Introduction to Integer Programming

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References

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- G. L. Nemhauser and L. A. Wolsey, Integer and Combinatorial Optimization, Wiley, 1999.

Some Definitions

- Integer linear Programming(ILP)
- Mixed Integer Program(MIP)
- Binary Integer Program(BIP)

An Example: Capital Budgeting

	Expenditure(million $)/yr			
Project	1	2	3	Returns (million $)
1	5	1	8	20
2	4	7	10	40
3	3	9	2	20
4	7	4	1	15
5	8	6	10	30
Available funds(million $)	25	25	25	

Formulation

- Decision variable x_j:

$$x_j = \begin{cases} 1, if\ project\ j\ is\ selected \\ 0, if\ project\ j\ is\ not\ selected \end{cases}$$

The ILP Model is:

Maximize z=20x_1+40x_2+20x_3+15x_4 + $30x_5$

Subject to:

5x_1+4x_2+3x_3+7x_4+8x_5 ≤ 25

x_1+7x_2+9x_3+4x_4+6x_5 ≤ 25

8x_1+10x_2+2x_3+x_4+10x_5 ≤ 25

x_1, x_2, x_3, x_4, x_5={0,1}

The Optimal Integer solution is:

x_1=x_2=x_3=x_4=1, x_5=0, z=95 (million $)

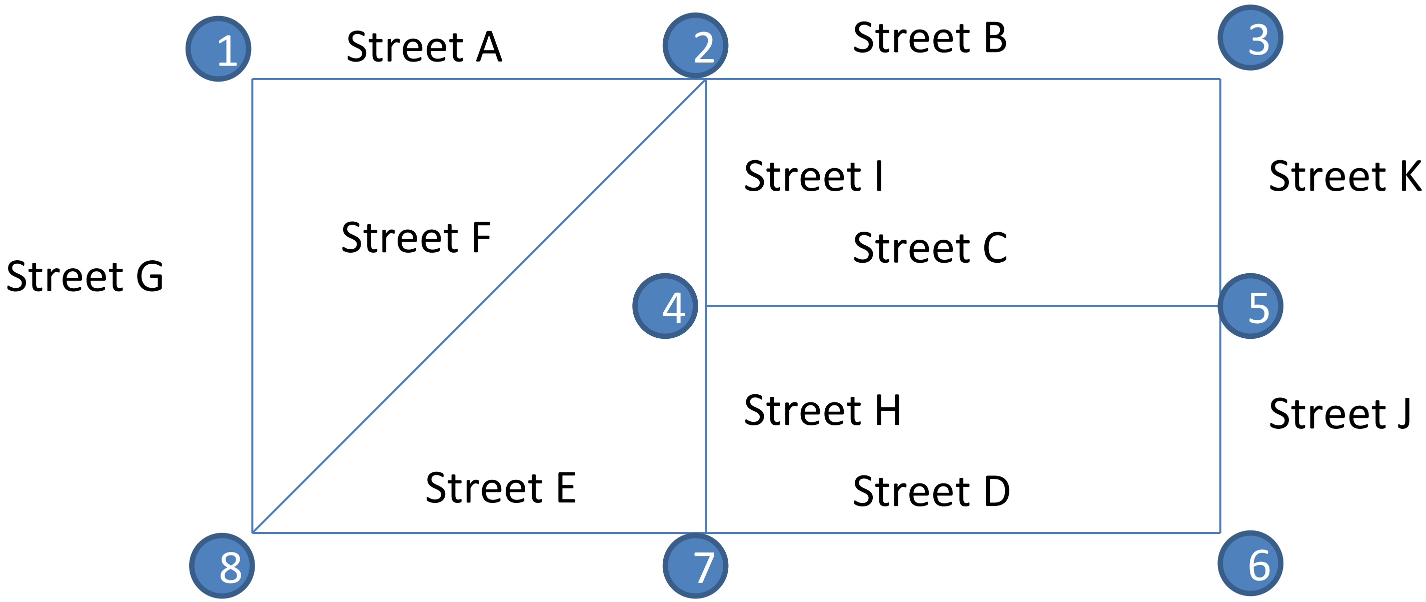
Relaxed LP solution:

x_1=0.5789, x_2=x_3=x_4=1, x_5=0.7368, z=108.68(million $)

Why rounding does not work?

Assignment: Write a concise formulation for the project selection problem.

