Logic

An Example of a Boolean Algebra

You know about algebra in math

- $(a+b)^2$, where
  - *a* and *b*, are variables whose values come from some set: the real-numbers.
  - +, -, *, /, ^, etc., are operators on these variables
  - the result of the expression is also a real number.
- We sometimes can show that one expression is the same (equal value) as another for all values of the variables …

$$(a+b)^2 = a^2 + 2ab + b^2$$

Logic is an algebra

- We have variables: typically *p, q, r, s, t, …*
- Variables can have a value from some set: true or false.
- We have operators: $\neg$, $\wedge$, $\vee$, $\rightarrow$, etc..
- We have expressions, whose resulting value is also true or false

$$(p \wedge q) \vee (\neg r)$$

- We also have expressions that are the same regardless of the values of the variables

$$\neg(p \wedge q) \equiv (\neg p) \vee (\neg q)$$

Propositions

- ***<u>Propositions</u>*** are *assertions* (stating a fact) about the world around us: they can either be true or false
- ***<u>Propositional variables</u>*** are variables whose value corresponds to the value of some proposition.
  - Let *p* be "Dr. Cobb is 5' 2'' tall"
    - Is *p* true or false?
  - Let *q* be "the moon is made of cheese"
    - Is *q* true or false?
  - Let *r* be "the earth's atmosphere is made mainly from oxygen and nitrogen".
    - Is *r* true or false?

What propositions are not

- Propositions are <u>not</u> questions …
  - Is Dr. Cobb 5’2’’ tall?
  - Is the moon made of cheese?
    - These have yes or no answers, not true or false
- Propositions are <u>not</u> commands (imperative)
  - Get up!
  - Go to work!
  - Finish your homework!

More proposition examples

- The sky is blue.
- A car is a vehicle.
- A human is a living being.
- CS 5333 is a prerequisite for a MS degree in CS at UTD.

What if the proposition is not clear?

- Dr. Cobb is 6’ 1’’? (not sure, depends if I stretch in the morning!)
- 5333 is a requirement for a CS MS degree? (it is *now*, but will it be 10 years from now)

- **<u>My point</u>**: all the propositions that we will use in the course will be either obviously true or obviously false
  - It is the *algebra* of proposition variables that we care about.

Basic Logic Operators

- There are three basic logic operators:
  - Negation: $\neg$
  - Conjunction: $\wedge$
  - Disjunction: $\vee$
- Any logic expression can be written using the above
- Other operators (e.g., implication, double implication, exclusive or) are used as a shorthand for expressions using the above.

Negation

- Negation (a unary operator)
  - Let *p* be "Dr. Cobb is 5'2'' tall"
  - Then $\neg p$ represents "Dr. Cobb ***is not*** 5'2'' tall"
  - Let *r* be "The moon is made of cheese"
  - Then $\neg$r represents "The moon ***is not*** made of cheese"

Truth Table

- **Truth table of an expression**: is a table showing all the possible values of the variables, and the corresponding values of an expression from those variables.

Truth table of
the expression: $\neg q$

q	$\neg q$
T	F
F	T

Conjunction

- The conjunction operator, $\wedge$, is also known as the "and" operator.
- The conjunction of p and q is written as

$$p \wedge q$$

  and is read "p and q"
- It is true only when both p and q are true

p	q	$p \wedge q$
T	T	T
T	F	F
F	T	F
F	F	F

Propositional expression example

- Let: b be "the sky is blue"
  g be "the grass is green"
- Each of b and g is obviously true ☺
- What about the values of each of the following:

$\neg b$

$b \wedge g$

$\neg b \wedge (b \wedge g)$

Do a truth table to find out different possibilities

b	g	$b \wedge g$	$\neg b$	$\neg b \wedge (b \wedge g)$
T	T	T	F	**F**
T	F	F	F	F
F	T	F	T	F
F	F	F	T	F

$\neg b \wedge (b \wedge g)$ is a **fallacy** (always false regardless of values of variables)

Disjunction

- The disjunction operator, $\vee$, is also known as the "or" operator.
- The disjunction of p and q is written as

$$p \vee q$$

  and is read "p or q"
- It is true if either p is true or q is true

p	q	$p \vee q$
T	T	T
T	F	T
F	T	T
F	F	F

Another propositional expression example

- Let: b be "the sky is blue"
  g be "the grass is green"
- Each of b and g is obviously true ☺
- What about the values of each of the following:

$b \vee g$

$\neg b \vee g$

$\neg b \vee \neg g$

BTW: order of evaluation? Parenthesis?

