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Affine Objects, Restrictive Morphisms and Quasi-coherent Sheaves

Master thesis

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Voor hooyo.

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1 Introduction

1.1 Motivation

Quasi-coherent sheaves, affine schemes and the inclusion of distinguised opens of an affine scheme have a special relationship. A sheaf of modules $\mathfrak{F}$ on any scheme X is quasi-coherent if and only if for any open affine subset $\mathrm{Spec}(R) \subset X$ and $f \in R$ we have the following coherence property $\Gamma(D(f); \mathfrak{F}) \otimes_{\Gamma(\mathrm{Spec}(R); \mathfrak{O})} \to \Gamma(\mathrm{Spec}(R); \mathfrak{F})$. A quasi-coherent sheaf. One can find out that a quasi-compact scheme is affine by proving that all higher cohomology of any quasi-coherent sheaf vanishes [Stacks, Tag 01XF].

1.2 Aim

This thesis will investigate, formalize and generalize this trinity. We will generalize affineness to a property that any object of any ringed site may have, which gives us a notion of 'affine' object, which we will call an **caffine**, shorthand for categorically affine, object. This definition will use the functor Λ which is a generalized version of the well-known functor $\widetilde{-}$ that sends modules of the global sections to a quasi-coherent module. We will define a property of morphisms in a ringed site, to replace the distinguised inclusions in the generalized trinity. Both definitions are defined purely in terms of quasi-coherent sheaves.

Our main results are the following. Any morphism between caffine objects will turn out to be restrictive. If a ringed site has 'enough caffines', then a sheaf module is quasi-coherent if and only if it is locally of the form $\Lambda(M)$ for some module M.

1.3 Outline

In chapter 2 we will build up all the necessary theory that is well-known, but not assumed to be familiar. We will start with some general categorical results and definitions. Then we will introduce the generalisation of topology for categories, the category of sites and the central notion of a sheaf. The induced topology on an over/slice category will get special attention because we will use it extensively.

In chapter 3 we will introduce restrictive morphisms and show how this notion interacts with affine schemes. It will turn out any morphism between affine schemes is restrictive.

In Chapter 4 we will define caffine objects and prove that any morphism between caffine objects is restrictive. Then we will show that the caffine objects in the category of schemes are exactly the affine schemes. Lastly we prove that a sheaf of modules is quasi-coherent if and only if it is locally of the form $\Lambda(M)$. We will also look at some examples of categories that lack any caffine objects.

In Chapter 5 I will expand on possible avenues of further research.

1.4 Assumptions

The intended audience are my peers, so some mathematical maturity is assumed. I assume familiar with the usual undergraduate curriculum in topology and algebra. Furthermore basic categorical notions such as limits and adjunctions are assumed. See the first 3 chapters of [6] for an introduction to these ideas.

2 Preliminaries

This chapter introduces all the well-known results and notions that are needed for the original material. We will start with a discussion of some purely categorical notions like over categories and presheaves. Secondly, we will introduce a notion of a topology on a category and look at some constructions that are relevant for us. Then we will introduce modules on ringed sites and see how they behave with respect to restricting along a functor. Lastly, the notion of a scheme is introduced and we establish some properties in the big and small site of a scheme.

2.1 Basic Category Theory

Some categorical notions like presheaves and over categories will be introduced in this section. See [1] and [4].

Definition 2.1.1 (Presheaf category)**.** Let $\mathcal{C}$ be a category. We define the category of presheaves on $\mathcal{C}$ as the category of contravariant functors to the category of sets Set. We will denote it by $\hat{\mathcal{C}}$.

Define the functor $h : \mathcal{C} \to \hat{\mathcal{C}}$ as follows: for $a \in \mathcal{C}$,

$$a \mapsto \mathrm{Hom}(-, a),$$

for $f : a' \to a$

$$f \mapsto f \circ -.$$

This functor is fully faithful by the Yoneda lemma.

Notation 2.1.2. Let $I, \mathcal{C}$ be categories. Let $L : I \to \mathcal{C}$ be a functor. The limit over this functor will be denoted by $\lim\limits_{i \in I} L(i)$. The colimit will be denoted by $\operatorname*{colim}\limits_{i \in I} L(i)$.

Definition 2.1.3 (Sections functor)**.** For any $a \in \mathrm{C}$ define the functor

$$\Gamma(a; -) : \hat{\mathrm{C}} \to \mathrm{Set}$$

by

$$\mathfrak{F} \to \mathfrak{F}(a).$$

For any presheaf $\mathfrak{G}$, we define

$$\Gamma(\mathfrak{G}; -) : \hat{\mathrm{C}} \to \mathrm{Set},$$

$$\mathfrak{F} \to \mathrm{Hom}(\mathfrak{G}, \mathfrak{F}).$$

We will mostly use when $\mathfrak{G}$ for the terminal presheaf, which will allow us to compute the global sections.

Definition 2.1.4 (Over/Under categories)**.** Let C and C' be categories. Let $F : \mathrm{C} \to \mathrm{C}'$ and $z \in \mathrm{C}'$. Define the category C_z and C^z as

$$\mathsf{Obj}(\mathrm{C}_{z,F}) := \{(a, w) \mid a \in \mathrm{C}, w : F(a) \to z\},$$

$$\mathrm{Hom}((a, w), (b, v)) := \{f : a \to b \mid v \circ F(f) = w\},$$

and

$$\mathsf{Obj}(\mathrm{C}^{z,F}) := \{(a, w) \mid a \in \mathrm{C}, w : z \to F(a)\},$$

$$\mathrm{Hom}((a, w), (b, v)) := \{f : a \to b \mid F(f) \circ w = v\}.$$

We get faithful functors $\mathrm{C}_z \to \mathrm{C} : (a, w) \to a$ and $\mathrm{C}^z \to \mathrm{C} : (a, w) \to a$. We will denote them both by u.

Definition 2.1.5 (direct image)**.** Let $f : \mathrm{C} \to \mathrm{D}$. Define the direct image functor $f_* : \hat{\mathrm{D}} \to \hat{\mathrm{C}}$ as

$$f_* = - \circ f.$$

When u is the forgetful functor $\mathrm{C}_a \to \mathrm{C}$ then we denote $u_*\mathfrak{F}$ sometimes by $\mathfrak{F}\big|_a$.

Definition 2.1.6 (inverse image of presheaves)**.** Let C, D be a categories. Let $f : \mathrm{C} \to \mathrm{D}$ be a functor. Define the inverse image functor $f^{-1} : \hat{\mathrm{C}} \to \hat{\mathrm{D}}$ as follows. Let $\mathfrak{F} \in \hat{\mathrm{C}}$. For any $d \in \mathrm{D}$

$$f^{-1}(\mathfrak{F})(d) = \operatorname*{colim}_{c \in \mathrm{C}^{d,f}} \Gamma(c; \mathfrak{F}).$$

See [Stacks, Tag 00VC].

Lemma 2.1.7. *The functor* f_* *is left adjoint to* f^{-1}.

Proof. See [Stacks, Tag 00VE]. ∎

Corollary 2.1.8. *The functor* f_* *commutes with arbitrary colimits.*

2.2 Topology

In this section we will define a notion of a topology on a category and look at the related notions of sheaves, sites and restriction of sites.

See [4] for more details.

2.2.1 Basic

Definition 2.2.1 (Sieve)**.** Let C be a category and $a \in \mathrm{C}$. Define the maximal sieve $\max(a)$ on a to be the collection of all morphisms to a. In formula,

$$\max(a) = \{f \in \mathrm{C} \mid \mathsf{Codom}(f) = a\}.$$

A sieve S on a is a subcollection of $\max(a)$ such that $f \circ g \in S$ for any $f \in S$ and any g.

Definition 2.2.2 (Sieve category)**.** Let C be a category and $a \in \mathrm{C}$. The sieve category $\mathrm{Sieves}(a)$ consists of all the sieves on a as objects and inclusions of sieves as morphisms.

Definition 2.2.3 (Pullback of sieve)**.** Let C be a category and $a, b \in \mathrm{C}$. Let S be a sieve on a. Let $f : b \to a$. The sieve f^*S on b is given by $f^*S = \{g \in \max b : fg \in S\}$ for any $c \in \mathrm{C}$. To show that this is actually a sieve on b, let $k : c \to c'$ and $h \in f^*S$. Hence $fh \in S$ and so $fhk \in S$. Conclude that $hk \in f^*S$. This defines a functor $f^* : \mathrm{Sieves}(a) \to \mathrm{Sieves}(b)$.

Definition 2.2.4 (Grothendieck Topology)**.** A Grothendieck topology $\mathcal{T}$ is a family $\mathcal{T}(a)$ of 'covering' sieves for every $a \in C$ with the following conditions:

1. $\max(a) \in \mathcal{T}(a)$
2. $f^*R \in \mathcal{T}(a')$ if $R \in \mathcal{T}(a)$ for all $f : a' \to a$
3. if $f^*R \in \mathcal{T}(a')$ for all $f \in S$ with $S \in \mathcal{T}(a)$ then $R \in \mathcal{T}(a)$

Definition 2.2.5 (Basis)**.** Let C be a category with pullbacks. A Grothendieck pretopology, or basis, $\mathcal{B}$ is a collection $\mathcal{B}(a)$ of 'covering' families $\{f_i : a_i \to a\}$ of morphisms for every $a \in C$ with the following conditions.

1. every isomorphism is a covering singleton family,
2. (Stability) The pullback of a covering family is a covering family. If $\{f_i : a_i \to a\}$ is covering and $g : b \to a$, then $\{f'_i : a_i \times_a b \to b\}$ is covering.
3. (Transitivity) If $\{f_i : a_i \to a\}$ is a covering family and $\{f_{ij} : a_{ij} \to a_i\}$ for every i, then $\{f_{ij} : a_{ij} \to a\}$ is a covering family.

Generating a topology from a basis: take any sieve containing a covering family to be a covering sieve.

Remark 2.2.6. Sometimes you have a set of families S you would like to generate the topology. So one can take the smallest basis and the topology containing these families. This will be called 'the topology generated by S'.

2.2.2 Sites

Definition 2.2.7 (Site)**.** A site $(C, \mathcal{T})$ is a category C with a Grothendieck topology $\mathcal{T}$.

Definition 2.2.8 (continuous functor)**.** Let $G : C \to D$ be a functor between sites. Let $c \in C$ and let R be a covering sieve on c. The functor G is said to preserves covers if $G(R)$ generates a covering sieve. It is enough to check that G sends covering families to covering families, if the topology of the sites is defined by a basis. If G also preserves pullbacks then we call it continuous. See [Stacks, Tag 00WU].

Definition 2.2.9 (cocontinuous functor)**.** The functor G is said to lift covers or be cocontinuous if for every $R \in \mathrm{Cov}G(c)$ there is some $S \in \mathrm{Cov}c$ such that $G(S) \subset R$. See [Stacks, Tag 00XJ].

Definition 2.2.10. The category Sites has as objects sites and the morphisms are continuous functors. When no confusion can arise then we will use C to denote the whole site $(\mathrm{C}, \mathcal{T})$.

Remark 2.2.11. In most resources a morphism of sites is defined to preserve all finite limits. To get the results that we want we only need preservation of pullbacks and we need the forgetful functor $\mathrm{C}_a \to \mathrm{C}$ to be a morphism of sites, hence this slightly weaker notion than usual.

