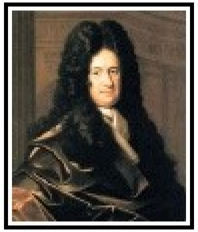
Unit – 5
Differentiation and Integration

- Introduction of Differentiation
- Rules of Differentiation
- Chain Rule
- Integration Concepts
- Introduction
- Fundamental Formulae
- Properties of Integration
- Integration by Substitution

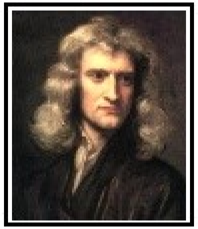
The History of Differentiation

Differentiation is part of the science of Calculus, and was first developed in the 17th century by two different Mathematicians.



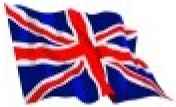
Gottfried Leibniz
(1646-1716)

Germany



Sir Isaac Newton
(1642-1727)

England



Differentiation, or finding the instantaneous rate of change, is an essential part of:

- Mathematics and Physics
- Chemistry
- Biology
- Computer Science
- Engineering
- Navigation and Astronomy

Differentiate means

'find out how fast something is changing in comparison with something else **at any one instant**'.

speed $= \dfrac{\Delta D}{\Delta T}$ 'rate of change of distance with respect to time'

acceleration $= \dfrac{\Delta S}{\Delta T}$ 'rate of change of speed with respect to time'

gradient $= \dfrac{\Delta y}{\Delta x}$ 'rate of change of y-coordinate with respect to x-coordinate'

Basic Differentiation

The instant rate of change of y with respect to x is written as $\frac{dy}{dx}$.

By long experimentation, it is possible to prove the following:

If $y = x^n$

then $\boxed{\frac{dy}{dx} = nx^{n-1}}$

$Y = x^3$

$dy/dx = 3x^{3-1} = 3x^2$

How to Differentiate:

- multiply by the power
- reduce the power by one

Note that $\frac{dy}{dx}$ describes both the rate of change and the gradient.

$$\frac{d(e^x)}{dx} = e^x$$

$$\frac{d(\ln(x))}{dx} = \frac{1}{x}$$

$$\frac{d(a^x)}{dx} = a^x \log a$$

d/dx(4^x) = 4^x log 4

$$\frac{d(\sin x)}{dx} = \cos x$$

$$\frac{d(\cos x)}{dx} = -\sin x$$

$$\frac{d(\tan x)}{dx} = \sec^2 x$$

$$\frac{d(\cot x)}{dx} = -\operatorname{cosec}^2 x$$

$$\frac{d(\sec x)}{dx} = \sec x \tan x$$

$$\frac{d(\operatorname{cosec} x)}{dx} = -\operatorname{cosec} x \cot x$$

1. The Constant Rule

If $y = c$ where c is a constant,

$$\frac{dy}{dx} = 0$$

e.g. $y = 10$ then $\frac{dy}{dx} = 0$

2. The Linear Function Rule

If $y = a + bx$

$$\frac{dy}{dx} = b$$

e.g. $y = 10 + 6x$ then $\frac{dy}{dx} = 6$

If $y = ax^n$, where a and n are constants

$$\frac{dy}{dx} = n.a.x^{n-1}$$

i) $y = 4x$ => $\frac{dy}{dx} = 4x^0 = 4$

ii) $y = 4x^2$ => $\frac{dy}{dx} = 8x$

iii) $y = 4x^{-2}$ => $\frac{dy}{dx} = -8x^{-3}$

If $y = f(x) \pm g(x)$

$$\frac{dy}{dx} = \frac{d[f(x)]}{dx} \pm \frac{d[g(x)]}{dx}$$

If y is the sum/difference of two or more functions of x:

differentiate the 2 (or more) terms separately, then add/subtract

(i) $y = 2x^2 + 3x$ then $\frac{dy}{dx} = 4x + 3$

(ii) $y = 5x + 4$ then $\frac{dy}{dx} = 5$

Examples of Basic Differentiation

Example 1

Find $\frac{dy}{dx}$ for $y = 3x^4 - 5x^3 + \frac{7}{x^2} + 9$

$$y = 3x^4 - 5x^3 + 7x^{-2} + 9$$

$$\therefore \frac{dy}{dx} = 12x^3 - 15x^2 - 14x^{-3}$$

$$= \underline{\underline{12x^3 - 15x^2 - \frac{14}{x^3}}}$$

this disappears because

$9 = 9x^0$

(multiply by zero)

The Sum Rule If f and g are both differentiable, then

$$\frac{d}{dx}[f(x) + g(x)] = \frac{d}{dx}f(x) + \frac{d}{dx}g(x)$$

The Difference Rule If f and g are both differentiable, then

$$\frac{d}{dx}[f(x) - g(x)] = \frac{d}{dx}f(x) - \frac{d}{dx}g(x)$$

The Product Rule If f and g are both differentiable, then

$$\frac{d}{dx}[f(x)g(x)] = f(x)\frac{d}{dx}[g(x)] + g(x)\frac{d}{dx}[f(x)]$$

(I .II)' = I.II' + II.I'

(First .second)' = first . Second' + second . first'

The Quotient Rule If f and g are differentiable, then

$$\frac{d}{dx}\left[\frac{f(x)}{g(x)}\right] = \frac{g(x)\frac{d}{dx}[f(x)] - f(x)\frac{d}{dx}[g(x)]}{[g(x)]^2}$$

(I / II)' = (II.I' - I.II ') /(II)^2

(First / *second)* = *(second . First' - first . Second')*/(Second)^2

Basic Derivatives Rules

Constant Rule: $\frac{d}{dx}(c) = 0$

Constant Multiple Rule: $\frac{d}{dx}[cf(x)] = cf'(x)$

Power Rule: $\frac{d}{dx}(x^n) = nx^{n-1}$

Sum Rule: $\frac{d}{dx}[f(x) + g(x)] = f'(x) + g'(x)$

Difference Rule: $\frac{d}{dx}[f(x) - g(x)] = f'(x) - g'(x)$

Product Rule: $\frac{d}{dx}[f(x)g(x)] = f(x)g'(x) + g(x)f'(x)$

Quotient Rule: $\frac{d}{dx}\left[\frac{f(x)}{g(x)}\right] = \frac{g(x)f'(x) - f(x)g'(x)}{[g(x)]^2}$

Chain Rule: $\frac{d}{dx}f(g(x)) = f'(g(x))g'(x)$

$$\frac{d}{dx}(c) = 0 \qquad \frac{d}{dx}(x^n) = nx^{n-1}$$

$$(cf)' = cf' \qquad (f+g)' = f' + g' \qquad (f-g)' = f' - g'$$

$$(fg)' = fg' + gf' \qquad \left(\frac{f}{g}\right)' = \frac{gf' - fg'}{g^2}$$

More Differentiation Rules

7. The Chain Rule

If $h(x) = g\left(f(x)\right)$ then

$$\boxed{h'(x) = g'\left(f(x)\right) \cdot f'(x)}$$

Note: $h(x)$ is a composite function.

Another Version:

If $y = h(x) = g\left(u\right)$, where $u = f(x)$, then

$$\boxed{\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}}$$

$$f(x)=(2x^2+8)^2$$

$$f'(x)=(2)(2x^2+8)^{(2-1)}\frac{d}{dx}[2x^2+8]$$ Apply the chain rule

$$f'(x)=(2)(2x^2+8)(4x)$$ Differentiate

$$f'(x)=(8x)(2x^2+8)$$ Multiply

$$f'(x)=16x^3+64x$$ Multiply

Starter

Differentiate the following:

$$y = (4x+5)^6$$

$$\frac{dy}{dx} = 24(4x+5)^5$$

$$f(x) = \frac{10}{(3x-1)^2}$$

$$f'(x) = \frac{-60}{(3x-1)^3}$$

$$y = 4x^3 + \sqrt{7-2x}$$

$$\frac{dy}{dx} = 12x^2 - \frac{1}{\sqrt{7-2x}}$$

7) $y = \sqrt{14 - 2x}$

8) $y = \sqrt[3]{x + 6}$

9) $y = \dfrac{1}{5x + 1}$

10) $y = \dfrac{1}{(7 - 3x)^2}$

11) $y = 8(x + 12)^5$

12) $y = 4(x^2 - 15)^3$

13) $y = \dfrac{5}{(x^2 - 1)^3}$

14) $y = (3x^2 + x)^8$

i) $y = (ax^2 + bx)^{½}$

let $v = (ax^2 + bx)$, so $y = v^{½}$

$$\frac{dy}{dx} = \frac{1}{2}\left(ax^2 + bx\right)^{-\frac{1}{2}}.(2ax + b)$$

ii) $y = (4x^3 + 3x – 7)^4$

let $v = (4x^3 + 3x – 7)$, so $y = v^4$

$$\frac{dy}{dx} = 4\left(4x^3 + 3x - 7\right)^3.\left(12x^2 + 3\right)$$

Application of differentiation in real life

use of differentiation in real life…

Differentiation is very important in many fields, and has made great contributions in these fields, which are still noticed nowadays.

The differentiation of calculus have many real- world applications from sports to engineering to astronomy to math and space travel.

uses in economics and science…

- Functions and derivatives of calculus are relate to relevant concepts in economics.

- Economic research uses calculus functional relationships, relation of income , market prediction etc Isaac Newton developed the use of calculus in his laws of motion and gravitation.

- Astronomical science deeply depend on calculus. Calculus is used to build tracks.

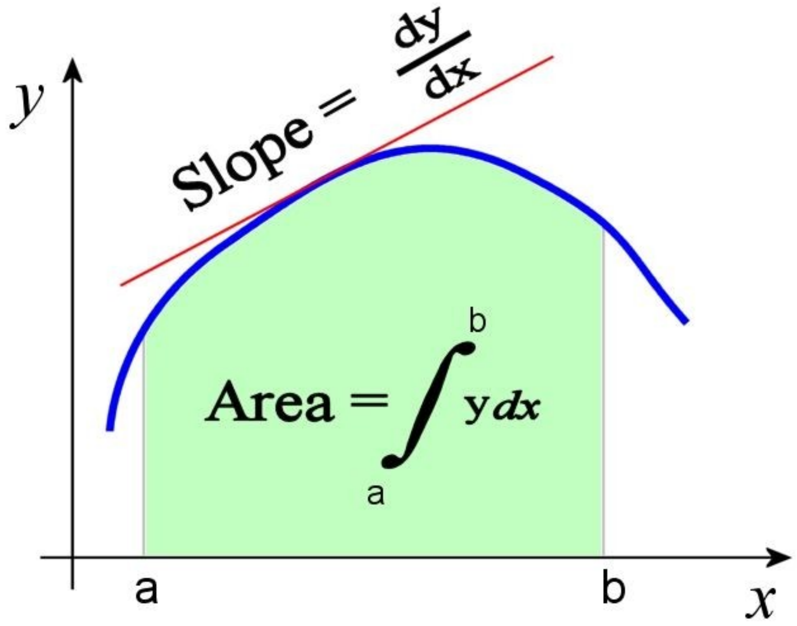
in engineering and biology…

- Differentiation initially developed for better navigation system.

- Engineers use Differentiation for building skyscrapers , bridges. In robotics calculus is used how robotic parts will work on given command.

- **Differentiation** is used for measuring growth rate of bacteria and certain species. The concentration of drugs in a living organism can be answered with calculus.