$\neg b \vee (b \wedge g)$

Another example: $g \wedge (\neg b \vee g)$

b	g	$\neg b$	$\neg b \vee g$	$g \wedge (\neg b \vee g)$
T	**T**	**F**	**T**	**T**
T	F	F	F	F
F	T	T	T	T
F	F	T	T	F

$$g = (g \wedge (x \vee g)) \text{ for any } x \text{ and } g \quad \text{(absorption law)}$$

Equivalence of Propositional Expressions

- $P_1 \equiv P_2$, where P_1 and P_2 are propositional expressions, means "for all possible values that can be assigned to the variables in P_1 and P_2, both expressions yield the same value."

- Basically, $\equiv$ is the same as $=$, except that it is applied to propositional expressions, rather than numbers

(examples follow)

Examples of $\equiv$

- $p \wedge \neg p \equiv \mathrm{F}$
  because, regardless of the value of p, $p \wedge \neg p$ is false
- $p \vee \mathrm{T} \equiv \mathrm{T}$
  because, regardless of the value of p, $p \vee \mathrm{T}$ yields the value T.
- $(p \wedge \neg p) \equiv \neg(p \vee \mathrm{T})$
  why?

Logical equivalences (algebraic laws)

Equivalence	*Name*
$p \wedge T \equiv p$ $p \vee F \equiv p$	Identity laws
$p \vee T \equiv T$ $p \wedge F \equiv F$	Domination laws
$p \wedge p \equiv p$ $p \vee p \equiv p$	Idempotent laws
$\neg(\neg p) \equiv p$	Double negation law
$p \wedge q \equiv q \wedge p$ $p \vee q \equiv q \vee p$	Commutative laws

Logical equivalences (continued…)

Equivalence	*Name*
$p \vee (q \vee r) \equiv (p \vee q) \vee r$ $p \wedge (q \wedge r) \equiv (p \wedge q) \wedge r$	Associative laws
$p \vee (q \wedge r) \equiv (p \vee q) \wedge (p \vee r)$ $p \wedge (q \vee r) \equiv (p \wedge q) \vee (p \wedge r)$	Distributive laws
$\neg(p \vee q) \equiv \neg p \wedge \neg q$ $\neg(p \wedge q) \equiv \neg p \vee \neg q$	De Morgan's laws
$p \vee (p \wedge q) \equiv p$ $p \wedge (p \vee q) \equiv p$	Absorption laws
$p \wedge \neg p \equiv F$ $p \vee \neg p \equiv T$	Negation laws

Distributive Rule

You can't mix disjunction and conjunction for associativity.

$$p \vee (q \wedge r) \not\equiv (p \vee q) \wedge r$$

Consider the specific case

p = T

r = F

q = who cares ....

what is the value of the left-hand side and the right-hand side?

Implication

- The implication operator, $\rightarrow$, is as follows

$$p \rightarrow q$$

- Truth table:

p	q	$p \rightarrow q$
T	T	T
T	F	F
F	T	T
F	F	T

- Note, it is NOT commutative

Implication (continued…)

- $p \rightarrow q$ is read "if p then q"

- When p is true, $p \rightarrow q$ has the same value as q

- When p is false, $p \rightarrow q$ is true ***regardless*** of q.

An example

- Let p = "Dr. Cobb is a faculty member
  in the E. J. School of Engineering"
  q = "Dr. Cobb is a faculty member at UTD"
- What is the value of p?
- What is the value of q?
- Is $p \rightarrow q$ true or false?
- Is $q \rightarrow p$ true or false?
- Repeat by changing "Dr. Cobb" above by "Dr. Andreescu".

Example

- Let p = "there is a lion in this room"
- Let q = "Dr. Cobb is a millionaire"
- $p \rightarrow q$, is it true or false?
- What about $q \rightarrow p$?

