

## Z-transform:

In mathematics and signal processing, the **Z-transform** converts a discrete-time signal, which is a sequence of real or complex numbers, into a complex frequency-domain (**z-domain** or **z-plane**) representation.

**Z-transform and Inverse Z-transform**

- The Z-transform of a discrete time signal  $x(n)$  is represented by  $X(z)$

$$x(n) \xrightarrow{\text{ZT}} X(z)$$

- $$X(z) = \sum_{n=-\infty}^{\infty} x(n) z^{-n}$$
- $x(t) \rightarrow \text{freq} \rightarrow \text{F.T.}, \text{L.T.}$   
 $x(n) \rightarrow \text{freq} \rightarrow \text{D.F.T.}, \text{Z.T.}$

- $$\text{F.T.} [x(t)] = \int_{-\infty}^{\infty} x(t) e^{-j\omega t} dt$$
- $$\text{D.F.T.} [x(n)] = \sum_{n=-\infty}^{\infty} x(n) e^{-j\omega n}$$

The Z-transform can be defined as either a *one-sided* or *two-sided* transform.

### Bilateral Z-transform

The *bilateral* or *two-sided* Z-transform of a discrete-time signal  $x[n]$  is the *formal power series*  $X(z)$  defined as

$$X(z) = \mathcal{Z}\{x[n]\} = \sum_{n=-\infty}^{\infty} x[n] z^{-n}$$

where  $n$  is an integer and  $z$  is, in general, a *complex number*:

$$z = Ae^{j\phi} = A \cdot (\cos \phi + j \sin \phi)$$

where  $A$  is the magnitude of  $z$ ,  $j$  is the *imaginary unit*, and  $\phi$  is the *complex argument* (also referred to as *angle* or *phase*) in *radians*.

### Unilateral Z-transform

Alternatively, in cases where  $x[n]$  is defined only for  $n \geq 0$ , the *single-sided* or *unilateral* Z-transform is defined as

$$X(z) = \mathcal{Z}\{x[n]\} = \sum_{n=0}^{\infty} x[n] z^{-n}$$

- Relation bet.<sup>n</sup> DFT & Z.T.

$$- X(z) = \sum_{n=-\infty}^{\infty} x(n) z^{-n}$$

$$z = r e^{j\omega}$$

$$\begin{aligned} - X(z) &= \sum_{n=-\infty}^{\infty} x(n) (r e^{j\omega})^{-n} \\ &= \sum_{n=-\infty}^{\infty} \frac{x(n) r^{-n}}{e^{-j\omega n}} \\ &= \text{DFT} [x(n) r^{-n}] \end{aligned}$$

$$\text{DFT} [x(n)] = \sum_{n=-\infty}^{\infty} x(n) e^{-j\omega n}$$

Inverse Z - transform

$$\Rightarrow X(z) = \text{DFT} [x(n) r^{-n}]$$

$$\Rightarrow \text{IDFT} [X(z)] = x(n) r^{-n}$$

$$\begin{aligned} \Rightarrow x(n) &= r^n [\text{IDFT} (X(z))] \\ &= r^n \left[ \frac{1}{2\pi} \int X(z) e^{j\omega n} d\omega \right] \\ &= \frac{1}{2\pi} \int X(z) (r e^{j\omega})^n d\omega \end{aligned}$$

$$\rightarrow z = r e^{j\omega}$$

$$\Rightarrow z = r e^{j\omega}$$

$$\Rightarrow dz = r e^{j\omega} j d\omega$$

$$\Rightarrow dz = j z d\omega$$

$$\Rightarrow d\omega = \frac{dz}{jz}$$

$$\Rightarrow x(n) = \frac{1}{2\pi} \int X(z) z^n \frac{dz}{jz}$$

$$\Rightarrow x(n) = \frac{1}{2\pi j} \int X(z) z^{n-1} dz$$

## Region of Convergence

Region of Convergence is the range of complex variable  $Z$  in the  $Z$ -plane. The  $Z$ -transformation of the signal is finite or convergent. So, ROC represents those set of values of  $Z$ , for which  $X(Z)$  has a finite value.

## Properties of ROC

- ROC does not include any pole.
- For right-sided signal, ROC will be outside the circle in  $Z$ -plane.
- For left sided signal, ROC will be inside the circle in  $Z$ -plane.
- For stability, ROC includes unit circle in  $Z$ -plane.
- For Both sided signal, ROC is a ring in  $Z$ -plane.
- For finite-duration signal, ROC is entire  $Z$ -plane.

### Example

Let us find the Z-transform and the ROC of a signal given as  $x(n)=\{7,3,4,9,5\}$ , where origin of the series is at 3.

