2 Preliminaries

2.1 Basic Category Theory

Definition 1 (Over/Under categories)**.** Let C and C' be categories. Let $F : C \to C'$ and $z \in C'$. Define the category C_z and C^z as

$$\mathrm{Obj}(C_z) := \{(a, w) \mid a \in C, w : F(a) \to z\},$$
$$\mathrm{Hom}((a, w), (b, v)) := \{f : a \to b \mid v \circ F(f) = w\},$$

and

$$\mathrm{Obj}(C^z) := \{(a, w) \mid a \in C, w : z \to F(a)\},$$
$$\mathrm{Hom}((a, w), (b, v)) := \{f : a \to b \mid F(f) \circ w = v\}.$$

faithful

We get faithfull functors $C_z \to C : (a, w) \to a$ and $C^z \to C : (a, w) \to a$. We will call both functors localization functors and denote them by u. We will suppress the functor F where there can be no confusion.

Definition 2 (Presheaf category)**.** Let C be a category. Let $a \in C$. Let $f : a' \to a$ We define

$$\hat{C} := [C^{op}, \mathrm{Set}],$$

and the functor $h : C \to \hat{C}$ as follows

$$a \mapsto \mathrm{Hom}(-, a),$$
$$f \mapsto f \circ -.$$

This functor is fully faithfull by the Yoneda lemma.

2.2 Topology

Definition 3 (Sieve). Let C be a category and $a \in C$. A sieve S on a is a subpresheaf of $h(a)$. ~~Explicitely~~, Explicitly

$$S(c) \subset \mathrm{Hom}(c, a)$$

such that

$$fg \in S(\mathrm{Dom}(g)), \forall f \in S(c), \forall g \in h(c).$$

I would phrase this in words: "For each $c \in C$, $S(c)$ is a subset of $\mathrm{Hom}(c,a)$ such that..."

The maximal sieve on a, which is $h(a)$ itself, will be denoted by $\max(a)$.

Definition 4 (Grothendieck Topology). A Grothendieck topology $\mathcal{T}$ is a family $\mathcal{T}(a)$ of 'covering' sieves for every $a \in C$ with the following conditions:

- $\max(a) \in \mathcal{T}(a)$
- $f^*R \in \mathcal{T}(a')$ if $R \in \mathcal{T}(a)$ for any $f : a' \to a$
- if $f^*R \in \mathcal{T}(a')$ for all $f \in S$ with $S \in \mathcal{T}(a)$ then $R \in \mathcal{T}(a)$

this is undefined

"any" sometimes means "some" and sometimes means "all". It's hard to tell when it will be unclear.

Remark. Note that if $f \in R$ then $f^*R = \max(a')$. So if $R \subset S$ and R is a covering (sieve) then S is a covering. Also $R \cap S$ is a covering if and only if R and S are coverings.

I don't see how these follow

Definition 5 (Basis). Let C be a category with pullbacks. A Grothendieck pretopology $\mathcal{B}$ is a collection $\mathcal{B}(a)$ of families $\{f_i : a_i \to a\}$ of 'covering' morphisms for every $a \in C$ with the following conditions.

- every isomorphism is a covering singleton family,
- (Stability) The pullback of a covering family is covering. If $\{f_i : a_i \to a\}$ is covering and $g : b \to a$, then $\{f_i' : a_i \times_a b \to b\}$ is covering.
- (Transitivity) If $\{f_i : a_i \to a\}$ is covering and $\{f_{ij} : a_{ij} \to a_i\}$ for every i, then $\{f_{ij} : a_{ij} \to a\}$ is covering.

Generating a topology from a basis: take ~~any~~ every sieve containing a covering family to be a covering sieve. Is this really all you have to do?

"is covering" sounds a little odd, like it's temporary. I would say "is a covering —" with sieve or family each time.

2.2.1 Sheaves

What are you going to do here? "We give an explicit definition of sheafification that works in every category" or something.

Definition 6 (Matching family). Let C be a category. Let $\mathfrak{F}$ be a presheaf on on C. Let $a \in C$ be an object. Let R be a sieve on a.

