Prelims

Let $Y=(X,\mathcal{T},\mathfrak{D})$ be a ringed site. Let $R=\Gamma(1;\mathfrak{D})$. Let $a,b,c,c'\in X$. Let \mathfrak{F} be a quasi-coherent module on Y_a . Let $M=\Gamma(a;\mathfrak{F})=\Gamma(1;\mathfrak{F})$. Let $f:b\to a$. We pick all these Some basic definitions and constructions.

Definition 1 (Over/Under categories). Let C and C' be categories. Let $F: C \to C'$ and $Z \in C'$. Define the category C_Z and C^Z as follows

Obj
$$(C_Z) := \{(X, u) \mid X \in C, u : F(X) \to Z\},$$

 $Hom((X, u), (Y, v)) := \{f : X \to Y \mid v \circ F(f) = u\},$

and

$$Obj(C^{Z}) := \{(X, u) \mid X \in C, u : Z \to F(X)\},$$

$$Hom((X, u), (Y, v)) := \{f : X \to Y \mid F(f)u = v\}.$$

We get faithfull functors $C_Z \to C: (X, \mathfrak{u}) \to X$ and $C^Z \to C: (X, \mathfrak{u}) \to X$. We will call both functors \mathfrak{u} and suppress the functor where there can be no confusion

Definition 2. Let M, N be an R-module. Let $g: M \to N$. Define

$$\lambda : R\text{-Mod} \to \underline{PMod}(Y)$$

by

$$\lambda(M)(a) = M \otimes_R \Gamma(a; \mathfrak{O}),$$

$$\lambda(M)(f) : Id \otimes \mathfrak{O}(f),$$

$$\lambda(g) = (a: g \otimes Id).$$

Definition 3. Define

$$\Lambda: R\text{-Mod} \to Mod(Y)$$

by $sh \circ \lambda$.

This functor is left adjoing to the global sections functor, which I will prove in the next episode.

Lemma 4. Let S be a subset of Hom(-,f). Then u(S) is a sieve on u(f) = b if and only if S is a sieve on f.

Proof. =>: Let $h: d \to b \in S$ and $k: e \to d$ be arbitrary. By assumption $u(hk) \in u(S)$. The functor u is faithfull, so $hk \in S$.

 \leq : Let h: d \rightarrow b \in u(S) and k: e \rightarrow d be arbitrary. By assumption hk \in S, hence $u(hk) \in u(S)$.

We will define the induced topology S on C_a . That u considered as a map on sieves commutes with the pullback of sieves is used and will not be proved.

Definition 5. Let $\mathcal{T}(u(f))$ be the set of covering sieves on $u(f) \in X$. By the previous lemma sieves on u(f) are sieves on f. Let $S(f) = \{R \mid u(R) \in \mathcal{T}(u(f)) \text{ be the induced } \}$

- topology. So u(R) is covering on u(f) if and only if R is covering on f.

 a) Since u commutes with pullback of sieves, we have $\max(u(f)) = u(\max(f)) = \max(f)$, hence $\max(f) \in S(f)$.

 What does max mean? Oh, the max max mean?
- b) Let R be a covering sieve on f. Let $h: b' \to a$ and $p: b' \to b$ with fp = h. Commutativity of u and pulling back implies that $u(p)^*u(R) = u(p^*R)$. Hence p^*R is covering since $u(p^*R)$ is.
- c) Let R be a covering sieve on f and Q be a sieve on f. Let h: $b' \to a$ and p: $b' \to b \in R$, hence with fp = h. Assume p^*Q is covering for every such p. Then $u(p^*Q) = u(p)^*u(Q)$ is covering for every p. We know that u(R) is covering hence u(Q) must be, which implies that Q is covering.

We proved that S is indeed a Grothendieck topology.

Main

Lemma 6. Let a be caffine. The global component of the sheafification morphism is equal to the unit of $\Lambda \dashv \Gamma(1; -)$ in C_a .

Proof. Let M be a $\Gamma(a; \mathfrak{D})$ -module. Consider the following maps, which you get by

repeatedly calling on an adjunction bijection. Let i be the universal sheafification morphism.

$$\Lambda(M) \to \Lambda(M)$$

 $i: \lambda(M) \to \Lambda(M)$ use sheafification adjunction

 $M \to \Gamma(\alpha; \Lambda(\mathfrak{M}))$ use $\lambda \dashv \Gamma(\alpha; -)$ What is this 372 ?

what bijection.

If you compose the two adjunction bijections used, you get the bijection of $\Lambda \dashv \Gamma(\alpha; -)$ by definition, so the last map is actually η_M . Hence $i_a = \eta_M$, which is an iso by assumption. assumption.