**Solution** – Applying the formula we have -

$$\begin{aligned} X(z) &= \sum_{n=-\infty}^{\infty} x(n)Z^{-n} \\ &= \sum_{n=-1}^3 x(n)Z^{-n} \\ &= x(-1)Z + x(0) + x(1)Z^{-1} + x(2)Z^{-2} + x(3)Z^{-3} \\ &= 7Z + 3 + 4Z^{-1} + 9Z^{-2} + 5Z^{-3} \end{aligned}$$

ROC is the entire Z-plane excluding  $Z = 0, \infty$

### Problem based on ROC in z-transform



#### Problem 01 :-

Determine z-transform and their ROC.  
of the following signal.

$$x[n] = \{ \underset{\uparrow}{1}, 2, 3, 4, 5 \}$$

Solution :-

$$x[n] = \left\{ \begin{array}{cccccc} 1, & 2, & 3, & 4, & 5 \\ \uparrow & 1 & 2 & 3 & 4 \end{array} \right\}$$

$$\mathcal{Z}[x[n]] = X(z) = \sum_{n=-\infty}^{\infty} x(n) \cdot z^{-n}$$

$n = 0 \text{ to } 4$

$$\therefore X(z) = \sum_{n=0}^4 x(n) \cdot z^{-n}$$

$$= x(0) + x(1) \cdot z^{-1} + x(2) \cdot z^{-2} + x(3) z^{-3} + x(4) \cdot z^{-4}$$

$$\underline{X(z)} = 1 + 2z^{-1} + 3z^{-2} + 4z^{-3} + 5z^{-4}$$

$$X(z) = 1 + 2z^{-1} + 3z^{-2} + 4z^{-3} + 5z^{-4}$$

$$\therefore X(z) = 1 + \frac{2}{z} + \frac{3}{z^2} + \frac{4}{z^3} + \frac{5}{z^4}$$


at  $z=0$ ,  $X(z) = \infty$ .

At  $z=\infty$ ,  $X(z) = 1 + 0 + 0 + 0 + 0 = 1$ .

The ROC of  $X(z)$  is available over the entire region of  $z$ -plane, except  $z=0$ .

Problem 02:-

Determine z-transform and their ROC of the following discrete time signal,

$$x[n] = \{ 1, 2, 3, 4, 5 \}$$


Solution:-

$$x[n] = \{ \underset{-4}{1}, \underset{-3}{2}, \underset{-2}{3}, \underset{-1}{4}, \underset{\uparrow_0}{5} \}$$

By the definition of z-transform,

$$z[x(n)] = X(z) = \sum_{n=-\infty}^{\infty} x(n) \cdot z^{-n}$$

$n = -4$  to  $0$ .

$$\therefore X(z) = \sum_{n=-4}^0 x(n) \cdot z^{-n}$$

$$= x(-4) \cdot z^4 + x(-3) \cdot z^3 + x(-2) \cdot z^2 + x(-1) z^1 + x(0)$$

$$X(z) = 1 \cdot z^4 + 2z^3 + 3z^2 + 4z + 5$$



$$\therefore \underline{X(z)} = z^4 + 2z^3 + 3z^2 + 4z + 5$$



R.O.C.

At  $z=0$ ,

$$X(z) = 5$$

At  $z=\infty$ ,

$$X(z) = \infty$$

The ROC of  $X(z)$  is available over entire region of  $z$ -plane, except  $z=\infty$ .

A finite sequence  $x(n)$  is defined as  $x(n) = \overset{x(0)}{5}, \overset{x(1)}{3}, \overset{x(2)}{-3}, \overset{x(3)}{0}, \overset{x(4)}{4}, \overset{x(5)}{-2}$  find  $X(z)$

$$Z.T[x(n)] = X(z) = \sum_{n=-\infty}^{\infty} x(n) z^{-n}$$

$$= \sum_{n=0}^5 x(n) z^{-n}$$

$$= x(0)z^0 + x(1)z^{-1} + x(2)z^{-2} + \frac{x(3)z^{-3}}{0} + x(4)z^{-4} + x(5)z^{-5}$$

$$\therefore X(z) = 5 + 3z^{-1} - 3z^{-2} + 4z^{-4} - 2z^{-5}$$

## Z-transform for finite sequence

A finite sequence  $x(n)$  is defined as  $x(n) = \{5, 3, -3, 0, 4, -2\}$   
find  $X(z)$  of given sequence.

$$- x(n) = \{ \overset{x(-2)}{5}, \overset{x(-1)}{3}, \overset{x(0)}{-3}, \overset{x(1)}{0}, \overset{x(2)}{4}, \overset{x(3)}{-2} \}$$

$$- X(z) = \sum_{n=-\infty}^{\infty} x(n) z^{-n} = \sum_{n=-2}^3 x(n) z^{-n}$$

$$= x(-2)z^2 + x(-1)z^1 + x(0)z^0 + x(1)z^{-1} + x(2)z^{-2} + x(3)z^{-3}$$
$$= 5z^2 + 3z^1 - 3z^0 + 0 \times z^{-1} + 4z^{-2} + (-2)z^{-3}$$

$$\boxed{X(z) = 5z^2 + 3z^1 - 3 + 4z^{-2} - 2z^{-3}}$$

A Finite duration sequence  $x(n) = \{5, 3, 0, 1, 2, 4\}$   
Find Z-transform of  $x(n)$ .

