

Affine Objects

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1 Preliminaries

I will introduce the basic notions here that are not assumed to be known.

1.1 Basic Category Theory

Definition 1 (Presheaf category). Let C be a category. Let $a \in C$. Let $f : a' \rightarrow a$. We define the category of presheaves on C as the category of contravariant functors to the category of sets Set . We will denote it by \hat{C} .

Define the functor $h : C \rightarrow \hat{C}$ as follows

$$a \mapsto \text{Hom}(-, a),$$

$$f \mapsto f \circ -.$$

This functor is fully faithful by the Yoneda lemma.

Notation 2. Let I, C be categories. Let $L : I \rightarrow C$ be a functor. The limit over this functor will be denoted by $\lim_{i \in I} L$. The colimit will be denoted by $\text{colim}_{i \in I} L$.

Definition 3 (Sections functor). For any $a \in C$ define the functor

$$\Gamma(c; -) : \hat{C} \rightarrow \text{Set}$$

by

$$\mathfrak{F} \mapsto \Gamma(a; \mathfrak{F}).$$

Let $L : I \rightarrow C$ be a small diagram. Define

$$\Gamma(\text{colim}_{i \in I} L(i); -) : \hat{C} \rightarrow \text{Set}$$

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by

$$\mathfrak{F} \rightarrow \text{Hom}(\text{colim}_{i \in I} L(i), \mathfrak{F}) = \lim_{i \in I} \text{Hom}(L(i), \mathfrak{F}).$$

By definition of a colimit these definitions coincide when a colimit exists in C .

Remark. The category \hat{C} is cocomplete so even if C does not have a terminal object, we can still compute the global sections as $\Gamma(1; -)$

Definition 4 (Over/Under categories). Let C and C' be categories. Let $F : C \rightarrow C'$ and $z \in C'$. Define the category C_z and C^z as

$$\begin{aligned} \text{Obj}(C_z) &:= \{(a, w) \mid a \in C, w : F(a) \rightarrow z\}, \\ \text{Hom}((a, w), (b, v)) &:= \{f : a \rightarrow b \mid v \circ F(f) = w\}, \end{aligned}$$

and

$$\begin{aligned} \text{Obj}(C^z) &:= \{(a, w) \mid a \in C, w : z \rightarrow F(a)\}, \\ \text{Hom}((a, w), (b, v)) &:= \{f : a \rightarrow b \mid F(f) \circ w = v\}. \end{aligned}$$

We get faithful functors $C_z \rightarrow C : (a, w) \rightarrow a$ and $C^z \rightarrow C : (a, w) \rightarrow a$. We will call both functors localization functors and denote them by u .

Definition 5 (direct image). Let $f : C \rightarrow D$. Define the direct image functor $f_* : \hat{D} \rightarrow \hat{C}$ as

$$f_* = - \circ f.$$

Definition 6 (Restriction). Let C, D be categories. Let $\mathfrak{F} \in \hat{D}$. Let $\alpha : C \rightarrow D$ be a functor. The restriction of \mathfrak{F} to C along α is defined to be $\mathfrak{F} \circ \alpha$.

1.2 Topology

Definition 7 (Sieve). Let C be a category and $a \in C$. Define the maximal sieve $\max(a)$ on a to be the set of all morphisms to a . In formula,

$$\max(a) = \{f \in C \mid \text{Codom}(f) = a\}.$$

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A sieve S is a subset of $\max(a)$ such that $gf \in S$ for any $f \in S$ and any g .

Remark. Let C be a category and $a, b \in C$. Let $f : b \rightarrow a \in C_a$.

Any morphism to b is also a morphism to f and vice versa. This observation yields us that $\text{Sieves}(b) = \text{Sieves}(f)$. Moreover composition in C and C_a are the same, so this identification respects pullback of sieves.

Definition 8 (Sieve category). Let C be a category and $a \in C$. The sieve category $\text{Sieves}(a)$ consists of all the sieves on a as objects and inclusions of sieves as morphisms.