From English to Logic

- "If I go to Harry's or go to the country I will not go shopping."
- Break the assertion into components, look for the logical operators.
- p = "go to Harry's"
  q = "go to the country"
  r = "go shopping"
- $(p \vee q) \rightarrow (\neg r)$

Truth Table

p	q	r	$(p \vee q) \rightarrow (\neg r)$
T	T	T	F
T	T	F	T
T	F	T	F
T	F	F	T
F	T	T	F
F	T	F	T
F	F	T	T
F	F	F	T

of possible operators

- Consider an operator of *n* variables
- How many rows are in its truth table?

- How many different truth tables with *x* rows can there be?

- Thus, how many different operators with *n* variables can exist?

Additional notes on $\rightarrow$

- $p \rightarrow q$, p is the *antecedent*, q is the *consequent*.
- $(p \rightarrow q) \equiv (\neg q \rightarrow \neg p)$
  - $\neg q \rightarrow \neg p$ is known as the *contrapositive*)
- The *converse of* $p \rightarrow q$ is $q \rightarrow p$

  (they are NOT the same)
- $p \rightarrow q$ in English is often stated as:
  - p is a sufficient condition for q
  - q is a necessary condition for p
  - p only if q

Logical Equivalences Involving Implications

$$p \to q \equiv \neg p \vee q$$

Note how → is written in terms of ∨ and ¬

$$p \to q \equiv \neg q \to \neg p$$

$$p \vee q \equiv \neg p \to q$$

$$p \wedge q \equiv \neg(p \to \neg q)$$

$$(p \to q) \wedge (p \to r) \equiv p \to (q \wedge r)$$

$$(p \to q) \vee (p \to r) \equiv p \to (q \vee r)$$

$$(p \to r) \wedge (q \to r) \equiv (p \vee q) \to r$$

$$(p \to r) \vee (q \to r) \equiv (p \wedge q) \to r$$

Proof of selected equivalences

- Consider the following two statements

$$(p \to r) \wedge (q \to r) \equiv (p \vee q) \to r$$

$$(p \to r) \vee (q \to r) \equiv (p \wedge q) \to r$$

- These can be proved using a truth table and considering all possible truth assignments.
- They can also be proved using some of the given equivalences

Proof of first statement

$$
\begin{aligned}
& (p \to r) \wedge (q \to r) && \{\to \text{ defined from } \vee \text{ and } \neg\} \\
\equiv\ & (\neg p \vee r) \wedge (\neg q \vee r) && \{\text{distribution}\} \\
\equiv\ & [(\neg p \vee r) \wedge \neg q] \vee [(\neg p \vee r) \wedge r] && \{\text{distribution on both}\} \\
\equiv\ & (\neg p \wedge \neg q) \vee (r \wedge \neg q) \vee (\neg p \wedge r) \vee (r \wedge r) && \{\text{idempotent}\} \\
\equiv\ & (\neg p \wedge \neg q) \vee (\neg p \wedge r) \vee (r \wedge \neg q) \vee r && \{\text{associativity}\} \\
\equiv\ & (\neg p \wedge \neg q) \vee (\neg p \wedge r) \vee [(r \wedge \neg q) \vee r] && \{\text{absorption}\} \\
\equiv\ & (\neg p \wedge \neg q) \vee (\neg p \wedge r) \vee r && \{\text{associativity}\} \\
\equiv\ & (\neg p \wedge \neg q) \vee [(\neg p \wedge r) \vee r] && \{\text{absorption}\} \\
\equiv\ & (\neg p \wedge \neg q) \vee r && \{\text{de Morgan's}\} \\
\equiv\ & (\neg (p \vee q)) \vee r && \{\to \text{ defined from } \vee \text{ and } \neg\} \\
\equiv\ & (p \vee q) \to r
\end{aligned}
$$

A lot simpler, and skipping over some “obvious steps” ....