$$- x(n) = \{ \overset{x(0)}{5}, \overset{x(1)}{3}, \overset{x(2)}{0}, \overset{x(3)}{1}, \overset{x(4)}{2}, \overset{x(5)}{4} \}$$

$$- X(z) = \sum_{n=-\infty}^{\infty} x(n) z^{-n} = \sum_{n=0}^5 x(n) z^{-n}$$

$$= x(0)z^0 + x(1)z^{-1} + x(2)z^{-2} + x(3)z^{-3} + x(4)z^{-4} + x(5)z^{-5}$$

$$= 5z^0 + 3z^{-1} + 0z^{-2} + 1 \times z^{-3} + 2 \times z^{-4} + 4 \times z^{-5}$$

$$= \boxed{5 + 3z^{-1} + z^{-3} + 2z^{-4} + 4z^{-5}}$$

## Linearity

It states that when two or more individual discrete signals are multiplied by constants, their respective Z-transforms will also be multiplied by the same constants.

Mathematically,

$$a_1x_1(n) + a_2x_2(n) = a_1X_1(z) + a_2X_2(z)$$

**Proof** – We know that,

$$X(Z) = \sum_{n=-\infty}^{\infty} x(n)Z^{-n}$$

$$= \sum_{n=-\infty}^{\infty} (a_1x_1(n) + a_2x_2(n))Z^{-n}$$

$$= a_1 \sum_{n=-\infty}^{\infty} x_1(n)Z^{-n} + a_2 \sum_{n=-\infty}^{\infty} x_2(n)Z^{-n}$$

$$= a_1X_1(z) + a_2X_2(z)$$

*Hence Proved*

Here, the ROC is  $ROC_1 \cap ROC_2$ .

## Time Shifting

Time Shifting property of Z transform

- If  $x(n) \xrightarrow{ZT} X(z)$

- The time shifting property states that

$$x(n-m) \xrightarrow{ZT} z^{-m} X(z)$$

$$x(n+m) \xrightarrow{ZT} z^m X(z)$$

Proof

$$- X(z) = Z.T. [x(n)]$$

$$= \sum_{n=-\infty}^{\infty} x(n) z^{-n}$$

$$- Z.T. [x(n-m)] = \sum_{n=-\infty}^{\infty} x(n-m) z^{-n}$$



$$\text{If } n-m=p \Rightarrow n=p+m$$

$$= \sum_{p=-\infty}^{\infty} x(p) z^{-(p+m)}$$

$$= \sum_{p=-\infty}^{\infty} x(p) z^{-p} z^{-m}$$

$$= z^{-m} \sum_{p=-\infty}^{\infty} x(p) z^{-p}$$

$$= z^{-m} x(z)$$

example

find Z-Transform  
of  $\delta(n-k)$

$$\delta(n) \xrightarrow{ZT} 1$$

$$\delta(n-k) \xrightarrow{ZT} z^{-k}$$

Time shifting property depicts how the change in the time domain in the discrete signal will affect the Z-domain, which can be written as;

$$x(n - n_0) \longleftrightarrow X(Z)Z^{-n}$$

Or 
$$x(n - 1) \longleftrightarrow Z^{-1}X(Z)$$

**Proof –**

Let 
$$y(P) = X(P - K)$$

$$\begin{aligned} Y(z) &= \sum_{p=-\infty}^{\infty} y(p)Z^{-p} \\ &= \sum_{p=-\infty}^{\infty} (x(p - k))Z^{-p} \end{aligned}$$

Let  $s = p - k$

$$\begin{aligned} &= \sum_{s=-\infty}^{\infty} x(s)Z^{-(s+k)} \\ &= \sum_{s=-\infty}^{\infty} x(s)Z^{-s}Z^{-k} \\ &= Z^{-k}[\sum_{s=-\infty}^{\infty} x(m)Z^{-s}] \\ &= Z^{-k}X(Z) \end{aligned}$$

*Hence Proved*

## Time Scaling

## Scaling property of Z Transform

$$- x(n) \xrightarrow{ZT} X(z)$$

Then scaling property of Z Transform states that

$$a^n x(n) \xrightarrow{ZT} X(z/a)$$

Proof

$$X(z) = Z.T. [x(n)]$$

$$= \sum_{n=-\infty}^{\infty} x(n) z^{-n}$$

$$Z.T. [a^n x(n)] = \sum_{n=-\infty}^{\infty} a^n x(n) z^{-n}$$

$$Z.T. [a^n x(n)] = \sum_{n=-\infty}^{\infty} a^n x(n) z^{-n}$$

$$= \sum_{n=-\infty}^{\infty} x(n) (a z^{-1})^n$$

$$= \sum_{n=-\infty}^{\infty} x(n) \left(\frac{1}{a z^{-1}}\right)^{-n}$$

$$= \sum_{n=-\infty}^{\infty} x(n) (a^{-1} z)^{-n}$$

$$= \boxed{X(z/a)}$$

e.g.  $x(n) = a^n u(n)$ , find  $X(z)$

$$u(n) \xrightarrow{ZT} \frac{z}{z-1}$$

$$a^n u(n) \xrightarrow{ZT} \frac{(z/a)}{(\frac{z}{a}) - 1} = \frac{z}{z-a}$$

Time Scaling property tells us, what will be the Z-domain of the signal when the time is scaled in its discrete form, which can be written as;

