $(A \cup B) \cap C$

This pair of venn diagrams is a counter example. Hence $A \cup (B \cap C) = (A \cup B) \cap C$ is not always true.

7 Question 7

There is a group of 191 students of which 10 are taking operation systems, databases and discrete mathematics; 36 are taking operation systems and databases; 20 are taking operation systems and discrete mathematics; 18 are taking discrete mathematics and databases; 65 are taking operation systems; 75 are taking databases and 63 are taking discrete mathematics.

i Draw a Venn diagram to illustrate the information above

ii How many are taking discrete mathematics or operation systems (or both) but not databases?

iii How many are taking none of the three subjects?

7.1 Answer

Let U be the universal set - the group of students,
OS the set of students taking Operating Systems,
DB the set of students taking Databases,
DM the set of students taking Discrete Mathematics.
From the details given above:
Total number of students:
$|U| = 191$
Number of students taking Operating Systems, Databases and Discrete Mathematics:
$|OS \cap DB \cap DM| = 10$
Number of students taking Operating Systems and Databases:
$|OS \cap DB| = 36$
Number of students taking Operating Systems and Discrete Mathematics:
$|OS \cap DM| = 20$
Number of students taking Discrete Mathematics and Databases:
$|DM \cap DB| = 18$
Number of students taking Operating Systems:
$|OS| = 65$
Number of students taking Databases:
$|DB| = 75$
Number of students taking Discrete Mathematics:
$|DM| = 63$