Definition 9 (Pullback of sieve). Let C be a category and $a, b \in C$. Let S be a sieve on a . Let $f : b \rightarrow a$.

The sieve f^*S on b is given by $f^*S(c) = \{g \in \text{Hom}(c, b) : fg \in S(c)\}$ for any $c \in C$.

To show that this is actually a subpresheaf of $h(b)$, let $k : c \rightarrow c'$ and $h \in f^*S(c')$. Hence $fh \in S(c')$ and so $fhk \in S(c)$. Conclude that $hk \in f^*S(c')$.

This defines a functor $f^* : \text{Sieves}(a) \rightarrow \text{Sieves}(b)$.

Definition 10 (Grothendieck Topology). A Grothendieck topology \mathcal{T} is a family $\mathcal{T}(a)$ of ‘covering’ sieves for every $a \in C$ with the following conditions:

1. $\max(a) \in \mathcal{T}(a)$
2. $f^*R \in \mathcal{T}(a')$ if $R \in \mathcal{T}(a)$ for all $f : a' \rightarrow a$
3. if $f^*R \in \mathcal{T}(a')$ for all $f \in S$ with $S \in \mathcal{T}(a)$ then $R \in \mathcal{T}(a)$

Definition 11 (Basis). Let C be a category with pullbacks. A Grothendieck pretopology \mathcal{B} is a collection $\mathcal{B}(a)$ of ‘covering’ families $\{f_i : a_i \rightarrow a\}$ of morphisms for every $a \in C$ with the following conditions.

1. every isomorphism is a covering singleton family,
2. (Stability) The pullback of a covering family is a covering family. If $\{f_i : a_i \rightarrow a\}$ is covering and $g : b \rightarrow a$, then $\{f'_i : a_i \times_a b \rightarrow b\}$ is covering.
3. (Transitivity) If $\{f_i : a_i \rightarrow a\}$ is a covering family and $\{f_{ij} : a_{ij} \rightarrow a_i\}$ for every i , then $\{f_{ij} : a_{ij} \rightarrow a\}$ is a covering family.

Generating a topology from a basis: take any sieve containing a covering family to be a covering sieve.

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Definition 12 (Site). A site (C, \mathcal{T}) is a category C with a Grothendieck topology \mathcal{T} . A morphism of sites $G : (C, \mathcal{T}) \rightarrow (D, \mathcal{S})$ is a functor $G' : C \rightarrow D$ such that

A functor $G' : C \rightarrow D$ is called cover-preserving if for every covering sieve R , the family $\{G'(f) | f \in R\}$ generates a \mathcal{S} -covering sieve.

The category **Sites** has as objects sites and as morphisms cover-preserving functors. When no confusion can arise then we will use C to denote the whole site (C, \mathcal{T}) .

1.2.1 Sheaves

Definition 13 (Matching family). Let C be a category. Let \mathfrak{F} be a presheaf on C . Let $\alpha \in C$ be an object. Let R be a sieve on α . A set $\{x_i\}_{i \in R}$ with $x_i \in \Gamma(\text{Dom}(i); \mathfrak{F})$ indexed by a sieve R and such that $x_{g \circ i} = \mathfrak{F}(g)(x_i)$ for any $g : b \rightarrow \text{Dom}(i)$ and $b \in C$ is called a ‘matching family’.

Definition 14 (Matching family/Morphisms). Let C be a category. Let \mathfrak{F} be a presheaf on C . Let $\alpha \in C$ be an object. Let R be a sieve on α . Define $\Gamma(R; \mathfrak{F}) = \text{Hom}(R, \mathfrak{F})$. An element $\varphi \in \Gamma(R; \mathfrak{F})$ is uniquely identified by the matching family $\{\varphi(i)\}_{i \in R}$ of images. Conversely, any matching family $\{x_i\}_{i \in R}$, with $x_i \in \Gamma(\text{Dom}(i); \mathfrak{F})$ indexed by R and such that $x_{g \circ i} = \mathfrak{F}(g)(x_i)$ for any $g : b \rightarrow \text{Dom}(i)$ and $b \in C$, uniquely identifies a map $\varphi : R \rightarrow \mathfrak{F}$. Namely, take $\varphi_\alpha(y) = x_y$.