$$a^n x(n) \longleftrightarrow X(a^{-1}Z)$$

**Proof -**

Let  $y(p) = a^p x(p)$

$$Y(P) = \sum_{p=-\infty}^{\infty} y(p) Z^{-p}$$

$$= \sum_{p=-\infty}^{\infty} a^p x(p) Z^{-p}$$

$$= \sum_{p=-\infty}^{\infty} x(p) [a^{-1} Z]^{-p}$$

$$= X(a^{-1} Z)$$

*Hence proved*

## Convolution

This depicts the change in Z-domain of the system when a convolution takes place in the discrete signal form, which can be written as -

$$x_1(n) * x_2(n) \longleftrightarrow X_1(Z) \cdot X_2(Z)$$

Proof -

$$\begin{aligned} X(Z) &= \sum_{n=-\infty}^{\infty} x(n) Z^{-n} \\ &= \sum_{n=-\infty}^{\infty} \left[ \sum_{k=-\infty}^{\infty} x_1(k) x_2(n-k) \right] Z^{-n} \\ &= \sum_{k=-\infty}^{\infty} x_1(k) \left[ \sum_n x_2(n-k) Z^{-n} \right] \\ &= \sum_{k=-\infty}^{\infty} x_1(k) \left[ \sum_{n=-\infty}^{\infty} x_2(n-k) Z^{-(n-k)} Z^{-k} \right] \end{aligned}$$

Let  $n-k = l$ , then the above equation can be written as -

$$\begin{aligned} X(Z) &= \sum_{k=-\infty}^{\infty} x_1(k) \left[ Z^{-k} \sum_{l=-\infty}^{\infty} x_2(l) Z^{-l} \right] \\ &= \sum_{k=-\infty}^{\infty} x_1(k) X_2(Z) Z^{-k} \\ &= X_2(Z) \sum_{k=-\infty}^{\infty} x_1(k) Z^{-k} \\ &= X_1(Z) \cdot X_2(Z) \end{aligned} \quad \text{Hence Proved}$$

Time reversal property of Z Transform

- If  $x(n) \xrightarrow{ZT} X(Z)$

Then time reversal property states that

$$x(-n) \xrightarrow{ZT} X(Z^{-1})$$

Proof

-  $X(Z) = Z.T. [x(n)]$

$$= \sum_{n=-\infty}^{\infty} x(n) Z^{-n}$$

-  $Z.T. [x(-n)] = \sum_{n=-\infty}^{\infty} x(-n) Z^{-n}$



$$- \text{Z.T} [x(-n)] = \sum_{n=-\infty}^{\infty} x(-n) z^{-n}$$

$$\text{If } p = -n$$

$$= \sum_{p=-\infty}^{\infty} x(p) z^p$$

$$= \sum_{p=-\infty}^{\infty} x(p) (z^{-1})^{-p}$$

$$= X(z^{-1})$$

e.g.  $x(n) = a^n u(-n)$ . find  $x(z)$

$$- x(n) = a^n u(-n)$$

$$= (1/a)^{-n} u(-n)$$

$$x(n) = (1/a)^n u(n) \xrightarrow{\text{ZT}} \frac{z}{z-1/a}$$

$$(1/a)^{-n} u(-n) \xrightarrow{\text{ZT}} \frac{z^{-1}}{z^{-1}-1/a} = \frac{1/2}{1/2-1/a} = \boxed{\frac{a}{a-z}}$$

Multiplication & Convolution property of Z-Transform

$$\text{If } x_1(n) \xrightarrow{\text{ZT}} x_1(z)$$

$$x_2(n) \xrightarrow{\text{ZT}} x_2(z)$$

- Multiplication property states that

$$x_1(n) x_2(n) \xrightarrow{\text{ZT}} x_1(z) * x_2(z)$$

- Convolution property states that

$$x_1(n) * x_2(n) \xrightarrow{\text{ZT}} x_1(z) x_2(z).$$

e.g.  $u(n-1) * \delta(n)$ , find Z-Transform

$$- x_1(n) = u(n-1) \xrightarrow{ZT} X_1(z) = \frac{1}{z-1}$$

$$- x_2(n) = \delta(n) \xrightarrow{ZT} X_2(z) = 1$$

$$- X(z) = ZT [x_1(n) * x_2(n)]$$

$$= X_1(z) X_2(z)$$

$$= \frac{1}{z-1}$$

e.g.  $u(n-2) * \delta(n-3)$ , find Z-Transform.

