Affine Objects

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1 Preliminaries

1.1 Topology

Definition 1 (Sieve)**.** Let C be a category and $a \in C$. A sieve S on a is a subpresheaf of $h(a)$. Explicitly, for each $c \in C$, $S(c)$ is a subset of $\mathrm{Hom}(c, a)$ such that $fg \in S(\mathrm{Dom}(g))$ for all $f \in S(c)$ and for all $g \in h(c)$.

The maximal sieve on a, which is $h(a)$, will be denoted by $\max(a)$.

Definition 2 (Sieve category)**.** Let C be a category and $a \in C$. The sieve category $\mathsf{Sieves}(a)$ is the subobject poset of the presheaf $h(a)$.

Definition 3 (Pullback of sieve)**.** Let C be a category and $a, b \in C$. Let S be a sieve on a. Let $f : b \to a$.

For any $c \in C$ the sieve f^*S on b is given by $f^*S(c) = \{g \in \mathrm{Hom}(c, b) : fg \in S(c)\}$.

To show that this is actually a subpresheaf of $h(b)$, let $k : c \to c'$ and $h \in f^*S(c')$. Hence $fh \in S(c')$ and so $fhk \in S(c)$. Conclude that $hk \in f^*S(c')$.

This defines a functor $f^* : \mathsf{Sieves}(a) \to \mathsf{Sieves}(b)$.

Definition 4 (Grothendieck Topology)**.** A Grothendieck topology $\mathcal{T}$ is a family $\mathcal{T}(a)$ of 'covering' sieves for every $a \in C$ with the following conditions:

- $\max(a) \in \mathcal{T}(a)$
- $f^*R \in \mathcal{T}(a')$ if $R \in \mathcal{T}(a)$ for all $f : a' \to a$
- if $f^*R \in \mathcal{T}(a')$ for all $f \in S$ with $S \in \mathcal{T}(a)$ then $R \in \mathcal{T}(a)$

Definition 5 (Basis)**.** Let C be a category with pullbacks. A Grothendieck pretopology $\mathcal{B}$ is a collection $\mathcal{B}(a)$ of families $\{f_i : a_i \to a\}$ of 'covering' morphisms for every $a \in C$ with the following conditions.

- every isomorphism is a covering singleton family,
- (Stability) The pullback of a covering family is a covering family. If $\{f_i : a_i \to a\}$ is covering and $g : b \to a$, then $\{f'_i : a_i \times_a b \to b\}$ is covering.
- (Transitivity) If $\{f_i : a_i \to a\}$ is a covering family and $\{f_{ij} : a_{ij} \to a_i\}$ for every i, then $\{f_{ij} : a_{ij} \to a\}$ is a covering family.

Generating a topology from a basis: take any sieve containing a covering family to be a covering sieve.

1.1.1 Sheaves

Definition 6 (Sheaves)**.** Let $(C, \mathcal{T})$ be a site. Let $\mathfrak{F} \in \hat{C}$.

A presheaf that admits a unique amalgamation for every matching family is called a sheaf. The category $\mathrm{Shv}(C)$ is the full subcategory in $\hat{C}$ all sheaves. Let i be the inclusion functor $\mathrm{Shv}(C) \to \hat{C}$.

In other words, we call $\mathfrak{F}$ a sheaf if the map

$$\mathfrak{F}(a) \to \mathfrak{F}(R)$$

$$a : x \mapsto \{\mathfrak{F}(i)(x)\}_{i \in R}$$

is an isomorphism.

