Affine Objects

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Abstract

Short summary of the contents of your thesis.

Acknowledgements

Put your acknowledgements here.

1 Introduction

Hoi, dit is de introductie.

2 Preliminaries

Let $Y = (X, \mathcal{T}, \mathfrak{O})$ be a ringed site. Let $R = \Gamma(1; \mathfrak{O})$. Let $a, b \in X$.

2.0.1 basic categorical definitions

Notation 1. Let C be a category.

Definition 2 (Presheaf categories and yoneda functors)**.** Let C be a category. Then we define

$$C^+ := [C^{op}, \mathsf{Set}],$$

$$C^- := [C^{op}, \mathsf{Set}^{op}] \cong [C, \mathsf{Set}],$$

and the functors

$$h_C : C \to C^+ := X \mapsto \mathrm{Hom}(-, X),$$
$$h^C : C \to C^- := X \mapsto \mathrm{Hom}(X, -).$$

Both these functors are fully faithfull by the Yoneda lemma.

Definition 3 (Over/Under categories)**.** Let C and C' be categories. Let $F : C \to C'$ and $Z \in C'$. Define the category C_Z and C^Z as follows

$$\mathrm{Obj}(C_Z) := \{(X, u) \mid X \in C, u : F(X) \to Z\},$$

$$\mathrm{Hom}((X, u), (Y, v)) := \{f : X \to Y \mid v \circ F(f) = u\},$$

and

$$\mathrm{Obj}(C^Z) := \{(X, u) \mid X \in C, u : Z \to F(X)\},$$
$$\mathrm{Hom}((X, u), (Y, v)) := \{f : X \to Y \mid F(f)u = v\}.$$

We get faithfull functors $C_Z \to C : (X, u) \to X$ and $C^Z \to C : (X, u) \to X$.

2.0.2 Presheaves

Definition 4 (Presheaves). Let A be any category. An A-valued presheaf $\mathfrak{F}$ is a functor $C^{op} \to A$. The category $[C^{op}, A]$ of all A-valued presheaves is denoted $C^+(A)$. If $A = \mathrm{Set}$, we will use C^+.

When it is obvious which presheaf is under consideration, then $\mathfrak{F}(f)$ we be denoted as f^*.

Definition 5 (Sections functor). For any $X \in C$ define the functor

$$\Gamma(X; -) : C^+(A) \to A$$

by

$$\mathfrak{F} \to \mathfrak{F}(X).$$

Let $L : I \to C$ be diagram and assume that $\operatorname*{colim}_{h_- \circ L}$ exists in $C^+(A)$. Define

$$\Gamma(\operatorname*{colim}_{i \in I} L(i); -) : C^+(A) \to A$$

by

$$\mathfrak{F} \to \mathrm{Hom}(\operatorname*{colim}_{i \in I} L(i), \mathfrak{F}) = \lim_{i \in I} \mathrm{Hom}(L(i), \mathfrak{F}).$$

By definition of the colimit these definitions coincide when the colimits exists in C.

Example 6. As we will see, $\mathfrak{C}$ is (co)complete so even if C does not have a terminal object, we can still compute the 'global sections'.

Lemma 7 (Complete). *If A complete (or cocomplete) then $C^+(A)$ is and all sections functors commute with arbitrary limits and colimits.*

Proof. Let $I \to C^+(A)$ be an diagram, with $i \mapsto \mathfrak{F}_i$. Then the presheaves $U \mapsto \lim_{i \in I} F_i(U)$ and $U \to \operatorname{colim}_{i \in i} F_i(U)$ are the limit and colimit of this diagram. ∎

Lemma 8 (Abelian/Grothendieck). *If* A *abelian (or Grothendieck) then* $C^+(A)$ *is.*

Proof. Let $f, g : \mathfrak{F} \to \mathfrak{G}$. Let $U \in C$. The sum $f + g$ will be defined such that all section functors are additive. Hence $(f + g)_U = f_U + g_U$, this completely determines $f + g$. Note that this makes $f + g$ into an actual morphism of presheaves because composition is bilinear in A.

Define $\mathrm{Ker}(f) \to \mathfrak{F}$ to be $U \mapsto \mathrm{Ker}(f_U)$ and $\mathrm{Coker}(f) \to \mathfrak{F}$ to be $U \mapsto \mathrm{Coker}(f_U)$. Since A is abelian we have $\mathrm{Im}(f) = \mathrm{Coim}(f)$. By the previous lemma $C^+(A)$ has direct sums. It also has a zero object which is the presheaf $U \mapsto 0$.