$$- x_1(n) = u(n-2) \xrightarrow{ZT} X_1(z) = z^{-2} \left( \frac{z}{z-1} \right) = \frac{1}{z} \left( \frac{1}{z-1} \right)$$

$$- x_2(n) = \delta(n-1) \xrightarrow{ZT} X_2(z) = z^{-1} = 1/z$$

$$- X(z) = ZT [x_1(n) * x_2(n)]$$

$$= X_1(z) X_2(z)$$

$$= \frac{1}{z} \left( \frac{1}{z-1} \right) \left( \frac{1}{z} \right)$$

$$= \frac{1}{z^2(z-1)}$$

## Successive Differentiation

Successive Differentiation property shows that Z-transform will take place when we differentiate the discrete signal in time domain, with respect to time. This is shown as below.

$$\frac{dx(n)}{dn} = (1 - Z^{-1})X(Z)$$

**Proof –**

Consider the LHS of the equation –  $\frac{dx(n)}{dn}$

$$= \frac{[x(n) - x(n-1)]}{[n - (n-1)]}$$

$$= x(n) - x(n-1)$$

$$= x(Z) - Z^{-1}x(Z)$$

$$= (1 - Z^{-1})x(Z)$$

*Hence Proved*

## Multiplication in Time

It gives the change in Z-domain of the signal when multiplication takes place at discrete signal level.

$$x_1(n) \cdot x_2(n) \longleftrightarrow \left(\frac{1}{2\pi j}\right)[X_1(Z) * X_2(Z)]$$

## Conjugation in Time

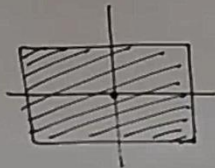
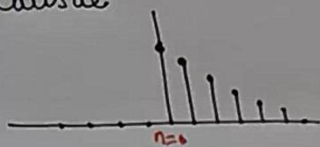
This depicts the representation of conjugated discrete signal in Z-domain.

$$X^*(n) \longleftrightarrow X^*(Z^*)$$

# Region of Convergence (ROC)

## Finite Duration Signals ✓

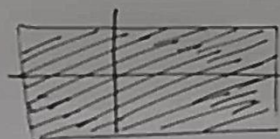
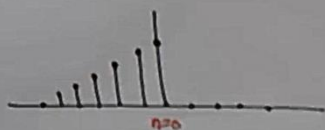
Causal



ROC

Entire z-plane  
except  $|z|=0$

Anti Causal



Entire z-plane  
except  $|z|=∞$

Two Sided

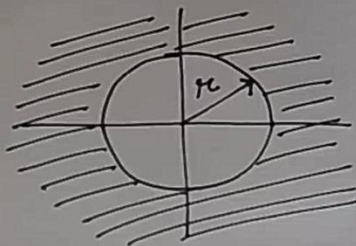
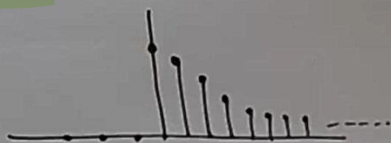


Entire z-plane  
except  $|z|=0$  &  $|z|=∞$

✓

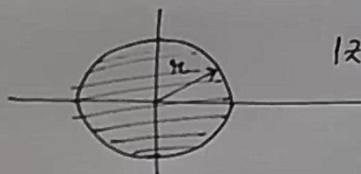
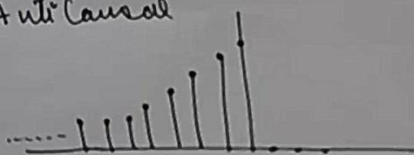
## Infinite Duration Signals

Causal



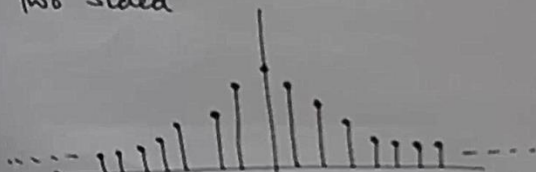
$|z| > r$

Anti Causal



$|z| < r$

Two Sided



$r_1 < |z| < r_2$

## RELATIONSHIP BETWEEN Z-TRANSFORM AND FOURIER TRANSFORM

Definition of z-transform,

$$Z[x(n)] = X(z) = \sum_{n=-\infty}^{\infty} x(n) \cdot z^{-n}.$$

Definition of Discrete Time Fourier Transform,

$$\text{DTFT}[x[n]] = X[e^{j\omega}] = \sum_{n=-\infty}^{\infty} x[n] \cdot e^{-j\omega n}.$$

$$\text{IF } \boxed{z = e^{j\omega}}$$

$$X[e^{j\omega}] = X(z) \Big|_{z=e^{j\omega}}.$$

**57. Determine the z-transform of the signal  $x(n) = \alpha^n u(n)$  and also the ROC and pole & zero locations of  $X(z)$  in the z-plane.**

**Solution:**

Given  $x(n) = \alpha^n u(n)$



By definition  $X(z) = \sum_{n=-\infty}^{\infty} x(n)z^{-n}$

$$= \sum_{n=-\infty}^{\infty} \alpha^n u(n) z^{-n}$$

$$= \sum_{n=0}^{\infty} (\alpha z^{-1})^n$$

Using geometric series,

$$\sum_{n=0}^{\infty} a^n = \frac{1}{1-a}$$

$$\therefore X(z) = \frac{1}{1-\alpha z^{-1}}; |z| > |\alpha|$$

$$= \frac{z}{z-\alpha}; |z| > |\alpha|$$

There is a pole at  $z = \alpha$  & zero at  $z = 0$ .

