Affine Objects

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1 Restrictive

This section will introduce the notion of a restrictive morphism We will see some examples, non-examples and results in the category of schemes and see that this notion is closely related to affinenes.

For a slightly different treatment see the chapter on quasi-coherent modules in [Vakil].

1.1 Restrictive

Definition 1.1.1 (Restrictive morphism)**.** Let $(C, \mathcal{T}, \mathfrak{O})$. A morphism $f : a \to b \in C$ is called restrictive if for every quasi-coherent module $\mathfrak{G}$ on C_b the morphism

$$\Gamma(b; \mathfrak{G}) \otimes_{\Gamma(b;\mathfrak{O})} \Gamma(a; \mathfrak{O}) \to \Gamma(a; \mathfrak{G})$$

is an isomorphism.

Example 1.1.2. In $\mathsf{Sch}_{\mathsf{Spec}(A)}$ the morphism $\mathrm{Spec}(A_f) \to \mathrm{Spec}(A)$ is restrictive. Let $\mathfrak{G}$ be a quasi-coherent sheaf on $\mathsf{Sch}_{\mathsf{Spec}(A)}$. This implies that $\mathfrak{G} = \Gamma(\widetilde{\mathrm{Spec}(A)}; \mathfrak{G})$. The morphism

$$\Gamma(\mathrm{Spec}(A); \mathfrak{G}) \otimes_{\Gamma(1;\mathfrak{O})} \Gamma(\mathrm{Spec}(A_f); \mathfrak{O}) \to \Gamma(\mathrm{Spec}(A_f); \mathfrak{G}) = \Gamma(A; \mathfrak{G})_f,$$

$$m \otimes r \to rm$$

is an isomorphism by basic commutative algebra.

Example 1.1.3. Let R be a ring. Consider the open immersion $U = \mathrm{Spec}(R[x, y]) \setminus \{(x, y)\} \to \mathrm{Spec}(R[x, y])$ and the quasi-coherent sheaf $\mathfrak{G} = \Lambda(\frac{R[x,y]}{xy})$. The global sections of this sheaf are $\frac{R[x,y]}{xy}$, because for affine schemes the counit is an isomorphism.

Define $U_1 = D(x) \to U$ and $U_2 = D(y) \to U$. Note that these cover U together. We have $\Gamma(U_1;\mathfrak{G}) = 0$ and $\Gamma(U_1;\mathfrak{G}) = 0$, since $\frac{R[x,y]}{xy}_x = 0$ and $\frac{R[x,y]}{xy}_y = 0$. Hence since $\mathfrak{G}$ is a sheaf, we get $\Gamma(U;\mathfrak{G}) = 0$.

The sections over U of $\Lambda(R[x,y])$ are (also) $R[x,y]$. See [Vakil] 4.4.1. We conclude that $\Gamma(1;\mathfrak{G}) \otimes_{\Gamma(1;\mathfrak{O})} \Gamma(U;\mathfrak{O}) \to \Gamma(U;\mathfrak{G})$ is not an isomorphism.

Lemma 1.1.4. *Let $(C, \mathfrak{T}, \mathfrak{O})$ be a ringed site. Let $f : b \to a \in C$ and $g : c \to b \in C$ be morphisms.*

1. *If fg and f are restrictive, then g is.*
2. *if f and g are restrictive, then fg.*

Proof. Let $\mathfrak{G}$ be a quasi-coherent sheaf on C. We have the following identities.

$$\begin{aligned}\mathfrak{G}(fg) \otimes_{\Gamma(a;\mathfrak{O})} \Gamma(c;\mathfrak{O}) &= (\mathfrak{G}(g)(\mathfrak{G}(f) \otimes_{\Gamma(a;\mathfrak{O})} \Gamma(b;\mathfrak{O}))) \otimes_{\Gamma(b;\mathfrak{O})} \Gamma(c;\mathfrak{O}) \\ &= \mathfrak{G}(g) \otimes_{\Gamma(b;\mathfrak{O})} \Gamma(c;\mathfrak{O}) \circ \mathfrak{G}(f) \otimes_{\Gamma(a;\mathfrak{O})} \Gamma(b;\mathfrak{O}) \otimes_{\Gamma(b;\mathfrak{O})} \Gamma(c;\mathfrak{G})\end{aligned}$$

