Affine Objects

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1 Preliminaries

This chapter introduces all the basic notions that are needed but not assumed to be known to the reader. We will start with a discussion of some purely categorical notions like slice categories and presheaves. Secondly, we will introduce a notion of a topology on a category and look at some constructions that are relevant for us. Then we will introduce modules on ringed sites. Lastly, the notion of a scheme is introduced.

1.1 Basic Category Theory

Some categorical notions like presheaves and slice categories will be introduced in this section. See [1] and [3].

Definition 1.1.1 (Presheaf category)**.** Let C be a category. Let $a \in \mathsf{C}$. Let $f : a' \to a$ We define the category of presheaves on C as the category of contravariant functors to the category of sets Set. We will denote it by $\hat{\mathsf{C}}$.

Define the functor $h : \mathsf{C} \to \hat{\mathsf{C}}$ as follows

$$a \mapsto \mathrm{Hom}(-, a),$$

$$f \mapsto f \circ -.$$

This functor is fully faithful by the Yoneda lemma.

Notation 1.1.2. Let I, C be categories. Let $L : \mathsf{I} \to \mathsf{C}$ be a functor. The limit over this functor will be denoted by $\lim\limits_{i \in \mathsf{I}} L(i)$. The colimit will be denoted by $\operatorname*{colim}\limits_{i \in \mathsf{I}} L(i)$.

Definition 1.1.3 (Sections functor)**.** For any $a \in \mathsf{C}$ define the functor

$$\Gamma(a; -) : \hat{\mathsf{C}} \to \mathsf{Set}$$

by

$$\mathfrak{F} \to \mathfrak{F}(a).$$

For any presheaf $\mathfrak{G}$, we define

$$\Gamma(\mathfrak{G}; -) : \hat{C} \to \mathsf{Set},$$
$$\mathfrak{F} \to \mathrm{Hom}(\mathfrak{G}, \mathfrak{F}).$$

We will mostly use this for the terminal presheaf, which will allow us to compute the global sections.

Definition 1.1.4 (Over/Under categories)**.** Let C and C' be categories. Let $F : C \to C'$ and $z \in C'$. Define the category C_z and C^z as

$$\mathrm{Obj}(C_z) := \{(a, w) \mid a \in C, w : F(a) \to z\},$$
$$\mathrm{Hom}((a, w), (b, v)) := \{f : a \to b \mid v \circ F(f) = w\},$$

and

$$\mathrm{Obj}(C^z) := \{(a, w) \mid a \in C, w : z \to F(a)\},$$
$$\mathrm{Hom}((a, w), (b, v)) := \{f : a \to b \mid F(f) \circ w = v\}.$$

We get faithful functors $C_z \to C : (a, w) \to a$ and $C^z \to C : (a, w) \to a$. We will call both functors localization functors and denote them by u.

Definition 1.1.5 (direct image)**.** Let $f : C \to D$. Define the direct image functor $f_* : \hat{D} \to \hat{C}$ as

$$f_* = - \circ f.$$

We call $f_*\mathfrak{F}$ the restriction of $\mathfrak{F}$ to C When f is the induced functor $C_b \to C_a$ coming from a morphism $g : b \to a$ the we sometimes denote it by $\mathfrak{F}\big|_b$.

Definition 1.1.6 (Restriction)**.** Let C, D be categories. Let $\mathfrak{F} \in \hat{D}$. Let $\alpha : C \to D$ be a functor. The restriction of $\mathfrak{F}$ to C along α is defined to be $\alpha_*\mathfrak{F}$.

1.2 Topology

In this section we will define a notion of a topology on a category and look at the related notions of sheaves, sites and restriction of sites.

See [3] for more details.

Definition 1.2.1 (Sieve)**.** Let C be a category and $a \in C$. Define the maximal sieve $\mathsf{max}(a)$ on a to be the collection of all morphisms to a. In formula,

$$\mathsf{max}(a) = \{f \in C \mid \mathsf{Codom}(f) = a\}.$$

A sieve S on a is a subcollection of $\mathsf{max}(a)$ such that $gf \in S$ for any $f \in S$ and any g.

Definition 1.2.2 (Sieve category)**.** Let C be a category and $a \in C$. The sieve category $\mathsf{Sieves}(a)$ consists of all the sieves on a as objects and inclusions of sieves as morphisms.

Definition 1.2.3 (Pullback of sieve)**.** Let C be a category and $a, b \in C$. Let S be a sieve on a. Let $f : b \to a$. The sieve f^*S on b is given by $f^*S = \{g \in \max b : fg \in S\}$ for any $c \in C$. To show that this is actually a sieve on b, let $k : c \to c'$ and $h \in f^*S$. Hence $fh \in S$ and so $fhk \in S$. Conclude that $hk \in f^*S$. This defines a functor $f^* : \mathsf{Sieves}(a) \to \mathsf{Sieves}(b)$.

Definition 1.2.4 (Grothendieck Topology)**.** A Grothendieck topology $\mathcal{T}$ is a family $\mathcal{T}(a)$ of 'covering' sieves for every $a \in C$ with the following conditions:

1. $\mathsf{max}(a) \in \mathcal{T}(a)$
2. $f^*R \in \mathcal{T}(a')$ if $R \in \mathcal{T}(a)$ for all $f : a' \to a$
3. if $f^*R \in \mathcal{T}(a')$ for all $f \in S$ with $S \in \mathcal{T}(a)$ then $R \in \mathcal{T}(a)$

Definition 1.2.5 (Basis)**.** Let C be a category with pullbacks. A Grothendieck pretopology, or basis, $\mathcal{B}$ is a collection $\mathcal{B}(a)$ of 'covering' families $\{f_i : a_i \to a\}$ of morphisms for every $a \in C$ with the following conditions.

1. every isomorphism is a covering singleton family,
2. (Stability) The pullback of a covering family is a covering family. If $\{f_i : a_i \to a\}$ is covering and $g : b \to a$, then $\{f_i' : a_i \times_a b \to b\}$ is covering.

3. (Transitivity) If $\{f_i : a_i \to a\}$ is a covering family and $\{f_{ij} : a_{ij} \to a_i\}$ for every i, then $\{f_{ij} : a_{ij} \to a\}$ is a covering family.

Generating a topology from a basis: take any sieve containing a covering family to be a covering sieve.

Remark 1.2.6. Sometimes you have a set of families S you would like to generate the topology. So one can take the smallest basis and subsequently topology containing these families. This will be called 'the topology generated by S'.

Definition 1.2.7 (continuous functor)**.** Let $G : C \to D$ be a functor between sites. Let $c \in C$ and let R be a covering sieve on c. The functor G is said to preserves covers if $G(R)$ generates a covering sieve. It is enough to check that G sends covering families to covering families, if the topology of the sites is defined by a basis. If G also preserves pullbacks then we call it continuous. See [5, Tag 00WV].

