Set Operations

Chapter 2

Cartesian

Definition: The *Cartesian product* of A with B, denoted A$\times$B, is the set of ordered pairs $\{<a, b> \mid a \in A \wedge b \in B\}$

Notation: $\displaystyle \mathop{\times}_{i=1}^{n} A_i = \{<a_1, a_2, \ldots, a_n> \mid a_i \in A_i\}$

Note: The Cartesian product of anything with $\varnothing$ is $\varnothing$. (why?)

Example

A = {a,b}, B = {1, 2, 3}

AxB = {<a, 1>, <a, 2>, <a, 3>, <b, 1>, <b, 2>, <b, 3>}

What is BxA? AxBxA?

If |A| = m and |B| = n, what is |AxB|?

Equality

Definition: Two sets A and B are *equal*, denoted $A = B$, iff

$$\forall x[x \in A \leftrightarrow x \in B].$$

Note: By a previous logical equivalence we have

$$\text{A} = \text{B iff } \forall x[(x \in A \rightarrow x \in B) \wedge (x \in B \rightarrow x \in A)]$$

or

$$\text{A} = \text{B iff } A \subseteq B \text{ and } B \subseteq A$$

Definitions

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- The *union* of A and B, denoted $A \cup B$, is the set

$$\{x \mid x \in A \vee x \in B\}$$

- The *intersection* of A and B, denoted $A \cap B$, is the set

$$\{x \mid x \in A \wedge x \in B\}$$

Note: If the intersection is void, A and B are said to be *disjoint*.

- The *complement* of A, denoted $\overline{A}$, is the set

$$\{x \mid \neg(x \in A)\}$$

Note: Alternative notation is A^c, and $\{x | x \notin A\}$.

More Definitions

- The *difference* of A and B, or the *complement* of B *relative to* A, denoted A - B, is the set

$$A \cap \overline{B}$$

Note: The (absolute) complement of A is U - A.

- The *symmetric difference* of A and B, denoted $A \oplus B$, is the set

$$(A - B) \cup (B - A)$$

Example

Examples: U = {0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10}

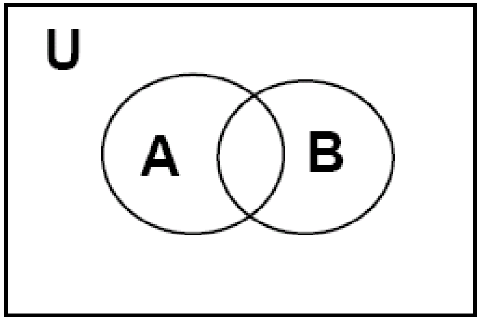
A= {1, 2, 3, 4, 5}, B = {4, 5, 6, 7, 8}. Then

- $A \cup B = \{1, 2, 3, 4, 5, 6, 7, 8\}$
- $A \cap B = \{4, 5\}$
- $\overline{A} = \{0, 6, 7, 8, 9, 10\}$
- $\overline{B} = \{0, 1, 2, 3, 9, 10\}$
- $A - B = \{1, 2, 3\}$
- $B - A = \{6, 7, 8\}$
- $A \oplus B = \{1, 2, 3, 6, 7, 8\}$

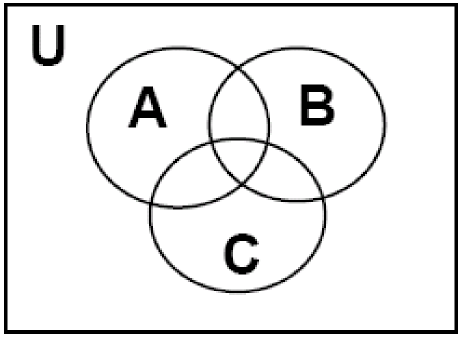
Venn Diagrams

A useful geometric visualization tool (for 3 or less sets)

- The Universe U is the rectangular box
- Each set is represented by a circle and its interior
- All possible combinations of the sets must be represented