From this both results follow. ∎

Remark 1.1.5. Knowing that fg and g are restrictive only implies that

$$\mathfrak{G}(f) \otimes_{\Gamma(a;\mathfrak{O})} \Gamma(b;\mathfrak{G}) \otimes_{\Gamma(b;\mathfrak{O})} \Gamma(c;\mathfrak{G})$$

is an isomorphism. Only when $\Gamma(c;\mathfrak{G})$ is faithfully flat over $\Gamma(b;\mathfrak{O})$ this would imply that f is restrictive.

Lemma 1.1.6 (Restrictive to affines)**.** *If $f : X \to \mathit{Spec}(R)$ is a restrictive open immersion. then X is affine.*

Proof. Since X is an open in Spec(R), we get a distinguised covering $\bigcup_i D(f_i) = X$ with $f_i \in R$. We will prove that $(f_i) = (1)$ in $S = \Gamma(X;\mathfrak{O})$ and then invoke the result from Ex. 2.1.7 in [H77].

Consider $M = \frac{R}{(f_i)}$ as an R-module and look at $\Lambda(M)$. By restrictiveness we get $M \otimes_R S = \Lambda(M)(S)$ and by $M \otimes_R R_{f_i} = \Lambda(M)(D(f_i)) = M_{f_i} = 0$. Hence $\Lambda(M)(S) = 0$ by the sheaf axiom. This implies that $(f_i) = (1)$ in S. ∎

Lemma 1.1.7. *Any morphism $Spec(S) \xrightarrow{f} Spec(R) \in \mathsf{Sch}_{Spec(R)}$ between affine schemes is restrictive.*

Proof. Let $\mathfrak{G}$ be a quasi-coherent module on $\mathsf{Sch}_{\mathrm{Spec}(R)}$. Set $M = \Gamma(\mathrm{Spec}(R); \mathfrak{G})$ We want to prove that

$$M \otimes_{\Gamma(\mathrm{Spec}(R);\mathfrak{O})} \Gamma(\mathrm{Spec}(S); \mathfrak{O}) \to \Gamma(\mathrm{Spec}(S); \mathfrak{G})$$

is an isomorphism.

Note that $\mathfrak{G} = \Lambda(M)$ since Spec(R) is affine. For the same reason, we get $\mathfrak{G}|_{\mathrm{Spec}(S)} = \Lambda(M \otimes \Gamma(\mathrm{Spec}(S); \mathfrak{O}))$ and ∎

Example 1.1.8 (Affine non-restrictive map)**.** One might expect(or want) that any property of all maps between affine schemes also hold for affine maps between any schemes. This is not the case for restrictiveness.

Consider the canonical inclusions $\mathbb{A}^1 \to \mathbb{P}^1$ and the shifted quasi-coherent module $\mathfrak{O}(-1)$. This module is even locally free of degree 1, this is often called an invertible module.

The global sections of the module $\mathfrak{O}(-1)$ are the elements of degree -1 in the global sections of $\mathfrak{O}$. There are no such elements, hence the global sections are the zero module.

On $\mathbb{A}^1$ all invertible modules are isomorphic to the structure sheaf. See [Vakil] 14.2.8 We conclude that the canonical inclusions cannot be restrictive.

Any inclusion $\mathrm{Spec}(\kappa(\mathfrak{p})) \to \mathbb{P}^1$ of a point is not restrictive which can be shown with the same argument.

This is a (more opaque) way of saying that on projective space not every quasi-coherent sheaf is generated by global sections.

Bibliography

[H77] Robin Hartshorne. *Algebraic Geometry*. Springer-Verlag New York, 1977.

[Vakil] Ravi Vakil. *The Rising Sea, Foundations of Algebraic Geometry*. `http://math.stanford.edu/~vakil/216blog/index.html`. Sept. 2015.