i Venn Diagram

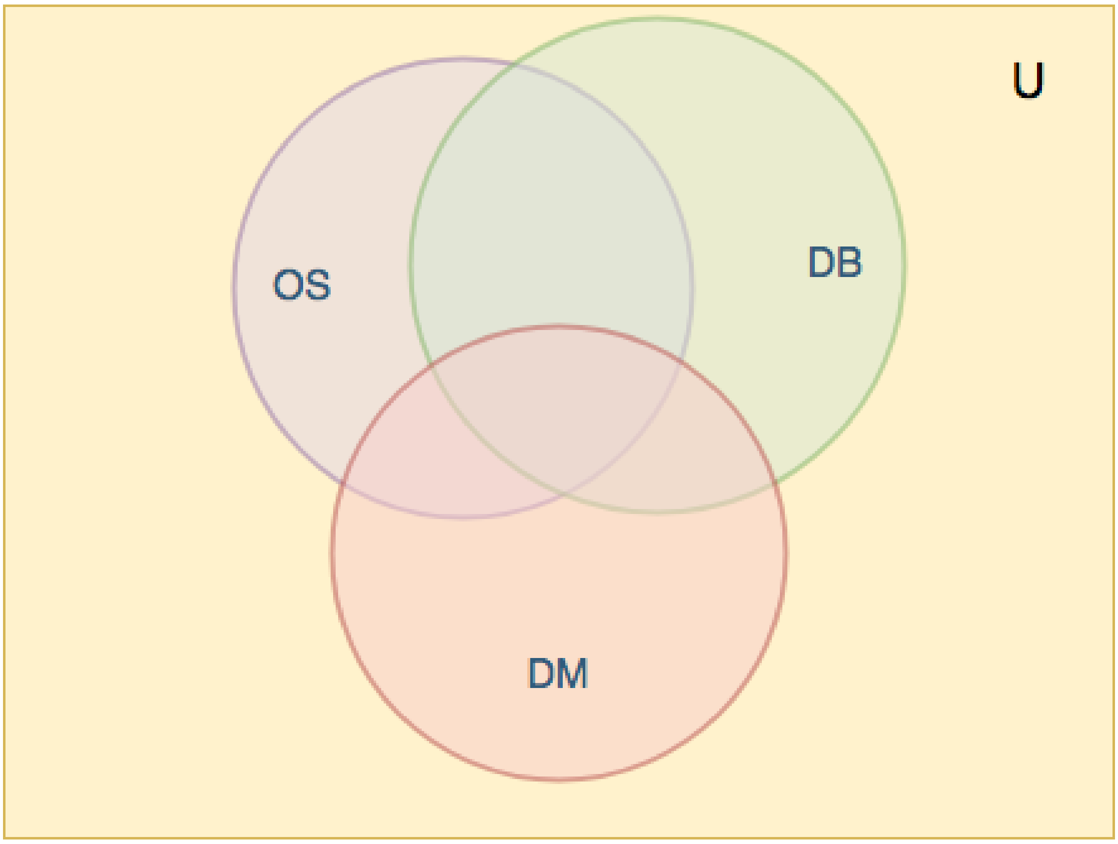

Figure 3: Venn Diagram - Answer 7 (i)

ii Number of students taking Discrete Mathematics or Operating Systems (or both) = $|OS \cup DM|$

$$\begin{aligned} |OS \cup DM| &= |OS| + |DM| - |OS \cap DM| \\ &= 65 + 63 - 20 \\ &= 108 \end{aligned}$$

Number of students taking Discrete Mathematics or Operating Systems but not Databases = $|OS \cup DM| - (|OS \cap DB| + |DM \cap DB|)$

$$\begin{aligned} |OS \cup DM| - (|OS \cap DB| + |DM \cap DB|) &= 108 - (36 + 18) \\ &= 108 - 54 \\ &= 54 \end{aligned}$$

$\therefore$ Answer = 54

iii Number of students taking none of the subjects is the difference between the Universal set (group of all students) and the union of all the three sets. However while calculating the cardinality of the union we need to remove the cardinality of the intersection of the sets. Also need to note that the students who are taking all the three courses are counted thrice in the union. Considering all these facts, the answer is:

$$\begin{aligned}|OS \cup DM \cup DB| &= |OS| + |DM| + |DB| - (|OS \cap DB| + |OS \cap DM| + \\ &\qquad |DM \cap DB| + (2 * |OS \cap DB \cap DM|)) \\ &= 65 + 75 + 63 - (36 + 20 + 18 + (2 * 10)) \\ &= 203 - 94 \\ &= 109\end{aligned}$$

$$\begin{aligned}|U| - |OS \cup DM \cup DB| &= 191 - 109 \\ &= 82\end{aligned}$$

$\therefore$ Answer = 82

8 Question 8

In the question below can you describe the relations/functions either by drawing a diagram, by a formula or by listing the ordered pairs. Explain your solution.

i Give an example of two sets A and B and a relation R from A to B which is not a function

ii Give an example of two sets A and B and a partial function f from A to B which is neither injective nor surjective.

iii Can you find a set A, $|A| = 4$ and define a bijective function between A and $P(A)$? If such a set doesn't exist give a reason.

8.1 Answer

i Consider the following sets A and B.
$A = \{a \mid a \in N\}$
$B = \{b \mid b \in N\}$
Let R be a relation from A to B such that
$R = \{(a, b) \mid (a, b) \in A \times B$ and b is divisible by a $\}$
This relation is not a function because for the same value of a there could be multiple values of b.
eg: For $a = 1$, $\{(1, 1), (1, 2), (1, 3) \cdots\}$ is a valid relation.
$\therefore$ This relation is not a function.

ii Consider the function,
$f : Q \to Z$

$f(x) = x^2$
This function is partial since squares of all fractions are not in the set of Integers.
It is not injective since negative and positive numbers have same squares.
It is not surjective because negative numbers in the set of Z are not in the range of this function.

iii It is not possible to have a bijective function between A and $P(A)$. By defenition, if a relation between two sets are bijective then their cardinalities are also equal. However in this case
$\mid A \mid \neq \mid P(A) \mid$
$\therefore$ It is not possible to define a bijective function between A and $P(A)$.

9 Question 9

Determine whether the function: $h : N \rightarrow N; h(x) = x^2 + 10$ is a partial/total, injective, surjective or bijective. Justify your answer.