A set $\{x_i\}_{i \in R}$ with $x_i \in \Gamma(\mathrm{Dom}(i); \mathfrak{F})$ indexed by a sieve R and such that $x_{g \circ i} = \mathfrak{F}(g)(x_i)$ for any $g : b \to \mathrm{Dom}(i)$ and $b \in C$ is called a 'matching family'.

Definition 7 (Matching family/Morphisms). Let C be a category. Let $\mathfrak{F}$ be a presheaf on on C. Let $a \in C$ be an object. Let R be a sieve on a. unnecessary new paragraph

Define $\Gamma(R; \mathfrak{F}) = \mathrm{Hom}(R, \mathfrak{F})$. An element $\varphi \in \Gamma(R; \mathfrak{F})$ is uniquely identified by the matching family $\{\varphi(i)\}_{i \in R}$ of images. Conversely, any matching family $\{x_i\}_{i \in R}$, with $x_i \in \Gamma(\mathrm{Dom}(i); \mathfrak{F})$ indexed by R and such that $x_{g \circ i} = \mathfrak{F}(g)(x_i)$ for any $g : b \to \mathrm{Dom}(i)$ and $b \in C$, uniquely identifies a map $\varphi : R \to \mathfrak{F}$. Namely, take $\varphi_a(y) = x_y$.

this is really nice.

Definition 8 (Amalgamation). An amalgamation of a matching family $\{x_i\}_R$ is an element $x \in \Gamma(1; \mathfrak{F})$ such that $\mathfrak{F}(i)(x) = x_i$.

When you consider the matching family as a morphism φ, an amalgamation is a morphism $\phi : h(a) \to \mathfrak{F}$ that extends φ.

Definition 9 (Sheaves). Let $(C, \mathcal{T})$ be a site. Let $\mathfrak{F} \in \hat{C}$.

A presheaf that admits a unique amalgamation of every matching family is called a sheaf. The category Shv(C) is the full subcategory in $\hat{C}$ consisting of all sheaves. Let i be the inclusion functor $\mathrm{Shv}(C) \to \hat{C}$.

these can be combined. "In other words..."

Definition 10 (Sheaves #2). Let $(C, \mathcal{T})$ be a site. Let $\mathfrak{F} \in \hat{C}$. Let R be a sieve on $a \in C$.

We call $\mathfrak{F}$ a sheaf if the map

$$\mathfrak{F}(a) \to \mathfrak{F}(R)$$
$$a : x \mapsto \{\mathfrak{F}(i)(x)\}_{i \in R}$$

is an isomorphism.

Definition 11 (Plus construction). Let $(C, \mathcal{T})$ be a site. Let $a, a' \in C$ and $f : a \to a'$. Let $\mathfrak{F} \in \hat{C}$. Define the functor $(-)^+ : \hat{C} \to \hat{C}$ as follows

On objects:

$$\mathfrak{F}^+(a) = \frac{\{(R, \varphi) \mid R \in \mathcal{T}(a), \varphi \in \Gamma(R; \mathfrak{F})\}}{\sim},$$

spell out

$$\mathfrak{F}^+(f)([[(R,\varphi)]]) = [(f^*R, h(f)\varphi)].$$

The eq. relation is defined as:

$$(R,\varphi) \sim (S,\phi)$$

if $\varphi = \phi$ on some $Q \subset R \cap S$. also a covering sieve?

Let $L : \mathfrak{F} \to \mathfrak{F}'$. Then

$$(L^+)_a([[(R,\varphi)]]) = [(R, L \circ \varphi)]$$

This functor comes with a natural transformation $\omega : \mathrm{Id} \to (-)^+$ defined by

$$\omega_{\mathfrak{F},a}(x) = [(\max(a), y]\ , y(i) = \mathfrak{F}(i)(x).$$

Definition 12. Define $\mathsf{sh} = (-)^+ \circ (-)^+$.

Lemma 13. *Let $Y = (C, \mathcal{T})$ be a site. The functor sh is left adjoint to the inclusion $Shv(Y) \to Shv(C)$ with unit*

$$\omega^2 : Id \xrightarrow{\omega} (-)^+ \xrightarrow{\omega^+} \mathsf{sh}$$

Proof. Cite an appropriate reference, e.g. SGL. ■

2.2.2 Relative topology

Definition 14 (Relative topology)**.** Let $(C, \mathcal{T})$ be a site. Let $a \in C$.