Set Covering Problem



Formulation

Decision variable:

$$x_j = \begin{cases} 1, a\ telephone\ is\ installed\ in\ location\ j \\ 0, otherwise \end{cases}$$

The ILP Model is:

Maximize z = $x_1+x_2+x_3+x_4+x_5+x_6+x_7+x_8$

Subject to:

$x_1+x_2 \geq 1$ (Street A)

$x_2+x_3 \geq 1$ (Street B)

$x_4+x_5 \geq 1$ (Street C)

$x_7+x_8 \geq 1$ (Street D)

$x_6+x_7 \geq 1$ (Street E)

$x_2+x_6 \geq 1$ (Street F)

$x_1+x_6 \geq 1$ (Street G)

$x_4+x_7 \geq 1$ (Street H)

$x_2+x_4 \geq 1$ (Street I)

$x_5+x_8 \geq 1$ (Street J)

$x_3+x_5 \geq 1$ (Street K)

x_j=(0,1), j=1,2,………,8

Optimum solution of problem requires installing four telephones at intersections 1,2,5 and 7.

Assignment: Write a concise formulation for this set covering problem

Either-Or Constraints

- Either $3x_1 + 2x_2 \leq 18$
  Or $x_1 + 4x_2 \leq 16$

  These two above inequalities are equivalent to:

  $3x_1 + 2x_2 \leq 18 + My$

  $x_1 + 4x_2 \leq 16 + M(1-y)$

  where, *y* is an auxiliary variable must be 0 or 1.

- Assignment:
  $y=\min(u_1, u_2)$, Write a formulation to find *y*.

- How to represent either-or variables?

0-1 Knapsack Problem

- What is knapsack problem?
- Cost and Profit of each item are given.
- Capacity or Budget constraint(s)
- To select items to maximize total profit

max$\left\{ \sum_{j=1}^{n} c_j x_j : \sum_{j=1}^{n} a_j \, x_j \leq b, x \in B^n \right\}$

- Multi-dimensional knapsack problem
- Why Knapsack constraint is so important?

Assignment Problem

Objective function:

Minimize $\sum_{i=1}^{m} \sum_{j=1}^{n} c_{ij} x_{ij}$

Subject to:

$\sum_{j=1}^{n} x_{ij}=1$ for i=1,…..,m

$\sum_{i=1}^{m} x_{ij} \leq 1$ for j=1,…..,n

$x_{ij}=\{0,1\}$

- Can you solve this problem without using any solver?
- Special characteristic of the formulation – totally unimodular

Traveling Salesman Problem

Decision variable:

x_{ij}=1 if j immediately follows i on the tour, x_{ij}=0 otherwise

Objective function:

Minimize $\sum_{(i,j)\in A} c_{ij}\, x_{ij}$

Subject to:

$\sum_{\{i:(i,j)\in A\}} x_{ij}=1$ for j$\in V$

$\sum_{\{j:(i,j)\in A\}} x_{ij}=1$ for i$\in V$

$\sum_{\{(i,j)\in A: i\in U, j\in \mathrm{U}\}} x_{ij} \leq |U|-1$ for $2\leq |U| \leq |V|-2$

- Complexity of the formulation
- Possible solution strategy

Choices in Model Formulation

- $max\{cx{:}Ax \leq b, x \in Z^n\}$

is a valid IP formulation if $S=\{x\in Z^n{:}Ax \leq b\}$

Example:

S={(0000),(1000).(0100),(0010),(0001),(0110),(0101),(0011)}

S={x∈ B^4:$93x_1$+$49x_2$+$37x_3$+$29x_4 \leq 111$}

S={x∈ B^4:$2x_1$+x_2+x_3+$x_4 \leq$2}

S={x∈ B^4:$2x_1$+x_2+x_3+$x_4 \leq$2

x_1+$x_2 \leq 1$

x_1+$x_3 \leq 1$

x_1+$x_4 \leq 1$}

- Which formulation is better and why?
- Why is formulation more important than solution algorithm?

Fixed Charge Problem

- Three telephone companies: Vodafone, Airtel and Idea.

Vodafone: flat Rs 16 per month plus Rs.0.25 a minute

Airtel: flat Rs 25 per month plus Rs. 0.21 a minute

Idea: flat Rs. 18 per month plus Rs. 0.22 a minute

Total call time per month = 200 minutes

Problem: how to distribute calls to these three operators to minimize the monthly telephone bill?