9.1 Answer

The function is total because $\forall x \in N$ has a corresponding $h(x) \in N$.
The function is injective. No two natural numbers have the same square.
The function is not surjective. Taking a counter example, $x \in N$ and $x < 10$ are not in the range of this function. Because the function is not surjective, it is not bijective either.

10 Question 10

Define an inverse function to the following functions. If such a function doesn't exist, give a reason.

a $f : Z \rightarrow Z, f(x) = 2x - 7$

b $f : Z \rightarrow Z, f(x) = x^2$

10.1 Answer

a $f : Z \rightarrow Z$
$f(x) = 2x - 7$
$y = 2x - 7$

Interchange x and y and solve for x.
$x = 2y - 7$
$2y = x + 7$
$y = (x + 7)/2$

Replace y with $f^{-1}(x)$
$f^{-1}(x) = (x + 7)/2$

However when x is even, $x + 7$ evaluates to odd number and $(x + 7)/2$ will be a fraction which is $\notin Z$.
$\therefore$ the inverse function $f^{-1}(x) = (x + 7)/2$ is not a total function.

b $f : Z \rightarrow Z$
$f(x) = x^2$
$y = x^2$

Interchange x and y and solve for x.
$x = y^2$
$y = \sqrt{x}$

Replace y with $f^{-1}(x)$
$f^{-1}(x) = \sqrt{x}$

11 Question 11

Let $f : N \rightarrow N$ be defined as $f(x) = 3x + 5$ for every $x \in N$. Calculate the following and write down either value or a function. Show your working.

a $f(f(f(1))) =$

b $(f \circ f)(x) = f(f(x)) =$

c Compose the function f n times recurrently $(f \circ f \circ f \ldots f \circ f)(x)$

11.1 Answer

a $f(f(f(1)))$
$= f(f(3 \times 1 + 5))$
$= f(f(8))$
$= f(3 \times 8 + 5)$
$= f(29)$

$$= 3 \times 29 + 5$$
$$= 92$$

b $(f \circ f)(x)$

$$\begin{aligned}
&= f(f(x)) \\
&= f(3x + 5) \\
&= 3(3x + 5) + 5 \\
&= 9x + 15 + 5 \\
&= 9x + 20
\end{aligned}$$

c $(f \circ f \circ f \dots f \circ f)(x)$ n times

When n = 1,

$$\begin{aligned}
f(x) &= 3x + 5 \\
&= 3^1 \times x + 3^0 \times 5
\end{aligned}$$

When n = 2,

$$\begin{aligned}
(f \circ f)(x) &= 9x + 15 + 5 \\
&= 3^2 \times x + 3^1 \times 5 + 3^0 \times 5
\end{aligned}$$

When n = 3,

$$\begin{aligned}
(f \circ f \circ f)(x) &= 27x + 45 + 15 + 5 \\
&= 3^3 \times x + 3^2 \times 5 + 3^1 \times 5 + 3^0 \times 5
\end{aligned}$$

Generalising the above pattern, we get

$$(f \circ f \circ f \dots f \circ f)(x) = 3^n \times x + 3^{n-1} \times 5 + 3^{n-2} \times 5 + \cdots + 3^1 \times 5 + 5$$

12 Question 12

i Give an example of a function $f : N \to N$ which is total and injective, but not surjective. If such a function doesn't exist, give a reason.

ii Give an example of a function $f : N \to N$ which is partial, injective and surjective. If such a function doesn't exist, give a reason.

12.1 Answer

i Consider the function $f : N \to N, f(x) = x^2$
This function is injective since all natural numbers have a unique square value. This function is not surjective because only natural numbers that are perfect squares are in the range of this function.

ii Such a functiond does not exist.
Lets assume that there is a partial function $f : N \rightarrow N$ which is injective. The function is injective and no two $x \in N$ has the same value for $f(x)$. Since both the domain and co-domain of this function has the same cardinality it is not possible in this case for the co-domain to be the same as range. Hence the function won't be surjective.