### ROC of Impulse function in Z-Transform

1.  $\delta(n) \xrightarrow{ZT} 1$

ROC! Entire Z plane, including  $z=0$  &  $z=\infty$

2.  $\delta(n-k) \xrightarrow{ZT} z^{-k}$

ROC! Entire Z plane, including  $z=\infty$  & excluding  $z=0$

3.  $\delta(n+k) \xrightarrow{ZT} z^k$

ROC! Entire Z plane, including  $z=0$  & excluding  $z=\infty$

$\delta(n-3)$  - Entire Z plane.

Includes  $z=\infty$

Excludes  $z=0$ .

## ROC of discrete time sequence in Z Transform

1. If sequence is purely right sided or causal, then

ROC: Entire Z plane except  $z=0$

$$x(n) = \{1, 2, 3, 4\} \xrightarrow{ZT} x(z) = 1 + 2z^{-1} + 3z^{-2} + 4z^{-3}$$

↑

2. If sequence is purely left sided or Anticausal, then

ROC: Entire Z plane except at  $z=0$

$$x(n) = \{1, 2, 3, 4\} \xrightarrow{ZT} x(z) = z + 2z^2 + 3z^3 + 4z^4$$

↑

Check may be  $\infty$

3. If sequence is a two sided, then

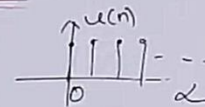
ROC: Entire Z plane except at  $z=0$  &  $z=\infty$

$$x(n) = \{1, 2, 1, 3, 1\} \xrightarrow{ZT} x(z) = z^2 + 2z + 1 + 3z^{-1} + z^{-2}$$

↑

## \* Z-Transform of Discrete unit step function. $u(n)$ \*

$$Z.T[u(n)]$$



$$Z.T[x(n)] = X(z) = \sum_{n=-\infty}^{\infty} x(n) z^{-n}$$

$$Z.T[u(n)] = \sum_{n=-\infty}^{\infty} u(n) z^{-n}$$

$$= \sum_{n=0}^{\infty} z^{-n}$$

$$= \sum_{n=0}^{\infty} \left(\frac{1}{z}\right)^n = \frac{1}{1 - z^{-1}} = \frac{z}{z-1}$$

$$\boxed{\sum_{n=0}^{\infty} a^n = \frac{1}{1-a}}$$

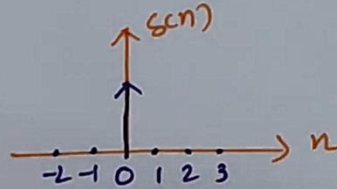
## Z-Transform of Unit Impulse function

$$- x(n) = \delta(n)$$

$$- X(z) = \text{Z.T.} [x(n)]$$

$$= \sum_{n=-\infty}^{\infty} x(n) z^{-n}$$

$$= \sum_{n=-\infty}^{\infty} \delta(n) z^{-n}$$



$$= \delta(-2)z^2 + \delta(-1)z^1 + \underline{\delta(0)z^0} + \delta(1)z^{-1} + \delta(2)z^{-2} + \dots$$

$$= 0 \times z^2 + 0 \times z^1 + 1 \times z^0 + 0 \times z^{-1} + 0 \times z^{-2} + \dots$$

$$= 1$$

## Z-Transform of Standard basic signals

- Find The Z-transform of the signals  $a^n u(n)$  &  $a^{-n} u(n)$

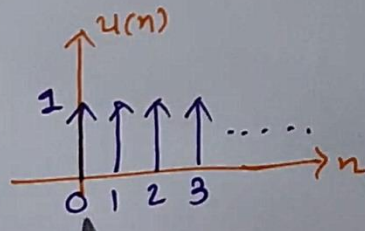
$$- x(n) = a^n u(n)$$

$$y(n) = a^{-n} u(n)$$

$$- X(z) = \text{Z.T.} [x(n)]$$

$$= \sum_{n=-\infty}^{\infty} x(n) z^{-n}$$

$$= \sum_{n=-\infty}^{\infty} a^n u(n) z^{-n}$$



$$\begin{aligned}
 - X(z) &= \sum_{n=0}^{\infty} (a^n)(z^{-n}) \\
 &= \sum_{n=0}^{\infty} (az^{-1})^n
 \end{aligned}$$

