Q2

(a) Prove the base case $n=1$

We can assume that a non-singular A have a unique LU factorization. We can also assume its leading submatrices as non-singular.

$$\underbrace{\left(\begin{array}{c|c} A_{TL} & A_{TR} \\ \hline A_{BL} & A_{BR} \end{array}\right)}_{A} = \underset{L}{\left(\begin{array}{c|c} L_{TL} & 0 \\ \hline L_{BL} & L_{BR} \end{array}\right)} \underbrace{\left(\begin{array}{c|c} U_{TL} & U_{TR} \\ \hline 0 & U_{BR} \end{array}\right)}_{U}$$

We can let $A = UL$ be the LU factorization of A where A_{TL}, L_{TL} and U_{TL} are $k \times k$. We can also notice that U cannot have a zero on the diagonal since then A would not have linearly independent columns. We can also see that now the $k \times k$ principle leading submatrix A_{TL} equals $A_{TL} = L_{TL}U_{TL}$ which is non-singular since L_{TL} has a unit diagonal and U_{TL} has no zeros on the diagonal. Also, since we chose k arbitrarily, all principle leading submatrices are also non-singular.

Hence, now we can do a proof by induction on n.

Base case: $n = 1$. Then A has the form $A = \left(\begin{array}{c} \alpha_{11} \\ \hline a_{21} \end{array}\right)$ where α_{11} is a scalar.

~~And~~ And, because principle leading submatrices are non-singular, $\alpha_{11} \neq 0$. Hence, $A = \underbrace{\left(\begin{array}{c} 1 \\ \hline a_{21}/\alpha_{11} \end{array}\right)}_{L} \underbrace{\alpha_{11}}_{U}$

is the LU factorization of A. In addition, this LU factorization is unique because the first element of

l must be 1.

2b

We can start this theorem by distributing the absolute value operation over the submatrices and multiplying out partitioned matrix-matrix multiplications

1. We can take error side as $\{\Delta A = 0\}$ for step 1a

2. In step 2, we can select error result for the computations perform so far. We will restrict these error result to the computation described by the loop-invariant yielding the error-invariant.

$$m(A_{TL}) = k \wedge \left(\frac{\check{L}_{TL}\check{U}_{TL} = A_{TL} + \Delta A_{TL}}{\check{L}_{BL}\check{U}_{TR} = A_{BL} + \Delta A_{BL}} \middle| \frac{\check{L}_{TL}\check{U}_{TR} = A_{TR} + \Delta A_{TR}}{} \right)$$

Let me write the above step clearly:-

$$m(A_{TL}) = k \wedge \left(\begin{array}{c|c} \check{L}_{TL}\check{U}_{TL} = A_{TL} + \Delta A_{TL} & L_{TL}\check{U}_{TR} = A_{TR} + \Delta A_{TR} \\ \hline \check{L}_{BL}\check{U}_{TR} = A_{BL} + \Delta A_{BL} & \end{array} \right) \wedge \left| \left(\begin{array}{c|c} \Delta A_{TL} & \Delta A_{TR} \\ \hline \Delta A_{BL} & \end{array} \right) \right| \leq \gamma_k$$

$$\left(\begin{array}{c|c} |\check{L}_{TL}||\check{U}_{TL}| & |\check{L}_{TL}||\check{U}_{TR}| \\ \hline |\check{L}_{BL}||\check{U}_{TL}| & \end{array} \right)$$

Hence, step 2 gives us the error invariant. We can do matrix-matrix multiplications in steps 4-7

③ Step 3, we can do the while loop such that while $m(A_{TL}) \leq m(A)$ do

④ In step 4, we can perform Partitions such that :-

$$A \rightarrow \left(\begin{array}{c|c} A_{TL} & A_{TR} \\ \hline A_{BL} & A_{BR} \end{array} \right), \quad L \rightarrow \left(\begin{array}{c|c} L_{TL} & 0 \\ \hline L_{BL} & L_{BR} \end{array} \right)$$

$$U \rightarrow \left(\begin{array}{c|c} U_{TL} & U_{TR} \\ \hline 0 & U_{BR} \end{array}\right), \quad \Delta A \rightarrow \left(\begin{array}{c|c} \Delta A_{TL} & \Delta A_{TR} \\ \hline \Delta A_{BL} & \Delta A_{BR} \end{array}\right)$$

where, A_{TL}, L_{TL}, U_{TL} and ΔA_{TL} are empty?

