Solution to Homework 1

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1 Problem 1

For these problems there are two possible solutions. Sometimes we write $g(n) = O(f(n))$, where $g(n)$ is a function and $O(f(n))$ is a set. So here the "=" actually means "$\in$ ". Then for the problems below, if you take "=" to be "$\in$" or "$\subseteq$", and only prove one direction of the equation, it's fine. Of course the perfect solution would be proving those two sets actually equal.

1.1 a.

True.

Proof. Suppose $g(n)$ is in $O(f(n))$, so by definition we have:

$$\exists c_1, n_1 \text{ s.t. } \forall n > n_1, 0 < g(n) < c_1 \cdot f(n)$$

Suppose $h(n)$ is in $O(g(n))$, so by definition we have:

$$\exists c_2, n_2 \text{ s.t. } \forall n > n_2, 0 < h(n) < c_2 \cdot g(n)$$

Combine them together, we have:

$$\exists c_3 = c_1 \cdot c_2, n_3 = \max(n_1, n_2) \text{ s.t. } \forall n > n_3, 0 < h(n) < c_3 \cdot f(n)$$

So we proved that $O(O(f(n))) \subseteq O(f(n))$

The other direction is trivial. Suppose $g(n)$ is in $O(f(n))$, and because $g(n) = O(g(n))$, so we have $g(n) \in O(O(f(n)))$, so we proved that $O(f(n)) \subseteq O(O(f(n)))$. $\square$

1.2 b.

True.

Proof. Suppose $g(n)$ is in $\Theta(f(n))$, so by definition we have:

$$\exists c_1, c_1^{'}, n_1 \text{ s.t. } \forall n > n_1, c_1 \cdot f(n) < g(n) < c_1^{'} \cdot f(n)$$

Suppose $h(n)$ is in $O(g(n))$, so by definition we have:

$$\exists c_2, n_2 \text{ s.t. } \forall n > n_2, 0 < h(n) < c_2 \cdot g(n)$$

Combine them together, we have:

$$\exists c_3 = c_1^{'} \cdot c_2, n_3 = \max(n_1, n_2) \text{ s.t. } \forall n > n_3, 0 < h(n) < c_3 \cdot f(n)$$

So we proved that $O(\Theta(f(n))) \subseteq O(f(n))$

Again the other direction is trivial and similar to problem (a). □

1.3 c.

False.
Counter-example: let $f(n) = n^2$, let $g(n) = n$, so $g(n) \in O(f(n))$, but $\Theta(g(n))$ is $\Theta(n)$, which is obviously not equal to $\Theta(f(n))$, which is $\Theta(n^2)$.

1.4 d.

True.
Intuitively, $O(\Omega(f(n)))$ and $\Omega(O(f(n)))$ both represent all the functions. A formal proof is as follows:

Proof.

1. For any function $g(n) > 0$, $h(n) \in \Omega(f(n))$ we have $g(n) + h(n) \in \Omega(f(n))$, and $g(n) \in O(g(n) + h(n))$. So this proves that every function $g(n) > 0$ is $O(\Omega(f(n)))$, no matter which $f(n)$ you choose.

2. Let $h(n) = 0$ be a constant function. Obviously $h(n) = O(f(n))$. For any $g(n) > 0$, $g(n) \in \Omega(h(n))$, so we have proved that any function $g(n) > 0$ is $\Omega(O(f(n)))$, no matter which $f(n)$ you choose.

Combining the above two statements, we have $O(\Omega(f(n))) = \Omega(O(f(n)))$ □

1.5 e.

True.

Proof. By definition we have:

$$\exists c_1, c_1^{'}, n_1 \text{ s.t. } \forall n > n_1, c_1 \cdot h(n) < f(n) < c_1^{'} \cdot h(n)$$

$$\exists c_2, c_2^{'}, n_2 \text{ s.t. } \forall n > n_2, c_2 \cdot h(n) < g(n) < c_2^{'} \cdot h(n)$$

So we have:

$$\exists c_3 = c_1 + c_2, c_3^{'} = c_1^{'} + c_2^{'}, n_3 = \max(n_1, n_2), \text{ s.t. } \forall n > n_3, c_3 \cdot h(n) < f(n) + g(n) < c_3^{'} \cdot h(n)$$

So by definition this means $f(n) + g(n) = \Theta(h(n))$ □

1.6 f.