Integration



Differentiation Formulas:

1. $\frac{d}{dx}(x) = 1$
2. $\frac{d}{dx}(ax) = a$
3. $\frac{d}{dx}(x^n) = nx^{n-1}$
4. $\frac{d}{dx}(\cos x) = -\sin x$
5. $\frac{d}{dx}(\sin x) = \cos x$
6. $\frac{d}{dx}(\tan x) = \sec^2 x$
7. $\frac{d}{dx}(\cot x) = -\csc^2 x$
8. $\frac{d}{dx}(\sec x) = \sec x \tan x$
9. $\frac{d}{dx}(\csc x) = -\csc x(\cot x)$
10. $\frac{d}{dx}(\ln x) = \frac{1}{x}$
11. $\frac{d}{dx}(e^x) = e^x$
12. $\frac{d}{dx}(a^x) = (\ln a)a^x$

Integration Formulas:

1. $\int 1\,dx = x + C$
2. $\int a\,dx = ax + C$
3. $\int x^n\,dx = \frac{x^{n+1}}{n+1} + C,\ n \neq -1$
4. $\int \sin x\,dx = -\cos x + C$
5. $\int \cos x\,dx = \sin x + C$
6. $\int \sec^2 x\,dx = \tan x + C$
7. $\int \csc^2 x\,dx = -\cot x + C$
8. $\int \sec x(\tan x)\,dx = \sec x + C$
9. $\int \csc x(\cot x)\,dx = -\csc x + C$
10. $\int \frac{1}{x}\,dx = \ln|x| + C$
11. $\int e^x\,dx = e^x + C$
12. $\int a^x\,dx = \frac{a^x}{\ln a} + C\ a > 0,\ a \neq 1$

$$\int x^n \, dx = \frac{1}{n+1} x^{n+1} + c$$

Example:

$$\int x^5 \, dx = \frac{1}{6} x^6 + c$$

$$\int x^4 + 3x - 9\,dx = \frac{1}{5}x^5 + \frac{3}{2}x^2 - 9x + c$$

$$\int x^3 + 4x^2 + 3 \, dx$$

$$= \frac{x^4}{4} + \frac{4x^3}{3} + 3x + c$$

3 is $3x^0$, so the integral is $3x^1$

Find $\displaystyle\int\left(2+5x-\frac{1}{(x-2)^2}\right)dx.$

Integration by Substitution

If you cannot see the answer for integral by inspection, then the method of substitution is deployed.

Example 1 Integrate $\int (2x-1)^5\,dx$

Let $u = 2x-1 \Rightarrow \dfrac{du}{dx} = 2$ Make dx the subject

$$dx = \frac{du}{2}$$

Replace $2x – 1$ by u and write dx in terms of du

$$\int (2x-1)^5\,dx = \int u^5\,\frac{du}{2} = \int \tfrac{1}{2}u^5\,du = \tfrac{1}{12}u^6 + c$$

$$= \tfrac{1}{12}(2x-1)^6 + c$$

$$\int \sqrt{3x-2}\,dx$$

$$\int u^{1/2}\,dx$$ Substitute u in for $3x$ - 2

$$\int u^{1/2}\frac{1}{3}\,du$$ Substitute du in for dx

$$\frac{1}{3}\int u^{1/2}\,du$$ Factor out the constant

Application of Integration

- The **Petronas Towers** in Kuala Lumpur experience high forces due to winds. **Integration** was used to design the building for strength.

- The **Sydney Opera House** is a very unusual design based on slices out of a ball. Many **differential equations** (one type of integration) were solved in the design of this building.

- Historically, one of the first uses of integration was in finding the **volumes of wine-casks** (which have a curved surface).

Conclusion

- Very important mathematical tool
- Used in many fields
- Important in business
- Helps to estimate things like
  - Marginal cost
  - Marginal revenue
  - Profit
  - Gross loss
  - Etc.