Definition 2.2.12 (Ringed sites)**.** A ringed site $(\mathrm{C}, \mathcal{T}, \mathfrak{O})$ is a site $(\mathrm{C}, \mathcal{T})$ with the sheaf of rings $\mathfrak{O}$ on it. A morphism of ringed sites is a morphism pair $(f, f^{\#}) : (\mathrm{C}, \mathcal{T}, \mathfrak{O}) \to (\mathrm{D}, \mathcal{S}, \mathfrak{U})$ with $f : (\mathrm{C}, \mathcal{T}) \to (\mathrm{D}, \mathcal{S})$ a morphism of sites and $f^{\#} : f_*\mathfrak{U} \to \mathfrak{O}$ a morphism of sheaf of rings. We will denote the pair $(f, f^{\#})$ by just f when no confusion can arise.

Example 2.2.13 (small site)**.** Let X be a topological space. Let the category $\mathrm{Open}(X)$ consist of the opens of X as objects and inclusion as the morphisms. Define a basis on this site consisting of the families $\{U_i \to U\}$ such that $\bigcup_i U_i = U_i$. A continouous map $f : X \to Y$ induces a morphism of sites $\mathrm{Open}(Y) \to \mathrm{Open}(X)$ by sending $U \mapsto f^{-1}(U)$. In this way you can embed the category of topological spaces into the category of sites.

If X is a ringed space we can turn the site $\mathrm{Open}(X)$ into a ringed site by setting the sheaf of rings of X to be the sheaf of rings of $\mathrm{Open}(X)$.

2.2.3 Sheaves

We will introduce the very important notion of a sheaf. See [4] for a more detailed treatment.

Definition 2.2.14 (Matching family)**.** Let C be a category. Let $\mathfrak{F}$ be a presheaf on on C. Let $a \in \mathrm{C}$ be an object. Let R be a sieve on a. A family $\{x_i\}_{i \in R}$ with $x_i \in \Gamma(\mathrm{Dom}(i); \mathfrak{F})$ indexed by a sieve R and such that $x_{g \circ i} = \mathfrak{F}(g)(x_i)$ for any $g : b \to \mathrm{Dom}(i)$ and $b \in \mathrm{C}$ is called a 'matching family'.

Definition 2.2.15 (Amalgamation)**.** An amalgamation of a matching family $\{x_i\}_R$ is an element $x \in \Gamma(a; \mathfrak{F})$ such that $\mathfrak{F}(i)(x) = x_i$.

Definition 2.2.16 (Sheaf)**.** Let $(\mathrm{C}, \mathcal{T})$ be a site. Let $\mathfrak{F} \in \hat{\mathrm{C}}$.

A presheaf that admits a unique amalgamation for every matching family is called a sheaf. The category $\mathrm{Shv}(\mathrm{C})$ is the full subcategory in $\hat{\mathrm{C}}$ of all sheaves.

Definition 2.2.17 (Plus construction)**.** Let $(\mathrm{C}, \mathcal{T})$ be a site. Let $a, a' \in \mathrm{C}$ and $f : a \to a'$. Let $\mathfrak{F} \in \hat{\mathrm{C}}$. Define the functor $(-)^{+} : \hat{\mathrm{C}} \to \hat{\mathrm{C}}$ as follows.

For all $a \in \mathrm{C}$,

$$F^{+}(a) = \frac{\{(R, \varphi) \mid R \in \mathcal{T}(a), \varphi \in \Gamma(R; \mathfrak{F})\}}{\sim},$$

for all morphisms $f \in \mathrm{C}$,

$$F^{+}(f)([(R, \varphi)]) = [(f^{*}R, \varphi h(f))].$$

The equivalence relation is defined as $(R, \varphi) \sim (S, \phi)$ if $\varphi = \phi$ on some covering sieve $Q \subset R \cap S$.

Let $L : \mathfrak{F} \to \mathfrak{F}'$. Then

$$(L^{+})_{a}([(R, \varphi)]) = [(R, L \circ \varphi)]$$

This functor comes with a natural transformation $\omega : \mathrm{Id} \to (-)^{+}$ defined by

$$\omega_{\mathfrak{F},a}(x) = [(\max(a), y]$$

where

$$y(i) = \mathfrak{F}(i)(x).$$

Lemma 2.2.18. *Let $\mathfrak{F}$ be a presheaf, $\mathfrak{G}$ a sheaf and $g : \mathfrak{F} \to \mathfrak{G}$ a morphism in $\hat{\mathrm{C}}$. Then g factors through ω_{F} via a unique g'.*

Proof. See [5, p. 24]. ∎

Lemma 2.2.19. *For every presheaf $\mathfrak{F}$, F^{+} is separated. If $\mathfrak{F}$ is separated, then F^{+} is a sheaf.*

Proof. See [5, p. 24]. ∎

Definition 2.2.20. Define $\mathrm{sh} = (-)^{+} \circ (-)^{+}$. This functor is left adjoint to the inclusion $\hat{\mathrm{C}} \to \mathrm{Shv}(\mathrm{C})$ with unit

$$\omega^{2}_{\mathfrak{F}} = \omega_{\mathfrak{F}^{+}} \circ \omega_{\mathfrak{F}}.$$

Lemma 2.2.21. *Let $f : \mathrm{C} \to \mathrm{D}$ be a continuous functor. Let $\mathfrak{O}$ be a sheaf on D Then $f_{*}\mathfrak{O}$ is a sheaf.*

Proof. See [Stacks, Tag 00WW]. ∎

2.2.4 Relative topology

We will look at what the induced topology on a over category looks like and what this implies for restriction of sheaves. See [Stacks, Tag 03A4] for a more detailed treatement.

Remark 2.2.22. Let C be a category and $a, b \in C$. Let $f : b \to a \in C_a$. The map $\max(f) \to \max(b)$ sending a morphism to f to its underlying morphism in C is a bijection. Moreover composition in C and C_a are the same, so this identification respects pullback of sieves. This observation yields us that $\mathrm{Sieves}(b) = \mathrm{Sieves}(f)$.

Whenever R is a sieve on b, we will denote the corresponding sieve on f by R_f.

Definition 2.2.23 (Relative topology)**.** Let $(C, \mathcal{T})$ be a site. Let $a \in C$.

Define the induced topology $\mathcal{T}_a$ on C_a by, for each $f \in C_a$

$$\mathcal{T}_a(f) = \mathcal{T}(\mathrm{Dom}(f)).$$

The identification from Remark 2.2.22 implies that $\mathcal{T}_a$ is a Grothendieck topology.

Definition 2.2.24 (Oversite)**.** Let $Y = (C, \mathcal{T})$ be a site. Let $a \in C$. Define the site Y_a to be the category C_a with the induced topology $\mathcal{T}_a$. We will denote it by just C_a.

Lemma 2.2.25. *The functor* $u : C_a \to C$ *is a morphism of sites that is cover lifting.*

Proof. Clearly u preserves pullbacks. By Remark 2.2.22 it is also immediate that u preserves covers and lifts covers. ∎

Definition 2.2.26 (Over ringed site)**.** Let $Y = (C, \mathcal{T}, \mathfrak{O})$ be a ringed site. Let $a \in C$. Define the ringed site Y_a to be the site C_a with the structure sheaf $u_* \mathfrak{O}$ where $u : C_a \to C$ is the forgetful functor. This makes sense because u is continuous, see Lemma 2.2.21.

Whenever u is used as a morphism between ringed sites it stands for the morphism of ringed sites (u, Id) as defined in Definition 2.2.12.

Definition 2.2.27. Let $(C, \mathcal{T}, \mathfrak{O})$ be a ringed site. Let $a, b \in C$ and $f : a \to b$.

Consider the induced functor $C_a \xrightarrow{F} C_b$. Let c be the pullback of some pair $d \to g \leftarrow e$. Then $F(c)$ is the pullback of the pair $F(d) \to F(g) \leftarrow F(e)$, so F preserves pullbacks.

Let $c \to b \in C_b$. Let R be a covering sieve on c. By Lemma 2.5.9 and the fact that $u \circ F = u$, we get that $F(R)$ generates a covering sieve, so F is continuous. With the same reasoning one can deduce that F lifts covers.

Let $\mathfrak{O}$ be the structure sheaf of C_a and $\mathfrak{U}$ be the structure sheaf of C_b. Note that $\Gamma(c \xrightarrow{g} b \xrightarrow{f} a; \mathfrak{O}) = \Gamma(c; \mathfrak{O}) = \Gamma(g; \mathfrak{U})$. Define the morphism of ringed sites $(F, \mathrm{Id}) : (C_a, \mathcal{T}_{\dashv}, \mathfrak{O}) \to (C_b, \mathcal{T}_{\lfloor}, \mathfrak{O})$, which we will denote by F or just f.

2.3 Modules

Presheaf modules and sheaf modules on a ringed site will be introduced in this section. Next we will introduce two main functors λ and Λ. The functors λ and Λ introduced here will be used extensively. See [Stacks, Tag 03A4] for more detail.

Definition 2.3.1 (Presheaf modules). Let $(C, \mathcal{T}, \mathfrak{O})$ be a ringed site.

A presheaf module on this ringed site is a presheaf of sets $\mathfrak{F}$ on C together with a map of presheaves

$$\mathfrak{O} \times \mathfrak{F} \to \mathfrak{F}$$

such that for every object $a \in C$ the map $\Gamma(a; \mathfrak{O}) \times \Gamma(a; \mathfrak{F}) \to \Gamma(a; \mathfrak{F})$ defines a $\Gamma(a; \mathfrak{O})$-module structure on $\Gamma(a; \mathfrak{F})$.

A morphism

$$\mathfrak{F} \to \mathfrak{G}$$

is a morphism of presheaf modules if

$$\begin{array}{ccc} \mathfrak{O} \times \mathfrak{F} & \longrightarrow & \mathfrak{F} \\ \downarrow & & \downarrow \\ \mathfrak{O} \times \mathfrak{G} & \longrightarrow & \mathfrak{G} \end{array}$$

commutes. The category of presheaf modules on C will be denoted $\mathrm{PMod}(\mathfrak{O})$. The full subcategory of sheaves in $\mathrm{PMod}(\mathfrak{O})$ will be denoted $\mathrm{Mod}(\mathfrak{O})$.

Definition 2.3.2. Let $\mathfrak{F}$ be a sheaf of modules on $(\mathrm{C}, \mathcal{T}, \mathfrak{O})$. It is called quasi-coherent if the following holds. For any object $a \in \mathrm{C}$ there exists a covering sieve S such that for any map $f : b \to a$ in S there exists a (global) presentation

$$\mathfrak{O}\big|_b^{\oplus I} \to \mathfrak{O}\big|_b^{\oplus J} \to \mathfrak{F}\big|_b \to 0$$

It is enough to have presentation for a generating set of S. Some also call this property that $\mathfrak{F}$ is locally presentable.

Quasi-coherent modules form a full subcategory of the category of sheaves of modules over $(\mathrm{C}, \mathcal{T}, \mathfrak{O})$ which are denoted by $\mathrm{Qcoh}(\mathfrak{O})$.

Lemma 2.3.3. *Let $f : \mathrm{C} \to \mathrm{D}$ be a morphism of ringed sites. Let $\mathfrak{F}$ be a quasi-coherent sheaf of modules on D. Then $f_*\mathfrak{F}$ is quasi-coherent.*

Proof. One can view f as a morphism of ringed topoi $(f_*, f^{-1}, f^\#)$. See [Stacks, Tag 082T]. ∎

Definition 2.3.4. Let $(\mathrm{C}, \mathcal{T}, \mathfrak{O})$ be a ringed site. Let $R = \Gamma(1; \mathfrak{O})$. Let M, N be an R-module.