$$
\begin{aligned}
&(p \rightarrow r) \wedge (q \rightarrow r) && \{\rightarrow \text{defined from} \vee \text{and} \neg\} \\
&\equiv (\neg p \vee r) \wedge (\neg q \vee r) && \{\text{commutativity, distribution of} \vee \text{over} \wedge, \\
& && \text{and commutativity again}\} \\
&\equiv (\neg p \wedge \neg q) \vee r && \{\text{de Morgan's}\} \\
&\equiv (\neg (p \vee q)) \vee r && \{\rightarrow \text{defined from} \vee \text{and} \neg\} \\
&\equiv (p \vee q) \rightarrow r
\end{aligned}
$$

Remarks

- We took advantage of the fact that, if we know $P \equiv Q$, and $Q \equiv R$, then we can conclude that $P \equiv R$.
- Also, if $p \equiv q$, and p occurs in an expression E, then

  $$E \equiv E'$$

  where E' is obtained from E by replacing p by q.

Generalized de Morgan's

$$\neg(p_1 \vee p_2 \vee \cdots \vee p_n) = (\neg p_1 \wedge \neg p_2 \wedge \cdots \wedge \neg p_n)$$

$$\neg(p_1 \wedge p_2 \wedge \cdots \wedge p_n) = (\neg p_1 \vee \neg p_2 \vee \cdots \vee \neg p_n)$$

Normal (Canonical) Forms

- Every propositional expression has a single *Disjunctive Normal Form*

- Every propositional expression also has a single *Conjunctive Normal Form*

Disjunctive Normal Form

- The following expression is in disjunctive normal form (DNF):

$$(p \wedge q \wedge \neg r) \vee (p \wedge \neg q \wedge \neg r) \vee (\neg p \wedge q \wedge r)$$

- I.e., it is a disjunction of *minterms* where
  - Each minterm is a conjunction of ***all*** variables
  - Each variable appears only once in each minterm, and the variable can be negated.
  - each minterm appears only once

Obtaining the DNF

- What is the DNF of $(p \vee q) \rightarrow (\neg r)$?

p	q	r	$(p \vee q) \rightarrow (\neg r)$
T	T	T	F
T	T	F	T
T	F	T	F
T	F	F	T
F	T	T	F
F	T	F	T
F	F	T	T
F	F	F	T

$$(p \vee q) \rightarrow (\neg r)$$

$$\equiv$$

$$(p \wedge q \wedge \neg r) \vee (p \wedge \neg q \wedge \neg r) \vee (\neg p \wedge q \wedge \neg r) \vee (\neg p \wedge \neg q \wedge r) \vee (\neg p \wedge \neg q \wedge \neg r)$$

Conjunctive Normal Form

- A propositional expression is in Conjunctive Normal Form (CNF) iff
  - it is a conjunction of *maxterms* where
  - each maxterm is a disjunction of ***all*** variables
  - each variable appears only once in each maxterm, and the variable can be negated.
  - each maxterm appears only once
- E.g.

$$(p \vee q \vee \neg r) \wedge (p \vee \neg q \vee \neg r) \wedge (\neg p \vee q \vee r)$$

Obtaining the CNF

- What is the CNF of $(p \vee q) \rightarrow (\neg r)$?

First, Truth Table of Negation

p	q	r	$\neg((p \vee q) \rightarrow (\neg r))$
T	T	T	T
T	T	F	F
T	F	T	T
T	F	F	F
F	T	T	T
F	T	F	F
F	F	T	F
F	F	F	F

$$\neg((p \vee q) \rightarrow (\neg r))$$

$$\equiv$$

$$(p \wedge q \wedge r) \vee (p \wedge \neg q \wedge r) \vee (\neg p \wedge q \wedge r)$$

Finally, apply De Morgan's

$$(p \vee q) \rightarrow (\neg r)$$

$$\equiv$$

$$\neg(\neg((p \vee q) \rightarrow (\neg r)))$$

$$\equiv$$

$$\neg((p \wedge q \wedge r) \vee (p \wedge \neg q \wedge r) \vee (\neg p \wedge q \wedge r))$$

$$\equiv$$

$$\neg(p \wedge q \wedge r) \wedge \neg(p \wedge \neg q \wedge r) \wedge \neg(\neg p \wedge q \wedge r)$$

$$\equiv$$

$$(\neg p \vee \neg q \vee \neg r) \wedge (\neg p \vee q \vee \neg r) \wedge (p \vee \neg q \vee \neg r)$$

- Is there another way to obtain the CNF from the original truth table of

$$(p \vee q) \rightarrow (\neg r)$$

???