13 Question 13

Determine whether the following pairs of statments are logically equivalent or not. Give a reason.

i $p \rightarrow (q \rightarrow r)$ and $(p \rightarrow q) \rightarrow r$

ii $p \rightarrow (q \rightarrow r)$ and $q \rightarrow (p \rightarrow r)$

13.1 Answer

i Plotting the truth table for the statement.

p	**q**	**r**	$\mathbf{p \rightarrow q}$	$\mathbf{q \rightarrow r}$	$\mathbf{p \rightarrow (q \rightarrow r)}$	$\mathbf{(p \rightarrow q) \rightarrow r}$
T	T	T	T	T	T	T
T	T	F	T	F	F	F
T	F	T	F	T	T	T
T	F	F	F	T	T	T
F	T	T	T	T	T	T
F	T	F	T	F	T	F
F	F	T	T	T	T	T
F	F	F	T	T	T	F

$p \rightarrow (q \rightarrow r)$ and $(p \rightarrow q) \rightarrow r$ do not have the same truth values and are not logically equivalent.

ii Plotting the truth table for the statement.

$\mathbf{p}$	$\mathbf{q}$	$\mathbf{r}$	$\mathbf{q \rightarrow r}$	$\mathbf{p \rightarrow r}$	$\mathbf{p \rightarrow (q \rightarrow r)}$	$\mathbf{q \rightarrow (p \rightarrow r)}$
T	T	T	T	T	T	T
T	T	F	F	F	F	F
T	F	T	T	T	T	T
T	F	F	T	F	T	T
F	T	T	T	T	T	T
F	T	F	F	T	T	T
F	F	T	T	T	T	T
F	F	F	T	T	T	T

$p \rightarrow (q \rightarrow r)$ and $q \rightarrow (p \rightarrow r)$ have the same truth values and are logically equivalent.

14 Question 14

Determine whether the following statement is a tautology, a contingency or a contradiction. Give a reason for your answer:
$(p \rightarrow q) \wedge (\neg p \rightarrow p) \rightarrow q$

14.1 Answer

Plotting the truth table for the statement.

$\mathbf{p}$	$\mathbf{q}$	$\mathbf{\neg p}$	$\mathbf{p \rightarrow q}$	$\mathbf{\neg p \rightarrow p}$	$\mathbf{(p \rightarrow q) \wedge (\neg p \rightarrow p)}$	$\mathbf{(p \rightarrow q) \wedge (\neg p \rightarrow p) \rightarrow q}$
T	T	F	T	T	T	T
T	F	F	F	T	F	T
F	T	T	T	F	F	T
F	F	T	T	F	F	T

From the truth table, $(p \rightarrow q) \wedge (\neg p \rightarrow p) \rightarrow q$ is true irrespective of the values of p and q.
$\therefore (p \rightarrow q) \wedge (\neg p \rightarrow p) \rightarrow q$ is a tautology.

15 Question 15

let P be the proposition "Roses are red' and Q be the proposition "Violets are blue'. Express each of the following propositions as logical expressions:

i If roses are not red, then violets are not blue.

ii Roses are red or violets are not blue.

iii Either roses are red or violets are blue (but not both).

15.1 Answer

p = "Roses are red'
q = "Violets are blue'

i $\neg p \to \neg q$

ii $p \lor \neg q$

iii $p \lor q$

16 Question 16

Verify each of the following equivalences by writing and equivalence proof. That is, start on one side and use known equivalences to get to the other side.

i $p \to (q \land r) \equiv (p \to q) \land (p \to r)$

ii $(p \to q) \land (p \lor q) \equiv q$

16.1 Answer

i To prove $p \to (q \land r) \equiv (p \to) \land (p \to r)$

$$\begin{aligned} p \to (q \land r) &\equiv \neg p \lor (q \land r) && \textit{Applying } p \to q \equiv \neg p \lor q \\ &\equiv (\neg p \lor q) \land (\neg p \lor r) && \textit{Applying Distibutive law} \\ &\equiv (p \to q) \land (p \to r) && \textit{Applying } p \to q \equiv \neg p \lor q \end{aligned} \tag{1}$$