Hence $C^+(A)$ is abelian as defined in tag 0109 of stacks.

Assume A is also Grothendieck. Note that the section functions $\Gamma(U; -)$ have adjoints : Then using this the family $\{c_! G : c \in C\}$ is a small generating family. Taking the colimit over this family provides us with a generator in $C^+(A)$.

Exactness and taking colimits both are determined pointwise, so directed colimits aka direct limits are exact because they are in A.

∎

Remark. So C^+ is a Grothendieck category.

Definition 9 (direct image). Let $f : C \to D$ and $X \in C$. Define the direct image f_* of $\mathfrak{F} \in C^+(A)$ to be

$$f_*(F) = \mathfrak{F} \circ f.$$

Lemma 10 (direct image commutes with limits). f_* *commutes with limits.*

Proof. Let $\mathfrak{G} = \lim_{i \in I} \mathfrak{F}_i$ be a limit of presheaves. Let $X \in C$. Then $f_* \mathfrak{G}(X) = \mathfrak{G}(f(X)) = \lim_{i \in I} \mathfrak{F}_i(f(X)) = (\lim_{i \in I} f_* \mathfrak{F}_i)(X)$. Hence f_* commutes with limits and this also holds for colimits. ∎

Definition 11 (Inverse image, direct image & push-forward)**.** Define the inverse image f^* of $\mathfrak{F} \in D^+(A)$ to be

$$f^*(F)(X) = \underset{(Y,u)\in C_X}{\operatorname{colim}} \mathfrak{F}(Y).$$

Define the pushforward $f_!$ of $\mathfrak{F} \in D^+(A)$ to be

$$f_!(F)(X) = \lim_{(Y,u)\in C^X} \mathfrak{F}(Y).$$

Now f^* is left adjoint to f^* and $f_!$ is right adjoint to f^* by construction. This is a general construction to get adjoints, which works now because the indexing categories are small and the target contains all small (co)limits.

See Stacks Tag 09YX for a different existence lemma for the push-forward.

Assume C has binary products.

Definition 12. Let $j_X : C_X \to C$ be the projection. Let $i_X : C \to C_X$ be defined by

$$V \mapsto (U \times V, p_0)$$

We will use $F\big|_X = j_{X*}\mathfrak{F}$.

Lemma 13. *Let* $\mathfrak{F} \in C^+(A)$ *and let* $\mathfrak{G} \in X^+(A)$

$$j_{X*}\mathfrak{F}(V \to X) = \mathfrak{F}(V),$$

$$j_X{}^*\mathfrak{G}(V) = \bigoplus_{s\in Hom(V,X)} \mathfrak{F}(V \xrightarrow{s} X)$$

$$j_{X!}\mathfrak{G}(V) = \mathfrak{F}(X \times V \to X)$$

Definition 14 (Internal Hom)**.** Let $\mathfrak{F}, \mathfrak{G} \in C^+(A)$. Define F^G to be the presheaf $X \mapsto \mathrm{Hom}(\mathfrak{F}\big|_X, \mathfrak{G}\big|_X)$. For $A = \mathrm{Set}$ or $A = R\text{-Mod}$, we have $F^G \in C^+(A)$.

If we assume that the representable sheaves are A-valued, then we can also define

$$F^G(X) = \mathrm{Hom}(\mathfrak{F} \times h_X, \mathfrak{G}).$$

These two definitions are equal because the following functions are inverses of each other:

$$r \mapsto (U : a \mapsto r(a,k)) \text{ where } U \xrightarrow{k} X,$$

$$s \mapsto (U : (a,b) \mapsto s(a)).$$

Definition 15 (Monoidal structure). Let $\mathfrak{F}, \mathfrak{G} \in C^{+}(A)$. Let A have a monoidal structure. Define $\mathfrak{F} \otimes \mathfrak{G}$ as $X \mapsto \mathfrak{F}(X) \otimes \mathfrak{G}(X)$

Lemma 16 (Adjunction/monoidal closed structure). *Let $\mathfrak{F}$ be fixed. Then $- \otimes \mathfrak{F}$ is left adjoint to $Hom(\mathfrak{F}, -)$.*

2.0.3 Topology

Definition 17 (Sieve). A sieve on $X \in C$ is a subpresheaf(or subobject or subfunctor) of the representable presheaf h_X. The maximal sieve will be denoted $\max(C)$.

