# Affine Objects

Mohamed Hashi

May 30, 2016

# 1 Preliminaries

This chapter introduces all the basic notions that are needed but not assumed to be known to the reader. We will start with a discussion of some purely categorical notions like slice categories and presheaves. Secondly, we will introduce a notion of a topology on a category and look at some constructions that are relevant for us. Then we will introduce modules on ringed sites. Lastly, the notion of a scheme is introduced.

## 1.1 Basic Category Theory

Some categorical notions like presheaves and slice categories will be introduced in this section. See [A10] and [MM92].

**Definition 1** (Presheaf category)**.** Let $\mathsf{C}$ be a category. Let $a \in \mathsf{C}$. Let $f : a' \to a$ We define the category of presheaves on $\mathsf{C}$ as the category of contravariant functors to the category of sets $\mathrm{Set}$. We will denote it by $\hat{\mathsf{C}}$.

Define the functor $h : \mathsf{C} \to \hat{\mathsf{C}}$ as follows

$$a \mapsto \mathrm{Hom}(-, a),$$

$$f \mapsto f \circ -.$$

This functor is fully faithful by the Yoneda lemma.

**Notation 2.** Let $\mathsf{I}, \mathsf{C}$ be categories. Let $L : \mathsf{I} \to \mathsf{C}$ be a functor. The limit over this functor will be denoted by $\lim_{i \in \mathsf{I}} L(i)$. The colimit will be denoted by $\operatorname{colim}_{i \in \mathsf{I}} L(i)$.

**Definition 3** (Sections functor)**.** For any $a \in \mathsf{C}$ define the functor

$$\Gamma(c; -) : \hat{\mathsf{C}} \to \mathsf{Set}$$

by

$$\mathfrak{F} \to \Gamma(a; \mathfrak{F}).$$

Let $L : I \to C$ be a small diagram. Define

$$\Gamma(\operatorname*{colim}_{i \in I} L(i); -) : \hat{C} \to \mathsf{Set}$$

by

$$\mathfrak{F} \to \operatorname{Hom}(\operatorname*{colim}_{i \in I} L(i), \mathfrak{F}) = \lim_{i \in I} \operatorname{Hom}(L(i), \mathfrak{F}).$$

By definition of a colimit these definitions coincide when a colimit exists in $C$.

*Remark.* The category $\hat{C}$ is cocomplete so even if $C$ does not have a terminal object, we can still compute the global sections as $\Gamma(1; -)$

**Definition 4** (Over/Under categories)**.** Let $C$ and $C'$ be categories. Let $F : C \to C'$ and $z \in C'$. Define the category $C_z$ and $C^z$ as

$$\operatorname{Obj}(C_z) := \{(a, w) \mid a \in C, w : F(a) \to z\},$$
$$\operatorname{Hom}((a, w), (b, v)) := \{f : a \to b \mid v \circ F(f) = w\},$$

and

$$\operatorname{Obj}(C^z) := \{(a, w) \mid a \in C, w : z \to F(a)\},$$
$$\operatorname{Hom}((a, w), (b, v)) := \{f : a \to b \mid F(f) \circ w = v\}.$$

We get faithful functors $C_z \to C : (a, w) \to a$ and $C^z \to C : (a, w) \to a$. We will call both functors localization functors and denote them by $u$.

**Definition 5** (direct image)**.** Let $f : C \to D$. Define the direct image functor $f_* : \hat{D} \to \hat{C}$ as

$$f_* = - \circ f.$$

**Definition 6** (Restriction)**.** Let $C, D$ be categories. Let $\mathfrak{F} \in \hat{D}$. Let $\alpha : C \to D$ be a functor. The restriction of $\mathfrak{F}$ to $C$ along $\alpha$ is defined to be $\alpha_* \mathfrak{F}$.

## 1.2 Topology

In this section we will define a notion of a topology on a category and look at the related notions of sheaves, sites and restriction of sites.

See [MM92] for more details.