Hence proved,
$p \to (q \land r) \equiv (p \to q) \land (p \to r)$

ii To prove $(p \to q) \land (p \lor q) \equiv q$

$$\begin{aligned} (p \to q) \land (p \lor q) &\equiv (\neg p \lor q) \land (p \lor q) && \textit{Applying } p \to q \equiv \neg p \lor q \\ &\equiv (q \lor \neg p) \land (q \lor p) && \textit{Applying Commutative Law } p \lor q \equiv q \lor p \\ &\equiv q \lor (\neg p \land p) && \textit{Applying Distributive Law} \\ &\equiv q \lor False && \textit{Applying } \neg p \land p = False \\ &\equiv q && \textit{Applying } p \lor False = p \end{aligned} \tag{2}$$

Hence proved,
$(p \to q) \land (p \lor q) \equiv q$

17 Question 17

Write each of the following statements in a symbolic form and determine whether they are logically equivalent. Include a truth table and a few words of explanation. "If you paid full price, you didnt́ buy it online. You didnt́ buy it online or you paid full price.'

17.1 Answer

Let,
p = You paid full price
q = You bought it online

Writing "If you paid full price, you didn't buy it online" in symbolic form we get,
$p \rightarrow \neg q$

Writing "You didn't buy it online or you paid full price" in symbolic form we get,
$\neg q \vee p$

Plotting the truth table

p	**q**	**¬q**	**p → ¬q**	**¬q ∨ p**
T	T	F	F	T
T	F	T	T	T
F	T	F	T	F
F	F	T	T	T

The statements have different truth values and are not logically equivalent.

18 Question 18

Write a negation (formal or informal) for each of the following statements. Be careful to avoid negations that are ambiguous.

i All dogs are friendly.

ii Some estimates are accurate.

iii $\forall x \in R,\ if\ x^2 \geq 1\ then\ x > 0$.

18.1 Answer

i All dogs are not friendly.

ii Some estimates are not accurate.

iii $\forall x \in R,\ if\ x^2 \geq 1\ then\ x < 0$.

19 Question 19

Give a direct proof of the fact that $a^2 - 5a + 8$ is even for any integer a.

19.1 Answer

Lets take both the cases where a is even and odd.

Case 1: a is even number.
a can be written as 2k.

$$\begin{aligned} a^2 - 5a + 8 &= (2k)^2 = 5(2k) + 8 \\ &= 4k^2 = 10k + 8 \\ &= 2(2k^2 - 5k + 4) \end{aligned} \tag{3}$$

$\therefore a^2 - 5a + 8$ is even when a is even.

Case 2: a is odd number.

a can be written as 2k + 1.

$$\begin{aligned} a^2 - 5a + 8 &= (2k+1)^2 - 5(2k+1) + 8 \\ &= 4k^2 + (2 \times 2k \times 1) + 1 - (10k + 5) + 8 \\ &= 4k^2 + 4k + 1 - 10k - 5 + 8 \\ &= 4k^2 + 4k - 10k + 1 - 5 + 8 \\ &= 4k^2 - 6k + 4 \\ &= 2(2k^2 - 3k + 2) \end{aligned} \tag{4}$$

$\therefore a^2 - 5a + 8$ is even when a is odd.

We proved the fact for both the cases where a is even and odd. Hence the fact $a^2 - 5a + 8$ is even for any integer a is true.

20 Question 20

Write down the contrapositive and the negation of the following implication.

i If a and b are integers, then ab is an integer.

ii If $x^2 + x - 2 < 0$, then $x > -2$ and $x < 1$.

20.1 Answer

i **Contrapositive:** If a and b are not integers, then ab is not an integer.
Negation: If a and b are integers, then ab is not an integer.

ii **Contrapositive:** If $x^2 + x - 2 > 0$, then $x < -2$ or $x > 1$.
Negation: If $x^2 + x - 2 < 0$, then $x < -2$ or $x > 1$

21 Question 21

Disprove the statement by giving a counterexample.

i $\forall a \in Z \, \forall b \in Z, \; if \; a < b \, then \, a^2 < b^2$.

ii $\forall a \in Z \, \forall b \in Z, (a^2 - b^2) = (a - b)^2$

iii $\forall a \in Z \, \forall b \in Z, \; if \; a(a - b) = 0 \, then \, a = b$.

