1 Minimum Knapsack Problem

We wish to solve the ILP.

$$\min \textstyle\sum_{i \in I} s_i x_i$$
$$\text{such that } \textstyle\sum_{i \in I} v_i x_i \geq D, x_i \in \{0,1\} \qquad \forall i \in I$$

View each $A \subseteq I$, such that $v(A) < D$, where we may achieve an additional value of $D_A = D - v(A)$. So, from v_i, $i \notin A$, we seek only $min(v_i, D_A)$. So, if $v(A) < D$ and $v(X) \geq D$ then we observe that

$$\textstyle\sum_{i \in X \setminus A}^{A} v_i \geq D_A$$

We may therefore write

$$\textstyle\sum_{i \in I} x_i s_i$$

$$\text{such that } \textstyle\sum_{i \in I \setminus A} v_i x_i \geq D_A, \forall A \subseteq I$$
$$x_i \in \{0,1\}, \forall i \in I$$

LP relaxation

$$\textstyle\sum_{i \in I} x_i s_i$$

$$\text{such that } \textstyle\sum_{i \in IA} v_i^A x_i \geq D_A, \forall A \subseteq I$$
$$, x_i \geq 0, \forall i \in I$$

So, for any $A \subseteq I$, the deficiency, if any can be satisfied by $I \setminus A$. The dual LP is

$$max \textstyle\sum_{A : A \subseteq I} D_A y_A$$

$$\text{such that } \textstyle\sum_{A \subseteq I; i \notin A} v_i^A y_A \leq s_i, y_A \geq 0, \forall A \subseteq I, \forall i \in I$$

On having collected a set $A \subseteq I$ already, we need to pick up the next item $i \in I$ and adding i to A, i.e., we update $A \leftarrow A \cup \{i\}$. This goes on until $v(A) \geq D$, when we output $X \leftarrow A$.

Initially $A = \phi$, and therefore any $i \in I$ can be selected. We show that the approximation ratio is 2.

So, when X is returned, $X \subseteq I$, and l was the last item selected, then

$$v(X) \geq D \text{ and } v(X \setminus \{l\}) < D$$

So,

$$\textstyle\sum_{i \in X} s_i = \sum_{i \in X} (\sum_{A \subseteq I, i \notin A} y_A v_i^A)$$

$$= \sum_{A \subseteq I} y_A \sum_{i \in X \setminus A} v_i^A \tag{1}$$

Now $v_i < D - v(A) = D_A$ in all but the last iteration, i.e., $v_i^A = min(v_i, D_A) = v_i$ when A was the set of items.

So,

$$\sum_{i \in X \setminus A} v_i^A = v_l^A + \sum_{i \in X \setminus A, i \neq l} v_i^A = v_l^A + v(X \setminus \{l\}) - v(A) \tag{2}$$

But $v_l^A \leq D_A$ and $v(X \backslash l) < D$

So that

$$v(X \setminus \{l\}) - v(A) < D - v(A) = D_A \tag{3}$$

So,

$$v_l^A + v(X \backslash l) - v(A) < 2D_A \tag{4}$$

$$\sum_{i \in X} s_i = \sum_{A \subseteq I} y_A \sum_{i \in X \setminus A} v_i^A < 2 \sum_{A \subseteq I} D_A y_A \leq 2OPT \tag{5}$$