**Definition 7** (Sieve)**.** Let $\mathsf{C}$ be a category and $a \in \mathsf{C}$. Define the maximal sieve $\mathsf{max}(a)$ on $a$ to be the set of all morphisms to $a$. In formula,

$$\mathsf{max}(a) = \{f \in \mathsf{C} \mid \mathsf{Codom}(f) = a\}.$$

A sieve $S$ is a subset of $\mathsf{max}(a)$ such that $gf \in S$ for any $f \in S$ and any $g$.

*Remark.* Let $\mathsf{C}$ be a category and $a, b \in \mathsf{C}$. Let $f : b \to a \in \mathsf{C}_a$.

Any morphism to $b$ is also a morphism to $f$ and vice versa. This observation yields us that $\mathsf{Sieves}(b) = \mathsf{Sieves}(f)$. Moreover composition in $\mathsf{C}$ and $\mathsf{C}_a$ are the same, so this identification respects pullback of sieves.

**Definition 8** (Sieve category)**.** Let $\mathsf{C}$ be a category and $a \in \mathsf{C}$. The sieve category $\mathsf{Sieves}(a)$ consists of all the sieves on $a$ as objects and inclusions of sieves as morphisms.

**Definition 9** (Pullback of sieve)**.** Let $\mathsf{C}$ be a category and $a, b \in \mathsf{C}$. Let $S$ be a sieve on $a$. Let $f : b \to a$.

The sieve $f^*S$ on $b$ is given by $f^*S(c) = \{g \in \mathrm{Hom}(c, b) : fg \in S(c)\}$ for any $c \in \mathsf{C}$.

To show that this is actually a subpresheaf of $h(b)$, let $k : c \to c'$ and $h \in f^*S(c')$. Hence $fh \in S(c')$ and so $fhk \in S(c)$. Conclude that $hk \in f^*S(c')$.

This defines a functor $f^* : \mathsf{Sieves}(a) \to \mathsf{Sieves}(b)$.

**Definition 10** (Grothendieck Topology)**.** A Grothendieck topology $\mathcal{T}$ is a family $\mathcal{T}(a)$ of 'covering' sieves for every $a \in \mathsf{C}$ with the following conditions:

1. $\mathsf{max}(a) \in \mathcal{T}(a)$
2. $f^*R \in \mathcal{T}(a')$ if $R \in \mathcal{T}(a)$ for all $f : a' \to a$
3. if $f^*R \in \mathcal{T}(a')$ for all $f \in S$ with $S \in \mathcal{T}(a)$ then $R \in \mathcal{T}(a)$

**Definition 11** (Basis)**.** Let C be a category with pullbacks. A Grothendieck pretopology $\mathcal{B}$ is a collection $\mathcal{B}(a)$ of 'covering' families $\{f_i : a_i \to a\}$ of morphisms for every $a \in C$ with the following conditions.

1. every isomorphism is a covering singleton family,
2. (Stability) The pullback of a covering family is a covering family. If $\{f_i : a_i \to a\}$ is covering and $g : b \to a$, then $\{f_i' : a_i \times_a b \to b\}$ is covering.
3. (Transitivity) If $\{f_i : a_i \to a\}$ is a covering family and $\{f_{ij} : a_{ij} \to a_i\}$ for every $i$, then $\{f_{ij} : a_{ij} \to a\}$ is a covering family.

Generating a topology from a basis: take any sieve containing a covering family to be a covering sieve.

**Definition 12** (Site)**.** A site $(C, \mathcal{T})$ is a category C with a Grothendieck topology $\mathcal{T}$. A morphism of sites $G : (C, \mathcal{T}) \to (D, \mathcal{S})$ is a functor $G' : C \to D$ such that

A functor $G' : C \to D$ is called cover-preserving if for every covering sieve R, the family $\{G'(f) | f \in R\}$ generates a S-covering sieve.

The category Sites has as objects sites and as morphisms cover-preserving functors. When no confusion can arise then we will use C to denote the whole site $(C, \mathcal{T})$.

### 1.2.1 Sheaves

We will introduce the very important notion of a sheaf. See [MM92] for a more detailed treatment.