Definition 1.2.8 (cocontinuous functor)**.** The functor G is said to lift covers or be cocontinuous if for every $R \in \mathrm{Cov}G(c)$ there is some $S \in \mathrm{Cov}c$ such that $G(S) \subset R$. See [5, Tag 00XJ].

Definition 1.2.9 (Site)**.** A site $(C, \mathcal{T})$ is a category C with a Grothendieck topology $\mathcal{T}$. A morphism of sites is a functor that preserves pullbacks and covers.

The category Sites has as objects sites and morphisms of sites as just defined. When no confusion can arise then we will use C to denote the whole site $(C, \mathcal{T})$.

1.2.1 Sheaves

We will introduce the very important notion of a sheaf. See [3] for a more detailed treatment.

Definition 1.2.10 (Matching family)**.** Let C be a category. Let $\mathfrak{F}$ be a presheaf on on C. Let $a \in C$ be an object. Let R be a sieve on a. A family $\{x_i\}_{i \in R}$ with $x_i \in \Gamma(\mathrm{Dom}(i); \mathfrak{F})$ indexed by a sieve R and such that $x_{g \circ i} = \mathfrak{F}(g)(x_i)$ for any $g : b \to \mathrm{Dom}(i)$ and $b \in C$ is called a 'matching family'.

Definition 1.2.11 (Amalgamation)**.** An amalgamation of a matching family $\{x_i\}_R$ is an element $x \in \Gamma(a; \mathfrak{F})$ such that $\mathfrak{F}(i)(x) = x_i$.

Definition 1.2.12 (Sheaf)**.** Let $(\mathsf{C}, \mathfrak{T})$ be a site. Let $\mathfrak{F} \in \hat{\mathsf{C}}$.

A presheaf that admits a unique amalgamation for every matching family is called a sheaf. The category $\mathsf{Shv}(\mathsf{C})$ is the full subcategory in $\hat{\mathsf{C}}$ of all sheaves.

Definition 1.2.13 (Plus construction)**.** Let $(\mathsf{C}, \mathfrak{T})$ be a site. Let $a, a' \in \mathsf{C}$ and $f : a \to a'$. Let $\mathfrak{F} \in \hat{\mathsf{C}}$. Define the functor $(-)^+ : \hat{\mathsf{C}} \to \hat{\mathsf{C}}$ as follows.

For all $a \in \mathsf{C}$,

$$F^+(a) = \frac{\{(R, \varphi) \mid R \in \mathfrak{T}(a), \varphi \in \Gamma(R; \mathfrak{F})\}}{\sim},$$

for all morphisms $f \in \mathsf{C}$,

$$F^+(f)([(R, \varphi)]) = [(f^*R, \varphi h(f))].$$

The equivalence relation is defined as $(R, \varphi) \sim (S, \phi)$ if $\varphi = \phi$ on some covering sieve $Q \subset R \cap S$.

Let $L : \mathfrak{F} \to \mathfrak{F}'$. Then

$$(L^+)_a([(R, \varphi)]) = [(R, L \circ \varphi)]$$

This functor comes with a natural transformation $\omega : \mathrm{Id} \to (-)^+$ defined by

$$\omega_{\mathfrak{F},a}(x) = [(\max(a), y]$$

where

$$y(i) = \mathfrak{F}(i)(x).$$

Lemma 1.2.14. *Let $\mathfrak{F}$ be a presheaf, $\mathfrak{G}$ a sheaf and $g : \mathfrak{F} \to \mathfrak{G}$ a morphism in $\hat{\mathsf{C}}$. Then g factors through ω_F via a unique g'.*

Proof. See [4, p. 2.10]. ∎

Lemma 1.2.15. *For every presheaf $\mathfrak{F}$, F^+ is separated.*

Proof. See [4, p. 2.11]. ∎

Lemma 1.2.16. *If $\mathfrak{F}$ is separated, then F^+ is a sheaf.*

Proof. See [4, p. 2.12]. ∎

Definition 1.2.17. Define $\mathrm{sh} = (-)^+ \circ (-)^+$. This functor is left adjoint to the inclusion $\hat{C} \to \mathsf{Shv}(C)$ with unit

$$\omega^2_{\mathfrak{F}} = \omega_{\mathfrak{F}^+} \circ \omega_{\mathfrak{F}}.$$

1.2.2 Relative topology

We will look at what the induced topology on a slice category looks like and what this implies for restriction of sheaves. See [5, Tag 03A4] for a more detailed treatement.

Remark 1.2.18. Let C be a category and $a, b \in C$. Let $f : b \to a \in C_a$. The map $\mathsf{max}(f) \to \mathsf{max}(b)$ sending a morphism to f to its underlying morphism in C is a bijection. Moreover composition in C and C_a are the same, so this identification respects pullback of sieves. This observation yields us that $\mathsf{Sieves}(b) = \mathsf{Sieves}(f)$.

Whenever R is a sieve on b, we will denote the corresponding sieve on f by R_f.

Definition 1.2.19 (Relative topology)**.** Let $(C, \mathcal{T})$ be a site. Let $a \in C$.

Define the induced topology $\mathcal{T}_a$ on C_a by, for each $f \in C_a$

$$\mathcal{T}_a(f) = \mathcal{T}(\mathrm{Dom}(f)).$$

The identification from Remark 1.2.18 implies that $\mathcal{T}_a$ is a Grothendieck topology.

Definition 1.2.20 (Oversite)**.** Let $Y = (C, \mathcal{T})$ be a site. Let $a \in C$. Define the site Y_a to be the category C_a with the induced topology T_a.

Definition 1.2.21 (Natural transformation s)**.** Let $(C, \mathcal{T})$ be a site. Let $a, b \in C$ and $f : b \to a$. Let $u : C_a \to C$.

Let $\{x_i \mid i \in R\}$ be a compatible family indexed by a sieve R on b. The same set $\{x_i \mid i \in R\}$ is a compatible family on f indexed by the same sieve R. This yields a natural isomorphism

$$s : u_* \circ (-)^+ \to (-)^+ \circ u_*,$$

by

$$s_{\mathfrak{F}} : u_*\mathfrak{F}^+ \to (u_*\mathfrak{F})^+$$

$$s_{\mathfrak{F},f}([\{x_i \mid i \in R\}]) = [\{x_i \mid i \in R_f\}].$$

We will treat s as an identification.