False.
Counter-example: let $f(n) = 2n$, $g(n) = n$, then obviously $f(n) = \Theta(g(n))$, but

$$\lim_{n\to+\infty} \frac{2^{f(n)}}{2^{g(n)}} = \lim_{n\to+\infty} 2^n = +\infty$$

so $2^{f(n)} = \omega(2^{g(n)})$

1.7 g.

False.
Counter-example: Let $f(n) = n$, $g(n) = n^2$, so $\min(f(n), g(n)) = f(n) = n$. But $f(n) + g(n) = n + n^2$ is $\omega(n)$, not $\Theta(n)$.

2 Problem 2

2.1 a.

Proof. **Base Case :** When $n = 1$, we have

$$\sum_{i=1}^{n} i \cdot r^{i-1} = 1$$

$$\frac{1 - r^{n+1} - (n+1)(1-r)r^n}{(1-r)^2} = 1$$

Induction : Suppose when $n = k$, the statement holds, now for $n = k+1$, we have:

$$\begin{aligned}
\sum_{i=1}^{k+1} ir^{i-1} &= \sum_{i=1}^{k} ir^{i-1} + (k+1)r^k \\
&= \frac{1 - r^{k+1} - (k+1)(1-r)r^k}{(1-r)^2} + (k+1)r^k \\
&= \frac{1 - r^{k+1} - (k+1)(1-r)r^k + (k+1)r^k(1-r)^2}{(1-r)^2} \\
&= \frac{1 - r^{k+2} - (k+2)(1-r)r^{k+1}}{(1-r)^2}
\end{aligned}$$

This finishes our proof. $\square$

2.2 b.

Proof. **Base Case :** $1 = 1, 2 = 2, 3 = 1 + 2, \cdots$
Induction : Suppose for $n \leq k$ the statement holds. Now for $n = k+1$, there are two situations:

1. If $k + 1$ itself is a Fibonacci number, then we are done;
2. Otherwise, $\exists i$, s.t. $F_i < k + 1 < F_{i+1}$. Let $a = k + 1 - F_i$, so $a \leq k$, so a can be represented as the sum of distinct unconsecutive Fibonacci numbers. Also notice that $a = k + 1 - F_i < F_{i+1} - F_i = F_{i-1}$, so F_{i-1} is not in the representation of a. So the representation of a plus F_i is the new representation for $k + 1$.

So we finish the proof. □

3 Problem 3

3.1 a.

Proof. Let $n_0 = 1$, let $c = 1$, we have

$$\forall n > n_0, n! = n(n-1)(n-2) \cdots 3 \cdot 2 \cdot 1 < c \cdot n \cdot n \cdot n \cdots n$$

So $n! = O(n^n)$ □

3.2 b.

Proof. Let $F(x) = \frac{-\frac{x^2}{4} + \frac{1}{2}x^2 \ln(x)}{\ln(2)}$, then we have $F^{'}(x) = x^2 \log(x)$. The rest follows immediately from the integration method. □

3.3 c.

Proof.

$$\sum_{i=0}^{k} \log(\frac{n}{2^i}) = \sum_{i=0}^{k} \log(2^{k-i}) = \sum_{i=0}^{k} (k - i) = \frac{k^2 - k}{2} = \Theta(k^2)$$

$$\log^2(n) = k^2$$

□

3.4 d.

This statement is false.