**Definition 13** (Matching family)**.** Let C be a category. Let $\mathfrak{F}$ be a presheaf on on C. Let $a \in C$ be an object. Let R be a sieve on $a$. A set $\{x_i\}_{i \in R}$ with $x_i \in \Gamma(\mathrm{Dom}(i); \mathfrak{F})$ indexed by a sieve R and such that $x_{g \circ i} = \mathfrak{F}(g)(x_i)$ for any $g : b \to \mathrm{Dom}(i)$ and $b \in C$ is called a 'matching family'.

**Definition 14** (Matching family/Morphisms)**.** Let C be a category. Let $\mathfrak{F}$ be a presheaf on on C. Let $a \in C$ be an object. Let R be a sieve on $a$. Define $\Gamma(R; \mathfrak{F}) = \mathrm{Hom}(R, F)$. An element $\varphi \in \Gamma(R; \mathfrak{F})$ is uniquely identified by the matching family $\{\varphi(i)\}_{i \in R}$ of images. Conversely, any matching family $\{x_i\}_{i \in R}$, with $x_i \in \Gamma(\mathrm{Dom}(i); \mathfrak{F})$ indexed by R and such that $x_{g \circ i} = \mathfrak{F}(g)(x_i)$ for any $g : b \to \mathrm{Dom}(i)$ and $b \in C$, uniquely identifies a map $\varphi : R \to \mathfrak{F}$. Namely, take $\varphi_a(y) = x_y$.

**Definition 15** (Amalgamation)**.** An amalgamation of a matching family $\{x_i\}_R$ is an element $x \in \Gamma(a; \mathfrak{F})$ such that $\mathfrak{F}(i)(x) = x_i$.

When you consider the matching family as a morphism $\varphi$, an amalgamation is a morphism $\phi : h(a) \to \mathfrak{F}$ that extends $\varphi$.

**Definition 16** (Separated presheaf)**.** A presheaf $\mathfrak{F}$ is separated if any matching family has at most one amalgamation.

**Definition 17** (Sheaves)**.** Let $(C, \mathfrak{T})$ be a site. Let $\mathfrak{F} \in \hat{C}$.

A presheaf that admits a unique amalgamation for every matching family is called a sheaf. The category $\mathsf{Shv}(C)$ is the full subcategory in $\hat{C}$ of all sheaves.

In other words, we call $\mathfrak{F}$ a sheaf if for each $a \in C$ and $R \in \mathfrak{T}(a)$ the map

$$\begin{aligned} \Gamma(a; \mathfrak{F}) &\to \Gamma(R; \mathfrak{F}) \\ x &\mapsto \{x_i \mid i \in R\} \end{aligned}$$

where $x_i = \mathfrak{F}(i)(x)$ is an isomorphism.

**Definition 18** (Plus construction)**.** Let $(C, \mathfrak{T})$ be a site. Let $a, a' \in C$ and $f : a \to a'$. Let $\mathfrak{F} \in \hat{C}$. Define the functor $(-)^+ : \hat{C} \to \hat{C}$ as follows.

For all $a \in C$,

$$F^+(a) = \frac{\{(R, \varphi) \mid R \in \mathfrak{T}(a), \varphi \in \Gamma(R; \mathfrak{F})\}}{\sim},$$

for all morphisms $f \in C$,

$$F^+(f)([(R, \varphi)]) = [(f^* R, \varphi h(f))].$$

The equivalence relation is defined as $(R, \varphi) \sim (S, \phi)$ if $\varphi = \phi$ on some covering sieve $Q \subset R \cap S$.