Lemma 1.2.22 (s and ω commute)**.** *Let $\mathfrak{F}$ be a presheaf on $(C, \mathcal{T})$. Let $f : b \to a \in C$. Let $u : C_a \to C$ be the localisation morphism. Then $\omega_{u_*\mathfrak{F}} = s_{\mathfrak{F}} \circ u_* \omega_{\mathfrak{F}}$.*

Proof. For any section $x \in \Gamma(b; \mathfrak{F})$. Let $x_i = \mathfrak{F}(i)(x)$ for any morphism $i \in C$. Note that $\max(f) = \max(b)$. This implies that the compatible family $\{x_i\}$ indexed by the maximal sieve on f is sent by s to the same set $\{x_i\}$ indexed by the maximal sieve on b. In diagram form, that

$$\begin{tikzcd}
u_*\mathfrak{F} \arrow[d, "u_*\omega_{\mathfrak{F}}"] \arrow[dr, "\omega_{u_*\mathfrak{F}}"] & \\
u_*\mathfrak{F}^+ \arrow[r, "s"] & (u_*\mathfrak{F})^+
\end{tikzcd}$$

commutes. ∎

Definition 1.2.23 (s^2)**.** Define

$$s^2 : u_* \circ (-)^+ \circ (-)^+ \to (-)^+ \circ (-)^+ \circ u_*$$

as

$$s^2_{\mathfrak{F}} = s_{\mathfrak{F}}{}^+ \circ s_{\mathfrak{F}^+}.$$

This is a natural isomorphism because s is one. Hence sheafifying and restricting commute via this iso

$$s^2 : u_* \circ sh_a \to sh_b \circ u_*.$$

Corollary 1.2.24. *Let $\mathfrak{F}$ be a presheaf on $(C, \mathcal{T})$. Let $f : b \to a \in C$. Let $\mathfrak{F}$ be a presheaf on C. Then $\omega^2_{u_*\mathfrak{F}} = s^2_{\mathfrak{F}} \circ (u_*\omega^2_{\mathfrak{F}})$.*

1.3 Modules

Presheaf modules and sheaf modules on a ringed site will be introduced in this section. The functors λ and Λ introduced here will be used extensively. See [5, Tag 03A4] for more detail.

Definition 1.3.1 (Presheaf modules)**.** Let $(\mathsf{C}, \mathfrak{T}, \mathfrak{O})$ be a ringed site. Let $R = \Gamma(1; \mathfrak{O})$.

A presheaf module on this ringed site is a presheaf of sets $\mathfrak{F}$ on C together with a map of presheaves

$$\mathfrak{O} \times \mathfrak{F} \to \mathfrak{F}$$

such that for every object $a \in \mathsf{C}$ the map $\Gamma(a; \mathfrak{O}) \times \Gamma(a; \mathfrak{F}) \to \Gamma(a; \mathfrak{F})$ defines a $\Gamma(a; \mathfrak{O})$-module structure on $\Gamma(a; \mathfrak{F})$.

A morphism

$$\mathfrak{F} \to \mathfrak{G}$$

is a morphism of presheaf modules if

$$\begin{tikzcd}
\mathfrak{O} \times \mathfrak{F} \arrow[r] \arrow[d] & \mathfrak{F} \arrow[d] \\
\mathfrak{O} \times \mathfrak{G} \arrow[r] & \mathfrak{G}
\end{tikzcd}$$

commutes. The category of presheaf modules on C will be denoted $\mathsf{PMod}(\mathfrak{O})$.

Definition 1.3.2. Let $\mathfrak{F}$ be a sheaf of modules on $(\mathsf{C}, \mathfrak{T}, \mathfrak{O})$. It is called quasi-coherent if the following holds. For any object $a \in \mathsf{C}$ there exists a covering sieve S such that for any map $f : b \to a$ in S there exists a presentation

$$\bigoplus_I \mathfrak{O}|_b \to \bigoplus_J \mathfrak{O}|_b \to \mathfrak{F}|_b \to 0$$

Quasi-coherent modules form a full subcategory of the category of sheafs of modules over $(\mathsf{C}, \mathfrak{T}, \mathfrak{O})$ which are denoted by $\mathsf{Qcoh}(\mathfrak{O})$.

Definition 1.3.3. Let $(\mathsf{C}, \mathcal{T}, \mathfrak{O})$ be a ringed site. Let $R = \Gamma(1; \mathfrak{O})$. Let M, N be an R-module.

Define

$$\lambda : R\text{-}\mathsf{Mod} \to \mathsf{PMod}(\mathfrak{O})$$

by for all $a \in \mathsf{C}$,

$$\lambda(M)(a) = M \otimes_R \Gamma(a; \mathfrak{O}),$$

for all $f : b \to a \in \mathsf{C}$,

$$\lambda(M)(f) = \mathrm{Id} \otimes \mathfrak{O}(f),$$

for all $g : M \to N \in R\text{-}\mathsf{Mod}$,

$$\lambda(g) = (a : g \otimes \mathrm{Id}).$$

Lemma 1.3.4. *Let $Y = (\mathsf{C}, \mathcal{T}, \mathfrak{O})$ be a ringed site. The functor λ is left adjoint to*

$$\Gamma(1; -) : \mathsf{PMod}(\mathfrak{O}) \to R\text{-}\mathsf{Mod}.$$

Proof. Let a be an object of C. Let M, N be R-modules. Let $\mathfrak{F}, \mathfrak{G} \in \mathsf{PMod}(\mathfrak{O})$ be presheaf modules.

Let $\varphi : \lambda(M) \to \mathfrak{G}$ be a morphism of presheaf modules. Let $\phi : M \to \Gamma(1; \mathfrak{G})$ be a morphism of presheaf modules.

Define

$$\begin{aligned} \alpha = H_{M,\mathfrak{G}} :& \mathrm{Hom}(\lambda(M), \mathfrak{G}) \to \mathrm{Hom}(M, \Gamma(1; \mathfrak{G})) \\ :& \varphi \mapsto \varphi_1 \end{aligned}$$

where φ_1 is the component of φ on the global sections.

Define

$$\beta = L_{M,\mathfrak{G}} : \mathrm{Hom}(M, \Gamma(1; \mathfrak{G})) \to \mathrm{Hom}(\lambda(M), \mathfrak{G})$$

by

$$\beta(\phi)_a = \phi \otimes_R \Gamma(a; \mathfrak{O}).$$

We will show that β and α are mutually inverse.

Let $d = \beta(\alpha(\varphi))$. Let $m \otimes g \in M \otimes_R \Gamma(a; \mathfrak{O})$. Let $p : \lambda(M)(1) \to \lambda(M)(a)$ be the projection map. Let $q : \mathfrak{G}(1) \to \mathfrak{G}(a)$ be the projection map. Then $d_a(m \otimes g) = \varphi_1(m) \otimes g$ and

$$\begin{aligned} \varphi_a(m \otimes g) &= g\varphi_a(m \otimes 1) \text{ by linearity} \\ &= g\varphi_a(p(m)) \\ &= gq(\varphi_1(m)) \text{ by naturality of } \varphi \\ &= g(\varphi_1(m) \otimes 1) \\ &= \varphi_1(m) \otimes g. \end{aligned}$$

Hence $d = \varphi$. In words, the natural transformations from presheaves of the from $\lambda(M)$ are uniquely determined by their global sections component.