Decision Variables:

x_1=Vodafone minutes per month

x_2= Airtel minutes per month

x_3= Idea minutes per month

y_1=1 if $x_1 > 0$, and 0 if x_1=0

y_2=1 if $x_2 > 0$, and 0 if x_2=0

y_3=1 if $x_3 > 0$, and 0 if x_3=0

Link Constraint: $x_j \leq My_j$, j=1,2,3

Formulation

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Minimize z=0.25x_1+0.21x_2+0.22x_3+16y_1+25y_2+18y_3

Subject to:

x_1+x_2+x_3 = 200
$x_1 \leq 200\ y_1$
$x_2 \leq 200\ y_2$
$x_3 \leq 200\ y_3$

$x_1, x_2, x_3 \geq 0$

$y_1, y_2, y_3 = \{0,1\}$

Optimum Solution yields x_3=200, y_3 =1 and all remaining variables are equal to zero.

y variables are called set-up variables, also found in facility location problems. If there is no set-up cost (flat rate), these variables are not needed.

Branch & Bound Algorithm

- You cannot solve an IP directly. Why?
- There are efficient solution algorithms for LP. So, use LP to solve the IP.
- LP relaxation of IP may give non-integer solution
- Add constraints and modify LP solution space to get to the integer solution

IP:

Maximize z =5x_1+ 4x_2

Subject to:

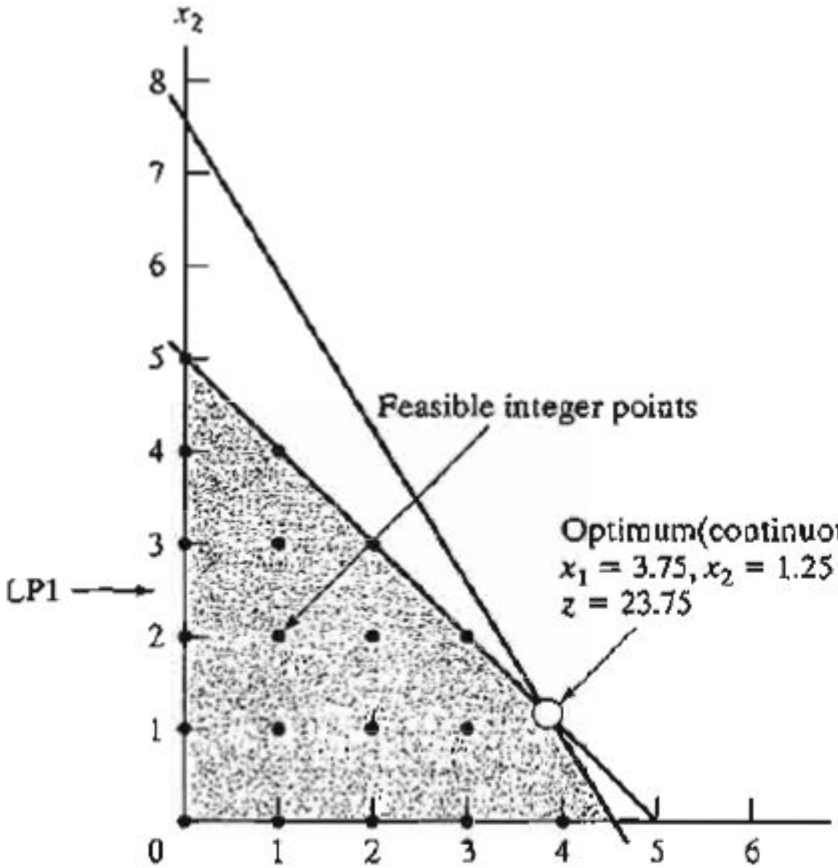
x_1+ $x_2 \leq 5$

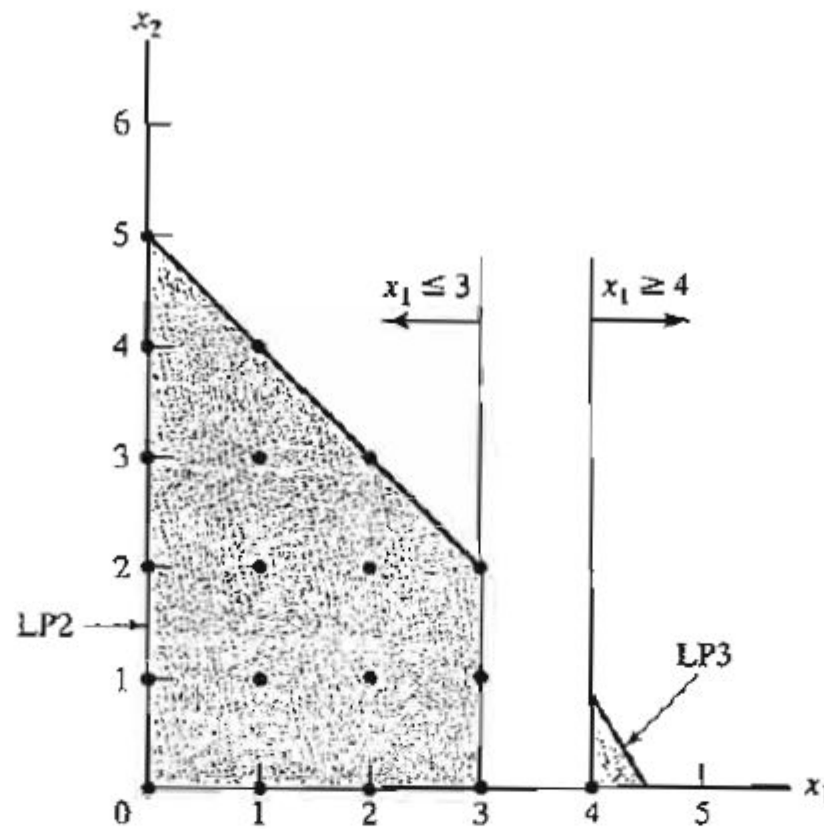
10x_1 + 6$x_2 \leq 45$

x_1, x_2 nonnegative integer

On solving the LP1 (x_1, x_2 $\geq$0), the optimum solution found is $x_1 = 3.75$, x_2=1.25, and z=23.75 (upper bound to the IP, but no lower bound has been found at this stage)

Because the optimum LP1 solution does not satisfy the integer requirements , the B&B algorithm modifies the solution space to reach the ILP optimum. How?





Source: Taha(2002)

Illustration of B&B Algorithm

- First , select one of the integer variable whose optimum value at LP1 is not integer.
- On selecting x_1(=3.75) ,the region 3< x_1 <4 of the LP1 solution space contains no integer values of x_1, and thus can be eliminated.
  This is equivalent to replace the original LP1 with two new LPs.

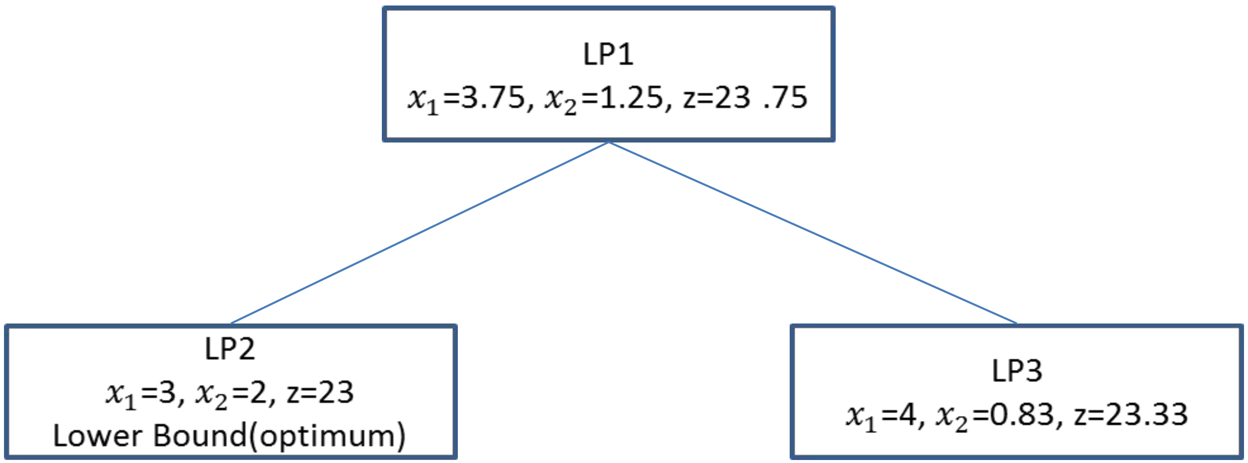
  LP2 : LP1 + $(x_1 \leq 3)$
  LP3 : LP1 + $(x_1 \geq 4)$

- Optimal solution of LP2: x_1=3, x_2=2, and z=23 (Integer-Feasible to IP);
  Set lower bound = 23, and remember the incumbent solution: (3,2)
  Fathom LP2; Why? Fathoming by integer feasibility

- Optimum solution of LP3:x_1=4, x_2=0.83, and z=23.33 (fathom by bound)