Let $L : \mathfrak{F} \to \mathfrak{F}'$. Then

$$(L^+)_a([(R, \varphi)]) = [(R, L \circ \varphi)]$$

This functor comes with a natural transformation $\omega : \mathrm{Id} \to (-)^+$ defined by

$$\omega_{\mathfrak{F},a}(x) = [(\mathsf{max}(a), y]$$

where

$$y(i) = \mathfrak{F}(i)(x).$$

**Lemma 19** (2.10 [ToposNotes]). *Let $\mathfrak{F}$ be a presheaf, $\mathfrak{G}$ a sheaf and $g : \mathfrak{F} \to \mathfrak{G}$ a morphism in $\hat{C}$. Then $g$ factors through $\omega_F$ via a unique $g'$.*

**Lemma 20** (2.11 [ToposNotes]). *For every presheaf $\mathfrak{F}$, $F^+$ is separated.*

**Lemma 21** (2.12 [ToposNotes]). *If $\mathfrak{F}$ is separated, then $F^+$ is a sheaf.*

**Definition 22.** Define $sh = (-)^+ \circ (-)^+$.

**Lemma 23** (Sheafification adjunction). *Let $(C, \mathcal{T})$ be a site. The functor $sh$ is left adjoint to the inclusion $\hat{C} \to \mathsf{Shv}(C)$ with unit*

$$\omega^2_{\mathfrak{F}} = \omega_{\mathfrak{F}^+} \circ \omega_{\mathfrak{F}}.$$

### 1.2.2 Relative topology

We will look at what the induced topology on a slice category looks like and what this implies for restriction of sheaves. See [Stacks, Tag 03A4] for a more detailed treatement.

**Definition 24** (Relative topology). Let $(C, \mathcal{T})$ be a site. Let $a \in C$.

Define the induced topology $\mathcal{T}_a$ on $C_a$ by, for each $f \in C_a$

$$\mathcal{T}_a(f) = \mathcal{T}(\mathrm{Dom}(f)).$$

The identification from Remark 1.2 implies that $\mathcal{T}_a$ is a Grothendieck topology.

**Definition 25** (Oversite). Let $Y = (C, \mathcal{T})$ be a site. Let $a \in C$. Define the site $Y_a$ to be the category $C_a$ with the induced topology $T_a$.

**Definition 26** (Natural transformation s). Let $(C, \mathcal{T})$ be a site. Let $a, b \in C$ and $f : b \to a$. Let $u : C_a \to C$.

Let $\{x_i \mid i \in R\}$ be a compatible family indexed by a sieve $R$ on $b$. The same set $\{x_i \mid i \in R\}$ is a compatible family on $f$ indexed by the same sieve $R$. This yields a natural isomorphism

$$s : u_* \circ (-)^+ \to (-)^+ \circ u_*,$$

by

$$s_{\mathfrak{F}} : u_*\mathfrak{F}^+ \to (u_*\mathfrak{F})^+$$

$$s_{\mathfrak{F},f}([\{x_i \mid i \in R\}]) = [\{x_i \mid i \in R\}].$$

We will treat $s$ as an identification.

**Lemma 27** (s and $\omega$ commute)**.** *Let $\mathfrak{F}$ be a presheaf on $(C, \mathcal{T})$. Let $f : b \to a \in C$. Let $u : C_a \to C$ be the localisation morphism. Then $\omega_{u_*\mathfrak{F}} = s_{\mathfrak{F}} \circ u_*\omega_{\mathfrak{F}}$.*

*Proof.* For any section $x \in \Gamma(b; \mathfrak{F})$. Let $x_i = \mathfrak{F}(i)(x)$ for any morphism $i \in C$. Note that $\max(f) = \max(b)$. This implies that the compatible family $\{x_i\}$ indexed by the maximal sieve on $f$ is sent by $s$ to the same set $\{x_i\}$ indexed by the maximal sieve on $b$. In diagram form, that

$$\begin{tikzcd}
u_*\mathfrak{F} \arrow[d, "u_*\omega_{\mathfrak{F}}"'] \arrow[dr, "\omega_{u_*\mathfrak{F}}"] & \\
u_*\mathfrak{F}^+ \arrow[r, "s"] & (u_*\mathfrak{F})^+
\end{tikzcd}$$

commutes. ∎

**Definition 28** ($s^2$)**.** Define

$$s^2 : u_* \circ (-)^+ \circ (-)^+ \to (-)^+ \circ (-)^+ \circ u_*$$

as

$$s^2_{\mathfrak{F}} = s_{\mathfrak{F}}{}^+ \circ s_{\mathfrak{F}^+}.$$

**Lemma 29.** *Let $\mathfrak{F}$ be a presheaf on $(C, \mathcal{T})$. Let $f : b \to a \in C$. Let $\mathfrak{F}$ be a presheaf on $C$. Then $\omega^2_{u_*\mathfrak{F}} = s^2_{\mathfrak{F}} \circ (u_*\omega^2_{\mathfrak{F}})$.*