Let $d = \alpha(\beta(\phi))$. Let $m \in M$. Then $d(m) = (\phi \otimes_R R)(m) = \phi(m)$. Hence $d = \phi$, which makes H and L mutual inverses.

Now we will show naturality in M and $\mathfrak{G}$. Let $g : N \to M$ and $h : \mathfrak{F} \to \mathfrak{G}$. Let $\rho \in \mathrm{Hom}(\lambda(N), \mathfrak{F})$. Let $k = H_{M,\mathfrak{G}}(h \circ \rho \circ \lambda(f))$. Let $l = h_1 \circ H_{N,\mathfrak{F}}(\rho) \circ f$.

Unfolding the definition for H shows that $k = h_1\rho_1 f$ and $l = h_1\rho_1 f$ as well. This proves naturality in M and $\mathfrak{G}$ and the adjunction between λ and $\Gamma(1; -)$ between R-Mod and PMod$(\mathfrak{O})$. ∎

Definition 1.3.5. Let $(C, \mathcal{T}, \mathfrak{O})$ Define

$$\Lambda : \text{R-Mod} \to \text{Mod}(\mathfrak{O})$$

by $sh \circ \lambda$.

It follows that we have the adjunction $\Lambda \dashv \Gamma(1; -)$.

This functor is the generalisation of [5, Tag 01BH] to general sites, so Λ on the site corresponding to a ringed space(we will define this later) will coincide with the stacks construction.

Lemma 1.3.6 (λ commutes with restriction)**.** *Let* $(C, \mathcal{T}, \mathfrak{O})$ *be a ringed site. Let* $a \in C$. *We have a natural isomorphism* $t : u_* \circ \lambda \Rightarrow \lambda \circ (- \otimes_{\Gamma(1;\mathfrak{O})} \Gamma(a; \mathfrak{O}))$.

Proof. Define the natural transformation $t : \lambda \circ (- \otimes_{\Gamma(1;\mathfrak{O})} \Gamma(a;\mathfrak{O})) \Rightarrow u_* \circ \lambda$, by for each $\Gamma(1;\mathfrak{O})$-module M and for each $f : b \to a \in C_a$,

$$t_{M,f} : M \otimes_{\Gamma(1;\mathfrak{O})} \Gamma(a;\mathfrak{O}) \otimes_{\Gamma(a;\mathfrak{O})} \Gamma(b;\mathfrak{O}) \to M \otimes_{\Gamma(1;\mathfrak{O})} \Gamma(b;\mathfrak{O}),$$

$$m \otimes r \otimes s \mapsto m \otimes rs.$$

Every component $t_{M,f}$ is an isomorphism by basic commutative algebra. ∎

Let C be a category. Let $a \in C$. Let ϵ be the counit of the adjunction $\lambda \dashv \Gamma(1;-)$ on C. Let ϵ_a be the counit of the adjunction $\lambda_a \dashv \Gamma(a;-)$ on C_a.

Lemma 1.3.7 (λ counit commute with restriction)**.** *We have $u_*\epsilon \cong \epsilon_a$ on presheaves of the form $\lambda_a(M \otimes \Gamma(b;\mathfrak{O}))$ with M a $\Gamma(1;\mathfrak{O})$-module via*

$$t \circ u_*\epsilon \circ t^{-1} = \epsilon_a.$$

Proof. Both maps are the identity map if you unfold them. ∎

Lemma 1.3.8 (Λ commutes with restriction)**.** *Let $(C, \mathfrak{T}, \mathfrak{O})$ be a ringed site. We have a natural isomorphism*

$$q : u_* \circ \Lambda \to \Lambda \circ (- \otimes_{\Gamma(1;\mathfrak{O})} \Gamma(a;\mathfrak{O})).$$

Proof. Define q to be the composition

$$\begin{aligned} u_* \circ \mathrm{sh} \circ \lambda &\overset{s^2\lambda}{\Rightarrow} \mathrm{sh} \circ u_* \circ \lambda \\ &\overset{\mathrm{sh}(t)}{\Rightarrow} \mathrm{sh} \circ \lambda \circ - \otimes_{\Gamma(1;\mathfrak{O})} \Gamma(a;\mathfrak{O})) \end{aligned}$$

By Definition 1.2.21 and lemma 1.3.6, t and s^2 are isomorphisms so q is an isomorphism as well. ∎

1.4 Schemes

We will recap the parts of scheme theory here that we use. See [6, 2] for thorough treatments of scheme theory.

Definition 1.4.1 (Spectrum of a ring)**.** Let R be a ring. The spectrum $\operatorname{Spec} R$ of R is the locally ringed space defined as follows. The underlying set is the set of prime ideals of R. The (Zariski) topology is generated by the basis of distinguished opens $D(f) = \{\mathfrak{p} \subset R | f \notin \mathfrak{p}\}$. The sheaf of rings is given on this basis by

$$D(f) \mapsto R_f.$$

A distinguished open $D(f)$ of $\mathrm{Spec}(R)$ viewed as locally ringed space is isomorphic to $\mathrm{Spec}(R_f)$, where the inclusion $D(f) \to \mathrm{Spec}(R)$ corresponds to the canonical map $R \to R_f$.

Definition 1.4.2 (Locus of a point)**.** Let $(X, \mathfrak{O})$ be a scheme. Define the locus of point $x \in X$ to be

$$\ker(x) = \ker(\Gamma(X; \mathfrak{O}) \to \kappa(x)).$$

Note that $\ker(x)$ is a prime ideal of $\Gamma(X; \mathfrak{O})$

Definition 1.4.3 (Support)**.** Let $(X, \mathfrak{O})$ be a locally ringed space. The support of a global section is

$$D_X(a) = \{x \in X \mid a \notin \ker(x)\}.$$

Definition 1.4.4 (Support of module)**.** Let $(X, \mathfrak{O})$ be a locally ringed space. Let $\mathfrak{M}$ be a module on this space. The support $\mathrm{Supp}(\mathfrak{M})$ of this module is the subspace

$$\{x \in X \mid \mathfrak{M}_x = 0\} \subset (X, \mathfrak{O}).$$

Definition 1.4.5 (Set cut out by ideal)**.** Let $(X, \mathfrak{O})$ be a scheme. Define the set cut out by an ideal to be

$$V_X(I) = \mathrm{Supp}(\Lambda_X(\frac{\mathfrak{O}(X)}{I})).$$

Lemma 1.4.6. *For any* $X \in \mathsf{LRSpaces}$ *and* $R \in \mathsf{Rng}$ *we have an isomorphism*

$$\mathrm{Hom}_{\mathsf{LRSpaces}}(X, \mathrm{Spec}(R)) \to \mathrm{Hom}_{\mathsf{Rng}}(R, \Gamma(X; \mathfrak{O}))$$

that is natural in X *and* R*. In short*

$$\mathsf{Spec} : \mathsf{Rng} \to \mathsf{LRSpaces}$$

is adjoint to

$$\Gamma(1; -) : \mathsf{LRSpaces} \to \mathsf{Rng}.$$

Proof. Sending a morphism of locally ringed spaces $X \to \operatorname{Spec} R$ to its global component $R \to \Gamma(X; \mathfrak{O})$ will turn out to be an isomorphism. The inverse is the following map.

