Functions

A function is a relation that maps each element x of a set A with one and only one element y of another set B. In other words, it is a relation between a set of inputs and a set of outputs in which each input is related with a unique output. A function is a rule that relates an input to exactly one output.

It is a special type of relation. A relation f from a set A to a set B is said to be a function if every element of set A has one and only one image in set B and no two distinct elements of B have the same mapped first element.

A and B are the non-empty sets.

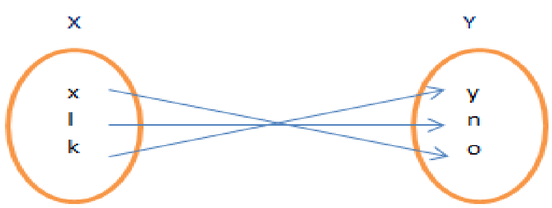
The whole set A is the domain and the whole set B is codomain.

The set of images of all the elements of A is the range of the function

Representation

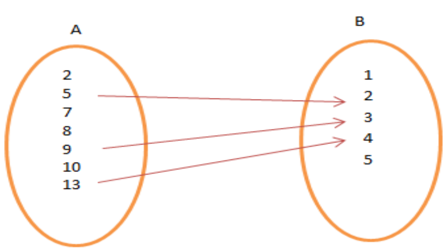
A function f: X →Y is represented as f(x) = y, where, (x, y) ∈ f and x ∈ X and y ∈ Y.

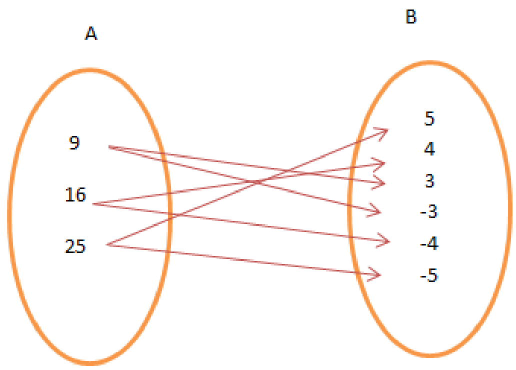
For any function f, the notation f(x) is read as "f of x" and represents the value of y when x is replaced by the number or expression inside the parenthesis. The element y is the image of x under f and x is the pre-image of y under f.



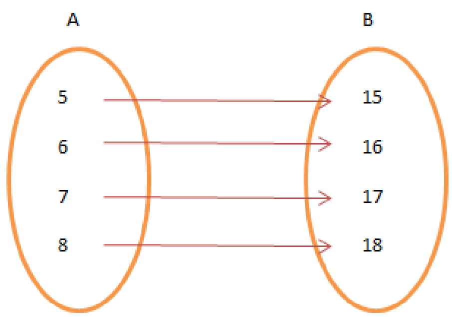
Every element of the set has an image which is unique and distinct. If we notice around, we can find many examples of functions.

Question 1: Which of the following is a function?1.





2.



3.

Answer : Figure 3 is an example of function since every element of A is mapped to a unique element of B and no two distinct elements of B have the same pre-image in A.

EX: Give the domain ,codomain and range of following functions:

1) f: A $\rightarrow$B, A= {1,2,3} B= {1,2,3,4,5,6,7} f(x) =2x
2) f: A $\rightarrow$N; f(x) = 3x+1 , where A= {1,2,3,4}
3) g:N$\rightarrow$N ; g(x) = x^2+1 , x $\in$N , x<5
4) f:N$\rightarrow$N,and f(x) = 5x-2 If the range of function f is {3,8,13} then find the domain of f.

   f(x)=3

   5x-2=3

   X=1

   f(x)=8 …… …….. ……..