*Proof.* Let $a \in C$. We have the following identities.

$$\begin{aligned}
\omega^2_{u_*\mathfrak{F}} &= \omega_{u_*\mathfrak{F}^+} \circ \omega_{u_*\mathfrak{F}} \text{ by definition} \\
&= \omega_{u_*\mathfrak{F}^+} \circ s_{\mathfrak{F}} \circ u_*\omega_{\mathfrak{F}} \text{ by applying Lemma 27} \\
&= {s_{\mathfrak{F}}}^+ \circ u_*{\omega_{\mathfrak{F}}}^+ \circ u_*\omega_{\mathfrak{F}} \text{ by naturality of } \omega \\
&= {s_{\mathfrak{F}}}^+ \circ u_*{\omega_{\mathfrak{F}}}^+ \circ s_{\mathfrak{F}} \circ u_*\omega_{\mathfrak{F}} \text{ by applying Lemma 27} \\
&= {s_{\mathfrak{F}}}^+ \circ s_{\mathfrak{F}^+} \circ u_*\omega_{\mathfrak{F}^+} \circ u_*\omega_{\mathfrak{F}} \text{ by naturality of } s \circ u_*\omega \\
&= s^2_{\mathfrak{F}} \circ u_*\omega^2_{\mathfrak{F}}
\end{aligned}$$

■

**Corollary 30.** *Let* $(C, \mathcal{T})$*. Let* $a, b \in C$*. Let* $f : b \to a \in C$*. Sheafifying and restricting commute via the iso*

$$s^2 : sh_b \circ u_* \to u_* \circ sh_a.$$

**Lemma 31** ($\lambda$ commutes with restriction)**.** *Let* $(C, \mathcal{T}, \mathfrak{O})$ *be a ringed site. Let* $a \in C$*. We have a natural isomorphism* $t : u_* \circ \lambda \Rightarrow \lambda \circ (- \otimes_{\Gamma(1;\mathfrak{O})} \Gamma(a; \mathfrak{O}))$*.*

*Proof.* Define the natural transformation $t : \lambda \circ (- \otimes_{\Gamma(1;\mathfrak{O})} \Gamma(a; \mathfrak{O})) \Rightarrow u_* \circ \lambda$, by for each $\Gamma(1; \mathfrak{O})$-module $M$ and for each $f : b \to a \in C_a$,

$$t_{M,f} : M \otimes_{\Gamma(1;\mathfrak{O})} \Gamma(a; \mathfrak{O}) \otimes_{\Gamma(a;\mathfrak{O})} \Gamma(b; \mathfrak{O}) \to M \otimes_{\Gamma(1;\mathfrak{O})} \Gamma(b; \mathfrak{O}),$$

$$m \otimes r \otimes s \mapsto m \otimes rs.$$

Every component $t_{M,f}$ is an isomorphism by basic commutative algebra. ■

Let $C$ be a category. Let $a \in C$. Let $\epsilon$ be the counit of the adjunction $\lambda \dashv \Gamma(1; -)$ on $C$. Let $\epsilon_a$ be the counit of the adjunction $\lambda_a \dashv \Gamma(a; -)$ on $C_a$.

