Prelims

Let $Y = (X, \mathcal{T}, \mathfrak{O})$ be a ringed site. Let $R = \Gamma(1; \mathfrak{O})$. Let $a, b, c, c' \in X$. Let $\mathfrak{F}$ be a quasi-coherent module on Y_a. Let $M = \Gamma(a; \mathfrak{F}) = \Gamma(1; \mathfrak{F})$. Let $f : b \to a$.

We pick all these things... then what?

Some basic definitions and constructions.

Definition 1 (Over/Under categories). Let C and C' be categories. Let $F : C \to C'$ and $Z \in C'$. Define the category C_Z and C^Z as follows

$$\mathrm{Obj}(C_Z) := \{(X, u) \mid X \in C, u : F(X) \to Z\},$$

$$\mathrm{Hom}((X, u), (Y, v)) := \{f : X \to Y \mid v \circ F(f) = u\},$$

and

$$\mathrm{Obj}(C^Z) := \{(X, u) \mid X \in C, u : Z \to F(X)\},$$

$$\mathrm{Hom}((X, u), (Y, v)) := \{f : X \to Y \mid F(f)u = v\}.$$

a little confusing to use X for the category and then for an object

So $u(X,u) = X$?

We get faithfull functors $C_Z \to C : (X, u) \to X$ and $C^Z \to C : (X, u) \to X$. We will call both functors u and suppress the functor where there can be no confusion

Definition 2. Let M, N be an R-module. Let $g : M \to N$. Define

$$\lambda : R\text{-Mod} \to \underset{?}{\underline{\mathrm{PMod}}}(Y)$$

Oh. ~~Modules of~~ presheaves of a modules? So this involves the "let $R = \Gamma(1, \mathfrak{O})$" from the beginning.

by

$$\lambda(M)(a) = M \otimes_R \Gamma(a; \mathfrak{O}),$$

$$\lambda(M)(f) : \mathrm{Id} \otimes \mathfrak{O}(f),$$

$$\lambda(g) = (a : g \otimes \mathrm{Id}).$$

Definition 3. Define

$$\Lambda : R\text{-Mod} \to \mathrm{Mod}(Y)$$

by $\mathrm{sh} \circ \lambda$.

This functor is left adjoing to the global sections functor, which I will prove in the next episode.

Lemma 4. *Let S be a subset of $Hom(-, f)$. Then $u(S)$ is a sieve on $u(f) = b$ if and only if S is a sieve on f.*

In what sense is this a set?

Proof. =>: Let $h : d \to b \in S$ and $k : e \to d$ be arbitrary. By assumption $u(hk) \in u(S)$. The functor u is faithfull, so $hk \in S$.

<=: Let $h : d \to b \in u(S)$ and $k : e \to d$ be arbitrary. By assumption $hk \in S$, hence $u(hk) \in u(S)$. ∎

We will define the induced topology $\mathcal{S}$ on C_a. That u considered as a map on sieves commutes with the pullback of sieves is used and will not be proved.

Definition 5. Let $\mathcal{T}(u(f))$ be the set of covering sieves on $u(f) \in X$. By the previous lemma sieves on $u(f)$ are sieves on f. Let $\mathcal{S}(f) = \{R \mid u(R) \in \mathcal{T}(u(f))$ be the induced topology. So $u(R)$ is covering on $u(f)$ if and only if R is covering on f.

What is R? A family of morphisms?

a) Since u commutes with pullback of sieves, we have $max(u(f)) = u(max(f)) = max(f)$, hence $max(f) \in \mathcal{S}(f)$.

What does max mean? Oh, the maximal sieve.

b) Let R be a covering sieve on f. Let $h : b' \to a$ and $p : b' \to b$ with $fp = h$. Commutativity of u and pulling back implies that $u(p)^*u(R) = u(p^*R)$. Hence p^*R is covering since $u(p^*R)$ is.

c) Let R be a covering sieve on f and Q be a sieve on f. Let $h : b' \to a$ and $p : b' \to b \in R$, hence with $fp = h$. Assume p^*Q is covering for every such p. Then $u(p^*Q) = u(p)^*u(Q)$ is covering for every p. We know that $u(R)$ is covering hence $u(Q)$ must be, which implies that Q is covering.

We proved that $\mathcal{S}$ is indeed a Grothendieck topology.

Main

Lemma 6. *Let a be caffine. The global component of the sheafification morphism is equal to the unit of $\Lambda \dashv \Gamma(1; -)$ in C_a.*

for which presheaves?

Proof. Let M be a $\Gamma(a; \mathfrak{O})$-module. Consider the following maps, which you get by repeatedly calling on an adjunction bijection. Let i be the universal sheafification morphism.

$$\Lambda(M) \to \Lambda(M)$$

$$i : \lambda(M) \to \Lambda(M) \text{ use sheafification adjunction}$$

$$M \to \Gamma(a; \Lambda(\mathfrak{M})) \text{ use } \lambda \dashv \Gamma(a; -)$$

what is this $\mathfrak{M}$?

what bijection?

If you compose the two adjunction bijections used, you get the bijection of $\Lambda \dashv \Gamma(a; -)$ by definition, so the last map is actually η_M. Hence $i_a = \eta_M$, which is an iso by assumption. ? ? ■

Lemma 7. *Sheafifying and restricting commute. In formula form*

$$sh_b \circ *\big|_b \cong *\big|_b \circ sh_a.$$

Proof. I will prove that we have a natural isomorphism

$$s : sh_b \circ *\big|_b \to *\big|_b \circ sh_a.$$

what is g?