   D_f= {1,2,3}

5) f(x) = $\frac{1}{x+3}$
   D_f= R-{-3}

R_f= R

6) f(x) = $\frac{x2-25}{x-5}$= $\frac{(x-5)(x+5)}{x-5}$ = x+5=10

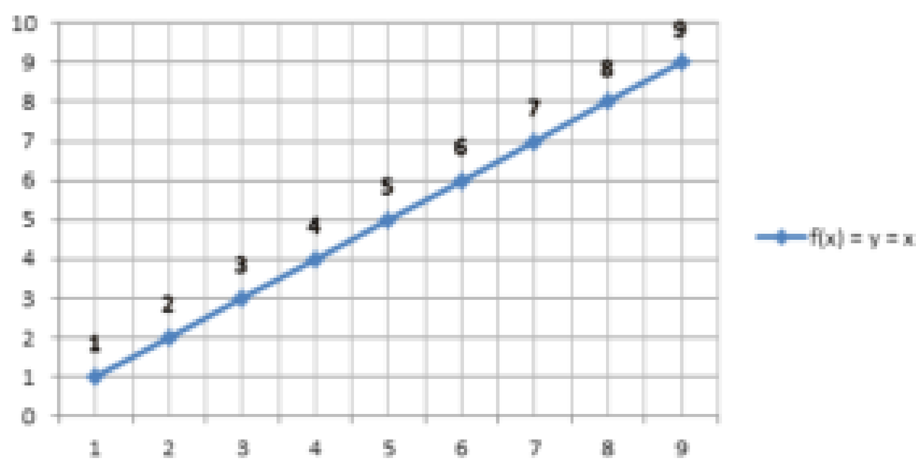
Domain = R-{5}
Range = R-{10}

7) f(x) = $\frac{7-x}{x-7}$ = -1($\frac{x-7)}{x-7}$ = -1
Domain = R-{7}
Range =-1

Types of functions

Identity Function

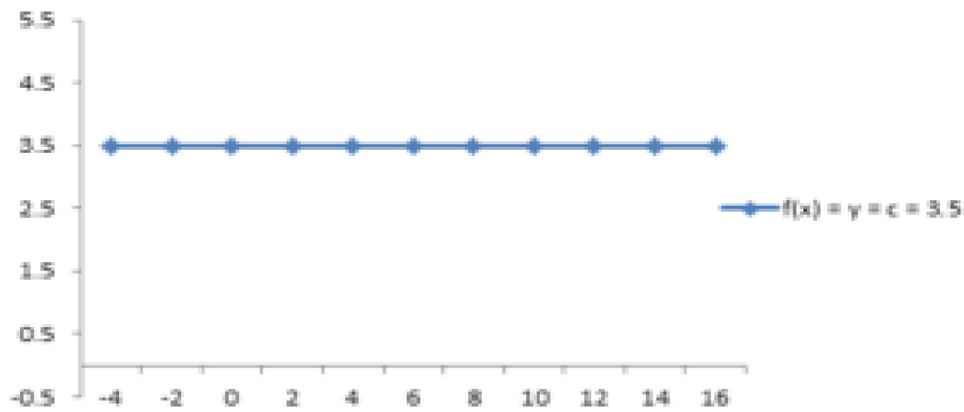
Let **R** be the set of real numbers. If the function f: **R**→**R** is defined as f(x) = y = x, for x ∈ R, then the function is known as Identity function. The domain and the range being **R.** The graph is always a straight line and passes through the origin.

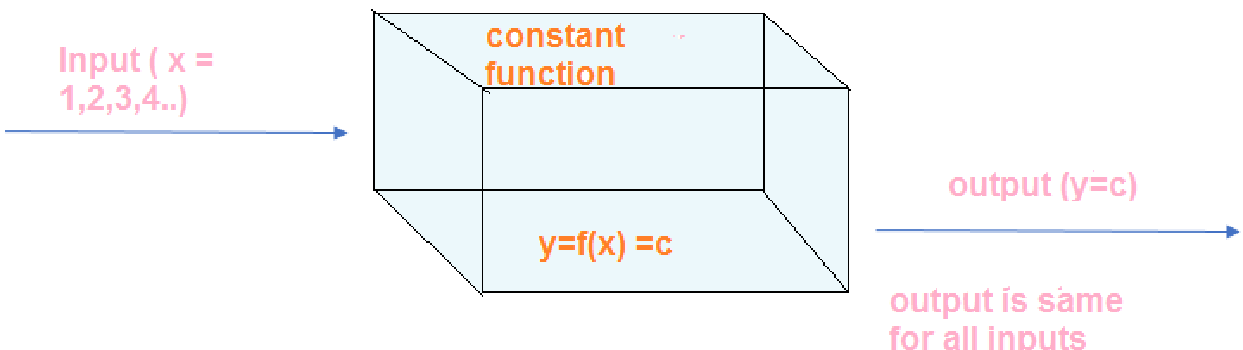


For example, f(2) = 2 is an identity function.