Definition 7 (Plus construction)**.** Let $(C, \mathcal{T})$ be a site. Let $a, a' \in C$ and $f : a \to a'$. Let $\mathfrak{F} \in \hat{C}$. Define the functor $(-)^+ : \hat{C} \to \hat{C}$ as follows

On objects:

$$(\mathfrak{F})^+(a) = \frac{\{(R, \varphi) \mid R \in \mathcal{T}(a), \varphi \in \Gamma(R; \mathfrak{F})\}}{\sim},$$

$$(\mathfrak{F})^+(f)([[(R, \varphi)]]) = [(f^*R, \varphi h(f))].$$

The equivalence relation is defined as:

$$(R, \varphi) \sim (S, \phi)$$

if $\varphi = \phi$ on some $Q \subset R \cap S$

Let $L : \mathfrak{F} \to \mathfrak{F}'$. Then

$$((L)^+)_a([(R, \varphi)]) = [(R, L \circ \varphi)]$$

This functor comes with a natural transformation $\omega : \mathrm{Id} \to (-)^+$ defined by

$$\omega_{\mathfrak{F},a}(x) = [(\max(a), y]$$

$$y(i) = \mathfrak{F}(i)(x).$$

Lemma 8. *Let $\mathfrak{F}$ be a presheaf, $\mathfrak{G}$ a sheaf and $g : \mathfrak{F} \to \mathfrak{G}$ a morphism in $\hat{C}$. Then g factors through $\omega_{\mathfrak{F}}$ via a unique g'.*

Lemma 9. *For every presheaf $\mathfrak{F}$, $(\mathfrak{F})^+$ is separated.*

Lemma 10. *If $\mathfrak{F}$ is separated, then $\mathfrak{F}$ is a sheaf.*

Definition 11. Define $sh = (-)^+ \circ (-)^+$.

Lemma 12 (Sheafification adjunction)**.** *Let $Y = (C, \mathcal{T})$ be a site. The functor sh is left adjoint to the inclusion $\hat{Y} \to Shv(C)$ with unit*

$$\omega^2_{\mathfrak{F}} = \omega_{(\mathfrak{F})^+} \circ \omega_{\mathfrak{F}}$$

1.1.2 Relative topology

Definition 13 (Relative topology)**.** Let $(C, \mathcal{T})$ be a site. Let $a \in C$.

Set $\mathcal{T}_a(f) = \{R^f : R \in \mathcal{T}(b)\}$. Define the induced topology $\mathcal{T}_a$ on C_a by, for each $f \in C_a$

$$\mathcal{T}_a(f) = Q^f(\mathcal{T}(\mathrm{Dom}(f))).$$

Lemma 14. *$\mathcal{T}_a$ defines a Grothendieck topology*

Proof. Axiom 1: Q^f is an equivalence of posets. So the terminal object is send to the terminal object. Hence $\max(f) \in \mathcal{T}_a(f)$.

Axiom 2 & 3 are consequences of: Q^f is an equivalence and Q^f commutes with sieve pullback. ∎

Definition 15 (Oversite)**.** Let $Y = (C, \mathcal{T})$ be a site. Let $a \in C$. Define the site Y_a to be the category C_a with the induced topology T_a.

Definition 16 (Natural transformation s)**.** Let $(C, \mathcal{T})$ be a site. Let $a, b \in C$ and $f : b \to a$.

Let $\{x_i\}$ be a compatible family indexed by a sieve R on b. The same set $\{x_i\}$ is a compatible family on f indexed by $Q^f(R)$. Define the natural transformation

$$s : u \circ (-)^+ \to (u \circ -)^+$$

$$s_{\mathfrak{F}} : u \circ (\mathfrak{F})^+ \to (u \circ \mathfrak{F})^+$$

$$s_{\mathfrak{F},f}([(x_i)_{i \in R}]) = [(x_i)_{i \in Q^f(R)}].$$

Lemma 17 (Restriction commutes with plus)**.** *The transformation* $s_{\mathfrak{F}}$ *is an isomorphism of functors.*

Proof. We use the variables from definition ?. The morphism $s_{\mathfrak{F},f}$ has an inverse, again sending the set to itself and applying Q^f on the indexing sieve. These are mutual inverses, so $s_{\mathfrak{F},f}$ is an isomorphism for each f. ∎

Lemma 18 (s and ω commute)**.** *Let* $\mathfrak{F}$ *be a presheaf on* $(C, \mathcal{T})$*. Let* $f : b \to a \in C$*. Then* $\omega_{u \circ \mathfrak{F}} = s_{\mathfrak{F}} \circ (u \circ \omega_{\mathfrak{F}})$*.*