Definition 15 (Amalgamation). An amalgamation of a matching family $\{x_i\}_R$ is an element $x \in \Gamma(\alpha; \mathfrak{F})$ such that $\mathfrak{F}(i)(x) = x_i$.

When you consider the matching family as a morphism φ , an amalgamation is a morphism $\phi : h(\alpha) \rightarrow \mathfrak{F}$ that extends φ .

Definition 16 (Separated presheaf). A presheaf \mathfrak{F} is separated if any matching family has at most one amalgamation.

Definition 17 (Sheaves). Let (C, \mathcal{T}) be a site. Let $\mathfrak{F} \in \hat{C}$.

A presheaf that admits a unique amalgamation for every matching family is called a sheaf. The category $\text{Shv}(C)$ is the full subcategory in \hat{C} of all sheaves.

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In other words, we call \mathfrak{F} a sheaf if for each $\mathfrak{a} \in \mathcal{C}$ and $\mathcal{R} \in \mathcal{T}(\mathfrak{a})$ the map

$$\begin{aligned} \Gamma(\mathfrak{a}; \mathfrak{F}) &\rightarrow \Gamma(\mathcal{R}; \mathfrak{F}) \\ x &\mapsto \{x_i \mid i \in \mathcal{R}\} \end{aligned}$$

where $x_i = \mathfrak{F}(i)(x)$ is an isomorphism.

Definition 18 (Plus construction). Let $(\mathcal{C}, \mathcal{T})$ be a site. Let $\mathfrak{a}, \mathfrak{a}' \in \mathcal{C}$ and $f : \mathfrak{a} \rightarrow \mathfrak{a}'$. Let $\mathfrak{F} \in \hat{\mathcal{C}}$. Define the functor $(-)^+ : \hat{\mathcal{C}} \rightarrow \hat{\mathcal{C}}$ as follows.

For all $\mathfrak{a} \in \mathcal{C}$,

$$\mathfrak{F}^+(\mathfrak{a}) = \frac{\{(\mathcal{R}, \varphi) \mid \mathcal{R} \in \mathcal{T}(\mathfrak{a}), \varphi \in \Gamma(\mathcal{R}; \mathfrak{F})\}}{\sim},$$

for all morphisms $f \in \mathcal{C}$,

$$\mathfrak{F}^+(f)([(\mathcal{R}, \varphi)]) = [(f^*\mathcal{R}, \varphi h(f))].$$

The equivalence relation is defined as $(\mathcal{R}, \varphi) \sim (\mathcal{S}, \phi)$ if $\varphi = \phi$ on some covering sieve $Q \subset \mathcal{R} \cap \mathcal{S}$.

Let $L : \mathfrak{F} \rightarrow \mathfrak{F}'$. Then

$$(L^+)_\mathfrak{a}([(\mathcal{R}, \varphi)]) = [(\mathcal{R}, L \circ \varphi)]$$

This functor comes with a natural transformation $\omega : \text{Id} \rightarrow (-)^+$ defined by

$$\omega_{\mathfrak{F}, \mathfrak{a}}(x) = [(\max(\mathfrak{a}), y)]$$

where

$$y(i) = \mathfrak{F}(i)(x).$$

Lemma 19 (2.10 [1]). Let \mathfrak{F} be a presheaf, \mathfrak{G} a sheaf and $g : \mathfrak{F} \rightarrow \mathfrak{G}$ a morphism in $\hat{\mathcal{C}}$. Then g factors through $\omega_{\mathfrak{F}}$ via a unique g' .

Lemma 20 (2.11 [1]). For every presheaf \mathfrak{F} , \mathfrak{F}^+ is separated.

Lemma 21 (2.12 [1]). If \mathfrak{F} is separated, then \mathfrak{F}^+ is a sheaf.

Definition 22. Define $\text{sh} = (-)^+ \circ (-)^+$.