**Lemma 32** ($\lambda$ counit commute with restriction)**.** *We have* $u_*\epsilon \cong \epsilon_a$ *on presheaves of the form* $\lambda_a(M \otimes \Gamma(b; \mathfrak{O}))$ *with* $M$ *a* $\Gamma(1; \mathfrak{O})$*-module via*

$$t \circ u_*\epsilon \circ t^{-1} = \epsilon_a.$$

*Proof.* Both maps are the identity map if you unfold them. ∎

**Lemma 33** ($\Lambda$ commutes with restriction)**.** *Let* $(\mathsf{C}, \mathfrak{T}, \mathfrak{O})$ *be a ringed site. We have a natural isomorphism*

$$q : u_* \circ \Lambda \to \Lambda \circ (- \otimes_{\Gamma(1;\mathfrak{O})} \Gamma(a;\mathfrak{O})).$$

*Proof.* Define $q$ to be the composition

$$\begin{aligned} u_* \circ \mathrm{sh} \circ \lambda &\overset{s^2\lambda}{\Rightarrow} \mathrm{sh} \circ u_* \circ \lambda \\ &\overset{\mathrm{sh}(t)}{\Rightarrow} \mathrm{sh} \circ \lambda \circ - \otimes_{\Gamma(1;\mathfrak{O})} \Gamma(a;\mathfrak{O})) \end{aligned}$$

By Definition 26 and lemma 31, $t$ and $s^2$ are isomorphisms so $q$ is an isomorphism as well. ∎

## 1.3 Modules

Presheaf modules and sheaf modules on a ringed site will be introduced in this section. The functors $\lambda$ and $\Lambda$ introduced here will be used extensively. See [Stacks, Tag 03A4] for more detail.

**Definition 34** (Presheaf modules)**.** Let $Y = (\mathsf{C}, \mathfrak{T}, \mathfrak{O})$ be a ringed site. Let $R = \Gamma(1;\mathfrak{O})$.

A presheaf module on $Y$ is a presheaf of sets $\mathfrak{F}$ on $\mathsf{C}$ together with a map of presheaves

$$\mathfrak{O} \times \mathfrak{F} \to \mathfrak{F}$$

such that for every object $a \in \mathsf{C}$ the map $\Gamma(a;\mathfrak{O}) \times \Gamma(a;\mathfrak{F}) \to \Gamma(a;\mathfrak{F})$ defines a $\Gamma(a;\mathfrak{O})$-module structure on $\Gamma(a;\mathfrak{F})$.

A morphism

$$\mathfrak{F} \to \mathfrak{G}$$

is a morphism of presheaf modules if

$$\begin{array}{ccc} \mathfrak{O} \times \mathfrak{F} & \longrightarrow & \mathfrak{F} \\ \downarrow & & \downarrow \\ \mathfrak{O} \times \mathfrak{G} & \longrightarrow & \mathfrak{G} \end{array}$$

commutes. The category of presheaf modules on $Y$ will be denoted $\mathsf{PMod}(Y)$.

**Definition 35.** Let $M, N$ be an R-module.

Define

$$\lambda : \mathsf{R\text{-}Mod} \to \mathsf{PMod}(Y)$$

by for all $a \in C$,

$$\lambda(M)(a) = M \otimes_R \Gamma(a; \mathfrak{O}),$$

for all $f : b \to a \in C$,

$$\lambda(M)(f) : \mathrm{Id} \otimes \mathfrak{O}(f),$$

for all $g : M \to N \in \mathsf{R\text{-}Mod}$,

$$\lambda(g) = (a : g \otimes \mathrm{Id}).$$

**Lemma 36.** *Let* $Y = (X, \mathcal{T}, \mathfrak{O})$ *be a ringed site. The functor* $\lambda$ *is left adjoint to*

$$\Gamma(1; -) : \mathsf{PMod}(Y) \to \mathsf{R\text{-}Mod}$$

.

*Proof.* Let $a$ be an object of $C$. Let $M, N$ be R-modules. Let $\mathfrak{F}, \mathfrak{G} \in \mathsf{PMod}(Y)$ be presheaf modules.

Let $\varphi : \lambda(M) \to \mathfrak{G}$ be a morphism of presheaf modules. Let $\phi : M \to \Gamma(1; \mathfrak{G})$ be a morphism of presheaf modules.

Define

$$\alpha = H_{M,\mathfrak{G}} : \mathrm{Hom}(\lambda(M), \mathfrak{G}) \to \mathrm{Hom}(M, \Gamma(1; \mathfrak{G}))$$

by

$$\alpha(\varphi) = \varphi_1,$$

where $\varphi_1$ is the component of $\varphi$ on the global sections.