Proof. For any section $x \in \Gamma(b; \mathfrak{F})$. Let $x_i = \mathfrak{F}(i)(x)$ for any morphism $i \in C$. Note that $Q^f(\max(f)) = \max(b)$. This implies that the compatible family $\{x_i\}$ indexed by the maximal sieve on f is sent by s to the same set $\{x_i\}$ indexed by the maximal sieve on b. In diagram form, that

$$\begin{tikzcd} u \circ \mathfrak{F} \arrow[d, "u \circ \omega_{\mathfrak{F}}"] \arrow[dr, "\omega_{u \circ \mathfrak{F}}"] & \\ u \circ (\mathfrak{F})^+ \arrow[r, "s"] & (u \circ \mathfrak{F})^+ \end{tikzcd}$$

commutes. ∎

Corollary 19. *Let* $(\mathsf{C}, \mathcal{T}, \mathfrak{O})$ *be a ringed site. Let* $b \in \mathsf{C}$. *Let* M *be a* $\Gamma(1; \mathfrak{O})$*-module. The morphism* $\omega^2_{\lambda(M),1,b} : \lambda(M)(b) \to \Lambda(M)(b)$ *is isomorphic to*

$$\omega^2_{\lambda(M \otimes \mathfrak{O}(b)),b,Id_b} : \lambda(\lambda(M \otimes \mathfrak{O}(b)) \to \Lambda(\lambda(M \otimes \mathfrak{O}(b)).$$

Definition 20. Let P be the plus functor and $U = u \circ -$. Define

$$s^2 : U \circ P \circ P \to P \circ P \circ U$$

as

$$s^2_{\mathfrak{F}} = P(s_{\mathfrak{F}}) \circ s_{P(\mathfrak{F})}.$$

Lemma 21. *Let* $\mathfrak{F}$ *be a presheaf on* $(\mathsf{C}, \mathcal{T})$. *Let* $f : b \to a \in \mathsf{C}$. *Let* $\mathfrak{F}$ *be a presheaf on* C. *Then* $\omega^2_{u \circ \mathfrak{F}} = s^2_{\mathfrak{F}} \circ (u \circ \omega^2_{\mathfrak{F}})$.

Proof. Let $a \in \mathsf{C}$. We have

$$\omega^2_{u \circ \mathfrak{F}} = \omega_{(u \circ \mathfrak{F})^+} \circ \omega_{u \circ \mathfrak{F}},$$

$$s^2_{\mathfrak{F}} = (s_{\mathfrak{F}})^+ \circ s_{(\mathfrak{F})^+},$$

$$u \circ \omega^2_{\mathfrak{F}} = u \circ (\omega_{(\mathfrak{F})^+} \circ \omega_{\mathfrak{F}}).$$

Let $x \in \Gamma(b; u \circ \mathfrak{F})$.

Then

$$\omega^2_{u \circ \mathfrak{F}, f}(x) = ((x_i)_{i \in \max(\mathrm{Dom}(j))_j})_{j \in \max(f)}$$

with $x_i = \mathfrak{F}i(x)$. We also have

$$u \circ \omega^2_{\mathfrak{F}, f}(x) = ((x_i)_{i \in \max(\mathrm{Dom}(j))_j})_{j \in \max(b)},$$

where $\mathrm{Dom}(j) \in \mathsf{C}_a$. Apply $s_{(\mathfrak{F})^+}$ on this to get

$$((x_i)_{i \in \max(\mathrm{Dom}(j))_j})_{j \in \max(f)},$$

where $\mathrm{Dom}(j) \in \mathsf{C}$. Lastly, apply $(s_{(\mathfrak{F})^+})^+$ to get

$$((x_i)_{i \in \max(\mathrm{Dom}(j))_j})_{j \in \max(f)}$$

where $\mathrm{Dom}(j) \in \mathsf{C}_a$. The statement of the lemma is established. ∎