Define

$$\lambda : R\text{-}\mathrm{Mod} \to \mathrm{PMod}(\mathfrak{O})$$

by for all $a \in \mathrm{C}$,

$$\lambda(M)(a) = M \otimes_R \Gamma(a; \mathfrak{O}),$$

for all $f : b \to a \in \mathrm{C}$,

$$\lambda(M)(f) = \mathrm{Id} \otimes \mathfrak{O}(f),$$

for all $g : M \to N \in R\text{-}\mathrm{Mod}$,

$$\lambda(g) = (a : g \otimes \mathrm{Id}).$$

Lemma 2.3.5. *Let $(\mathrm{C}, \mathcal{T}, \mathfrak{O})$ be a ringed site. Let $R = \Gamma(1; \mathfrak{O})$. The functor λ is left adjoint to*

$$\Gamma(1; -) : \mathrm{PMod}(\mathfrak{O}) \to R\text{-}\mathrm{Mod}.$$

Proof. Let M, N be R-modules. Let $\mathfrak{F}, \mathfrak{G} \in \mathrm{PMod}(\mathfrak{O})$ be presheaf modules.

Let $\varphi : \lambda(M) \to \mathfrak{G}$ be a morphism of presheaf modules. Let $\phi : M \to \Gamma(1; \mathfrak{G})$ be a morphism of R-modules.

Define

$$\begin{aligned}\alpha : &\mathrm{Hom}(\lambda(M), \mathfrak{G}) \to \mathrm{Hom}(M, \Gamma(1; \mathfrak{G}))\\ :&\varphi \mapsto \varphi_1\end{aligned}$$

where φ_1 is the component of φ on the global sections.

Define

$$\beta : \mathrm{Hom}(M, \Gamma(1; \mathfrak{G})) \to \mathrm{Hom}(\lambda(M), \mathfrak{G})$$

by, for each $a \in C$

$$\beta(\phi)_a = \phi \otimes_R \Gamma(a; \mathfrak{O}).$$

We will show that β and α are mutually inverse.

Let $\varphi \in \mathrm{Hom}(\lambda(M), \mathfrak{G})$ and $a \in C$. Let $d = \beta(\alpha(\varphi))$. Let $m \otimes g \in M \otimes_R \Gamma(a; \mathfrak{O})$. Let $p : \lambda(M)(1) \to \lambda(M)(a)$ be the projection map. Let $q : \mathfrak{G}(1) \to \mathfrak{G}(a)$ be the projection map. Then $d_a(m \otimes g) = \varphi_1(m) \otimes g$ and

$$\begin{aligned}\varphi_a(m \otimes g) &= g\varphi_a(m \otimes 1) \text{ by linearity}\\ &= g\varphi_a(p(m))\\ &= gq(\varphi_1(m)) \text{ by naturality of } \varphi\\ &= g(\varphi_1(m) \otimes 1)\\ &= \varphi_1(m) \otimes g.\end{aligned}$$

Hence $d = \varphi$. In words, the natural transformations from presheaves of the from $\lambda(M)$ are uniquely determined by their global sections component.

Let $d = \alpha(\beta(\phi))$. Let $m \in M$. Then $d(m) = (\phi \otimes_R R)(m) = \phi(m)$. Hence $d = \phi$, which makes α and β mutual inverses. Naturality is straightforward to check and is omitted. ∎

Definition 2.3.6. Let $(\mathrm{C}, \mathcal{T}, \mathfrak{O})$ be a ringed site. Define

$$\Lambda : \mathsf{R}\text{-}\mathrm{Mod} \to \mathrm{Mod}(\mathfrak{O})$$

by $\mathsf{sh} \circ \lambda$.

It follows that we have the adjunction $\Lambda \dashv \Gamma(1; -)$.

This functor is the generalisation of [Stacks, Tag 01BH] to general sites, so Λ on $\mathrm{Open}(X)$ for some ringed space X will coincide with the construction defined in the Stacks Project.

2.4 Restricting and module functors

Lemma 2.4.1. *Let* $f : (\mathrm{D}, \mathcal{S}, \mathfrak{U}) \to (\mathrm{C}, \mathcal{T}, \mathfrak{O})$ *be a morphism of ringed sites. Suppose* $f^{\#} : f_* \mathfrak{O} \to \mathfrak{U}$ *is an isomorphism. We get a natural isomorphism* $f_* \circ \lambda \Rightarrow \lambda \circ (- \otimes_{\Gamma(1;\mathfrak{O})} \Gamma(1; \mathfrak{U}))$.

Proof. Define the natural transformation $t : \lambda \circ (- \otimes_{\Gamma(1;\mathfrak{O})} \Gamma(1;\mathfrak{U})) \Rightarrow u_* \circ \lambda$, by for each $\Gamma(1;\mathfrak{O})$-module M and for each $a \in \mathrm{D}$,

$$t_{M,a} : M \otimes_{\Gamma(1;\mathfrak{O})} \Gamma(1;\mathfrak{U}) \otimes_{\Gamma(1;\mathfrak{U})} \Gamma(a;\mathfrak{U}) \to M \otimes_{\Gamma(1;\mathfrak{O})} \Gamma(a;\mathfrak{O}),$$

$$m \otimes r \otimes s \mapsto m \otimes r f^{\#,-1}(s).$$

Every component $t_{M,f}$ is an isomorphism by basic commutative algebra and the fact that $f^{\#}$ is an isomorphism ∎

Lemma 2.4.2. *Let* $F : (\mathrm{C}, \mathcal{T}) \to (\mathrm{D}, \mathcal{S})$ *be a morphism of sites that lift covers. Then* $F_* \circ (-)^+ = (-)^+ \circ F_*$ *and hence also* $F_* \circ \mathsf{sh} = \mathsf{sh} \circ F_*$.

Proof. Let $c \in \mathrm{C}$. Let $R \in \mathrm{Cov}c$. Let $\mathfrak{G}$ be a presheaf on D. Let $s = \{s_i \mid i \in R\}$ be a matching family in $F_*\mathfrak{G}$ on c indexed by R. So $s_i \in \Gamma(F(\mathrm{Dom}(i)); \mathfrak{G})$. By the cover preserving assumption we know $F(R)$ generates a covering sieve S on $F(c)$. Define the natural transformation $T : F_* \circ (-)^+ \to (-)^+ \circ F_*$ by

$$s \mapsto \{t_j \mid j \in S\}$$

where $t_j = F(h)s_i$ for some factorisation $j = F(i)h$. Such a factorisation always exists since S is generated by $F(R)$. This assignment is well-defined since F preserves pullbacks, see proof of [3, Lemma 2.3.3].

We will prove that T is injective. Let $s' = \{s'_i \mid i \in R'\}$. Assume $T(s) = T(s')$ then $\{t_j \mid j \in S\} = \{t'_{j'} \mid j' \in S'\}$, hence there exists covering sieve $Q \subset S \cap S'$ such that $t_k = t'_k$ for every $k \in Q$. Since F lifts covers, there exists covering sieve $P \subset R \cap R'$ on c such that $F(P)$ generates a sieve $P' \subset Q$. Hence $t_k = F(h)s_k = F(h)s'_k = t'_k$ with $k \in P'$ and $k = F(k)h$. Take $h = \mathrm{Id}$ to conclude that $s = s'$ on P, hence they are in the same equivalence class.

Now comes surjectivity. Let $t = \{t_j \mid j \in S\} \in ..$ on $F(c)$. We get a covering sieve R such that $F(R) \subset S$. Consider this matching family now on the covering sieve generated by $F(R)$, which amounts to taking a different representing element from the equivalence class. So now $F(R)$ generates S. For any $j \in S$ we get a factorisation $j = F(i)h$ for some $i \in R$. We want to have $s_i = t_{F(i)}$. This produces a matching family because restriction in $F_*\mathfrak{G}$ is exactly the same as in $\mathfrak{G}$. Note that now $T(s) = t$. Hence we got surjectivity. ∎

Corollary 2.4.3. *We have* $F_*\omega = \omega$ *and hence also* $F_*\omega^2 = \omega^2$

Lemma 2.4.4 (Λ commutes with restriction)**.** *Let* $(C, \mathcal{T}, \mathfrak{O})$ *be a ringed site. Let* $f : (C, \mathcal{T}) \to (D, \mathcal{S})$ *be a morphism of sites that lift covers. Let* $f^\#$ *be an isomorphism.*

We have a natural isomorphism

$$f_* \circ \Lambda \Rightarrow \Lambda \circ (- \otimes_{\Gamma(1;\mathfrak{O})} \Gamma(1;\mathfrak{U})).$$

Proof. Follows from Lemmas 2.4.1 and 2.4.2. ∎

Lemma 2.4.5. *Let* $f : (D, \mathcal{S}, \mathfrak{U}) \to (C, \mathcal{T}, \mathfrak{O})$ *be a morphism of ringed sites that is cover lifting. Suppose* $f^\#$ *be an isomorphism. Let* $\mathfrak{F}$ *be a quasi-coherent module on* C*. Let* $M = \Gamma(1;\mathfrak{F})$*. Consider* $\epsilon : \Lambda(M) \to \mathfrak{F}$*. Then* $f_*\epsilon : \Lambda(M \otimes_{\Gamma(1;\mathfrak{O})} \Gamma(1;\mathfrak{U}))) \to f_*\mathfrak{F}$ *is the counit of the adjunction* $\Lambda \dashv \Gamma(1;-)$ *on* D*.*

Proof. Let $a \in D$. We will show that $f_*\epsilon$ corresponds to the same morphism $\lambda(M \otimes_{\Gamma(1;\mathfrak{O})}) \to f_*\mathfrak{F}$ as the counit. By the universal property of the sheafification, this implies that $f_*\epsilon$ is the counit.

We have $(f_*\epsilon)_a(\omega^2(m \otimes r)) = rm$, which is the same as for the counit. ■

Example 2.4.6. Let $(\mathcal{C}, \mathcal{T}, \mathfrak{O})$ be a ringed site. Let $a \in \mathcal{C}$. Consider the morphism of ringed sites $u : \mathcal{C}_a \to \mathcal{C}$. Note that $u^\# = \mathrm{Id}$ and by Lemma 2.2.25 it lifts covers. So all the results from this section apply. This is the most important case.

2.5 Schemes

We will recap some definitions in scheme theory that we use later. See [7, 2] for thorough treatments of scheme theory.

Definition 2.5.1 (Spectrum of a ring). Let R be a ring. The spectrum $\operatorname{Spec} R$ of R is the locally ringed space defined as follows. The underlying set is the set of prime ideals of R. The (Zariski) topology is generated by the basis of distinguised opens $D(f) = \{\mathfrak{p} \subset R | f \notin \mathfrak{p}\}$. The sheaf of rings is given on this basis by

$$D(f) \mapsto R_f.$$

A distinguised open $D(f)$ of $\operatorname{Spec}(R)$ viewed as locally ringed space is isomorphic to $\operatorname{Spec}(R_f)$, where the inclusion $\operatorname{Spec}(R_f) \to \operatorname{Spec}(R)$ corresponds to the canonical map $R \to R_f$.

The functor

$$\operatorname{Spec} : \mathrm{Rng} \to \mathrm{LRSpaces}$$

is left adjoint to

$$\Gamma(1; -) : \mathrm{LRSpaces} \to \mathrm{Rng},$$

see [Stacks, Tag 01I1].

Definition 2.5.2 (Scheme). We call the locally ringed space $\operatorname{Spec}(R)$ an affine scheme.

A scheme S is a locally ringed space that admits a covering of affine schemes. A morphism of schemes is a morphism of locally ringed spaces. The category of schemes we will denote by Sch.