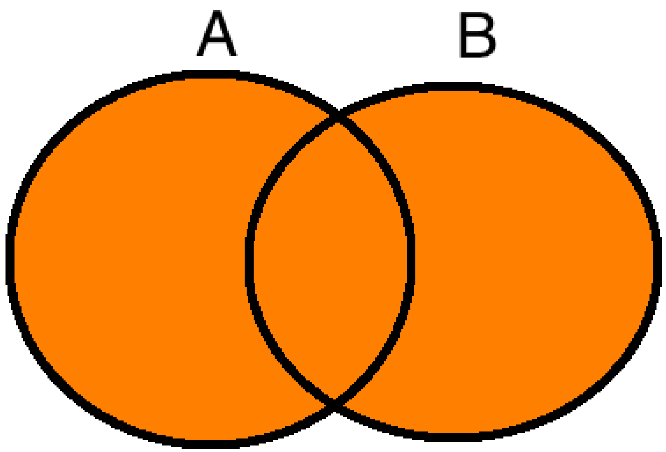
For 2 sets



For 3 sets

Shade the appropriate region to represent the given set operation.

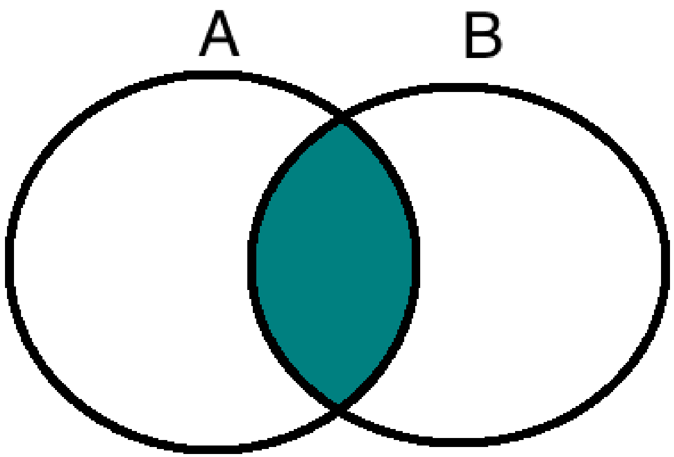
Venn Diagrams Examples



A union B

What about intersection?

