## **Prelims**

Let  $Y=(X, \mathfrak{I}, \mathfrak{O})$  be a ringed site. Let  $R=\Gamma(1;\mathfrak{O})$ . Let  $\mathfrak{a}, \mathfrak{b}, \mathfrak{c}, \mathfrak{c}' \in X$ . Let  $\mathfrak{F}$  be a quasi-coherent module on  $Y_{\mathfrak{a}}$ . Let  $M=\Gamma(\mathfrak{a};\mathfrak{F})=\Gamma(1;\mathfrak{F})$ . Let  $\mathfrak{f}:\mathfrak{b}\to\mathfrak{a}$ .

Some basic definitions and constructions.

**Definition 1** (Over/Under categories). Let C and C' be categories. Let  $F: C \to C'$  and  $Z \in C'$ . Define the category  $C_Z$  and  $C^Z$  as follows

$$Obj(C_{Z}) := \{(X, u) \mid X \in C, u : F(X) \to Z\},$$

$$Hom((X, u), (Y, v)) := \{f : X \to Y \mid v \circ F(f) = u\},$$

and

$$Obj(C^{Z}) := \{(X, u) \mid X \in C, u : Z \to F(X)\},$$

$$Hom((X, u), (Y, v)) := \{f : X \to Y \mid F(f)u = v\}.$$

We get faithfull functors  $C_Z \to C: (X, \mathfrak{u}) \to X$  and  $C^Z \to C: (X, \mathfrak{u}) \to X$ . We will call both functors  $\mathfrak{u}$  and suppress the functor where there can be no confusion

**Definition 2.** Let M, N be an R-module. Let  $g: M \to N$ . Define

$$\lambda:R\text{-}Mod\to PMod(Y)$$

by

$$\lambda(M)(\alpha) = M \otimes_R \Gamma(\alpha; \mathfrak{O}),$$
$$\lambda(M)(f) : Id \otimes \mathfrak{O}(f),$$
$$\lambda(g) = (\alpha : g \otimes Id).$$

Definition 3. Define

$$\Lambda : R\text{-Mod} \rightarrow Mod(Y)$$

by  $sh \circ \lambda$ .

This functor is left adjoing to the global sections functor, which I will prove in the next episode.

**Lemma 4.** Let S be a subset of Hom(-,f). Then u(S) is a sieve on u(f) = b if and only if S is a sieve on f.

*Proof.* =>: Let  $h: d \to b \in S$  and  $k: e \to d$  be arbitrary. By assumption  $u(hk) \in u(S)$ . The functor u is faithfull, so  $hk \in S$ .

<=: Let  $h: d \to b \in \mathfrak{u}(S)$  and  $k: e \to d$  be arbitrary. By assumption  $hk \in S$ , hence  $\mathfrak{u}(hk) \in \mathfrak{u}(S)$ .

We will define the induced topology S on  $C_{\alpha}$ . That u considered as a map on sieves commutes with the pullback of sieves is used and will not be proved.

**Definition 5.** Let  $\mathfrak{T}(\mathfrak{u}(f))$  be the set of covering sieves on  $\mathfrak{u}(f) \in X$ . By the previous lemma sieves on  $\mathfrak{u}(f)$  are sieves on f. Let  $\mathfrak{S}(f) = \{R \mid \mathfrak{u}(R) \in \mathfrak{T}(\mathfrak{u}(f)) \text{ be the induced topology. So } \mathfrak{u}(R) \text{ is covering on } \mathfrak{u}(f) \text{ if and only if } R \text{ is covering on } f.$ 

- a) Since u commutes with pullback of sieves, we have max(u(f)) = u(max(f)) = max(f), hence  $max(f) \in S(f)$ .
- b) Let R be a covering sieve on f. Let  $h:b'\to a$  and  $p:b'\to b$  with fp=h. Commutativity of u and pulling back implies that  $\mathfrak{u}(p)^*\mathfrak{u}(R)=\mathfrak{u}(p^*R)$ . Hence  $p^*R$  is covering since  $\mathfrak{u}(p^*R)$  is.
- c) Let R be a covering sieve on f and Q be a sieve on f. Let  $h:b'\to a$  and  $p:b'\to b\in R$ , hence with fp=h. Assume  $p^*Q$  is covering for every such p. Then  $\mathfrak{u}(p^*Q)=\mathfrak{u}(p)^*\mathfrak{u}(Q)$  is covering for every p. We know that  $\mathfrak{u}(R)$  is covering hence  $\mathfrak{u}(Q)$  must be, which implies that Q is covering.

We proved that S is indeed a Grothendieck topology.

## Main

**Lemma 6.** Let  $\alpha$  be caffine. The global component of the sheafification morphism is equal to the unit of  $\Lambda \dashv \Gamma(1;-)$  in  $C_{\alpha}$ .