Definition 2.5.3 (Sheaf of algebras). A sheaf of algebras $\mathfrak{F}$ on a ringed site $(\mathcal{C}, \mathcal{T}, \mathfrak{O})$ is a sheaf of rings that comes with a (structure) morphism of sheaves of rings $\mathfrak{O} \to \mathfrak{F}$.

Definition 2.5.4 (Relative spec)**.** Let X be a scheme. Let $\mathfrak{S}$ be a sheaf of algebras on X that is quasi-coherent as a sheaf of modules.

Define the relative spectrum of $\mathfrak{S}$ over X to be the scheme

$$\text{Rspec } \mathfrak{S} \to X$$

that you get by glueing the spectra $\mathrm{Spec}(\Gamma(V; \mathfrak{S})) \to V \subset X$ for every affine open V. See [Stacks, Tag 01LW].

Definition 2.5.5 (Tilde functor)**.** Let $\mathrm{Spec}(R)$ be an affine scheme. Let M be a R-module. Define $\widetilde{M}$ to be the unique sheaf on $\mathrm{Spec}(R)$ with

$$\widetilde{M} : D(f) \mapsto M_f.$$

See [Stacks, Tag 01HR].

Remark 2.5.6. This functor(with this notation) is commonly used in algebraic geometry texts. By [Stacks, Tag 01I7] and uniqueness of left adjoints Λ and $\widetilde{-}$ are canonically isomorphic on spaces, hence on small Zariski sites. We will only use Λ in subsequent sections.

Definition 2.5.7. A morphism $f : X \to Y$ of schemes is called affine if for any affine open $U \subset Y$the open subscheme $f^{-1}(U)$ is affine.

Definition 2.5.8. Let X be a scheme. Let $f \in \Gamma(X; \mathfrak{O})$. Define X_f to be the pullback of $D(f) \to \mathrm{Spec}(\Gamma(X; \mathfrak{O})) \leftarrow X$.

Lemma 2.5.9. *Let* X *be a scheme. If* X_{f_i} *is affine for all* i *and* $(f_i)_{i \in I} = (1)$ *then* X *is affine.*

Proof. Let $\bigcup_i D(f_i) = \mathrm{Spec}(\Gamma(X; \mathfrak{O}))$. Consider the canonical $\epsilon : X \to \mathrm{Spec}(\Gamma(X; \mathfrak{O}))$. By definition $\epsilon^{-1}(D(f_i)) = X_{f_i}$. By assumption X_{f_i} is affine. By [Stacks, Tag 01S8]] the morphism ϵ is affine, hence X is affine. ∎

2.6 Small Zariski site

In this section we will work on the small Zariski site corresponding to a scheme. We will prove that the unit η and ϵ of the adjunction $\Lambda \dashv \Gamma(1; -)$ are isomorphisms in this case.

Lemma 2.6.1. *Let* R *be a ring. Let* M *be a* R*-module. Consider* $\lambda(M), \Lambda(M)$ *as sheaf modules on* $\mathrm{Open}(\mathrm{Spec}(R))$*. Then*

$$\omega^2_{\lambda(M),\mathrm{Spec}(R)} : \Gamma(\mathrm{Spec}(R); \lambda(M)) \to \Gamma(\mathrm{Spec}(R); \Lambda(M))$$

is an isomorphism.

Proof. Let $\mathfrak{p} \subset R$ be a prime ideal. As stated in Definition 2.3.6, we may use results from [Stacks, Tag 01BH] in this setting. We will use that $\Lambda(M)_\mathfrak{p} = M \otimes_R R_\mathfrak{p}$. By naturality, localized at $\mathfrak{p}$, the map ω^2 sends m to $m \otimes 1 \in M \otimes_R R_\mathfrak{p}$, hence is the inverse of the multiplication map which is an isomorphism. Hence globally ω^2 is an isomorphism. ■

Corollary 2.6.2. *Let* X *be a scheme. Consider* $\lambda(M), \Lambda(M)$ *as sheaf modules on* $\mathrm{Open}(X)$*. Let* $\mathrm{Spec}(R)$ *be an open subset of* X*. Let* M *be a* $\Gamma(X; \mathfrak{O})$*-module. Then* $\omega^2_{\lambda(M),\mathrm{Spec}(R)} : \Gamma(\mathrm{Spec}(R); \lambda(M)) \to \Gamma(\mathrm{Spec}(R); \Lambda(M))$ *is an isomorphism.*

Remark 2.6.3. Let us be in the situation of Definition 2.2.15. Note that η_M is the composition of the unit of $\lambda \dashv \Gamma(1; -)$ and $sh \dashv i$. The first unit is the identity and the second is $\omega^2_{\lambda(M),\mathrm{Spec}(R)}$.

Corollary 2.6.4. *Consider the adjunction* $\Lambda(-) \dashv \Gamma(1; -)$ *on* $\mathrm{Open}(\mathrm{Spec}(R))$*. The unit* $\eta_M : M \to \Gamma(\mathrm{Spec}(R); \Lambda(M))$ *is an isomorphism.*

Lemma 2.6.5. *Let* $\mathfrak{F}$ *be a sheaf of modules on scheme* X*.* $\mathfrak{F}$ *is quasi-coherent on* X *if and only if for any open* $\mathrm{Spec}(R) \subset X$ *the sheaf* $\mathfrak{F}|_{\mathrm{Spec}(R)}$ *is isomorphic to* $\Lambda(M)$ *with* $M = \Gamma(\mathrm{Spec}(R); \mathfrak{F})$*.*

Proof. $\Rightarrow$: By assumption we get local presentations indexed by a covering. Let $\bigcup_{i \in I} U_i = X$ be this covering. Assume without loss of generality that it is an affine open covering. Let $U_i = \mathrm{Spec}(R_i)$. Let $\mathfrak{O}_{U_i}^{\bigoplus K} \to \mathfrak{O}_{U_i}^{\bigoplus J} \to \mathfrak{F}|_{U_i} \to 0$ be one of the given presentations. Taking global sections gives us an exact sequence

$$R_i^{\bigoplus K} \to R_i^{\bigoplus J} \to \Gamma(U_i; \mathfrak{F}) \to 0.$$

Tensoring it with the localisation $R_{i,f}$ for any $f \in R_i$ yields

$$R_{i,f}^{\bigoplus K} \to R_{i,f}^{\bigoplus J} \to \Gamma(U_i; \mathfrak{F}) \otimes R_{i,f} \to 0.$$

Taking sections at $D(f)$ from the sheaf sequence yields

$$R_{i,f}^{\bigoplus K} \to R_{i,f}^{\bigoplus J} \to \Gamma(D(f);\mathfrak{F}) \to 0.$$

Hence $\mathfrak{F}|_{U_i}$ is the unique sheaf with $D(f) \mapsto \Gamma(U_i;\mathfrak{F})_f$, which we defined to be $\Lambda(\Gamma(U_i;\mathfrak{F}))$. By the affine communication lemma, this property holds for any affine and not just for the affines in this covering.

$\Leftarrow$: Let $M = \Gamma(X;\mathfrak{F})$. Take a presentation $R^{\bigoplus I} \to R^{\bigoplus J} \to M$ and apply $\Lambda(-)$. Then note that $\Lambda(-)$ commutes with arbitrary colimits since it is a left adjoint, see lemma ?. We have $\Lambda(R) = \mathfrak{O}_{\mathrm{Spec}(R)}$ so we get a presentation $\mathfrak{O}_{\mathrm{Spec}(R)}^{\bigoplus I} \to \mathfrak{O}_{\mathrm{Spec}(R)}^{\bigoplus J} \to \mathfrak{F}$ on every affine open subset $\mathrm{Spec}(R) \subset X$, hence $\mathfrak{F}$ is quasi-coherent. ■

Corollary 2.6.6. *Consider the adjunction $\Lambda(-) \dashv \Gamma(1;-)$. For quasi-coherent sheaf $\mathfrak{F}$ the counit $\epsilon_{\mathfrak{F}} : \Lambda(M) \to \mathfrak{F}$ is an isomorphism.*

2.7 Big Zariski Site

In this section we will introduce the big Zariski ringed site and look at how quasi-coherence and Λ behave on this site. We will prove that the category of quasi-coherent sheaves on an affine scheme $\mathrm{Spec}(R) \in \mathrm{Sch}$ is equivalent to the category of R-modules.

Definition 2.7.1 (Big Zariski site)**.** Define the big Zariski site to be $(\mathrm{Sch}, \mathcal{T}, \mathfrak{O})$ with the following components. The underlying category is Sch. The topology T is generated by the basis cosisting of the covering families $\{X_i \xrightarrow{f_i} X\}$ where f_i is an open immersion and $\bigcup_i f_i(X_i) = X$. The sheaf of rings $\mathfrak{O}$ sends $(U, \mathfrak{Q}) \to (X, \mathfrak{O})$ to $\Gamma(U;\mathfrak{Q})$.

We will mostly be interested in the site Sch_X for a scheme X.

Definition 2.7.2 (k)**.** Let X be a scheme. Define the functor $k : \mathrm{Open}(X) \to \mathrm{Sch}_X$ by $U \mapsto ((U, \mathfrak{O}_U), i)$ where $i : U \to X$ is the inclusion of the open subscheme into X.

We will show that it preserves limits and covers. The terminal $X \in \mathrm{Open}(X)$ is sent to the terminal $X \to X$. Let $U \to V$ and $W \to V$ be two morphism in $\mathrm{Open}(X)$. We have $k(U \cap W) = U \cap W \to X$ which is the pullback of $k(U) \to k(V)$ and $k(W) \to k(V)$.

Let $S = \{D(f_i) \to \mathrm{Spec}(R)\}$ be one of the generating family in $\mathrm{Open}(X)$. Note that $k(D(f_i))$ is isomorphic to the object $\mathrm{Spec}(R_{f_i}) \to \mathrm{Spec}(R)$. Hence $k(S)$ generates a covering sieve on Sch and hence on Sch_X. We established that k is a morphism of sites.

Consider $(\mathrm{Open}(X), \mathcal{T}, \mathfrak{O})$ and $(\mathrm{Sch}_X, \mathcal{S}, \mathfrak{U})$. Note that $k_*\mathfrak{U} = \mathfrak{O}$ by construction. Define the morphism of ringed sites $(k, \mathrm{Id}) : (\mathrm{Open}(X), \mathcal{T}, \mathfrak{O}) \to (\mathrm{Sch}_X, \mathcal{S}, \mathfrak{U})$. We will denote this by k.

Lemma 2.7.3. *Let X be a scheme. The morphism of sites $k : \mathrm{Open}(X) \to \mathrm{Sch}_X$ is cover lifting.*

Proof. Let $U \in \mathrm{Open}(X)$. Let $W = \{U_i \xrightarrow{f_i} k(U)\}$ be a covering family on $k(U)$. Then $V = \{f_i(U_i) \to U\}$ is a covering family on U such that $k(V)$ generates the same sieve as W. Hence k is cover lifting. ■

Lemma 2.7.4. *Let X be a scheme. Consider $k : \mathrm{Open}(X) \to \mathrm{Sch}_X$. Then $k_*\Lambda = \Lambda$.*

Proof. By definition $k^\#$ is an isomorphism. By Lemma 2.7.3 we get that k is cover lifting. Then we can apply Lemma 2.4.4. Let $\mathfrak{U}$ be the structure sheaf of Sch_X. Let $\mathfrak{O}$ be the structure sheaf of $\mathrm{Open}(X)$. Note that the structure morphism $\Gamma(1; k_*\mathfrak{U}) \to \Gamma(1; \mathfrak{O})$ is an isomorphism, so Lemma 2.4.4 implies this result. ■

Lemma 2.7.5. *Let $X = \mathrm{Spec}(R)$ be a scheme. Let $\mathfrak{F}$ be a quasi-coherent sheaf on Sch_X. Let $M = \Gamma(X; \mathfrak{F})$. Then $\epsilon_{\mathfrak{F}} : \Lambda(M) \to \mathfrak{F}$ is an isomorphism.*

Proof. By Lemma 2.3.3, k_* sends quasi-coherent sheaves to quasi-coherent sheaves. so $k_*\mathfrak{F} = \Lambda(M)$, where $k : \mathrm{Open}(X) \to \mathrm{Sch}_X$. Hence $\Gamma(D(f); \mathfrak{F}) = \Gamma(D(f); \Lambda\mathfrak{M}) = M_f$ for any $f \in R$.