The topology $\mathcal{T}$ induces a topology $\mathcal{T}_a$ on C_a as follows. Let $f : b \to a \in C_a$. Let $R \in \mathcal{T}(b)$. Define the sieve $R_f \subset h(f)$ as follows. ~~Let~~ for each $g : b' \to a \in C_a$, set

$$R_f(g) = \{p : b' \to b \in R(b') \mid g = f \circ p\}.$$

This is a sieve because if $p \in R_f(g)$ and $h : g' \to g$ arbitrary, then $gh = fph$ so $ph \in R_f(gh)$.

Set $\mathcal{T}_a(f) = \{R_f : R \in \mathcal{T}(b)\}$.

Remark. Every sieve S on $f : b \to a$ can be considered as a sieve S' on b by defining $S'(b') = \bigcup_g S(g)$ with $g : b' \to b$. This is clearly a sieve. We have $S'_f = S$ and $(S_f)' = S$. Hence $S \in \mathcal{J}(b)$ if and only if $S_f \in \mathcal{J}_a(\cancel{b})$. f ?

always a suspicious word choice

Lemma 15. $\mathcal{J}_a$ *defines a Grothendieck topology*

Proof. We will prove the axioms one by one.

- Axiom 1

Let $p : b' \to b \in h(f)(g)$ with $g : b' \to a$. Then $g = f \circ p$ hence $p \in \max(b)_f(g)$ so $\max(b)_f = h(f) = \max(f)$. This proves that $\max(f) \in \mathcal{J}_a(f)$.

- Axiom 2

Really this means "let $S \in \mathcal{J}_a(f)$ and $R = S'$, so that $S = R_f$."

Let $p : b' \to b \in h(f)g$ with $g : b' \to a$. Let $R_f \in \mathcal{J}_a(f)$. We have to show that $p^*R_f \in \mathcal{J}_a(g)$. We will prove $p^*R_f = (p^*R)_g$, which implies the desired result.

because now you're referring to R.

Let $h \in p^*R_f(t)$ for some $t \in C_a$. Then $ph \in R_f(t)$, so $ph \in R(\mathrm{Dom}(t))$ and $t = fph$. This implies $h \in p^*R(\mathrm{Dom}(t))$ and since $g = fp$ also that $t = gh$. Hence $h \in (p^*R)_g(t)$.

Let $h \in (p^*R)_g(t)$ for some $t \in C_a$. Then $h \in p^*R(\mathrm{Dom}(t))$ and $t = gh$. So we get $ph \in R(\mathrm{Dom}(t))$ and $t = fph$, so $ph \in R_f(t)$. Hence also $h \in p^*R_f(t)$.

- Axiom 3

Let $R_f \in \mathcal{J}_a(f)$. Let S_f be any sieve on f for some sieve S on b. Any sieve on f can be written as S_f for some S by the above remark. Assume that $p^*S_f \in \mathcal{J}_a(g)$ for any $p : b' \to b \in R_f(g)$ with $g : b' \to a$.

By assumption $p^*S_f = (p^*S)_g$ is covering for any $p \in R_f(g)$, hence p^*S is covering for any $p \in R(b')$, hence S is covering so S_f is. ∎

Lemma 16. *Let* C *be a category. Let* $a, b \in C$. *Let* $(x_i)_T$ *be a matching family for some presheaf* $\mathcal{F}$ *on* b *indexed by sieve* T. *For any* $f : b \to a$ *the family* $(x_i)_{T_f}$ *is matching again.*

Proof. Let $u : C_a \to C$ be the localization functor. Only when a domain has an 'a' as subscript, is it taken in C_a.

We have $x_i \in \Gamma(\mathrm{Dom}(i); \mathfrak{F})$. Hence also $x_i \in \Gamma(fi; \mathfrak{F}u) = \Gamma(\mathrm{Dom}(i)_a; \mathfrak{F}u)$, where now i is considered as a morphism in C_a. Note that

$$(\mathfrak{F}u)(p)(x_i) = \mathfrak{F}(p)(x_i) = x_{i \circ p}$$

for any $p : c \to \mathrm{Dom}(i)$, since $(x_i)_T$ is a matching family in C. ■

Lemma 17. *Let* $Y = (C, \mathcal{T})$. *Let* $a, b \in C$. *Let* $f : b \to a$. *Sheafifying and restricting commute. In formula form*

$$\mathrm{sh}_b \circ *|_b \cong *|_b \circ \mathrm{sh}_a.$$

Putting the map here * would be very helpful.