Constant Function

If the function f: **R**→**R** is defined as f(x) = y = c, for x ∈ R and c is a constant in **R**, then such function is known as Constant function. The domain of the function f is **R** and its range is a constant, c**.** Plotting a graph, we find a straight line parallel to the x-axis.





Y=F(x)=2

Solved examples of constant Functions

1. which is below function is a constant function?
a. y=f(x)=x
b. y=f(x)=11
c. y=f(x)=π

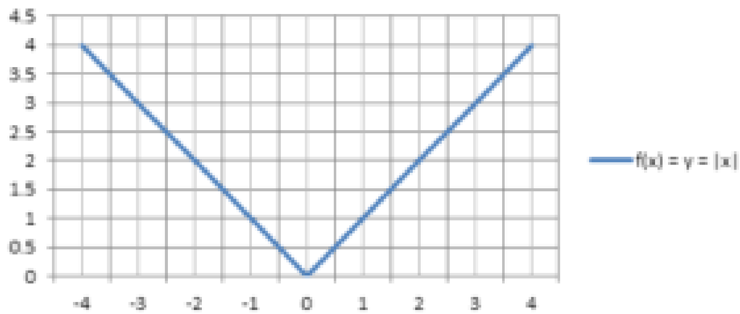
Modulus Function

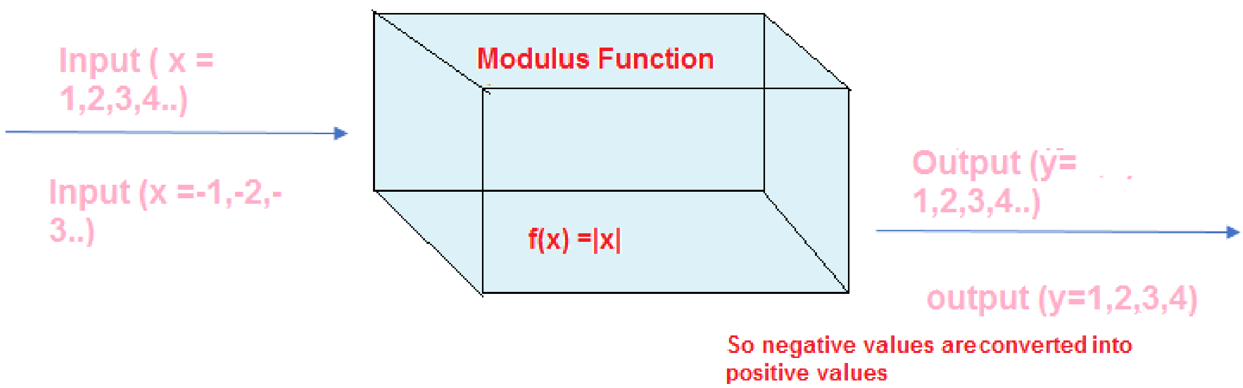
The absolute value of any number, c is represented in the form of |c|. If any function f: **R**→ **R** is defined by f(x) = |x|, it is known as Modulus Function. For each non-negative value of x, f(x) = x and for each negative value of x, f(x) = -x, i.e.,

f(x) = {x, if x ≥ 0;

− x, if x < 0.

Its graph is given as, where the domain and the range are **R.**



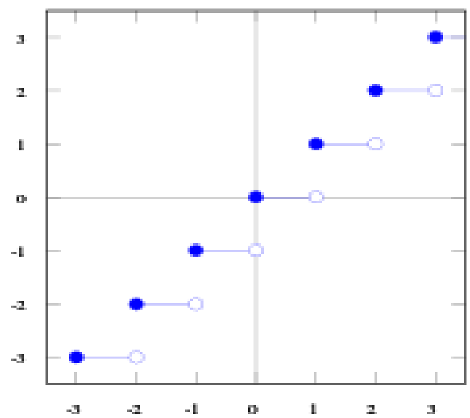


Example
|-3| =3
|-2|=2
|2|=2

|y-1| = y-1

Greatest Integer Function

If a function f: **R**→ **R** is defined by f(x) = [x], x ∈ X. It round-off to the real number to the integer less than the number. Suppose, the given interval is in the form of (k, k+1), the value of greatest integer function is k which is an integer. For example: [-2.5] =-3 [5.12] = 5. The graphical representation is



The Greatest Integer Function

- $y = [x]$ means "the greatest integer not greater than x"
- For example, [2.4], means "the greatest integer not greater than 2.4."
  So, [2.4] = 2.
- Here are some other examples:
  - [3] = 3
  - [-2.2] = -3
  - [5.8] = 5

<u>Limit</u>

How to Solve Limits by Direct substituition

EX: $\lim_{x\to 1} 6x^2+4x+5=15$

Ex: $\lim_{x\to 1} \frac{x2-4}{x+3} = -\frac{3}{4}$

How to Solve Limits by Factoring

You can use the algebraic technique of factoring to solve “real” limit problems. All algebraic methods involve the same basic idea. When substitution doesn’t work in the original function — usually because of a hole in the function — you can use algebra to manipulate the function until substitution does work (it works because your manipulation plugs up the hole).