Let $\varphi : R \to \Gamma(X; \mathfrak{O})$ be given. We need to construct a morphism of locally ringed spaces $(f, f^{\#})$. Define $f(x) = \ker x$. For distinguised open $D(f) \subset \operatorname{Spec}(R)$, define $f^{\#}_{D(f)}(\frac{s}{f}) = \frac{s}{f}$. This makes sense because $f \in \Gamma(X; \mathfrak{O})$ is invertible in $D_X(f)$

These maps are mutually inverses and we have naturality. See [5, Tag 01I1] ∎

Definition 1.4.7 (Scheme)**.** We call the locally ringed space $\operatorname{Spec}(R)$ an affine scheme.

A scheme S is a locally ringed space that admits a covering of affine schemes. A morphism of schemes is a morphism of locally ringed spaces. The category of schemes we will denote by Sch.

Definition 1.4.8 (Tilde functor)**.** Let $\operatorname{Spec}(R)$ be an affine scheme. Let M be a R-module. Define $\widetilde{M}$ to be the unique sheaf on $\operatorname{Spec}(R)$ with

$$\widetilde{M} : D(f) \mapsto M_f.$$

See [5, Tag 01HR].

Remark 1.4.9*.* This functor(with this notation) is commonly used in algebraic geometry texts. We will show that it is equal to Λ on the small Zariski site and then only use Λ.

Lemma 1.4.10. *Let $\mathfrak{F}$ be a sheaf of modules on scheme X. Then $\mathfrak{F}$ is quasi-coherent if and only if for any $D(f) = \operatorname{Spec}(S_f) \to \operatorname{Spec}(S)$ opens in X with $f \in S$ the induced map*

$$\Gamma(\operatorname{Spec}(S); \mathfrak{F}) \otimes_S S_f \to \Gamma(D(f); \mathfrak{F})$$

is an isomorphism.

Proof. $\Rightarrow$:

Assume $\mathfrak{F}$ is quasi-coherent. Let $\operatorname{Spec}(S) \subset X$ be a open affine subset. By Lemma 1.5.7 we have $\mathfrak{F}|_{\operatorname{Spec}(S)} = \widetilde{M}$ for some S-module M. By construction $\Gamma(D(f); \mathfrak{F}) = M_f$ and hence the induced map is an isomorphism by basic commutative algebra.

$\Leftarrow$:

Assume that the induced map is an isomorphism for every $f \in S$ with $\operatorname{Spec}(S) \subset X$ open affine subset. Choose collection f_i such that $(f_i) = (1)$. Then $\Gamma(D(f_i); \mathfrak{F}) =$

$\Gamma(\mathrm{Spec}(S);\mathfrak{F}) \otimes_S S_{f_i}$ by assumption. Hence $F\big|_{\mathrm{Spec}(S)} = \widetilde{M}$ with $M = \Gamma(\mathrm{Spec}(S);\mathfrak{F})$. By Lemma 1.5.7 this implies that $\mathfrak{F}$ is quasi-coherent. ∎

Definition 1.4.11 (Sheaf of algebras)**.** A sheaf of algebras $\mathfrak{F}$ on a ringed site $(C, \mathcal{T}, \mathfrak{O})$ is a sheaf of rings that comes with a (structure) morphism of sheaf of rings $\mathfrak{O} \to \mathfrak{F}$.

Definition 1.4.12 (Relative spec)**.** Let X be a scheme. Let $\mathfrak{S}$ be a sheaf of algebras on X that is quasi-coherent as a sheaf of modules.

Define the relative spectrum of $\mathfrak{S}$ over X to be the scheme

$$\mathrm{Rspec}\ \mathfrak{S} \to X$$

that you get by glueing the spectra $\mathrm{Spec}(\Gamma(V;\mathfrak{S})) \to V \subset X$ for every affine open V. See [5, Tag 01LW].

1.5 Schemes and sites

In this section we will introduce the big Zariski ringed site and look at how quasi-coherence and Λ behave on these sites.

Definition 1.5.1 (Big Zariski site)**.** Define the big Zariski site to be $(\mathsf{Sch}, \mathcal{T}, \mathfrak{O})$ with the following components. The underlying category is Sch. The topology T is generated by the basis cosisting of the covering families $\{X_i \xrightarrow{f_i} X\}$ where f_i is an open immersion and $\bigcup_i f_i(X_i) = X$. The sheaf of rings $\mathfrak{O}$ sends $(U, \mathfrak{Q}) \to (X, \mathfrak{O})$ to $\Gamma(U;\mathfrak{Q})$.

We will mostly be interested in the sliced site Sch_X for a scheme X.

Definition 1.5.2 (k)**.** Let X be a scheme. Define the functor $k : \mathsf{Open}(X) \to \mathsf{Sch}_X$ by $U \mapsto ((U, \mathfrak{O}_U), i)$ where $i : U \to X$ is the inclusion of the open subscheme into X.

We will show that it preserves limits and covers. The terminal $X \in \mathsf{Open}(X)$ is send to the terminal $X \to X$. Let $U \to V$ and $W \to V$ be two morphism in $\mathsf{Open}(X)$. We have $k(U \cap W) = U \cap W \to X$ which is the pullback of $k(U) \to k(V)$ and $k(W) \to k(V)$.

Let $S = \{D(f_i) \to \mathrm{Spec}(R)\}$ be one of the generating family in $\mathsf{Open}(X)$. Note that $k(D(f_i))$ is isomorphic to the object $\mathrm{Spec}(R_{f_i}) \to \mathrm{Spec}(R)$. Hence $k(S)$ generates a covering sieve on Sch and hence on Sch_X.

So k is morphism of sites.