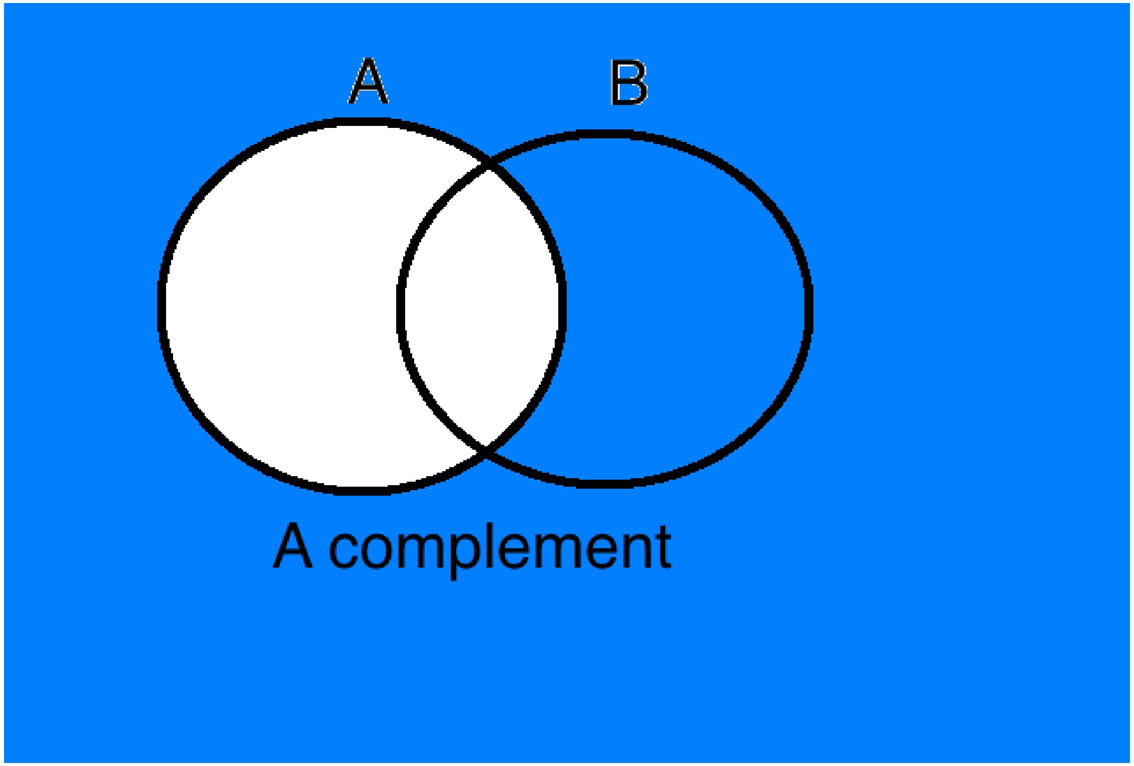
Venn Diagrams Examples



A intersects B

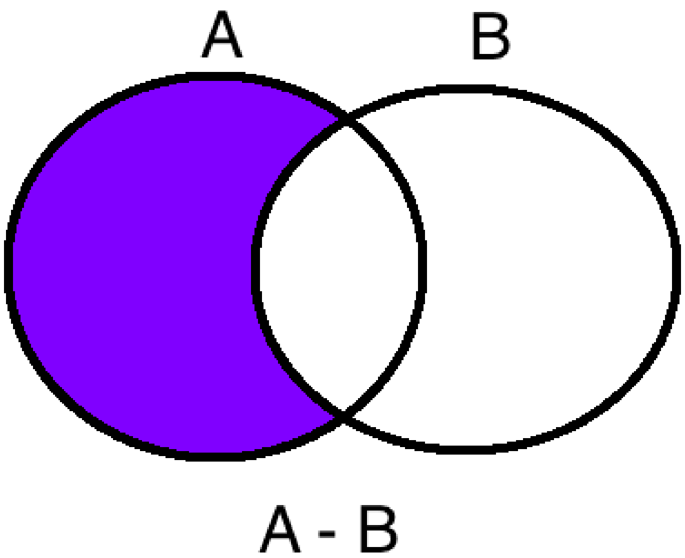
What about complement?

Venn Diagrams Examples



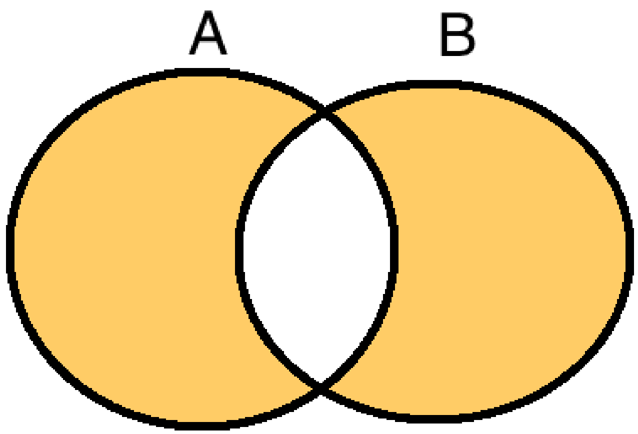
What about set difference?

Venn Diagrams Examples



What about symmetric difference?

Venn Diagrams Examples



Symmetric difference of A and B

Can you express symmetric difference using only union, intersection, and complement?