Lemma 23 (Sheafification adjunction). Let $(\mathcal{C}, \mathcal{T})$ be a site. The functor sh is left adjoint to the inclusion $\hat{\mathcal{C}} \rightarrow \text{Shv}(\mathcal{C})$ with unit

$$\omega_{\mathfrak{F}}^2 = \omega_{\mathfrak{F}^+} \circ \omega_{\mathfrak{F}}.$$

1.2.2 Relative topology

Definition 24 (Relative topology). Let (C, \mathcal{T}) be a site. Let $a \in C$.

Define the induced topology \mathcal{T}_a on C_a by, for each $f \in C_a$

$$\mathcal{T}_a(f) = \mathcal{T}(\text{Dom}(f)).$$

The identification from remark ? implies that \mathcal{T}_a is a Grothendieck topology.

Definition 25 (Oversite). Let $Y = (C, \mathcal{T})$ be a site. Let $a \in C$. Define the site Y_a to be the category C_a with the induced topology \mathcal{T}_a .

Definition 26 (Natural transformation s). Let (C, \mathcal{T}) be a site. Let $a, b \in C$ and $f : b \rightarrow a$. Let $u : C_a \rightarrow C$.

Let $\{x_i \mid i \in R\}$ be a compatible family indexed by a sieve R on b . The same set $\{x_i \mid i \in R\}$ is a compatible family on f indexed by the same sieve R . This yields a natural isomorphism

$$s : u_* \circ (-)^+ \rightarrow (-)^+ \circ u_*,$$

by

$$\begin{aligned} s_{\mathfrak{F}} : u_* \mathfrak{F}^+ &\rightarrow (u_* \mathfrak{F})^+ \\ s_{\mathfrak{F}, f}(\{[x_i \mid i \in R]\}) &= \{[x_i \mid i \in R]\}. \end{aligned}$$

We will treat s as an identification.

Lemma 27 (Restriction commutes with plus). *The transformation $s_{\mathfrak{F}}$ is an isomorphism of functors.*

Proof. We use the variables from definition ?. The morphism $s_{\mathfrak{F}, f}$ has an inverse, again sending the set to itself and applying Q^f on the indexing sieve. These are mutual inverses, so $s_{\mathfrak{F}, f}$ is an isomorphism for each f . ■

Lemma 28 (s and ω commute). *Let \mathfrak{F} be a presheaf on (C, \mathcal{T}) . Let $f : b \rightarrow a \in C$. Let $u : C_a \rightarrow C$ be the localisation morphism. Then $\omega_{u_* \mathfrak{F}} = s_{\mathfrak{F}} \circ u_* \omega_{\mathfrak{F}}$.*

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Proof. For any section $x \in \Gamma(b; \mathfrak{F})$. Let $x_i = \mathfrak{F}(i)(x)$ for any morphism $i \in C$. Note that $\max(f) = \max(b)$. This implies that the compatible family $\{x_i\}$ indexed by the maximal sieve on f is sent by s to the same set $\{x_i\}$ indexed by the maximal sieve on b . In diagram form, that

$$\begin{array}{ccc} u_* \mathfrak{F} & & \\ \downarrow u_* \omega_{\mathfrak{F}} & \searrow \omega_{u_* \mathfrak{F}} & \\ u_* \mathfrak{F}^+ & \xrightarrow{s} & (u_* \mathfrak{F})^+ \end{array}$$

commutes. ■

Definition 29 (s^2). Define

$$s^2 : u_* \circ (-)^+ \circ (-)^+ \rightarrow (-)^+ \circ (-)^+ \circ u_*$$

as

$$s_{\mathfrak{F}}^2 = s_{\mathfrak{F}}^+ \circ s_{\mathfrak{F}^+}.$$

Lemma 30. *Let \mathfrak{F} be a presheaf on (C, \mathcal{T}) . Let $f : b \rightarrow a \in C$. Let \mathfrak{F} be a presheaf on C . Then $\omega_{u_* \mathfrak{F}}^2 = s_{\mathfrak{F}}^2 \circ (u_* \omega_{\mathfrak{F}}^2)$.*