Lemma 1.5.3. *Let* R *be a ring. Let* M *be a* R*-module.* $\omega^2 : \Gamma(\mathrm{Spec}(R); \lambda\mathfrak{M}) \to \Gamma(\mathrm{Spec}(R); \Lambda\mathfrak{M})$ *is an isomorphism.*

Proof. Let $\mathfrak{p} \subset R$ be a prime ideal. As stated in Definition 1.3.5, we may use results from [5, Tag 01BH] in this setting. We will use that $\Lambda(M)_\mathfrak{p} = M \otimes_R R_\mathfrak{p}$. By naturality, localized at $\mathfrak{p}$, the map ω^2 sends m to $m \otimes 1 \in M \otimes_R R_\mathfrak{p}$, hence is the inverse of the multiplication map which is an isomorphism. Hence globally ω^2 is an isomorphism. ∎

Corollary 1.5.4. *Let* X *be a scheme. Let* $\mathrm{Spec}(R)$ *be an open subset of* X*. Let* M *be a* $\Gamma(X; \mathfrak{O})$*-module. Then* $\omega^2 : \Gamma(\mathrm{Spec}(R); \lambda\mathfrak{M}) \to \Gamma(\mathrm{Spec}(R); \Lambda\mathfrak{M})$ *is an isomorphism.*

Lemma 1.5.5. *Let* R *be a ring. Let* M *be a* R*-module.* $\widetilde{M} = \Lambda(M)$ *on the small Zariski site coming from* $\mathrm{Spec}(R)$*.*

Proof. Note that $D(f)$ is affine for any $f \in R$. So by lemma ? $\Gamma(D(f); \Lambda\mathfrak{M}) = \Gamma(D(f); \lambda\mathfrak{M}) = M \otimes R_f$. By construction $\Gamma(D(f); \widetilde{\mathfrak{M}}) = M_f$. So for every $D(f)$ we get an isomorphism $\Gamma(D(f); \Lambda\mathfrak{M}) \to \Gamma(D(f); \widetilde{\mathfrak{M}})$. These distinguised opens form a basis of the topology and our isos are compatible in the usual sense, so there is an unique sheaf isomorphism extending this collection of isomorphism.

This sheaf isomorphism actually is the map $\Lambda(M) \to \widetilde{M}$ coming from $M \to \Gamma(\mathrm{Spec}(R); \widetilde{\mathfrak{M}})$ via the adjunction $\Lambda \dashv \Gamma(\mathrm{Spec}(R); -)$. Both maps send $m \otimes \frac{a}{f}$ to $\frac{am}{f}$ on $D(f)$. ∎

See [5, Tag 01I7].

Remark 1.5.6. The next result will prove that our definition of quasi-coherence in the sense of Definition 1.3.2 coincides with the usual definition for schemes using $\widetilde{-}$. See [6, definition 13.2.2] for the usual definition.

Lemma 1.5.7. *Let* $\mathfrak{F}$ *be a sheaf of modules on scheme* X*.* $\mathfrak{F}$ *is quasi-coherent on* X *if and only if for any open* $\mathrm{Spec}(R) \subset X$ *the sheaf* $\mathfrak{F}|_{\mathrm{Spec}(R)}$ *is isomorphic to* $\widetilde{M}$ *with* $M = \Gamma(\mathrm{Spec}(R); \mathfrak{F})$*.*

Proof. $\Rightarrow$: By assumption we get local presentations indexed by a covering. Let $\bigcup_{i \in I} U_i = X$ be this covering. Assume without loss of generality that it is an affine open covering. Let $U_i = \mathrm{Spec}(R_i)$. Let $\mathfrak{O}_{U_i}^{\bigoplus K} \to \mathfrak{O}_{U_i}^{\bigoplus J} \to \mathfrak{F}|_{U_i} \to 0$ be one of the given presentations. Taking global sections gives us an exact sequence

$$R_i^{\bigoplus K} \to R_i^{\bigoplus J} \to \Gamma(U_i; \mathfrak{F}) \to 0.$$

Tensoring it with the localisation $R_{i,f}$ for any $f \in R_i$ yields

$$R_{i,f}^{\bigoplus K} \to R_{i,f}^{\bigoplus J} \to \Gamma(U_i; \mathfrak{F}) \otimes R_{i,f} \to 0.$$

Taking sections at $D(f)$ from the sheaf sequence yields

$$R_{i,f}^{\bigoplus K} \to R_{i,f}^{\bigoplus J} \to \Gamma(D(f); \mathfrak{F}) \to 0.$$

Hence $\mathfrak{F}|_{U_i}$ is the unique sheaf with $D(f) \mapsto \Gamma(U_i; \mathfrak{F})_f$, which we defined to be $\widetilde{\Gamma(U_i; \mathfrak{F})}$. By the affine communication lemma, this property holds for any affine and not just for the affines in this covering.

$\Leftarrow$: Let $M = \Gamma(X; \mathfrak{F})$. Take a presentation $R^{\bigoplus I} \to R^{\bigoplus J} \to M$ and apply $\widetilde{-}$. Then note that $\widetilde{-}$ commutes with arbitrary colimits since it is a left adjoint, see lemma ?. We have $\widetilde{R} = \mathfrak{O}_{\mathrm{Spec}(R)}$ so we get a presentation $\mathfrak{O}_{\mathrm{Spec}(R)}^{\bigoplus I} \to \mathfrak{O}_{\mathrm{Spec}(R)}^{\bigoplus J} \to \mathfrak{F}$ on every affine open subset $\mathrm{Spec}(R) \subset X$, hence $\mathfrak{F}$ is quasi-coherent. ■

Lemma 1.5.8. *Let* $X = \mathrm{Spec}(R)$ *be a scheme. Let* $\mathfrak{F}$ *be a quasi-coherent sheaf on* Sch_X. *Let* $M = \Gamma(X; \mathfrak{F})$.

Restricting a quasi-coherent sheaf gives a quasi-coherent sheaf, so $k_* \mathfrak{F} = \Lambda(M)$ *by lemma ?. Hence* $\Gamma(D(f); \mathfrak{F}) = \Gamma(D(f); \Lambda\mathfrak{M}) = M_f$ *for any* $f \in R$.