Define

$$\beta = L_{M,\mathfrak{G}} : \mathrm{Hom}(M, \Gamma(1;\mathfrak{G})) \to \mathrm{Hom}(\lambda(M), \mathfrak{G})$$

by

$$\beta(\phi)_a = \phi \otimes_R \Gamma(a;\mathfrak{O}).$$

We will show that $\beta$ and $\alpha$ are mutually inverse.

Let $d = \beta(\alpha(\varphi))$. Let $m \otimes g \in M \otimes_R \Gamma(a;\mathfrak{O})$. Let $p : \lambda(M)(1) \to \lambda(M)(a)$ be the projection map. Let $q : \mathfrak{G}(1) \to \mathfrak{G}(a)$ be the projection map. Then $d_a(m \otimes g) = \varphi_1(m) \otimes g$ and

$$\begin{aligned}
\varphi_a(m \otimes g) &= g\varphi_a(m \otimes 1) \text{ by linearity} \\
&= g\varphi_a(p(m)) \\
&= gq(\varphi_1(m)) \text{ by naturality of } \varphi \\
&= g(\varphi_1(m) \otimes 1) \\
&= \varphi_1(m) \otimes g.
\end{aligned}$$

Hence $d = \varphi$. In words, the natural transformations from presheaves of the from $\lambda(M)$ are uniquely determined by their global sections component.

Let $d = \alpha(\beta(\phi))$. Let $m \in M$. Then $d(m) = (\phi \otimes_R R)(m) = \phi(m)$. Hence $d = \phi$, which makes $H$ and $L$ mutual inverses.

*Naturality in $M$ and $\mathfrak{G}$*

Let $g : N \to M$ and $h : \mathfrak{F} \to \mathfrak{G}$. Let $\rho \in \mathrm{Hom}(\lambda(N), \mathfrak{F})$. Let $k = H_{M,\mathfrak{G}}(h \circ \rho \circ \lambda(f))$. Let $l = h_1 \circ H_{N,\mathfrak{F}}(\rho) \circ f$.

Unfolding the definition for $H$ shows that $k = h_1\rho_1 f$ and $l = h_1\rho_1 f$ as well. This proves naturality in $M$ and $\mathfrak{G}$ and the adjunction between $\lambda$ and $\Gamma(1;-)$. ■

**Definition 37.** Let $(C, \mathfrak{T}, \mathfrak{O})$ Define

$$\Lambda : \mathsf{R\text{-}Mod} \to \mathsf{Mod}(O)$$

by $\mathrm{sh} \circ \lambda$.

It follows that we have the adjunction $\Lambda \dashv \Gamma(1; -)$.

**Definition 38.** Let $\mathfrak{F}$ be a sheaf of modules on $(C, \mathcal{T}, \mathfrak{O})$. It is called quasi-coherent if the following holds. For any object $a \in C$ there exists a covering sieve $S$ such that for any map $f : b \to a$ in $S$ there exists a presentation

$$\bigoplus_I \mathfrak{O}\big|_b \to \bigoplus_J \mathfrak{O}\big|_b \to \mathfrak{F}\big|_b \to 0$$

Quasi-coherent modules form a full subcategory of the category of sheafs of modules over $(C, \mathcal{T}, \mathfrak{O})$ which are denoted by $\mathsf{Qcoh}(\mathfrak{O})$.

## 1.4 Schemes

**Definition 39** (Spectrum of a ring)**.** Let $R$ be a ring. The spectrum $\mathrm{Spec}R$ of $R$ is the ringed space defined as follows. The underlying set is the set of prime ideals of $R$. The (zariski) topology is generated by the basis $D(f) = \{\mathfrak{p} \subset R | f \notin \mathfrak{p}\}$. The sheaf of rings is given by

$$D(f) \mapsto R_f.$$

**Definition 40** (Distinguised open)**.** Let $\mathrm{Spec}R$ be a affine scheme. The set

$$D(f) = \{\mathfrak{p} \subset R | f \notin \mathfrak{p}\}$$

for a global section $f$ is called a distinguised open. The open $D(f)$ is isomorphic to $\mathrm{Spec}(R_f)$ as a locally ringed space.