Let $D(f_i)|_{\mathfrak{O}}^{\bigoplus J} \xrightarrow{\alpha_i} D(f_i)|_{\mathfrak{O}}^{\bigoplus K} \to D(f_i)|_{\mathfrak{F}} \to 0$ be a presentation with $(f_i) = (1)$. Note that the presheaf cokernel $\mathrm{Coker}(\alpha_i)$ of the sheaf morphism is $\lambda(\mathrm{Coker}(\alpha_{i,D(f_i)}))$ where $\alpha_{i,D(f_i)}$ is the component at $D(f_i)$ of α_i. So $D(f_i)|_{\mathfrak{F}} = \Lambda(M_f)$ since ω^2 is iso for affines.

Hence

$$\begin{aligned}
\Gamma(D(f) \times_X \mathrm{Spec}(S); \mathfrak{F}) &= \Gamma(D(f) \times_X \mathrm{Spec}(S); \Lambda\mathfrak{F}) \\
&= \Gamma(\mathrm{Spec}(S_f); \Lambda\mathfrak{M}_f) \\
&= M_f \otimes S_f \\
&= M \otimes S_f.
\end{aligned}$$

By the sheaf property it follows that $\Gamma(\mathrm{Spec}(S); \mathfrak{F}) = \Gamma(X; \mathfrak{F}) \otimes S$, hence the counit $\Lambda(M) \to \mathfrak{F}$ is an isomorphism. ■

Lemma 2.7.6. *Let* $X = \mathrm{Spec}(R)$ *be a scheme. Let* M *be a* R*-module. We work on the ringed site* Sch_X*. The unit* $\eta : M \to \Gamma(X; \Lambda(M))$ *is an isomorphism*

Proof. Let $\Lambda_a(M)$ be $\Lambda(M)$ over $\mathrm{Open}(X)$. By Lemma 2.7.4, we have $\Lambda_a(M) = k_* \Lambda(M)$. Hence $\Gamma(X; \Lambda(M)) = \Gamma(X; \Lambda_a(M))$. Moreover $\Gamma(X; \Lambda(M)) = M$ by Corollary 2.6.4. ∎

3 Restrictive morphisms

This section will introduce the notion of a restrictive morphism. We will see some examples, non-examples and results in the category of schemes and see that this notion is closely related to affinenes. We will always work over small sites in this section. For some of the examples and results see the chapter on quasi-coherent modules in [7].

Definition 3.0.1 (Restrictive morphism)**.** Let $(\mathrm{C}, \mathcal{T}, \mathfrak{O})$. A morphism $f : a \to b \in \mathrm{C}$ is called restrictive if for every quasi-coherent module $\mathfrak{G}$ on C_b the morphism

$$\widehat{f} : \Gamma(b; \mathfrak{G}) \otimes_{\Gamma(b;\mathfrak{O})} \Gamma(a; \mathfrak{O}) \to \Gamma(a; \mathfrak{G}) \tag{3.1}$$

is an isomorphism.

Remark 3.0.2. Let us be in the setting of Definition 3.0.1. Assume $\mathfrak{G} = \Lambda(M)$ where $M = \Gamma(1; \mathfrak{G})$. Consider $\omega^2_{\mathfrak{G},a} : \lambda(M)(a) \to \Lambda(M)(a)$. This is map is equal to $\widehat{f}$.

Example 3.0.3. In $\mathrm{Sch}_{\mathrm{Spec}(A)}$ the morphism $\mathrm{Spec}(A_f) \to \mathrm{Spec}(A)$ is restrictive. Let $\mathfrak{G}$ be a quasi-coherent sheaf on $\mathrm{Sch}_{\mathrm{Spec}(A)}$. This implies that $\mathfrak{G} = \Lambda(\Gamma(\mathrm{Spec}(A); \mathfrak{G}))$, see ?. The morphism

$$\Gamma(\mathrm{Spec}(A); \mathfrak{G}) \otimes_{\Gamma(1;\mathfrak{O})} \Gamma(\mathrm{Spec}(A_f); \mathfrak{O}) \to \Gamma(\mathrm{Spec}(A_f); \mathfrak{G}) = \Gamma(A; \mathfrak{G})_f,$$

$$m \otimes r \to rm$$

is an isomorphism by basic commutative algebra.

Example 3.0.4. Let R be a ring. Consider the open immersion $U = \mathrm{Spec}(R[x, y]) \setminus \{(x, y)\} \to \mathrm{Spec}(R[x, y])$ and the quasi-coherent sheaf $\mathfrak{G} = \Lambda(\frac{R[x,y]}{xy})$. The global sections of this sheaf are $\frac{R[x,y]}{xy}$, as shown in Lemma 2.6.1.

Define $U_1 = D(x) \to U$ and $U_2 = D(y) \to U$. Note that these cover U together. We have $\Gamma(U_1; \mathfrak{G}) = 0$ and $\Gamma(U_1; \mathfrak{G}) = 0$, since ${\frac{R[x,y]}{xy}}_x = 0$ and ${\frac{R[x,y]}{xy}}_y = 0$. Hence since $\mathfrak{G}$ is a sheaf, we get $\Gamma(U; \mathfrak{G}) = 0$.

The sections over U of $\Lambda(R[x,y])$ are (also) $R[x,y]$. See [7, p. 4.4.1]. We conclude that $\Gamma(1;\mathfrak{G}) \otimes_{\Gamma(1;\mathfrak{O})} \Gamma(U;\mathfrak{O}) \to \Gamma(U;\mathfrak{G})$ is not an isomorphism.

Lemma 3.0.5. *Let* $(C,\mathcal{T},\mathfrak{O})$ *be a ringed site. Let* $f : b \to a \in C$ *and* $g : c \to b \in C$ *be morphisms.*

1. *If* fg *and* f *are restrictive, then* g *is.*
2. *if* f *and* g *are restrictive, then* fg.

If $\Gamma(b;\mathfrak{O}) \to \Gamma(c;\mathfrak{O})$ *is faithfully flat then* fg *and* g *restrictive implies* f *is restrictive.*

Proof. Consider the following diagram

$$\begin{tikzcd}
\Gamma(a;\mathfrak{G}) \otimes_{\Gamma(a;\mathfrak{O})} \Gamma(c;\mathfrak{O}) \arrow[r, "\widehat{f}\otimes_{\Gamma(b;\mathfrak{O})}\Gamma(c;\mathfrak{O})"] \arrow[d, equal] & \Gamma(b;\mathfrak{G}) \otimes_{\Gamma(b;\mathfrak{O})} \Gamma(c;\mathfrak{O}) \arrow[d, "\widehat{g}"] \\
\Gamma(a;\mathfrak{G}) \otimes_{\Gamma(a;\mathfrak{O})} \Gamma(c;\mathfrak{O}) \arrow[r, "\widehat{fg}"] & \Gamma(c;\mathfrak{G}),
\end{tikzcd}$$

where $\widehat{-}$ is as in lemma ?.

This diagram commutes: going either direction sends $g \otimes r$ to rg. The results follows from commutativity. ∎

Lemma 3.0.6 (coproduct). *Let* X_1, X_2, Y *be a schemes.* $X_1 \xrightarrow{f_1} Y$ *and* $X_2 \xrightarrow{f_2} Y$ *are restrictive morphisms if and only if the corresponding morphism* $X_1 \sqcup X_2 \xrightarrow{f_1 \sqcup f_2} Y$ *is restrictive.*

Proof. $\Rightarrow$:

Note that $\Gamma(X_1 \sqcup X_2; -) = \Gamma(X_1; -) \times \Gamma(X_2; -)$ by the sheaf property. We will show that

$$\widehat{f_1 \sqcup f_2} : \Gamma(Y;\mathfrak{G}) \otimes_{\Gamma(Y;\mathfrak{O})} \Gamma(X_1 \sqcup X_2;\mathfrak{O}) \to \Gamma(X_1 \sqcup X_2;\mathfrak{G})$$

is an isomorphism. Tensor commutes over products, so this becomes

$$\widehat{f_1} \times \widehat{f_2} : (\Gamma(Y;\mathfrak{G}) \otimes_{\Gamma(Y;\mathfrak{O})} \Gamma(X_1;\mathfrak{O})) \times (\Gamma(Y;\mathfrak{G}) \otimes_{\Gamma(Y;\mathfrak{O})} \Gamma(X_2;\mathfrak{O})) \to \Gamma(X_1;\mathfrak{G}) \times \Gamma(X_2;\mathfrak{G}).$$

By assumption $\widehat{f_1}$ and $\widehat{f_2}$ are isos, so their product is.

$\Leftarrow$: By Lemma 3.0.7, the canonical morphism $X_i \to X_1 \sqcup X_2$ is restrictive. By assumption $X_1 \sqcup X_2 \to Y$ is restrictive. Composing these morphisms yields f_i, by Lemma 3.0.5 this yields a restrictive morphism. ∎

Lemma 3.0.7. *Let* X, Y *be a schemes. The canonical morphism* $X \to X \sqcup Y$ *is restrictive.*

Proof. Let $\mathfrak{G}$ be a quasi-coherent sheaf on $X \sqcup Y$. By the sheaf property $\Gamma(X \sqcup Y;\mathfrak{G}) = \Gamma(X;\mathfrak{G}) \times \Gamma(Y;\mathfrak{G})$. The same holds also for $\mathfrak{O}$.

We are considering the morphism

$$\Gamma(X \sqcup Y;\mathfrak{G}) \otimes_{\Gamma(X\sqcup Y;\mathfrak{O})} \Gamma(X;\mathfrak{O}) \to \Gamma(X;\mathfrak{G}).$$

By the previous remark about disjoint unions and the sheaf property and some basic commutative algebra one sees that this becomes

$$(\Gamma(X;\mathfrak{G}) \otimes_{\Gamma(X\sqcup Y;\mathfrak{O})} \Gamma(X;\mathfrak{O})) \times (\Gamma(Y;\mathfrak{G}) \otimes_{\Gamma(X\sqcup Y;\mathfrak{O})} \Gamma(X;\mathfrak{O})) \to \Gamma(X;\mathfrak{G}).$$

Since $\Gamma(Y;\mathfrak{G}) \otimes_{\Gamma(X\sqcup Y;\mathfrak{O})} \Gamma(X;\mathfrak{O}) = 0$, we are left with

$$(\Gamma(X;\mathfrak{G}) \otimes_{\Gamma(X\sqcup Y;\mathfrak{O})} \Gamma(X;\mathfrak{O})) \to \Gamma(X;\mathfrak{G})$$
$$g \otimes r \to rg.$$

Note that $\Gamma(X;\mathfrak{G})$ already is an $\Gamma(X;\mathfrak{O})$-module and conclude that hence this morphism is an isomorphism. ∎

Lemma 3.0.8 (Restrictive to affines)**.** *If* $f : X \to \mathrm{Spec}(R)$ *is a restrictive open immersion, then* X *is affine.*

Proof. Since X is an open in $\operatorname{Spec}(R)$, we get a distinguised covering $\bigcup_i D(f_i) = X$ with $f_i \in R$ and $i \in I$. We will prove that $(f_i|_X)_{i\in I} = (1)$ in $S = \Gamma(X; \mathfrak{O})$.