Definition 18 (Grothendieck Topology). A Grothendieck topology $\mathcal{T}$ is a family $\mathcal{T}(X)$ of 'covering' sieves for every $X \in C$ with the following conditions:

- $\max(X) \in \mathcal{T}(X)$
- $f^*R \in \mathcal{T}(X')$ if $R \in \mathcal{T}(X)$ for any $f : X' \to X$
- if $f^*R \in \mathcal{T}(X')$ for all $f \in S$ with $S \in \mathcal{T}(X)$ then $R \in \mathcal{T}(X)$

Note that if $f \in R$ then $f^*R = \max(X')$. So if $R \subset S$ and R is covering then S is covering. Also $R \cap S$ is covering if and only if R and S are covering.

Definition 19 (Basis). Let C have pullbacks. A Grothendieck pretopology $\mathcal{B}$ is a collection $\mathcal{B}(X)$ of families $\{f_i : X_i \to X\}$ of 'covering' morphisms for every $X \in C$ with the following conditions:

- every isomorphism is a covering singleton family.
- (Stability) The pullback of a covering family is covering. If $\{f_i : X_i \to X\}$ is covering and $g : Y \to X$, then $\{f_i' : X_i \times_X Y \to Y\}$ is covering.
- (Transitivity) If $\{f_i : X_i \to X\}$ is covering and $\{f_{ij} : X_{ij} \to X_i\}$ for every i, then $\{f_{ij} : X_{ij} \to X\}$ is covering.

Generating a real topology: take any sieve containing a covering family to be a covering sieve. Any sieve is generated by itself as covering family, in this way any topology can be interpreted as a pretopology. This enables one to use the pullbacks in proofs.

Definition 20. A site $(C, \mathcal{T})$ is a category C with the Grothendieck topology $\mathcal{T}$. If C has pullbacks, then we consider $\mathcal{T}$ always as a pretopology.

Definition 21 (Cocontinuous functor).

Lemma 22. *Let* $(C, \mathcal{T}) \xrightarrow{g} (D, \mathcal{S})$. *Let* $\mathfrak{F}$ *be a presheaf on* D. *If* g *is cocontinuous, then*

$$g_! \mathfrak{F}^+ \cong g_! \mathfrak{F}^+.$$

Proof. Let X be an object. The two presheafs reduce to

$$\lim_{R \in S(g(X))} \mathrm{Hom}(R, F) \to \lim_{K \in T(X)} \mathrm{Hom}(g(K), F).$$

The poset of covering sieves on X is send to a dense poset of $g(X)$ so the limits are isomorphic and this isomorphism is natural. ∎

2.0.4 Ringed sites

2.0.5 Sheaves

Definition 23 (Sheaves of sets). Let $(C, \mathcal{T})$ be a site. Let $\mathfrak{F} \in C^+$. A compatible family on X is a family of elements $x_f \in \mathfrak{F}(X_f)$ indexed by a sieve R on X, where $X_f = \mathrm{Dom}(f)$ and such that $g^*(x_f) = x_{fg}$. This is the same as a morphism $R \to \mathfrak{F}$ as presheaves.

An amalgamation of a compatible family $(x_f)_R$ on X is an element $x \in \mathfrak{F}(X)$ such that $f^*(x) = x_f$. Hence given an morphism $X \to \mathfrak{F}$ that extends the morphism $R \to \mathfrak{F}$ defined by the compatible family.

A presheaf that admits a unique amalgamation of every compatible family is called a sheaf. The category $\mathrm{Shv}(C)$ is the full subcategory on these sheaves. Let i be the inclusion functor $\mathrm{Shv}(C) \to C^+$.

Definition 24 (Sheaves #2)**.** Let $(\mathsf{C}, \mathcal{T})$ be a site. Let $\mathfrak{F} \in \mathsf{C}^{+}\mathsf{A}$. Let A be (small) complete. Define $\mathfrak{F}(R)$ for a sieve R on X to be

$$\mathrm{Hom}(R, F).$$

We call $\mathfrak{F}$ a sheaf if the map

$$\mathfrak{F}(X) \to \mathfrak{F}(R)$$

is an isomorphism.

This is just shorthand notation for the above definition. There is a bijection between $\mathrm{Hom}(R, F)$ and matching families and the map sends a section to the unique matching family indexed by R it is an amalgamation of.