**Definition 41** (Locus of a point)**.** Let $(X, \mathfrak{O})$ be a scheme. Define the locus of a global section $x \in \Gamma(1; \mathfrak{O})$ to be

$$\ker(x) = \ker(\mathfrak{O}(X) \to \kappa(x)).$$

**Lemma 42.** *The functor*

$$\mathit{Spec} : \mathsf{Rng} \to \mathsf{LRSpaces}$$

*is left adjoint to*

$$\Gamma(1; -) : \mathsf{LRSpaces} \to \mathsf{Rng}.$$

*With unit*

$$F = \eta : (X, \mathfrak{O}) \to \mathit{Spec}(\Gamma(1; \mathfrak{O})).$$
$$x \mapsto \ker(x),$$

*Proof.* ∎

**Definition 43** (Affine scheme)**.** We call the ringed space $\mathrm{Spec}(R)$ an affine scheme.

**Definition 44** (Scheme)**.** A scheme $S$ is a locally ringed space that admits a covering of affine schemes. A morphism of schemes is a morphism of locally ringed spaces. The category of schemes we will denote by $\mathsf{Sch}$.

# 2 Restrictive

## 2.1 Restrictive

**Definition 45** (Restrictive functor)**.** A functor $f : (C, \mathcal{T}, \mathfrak{O}) \to (D, \mathcal{S}, \mathfrak{U})$ between ringed sites is called restrictive if for every quasi-coherent module $\mathfrak{G}$ on $(D, \mathcal{S}, \mathfrak{U})$ the unit $\eta$ of $f^{-1} \dashv f_*$ induces an isomorphism

$$\begin{aligned} \eta_{\mathfrak{G}} :& \mathfrak{G} \to f_* f^{-1} \mathfrak{G}, \\ \eta_{\mathfrak{G},1} :& \Gamma(1; \mathfrak{G}) \to \Gamma(1; f_* f^{-1} \mathfrak{G}) \\ \eta_{\mathfrak{G},1} \otimes_{\Gamma(1;\mathfrak{O})} \Gamma(1; \mathfrak{U}) :& \Gamma(1; \mathfrak{G}) \otimes_{\Gamma(1;\mathfrak{O})} \Gamma(1; \mathfrak{U}) \to \Gamma(1; f_* f^{-1} \mathfrak{G}). \end{aligned}$$

**Definition 46** (Restrictive morphism)**.** A morphism $f : a \to b \in C$ is called restrictive if the induced functor $C_a \to C_b$ is restrictive.

**Example 47.** In Sch, the morphism $\mathrm{Spec}(A_f) \to \mathrm{Spec}(A)$ is restrictive.

**Lemma 48.** *The composition of two restrictive functors is restrictive. If the composition* $gf$ *is restrictive, then* $g$ *is restrictive*

*Proof.* ■

**Non-Example 49.** The open immersion $\mathrm{Spec}(R^2) \setminus 0 \to \mathrm{Spec}(R^2)$ is not restrictive. The quasi-coherent sheaf $\Lambda(\frac{R[x,y]}{xy})$ fails to satisfy the condition from the definition.

**Non-Example 50** (Affine non-restrictive map)**.** Both canonical inclusions $\mathbb{A}^1 \to \mathbb{P}^1$ are not restrictive. Look at the quasi-coherent module $\mathfrak{O}(-1)$. There are no global sections but on every affine chart this invertible sheaf is trivial.

**Non-Example 51.** Any inclusion $\mathrm{Spec}(\kappa(\mathfrak{p})) \to \mathbb{P}^1$ is not restrictive. Look at $\mathfrak{O}(-1)$.

**Lemma 52** (Restrictive to affines)**.** *If* $f : X \to$ *Spec*$(R)$ *is a restrictive open immersion. then* $X$ *is affine.*

*Proof.* ∎

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