(5) In step 5a, we can perform the repartition step such that

A, L, U partitioned as $\left(\begin{array}{c|c} \Delta A_{TL} & \Delta A_{TR} \\ \hline \Delta A_{BL} & \Delta A_{BR} \end{array}\right)$

$$\rightarrow \left(\begin{array}{c|c|c} \Delta A_{00} & \delta a_{01} & \Delta A_{02} \\ \hline \delta a_{10}^T & \delta \alpha_{11} & \delta a_{12}^T \\ \hline \Delta A_{20} & \delta a_{21} & \Delta A_{22} \end{array}\right)$$

where $\delta\alpha_{11}$ is a scalar.

Step 5b we can do A, L, U partitioned as follows

$$\left(\begin{array}{c|c} \Delta A_{TL} & \Delta A_{TR} \\ \hline \Delta A_{BL} & \Delta A_{BR} \end{array}\right) \leftarrow \left(\begin{array}{c|c|c} \Delta A_{00} & \delta a_{01} & \Delta A_{02} \\ \delta a_{10}^T & \delta \alpha_{11} & \delta a_{12}^T \\ \Delta A_{20} & \delta a_{21} & \Delta A_{22} \end{array}\right)$$

6) When combining step 5a and 5b step 6 becomes :-

$$
\left\{
\begin{array}{l}
m(A_{00}) = k \\
\wedge \left(\begin{array}{c|c}
\check{L}_{00}\check{U}_{00} = A_{00} + \Delta A_{00} & \\ \hline
l^T_{10}\check{U}_{00} = a^T_{10} + \delta a^T_{10} & \\ \hline
\check{L}_{20}\check{U}_{00} = A_{20} + \Delta A_{02} &
\end{array} \right. \\
\left. \begin{array}{l|l}
\check{L}_{00}\check{U}_{01} = a_{01} + \delta a_{01} & \check{L}_{00}\check{U}_{02} = A_{02} + \Delta A_{02} \\ \hline
&
\end{array} \right) \\
\wedge \left| \left(\begin{array}{c|c|c}
\Delta A_{00} & \delta a_{01} & \Delta A_{02} \\ \hline
\delta a^T_{10} & & \\ \hline
\Delta A_{20} & &
\end{array} \right) \right| \leq \\
\gamma_k \left(\begin{array}{c|c|c}
|\check{L}_{00}||\check{U}_{00}| & |\check{L}_{00}||\check{U}_{01}| & |\check{L}_{00}||\check{U}_{02}| \\ \hline
|l^T_{10}||\check{U}_{00}| & & \\ \hline
|\check{L}_{20}||\check{U}_{00}| & &
\end{array} \right)
\end{array}
\right\}
$$

(6) Step 6 also represents inductive manipulation hypothesis.

(7) We can further perform algebraic manipulation in step 7:-

$$m\left(\begin{array}{c|c} A_{00} & a_{01} \\ \hline a_{10}^T & \alpha_{11} \end{array}\right) = k + 1 \wedge$$

$$\left(\begin{array}{l|l} \check{L}_{00}\check{U}_{00} = A_{00} + \Delta A_{00} & \\ \hline \check{l}_{10}^T \check{U}_{00} = a_{10}^T + \delta a_{10}^T & \\ \check{L}_{20}\check{U}_{00} = A_{20} + \Delta A_{20} & \end{array}\right.$$

$$\begin{array}{l|l} \check{L}_{00}\check{U}_{01} = a_{01} + \delta a_{01} & \\ \hline \check{l}_{10}^T \check{U}_{01} + \check{U}_{11} = \alpha_{11} + \delta\alpha_{11} & \\ \check{L}_{20}\check{U}_{01} + \check{l}_{21}\check{U}_{11} = a_{21} + \delta a_{21} & \end{array}$$

$$\left.\begin{array}{l} \check{L}_{00}\check{U}_{02} = A_{02} + \Delta A_{02} \\ \hline l_{10}^T \check{U}_{02} + \check{U}_{12}^T = a_{12}^T + \delta a_{12}^T \end{array}\right) \wedge$$

$$\left\|\left(\begin{array}{c|c|c} \Delta A_{00} & \delta a_{01} & \Delta A_{02} \\ \hline \delta a_{10}^T & \delta \alpha_{11} & \delta a_{12}^T \\ \Delta A_{20} & \delta a_{21} & \end{array}\right)\right\| \leq$$

$$\gamma_{k+1} \begin{pmatrix} |\check{L}_{00}||\check{U}_{00}| & |\check{L}_{00}||\check{u}_{01}| & |\check{L}_{00}||\check{U}_{02}| \\ |l^T_{10}||\check{U}_{00}| & |l^T_{10}||\check{u}_{01}| + |\check{v}_{11}| & |l^T_{10}||\check{U}_{02}| + |\check{u}^T_{12}| \\ |\check{L}_{20}||\check{U}_{00}| & |\check{L}_{20}||\check{u}_{01}| + |\check{l}_{21}||\check{v}_{11}| & \end{pmatrix},$$

Hence, algebraic manipulation of step 7 yields the above step which is the predicate that must be true after the execution of the computational statements.