Let $\mathfrak{F}$ be a presheaf on Y_a. Let $\mathfrak{H} = sh(F|_b)$ and $\mathfrak{K} = sh(F)|_b$. Let T be a covering sieve on g in Y_b and $j \in T$. Let S_j be a covering sieve on $\mathrm{Dom}(j)$ in Y_b and $i \in S_j$.

in what sense?

It's hard to see where you're going.

Let $x = (x_{i,j}) \in sh(F|_b)$ be indexed by S_j and T. We have $x_{i,j} \in \Gamma(\mathrm{Dom}(i); \mathfrak{F})$. Define $s_g(x) = (x_{u(i),u(j)})$ with indexing covering sieves $u(S_j), u(T)$.

In what sense?

Let $x \sim y$. Let R be the covering sieve on which they are the same. Then $s_g(x) \sim s_g(y)$ because they are the same on $u(R)$. Hence this map is well-defined.

This is not usually how you use "Let". Do you mean "suppose"?

Let $s_g(x) = s_g(y)$. Then there is some covering sieve R on fg on which they agree. Consider $u^*(R)$ as a covering sieve on g and its is clear that x and y must agree on it, hence the map is injective.

Let $y = (y_{k,l})$ be an element of $\Gamma(c; \mathfrak{K})$ which is indexed by V, W. Then $s_g(y') = y$ where y' has the same elements as y but is indexed by $u(V), u(W)$, so $y' \in \Gamma(c; \mathfrak{H})$. Hence s_g is surjective.

Let $h : c' \to b$ and $p : c' \to c$, such that $gp = h$. We will show that $s_h\mathfrak{H}(t) = \mathfrak{K}(t)s_g$. See below diagram. Let $x = (x_{i,j}) \in \Gamma(c; \mathfrak{H})$ with indexing covering sieves S_j and T. Then

It sounds like you're using a particular sheafification construction. It would be helpful to let the reader know what your notation ("indexed by", "$\sim$") means.

$\mathfrak{K}(t)(s_g(x)) = (x_{k,l})$ with indexing covering sieves t^*S_l and t^*T. The other one becomes $s_h(\mathfrak{H}(t)(x)) = (x_{k,l})$ with indexing covering sieves t^*S_l and t^*T. Hence s is natural.

$$\begin{tikzcd}
\Gamma(h;\mathfrak{H}) \arrow[r, "s_h"] & \Gamma(h;\mathfrak{K}) \\
\Gamma(g;\mathfrak{H}) \arrow[r, "s_g"] \arrow[u, "\mathfrak{H}(t)"] & \Gamma(g;\mathfrak{K}) \arrow[u, "\mathfrak{K}(t)"']
\end{tikzcd}$$

∎

Proposition 8. *The adjunct of* f

What are the hypotheses here?

$$\Gamma(a;\mathfrak{F}) \otimes_{\Gamma(a;\mathfrak{O})} \Gamma(b;\mathfrak{O}) \to \Gamma(b;\mathfrak{F})$$

is an isomorphism.

Consider

$$\begin{tikzcd}
\mathrm{PMod}(Y_a) \arrow[r, "sh_a"] \arrow[d, "*|_b"] & \mathrm{Mod}(Y_a) \arrow[d, "*|_b"] & \Gamma(a;\mathfrak{O})\text{-Mod} \arrow[l, "\Lambda"'] \arrow[d, "-\otimes\Gamma(b;\mathfrak{O})"] \arrow[ll, bend right, "\lambda"'] \\
\mathrm{PMod}(Y_b) \arrow[r, "sh_a"] & \mathrm{Mod}(Y_b) & \Gamma(b;\mathfrak{O})\text{-Mod} \arrow[l, "\Lambda"'] \arrow[ll, bend left, "\lambda"]
\end{tikzcd}$$

By a previous lemma, the left square commutes. By definition the two 'triangles' commute too and the outer square commute, hence the right square also commutes. Therefore $M \otimes \Gamma(b;\mathfrak{O}) \cong \Gamma(b;\mathfrak{F})$. This is the proof you wrote down friday.

The requirement is not to find any isomorphism but a specific one. So I think this is not enough and we need to do some bookkeeping and see if the witnessing isomorphism is our map.

That's true, well-spotted!

Consider

$$
\begin{tikzcd}
\Lambda(M)|_b & & \Lambda(M \otimes \Gamma(b;\mathfrak{D})) \arrow[ll, "s"'] \\
\lambda(M \otimes \Gamma(b;\mathfrak{D})) \arrow[u, "i|_b"] \arrow[urr, "j"] & &
\end{tikzcd}
$$

The natural transformation j is the universal sheafification morphism coming from sh_b. We have seen that $\Gamma(b;j)$ and s are isomorphisms. *this is really a different s, right?*

Let $g : c \to b$. Let $x \underset{?}{=} m \otimes r \in \lambda(M \otimes \Gamma(c;\mathfrak{D}))$. Then $j_g(x) = (x_i)$ indexed by the maximal sieve on g and $i_g(x) = i_{fg}(x) = (x_i)$ indexed by the maximal sieve on gf. Hence we get $s_g(j_g(x)) = i_g(x)$, so the triangle commutes. Evaluating everything on the terminal, in this case on b, shows that two out of three maps are isomorphisms, hence i_b is an isomorphism.

You have a lot of details here, but I feel like I'm reading half a conversation. Here's what I think would help:

- *Putting in the setup for each proposition into its statement. This prevents name-space collisions and makes it clearer which hypotheses ~~th~~ you're using.*
- *Make the use of fonts and letters consistent. Is f an object, or a morphism? Is u a morphism, or a functor? Is X an object, or a category? When you write "Let g : c → b", which category do these live in?*
- *Describe the approach you will take before getting into the calculations. "We will construct ~~a natural transformation~~ each component of s separately, prove it is bijective, and then show that the resulting family is natural."*