Evaluate $\lim_{x \to 5} \frac{x^2 - 25}{x - 5}$.

☐ Try plugging 5 into *x* — you should *always* try substitution first.

You get $\frac{0}{0}$ — no good, on to plan B.

☐ Factor:

$$\lim_{x \to 5} \frac{x^2 - 25}{x - 5}$$
$$= \lim_{x \to 5} \frac{(x-5)(x+5)}{x - 5}$$

☐ Cancel the (*x* – 5) from the numerator and denominator.

$$= \lim_{x \to 5} (x+5)$$

☐ Now substitution will work.

= 5 + 5

= 10

So, $\lim_{x \to 5} \frac{x^2 - 25}{x - 5} = 10$

By the way, the function you got after canceling the $(x - 5)$, namely $(x + 5)$, is identical to the original function $\frac{x^2 - 25}{x - 5}$, except that the hole in the original function at (5,10) has been plugged.

Ex: $\lim_{x \to 1} \frac{x2+x-2}{x2+2x-3}$ (Self practice)

Ex: $\lim_{x \to 1} \frac{x2-5x+6}{x2-4}$ (Self practice)

2. $\lim_{x \to 0} \frac{2x^2+3x+4}{x^2+3x+2}$

Solution:

Given:

$$\lim_{x \to 0} \frac{2x^2+3x+4}{x^2+3x+2}$$

Let us substitute the value of x directly in the given limit, we get

$$\lim_{x \to 0} \frac{2x^2+3x+4}{x^2+3x+2} = \frac{2(0^2)+3(0)+4}{0^2+3(0)+2}$$

$$= 4 / 2$$

$$= 2$$

∴ The value of the given limit is 2.

$$\lim_{x\to 2}\frac{x^3-8}{x^2-4} = \frac{(2)^3-8}{(2)^2-4}$$

$$= \frac{8-8}{(4)-4}$$

$$= \frac{0}{0}$$ [Since, it is of the form indeterminate]

By using factorization method:

$$\lim_{x\to 2}\frac{x^3-8}{x^2-4} = \lim_{x\to 2}\frac{(x^3-8)}{(x^2-4)}$$

$$= \lim_{x\to 2}\frac{(x^3-2^3)}{(x^2-2^2)}$$

$$= \lim_{x\to 2}\frac{(x-2)(x^2+2^2+2x)}{(x+2)(x-2)}$$

[By using the formula, $(a^3 - b^3) = (a - b)(a^2 + b^2 + ab)$ & $(a^2 - b^2) = (a + b)(a - b)$]

$$= \lim_{x\to 2}\frac{(x^2+2^2+2x)}{(x+2)}$$

$$= \frac{(2^2+2^2+2(2))}{(2+2)}$$

$$= \frac{3.4}{(4)}$$

$$= 3$$

∴ The value of the given limit is 3.

By substituting the value of x, we get

$$\lim_{x\to 3}\frac{x^2-4x+3}{x^2-2x-3} = \frac{(3)^2-4(3)+3}{(3)^2-2(3)-3}$$

$$= \frac{12-12}{(-9)+9}$$

$$= \frac{0}{0}$$ [Since, it is of the form indeterminate]

By using factorization method:

$$\lim_{x\to 3}\frac{x^2-4x+3}{x^2-2x-3} = \lim_{x\to 3}\frac{(x^2-4x+3)}{(x^2-2x-3)}$$

$$= \lim_{x\to 3}\frac{(x^2-3x-x+3)}{(x^2-3x+x-3)}$$

$$= \lim_{x\to 3}\frac{x(x-3)-1(x-3)}{x(x-3)+1(x-3)}$$

$$= \lim_{x\to 3}\frac{(x-3)(x-1)}{(x-3)(x+1)}$$

$$= \lim_{x\to 3}\frac{(x-1)}{(x+1)}$$

$$= \frac{(3-1)}{(3+1)}$$

$= 2/4$

$= 1/2$

Solving limit by rationalization

5. $\lim_{x \to 2} \frac{\sqrt{3-x}-1}{2-x}$

Solution:

Given:

The limit $\lim_{x \to 2} \frac{\sqrt{3-x}-1}{2-x}$

We need to find the limit of the given equation when x => 0

Now let us substitute the value of x as 0, we get an indeterminate form of 0/0.