Corollary 22. *Let* $Y = (C, \mathcal{T})$. *Let* $a, b \in C$. *Let* $f : b \to a$. *Sheafifying and restricting commute via the iso*

$$s^2 : sh_b \circ *\big|_b \to *\big|_b \circ sh_a.$$

Lemma 23 (λ commutes with restriction)**.** *Let* $(C, \mathcal{T}, \mathfrak{O})$ *be a ringed site. Let* $f : b \to a \in C$. *The functors* $u \circ \lambda$ *and* $\lambda \circ (- \otimes \Gamma(a; \mathfrak{O}))$ *are the same functor.*

Lemma 24 (λ counit commute with restriction)**.** *Let* C *be a category. Let* $a \in C$. *Let* ϵ *be the counit of the adjunction* $\lambda \dashv \Gamma(1; -)$ *on* C. *Let* ϵ_a *be the counit of the adjunction* $\lambda_a \dashv \Gamma(1; -)$ *on* C_a. *Let* $\mathfrak{F}$ *be a presheaf module on* C. *By lemma ?,* $\lambda(\Gamma(1; \mathfrak{F}))\big|_a = \lambda(\Gamma(1; \mathfrak{F}) \otimes \Gamma(a; \mathfrak{O}))$. *We have* $u \circ (\epsilon) = \epsilon_a$ *on this presheaf module.*

Proof. Let $f : b \to a \in C_a$. Let $m \otimes r \in \lambda(\Gamma(1; \mathfrak{F}) \otimes \Gamma(a; \mathfrak{O}))(f) = \Gamma(1; \mathfrak{F}) \otimes \Gamma(b; \mathfrak{O})$. Then both maps do $m \otimes r \mapsto rm$. ∎

Lemma 25 (λ commutes with restriction)**.** *Let* $(C, \mathcal{T}, \mathfrak{O})$ *be a ringed site. Let* $f : b \to a \in C$. *The functors* $u \circ \Lambda$ *and* $\Lambda \circ (- \otimes \Gamma(a; \mathfrak{O}))$ *are isomorphic functors.*

Lemma 26 (Λ counit commute with restriction)**.** *Let* C *be a category. Let* $a \in C$. *Let* ϵ *be the counit of the adjunction* $\Lambda \dashv \Gamma(1; -)$ *on* C. *Let* ϵ_a *be the counit of the adjunction* $\Lambda_a \dashv \Gamma(1; -)$ *on* C_a. *Let* $\mathfrak{F}$ *be a presheaf module on* C. *By lemma ?,* $\Lambda(\Gamma(1; \mathfrak{F}))\big|_a = \Lambda(\Gamma(1; \mathfrak{F}) \otimes \Gamma(a; \mathfrak{O}))$. *We have* $u \circ (\epsilon) = \epsilon_a$ *on this presheaf module.*

Proof. ∎

Lemma 27. *Let* $(C, \mathcal{T})$ *be a site. Let* $a \in C$. *Consider the conunit* ϵ *of* $\Gamma(1; -) \dashv \Lambda(-)$ *on* C *and the counit* ϵ_a *of the same adjunction on* C_a. *Then* $\epsilon\big|_a = \epsilon_a$.

Proof. Let $\mathfrak{F}$ be a sheaf module on C. Let $f : b \to a \in C$. We want to show that $\epsilon_{\mathfrak{F},a,f} : \Lambda(\Gamma(1; \mathfrak{F})) \to \mathfrak{F}$ is the same map as $\epsilon_{\mathfrak{F}}\big|_{af} : \Lambda(\Gamma(1; \mathfrak{F})) \to \mathfrak{F}$

$$\alpha : \lambda(\Gamma(1; \mathfrak{F})) \to \mathfrak{F},$$

$$m \otimes r \mapsto mr$$

composed with $\omega^2_{\lambda(\Gamma(1;\mathfrak{F}))}$. ∎

2 Caffine objects

2.1 Caffine objects

Definition 28 (Caffine object)**.** Let $Y = (C, \mathcal{T}, \mathfrak{O})$ be a ringed site. Let $a \in C$ be an object. We call a caffine if the unit η and co-unit ϵ of the adjunction $\Gamma(1; -) \dashv \Lambda(-)$ on Y_a are natural isomorphisms.