Proof. Let $a \in C$. We have the following identities.

$$\begin{aligned} \omega_{u_* \mathfrak{F}}^2 &= \omega_{u_* \mathfrak{F}^+} \circ \omega_{u_* \mathfrak{F}} \text{ by definition} \\ &= \omega_{u_* \mathfrak{F}^+} \circ s_{\mathfrak{F}} \circ u_* \omega_{\mathfrak{F}} \text{ by applying lemma ?} \\ &= s_{\mathfrak{F}}^+ \circ u_* \omega_{\mathfrak{F}}^+ \circ u_* \omega_{\mathfrak{F}} \text{ by naturality of } \omega \\ &= s_{\mathfrak{F}}^+ \circ u_* \omega_{\mathfrak{F}}^+ \circ s_{\mathfrak{F}} \circ u_* \omega_{\mathfrak{F}} \text{ by applying lemma ?} \\ &= s_{\mathfrak{F}}^+ \circ s_{\mathfrak{F}^+} \circ u_* \omega_{\mathfrak{F}^+} \circ u_* \omega_{\mathfrak{F}} \text{ by naturality of } s \circ u_* \omega \\ &= s_{\mathfrak{F}}^2 \circ u_* \omega_{\mathfrak{F}}^2 \end{aligned}$$

■

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Corollary 31. *Let (C, \mathcal{T}) . Let $a, b \in C$. Let $f : b \rightarrow a \in C$. Sheafifying and restricting commute via the iso*

$$s^2 : sh_b \circ u_* \rightarrow u_* \circ sh_a.$$

Lemma 32 (λ commutes with restriction). *Let $(C, \mathcal{T}, \mathfrak{D})$ be a ringed site. Let $a \in C$. We have a natural isomorphism $t : u_* \circ \lambda \Rightarrow \lambda \circ (- \otimes_{\Gamma(1; \mathfrak{D})} \Gamma(a; \mathfrak{D}))$.*

Proof. Define the natural transformation $t : \lambda \circ (- \otimes_{\Gamma(1; \mathfrak{D})} \Gamma(a; \mathfrak{D})) \Rightarrow u_* \circ \lambda$, by for each $\Gamma(1; \mathfrak{D})$ -module M and for each $f : b \rightarrow a \in C_a$,

$$\begin{aligned} t_{M, f} : M \otimes_{\Gamma(1; \mathfrak{D})} \Gamma(a; \mathfrak{D}) \otimes_{\Gamma(a; \mathfrak{D})} \Gamma(b; \mathfrak{D}) &\rightarrow M \otimes_{\Gamma(1; \mathfrak{D})} \Gamma(b; \mathfrak{D}), \\ m \otimes r \otimes s &\mapsto m \otimes rs. \end{aligned}$$

Every component $t_{M, f}$ is an isomorphism by basic commutative algebra. ■

Let C be a category. Let $a \in C$. Let ϵ be the counit of the adjunction $\lambda \dashv \Gamma(1; -)$ on C . Let ϵ_a be the counit of the adjunction $\lambda_a \dashv \Gamma(a; -)$ on C_a .

Lemma 33 (λ counit commute with restriction). *We have $u_* \epsilon \cong \epsilon_a$ on presheaves of the form $\lambda_a(M \otimes \Gamma(b; \mathfrak{D}))$ with M a $\Gamma(1; \mathfrak{D})$ -module via*

$$t \circ u_* \epsilon \circ t^{-1} = \epsilon_a.$$

Proof. Both maps are the identity map if you unfold them. ■

Lemma 34 (Λ commutes with restriction). *Let $(C, \mathcal{T}, \mathfrak{D})$ be a ringed site. We have a natural isomorphism*

$$q : u_* \circ \Lambda \rightarrow \Lambda \circ (- \otimes_{\Gamma(1; \mathfrak{D})} \Gamma(a; \mathfrak{D})).$$