Then we invoke the result in Lemma 2.5.9 that states the following. For a scheme Y let Y_f be the support of $f \in \Gamma(Y; \mathfrak{O})$ as in lemma ?. if Y_{g_j} are affine and $(g_j)_{j\in J} = (1)$ then Y is affine.

Note that $D(f_i) = X_{f_i|_X}$. Consider $M = \frac{R}{(f_i)}$ as an R-module and look at $\Lambda(M)$. By restrictiveness we get $M \otimes_R S = \Lambda(M)(S)$ and by $M \otimes_R R_{f_i} = \Lambda(M)(D(f_i)) = M_{f_i} = 0$. Hence $\Lambda(M)(S) = 0$ by the sheaf axiom. This implies that $(f_i|_X)_{i\in I} = (1)$ in S. So X is affine. ∎

Lemma 3.0.9. *Any morphism* $\operatorname{Spec}(S) \xrightarrow{f} \operatorname{Spec}(R) \in \mathrm{Sch}_{\operatorname{Spec}(R)}$ *between affine schemes is restrictive.*

Proof. Let $\mathfrak{G}$ be a quasi-coherent module on $\mathrm{Sch}_{\operatorname{Spec}(R)}$. Set $M = \Gamma(\operatorname{Spec}(R); \mathfrak{G})$. We want to prove that

$$\widehat{f} : M \otimes_R S \to \Gamma(\operatorname{Spec}(S); \mathfrak{G})$$

is an isomorphism.

By Lemma 2.7.5, we get $\mathfrak{G} = \Lambda(M)$. As said in Remark 3.0.2, in this case $\widehat{f} = \omega^2_{\operatorname{Spec}(S)}$. By Definition 2.2.15, we know that ω^2 is an isomorphism at affine schemes. ∎

Example 3.0.10 (Affine non-restrictive map)**.** One might expect(or want) that any property of all maps between affine schemes also hold for affine maps between any schemes. This is not the case for restrictiveness, so it is not local on the target.

Consider the canonical inclusions $\mathbb{A}^1 \to \mathbb{P}^1$ and the shifted quasi-coherent module $\mathfrak{O}(-1)$. This module is locally free of degree 1, this is often called an invertible module.

The global sections of the module $\mathfrak{O}(-1)$ are the elements of degree -1 in the global sections of $\mathfrak{O}$. There are no such elements, hence the global sections are the zero module.

On $\mathbb{A}^1$ all invertible modules are isomorphic to the structure sheaf. See [7, p. 14.2.8].

Similary any inclusion $\operatorname{Spec}(\kappa(\mathfrak{p})) \to \mathbb{P}^1$ of a point is not restrictive which can be shown with the same argument.

This is a (more opaque) way of saying that on projective space not every quasi-coherent sheaf is generated by global sections.

4 Caffine objects

In this chapter we will define the property affineness for an object in any ringed sites. This will be called ‘caffine’. We will prove that caffine and affine are equivalent in the ringed site of schemes. Lastly, we introduce the notion of enough affines. If this conditions holds then a sheaf is quasi-coherent if and only if it is locally $\Lambda(M)$ for any caffine object and some (non-)examples.

4.1 Caffine objects

In this section we will introduce caffine objects, see some examples and non-examples and prove some properties. Most of these results will be generalisations of their counterpart for affine schemes.

Definition 4.1.1 (Caffine object)**.** Let $(\mathrm{C}, \mathcal{T}, \mathfrak{O})$ be a ringed site. Let $a \in \mathrm{C}$ be an object. We call a caffine if the unit η and co-unit ϵ of the adjunction $\Lambda \dashv \Gamma(a; -)$ on C_a are natural isomorphisms for any $\Gamma(a; \mathfrak{O})$-module M and any quasi-coherent $\mathfrak{F}$ on C_a

In other words, that $\Lambda : \Gamma(a; \mathfrak{O})\text{-}\mathrm{Mod} \to \mathrm{Qcoh}(\mathfrak{O})$ is an equivalence of categories with $\Gamma(a; -)$ as pseudo-inverse and the unit and co-unit as witnessing natural isomorphisms.

Example 4.1.2 (Examples of caffine objects)**.** The main example to keep in mind is $\mathrm{Spec}(R) \in \mathrm{Sch}$. See Lemmas 2.7.5 and 2.7.6 for proofs that the unit and co-unit are isomorphisms.

Example 4.1.3. The scheme $\mathbb{P}^1$ is not caffine in Sch. The counit at the quasi-coherent sheaf of modules $\mathfrak{O}(-1)$ has signature $\epsilon : \Lambda(0) \to \mathfrak{O}(-1)$. Since $\mathfrak{O}(-1)$ is locally free of degree 1, this cannot be an isomorphism.

Definition 4.1.4 (caffine cover)**.** Let $(\mathrm{C}, \mathcal{T}, \mathfrak{O})$ be a ringed site. A family of maps $\{a_i \to a\}$ is called a caffine covering of a if every a_i is caffine and the family is a covering family.

Definition 4.1.5. We say that a ringed site $(C, \mathcal{T}, \mathfrak{O})$ has enough affines if any object admits a caffine covering.

Lemma 4.1.6. *Let $(C, \mathcal{T}, \mathfrak{O})$ be a ringed site. Let $a \in C$. Let $\{b_i \to a\}$ be a caffine covering on a. Assume every map $b_i \xrightarrow{i} a$ is restrictive. Then the counit ϵ of the adjunction $\Lambda_a(-) \dashv \Gamma(a; -)$ is a natural isomorphism.*

Proof. Let $\mathfrak{F}$ be a quasi-coherent sheaf module. Set $M = \Gamma(a; \mathfrak{F})$. Set $M_i = \Gamma(b_i; \mathfrak{F})$. Set $\beta_i = i_* \epsilon_{\mathfrak{F},a}$. By Definition 2.2.27 the induced functor of i satisfies all requirements of Lemma 2.4.5, so we have $i_* \epsilon_{\mathfrak{F},a} \cong \epsilon_{i_*\mathfrak{F},b_i}$. Since b_i is caffine β_i is an isomorphism. Hence ϵ is an isomorphism between sheaves, since it is an isomorphism restricted along every morphism from a covering. ∎

Lemma 4.1.7. *Let $(C, \mathcal{T}, \mathfrak{O})$ be a ringed site. Let $a \in C$. Let M be a $\Gamma(a; \mathfrak{O})$-module. The component*

$$\omega^2_{\lambda(M),a} : \lambda(M)(a) \to \Lambda(M)(a)$$

at Id_a of the sheafification morphism

$$\omega^2_{\lambda(M)} : \lambda(M) \to \Lambda(M)$$

is equal to the unit of $\Lambda \dashv \Gamma(1; -)$ in C_a.

Proof. Consider the following maps, which you get by repeatedly calling on an adjunction.

$$\mathrm{Id} : \Lambda(M) \to \Lambda(M)$$
$$\omega^2_{\Lambda(M)} : \lambda(M) \to \Lambda(M) \text{ use sheafification adjunction}$$
$$\omega^2_{\lambda(M),a} M \to \Gamma(a; \Lambda(M)) \text{ take sections at } a$$

We took the adjunct of Id with respect to the sheafification adjunction as the first step. Then we took the adjunct of the result wrt the λ ajunction. Hence we get the adjunct of Id wrt the Λ adjunction. so the last map is actually the unit of the Λ adjunction. ∎

Corollary 4.1.8. *Let $(C, \mathcal{T}, \mathfrak{O})$ be a ringed site. Let $a \in C$ be caffine. Then $\omega^2_{\lambda(M),a}$ is an isomorphism for any $\mathfrak{O}(a)$-module M.*

Theorem 4.1.9 (Morphism between caffines is restrictive). *Let* $Y = (\mathcal{C}, \mathcal{T}, \mathfrak{O})$. *Let* $f : b \to a \in \mathcal{C}$ *be a morphism between caffine objects, then* f *is restrictive.*

Proof. Let $\mathfrak{F}$ be a quasi-coherent module on Y_a. Let $M = \Gamma(a; \mathfrak{F})$. Since a is caffine, we have $\mathfrak{F} = \Lambda(M)$.

We have to show that

$$\widehat{f} : \Gamma(a; \mathfrak{F}) \otimes_{\Gamma(a;\mathfrak{O})} \Gamma(b; \mathfrak{O}) \to \Gamma(b; \mathfrak{F})$$

is an isomorphism, where $\widehat{f}$ as defined in Definition 3.0.1.

Note that $\widehat{f}$ is the component at b of the natural transformation $\omega^2_{\Lambda(\Gamma(1;\mathfrak{F}))}$, see Remark 3.0.2. Since b is caffine, this component is an isomorphism by Corollary 4.1.8. ■

4.2 Caffine schemes

Let $X \in \mathrm{Sch}$.

Set $R = \Gamma(X; \mathfrak{O})$. We will prove that the canonical morphism

$$\epsilon_X : X \to \mathrm{Spec}(R)$$

is an isomorphism if X is a caffine object. This morphism sends $x \in X$ to $\ker x$ as defined later.

We will start this section with some general results that work for any scheme.

Definition 4.2.1 (Support of module). Let $\mathfrak{M}$ be a $\mathfrak{O}$-module. The support $\mathrm{Supp}(\mathfrak{M})$ of this module is the subspace

$$\{x \in X \mid \mathfrak{M}_x = 0\} \subset X.$$

Definition 4.2.2 (Set cut out by ideal). Define the set cut out by an ideal I of R to be

$$V_X(I) = \mathrm{Supp}(\Lambda_X(\frac{R}{I})).$$

Definition 4.2.3. For a global section $a \in R$ define

$$D_X(a) = \{x \in X \mid a \notin \ker(x)\}.$$

The following lemma will be the tool we use to prove that X is affine if it is caffine.

Lemma 4.2.4. *The following are equivalent:*

1. X *is affine*
2. ϵ_X *is an isomorphism*
3. ϵ_X *is a homeomorphism*

Proof. $1 \Rightarrow 2$: Assume that $X = \operatorname{Spec} R$. Then $\ker \mathfrak{p} = \mathfrak{p}$ for $\mathfrak{p} \in X$, so the topological part of ϵ_X is an homeomorphism. The sheaf part of ϵ_X sends the global section $1 \in \mathfrak{O}(X)$ to $1 \in R$, since the sheaf morphism is a $\mathfrak{O}_{\operatorname{Spec}(R)}$-algebra map, it has to be the identity.

$2 \Rightarrow 3$: By definition.

$3 \Rightarrow 1$: Let $\operatorname{Spec} A_i = U_i \subset X$ be open affines and suppose $\bigcup_i U_i = X$. Assume it is a finite affine cover, which can be done since X is quasi-compact. Using our base, we get a cover of $U_i = \bigcup_j D_X(a_{ij})$ with a_{ij} global sections. Observe that $D_X(a_{ij}) \subset U_i$, hence $D_{U_i}(a_{ij}|_{U_i}) = D_X(a_{ij})$ which makes them affine. Continuing like this, we get a finite cover of affines $D_X(a_{ij})$ of X. Since

$$\epsilon_X(X) = \epsilon_X(\bigcup_{ij} D_X(a_{ij})) = \bigcup_{ij} D_{\operatorname{Spec}(R)}(a_{ij}) = \operatorname{Spec} R,$$

we have $(a_{ij}) = (1)$. Now both requirements of Lemma 2.5.9 are satisfied, hence X is affine. ∎

Lemma 4.2.5 (Stalks)**.** *Let* M *be a* $\Gamma(X; \mathfrak{O})$*-module. Let* $x \in X$*. Then* $\Lambda(M)_x = M\mathfrak{O}_x$.

