CSE431/531, Problem Set 1 Due Wed 9/24, in class

Note: There are 6 problems with a total of 130 points. You are required to do all the problems. In the problems, the base of $\log n$ is 2.

1. ($7 \times 5 = 35$ points) Let $f(n)$ and $g(n)$ be two functions from N^+ to R^+. Prove or disprove the following assertions. To disprove, you only need to give a counter example for functions $f(n)$ and/or $g(n)$ which make the assertion false.

   (a) $O(O(f(n))) = O(f(n))$
   (b) $O(\Theta(f(n))) = O(f(n))$
   (c) $\Theta(O(f(n))) = \Theta(f(n))$
   (d) $\Omega(O(f(n))) = O(\Omega(f(n)))$
   (e) If $f(n) = \Theta(h(n))$ and $g(n) = \Theta(h(n))$, then $f(n) + g(n) = \Theta(h(n))$
   (f) If $f(n) = \Theta(g(n))$, then $2^{f(n)} = \Theta(2^{g(n)})$
   (g) $f(n) + g(n) = \Theta(\min(f(n) + g(n)))$

2. ($2 \times 5 = 10$ points) Use mathematical induction to prove the following.

   (a) $\sum_{i=1}^{n} ir^{i-1} = \frac{1-r^{n+1}-(n+1)(1-r)r^n}{(1-r)^2}$ for all $n \geq 1$, where $0 \leq r < 1$.
   (b) Every integer $n \geq 1$ can be represented as the sum of distinct Fibonacci numbers, no two of which are consecutive in the Fibonacci sequence.

3. ($4 \times 5 = 20$ points) Prove or disprove the following assertions.

   (a) $n! = O(n^n)$
   (b) $\sum_{i=1}^{n} i \log i = \Theta(n^2 \log n)$
   (c) If $n = 2^k$, then $\sum_{i=0}^{k} \log(n/2^i) = \Theta(\log^2 n)$
   (d) $n^n = O(2^n)$

4. (15 points) Rank the following functions in asymptotically increasing order based on O-notation and justify your ordering: $n!$, $(lgn)^{lg(lgn)}$, $[lg(lgn)]^{lgn}$, $2^{n^{0.001}}$, $n^{1/lgn}$, $lg^*(lgn)$, $2^{\sqrt{2lgn}}$, 2^{2^n}, n^5, $\sqrt{lgn}$.

5. ($8 \times 5 = 40$ points) Find a closed form for each $T(n)$. You may assume that $T(1) = 1$.

   (a) $T(n) = T(n-1) + 2^n$
   (b) $T(n) = 4T(n/3) + n^2$
   (c) $T(n) = 6T(n/7) + n$
   (d) $T(n) = T(\sqrt{n}) + logn$
   (e) $T(n) = 2 + \sum_{i=1}^{n-1} T(i)$
   (f) $T(n) = 3T(n/2) + n \log n$
   (g) $T(n) = 2T(n/2) + n/\log n$
   (h) $T(n) = \sqrt{n}T(\sqrt{n}) + n$

6. (10 points) The sequence $< a_n >$ is defined for $n \geq 0$ by $a_0 = 2$, $a_1 = 5$, and $a_n = 5a_{n-1} - 6a_{n-2}$ for $n > 1$. The first few elements of the sequence are 2, 5, 13, 35, 97. Find a closed form for a_n.