Let us rationalizing the given equation, we get

$$\lim_{x \to 2} \frac{\sqrt{3-x}\ -1}{2-x} = \lim_{x \to 2} \frac{(\sqrt{3-x}\ -1)}{(2-x)} \frac{(\sqrt{3-x}+1)}{(\sqrt{3-x}+1)}$$

[By using the formula: $(a+b)(a-b) = a^2 - b^2$]

$$= \lim_{x \to 2} \frac{(3-x-1)}{(2-x)(\sqrt{3-x}+1)}$$

$$= \lim_{x \to 2} \frac{(2-x)}{(2-x)(\sqrt{3-x}+1)}$$

$$= \lim_{x \to 2} \frac{1}{(\sqrt{3-x}+1)}$$

Now we can see that the indeterminate form is removed,

So, now we can substitute the value of x as 0, we get

$$\lim_{x \to 2} \frac{\sqrt{3-x}\ -1}{2-x} = \frac{1}{1+1}$$

$$= \frac{1}{2}$$

1. $\lim_{x\to 0}\frac{\sqrt{1+x+x^2}-1}{x}$

Solution:

Given:

The limit $\lim_{x\to 0}\frac{\sqrt{1+x+x^2}-1}{x}$

We need to find the limit of the given equation when x => 0

Now let us substitute the value of x as 0, we get an indeterminate form of 0/0.

Let us rationalizing the given equation, we get

$$\lim_{x\to 0}\frac{\sqrt{1+x+x^2}-1}{x}=\lim_{x\to 0}\frac{(\sqrt{1+x+x^2}-1)}{x}\frac{(\sqrt{1+x+x^2}+1)}{(\sqrt{1+x+x^2}+1)}$$

[By using the formula: (a + b) (a - b) = a^2 - b^2]

$$=\lim_{x\to 0}\frac{1+x+x^2-1}{x(\sqrt{1+x+x^2}+1)}$$

$$=\lim_{x\to 0}\frac{x(1+x)}{x(\sqrt{1+x+x^2}+1)}$$

$$=\lim_{x\to 0}\frac{(1+x)}{(\sqrt{1+x+x^2}+1)}$$

Now we can see that the indeterminate form is removed,

So, now we can substitute the value of x as 0, we get

$$\lim_{x\to 0}\frac{\sqrt{1+x+x^2}-1}{x}=\frac{1}{1+1}$$

$$=\frac{1}{2}$$

$$\lim_{x\to 0}\frac{(\sqrt{1+x}-1)}{x}$$

$$=\lim_{x\to 0}\frac{(\sqrt{1+x}-1)}{x}\times\frac{(\sqrt{1+x}+1)}{(\sqrt{1+x}+1)}$$

$$=\lim_{x\to 0}\frac{(1+x-1)}{x(\sqrt{1+x}+1)}$$

$$=\lim_{x\to 0}\frac{x}{x(\sqrt{1+x}+1)}$$

$$=\lim_{x\to 0}\frac{1}{(\sqrt{1+x}+1)}$$

$$=\frac{1}{\sqrt{1}+1}=\frac{1}{2}$$

$$\lim_{x \to 1} \frac{\left(\sqrt{5x - 4} - \sqrt{x}\right)}{x - 1}$$

$$= \lim_{x \to 1} \frac{\left(\sqrt{5x - 4} - \sqrt{x}\right)}{x - 1} \times \frac{\left(\sqrt{5x - 4} + \sqrt{x}\right)}{\left(\sqrt{5x - 4} + \sqrt{x}\right)}$$

$$= \lim_{x \to 1} \frac{\left((5x - 4) - x\right)}{(x - 1)\left(\sqrt{5x - 4} + \sqrt{x}\right)}$$

$$= 4 \lim_{x \to 1} \frac{(x - 1)}{(x - 1)\left(\sqrt{5x - 4} + \sqrt{x}\right)}$$