We know

$$\sum_{n=0}^{\infty} a^n = \frac{1}{1-a}$$

$$X(z) = \frac{1}{1-az^{-1}} = \frac{1}{1-(a/2)} = \frac{z}{z-a}$$

$$y(n) = a^{-n} u(n)$$

$$- Y(z) = \text{Z.T.} [y(n)]$$

$$= \sum_{n=-\infty}^{\infty} y(n) z^{-n}$$

$$= \sum_{n=-\infty}^{\infty} a^{-n} u(n) z^{-n}$$



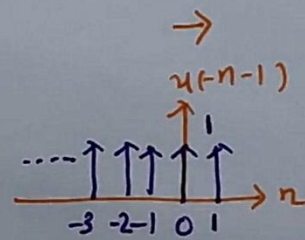
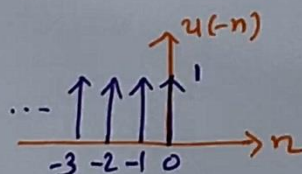
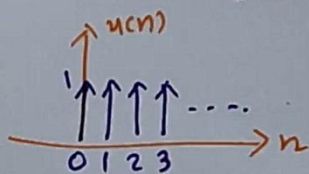
## Z-Transform of standard basic signal

Find the Z-transform of the sequence  $x(n) = -a^n u(-n-1)$

$$X(z) = Z.T. [x(n)]$$

$$= \sum_{n=-\infty}^{\infty} x(n) z^{-n}$$

$$= \sum_{n=-\infty}^{\infty} -a^n u(-n-1) z^{-n}$$



$$X(z) = \sum_{n=-\infty}^{\infty} -a^n z^{-n}$$

$$= - \sum_{n=-\infty}^{\infty} (az^{-1})^n$$

$$\left[ \sum_{n=0}^{\infty} a^n = \frac{1}{1-a} \right]$$

- To swap limits place  $n = -n$ .

$$X(z) = - \sum_{n=1}^{\infty} (az^{-1})^{-n}$$

$$= - \sum_{n=1}^{\infty} (za^{-1})^n$$



$$z = - \left[ \sum_{n=0}^{\infty} (za^{-1})^n - 1 \right]$$

$$z = - \left[ \frac{1}{1-za^{-1}} - 1 \right]$$

$$z = - \left[ \frac{a}{a-z} - 1 \right]$$

$$z = - \left[ \frac{a-a+z}{a-z} \right]$$

$$z = \boxed{\frac{z}{z-a}}$$

$$\uparrow x(n) = -a^n u(-n-1)$$

$$a^n u(n) \xrightarrow{\text{ZT}} \frac{z}{z-a}$$

$$-a^n u(-n-1) \xrightarrow{\text{ZT}} \frac{z}{z-a}$$

$$a^{-n} u(n) \xrightarrow{\text{ZT}} \frac{az}{az-1}$$

$$-a^{-n} u(-n-1) \xrightarrow{\text{ZT}} \frac{az}{az-1}$$

$$Y(z) = \sum_{n=0}^{\infty} a^{-n} z^{-n}$$

$$= \sum_{n=0}^{\infty} (az)^{-n}$$

We know  $\sum_{n=0}^{\infty} a^n = \frac{1}{1-a}$

$$- Y(z) = \frac{1}{1-(az)^{-1}} = \frac{1}{1-1/az}$$

$$= \boxed{\frac{az}{az-1}}$$

↑  
 $y(n) = a^{-n} u(n)$

### Z-transform for finite sequence

A finite sequence  $x(n)$  is defined as  $x(n) = \{5, 3, -3, 0, 4, -2\}$   
find  $X(z)$  of given sequence.

$$- x(n) = \{ \overset{x(-2)}{5}, \overset{x(-1)}{3}, \overset{x(0)}{-3}, \overset{x(1)}{0}, \overset{x(2)}{4}, \overset{x(3)}{-2} \}$$

$$- X(z) = \sum_{n=-\infty}^{\infty} x(n) z^{-n} = \sum_{n=-2}^3 x(n) z^{-n}$$

$$= x(-2)z^2 + x(-1)z^1 + x(0)z^0 + x(1)z^{-1} + x(2)z^{-2} + x(3)z^{-3}$$
$$= 5z^2 + 3z^1 - 3z^0 + 0 \times z^{-1} + 4z^{-2} + (-2)z^{-3}$$

$$\boxed{X(z) = 5z^2 + 3z^1 - 3 + 4z^{-2} - 2z^{-3}}$$

A Finite duration sequence  $x(n) = \{5, 3, 0, 1, 2, 4\}$   
Find Z-transform of  $x(n)$ .

$$- x(n) = \{ \overset{x(0)}{5}, \overset{x(1)}{3}, \overset{x(2)}{0}, \overset{x(3)}{1}, \overset{x(4)}{2}, \overset{x(5)}{4} \}$$

$$- X(z) = \sum_{n=-\infty}^{\infty} x(n) z^{-n} = \sum_{n=0}^5 x(n) z^{-n}$$

$$= x(0)z^0 + x(1)z^{-1} + x(2)z^{-2} + x(3)z^{-3} + x(4)z^{-4} + x(5)z^{-5}$$

$$= 5z^0 + 3z^{-1} + 0z^{-2} + 1 \times z^{-3} + 2 \times z^{-4} + 4 \times z^{-5}$$

$$= \boxed{5 + 3z^{-1} + z^{-3} + 2z^{-4} + 4z^{-5}}$$



### Examples of Inverse Z transform

Find the signal  $x(n)$  for which the Z transform is

$$X(z) = 4z^4 - z^3 - 3z + 4z^{-1} + 3z^{-2}$$

$$- x(n) = Z^{-1}[X(z)] = \frac{1}{2\pi j} \int X(z) z^{n-1} dz$$

$$X(z) = \overset{\uparrow}{4} z^4 - \overset{\uparrow}{1} z^3 + \overset{\uparrow}{0} z^2 - \overset{\uparrow}{3} z + \overset{\uparrow}{0} z^0 + \overset{\uparrow}{4} z^{-1} + \overset{\uparrow}{3} z^{-2}$$