Proof.

$$\lim_{n \to +\infty} \frac{2^n}{n^n} = \lim_{n \to +\infty} \frac{2}{1} \cdot \frac{2}{2} \cdot \frac{2}{3} \cdot \frac{2}{4} \cdot \frac{2}{5} \cdots < \frac{4}{3} \lim_{n \to +\infty} (\frac{2}{4})^{n-3} = 0$$

So $n^n = \omega(2^n)$ □

4 Problem 4

In increasing order ($f(n)$ appears before $g(n)$ means $f(n) = O(g(n))$):

$n^{\frac{1}{\log(n)}}, \log^*(\log(n)), \sqrt{\log(n)}, (\log(n))^{\log(\log(n))}, 2^{\sqrt{2\log(n)}}, n^5, (\log(\log(n)))^{\log(n)}, 2^{n^{0.0001}}, n!, 2^{2^n}$

Proof. 1. $n^{\frac{1}{\log(n)}} = 2^{\log(n^{\frac{1}{\log(n)}})} = 2^1 = 2$, this is a constant, so

$$lim_{n\to+\infty} \frac{2}{\log^*(\log(n))} = 0$$

2. to prove $\log^*(\log(n)) = O(\sqrt{\log(n)})$ is not very easy. It seems trivial, but the function $\log^*(n)$ does not have a closed form. So we need to prove it by mathematical induction. First we prove a lemma, that for all $m > 16$, $\log^*(m) \le \sqrt{m}$:
Base Case when $m = 17$, the statement holds, obviously;
Induction suppose when $m \le k$, the statement holds. for $m = k+1$, there are two cases:

   (a) If $\log^*(k+1) = \log^*(k)$, then $\log^*(k+1) = \log^*(k) \le \sqrt{k} < \sqrt{k+1}$

   (b) If $\log^*(k+1) = \log^*(k)+1$. This only happens when $k+1 = 2^{p+1}$, for some p. Now we let $m = 2^p$, by induction hypothesis we have $\log^*(2^p) \le \sqrt{2^p}$. So we have:

$$\begin{aligned}\sqrt{k+1} = \sqrt{2^{p+1}} &= \sqrt{2^{p+1}} - \sqrt{2^p} + \sqrt{2^p} \\ &\ge \sqrt{2^{p+1}} - \sqrt{2^p} + \log^*(2^p) \\ &= \sqrt{2^{p+1}} - \sqrt{2^p} + \log^*(2^{p+1}) - 1 \\ &\ge \log^*(2^{p+1})\end{aligned}$$

   where the last $\ge$ is due to the fact that $\sqrt{2^{p+1}} - \sqrt{2^p} > 1$, when $p > 4$. And

$$\log^*(2^p) = \log^*(2^{p+1}) - 1$$

   is for the following reason: the $\log^*$ function increases by 1 at 2^{p+1}, so the last time it increase by 1 is at $p+1$. And for a number x within the range from $p+1$ to $2^{p+1}-1$, $\log^*(x)$ remains the same value. And when $p > 4$, obviously 2^p falls within this range.

   This finishes the proof of our lemma. Let $\log(n) = m$, by the lemma we have $\log^*(m) \le \sqrt{m}$, so we have:

$$\exists c = 1, n_0 = 65536, \text{ s.t. } \forall n > n_0, \log^*(\log(n)) \le c \cdot \sqrt{\log(n)}$$

3.

$$\lim_{n\to+\infty} \frac{\sqrt{\log(n)}}{(\log(n))^{\log(\log(n))}} = \lim_{n\to+\infty} (\log(n))^{\frac{1}{2}-\log(\log(n))} = 0$$

4. we prove the next five relations in a similar way. Let $m = \log(n)$, we have:

$$(\log(\log(n)))^{\log(n)} = (\log(m))^m = 2^{(\log(m))^2}$$

$$2^{\sqrt{2\log(n)}} = 2^{\sqrt{2m}}$$

$$n^5 = 2^{5m}$$

$$(\log(n))^{\log(\log(n))} = m^{\log(m)} = 2^{m\log(\log(m))}$$

$$2^{n^{0.0001}} = 2^{2^{0.0001m}}$$

   Notice that they are all in base-2 exponential form. So:

   (a) $(\log(m))^2$ is a polylog function, it is asymptotically smaller than any polynomial, so $(\log(m))^2 < \sqrt{2m}$, when $m > m_1$;
   (b) $\sqrt{2m} < 5m$ is obvious;
   (c) $5m < m\log(\log(m))$, when $m > m_2$;
   (d) $m\log(\log(m))$ is polynomially bounded, and $2^{0.0001m}$ is exponential, so $m\log(\log(m)) < 2^{0.0001m}$, when $m > m_3$.