Proof. <!– This proof is not improved from last time, left it out for now. Will rewrite it using new parts about relative topology. –>

Okay.

■

It would be nice to have a sentence at the beginning of the section explaining that this lemma is the goal of all the details that precede it.

2.3 Modules

Definition 18 (Presheaf modules)**.** Let $Y = (C, \mathcal{T}, \mathfrak{O})$ be a ringed site. Let $R = \Gamma(1; \mathfrak{O})$.

Have you defined this?

A presheaf module on Y is a presheaf of sets $\mathfrak{F}$ on C together with a map of presheaves

$$\mathfrak{O} \times \mathfrak{F} \to \mathfrak{F}$$

such that for every object $a \in C$ the map $\Gamma(a; \mathfrak{O}) \times \Gamma(a; \mathfrak{F}) \to \Gamma(a; \mathfrak{F})$ defines a $\Gamma(a; \mathfrak{O})$-module structure on $\Gamma(a; \mathfrak{F})$.

A morphism

$$\mathfrak{F} \to \mathfrak{G}$$

is a morphism of presheaf modules if

In general, if $\mathcal{F}, \mathcal{G}$ are presheaves on $\mathcal{C}$ we mean $\Gamma(\mathcal{F}, \mathcal{G}) = \mathrm{Hom}(\mathcal{F}, \mathcal{G})$, so maybe you should just use Hom?

* You have, for any presheaf P on Y_a, a morphism $P \to P^{sh}$. Restrict this to a morphism

$P|_b \to P^{sh}|_b$ of presheaves on b.

The latter is a sheaf, so you get a morphism

$(P|_b)^{sh} \to (P^{sh})|_b$ by the universal property.

This is the map that should be an iso.

$$\begin{tikzcd} \mathfrak{O} \times \mathfrak{F} \arrow[r] \arrow[d] & \mathfrak{F} \arrow[d] \\ \mathfrak{O} \times \mathfrak{G} \arrow[r] & \mathfrak{G} \end{tikzcd}$$

commutes. The category of presheaf modules on Y will be denoted PMod(Y).

Definition 19. Let M, N be an R-module. Let $f : b \to a \in C$. Let $g : M \to N \in$ R-Mod. Define

These are supposed to be free variables below:

$$\lambda : \text{R-Mod} \to \text{PMod}(Y)$$

by

$$\lambda(M)(a) = M \otimes_R \Gamma(a; \mathfrak{O}),$$ "for all $a \in C$"

$$\lambda(M)(f) = \mathrm{Id}_M \otimes \mathfrak{O}(f),$$ "for all $f : b \to a$ in C"

$$\lambda(g) = \underbrace{(a : g \otimes \mathrm{Id})}.$$

I don't know what this means. I would write $\lambda(g)_a = g \otimes \mathrm{Id}_{\mathfrak{O}(a)}$.

Lemma 20. *Let* $Y = (X, \mathcal{T}, \mathfrak{O})$ *be a* ringed *site. The functor* λ *is left adjoint to*

$$\Gamma(1; -) : \underline{PMod}(Y) \to \text{R-Mod}$$

font?

.

Proof. Let a be an object of C. Let M, N be R-modules. Let $\mathfrak{F}, \mathfrak{G} \in$ PMod(Y) be presheaf modules.

Let $\varphi : \lambda(M) \to \mathfrak{G}$ be a morphism of presheaf modules. Let $\phi : M \to \Gamma(1; \mathfrak{G})$ be a morphism of ~~presheaf~~ R-modules. ?

Define

$$\alpha = H_{M,\mathfrak{G}} : \mathrm{Hom}(\lambda(M), \mathfrak{G}) \to \mathrm{Hom}(M, \Gamma(1; \mathfrak{G}))$$

by

$$\alpha(\varphi) = \varphi_1,$$

where φ_1 is the component of φ on the globalsections.

two words. Also, this is really composition with φ: $\mathrm{Hom}(1, \lambda(M)) \to \mathrm{Hom}(1, \mathfrak{G}(M))$

Define

$$\beta = L_{M,\mathfrak{G}} : \mathrm{Hom}(M, \Gamma(1;\mathfrak{G})) \to \mathrm{Hom}(\lambda(M), \mathfrak{G})$$

by

$$\beta(\phi)_a = \phi \otimes_R \Gamma(a;\mathfrak{O}).$$

We will show that β and α are mutually inverse.