Proof. See [Stacks, Tag 01BH]. ∎

Lemma 4.2.6. *For* $x \in X$ *TFAE:*

1. $x \in V_X(I)$
2. $I\mathfrak{O}_x \neq \mathfrak{O}_x$
3. $I \subset \ker(x)$.

Proof. $1 \Rightarrow 2$:

Assume $x \in V_X(I)$. We have $\Lambda_X(\frac{R}{I}) = \frac{\Lambda_X(R)}{\Lambda_X(I)}$ and taking stalks is exact so $\frac{\Lambda_X(R)}{\Lambda_X(I)}_x = \frac{\Lambda_X(R)_x}{\Lambda_X(I)_x}$. Lastly by Lemma 4.2.5 we get $\Lambda_X(\frac{\mathfrak{O}(X)}{I})_x = \frac{\mathfrak{O}_x}{I\mathfrak{O}_x}$. Since $x \in V_X(I)$ we know $\frac{\mathfrak{O}_x}{I\mathfrak{O}_x} \neq 0$, hence $I\mathfrak{O}_x \neq \mathfrak{O}_x$.

$2 \Rightarrow 3$:

Assume $I\mathfrak{O}_x \neq \mathfrak{O}_x$. Then $I\mathfrak{O}_x$ is proper hence contained in the unique maximal ideal of the local ring $\mathfrak{O}_x$, therefore $I \mapsto 0$ in $k(x)$ or equivalently $I \subset \ker(x)$.

$3 \Rightarrow 1$:

Assume $I \subset \ker(x)$. Then I maps into $\mathfrak{m}_x$, hence $I\mathfrak{O}_x \subset \mathfrak{m}_x$. Therefore

$$\frac{\mathfrak{O}_x}{\Lambda_X(I)_x} = \frac{\mathfrak{O}_x}{I\mathfrak{O}_x} \neq 0.$$

■

Corollary 4.2.7. *If* $y \in I$ *then* $D_X(y) \cap V_X(I) = \emptyset$.

Proof. Assume $y \in I$. Let $z \in V_X(I)$, then $y \in \ker(z)$ by Lemma 4.2.6. This implies $z \notin D_X(y)$. ■

Corollary 4.2.8. $V_X(I) \cup V_X(J) = V_X(IJ)$

Proof. Let $z \in V_X(I) \cup V_X(J)$. Then $I \subset \ker(z)$ and $J \subset \ker(z)$ by Lemma 4.2.6, hence $IJ \subset \ker(z)$. Apply Lemma 4.2.6 again to get $z \in V_X(IJ)$. Let $z \in V_X(IJ)$. Then $IJ \subset \ker(z)$ by Lemma 4.2.6. The ideal $\ker(z)$ is prime, so $I \subset \ker(z)$ or $J \subset \ker(z)$. Invoke Lemma 4.2.6 again to get $z \in V_X(I) \cup V_X(J)$. ■

Lemma 4.2.9. *Let* $I \subset \Gamma(X; \mathfrak{O})$ *be an ideal. The set* $V_X(I)$ *is closed.*

Proof. Consider the exact sequence

$$\mathfrak{O}(X) \to \frac{\mathfrak{O}(X)}{I} \to 0.$$

The functor Λ is a left adjoint hence right exact so

$$\mathfrak{O} \xrightarrow{f} \Lambda(\frac{\mathfrak{O}(X)}{I}) \to 0$$

is exact. Hence the sequence

$$\mathfrak{O}_x \xrightarrow{f_x} \Lambda(\frac{\mathfrak{O}(X)}{I})_x \to 0$$

is exact. So we get that $f_x(1_x)$ generates $\Lambda(\frac{\mathfrak{O}(X)}{I})_x$.

Note that $f_x(1_x) = f(1)_x$ by definition of f_x, hence $f(1)_x$ is a generating element. Hence $\Lambda(\frac{\mathfrak{O}(X)}{I})_x \neq 0$ if and only if $f(1)_x \neq 0$.

This implies $V_X(I) = \text{Supp}(f(1))$ which makes $V_X(I)$ closed as the support of a global section. ■

From now on assume that X is a caffine object in Sch.

Lemma 4.2.10. *Every closed set $W \subset X$ can be written as $V_X(I)$ for some ideal $I \subset \Gamma(x; \mathfrak{O})$.*

Proof. Let $\mathfrak{I}$ be some ideal sheaf inducing a closed subscheme structure on W. This is by definition a quasi-coherent module. Let $I = \Gamma(x; \mathfrak{I})$. Since X is caffine, we get $\mathfrak{I} = \Lambda(I)$. Let $\mathfrak{O}_W$ be the structure sheaf of this closed subscheme. Consider the closed immersion $W \xrightarrow{i} X$. By construction $\mathfrak{I} \to \mathfrak{O} \to i_*\mathfrak{O}_W \to 0$ is exact, hence $i_*\mathfrak{O}_W = \frac{\mathfrak{O}}{\mathfrak{I}}$. Hence $V_X(I) = \text{Supp}\ (i_*\mathfrak{O}_W) = W$. ■

Lemma 4.2.11. *The sets $D_X(a)$ form a basis for the topology of X, with $a \in R$.*

Proof. Let $x \in D_X(a)$. Then $a \notin \ker x$, hence $\frac{1}{a}$ exists in $\mathfrak{O}_x$. We get an open neighbourhood V of x such that $\frac{1}{a} \in \Gamma(V; \mathfrak{O})$. Let $y \in V$, then clearly $a \notin \ker y$. We have shown that $x \in V \subset D_X(a)$, hence $D_X(a)$ is open.

Let $U \subset X$ be any open. Let $x \in U$. By Lemma 4.2.10 we get I such that $V_X(I) = U^c$. It follows that $x \notin V_X(I)$ and $I \not\subset \ker(x)$ by Lemma 4.2.6. So we get a $g \in I$ with $g \notin \ker(x)$. We get $x \in D_X(g)$ and $D_X(g) \subset U$, because $D_X(g) \cap V_X(I) = \emptyset$ by Corollary 4.2.7.

Let $a, b \in R$. Note that $D_X(ab) = D_X(a) \cap D_X(b)$ since $\ker(x)$ is a prime ideal. So the opens $D_X(a)$ form a basis. ■

Lemma 4.2.12. *The map $\epsilon_X : X \to \text{Spec}(R)$ is surjective.*

Proof. Let $\mathfrak{p} \in \operatorname{Spec} R$ be a point in the target of ϵ_X. Then $\Lambda_X(\kappa(\mathfrak{p}))$ is a quasi-coherent sheaf of modules. In fact $\kappa(\mathfrak{p}) \otimes_{\Gamma(X;\mathfrak{O})} \mathfrak{O}(U)$ is a $\mathfrak{O}(U)$ algebra for any open $U \subset X$, hence $\Lambda_X(\kappa(\mathfrak{p}))$ is a quasi-coherent sheaf of algebras. Hence we can compute the relative spec $\operatorname{Rspec}(\Lambda_X(\kappa(\mathfrak{p}))) \xrightarrow{h} X$.

Since X is caffine we have $\Lambda_X(\kappa(\mathfrak{p}))(X) = \kappa(\mathfrak{p})$. So

$$\operatorname{Rspec}(\Lambda_X(\kappa(\mathfrak{p}))) \xrightarrow{\epsilon_X \circ h} \operatorname{Spec}(R)$$

factors as

$$\operatorname{Rspec}(\Lambda_X(\kappa(\mathfrak{p}))) \to \operatorname{Spec}(\kappa(\mathfrak{p})) \to \operatorname{Spec}(R).$$

This gives us a commutative square

$$\begin{tikzcd}
X \arrow[r, "\epsilon_X"] & \operatorname{Spec} R \\
\operatorname{Rspec}(\Lambda_X(\kappa(\mathfrak{p}))) \arrow[u] \arrow[r] & \operatorname{Spec} \kappa(\mathfrak{p}) \arrow[u]
\end{tikzcd}$$

Since $\Lambda_X(\kappa(\mathfrak{p}))$ is not the zero sheaf hence the structure sheaf of $\operatorname{Rspec}(\Lambda_X(\kappa(\mathfrak{p})))$ is non-zero. This implies that $\operatorname{Rspec}(\Lambda_X(\kappa(\mathfrak{p})))$ is not the empty scheme. Therefore the point $\mathfrak{p}$ is in the image of $\operatorname{Rspec}(\Lambda_X(\kappa(\mathfrak{p}))) \to \operatorname{Spec}(\kappa(\mathfrak{p})) \to \operatorname{Spec}(R)$, hence also in the image of ϵ_X. ∎

Lemma 4.2.13. *Let $\mathfrak{p} \subset R$ be a prime ideal. The closed set $V_X(\mathfrak{p})$ is irreducible. This implies that ϵ_X is injective.*

Proof. Let $z \in X$. Assume that $\epsilon_X(z) = \mathfrak{p}$. So $\ker z = \mathfrak{p}$. Let $y \in V_X(\mathfrak{p})$. Then $\ker(z) \subset \ker(y)$, hence if $y \in D_X(a)$ then $z \in D_X(a)$. Therefore z is contained in any open subset of $V_X(\mathfrak{p})$, hence it is irreducible. Note that z is its unique generic point since the biggest closed subset of $V_X(\mathfrak{p})$ not containing z is the empty set. This shows that ϵ_X is injective. ∎

Lemma 4.2.14. *The counit ϵ_X is open, hence a homeomorphism.*

Proof. Let $a \in R$. Let $x \in X$. Let $\mathfrak{p} = \epsilon_X(x)$. Since ϵ_X is a morphism of locally ringed spaces, if $a \notin \ker x$ then $a \notin \ker \mathfrak{p} = \mathfrak{p}$. Note that $\ker -$ is defined for arbitrary schemes so also works on $\mathrm{Spec}(R)$. Hence $\epsilon_X(D_X(a)) = \{\epsilon_X(x) \mid a \notin \ker(x)\} = \epsilon_X(X) \cap D_{\mathrm{Spec}\,R}(a) = D_{\mathrm{Spec}\,R}(a)$. Our map ϵ_X is continuous and open, so a homeomorphism. ∎

Proposition 4.2.15. *The caffine scheme X is affine.*

Proof. By Lemmas 4.2.12 to 4.2.14 ϵ is an homeomorphism and so by Lemma 4.2.4 it follows that X is affine and ϵ is an isomorphism. ∎

4.3 Enough caffines

For schemes a quasi-coherent module often is defined by a module that over affine schemes is of the form $\Lambda(M)$ for some module M. We will look how often this generalizes, which we will do through the notion of 'enough caffines'. We will see some examples where it does not work as in Sch and some results about when it does.

Definition 4.3.1. We say that a ringed site $(C, \mathcal{T}, \mathfrak{O})$ has enough affines if any object admits a caffine covering.

Lemma 4.3.2. *Let C be a ringed site that has enough affines. Let $\mathfrak{F}$ be a sheaf of modules on C. Then $\mathfrak{F}$ is quasi-coherent if and only if for each caffine $a \in C$ we have $\mathfrak{F}|_a \cong \Lambda(M)$ for some $\Gamma(a; \mathfrak{O})$-module M.*

Proof. $\Rightarrow$: This is Lemma 2.3.3.