   Notice in the above argument I didn't use the big-O notation because $f(n) = O(g(n))$ does not imply $2^{f(n)} = O(2^{g(n)})$. But if $f(n) < g(n)$, we can have $2^{f(n)} = O(2^{g(n)})$. So simply let $m_0 = \max(m_1, m_2, m_3)$ and let $c = 1$, we have all the above relations proved by definition.

5. $n! = \omega(2^n)$, $n > n^{0.0001}$, so $n! = \omega(2^{n^{0.0001}})$
6. $n! = o(n^n) = o(2^{n\log(n)})$, and $n\log(n) < 2^n$. So $n! = o(2^{2^n})$.

$\square$

5 Problem 5

5.1 a.

We have

$$T(n) = T(n-1) + 2^n$$

$$T(n-1) = T(n-2) + 2^{n-1}$$

$$T(n-2) = T(n-3) + 2^{n-2}$$

$$\cdots\cdots$$

$$T(2) = T(1) + 2^2$$

Adding them together, we have

$$T(n) = T(1) + \sum_{i=2}^{n} 2^i = 2^{n+1} - 3$$

5.2 b.

Directly apply Master Theorem, where $a = 4$, $b = 3$, and $n^2 = \Omega(n^{\log_3(4)})$, also $4 \cdot (\frac{n}{3})^2 \leq \frac{4}{9}n^2$, so Case 3 applies. So $T(n) = \Theta(n^2)$.

5.3 c.

Again we use the Master Theorem. Here $a = 6$, $b = 7$, and $n = \Omega(n^{\log_7(6)})$, also $6 \cdot \frac{n}{7} \leq \frac{6}{7}n$, so Case 3 applies. So $T(n) = \Theta(n)$.

5.4 d.

We have:

$$T(n) = T(\sqrt{n}) + \log(n)$$

$$T(\sqrt{n}) = T(\sqrt[4]{n}) + \log(\sqrt{n}) = T(\sqrt[4]{n}) + \frac{1}{2}\log(n)$$

$$T(\sqrt[4]{n}) = T(\sqrt[8]{n}) + \log(\sqrt[4]{n}) = T(\sqrt[8]{n}) + \frac{1}{2^2}\log(n)$$

$$\cdots\cdots$$

$$T(\sqrt[2^k]{n}) = T(\sqrt[2^{k+1}]{n}) + \log(\sqrt[2^k]{n}) = T(\sqrt[2^{k+1}]{n}) + \frac{1}{2^k}\log(n)$$

Add them together, we have:

$$T(n) = T(\sqrt[2^{k+1}]{n}) + \sum_{i=0}^{k+1}\frac{1}{2^i}\log(n)$$

Taking the limit on both side, we have:

$$lim_{k\to+\infty}T(n) = \lim_{k\to+\infty}T(\sqrt[2^{k+1}]{n}) + \lim_{k\to+\infty}\sum_{i=0}^{k+1}\frac{1}{2^i}\log(n) = T(1) + 2\log(n)$$

So we have

$$T(n) = 1 + 2\log(n)$$

5.5 e.

From

$$T(n) = 2 + \sum_{i=1}^{n-1}T(i)$$

We have

$$T(n+1) = 2 + \sum_{i=1}^{n}T(i)$$

So we have

$$T(n+1) - T(n) = \sum_{i=1}^{n}T(i) - \sum_{i=1}^{n-1}T(i) = T(n)$$

So

$$T(n+1) = 2T(n)$$

when $n \geq 2$. This is because we use a term $\sum_{i=1}^{n-1} T(i)$ in the above equations, and $\sum_{i=1}^{1-1} T(i)$ is not defined, so n must start from 2. Since $T(2) = 2 + \sum_{i=1}^{1} T(i) = 3$ and we have

$$T(n) = 2T(n-1)$$

$$T(n-1) = 2T(n-2)$$

$$T(n-2) = 2T(n-3)$$

$$\cdots\cdots$$

$$T(3) = 2T(2)$$

Multiply them together, we have

$$T(n) = 2^{n-2}T(2) = 3 \cdot 2^{n-2}$$

5.6 f.