Let $d = \beta(\alpha(\varphi))$. Let $m \otimes g \in M \otimes_R \Gamma(a;\mathfrak{O})$. Let $p : \lambda(M)(1) \to \lambda(M)(a)$ be the projection map. Let $q : \mathfrak{G}(1) \to \mathfrak{G}(a)$ be the projection map. Then $d_a(m \otimes g) = \varphi_1(m) \otimes g$ and

$$\begin{aligned}
\varphi_a(m \otimes g) &= g\varphi_a(m \otimes 1) && \text{by } \gamma \text{ linearity} \\
&= g\varphi_a(p(m)) \\
&= gq(\varphi_1(m)) && \text{naturality of } \varphi \\
&= g(\varphi_1(m) \otimes 1) \\
&= \varphi_1(m) \otimes g.
\end{aligned}$$

Hence $d = \varphi$. In words, the natural transformations from presheaves of the from $\lambda(M)$ are unique ly determined by their global sections component.

Let $d = \alpha(\beta(\phi))$. Let $m \in M$. Then $d(m) = (\phi \otimes_R R)(m) = \phi(m)$. Hence $d = \phi$, which makes H and L mutual inverses.

Naturality in M *and* $\mathfrak{G}$

Let $g : N \to M$ and $h : \mathfrak{F} \to \mathfrak{G}$. Let $\rho \in \mathrm{Hom}(\lambda(N), \mathfrak{F})$. Let $k = H_{M,\mathfrak{G}}(h \circ \rho \circ \lambda(f))$. Let $l = h_1 \circ H_{N,\mathfrak{F}}(\rho) \circ f$.

Unfolding the definition for H shows that $k = h_1 \rho_1 f$ and $l = h_1 \rho_1 f$ as well. This proves naturality in M and $\mathfrak{G}$ and the adjunction between λ and $\Gamma(1;-)$. ∎

Definition 21. Define

$$\Lambda : \text{R-Mod} \to \mathrm{Mod}(Y)$$

by $sh \circ \lambda$.

It follows from lemma .. that we have the adjunction $\Lambda \dashv \Gamma(1;-)$.

3 Caffine objects

Definition?

3.1 Restrictive maps between caffine objects

Lemma 22. *Let $(\mathsf{C}, \mathcal{T}, \mathfrak{O})$ be a ringed site. Let a be caffine. Let M be a $\Gamma(\mathsf{a};\mathfrak{O})$-module. The component $\omega^2\lambda(\mathsf{M}), \mathsf{a}$ at a of the sheafification morphism $\omega^2_{\Lambda(\mathsf{M})}$: $\lambda(\mathsf{M}) \to \Lambda(\mathsf{M})$ is equal to the unit of $\Lambda \dashv \Gamma(1;-)$ in C_{a}.*

Proof. Consider the following maps, which you get by repeatedly calling on an adjunction.

$$\mathrm{Id} : \Lambda(\mathsf{M}) \to \Lambda(\mathsf{M})$$

$$\omega^2_{\Lambda(\mathsf{M})} : \lambda(\mathsf{M}) \to \Lambda(\mathsf{M}) \text{ use sheafification adjunction, see lemma ..}$$

$$\omega^2_{\lambda(\mathsf{M}),\mathsf{a}} : \mathsf{M} \to \Gamma(\mathsf{a}; \Lambda(\mathfrak{M})) \text{ take sections at } \mathsf{a}$$

spell out

We took the adjunct of Id wrt the sheafification adjunction as the first step. Then we took the adjunct of the result wrt the λ ajunction. Hence we get the adjunct of Id wrt the Λ adjunction. so the last map is actually the unit of the Λ adjunction. This map is an isomorphism because we assume a to be caffine. ■

Theorem 23 (Morphism between caffines is restrictive). *Let $\mathsf{Y} = (\mathsf{C}, \mathcal{T}, \mathfrak{O})$. Let $\mathsf{f} : \mathsf{b} \to \mathsf{a} \in \mathsf{C}$ be a morphism between caffine objects, then f is restrictive.*

definition?