XOR

- p exclusive or q is written as

$$p \oplus q$$

- It means either p is true, or q is true, <u>but not both</u>
- Truth table:

p	q	$p \oplus q$
T	T	F
T	F	T
F	T	T
F	F	F

Example

- This afternoon, either I will go to the movies or I will go to a football game
- p = "I am at the movies"
- q = "I am at the football game"
- The following is a true statement

$$p \oplus q$$

- Why is the above statement true?

Double Implication

- p "if and only if" q is written as

$$p \leftrightarrow q$$

- It is really a shorthand

$$p \leftrightarrow q \equiv (p \rightarrow q) \wedge (q \rightarrow p)$$

- Truth table:

p	q	$p \leftrightarrow q$
T	T	T
T	F	F
F	T	F
F	F	T

Example

- p = "The moon is made of cheese"
- q = "Horses have four legs"
- The following is a true statement, why?

  $$p \leftrightarrow (\neg q)$$

- Note that $p \not\equiv (\neg q)$, why?

- The following is a propositional expression (i.e. it returns a value that is true or false)

  $$p \leftrightarrow q$$

  and its value is ….?

Propositional Equivalences

- A *tautology* is a propositional expression that regardless of the values of its variables it always returns true.
- A *fallacy* is a propositional expression that regardless of the values of its variables it always returns false.
- A *contingency* is a logical expression that is neither a tautology nor a fallacy

Examples

$p \wedge \neg p$ is a fallacy

T is a tautology :)

F is a fallacy :)

$p \vee T$ is a tautology

$\big((p \vee q) \wedge p\big) \to p$ is a tautology (why?)

Tautology and Equivalence

- Saying that a propositional expression P is a tautology is the same as saying that
  $$P \equiv T$$
- Saying that a propositional expression P is a fallacy is the same as saying that
  $$P \equiv F$$

Predicates

- A *propositional* *function* (a.k.a. a *predicate*) is a generalization of propositions, in which the value of a proposition depends on an argument (i.e. variable)

- *Predicates become propositions* once every one of its variables is bound by
  a) assigning it a value from the *Universe of Discourse* (typically denoted by U), or by
  b) quantifying it (we cover this later)

Examples

- Let U = Z, the integers = {. . . -2, -1, 0 , 1, 2, . . .}
- *P(x): x* > 0 is the predicate. Note: it has no truth value until the variable *x* is bound.
- Examples of propositions where *x* is assigned a value:
  - *P*(-3) is false,
  - *P*(0) is false,
  - *P*(3) is true.
- The collection of integers for which *P*(*x*) is true are the positive integers

Same example, continued …

- $P(y) \vee \neg P(0)$ *is not a proposition. The variable* y has not been bound. However, $P(3) \vee \neg P(0)$ *is* a proposition (which is true).
- Let R be the three-variable predicate
  $R(x, y, z)$: $\quad x + y > z$
- Find the truth value of

  $R(2, -1, 5)$, $R(3, 4, 7)$, $R(x, 3, z)$

Quantifiers

- We turn a predicate into a proposition if all its variables are bound by a quantifier.

- Typical quantifiers:
  - Universal
  - Existential

Quantifiers: Universal

- $P(x)$ is true for every x in the universe of discourse is written as

  $$\forall x\, P(x)$$

  It is read 'For all x, $P(x)$', also as 'For every x, $P(x)$'

- The variable x is bound by the universal quantifier, producing a proposition.

Example

- U={1,2,3}, $P(x)$: $x > 0$

$$\forall x\, P(x) = P(1) \wedge P(2) \wedge P(3)$$

- U={-2,1,2,3}, $P(x)$: $x > 0$

$$\forall x\, P(x) = P(-2) \wedge P(1) \wedge P(2) \wedge P(3)$$

- What if U is infinite, e.g., U = natural numbers? Does $\forall x$ P(x) make sense then?

Quantifiers: Existential

- $P(x)$ is true for at least one value of x in the universe of x

$$\exists x\ P(x)$$

- It is read as 'There is an x such that $P(x)$,' 'For some x, P(x)', 'For at least one x, P(x)', 'I can find an x such that $P(x)$.'