$\Leftarrow$: Since C has enough affines, every object has a cover of caffine objects. For any caffine $a \in C$, we have $\mathfrak{F}|_a \cong \Lambda(M)$ so $\mathfrak{F}|_a$ has a global presentation. Hence $\mathfrak{F}$ is quasi-coherent. ∎

Lemma 4.3.3. *Any ringed site $(C, \mathcal{T}, \mathfrak{O})$ that has a finite poset as underlying category, has enough caffines.*

Proof. Let $x_0 \in C$. If x_0 is covered by the maximal sieve only or the maximal sieve and the empty sieve, it is caffine and we are done. Assume otherwise. Let $S = \{y_i \to x_0\}$ be a non-maximal, non-empty cover of x_0. Then S does not contain isomorphisms.

We can associate to any non-maximal non-empty covering sieve S of an element x_0, the set of all NA-chains $x_0 \leftarrow x_1 \leftarrow \ldots \leftarrow x_n$. An NA-chain, associated to R, is a chain of maps ending in x_0 such that $x_i \leftarrow x_{i+1}$ is contained in a non-maximal, non-empty cover of x_i, where $x_0 \leftarrow x_1$ is contained in R.

By finiteness of C, any chain of maps is bounded by the size of C or contains a cycle. If a chain contains a cycle, it contains isomorphisms. By construction, no isomorphism can be present in a NA-chain. Therefore the length of any NA-chain is bounded by $\|C\|$.

Let H be a NA-chain associated to S of maximal length m. Then the last map $\ldots \leftarrow h \leftarrow g$ in H has a caffine object g as domain, because H cannot be increased and so g has no non-maximal, non-empty coverings which makes it caffine. Also the non-maximal, non-empty covering of h where this map appears must be a caffine covering by applying the same reasoning to the other objects occurring in it. Hence all objects occurring at the $(m-1)$th place in any NA-chain admits a caffine cover. Let $i \leqslant m-1$. Assume all elements at the $(i-1)$th place admit a caffine cover. Let b be a object occurring at the $(i-1)$th place in a chain. It is either caffine or all objects in any non-maximal, non-empty cover occur at the ith place in some chain hence admit a caffine cover. Therefore any non maximal, non empty cover on b can be refined to a caffine cover. This provides us with a caffine cover of b. By reversed induction, x_0 admits a caffine cover. ∎

Example 4.3.4. The following will be an example of a category that does not have enough caffines.

The category C is $\mathbb{Z} \times \mathbb{Z}$ as a product partial order. An element (i, j) is only non-trivially covered by $\{(i, j-1) \to (i, j), (i-1, j) \to (i, j)\}$. Let k be any field. Let $R = k[x_{ij} | i, j \in \mathbb{Z}]$. Define the structure sheaf as $\mathfrak{O}(i, j) = R[x_{kl}^{-1} | i \leqslant k \,\&\, j \leqslant l]$.

Fix $(a, b) \in C$. Consider the over category $C_{(a,b)}$ at this point. Let $(i, j) \to (a, b)$ be an object of $C_{(a,b)}$. So $i \leqslant a$ and $j \leqslant b$. Define the presheaf of modules $F(i, j) = \mathfrak{O}(i, j)/(x_{a-1,b}x_{a,b-1})$ on X. We have $a > i$ or $b > j$ or ($i = a$ and $j = b$). If $a > i$ or $b > j$, then $x_{a-1,b}$ or $x_{a,b-1}$ is invertible in $\mathfrak{O}(i, j)$, hence $F(i, j) = 0$ in both cases. This presheaf is zero everywhere except at (a, b), because both $x_{a-1,b}$ and $x_{a,b-1}$ are not invertible in $\mathfrak{O}(a, b)$ hence sheafifies to the zero sheaf. In other words: $\Lambda(\frac{\mathfrak{O}(a,b)}{(x_{a-1,b}x_{a,b-1})}) =$

0, Hence (a, b) is not affine, which shows that C has no affine objects.

Example 4.3.5. Let C be as in Example 4.3.4. Consider $G = \mathfrak{O}(i,j)[y_{k,l} | k \leqslant i \;\&\; l \leqslant j]$. Let $\bigoplus_{k\in I}\mathfrak{O} \xrightarrow{\alpha} G$ be any sheaf map. Let $\alpha(e_k)$ be the image of the generators $e_k \in \bigoplus_{k\in I}\mathfrak{O}$ in the global sections. The section $y_{1,1} \in G(1,1)$ cannot be written as a finite sum $\sum_k \lambda_k\alpha(e_k)$ for scalars $\lambda_k \in \mathfrak{O}(i,j)$ for any (i,j). This shows that α is not surjective hence G is not quasi-coherent(locally presentable).

Example 4.3.6. This is the example in [Stacks, Tag 01BL]. We state it here because we will categorify it to yield a category that has not enough affines.

Let $L = (\mathbb{R}, \mathfrak{O}_R)$ be the real line with the euclidean topology and the sheaf of continuous real valued functions as structure sheaf. Let

$$X = \frac{\bigcup_{i=0}^{\infty} L_i}{\sim}$$

with $[i,x] \sim [j,y]$ if and only if $i = j$ and $x = y$ or $y = x = 0$. The real lines are glued to each other at zero. Define the open $U_n \subset X$ as $U_n \cap L_i = (-\frac{1}{n}, \frac{1}{n})$. These opens form a basis of neighbourhoods of 0. Let $f : \mathbb{R} \to \mathbb{R}$ be any continuous function such that $f(x) = 0$ if $x \in (-1, 1)$ and $f(x) = 1$ if $x \in (-\infty, -2) \cup (2, \infty)$. Let $f_n(x) = f(nx)$.

Define the sheaf map

$$\bigoplus_i \mathfrak{O}_R \xrightarrow{\alpha} \bigoplus_{ij} \mathfrak{O}_R,$$

$$e_i \mapsto \sum_j f_i 1_{L_j} e_{ij}.$$

To proof that this is well-defined, we need to show that the sum $\sum_j f_i 1_{L_j} e_{ij}$ is locally finite for every i. Let $[k,y] \in X$. If $y \neq 0$, then

$$W_{[k,y]} = \{[k,z] \in X \mid z \in (y-\delta, y+\delta)\} \subset L_k$$

is open in X and $\alpha_{W_{[k,y]}}(e_i) = f_i e_{ik}$ for any $\delta < |\frac{y}{2}|$. If $y = 0$, then $\alpha_{U_n}(e_i) = 0$ if $n > i$ because f_i is zero on U_n. Hence we found a cover on which our sum is locally finite, which makes α well-defined.

The adjunction $\Lambda \dashv \Gamma(X; -)$ implies that Λ commutes with arbitrary colimits. Moreover

$$\mathfrak{O}_X \cong \Lambda(\Gamma(X; \mathfrak{O})),$$

so

$$\bigoplus_i \mathfrak{O}_R \cong \Lambda(\bigoplus_i \Gamma(X;\mathfrak{O})).$$

This shows that α is a morphism between objects that are of the form $\Lambda(M)$.

Let $\beta : \bigoplus_i \Gamma(U;\mathfrak{O}) \to \bigoplus_{ij} \Gamma(U;\mathfrak{O}))$ for some open U. Then $\Lambda(\beta)(e_i) = \sum_{j\in J_i} a_{ij}e_{ij}$ where J_i is finite for every i.

Assume that $\alpha = \Lambda(\beta)$ over some neighbourhood U of 0. Then there exists a m such that $U_m \subset U$. Let $k > 2m$. Then $f_k \neq 0$ on U_m, hence $f_k 1_{L_j} \neq 0$ on U_m for every j and so no coëfficients vanish of $\alpha_{U_m}(e_k) = \sum_j f_k 1_{L_j} e_{kj}$. This contradicts $\alpha = \Lambda(\beta)$. Hence α does not come from a module map locally.

Remark 4.3.7. This is an example of a ringed space where Λ is not an equivalence locally, since it shows that Λ is not full. We will categorify this example next.

Definition 4.3.8 (Neighbourhood site)**.** Let X be a topological space and $\mathrm{Open}(X)$ the corresponding category of open subsets. Let $y \in X$. Define the category $N(y)$ to be the full subcategory of $\mathrm{Open}(X)$ of all neighbourhoods U of y. A family $\{U_i \to U\}$ is covering on U in $N(y)$ if it is covering in $\mathrm{Open}(X)$. Note that $i : N(y) \to \mathrm{Open}(X)$ is a morphism of sites.

Assume that X is a ringed space coming with a structure sheaf $\mathfrak{O}$. Define $\mathfrak{O}_{X,y} = i_*\mathfrak{O}$.

Example 4.3.9. This is will be a categorification of Example 4.3.6. This will yield us a category where Λ is not full over any object, so has no caffines. Let X, f_j and U_n be as in Example 4.3.6. Set $y = 0$ and consider the ringed site $N(y)$. Note that $\bigoplus_i \mathfrak{O}_{X,y}$ is quasi-coherent.

Define the sheaf map

$$\bigoplus_i O_{X_y} \xrightarrow{\alpha} \bigoplus_{ij} O_{X_y},$$

$$e_i \mapsto \sum_j f_i 1_{L_j} e_{ij}.$$

Fix i. We will prove that $\alpha_X(e_i)$ is a well-defined global section. Choose $m > i$. Let $V_k = L_k \cup U_m$ and note that $\bigcup_k V_k = X$. By construction f_i is zero on U_m, hence $f_i 1_{L_j}$

is zero on V_k if $k \neq j$ and so $\sum_j f_i 1_{L_j} e_{ij} = f_i 1 L_k e_{ik}$ on V_k. This shows that $\alpha_X(e_i)$ is a well-defined section on any element of the cover $\{V_k \to X\}$ and this family is matching since the sections are functions and the 'restriction' maps are actual restriction.

Let $V \in N(y)$. Assume there exists $\beta : \bigoplus_i \Gamma(V, O_{X,y}) \to \bigoplus_{ij} \Gamma(V, O_{X,y})$ such that $\Lambda(\beta) = \alpha_V$. Then $\alpha_V(e_i) = \sum_j f_i 1_{L_j} e_{ij}$ is not just locally finite over some cover that may depend on i, but actually finite globally on V for all i. So almost all $f_i 1_{L_j}$ are zero on V. Note that $y \in V$, so $U_d \subset V$ for some d. Let $i > 2d$, then $f_i \neq 0$ on $(-\frac{1}{d}, \frac{1}{d})$ and so $f_i 1_{L_j} \neq 0$ on U_d for any j. Hence $\alpha_V(e_i) = \sum_j f_i 1_{L_j} e_{ij}$ is not a finite sum for $i > 2d$. This contradicts our assumption, so α does not come

The restriction of any quasi-coherent sheaf is quasi-coherent. Observe that α, and its restrictions, is a morphism between quasi-coherent sheaves but does not come from a map of modules by the previous contradiction. Therefore Λ is not full for any V and so no object V is caffine in $N(y)$.

5 Further research

All the loose ends and open questions are collected in this section.

The vanishing theorem of Serre tells us that all higher cohomology of quasi-coherent sheaves are zero [Stacks, Tag 01XB].

Question 5.0.1. *Let* C *be a ringed site. Let* $a \in C$ *be a caffine object. Let* $\mathfrak{F}$ *be a quasi-coherent sheaf on* C_a. *Does* $H^i(a, \mathfrak{F}) = 0$ *for all* $i > 0$ *hold?*

Recall the definition of a restrictive morphism from Definition 3.0.1.

Question 5.0.2. *Is the pullback of a restrictive morphism restrictive?*

It would be nice to have methods to see whether an object is caffine. We used some counting arguments in the finite case in section 3.3, which might be extendable for objects that are quasi-compact in some sense categorical sense.

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