*Proof.* Let M be a  $\Gamma(a; \mathfrak{O})$ -module. Consider the following maps, which you get by

repeatedly calling on an adjunction bijection. Let i be the universal sheafification morphism.

$$\Lambda(M) \rightarrow \Lambda(M)$$

 $i: \lambda(M) \to \Lambda(M)$  use sheafification adjunction

$$M \to \Gamma(\alpha; \Lambda(\mathfrak{M}))$$
 use  $\lambda \dashv \Gamma(\alpha; -)$ 

If you compose the two adjunction bijections used, you get the bijection of  $\Lambda \dashv \Gamma(\alpha; -)$  by definition, so the last map is actually  $\eta_M$ . Hence  $i_\alpha = \eta_M$ , which is an iso by assumption.

Lemma 7. Sheafifying and restricting commute. In formula form

$$sh_b \circ *|_b \cong *|_b \circ sh_a$$
.

*Proof.* I will prove that we have a natural isomorphism

$$s: sh_b \circ *|_b \to *|_b \circ sh_a$$
.

Let  $\mathfrak{F}$  be a presheaf on  $Y_a$ . Let  $\mathfrak{H} = \mathfrak{sh}(F|_b)$  and  $\mathfrak{K} = \mathfrak{sh}(F)|_b$ . Let T be a covering sieve on g in  $Y_b$  and  $j \in T$ . Let  $S_j$  be a covering sieve on Dom(j) in  $Y_b$  and  $i \in S_j$ .

Let  $x = (x_{i,j}) \in sh(F|_b)$  be indexed by  $S_j$  and T. We have  $x_{i,j} \in \Gamma(Dom(i); \mathfrak{F})$ . Define  $s_g(x) = (x_{u(i),u(j)})$  with indexing covering sieves  $u(S_j), u(T)$ .

Let  $x \sim y$ . Let R be the covering sieve on which they are the same. Then  $s_g(x) \sim s_g(y)$  because they are the same on u(R). Hence this map is well-defined.

Let  $s_g(x) = s_g(y)$ . Then there is some covering sieve R on fg on which they agree. Consider  $u^*(R)$  as a covering sieve on g and its is clear that x and y must agree on it, hence the map is injective.

Let  $y=(y_{k,l})$  be an element of  $\Gamma(c;\mathfrak{K})$  which is indexed by V,W. Then  $s_g(y')=y$  where y' has the same elements as y but is indexed by  $\mathfrak{u}(V),\mathfrak{u}(W)$ , so  $y'\in\Gamma(c;\mathfrak{H})$ . Hence  $s_g$  is surjective.

Let  $h: c' \to b$  and  $p: c' \to c$ , such that gp = h. We will show that  $s_h \mathfrak{H}(t) = \mathfrak{K}(t) s_g$ . See below diagram. Let  $x = (x_{i,j}) \in \Gamma(c; \mathfrak{H})$  with indexing covering sieves  $S_i$  and T. Then  $\mathfrak{K}(t)(s_g(x))=(x_{k,l})$  with indexing covering sieves  $t^*S_l$  and  $t^*T$ . The other one becomes  $s_h(\mathfrak{H}(t)(x))=(x_{k,l})$  with indexing covering sieves  $t^*S_l$  and  $t^*T$ . Hence s is natural.

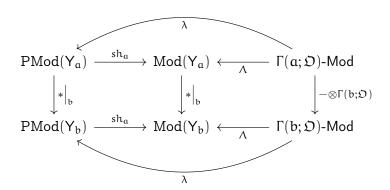
$$\begin{array}{ccc} \Gamma(h;\mathfrak{H}) & \xrightarrow{\quad s_h \quad} \Gamma(h;\mathfrak{K}) \\ \\ \mathfrak{H}(t) & & & \mathfrak{K}(t) \\ \\ \Gamma(g;\mathfrak{H}) & \xrightarrow{\quad s_g \quad} \Gamma(g;\mathfrak{K}) \end{array}$$

Proposition 8. The adjunct of f

$$\Gamma(\mathfrak{a};\mathfrak{F})\otimes_{\Gamma(\mathfrak{a};\mathfrak{O})}\Gamma(\mathfrak{b};\mathfrak{O})\to\Gamma(\mathfrak{b};\mathfrak{F})$$

is an isomorphism.

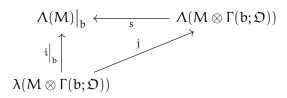
Consider



By a previous lemma, the left square commutes. By definition the two 'triangles' commute too and the outer square commute, hence the right square also commutes. Therefore  $M \otimes \Gamma(b;\mathfrak{D}) \cong \Gamma(b;\mathfrak{F})$ . This is the proof you wrote down friday.

The requirement is not to find any isomorphism but a specific one. So I think this is not enough and we need to do some bookkeeping and see if the witnessing isomorphism is our map.

Consider



The natural transformation j is the universal sheafification morphism coming from  $sh_b$ . We have seen that  $\Gamma(b;j)$  and s are isomorphisms

Let  $g:c\to b$ . Let  $x=m\otimes r\in \lambda(M\otimes \Gamma(c;\mathfrak{O}))$ . Then  $j_g(x)=(x_i)$  indexed by the maximal sieve on g and  $i_g(x)=i_{fg}(x)=(x_i)$  indexed by the maximal sieve on gf. Hence we get  $s_g(j_g(x))=i_g(x)$ , so the triangle commutes. Evaluating everything on the terminal, in this case on b, shows that two out of three maps are isomorphisms, hence  $i_b$  is an isomorphism.