- The variable x is bound by the existential quantifier, producing a proposition.

Example

- U={1,2,3}, $P(x)$: $x > 0$

  $\exists x\, P(x) = P(1) \vee P(2) \vee P(3)$

- U={-2,1,2,3}, $P(x)$: $x > 0$

  $\exists x\, P(x) = P(-2) \vee P(1) \vee P(2) \vee P(3)$

- What if U is infinite, e.g., U = natural numbers? Is it well defined then?

Quantifier: Unique Existential

- P(x) is true for one and only one x in the universe of discourse. Notation:

$$\exists ! x P(x)$$

- It is read as: 'There is a unique x such that P(x),' 'There is one and only one x such that P(x),' 'One can find only one x such that P(x).'

Truth Table

- Assume U = {0, 1, 2}. Consider all possible values of each Q(0), Q(1), and Q(2). What are the values of the qualifiers?

Q(0)	Q(1)	Q(2)	∀x Q(x)	∃x Q(x)	∃!x Q(x)
T	T	T	T	T	F
T	T	F	F	T	F
T	F	T	F	T	F
T	F	F	F	T	T
F	T	T	F	T	F
F	T	F	F	T	T
F	F	T	F	T	T
F	F	F	F	F	F

- A particular row would correspond to a specific definition of Q.

Just for fun..

- Can you write a definition of the unique existential quantifier using the universal and existential quantifiers?

$$\exists! x\ P(x) \equiv ???$$

Lewis Carroll

- Consider the following statements

- All lions are fierce.
- Some lions do not drink coffee
- Some fierce creatures do not drink coffee.

- We wish to turn this into symbols using an appropriate universe

Continued …

- Let the universe be all living creatures.

- Consider the following predicates:
- $L(x)$: x is a lion
- $C(x)$: x drinks coffee
- $F(x)$: x is fierce

Continued …

- $L(x)$: x is a lion, $C(x)$: x drinks coffee, $F(x)$: x is fierce
- All lions are fierce.

$$\forall x\,(L(x) \rightarrow F(x))$$

- Recall, the universe is over ALL creatures, so *x* ranges over all creatures
- We must thus restrict ourselves ONLY to those creatures that are lions.

Continued …

$$\forall x\,(L(x) \rightarrow F(x))$$

- Consider instead

  $$\forall x(L(x) \wedge F(x))$$

  what is the result of L(x) $\wedge$ F(x) when x is a lion?
  what about when x is not a lion? (e.g. an elephant)

$$(L(lion_1) \wedge F(lion_1)) \;\wedge\; (L(elephant_1) \wedge F(elephant_1)) \;\wedge\; etc\; \ldots$$

- Thus, does the whole expression $\forall x(L(x) \wedge F(x))$ depend in any way on the lions?
  - Thus, is $\forall x(L(x) \wedge F(x))$ the correct expression for what we want?

Continued …

- Some lions do not drink coffee

$$\exists x(L(x) \wedge \neg C(x))$$

- Some fierce creatures do not drink coffee.

$$\exists x(F(x) \wedge \neg C(x))$$

- For the first one, why is the following **NOT** what we want:

$$\exists x(L(x) \rightarrow \neg C(x))$$

$$\big(L(lion_1) \rightarrow \neg C(lion_1)\big) \vee \big(L(elephant_1) \rightarrow \neg C(elephant_1)\big) \wedge etc \ \ldots$$

