MAXIMUM NUMBER OF LOCAL MAXIMA ON HAMMING GRAPH

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Let us begin with a few key definitions. For a graph G, we denote by $V(G)$ the vertex set, and by q the cardinality of the vertex set.

Definition 1.1. (Complete graph). A graph G is said to be complete if each pair of vertices in $V(G)$ are connected by exactly one edge.

Definition 1.2. (Distance metric). Define the distance d between two vertices $d(v_i, v_j), v_i, v_j \in V(G)$ as the smallest number of edges between v_i and v_j.

Definition 1.3. (Local maximum.) Let G be a graph and let $F : V(G) \to \mathbb{R}$ be some function. A vertex $v_0 \in V(G)$ is said to be a local maximum for f if,

$$f(v_0) > f(v) \text{ for all } v \in \mathbb{M}_1$$

where $\mathbb{M}_1 := \{v \in V(G) : d(v_0, v) = 1\}$.

Proposition 1.4. (Number of local maxima in a complete graph).*Let G be a complete graph. The number of local maxima for a function $f : V(G) \to \mathbb{R}$ is at most one.*

Proof. Immediate from Definition 1.1. – 1.3. We provide a proof by contradiction below for the interested reader as follows. Suppose G has 2 maxima, v_0, v_1. From Definition 1.1 and 1.2, $d(v_i, v_j) = 1$ for all $v_i, v_j \in V(K)$; this implies $d(v_0, v_1) = 1$. This is a contradiction of Definition 1.3; a complete graph cannot have more than one local maximum. □

Definition 1.5. (Cartesian product of graphs). The Cartesian product of graphs G and H, $G\square H$, is the Cartesian product of the vertex sets of G and H:

$$G\square H = (V(G) \times V(H), E(G \times H))$$

where two vertices (g, h) and (g', h') are adjacent if and only if either:

(1) $g = g'$ and h is adjacent to h' in H; or
(2) $h = h'$ and g is adjacent to g' in G.

Definition 1.6. (Hamming graph). The Hamming graph $H(L, q)$ is the L-fold Cartesian product of the complete graph G with vertex set cardinality q.

Proposition 1.7. (Number of edges in a Cartesian product of graphs). *Let G and H be graphs, and $G \square H$ their Cartesian product. Then,*

$$|E(G \square H)| = |E(G)| \cdot |V(H)| + |E(H)| \cdot |V(G)|$$

Proof. The cardinality of a set S that is the Cartesian product of two sets A, B is the product of the cardinalities of A and B, $|S| = |A| \cdot |B|$. It follows that $|V(G \square H)| = |V(G)| \cdot |V(H)|$. Insert good proof here. $\square$

Proposition 1.8. (Number of local maxima in a Cartesian product of complete graphs.) *Let G and H be graphs, and $G \square H$ their Cartesian product. The number of local maxima for a function $f : V(G \square H) \to \mathbb{R}$ is at most $|V(G)|$ if $|V(G)| < |V(H)|$ or $|V(H)|$ if $|V(G)| > |V(H)|$.*

Proof. By Fiending. $\square$

Theorem A. *Let H(L,q) be a Hamming graph. The number of local maxima for a function $f : V(H) \to \mathbb{R}$ is at most —.*

Proof. By fiending. $\square$

Definition 1. (Complete graph). A graph $K(q)$ is complete if each pair of vertices $v, u \in V(K)$ are connected by an edge. Here, q is the cardinality of the vertex set, i.e. $q = |V(K)|$. We note that a complete graph has $\binom{q}{2}$ edges (proof not provided).

Definition 2. (Cartesian product of graphs). The Cartesian product of graphs G and H, $G \square H$, is the Cartesian product of the vertex sets of G and H:

$$G \square H = (V(G) \times V(H), E(G \times H))$$

where 2 vertices (g, h) and (g', h') are adjacent if and only if either:

(1) $g = g'$ and h is adjacent to h' in H; or
(2) $h = h'$ and g is adjacent to g' in G.

Definition 3. (Hamming Graph). The Hamming graph $H(L, q)$ is the L-fold Cartesian product of the complete graph $K(q)$.

Definition 4. (Graph function). A graph function on a graph $G(V, E)$ is a function $f : V \to \mathbb{R}$ from the vertices to the real numbers.

Definition 5. (Maximum vertex). A vertex V_0 is a maximum if and only if V_0 has the maximum value of f among its 1 unit distance neighbours $\mathcal{V}_1$, where $\mathcal{V}_1 = \{d(V_0, V_i) = 1 \; \forall V_i \in V(G)\}$.

Proposition 1. *The complete graph $K(q)$ can have a maximum of 1 maximum node.*

Proof. (By induction). By Definition 1, $d(v, u) = 1 \; \forall v, u \in V(K)$. In conjunction with Definition 5, it follows by induction that since all vertices in $K(q)$ have distance 1, there can only be a maximum of 1 maximum.

(By contradiction.) Suppose $K(q)$ has 2 maxima, v_0, v_1. By Definition 1, $d(v, u) = 1 \; \forall v, u \in V(K)$; this implies $d(v_0, v_1) = 1$. This is a contradiction of Definition 5. $\square$