Let $D(f_i)|_{\mathfrak{O}}^{\bigoplus J} \xrightarrow{\alpha_i} D(f_i)|_{\mathfrak{O}}^{\bigoplus K} \to D(f_i)|_{\mathfrak{F}} \to 0$ *be a presentation with* $(f_i) = (1)$. *Note that the presheaf cokernel* $\mathit{Coker}(\alpha_i)$ *of the sheaf morphism is* $\lambda(\mathit{Coker}(\alpha_{i,D(f_i)}))$ *where* $\alpha_{i,D(f_i)}$ *is the component at* $D(f_i)$ *of* α_i. *So* $D(f_i)|_{\mathfrak{F}} = \Lambda(M_f)$ *since* ω^2 *is iso for affines.*

Hence

$$\begin{aligned}
\Gamma(D(f) \times_X \mathrm{Spec}(S); \mathfrak{F}) &= \Gamma(D(f) \times_X \mathrm{Spec}(S); \Lambda\mathfrak{F}) \\
&= \Gamma(\mathrm{Spec}(S_f); \Lambda\mathfrak{M}_f) \\
&= M_f \otimes S_f \\
&= M \otimes S_f.
\end{aligned}$$

By the sheaf property it follows that $\Gamma(\mathrm{Spec}(S); \mathfrak{F}) = \Gamma(X; \mathfrak{F}) \otimes S$, *hence the counit* $\Lambda(M) \to \mathfrak{F}$ *is an isomorphism.*

2 Restrictive

This section will introduce the notion of a restrictive morphism We will see some examples, non-examples and results in the category of schemes and see that this notion is closely related to affinenes.

For some of the examples and results see the chapter on quasi-coherent modules in [6].

Definition 2.0.1 (Restrictive morphism)**.** Let $(\mathsf{C}, \mathcal{T}, \mathfrak{O})$. A morphism $f : a \to b \in \mathsf{C}$ is called restrictive if for every quasi-coherent module $\mathfrak{G}$ on C_b the morphism

$$\widehat{f} : \Gamma(b; \mathfrak{G}) \otimes_{\Gamma(b;\mathfrak{O})} \Gamma(a; \mathfrak{O}) \to \Gamma(a; \mathfrak{G})$$

is an isomorphism.

Remark 2.0.2. Assume we are in the context of Definition 2.0.1. Assume $G = \Lambda(M)$ for some $\Gamma(b; \mathfrak{O})$-module, then $\widehat{f} = \omega^2_a$ for the sheafification transformation $\omega^2 : \lambda(M) \to \Lambda(M)$.

Example 2.0.3. In $\mathsf{Sch}_{\mathsf{Spec}(A)}$ the morphism $\mathsf{Spec}(A_f) \to \mathsf{Spec}(A)$ is restrictive. Let $\mathfrak{G}$ be a quasi-coherent sheaf on $\mathsf{Sch}_{\mathsf{Spec}(A)}$. This implies that $\mathfrak{G} = \Lambda(\Gamma(\mathsf{Spec}(A); \mathfrak{G}))$, see ?. The morphism

$$\Gamma(\mathsf{Spec}(A); \mathfrak{G}) \otimes_{\Gamma(1;\mathfrak{O})} \Gamma(\mathsf{Spec}(A_f); \mathfrak{O}) \to \Gamma(\mathsf{Spec}(A_f); \mathfrak{G}) = \Gamma(A; \mathfrak{G})_f,$$

$$m \otimes r \to rm$$

is an isomorphism by basic commutative algebra.

Example 2.0.4. Let R be a ring. Consider the open immersion $U = \mathsf{Spec}(R[x, y]) \setminus \{(x, y)\} \to \mathsf{Spec}(R[x, y])$ and the quasi-coherent sheaf $\mathfrak{G} = \Lambda(\frac{R[x,y]}{xy})$. The global sections of this sheaf are $\frac{R[x,y]}{xy}$, as shown in

Define $U_1 = D(x) \to U$ and $U_2 = D(y) \to U$. Note that these cover U together. We have $\Gamma(U_1; \mathfrak{G}) = 0$ and $\Gamma(U_1; \mathfrak{G}) = 0$, since $\frac{R[x,y]}{xy}_x = 0$ and $\frac{R[x,y]}{xy}_y = 0$. Hence since $\mathfrak{G}$ is a sheaf, we get $\Gamma(U; \mathfrak{G}) = 0$.

The sections over U of $\Lambda(R[x,y])$ are (also) $R[x,y]$. See [6, p. 4.4.1]. We conclude that $\Gamma(1; \mathfrak{G}) \otimes_{\Gamma(1;\mathfrak{O})} \Gamma(U; \mathfrak{O}) \to \Gamma(U; \mathfrak{G})$ is not an isomorphism.

Lemma 2.0.5. *Let* $(C, \mathfrak{T}, \mathfrak{O})$ *be a ringed site. Let* $f : b \to a \in C$ *and* $g : c \to b \in C$ *be morphisms.*

1. *If* fg *and* f *are restrictive, then* g *is.*
2. *if* f *and* g *are restrictive, then* fg.

If $\Gamma(b; \mathfrak{O}) \to \Gamma(c; \mathfrak{O})$ *is faithfully flat then* fg *and* g *restrictive implies* f *is restrictive*

Proof. Consider the following diagram

$$\begin{tikzcd}
\Gamma(a; \mathfrak{G}) \otimes_{\Gamma(a;\mathfrak{O})} \Gamma(c; \mathfrak{O}) \arrow[r, "\widehat{f} \otimes_{\Gamma(b;\mathfrak{O})} \Gamma(c;\mathfrak{O})"] \arrow[d, equal] & \Gamma(b; \mathfrak{G}) \otimes_{\Gamma(b;\mathfrak{O})} \Gamma(c; \mathfrak{O}) \arrow[d, "\widehat{g}"] \\
\Gamma(a; \mathfrak{G}) \otimes_{\Gamma(a;\mathfrak{O})} \Gamma(c; \mathfrak{O}) \arrow[r, "\widehat{fg}"] & \Gamma(c; \mathfrak{G}),
\end{tikzcd}$$

where $\widehat{-}$ is the adjunct map of $\mathfrak{G}(-)$ with respect to the extension/restriction of scalars adjunction.

This diagram commutes: going either direction sends $g \otimes r$ to rg. The results follows from commutativity. ∎

Lemma 2.0.6 (coproduct). *Let* X_1, X_2, Y *be a schemes.* $X_1 \xrightarrow{f_1} Y$ *and* $X_2 \xrightarrow{f_2} Y$ *are restrictive morphisms if and only if the corresponding morphism* $X_1 \sqcup X_2 \xrightarrow{f_1 \sqcup f_2} Y$ *is restrictive.*

Proof. $\Rightarrow$:

Note that $\Gamma(X_1 \sqcup X_2; -) = \Gamma(X_1; -) \times \Gamma(X_2; -)$ by the sheaf property. We will show that

$$\widehat{f_1 \sqcup f_2} : \Gamma(Y; \mathfrak{G}) \otimes_{\Gamma(Y;\mathfrak{O})} \Gamma(X_1 \sqcup X_2; \mathfrak{O}) \to \Gamma(X_1 \sqcup X_2; \mathfrak{G})$$

is an isomorphism. Tensor commutes over products, so this becomes

$$\widehat{f_1} \times \widehat{f_2} : (\Gamma(Y; \mathfrak{G}) \otimes_{\Gamma(Y;\mathfrak{O})} \Gamma(X_1; \mathfrak{O})) \times (\Gamma(Y; \mathfrak{G}) \otimes_{\Gamma(Y;\mathfrak{O})} \Gamma(X_2; \mathfrak{O})) \to \Gamma(X_1; \mathfrak{G}) \times \Gamma(X_2; \mathfrak{G}).$$

By assumption $\widehat{f_1}$ and $\widehat{f_2}$ are isos, so their product is.