Example 29 (Examples of caffine objects)**.** The main example to keep in mind is $\mathsf{Spec}(R) \in \mathsf{Sch}$.

Definition 30 (caffine cover)**.** Let $(X, \mathcal{T}, \mathfrak{O})$ be a ringed site. A family of maps $\{a_i \to a\}$ is called a caffine covering of a if every a_i is caffine and the family is a covering family.

Definition 31. We say that a ringed site $(C, \mathcal{O}, \mathfrak{T})$ has enough affines if any object admits a caffine covering.

Lemma 32. *Let $(C, \mathcal{T}, \mathfrak{O})$ be a ringed site. Let $a \in C$. Let $\{b_i \to a\}$ be a caffine covering on a. Assume every map $b_i \to a$ is restrictive. Then the counit ϵ of the adjunction $\Gamma(1; -) \dashv \Lambda(-)a$ is a natural isomorphism.*

Proof. Let $\mathfrak{F}$ be a quasi-coherent sheaf module. Set $M = \Gamma(a; \mathfrak{F})$. Set $M_i = \Gamma(b_i; \mathfrak{F})$. Set $\beta_i = \epsilon_{\mathfrak{F},a}|_{b_i}$. By lemma ? $\beta_i \cong \epsilon_{b_i}$, hence $beta_i$ is an isomorphism.

■

Lemma 33. *Let $(C, \mathcal{T}, \mathfrak{O})$ be a ringed site. Let $a \in C$. Let M be a $\Gamma(a; \mathfrak{O})$-module. The component*

$$\omega^2_{\lambda(M),a} : \lambda(M) \to \Lambda(M)$$

at Id_a of the sheafification morphism

$$\omega^2_{\Lambda(M)} : \lambda(M) \to \Lambda(M)$$

is equal to the unit of $\Lambda \dashv \Gamma(1;-)$ *in* C_a.

Proof. Consider the following maps, which you get by repeatedly calling on an adjunction.

$$\mathrm{Id} : \Lambda(M) \to \Lambda(M)$$
$$\omega^2_{\Lambda(M)} : \lambda(M) \to \Lambda(M) \text{ use sheafification adjunction, see lemma ..}$$
$$\omega^2_{\lambda(M),a} M \to \Gamma(a; \Lambda(\mathfrak{M})) \text{ take sections at } a$$

We took the adjunct of Id with respect to the sheafification adjunction as the first step. Then we took the adjunct of the result wrt the λ ajunction. Hence we get the adjunct of Id wrt the Λ adjunction. so the last map is actually the unit of the Λ adjunction. ∎

Corollary 34. *Let* $(C, \mathcal{T}, \mathfrak{O})$ *be a ringed site. Let* $a \in C$ *be caffine. Then* $\omega^2_{\lambda(M),a}$ *is an isomorphism for any* $\mathfrak{O}(a)$*-module* M.

Theorem 35 (Morphism between caffines is restrictive)**.** *Let* $Y = (C, \mathcal{T}, \mathfrak{O})$. *Let* $f : b \to a \in C$ *be a morphism between caffine objects, then* f *is restrictive.*

Proof. Let $\mathfrak{F}$ be a quasi-coherent module on Y_a. Let $M = \Gamma(a; \mathfrak{F})$. Since a is caffine, we have $\mathfrak{F} = \Lambda(M)$.

We have to show that the adjunct, along the extension of scalars adjunction, of $\mathfrak{F}(f)$

$$\Gamma(a; \mathfrak{F}) \otimes_{\Gamma(a;\mathfrak{O})} \Gamma(b; \mathfrak{O}) \to \Gamma(b; \mathfrak{F})$$

is an isomorphism.

This adjunct is the component at b of the natural transformation $\omega^2_{\lambda(\Gamma(1;\mathfrak{F})),a}$. Since b is caffine, this component is an isomorphism.

∎