Sets

Chapter 2

Definition

A ***set*** is a collection or group of objects or *elements* or *members*. (Cantor 1895)

- A set is said to *contain* its elements.

- There must be an underlying universal set U, either specifically stated or understood.

Notation

- list the elements between braces:

$$S = \{a, b, c, d\} = \{b, c, a, d, d\}$$

(Note: listing an object more than once does not change the set. Ordering means nothing.)

- specification by predicates:

$$S = \{x \mid P(x)\},$$

S contains all the elements from U which make the predicate P true.

- brace notation with ellipses:

$$S = \{\ldots, -3, -2, -1\},$$

the negative integers.

Common Universal Sets

- R = reals

- N = natural numbers = {0,1, 2, 3, . . . }, the *counting* numbers

- Z = all integers = {. . , -3, -2, -1, 0, 1, 2, 3, 4, . .}

Notation:

x is a member of S or x is an element of S:

$$x \in S.$$

x is not an element of S:

$$x \notin S.$$

Subsets

Definition: The set A is a *subset* of the set B, denoted A $\subseteq$ B, iff

$$\forall x[x \in A \rightarrow x \in B]$$

Definition: The *void* set, the *null* set, the *empty* set, denoted $\varnothing$, is the set with no members.

Note: the assertion $x \in \varnothing$ is always false. Hence

$$\forall x[x \in \varnothing \rightarrow x \in B]$$

is always true(vacuously). Therefore, $\varnothing$ is a subset of every set.

Note: A set B is always a subset of itself.

Subset examples

- Let N be the set of natural numbers
- Let Z be the set of integers
- Let P = {x | x is a positive integer}
- Let G = {x | x is a negative integer}
- Let H = {x | x is an integer at most 0}
- Is $P \subseteq N$?
- Is $N \subseteq P$?
- Is $G \subseteq N$?
- Is $H \subseteq N$?

Special Subsets

Definition: If $A \subseteq B$ but $A \neq B$ the we say A is a *proper* subset of B, denoted $A \subset B$ (in some texts).

Definition: The set of all subset of a set A, denoted P(A), is called the *power set* of A.

Example: If A = {a, b} then

$$P(A) = \{\varnothing, \{a\}, \{b\}, \{a,b\}\}$$

Cardinality

Definition: The number of (distinct) elements in A, denoted $|A|$, is called the *cardinality* of A.

If the cardinality is a natural number (in N), then the set is called *finite*, else *infinite*.

Example:

$A = \{a, b\}$,

$$|\{a, b\}| = 2,$$

$$|P(\{a, b\})| = 4.$$

A is finite and so is P(A).

Useful Fact: $|A|=n$ implies $|P(A)| = 2^n$

Notes

- The power-set of A is often written as 2^A
- A set can be an element or a subset of another set

Example:

$A = \{\varnothing, \{\varnothing\}\}.$

A has two elements and hence four subsets:

$$\varnothing, \{\varnothing\}, \{\{\varnothing\}\}, \{\varnothing, \{\varnothing\}\}$$

Note that $\varnothing$ is both a member of A and a subset of A!

- Can you write down the power set of A?

Paradoxes ☺

Russell's paradox: Let S be the set of all sets which are not members of themselves. Is S a member of itself?

Another paradox: Henry is a barber who shaves all people who do not shave themselves. Does Henry shave himself?