Proof. Let $\mathfrak{F}$ be a quasi-coherent module on Y_{a}. Let $\mathsf{M} = \Gamma(\mathsf{a}; \mathfrak{F})$. Since a is caffine, we have $\mathfrak{F} = \Lambda(\mathsf{M})$.

We have to show that the adjunct of f

It's the adjunct of $\mathcal{F}(a) \to \mathcal{F}(b)$ along the ordinary ~~tensor adj~~ extension of scalars adjunction.

$$\Gamma(\mathsf{a}; \mathfrak{F}) \otimes_{\Gamma(\mathsf{a};\mathfrak{O})} \Gamma(\mathsf{b}; \mathfrak{O}) \to \Gamma(\mathsf{b}; \mathfrak{F})$$

unlike how we first wrote it down, this may be more readable as

$$\begin{tikzcd}
\Gamma(a;\mathcal{O})\text{-Mod} \arrow[r, "\lambda"] \arrow[d, "\otimes"] & \mathrm{PMod}(Y_a) \arrow[r, "sh"] \arrow[d, "*|_b"] & \mathrm{Mod}(Y_a) \arrow[d, "*|_b"] \arrow[ll, bend right, "\Gamma(a;-)"'] \\
\Gamma(b;\mathcal{O})\text{-Mod} \arrow[r, "\lambda"] & \mathrm{PMod}(Y_b) \arrow[r, "sh"] & \mathrm{Mod}(Y_b) \arrow[ll, bend left, "\Gamma(b;-)"]
\end{tikzcd}$$

not a big deal though

is an isomorphism.

Consider

$$\begin{tikzcd}
\mathrm{PMod}(Y_a) \arrow[r, "sh_a"] \arrow[d, "*|_b"] & \mathrm{Mod}(Y_a) \arrow[d, "*|_b"] & \Gamma(a;\mathfrak{D})\text{-Mod} \arrow[l, "\Lambda"] \arrow[d, "-\otimes\Gamma(b;\mathfrak{D})"] \arrow[ll, bend right, "\lambda"'] \\
\mathrm{PMod}(Y_b) \arrow[r, "sh_a"] & \mathrm{Mod}(Y_b) & \Gamma(b;\mathfrak{D})\text{-Mod} \arrow[l, "\Lambda"] \arrow[ll, bend left, "\lambda"]
\end{tikzcd}$$

By a previous lemma, the left square commutes. By definition the two 'triangles' commute too and the outer square commute, hence the right square also commutes. Therefore $M \otimes \Gamma(b;\mathfrak{D}) \cong \Gamma(b;\mathfrak{F})$. ~~This is the proof you wrote down friday.~~

this "therefore" isn't clear

The requirement is not to find any isomorphism but a specific one. So I think this is not enough and we need to do some bookkeeping and see if the witnessing isomorphism is our map. You should just have to see where elts of the form $m \otimes 1$ go.

Let i be the morphism of presheaves at $\lambda(M)$ of the natural transformation ω^2 coming with sh_a as defined in lemma ?. Let j be the morphism at $\lambda(M \otimes \Gamma(b;\mathfrak{D}))$ of the natural transformation ω^2 coming with sh_a as defined in lemma ? .

This would be easier if you put the data of the map in Lemma 17.

Consider

$$\begin{tikzcd}
\Lambda(M)|_b & \Lambda(M \otimes \Gamma(b;\mathfrak{D})) \arrow[l, "s_{\lambda(M)}"] \\
\lambda(M \otimes \Gamma(b;\mathfrak{D})) \arrow[u, "i|_b"] \arrow[ur, "j"] &
\end{tikzcd}$$

We have seen that the component j_b at b, the global component, is an isomorphism in lemma ?. since b is caffine and that $s_{\lambda(M)}$ is an isomorphism as constructed in lemma ?.

We will prove commutativity of the triangle. Let $g : c \to b \in Y_b$. Let $\mathfrak{M} = \lambda(M \otimes \Gamma(c;\mathfrak{D}))$. Let $x = m \otimes r \in \mathfrak{M}$.