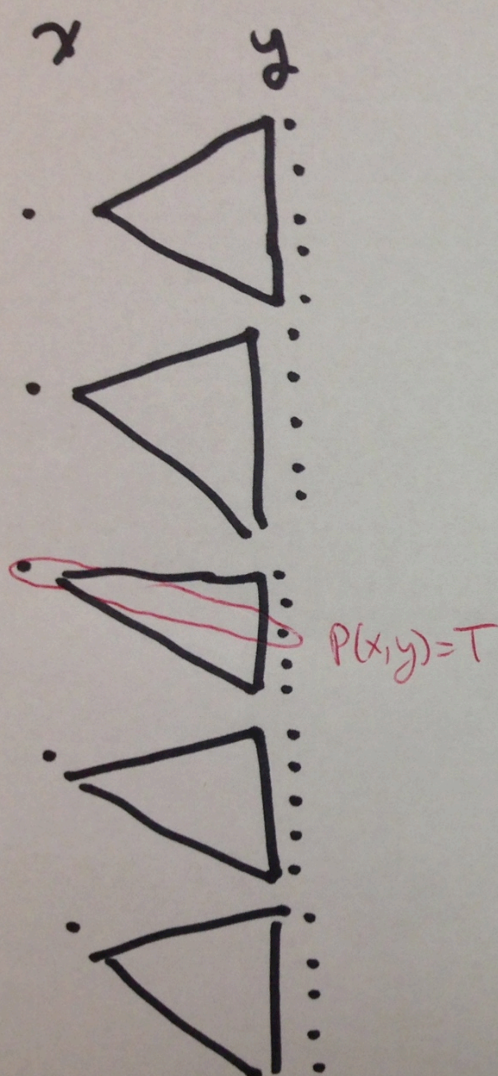
Nested Quantifiers

- Read from left to right:
  $\exists x \exists y P(x, y)$ is the same as $\exists x(\exists y P(x, y))$
- Example: Let U = R (real numbers), and
  $P(x,y)$: $xy = 0$

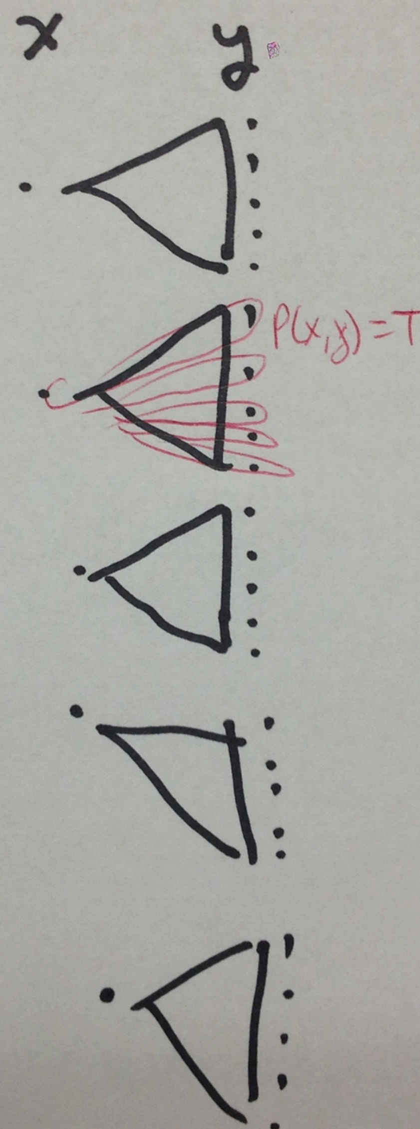
$$\exists x \exists y P(x, y)$$
$$\exists x \forall y P(x, y)$$
$$\forall x \exists y P(x, y)$$
$$\forall x \forall y P(x, y)$$

  which ones are true, which false?
- What if $P(x,y)$: $x/y = 1$?