Step 8 shows the relations for submatrices that are satisfied by an inductive proof, since $\gamma_k \le \gamma_{k+1}$. Therefore, ~~it~~ step 8 shows that there exist $\delta\alpha_{11}, \delta a^T_{12}$, and δa_{21}. Hence we examine the the error introduced by the computational update in the following 1-8 steps to determine how error is ~~computed~~

contributed in each of these variables :-

a) Hence to determine error for $\delta\alpha_{11}$:
The assignment $\upsilon_{11} := \alpha_{11} - l^T_{10} u_{01}$
is conducted in a state where $m(A_{00}) = k$. Theorem 3.12 R2-F (from book) states that there exists δv_{11} such that

$$l^T_{10} u_{01} + \check{\upsilon}_{11} = \alpha_{11} + \delta v_{11}, \text{ where}$$

$$|\delta v_{11}| \leq \gamma_k (|l^T_{10}||u_{01}| + |\check{\upsilon}_{11}|) \leq$$

$$\gamma_{k+1} (|l^T_{10}||u_{01}| + |\check{\upsilon}_{11}|), \text{ hence we}$$

choose $\delta\alpha_{11} := \delta v_{11}$

b) Determine error for δa^T_{12} := The assignment $\upsilon^T_{12} := a^T_{12} - l^T_{10} U_{02}$ executed in a step where $m(A_{00}) = k$ and Theorem 5.1 R2-F (book) implies that there exists δv_{12} such that

$$l_{10}^T U_{02} + \check{u}_{12}^T = a_{12}^T + \delta v_{12}^T,$$

where $|\delta v_{12}^T| \le \gamma_k (|l_{10}^T||U_{02}| + |\check{u}_{12}^T|) \le \gamma_{k+1} (|l_{10}^T||U_{02}| + |\check{u}_{12}^T|)$;

hence, $\delta a_{12}^T := \delta v_{12}^T$ is the desired update.

c) Determine error for δa_{21}: the assignment $l_{21} := (a_{21} - L_{20} u_{01})/v_{11}$ executed in a state where $m(A_{00}) = k$ and Theorem 5.1 $R_{4\text{-}F}$ (book) implies that there exist δw_{21} such that

$L_{20} u_{01} + v_{11} l_{21} = a_{21} + \delta w_{21}$, where $|\delta w_{21}| \le \gamma_{k+1} (|L_{20}||u_{01}| + |l_{21}||v_{11}|)$; therefore $\delta a_{21} := \delta w_{21}$ is the desired update.

Step 8 is as follows:-

$\upsilon_{11} := \alpha_{11} - l_{10}^T \upsilon_{01}$
$\upsilon_{12}^T := a_{12}^T - l_{10}^T U_{02}$
$l_{21} := (a_{21} - L_{20} \upsilon_{01}) / \upsilon_{11}$

$\check{\upsilon}_{11} + \delta\alpha_{11} = \alpha_{11} - \check{l}_{10}^T \upsilon_{01}$
$\wedge\ |\delta\alpha_{11}| \le \gamma_{n+1}(|l_{10}^T||\upsilon_{01}| + |\check{\upsilon}_{11}|)$
Theorem from book (3.12 R2-F)

$\check{\upsilon}_{12}^T + \delta \alpha_{12}^T = a_{12}^T - l_{10}^T U_{02}$
$\wedge\ |\delta \alpha_{12}^T| \le \gamma_{n+1}(|l_{10}^T||U_{02}| + |\check{\upsilon}_{12}^T|)$

Theorem from book (th 5.12 R2-F)

$\check{l}_{21} \upsilon_{11} + \delta\alpha_{21} = a_{21} - L_{20} \upsilon_{01}$
$\wedge\ |\delta\alpha_{21}| \le \gamma_{n+1}(|L_{20}||\check{\upsilon}_{01}| +$
$|l_{21}||\upsilon_{11}|)$ theorem 5.1 R4-F.

This completes the proof of the inductive step for an upper triangular matrix from the result for solving with lower triangular matrix.