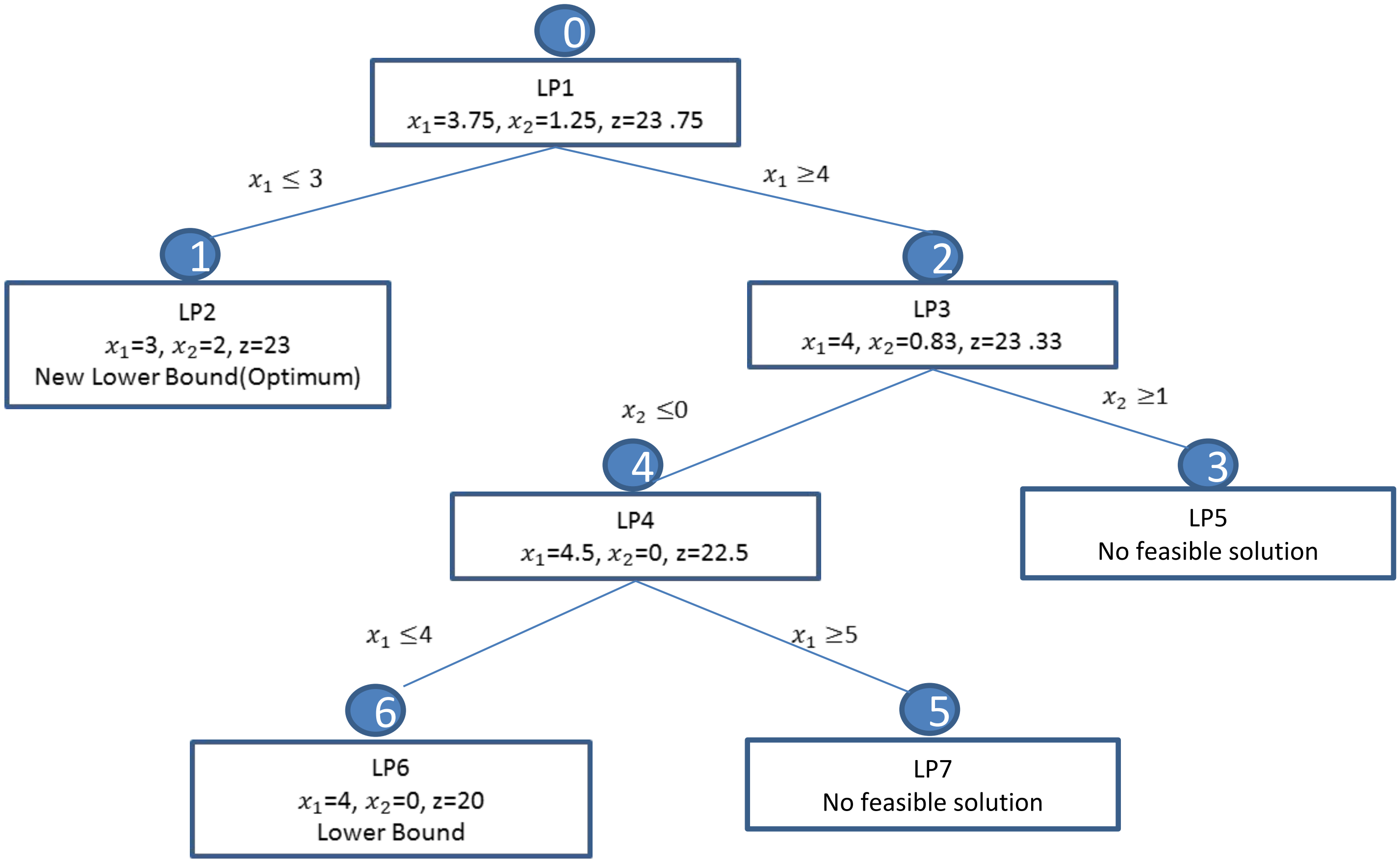
- Optimal solution to the IP: *z*=23 (the lower bound from B&B), solution (3,2)

B&B tree



LP2 provide a lower bound on optimum objective value of the original ILP

B&B tree if LP3 is solved before LP2



Summary of the B&B Algorithm (maximization problem)

Step 1: Fathoming/Bounding conditions::

a) Optimal value of LP is inferior to the current lower bound(LB).

b) LP yield a *feasible integer* solution superior to the current incumbent.

c) LP is infeasible

Two cases:

case1:if LP fathomed and better integer solution found, update the current lower bound.

If all subproblems are fathomed, STOP. Otherwise, repeat step1 for other subproblem.

case2:If LP is not fathomed, go to step 2 for branching.

Step 2: (Branching)Select one of the integer variable x whose optimum value x^* in relaxed LP is not integer. Eliminate region : $[x^*] < x < [x^*] + 1$ by creating two subproblems corresponding to

$x \leq [x^*]$ and $x \geq [x^*] + 1$

Consider the next LP and go to Step 1.

- B&B algorithms for minimization problem

Thank you for your attention