$\Leftarrow$: By Lemma 2.0.7, the canonical morphism $X_i \to X_1 \sqcup X_2$ is restrictive. By assumption $X_1 \sqcup X_2 \to Y$ is restrictive. Composing these morphisms yields f_i, by Lemma 2.0.5 this morphism is restrictive. ∎

Lemma 2.0.7. *Let* X, Y *be a schemes. The canonical morphism* $X \to X \sqcup Y$ *is restrictive.*

Proof. Let $\mathfrak{G}$ be a quasi-coherent sheaf on $X \sqcup Y$. By the sheaf property $\Gamma(X \sqcup Y; \mathfrak{G}) = \Gamma(X; \mathfrak{G}) \times \Gamma(Y; \mathfrak{G})$. The same holds also for $\mathfrak{O}$.

We are considering the morphism

$$\Gamma(X \sqcup Y; \mathfrak{G}) \otimes_{\Gamma(X \sqcup Y;\mathfrak{O})} \Gamma(X; \mathfrak{O}) \to \Gamma(X; \mathfrak{G}).$$

By the previous remark about disjoint unions and the sheaf property and some basic commutative algebra one sees that this becomes

$$(\Gamma(X; \mathfrak{G}) \otimes_{\Gamma(X \sqcup Y;\mathfrak{O})} \Gamma(X; \mathfrak{O})) \times (\Gamma(Y; \mathfrak{G}) \otimes_{\Gamma(X \sqcup Y;\mathfrak{O})} \Gamma(X; \mathfrak{O})) \to \Gamma(X; \mathfrak{G}).$$

Since $\Gamma(Y; \mathfrak{G}) \otimes_{\Gamma(X \sqcup Y;\mathfrak{O})} \Gamma(X; \mathfrak{O}) = 0$, we are left with

$$(\Gamma(X; \mathfrak{G}) \otimes_{\Gamma(X \sqcup Y;\mathfrak{O})} \Gamma(X; \mathfrak{O})) \to \Gamma(X; \mathfrak{G})$$
$$g \otimes r \to rg.$$

Note that $\Gamma(X; \mathfrak{G})$ already is an $\Gamma(X; \mathfrak{O})$-module and conclude that hence this morphism is an isomorphism. ∎

Lemma 2.0.8 (Restrictive to affines)**.** *If* $f : X \to \mathrm{Spec}(R)$ *is a restrictive open immersion, then* X *is affine.*

Proof. Since X is an open in $\mathrm{Spec}(R)$, we get a distinguised covering $\bigcup_i D(f_i) = X$ with $f_i \in R$ and $i \in I$. We will prove that $(f_i|_X)_{i\in I} = (1)$ in $S = \Gamma(X; \mathfrak{O})$.

Then we invoke the result in [2, Ex. 2.1.7] that states the following. For a scheme Y let Y_f be the support of $f \in \Gamma(X; \mathfrak{O})$ as in Definition 1.4.3. For a scheme Y if X_{g_j} are affine and $(g_j)_{j\in J} = (1)$ then X is affine.

Note that $D(f_i) = X_{f_i|_X}$. Consider $M = \frac{R}{(f_i)}$ as an R-module and look at $\Lambda(M)$. By restrictiveness we get $M \otimes_R S = \Lambda(M)(S)$ and by $M \otimes_R R_{f_i} = \Lambda(M)(D(f_i)) = M_{f_i} = 0$. Hence $\Lambda(M)(S) = 0$ by the sheaf axiom. This implies that $(f_i|_X)_{i\in I} = (1)$ in S. So X is affine. ∎

Lemma 2.0.9. *Any morphism* $\mathrm{Spec}(S) \xrightarrow{f} \mathrm{Spec}(R) \in \mathrm{Sch}_{\mathrm{Spec}(R)}$ *between affine schemes is restrictive.*

Proof. Let $\mathfrak{G}$ be a quasi-coherent module on $\mathrm{Sch}_{\mathrm{Spec}(R)}$. Set $M = \Gamma(\mathrm{Spec}(R); \mathfrak{G})$. We want to prove that

$$\widehat{f} : M \otimes_R S \to \Gamma(\mathrm{Spec}(S); \mathfrak{G})$$

is an isomorphism.

By Lemma 1.5.8, we get $\mathfrak{G} = \Lambda(M)$. As said in Remark 2.0.2, in this case $\widehat{f} = \omega^2_{\mathrm{Spec}(S)}$. By Lemma 1.5.3, we know that ω^2 is an isomorphism at affine schemes. ∎

Example 2.0.10 (Affine non-restrictive map)**.** One might expect(or want) that any property of all maps between affine schemes also hold for affine maps between any schemes. This is not the case for restrictiveness, so it is not local on the target.

Consider the canonical inclusions $\mathbb{A}^1 \to \mathbb{P}^1$ and the shifted quasi-coherent module $\mathfrak{O}(-1)$. This module is locally free of degree 1, this is often called an invertible module.

The global sections of the module $\mathfrak{O}(-1)$ are the elements of degree -1 in the global sections of $\mathfrak{O}$. There are no such elements, hence the global sections are the zero module.

On $\mathbb{A}^1$ all invertible modules are isomorphic to the structure sheaf. See [6, p. 14.2.8].

Similary any inclusion $\mathrm{Spec}(\kappa(\mathfrak{p})) \to \mathbb{P}^1$ of a point is not restrictive which can be shown with the same argument.

This is a (more opaque) way of saying that on projective space not every quasi-coherent sheaf is generated by global sections.

Bibliography

[1] Steve Awodey. *Category Theory*. Oxford University Press, 2010.

[2] Robin Hartshorne. *Algebraic Geometry*. Springer-Verlag New York, 1977.

[3] Saunders Mac Lane and Ieke Moerdijk. *Sheaves in Geometry and Logic*. Springer New York, 1992.

[4] Ieke Moerdijk and Jaap van Oosten. *Topos Theory*. URL: http://www.staff.science.uu.nl/~ooste110/syllabi/toposmoeder.pdf (visited on 09/15/2015).

[5] The Stacks Project Authors. *Stacks Project*. http://stacks.math.columbia.edu. 2016.

[6] Ravi Vakil. *The Rising Sea, Foundations of Algebraic Geometry*. http://math.stanford.edu/~vakil/216blog/index.html. Sept. 2015.