Proof. Define q to be the composition

$$\begin{aligned} u_* \circ sh \circ \lambda &\xrightarrow{s^2 \lambda} sh \circ u_* \circ \lambda \text{ by lemma ?} \\ &\xrightarrow{sh(t)} sh \circ \lambda \circ (- \otimes_{\Gamma(1; \mathfrak{D})} \Gamma(a; \mathfrak{D})) \text{ by lemma ?} \end{aligned}$$

From lemma ? and lemma ?, t and s^2 are isomorphisms so q is an isomorphism as well. ■

1.3 Modules

Definition 35 (Presheaf modules). Let $Y = (C, \mathcal{T}, \mathfrak{D})$ be a ringed site. Let $R = \Gamma(1; \mathfrak{D})$.

A presheaf module on Y is a presheaf of sets \mathfrak{F} on C together with a map of presheaves

$$\mathfrak{D} \times \mathfrak{F} \rightarrow \mathfrak{F}$$

such that for every object $\alpha \in C$ the map $\Gamma(\alpha; \mathfrak{D}) \times \Gamma(\alpha; \mathfrak{F}) \rightarrow \Gamma(\alpha; \mathfrak{F})$ defines a $\Gamma(\alpha; \mathfrak{D})$ -module structure on $\Gamma(\alpha; \mathfrak{F})$.

A morphism

$$\mathfrak{F} \rightarrow \mathfrak{G}$$

is a morphism of presheaf modules if

$$\begin{array}{ccc} \mathfrak{D} \times \mathfrak{F} & \longrightarrow & \mathfrak{F} \\ \downarrow & & \downarrow \\ \mathfrak{D} \times \mathfrak{G} & \longrightarrow & \mathfrak{G} \end{array}$$

commutes. The category of presheaf modules on Y will be denoted $\text{PMod}(Y)$.

Definition 36. Let M, N be an R -module.

Define

$$\lambda : R\text{-Mod} \rightarrow \text{PMod}(Y)$$

by for all $\alpha \in C$,

$$\lambda(M)(\alpha) = M \otimes_R \Gamma(\alpha; \mathfrak{D}),$$

for all $f : b \rightarrow \alpha \in C$,

$$\lambda(M)(f) : \text{Id} \otimes \mathfrak{D}(f),$$

for all $g : M \rightarrow N \in R\text{-Mod}$,

$$\lambda(g) = (\alpha : g \otimes \text{Id}).$$

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Lemma 37. *Let $Y = (X, \mathcal{T}, \mathfrak{O})$ be a ringed site. The functor λ is left adjoint to*

$$\Gamma(1; -) : \text{PMod}(Y) \rightarrow \text{R-Mod}$$

.

Proof. Let \mathfrak{a} be an object of C . Let M, N be R -modules. Let $\mathfrak{F}, \mathfrak{G} \in \text{PMod}(Y)$ be presheaf modules.

Let $\varphi : \lambda(M) \rightarrow \mathfrak{G}$ be a morphism of presheaf modules. Let $\phi : M \rightarrow \Gamma(1; \mathfrak{G})$ be a morphism of presheaf modules.

Define

$$\alpha = H_{M, \mathfrak{G}} : \text{Hom}(\lambda(M), \mathfrak{G}) \rightarrow \text{Hom}(M, \Gamma(1; \mathfrak{G}))$$

by

$$\alpha(\varphi) = \varphi_1,$$

where φ_1 is the component of φ on the global sections.

Define

$$\beta = L_{M, \mathfrak{G}} : \text{Hom}(M, \Gamma(1; \mathfrak{G})) \rightarrow \text{Hom}(\lambda(M), \mathfrak{G})$$

by

$$\beta(\phi)_\mathfrak{a} = \phi \otimes_R \Gamma(\mathfrak{a}; \mathfrak{O}).$$

We will show that β and α are mutually inverse.