21.1 Answer

i $\forall a \in Z \, \forall b \in Z, \; if \; a < b \, then \, a^2 < b^2$.

Counter example:
Let $a = -1$ and $b = 1$
$a^2 = 1$
$b^2 = 1$
In this case a^2 is not less than b^2
Hence the statement is disproved.

ii $\forall a \in Z \, \forall b \in Z, (a^2 - b^2) = (a - b)^2$
Counter example:
Let $a = 2$ and $b = 1$
$a^2 - b^2 = 2^2 - 1^2 = 3$

$(a-b)^2 = (2-1)^2 = 1$
$a^2 - b^2) \neq (a-b)^2$
Hence the statement is disproved.

iii $\forall a \in Z\, \forall b \in Z,\ if\ a(a-b) = 0\ then\ a = b$
Counter example:
Let $a = 0$ and $b = 4$
$a(a-b) = 0(0-4) = 0$
However in this case $a \neq b$
Hence the statement is disproved.

22 Question 22

Rewrite the definition of a surjective function $f : A \to B$ using $\forall$ and $\exists$.
Write down the negation of that definition explaining when function is not surjective.

22.1 Answer

A function $f : A \to B$ is surjective iff $\forall b \in B,\ \exists a \in A \mid f(a) = b$

23 Question 23

Prove that for all integers $n, n \geq 1, 1 + 3 + 5 + \cdots + (2n-1) = n^2$.

23.1 Answer

To prove, for all integers $n, n \geq 1, 1 + 3 + 5 + \cdots + (2n-1) = n^2$

Base step: $n = 1$

$$\begin{aligned} 2n - 1 &= 1 \\ n^2 &= 1 \\ \therefore 1 \ldots 2n - 1 &= n^2 \end{aligned} \tag{5}$$

Inductive step: If the statement is true for $n = k$, where $k \geq 1$ then it is true for $n = k+1$, where $k+1 \geq 1$
When $n = k$,
$1 + 3 + \cdots + (2k-1) = k^2$

We need to prove that the same is true for k+1.

$$\begin{aligned}
n &= k+1 \\
1+3+\cdots+2(k+1)-1 &= (k+1)^2 \\
1+3+\cdots+2k+2-1 &= k^2+2k+1 \\
1+3+\cdots+2k+1 &= k^2+2k+1 \\
Can\ be\ re-written\ as & \\
(1+3+\cdots+2k-1)+2k+1 &= k^2+2k+1 \\
Replacing\ (1+3+\cdots+2k-1)\ with\ k^2 & \\
k^2+2k+1 &= k^2+2k+1
\end{aligned} \tag{6}$$

Hence we proved that if $1+3+5+\cdots+(2n-1)=n^2$ is true for $n=k$ then it is true for $n=k+1$ as well.

We proved the base case and inductive case,
$\therefore forallintegersn, n \geq 1, 1+3+5+\cdots+(2n-1)=n^2$ is proved.

24 Question 24

Let $s(x)$ denote the statement "x is a student',
$h(x)$ denote the statement "x is happy'.
Formalise each of the following statements. The domain for all variables it the set of all people.

i "Some student is happy.'

ii "Not all students are happy.'

iii "Every student is happy.'

iv "There is a sad student.'

v "All students are sad.'

24.1 Answer

i $\exists x$ such that $s(x)$ is true and $h(x)$ is true

ii $\exists x$ such that $s(x)$ is true and $h(x)$ is false

iii $\exists x$ such that if $s(x)$ is true, $h(x)$ is true iv $\exists! x$ such that $s(x)$ is true and $h(x)$ is false

v $\exists x$ such that $\forall s(x)$ is true, $h(x)$ is false