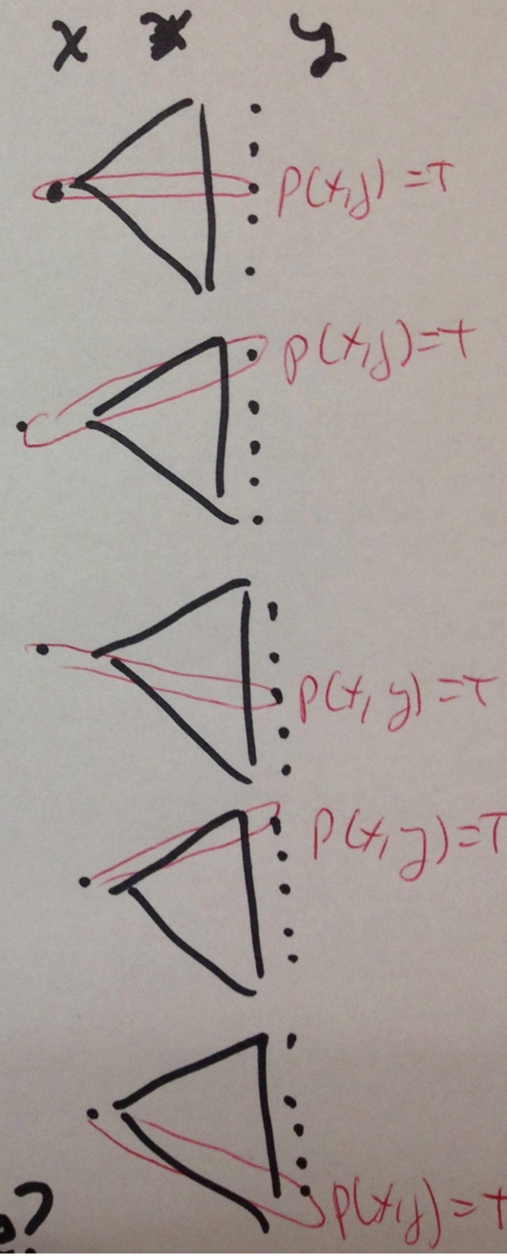
$\exists x \exists y\, P(x,y)$



$\exists x \forall y\, P(x,y)$



$\forall x \exists x\, P(x,y)$



what do we need for the quantifier to be true?

Another example

- Let $U = \{1,2,3\}$.
- Find an expression equivalent to $\forall x \exists y P(x,y)$ where the variables are bound by substitution and not quantification.

- Solution: Expand from outside in or inside out

$$\forall x \exists y P(x,y) \equiv$$

$$\forall x(\exists y P(x,y)) \equiv$$

$$\exists y P(1,y) \land \exists y P(2,y) \land \exists y P(3,y) \equiv$$

$$(P(1,1) \lor P(1,2) \lor P(1,3)) \land$$

$$(P(2,1) \lor P(2,2) \lor P(2,3)) \land$$

$$(P(3,1) \lor P(3,2) \lor P(3,3))$$

Scope

- The scope of a quantifier is the part of the expression in which the variable is bound by the quantifier.
- E.g.,

  $\forall x(P(x) \wedge Q(x))$ is a proposition

  $\forall x(P(x)) \wedge Q(x)$ is ***not*** a proposition because

  x in Q(x) is not bound a quantifier

Equivalences using Quantifiers

- A proposition using quantifiers is equivalent to another proposition when, *for all universes* (appropriate for the predicates) *and for all possible values of each predicate*, the value of the two expressions is the same.

$$\forall x\left(P(x) \wedge \neg P(x)\right) \equiv F \quad \text{(LHS is an * unsatisfiable * assertion)}$$

$$\forall x\left(P(x) \wedge P(x)\right) \equiv \forall x P(x)$$

$$\forall x\left(P(x) \vee \neg P(x)\right) \equiv T \quad \text{(LHS is a * valid * assertion)}$$

Quantifiers and negations

- For one variable:

$$\neg \forall x P(x) \equiv \exists x \neg P(x)$$

$$\neg \exists x P(x) \equiv \forall x \neg P(x)$$

- Why? (think of the finite case with a few values)

Continued …

- For multiple variables

Statement	Equivalent Expression
$\neg\exists x \exists y\, P(x,y)$	$\forall x \forall y \neg P(x,y)$
$\neg\exists x \forall y\, P(x,y)$	$\forall x \exists y \neg P(x,y)$
$\neg\forall x \forall y\, P(x,y)$	$\exists x \exists y \neg P(x,y)$
$\neg\forall x \exists y\, P(x,y)$	$\exists x \forall y \neg P(x,y)$

Which are correct below?

- $\forall x \forall y\, P(x, y) \equiv \forall y \forall x\, P(x, y)$???
- $\forall x \forall y (P(x) \wedge Q(y)) \equiv \forall x P(x) \wedge \forall y Q(y)$???
- $\exists\, x\, \exists y (P(x) \vee Q(y)) \equiv \exists x P(x) \vee \exists y Q(y)$???
- $\forall x \exists y P(x, y) \equiv \exists y \forall x P(x, y)$???
- $\forall x \exists y P(x, y) \equiv \exists x \forall y P(x, y)$???