Let $d = \beta(\alpha(\varphi))$. Let $\mathfrak{m} \otimes g \in M \otimes_R \Gamma(\mathfrak{a}; \mathfrak{O})$. Let $p : \lambda(M)(1) \rightarrow \lambda(M)(\mathfrak{a})$ be the projection map. Let $q : \mathfrak{G}(1) \rightarrow \mathfrak{G}(\mathfrak{a})$ be the projection map. Then $d_\mathfrak{a}(\mathfrak{m} \otimes g) = \varphi_1(\mathfrak{m}) \otimes g$ and

$$\begin{aligned} \varphi_\mathfrak{a}(\mathfrak{m} \otimes g) &= g\varphi_\mathfrak{a}(\mathfrak{m} \otimes 1) \text{ by linearity} \\ &= g\varphi_\mathfrak{a}(p(\mathfrak{m})) \\ &= gq(\varphi_1(\mathfrak{m})) \text{ by naturality of } \varphi \\ &= g(\varphi_1(\mathfrak{m}) \otimes 1) \\ &= \varphi_1(\mathfrak{m}) \otimes g. \end{aligned}$$

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Hence $d = \varphi$. In words, the natural transformations from presheaves of the form $\lambda(M)$ are uniquely determined by their global sections component.

Let $d = \alpha(\beta(\phi))$. Let $m \in M$. Then $d(m) = (\phi \otimes_R R)(m) = \phi(m)$. Hence $d = \phi$, which makes H and L mutual inverses.

Naturality in M and \mathfrak{G}

Let $g : N \rightarrow M$ and $h : \mathfrak{F} \rightarrow \mathfrak{G}$. Let $\rho \in \text{Hom}(\lambda(N), \mathfrak{F})$. Let $k = H_{M, \mathfrak{G}}(h \circ \rho \circ \lambda(f))$. Let $l = h_1 \circ H_{N, \mathfrak{F}}(\rho) \circ f$.

Unfolding the definition for H shows that $k = h_1 \rho_1 f$ and $l = h_1 \rho_1 f$ as well. This proves naturality in M and \mathfrak{G} and the adjunction between λ and $\Gamma(1; -)$. ■

Definition 38. Let $(C, \mathcal{T}, \mathfrak{D})$ Define

$$\Lambda : R\text{-Mod} \rightarrow \text{Mod}(O)$$

by $sh \circ \lambda$.

It follows that we have the adjunction $\Lambda \dashv \Gamma(1; -)$.

1.4 Schemes

Definition 39 (Spectrum of a ring). Let R be a ring. The spectrum $\text{Spec} R$ of R is the ringed space defined as follows. The underlying set is the set of prime ideals of R . The (zariski) topology is generated by the basis $D(f) = \{\mathfrak{p} \subset R \mid f \notin \mathfrak{p}\}$. The sheaf of rings is given by

$$D(f) \mapsto R_f.$$

Definition 40 (Distinguished open). Let $\text{Spec} R$ be a affine scheme. The set

$$D(f) = \{\mathfrak{p} \subset R \mid f \notin \mathfrak{p}\}$$

for a global section f is called a distinguished open. The open $D(f)$ is isomorphic to $\text{Spec}(R_f)$ as a locally ringed space.

Definition 41 (Locus of a point). Let (X, \mathfrak{D}) be a scheme. Define the locus of a global section $x \in \Gamma(1; \mathfrak{D})$ to be

$$\ker(x) = \ker(\mathfrak{D}(X) \rightarrow \kappa(x)).$$

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Lemma 42. *The functor*

$$\text{Spec} : \mathbf{Rng} \rightarrow \mathbf{LRSpaces}$$

is left adjoint to

$$\Gamma(1; -) : \mathbf{LRSpaces} \rightarrow \mathbf{Rng}.$$

With unit

$$F = \eta : (X, \mathfrak{O}) \rightarrow \text{Spec}(\Gamma(1; \mathfrak{O})).$$

$$x \mapsto \ker(x),$$

Proof.

■

Definition 43 (Affine scheme). We call the ringed space $\text{Spec}(R)$ an affine scheme.

Definition 44 (Scheme). A scheme S is a locally ringed space that admits a covering of affine schemes. A morphism of schemes is a morphism of locally ringed spaces. The category of schemes we will denote by \mathbf{Sch} .

Bibliography

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