$$= 4 \lim_{x \to 1} \frac{1}{\sqrt{5x - 4} + \sqrt{x}}$$

$$= 4 \times \frac{1}{\sqrt{5 - 4} + \sqrt{1}}$$

$$= 4 \times \frac{1}{\sqrt{1} + \sqrt{1}}$$

$$= \frac{4}{2} = 2$$

Some standard limits

$$\lim_{x \to 0} \frac{e^x - 1}{x} = 1$$

$$\lim_{x \to 0} \frac{a^x - 1}{x} = \log_e a$$

$$\lim_{x \to 0} \frac{\log(1+x)}{x} = 1$$

$$\lim_{x \to \infty} \left(1 + \frac{1}{x}\right)^x = e$$

$$\lim_{x \to 0} (1 + x)^{\frac{1}{x}} = e$$

$$\lim_{x \to \infty} \left(1 + \frac{a}{x}\right)^x = e^a$$

Continuity

Definition of Continuity

A function f is continuous at $x = a$ when:

1. $f(a)$ is defined
2. $\lim_{x \to a} f(x)$ exist
3. $\lim_{x \to a} f(x) = f(a)$

If any one of the conditions is not met, the function is not continuous at $x = a$.

Let f be a function that is defined for all values of x close to $x = a$ with the possible exception of a itself. Then

$$\lim_{x \to a} f(x) = L \quad \text{if and only if} \quad \lim_{x \to a^+} f(x) = \lim_{x \to a^-} f(x) = L$$

PROBLEM 1 : Determine if the following function is continuous at x=1 .

Function f is defined at x=1 since

$$f(x) = \begin{cases} 3x-5, & \text{if } x \neq 1 \\ 2 \quad , & \text{if } x = 1 \end{cases}$$

i.) $f(1) = 2$.

The limit

$$\lim_{x \to 1} f(x) = \lim_{x \to 1} (3x-5)$$

= 3 (1) - 5

= -2 ,

i.e.,

ii.) $\lim_{x \to 1} f(x) = -2$.

But

iii.) $\lim_{x \to 1} f(x) \neq f(1)$,

so condition iii.) is not satisfied and function f is NOT continuous at x=1 .

PROBLEM 2 : Determine if the following function is continuous at x=-2 .

$$f(x) = \begin{cases} x^2+2x, & \text{if } x \leq -2 \\ x^3-6x, & \text{if } x > -2 \end{cases}$$

SOLUTION 2 : Function f is defined at x=-2 since

i.) $f(-2) = (-2)^2 + 2(-2) = 4-4 = 0$.

The left-hand limit

$$\lim_{x \to -2^-} f(x) = \lim_{x \to -2^-} (x^2+2x)$$

$= (-2)^2 + 2(-2)$

= 4 - 4

= 0 .

The right-hand limit

$$\lim_{x \to -2^+} f(x) = \lim_{x \to -2^+} (x^3 - 6x)$$

$= (-2)^3 - 6(-2)$

$= -8 + 12$

$= 4$.

Since the left- and right-hand limits are not equal, ,

ii.) $\lim_{x \to -2} f(x)$ does not exist,

and condition ii.) is not satisfied. Thus, function *f* is NOT continuous at $x=-2$.

PROBLEM 3 : Determine if the following function is continuous at $x=0$.

$$f(x) = \begin{cases} \dfrac{x-6}{x-3}, & \text{if } x < 0 \\ 2, & \text{if } x = 0 \\ \sqrt{4+x^2}, & \text{if } x > 0 \end{cases}$$

SOLUTION 3 : Function *f* is defined at $x=0$ since

i.) $f(0) = 2$.

The left-hand limit

$$\lim_{x \to 0^-} f(x) = \lim_{x \to 0^-} \frac{x-6}{x-3}$$

$$= \frac{-6}{-3}$$

$= 2$.

The right-hand limit

$$\lim_{x \to 0^+} f(x) = \lim_{x \to 0^+} \sqrt{4 + x^2}$$

$$= \sqrt{4 + (0)^2}$$

$$= \sqrt{4}$$

$$= 2 \ .$$

Thus, $\lim_{x \to 0} f(x)$ exists with

ii.) $\lim_{x \to 0} f(x) = 2$.

iii.) $\lim_{x \to 0} f(x) = 2 = f(0)$,

all three conditions are satisfied, and *f* is continuous at $x=0$.