Lemma 7. Sheafifying and restricting commute. In formula form

$$sh_b \circ *|_b \cong *|_b \circ sh_a.$$

Proof. I will prove that we have a natural isomorphism

$$s: sh_b \circ *|_b \to *|_b \circ sh_a.$$

Let \mathfrak{F} be a presheaf on Y_a . Let $\mathfrak{H} = \operatorname{sh}(F|_b)$ and $\mathfrak{K} = \operatorname{sh}(F)|_b$. Let T be a covering sieve on g in Y_b and $j \in T$. Let S_j be a covering sieve on $\operatorname{Dom}(j)$ in Y_b and $i \in S_j$.

Let $x = (x_{i,j}) \in \operatorname{sh}(F|_b)$ be indexed by S_j and T. We have $x_{i,j} \in \Gamma(\operatorname{Dom}(i);\mathfrak{F})$. Define $X_i \in S_j$.

 $s_{\mathfrak{g}}(x) = (x_{u(\mathfrak{i}),u(\mathfrak{j})}) \text{ with indexing covering sieves } u(S_{\mathfrak{j}}), u(T).$

Let $x \sim y$. Let R be the covering sieve on which they are the same. Then $s_g(x) \sim s_g(y)$ because they are the same on u(R). Hence this map is well-defined.

Let $s_g(x) = s_g(y)$. Then there is some covering sieve R on fg on which they agree. Consider $u^*(R)$ as a covering sieve on g and its is clear that x and y must agree on it, hence the map is injective.

Let $y = (y_{k,l})$ be an element of $\Gamma(c; \mathfrak{K})$ which is indexed by V, W. Then $s_g(y') = y$ where y' has the same elements as y but is indexed by u(V), u(W), so $y' \in \Gamma(c; \mathfrak{H})$. Hence s_q is surjective.

Let $h: c' \to b$ and $p: c' \to c$, such that gp = h. We will show that $s_h \mathfrak{H}(t) = \mathfrak{K}(t) s_g$. See below diagram. Let $x = (x_{i,j}) \in \Gamma(c; \mathfrak{H})$ with indexing covering sieves S_j and T. Then

It's sounds like you're using a particular sheafification construction. It would be helpful to let the reader know what your notation ("indexed by", """) means.

 $\mathfrak{K}(t)(s_g(x)) = (x_{k,l})$ with indexing covering sieves t^*S_l and t^*T . The other one becomes $s_h(\mathfrak{H}(t)(x)) = (x_{k,l})$ with indexing covering sieves t^*S_l and t^*T . Hence s is natural.

$$\Gamma(h;\mathfrak{H}) \xrightarrow{s_h} \Gamma(h;\mathfrak{K})$$

$$\mathfrak{H}(t) \uparrow \qquad \qquad \mathfrak{H}(t) \uparrow$$

$$\Gamma(g;\mathfrak{H}) \xrightarrow{s_g} \Gamma(g;\mathfrak{K})$$

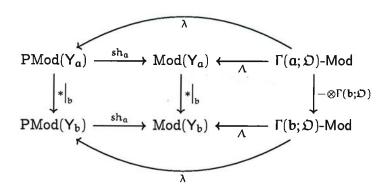
Proposition 8. The adjunct of f

What are the hypotheses

$$\Gamma(\mathfrak{a};\mathfrak{F})\otimes_{\Gamma(\mathfrak{a};\mathfrak{O})}\Gamma(\mathfrak{b};\mathfrak{O})\to\Gamma(\mathfrak{b};\mathfrak{F})$$

is an isomorphism.

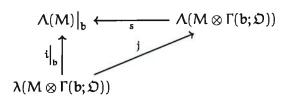
Consider



By a previous lemma, the left square commutes. By definition the two 'triangles' commute too and the outer square commute, hence the right square also commutes. Therefore $M \otimes \Gamma(b; \mathfrak{D}) \cong \Gamma(b; \mathfrak{F})$. This is the proof you wrote down friday.

The requirement is not to find any isomorphism but a specific one. So I think this is not enough and we need to do some bookkeeping and see if the witnessing isomorphism is our map.

Consider



The natural transformation j is the universal sheafification morphism coming from sh_b . We have seen that (b;j) and s are isomorphisms is really a different s, right? Let $g:c\to b$. Let $x = m\otimes r \in \lambda(M\otimes \Gamma(c;\mathfrak{D}))$. Then $j_g(x)=(x_i)$ indexed by the maximal sieve on g and $i_g(x)=i_{fg}(x)=(x_i)$ indexed by the maximal sieve on gf. Hence we get $s_g(j_g(x))=i_g(x)$, so the triangle commutes. Evaluating everything on the terminal, in this case on b, shows that two out of three maps are isomorphisms, hence i_b is an isomorphism.

You have a lot of details here, but I feel like I'm reading half a conversation. Here's what I think would help:

- Putting in the setup for each proposition into its statement. This prevents name-space collisions and makes it clearer which hypotheses the you're using.
- Make the use of fonts and letters consistent. Is fan object, or a morphism? Is u a morphism, or a functor? Is X an object, or a category? When you write "Let g: c -> b", which category do these live in?
- Describe the approach you will take before getting into the calculations. "We will construct a natural transformation each component of 5 separately, prove it is bijective, and then show that the resulting family is natural."