Apply the Master Theorem, where $a = 3$, $b = 2$, and $n\log(n) = O(n^{\log_2(3)})$, so Case 1 applies. So $T(n) = \Theta(n^{\log(3)})$

5.7 g.

Here we cannot apply the Master Theorem, because $\frac{n}{\log(n)}$ is not $O(n^{1-\epsilon})$, for any ϵ. So we have to try another way.
Suppose $T(2) = a$ is given. Let $n = 2^m$, we have:

$$T(2^m) = 2T(2^{m-1}) + \frac{2^m}{m}$$

Divide the equation by 2^m, we have:

$$\frac{T(2^m)}{2^m} = \frac{T(2^{m-1})}{2^{m-1}} + \frac{1}{m}$$

Define $U(m) = \frac{T(2^m)}{2^m}$, so $U(1) = \frac{a}{2}$, and we have:

$$U(m) = U(m-1) + \frac{1}{m}$$

$$U(m-1) = U(m-2) + \frac{1}{m-1}$$

$$U(m-2) = U(m-3) + \frac{1}{m-2}$$

$$\cdots\cdots$$

$$U(2) = U(1) + \frac{1}{2}$$

Adding them together we get:

$$U(m) = U(1) + \sum_{i=2}^{m} \frac{1}{i} = \Theta(\log(m))$$

where the last "=" is because $H(m) = \Theta(\log(m))$, where $H(m)$ is the Harmonic series. So we have:

$$T(n) = T(2^m) = 2^m \cdot U(m) = n \cdot \Theta(\log(\log(n))) = \Theta(n \log(\log(n)))$$

5.8 h.

Suppose $T(2) = a$ is given. Divide the original recursion formula by n, we have:

$$\frac{T(n)}{n} = \frac{T(\sqrt{n})}{\sqrt{n}} + 1$$

Define $U(n) = \frac{T(n)}{n}$, so $U(2) = \frac{a}{2}$, and

$$U(n) = U(\sqrt{n}) + 1$$

Define $m = \log(n)$, so $n = 2^m$, so we have:

$$U(2^m) = U(2^{\frac{1}{2}m}) + 1$$

Define $V(m) = U(2^m)$, so $V(1) = U(2) = \frac{a}{2}$, and

$$V(m) = V(\frac{1}{2}m) + 1$$

and so we have:

$$V(\frac{1}{2}m) = V(\frac{1}{4}m) + 1$$
$$V(\frac{1}{4}m) = V(\frac{1}{8}m) + 1$$
$$V(\frac{1}{8}m) = V(\frac{1}{16}m) + 1$$
$$\cdots\cdots$$
$$V(\frac{1}{\frac{m}{2}}m) = V(\frac{1}{m}m) + 1$$

We have $\log(m)$ many of such equations(think why?). Adding them together, we have:

$$V(m) = V(1) + \log(m) = \frac{a}{2} + \log(m) = \frac{a}{2} + \log(\log(n))$$

Remember that $V(m) = U(2^m)$, and $U(2^m) = U(n)$, so we have:

$$U(n) = V(m) = \frac{a}{2} + \log(\log(n))$$

So we have:

$$T(n) = \frac{a}{2}n + n\log(\log(n))$$

6 Problem 6

Directly use the characteristic equation method. The equation is:

$$x^2 = 5x - 6$$

the roots are 2 and 3. So a_n is in the form $a_n = A \cdot 2^n + B \cdot 3^n$. And we have $a_0 = 2$, $a_1 = 5$, so we have

$$A + B = 2$$

$$2A + 3B = 5$$

So $A = 1$, $B = 1$. So $a_n = 2^n + 3^n$