$\uparrow$   $x(-4)$     $\uparrow$   $x(-3)$     $\uparrow$   $x(-2)$     $\uparrow$   $x(-1)$     $\uparrow$   $x(0)$     $\uparrow$   $x(1)$     $\uparrow$   $x(2)$

$$- X(z) = \sum_{n=-\infty}^{\infty} x(n) z^{-n}$$

$$= \sum_{n=-4}^2 x(n) z^{-n}$$

$$- x(n) = \{ 4, -1, 0, -3, 0, 4, 3 \}$$

$\uparrow$

If  $X(z) = 2z^2 + 3$ , find  $x(n)$

$$- X(z) = \underline{2z^2} + \underline{0 \times z^1} + \underline{3 \times z^0}$$

$$- X(z) = \sum_{n=-\infty}^{\infty} x(n) z^{-n}$$

$$= \sum_{n=-2}^0 x(n) z^{-n}$$

$$- x(n) = \{ 2, 0, 3 \}$$

$\uparrow$



If  $x(z) = 3 + 2z^{-1} + 3z^{-3}$ , find  $x(n)$

$$- x(z) = \frac{3 \times z^0}{x(0)} + \frac{2 \times z^{-1}}{x(1)} + \frac{0 \times z^{-2}}{x(2)} + \frac{3 \times z^{-3}}{x(3)}$$

$$- x(z) = \sum_{n=-\infty}^{\infty} x(n) z^{-n} = \sum_{n=0}^3 x(n) z^{-n}$$

$$- x(n) = \{ \underset{\uparrow}{3}, 2, 0, 3 \}$$

**4.18.** Using the power series expansion technique, find the inverse  $z$ -transform of the following  $X(z)$ :

$$(a) \quad X(z) = \frac{z}{2z^2 - 3z + 1} \quad |z| < \frac{1}{2}$$

$$(b) \quad X(z) = \frac{z}{2z^2 - 3z + 1} \quad |z| > 1$$

(a) Since the ROC is  $|z| < \frac{1}{2}$ ,  $x[n]$  is a left-sided sequence. Thus, we must divide to obtain a series in power of  $z$ . Carrying out the long division, we obtain

$$\begin{array}{r} z + 3z^2 + 7z^3 + 15z^4 + \dots \\ 1 - 3z + 2z^2 \overline{) z} \\ \underline{z - 3z^2 + 2z^3} \phantom{+ 15z^4 + \dots} \\ 3z^2 - 2z^3 \phantom{+ 15z^4 + \dots} \\ \underline{3z^2 - 9z^3 + 6z^4} \phantom{+ \dots} \\ 7z^3 - 6z^4 \phantom{+ \dots} \\ \underline{7z^3 - 21z^4 + 14z^5} \phantom{+ \dots} \\ 15z^4 \dots \end{array}$$

Thus,

$$X(z) = \dots + 15z^4 + 7z^3 + 3z^2 + z$$

and so by definition (4.3) we obtain

$$x[n] = \{\dots, 15, 7, 3, 1, 0\}$$

↑

(b) Since the ROC is  $|z| > 1$ ,  $x[n]$  is a right-sided sequence. Thus, we must divide so as to obtain a series in power of  $z^{-1}$  as follows:

$$\begin{array}{r} \frac{1}{2}z^{-1} + \frac{3}{4}z^{-2} + \frac{7}{8}z^{-3} + \dots \\ 2z^2 - 3z + 1 \overline{) z} \\ \underline{z - \frac{3}{2} - \frac{1}{2}z^{-1}} \phantom{+ \dots} \\ \frac{3}{2} - \frac{1}{2}z^{-1} \phantom{+ \dots} \\ \underline{\frac{3}{2} - \frac{9}{4}z^{-1} + \frac{3}{4}z^{-2}} \phantom{+ \dots} \\ \frac{7}{4}z^{-1} - \frac{3}{4}z^{-2} \phantom{+ \dots} \\ \vdots \end{array}$$

Thus,

$$X(z) = \frac{1}{2}z^{-1} + \frac{3}{4}z^{-2} + \frac{7}{8}z^{-3} + \dots$$

and so by definition (4.3) we obtain

$$x[n] = \{0, \frac{1}{2}, \frac{3}{4}, \frac{7}{8}, \dots\}$$