Set Identities

Identity	Name
$A \cup \phi = A$ $A \cap U = A$	Identity Laws
$A \cap \phi = \phi$ $A \cup U = U$	Domination Laws
$A \cup A = A$ $A \cap A = A$	Idempotent Laws
$\overline{(\overline{A})} = A$	Complement Laws
$A \cup B = B \cup A$ $A \cap B = B \cap A$	Commutative Laws

some more …

Identity	Name
$A \cup (B \cup C) = (A \cup B) \cup C$ $A \cap (B \cap C) = (A \cap B) \cap C$	Associative laws
$A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$ $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$	Distributive laws
$\overline{A \cup B} = \overline{A} \cap \overline{B}$ $\overline{A \cap B} = \overline{A} \cup \overline{B}$	DeMorgan's laws
$A \cup (A \cap B) = A$ $A \cap (A \cup B) = A$	Absorption laws
$A \cup \overline{A} = U$ $A \cap \overline{A} = \phi$	Complement laws

How to prove set identities?

- Several methods:
  - Use basic definition of sets, and apply logic
  - Use existing set identities
  - Use set table membership

Example with definitions and logic

The complement of the union is the intersection of the complements:

$$\overline{A \cup B} = \overline{A} \cap \overline{B}$$

Proof: To show:

$$\forall x [x \in \overline{A \cup B} \leftrightarrow x \in \overline{A} \cap \overline{B}]$$

To show two sets are equal we show for all x that x is a member of one set if and only if it is a member of the other.

continued…

- We will use “universal generalization”
  - We will perform our proof of $P(x)$ using an “arbitrary” x
  - Since we did not assume anything specific about x, we can then conclude that
    - $\forall x\, P(x)$

$x \in \overline{A \cup B}$	Def. of complement
$\equiv x \notin (A \cup B)$	Def. of $\notin$
$\equiv \neg(x \in (A \cup B))$	Def. of union
$\equiv \neg(x \in A \vee x \in B)$	DeMorgan's Laws
$\equiv \neg(x \in A) \wedge \neg(x \in B)$	Def. of $\notin$
$\equiv x \notin A \wedge x \notin B$	Def. of complement
$\equiv x \in \bar{A} \wedge x \in \bar{B}$	Def. of intersection
$\equiv x \in (\bar{A} \cap \bar{B})$	

Example using set identities

Show that $A - B = A \cap \overline{(A \cap B)}$

	$A \cap \overline{(A \cap B)}$	De Morgan's laws
$=$	$A \cap (\overline{A} \cup \overline{B})$	Distributive Rule
$=$	$(A \cap \overline{A}) \cup (A \cap \overline{B})$	Complement Law
$=$	$\phi \cup (A \cap \overline{B})$	Identity Law
$=$	$(A \cap \overline{B})$	Definition of difference
$=$	$A - B$	

Membership Table

$A \quad B$	$A \cup B$	$\overline{A}$	$\overline{B}$	$\overline{A \cup B}$	$\overline{A} \cap \overline{B}$
1 1	1	0	0	0	0
1 0	1	0	1	0	0
0 1	1	1	0	0	0
0 0	0	1	1	1	1

Computer Representation

- Assume that the universal set U is finite and a reasonable size n.
- Order the set.
- Represent a subset A of U to be the bit string of length n which has a 1 in the i^{th} position if the i^{th} element in the ordering is in set A and a 0 otherwise.

Example

- U = {1,2,3,4,5,6,7,8,9,10} be ordered in the usual fashion.
- The set S = {1,5,8}
- Has the representation

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Union and Intersection of Indexed Collections

Union and intersection are associative

Let $A_1, A_2, \ldots, A_n$ be an indexed collection of sets.

$$\bigcup_{i=1}^{n} A_i = A_1 \cup A_2 \cup \ldots \cup A_n$$

and

$$\bigcap_{i=1}^{n} A_i = A_1 \cap A_2 \cap \ldots \cap A_n$$

$$A_i = [i, \infty), 1 \le i < \infty$$

Example $\bigcup_{i=1}^{n} A_i = [1, \infty)$

$$\bigcap_{i=1}^{n} A_i = [n, \infty)$$