Definition 25 (Plus construction Shapiro)**.** Let $(\mathsf{C}, \mathcal{T})$ be a site. Let $\mathfrak{F} \in \mathsf{C}^{+}$. Define the category I whose objects are pairs (R, φ) with $R \in \mathcal{T}(X)$ and $R \xrightarrow{\varphi} F$. A morphism between $(R, \varphi) \to (S, \phi)$ are inclusions $R \to S$ such that $\varphi = \phi$ restricted to R.

Then

$$\mathfrak{F}^{+}(X) = \lim_{(R,\varphi)\in \mathsf{I}} F(R).$$

More concretely, $\mathfrak{F}^{+}(X)$ is the set of all objects (R, φ) with the equivalence relation that $(R, \varphi) \sim (S, \phi)$ if $\varphi = \phi$ on $R \cap S$. Or equivalently the set all compatible families with the equivalence relation that $(x_f)_R \sim (y_g)_S$ if $(x_f)_{R\cap S} = (y_g)_{R\cap S}$.

We have the map

$$\eta : \mathfrak{F}(X) \to \prod f \in R\mathfrak{F}(\mathrm{Dom}(f))$$
$$x \mapsto (X, x).$$

This defines a natural transformation from Id to $-^{+}$.

Definition 26 (Plus construction Moerdijk)**.**

Lemma 27. *$\mathfrak{F}^{+}$ is separated*

Lemma 28. *If $\mathfrak{F}$ is separated then $\mathfrak{F}^{+}$ is a sheaf.*

Lemma 29. *Let $(\mathsf{C}, \mathcal{S}) \xrightarrow{g} (\mathsf{D}, \mathcal{T})$. Let $\mathfrak{F}$ be a presheaf on D. Then*

$$\mathfrak{F}^{+}g \cong \mathfrak{F}g^{+}.$$

Proof. ∎

Definition 30 (Sheafification)**.** Define $\mathfrak{a}(\mathfrak{F}) = \mathfrak{F}^{++} : \mathsf{C}^{+} \to \mathrm{Shv}(\mathsf{C})$. This is a left adjoint to the inclusion functor.

The functor $\mathfrak{a}(-)$ commutes with finite limits. There are two different proofs: Moerdijk and stack+Shapiro. Shapiro defines the plus with as a colimit over a directed set, hence this commutes with limits.

Theorem 31. *The following this are equivalent for a category* C*.*

- *A Grothendieck topology*
- *A full subcategory* $\mathsf{E} \subset \mathsf{C}^{+}$ *such that the the inclusion functor has a left adjoint that preserves finite limits.*

Remark (Properties)*.*

- The restriction of a sheaf is a sheaf

3 Restrictive

Definition 32 (Restrictive functor). A functor $f : (C, \mathcal{T}, \mathfrak{O}) \to (D, \mathcal{S}, \mathfrak{U})$ between ringed sites is called restrictive if for every quasi-coherent module $\mathfrak{G}$ the co-unit induces an isomorphism

$$\mathfrak{G} \to f_* f^* \mathfrak{G},$$

$$\Gamma(1; \mathfrak{G}) \to \Gamma(1; f_* f^* \mathfrak{G}) \cong \Gamma(1; f^* \mathfrak{G})$$

$$\Gamma(1; \mathfrak{G}) \otimes_{\Gamma(1;\mathfrak{O})} \Gamma(1; \mathfrak{U}) \to f_* f^* \mathfrak{G}.$$

Note that

Definition 33 (Restrictive morphism). A morphism $f : X \to Y \in C$ is called restrictive if the induced functor

$$C_X \to C_Y$$

is restrictive.

Example 34. In Sch, the morphism Spec $A_f \to$ Spec A is restrictive.

Non-Example 35. The open immersion Spec $R^2 \setminus 0 \to$ Spec R^2 is not restrictive. The quasi-coherent sheaf $\Lambda(\frac{R[x,y]}{xy})$ fails to satisfy the condition from the definition.

Lemma 36 (Restrictive to affines). *If* $f : X \to$ *Spec* R *is a restrictive open immersion. then* X *is affine.*

Proof. ∎

Non-Example 37 (Affine non-restrictive map). Both canonical inclusions $\mathbb{A}^1 \to \mathbb{P}^1$ are not restrictive. Look at the quasi-coherent module $\mathfrak{O}(-1)$. There are no global sections but on every affine chart this invertible sheaf is trivial.

Non-Example 38. Any inclusion Spec $\kappa(\mathfrak{p}) \to \mathbb{P}^1$ is not restrictive. Look at $\mathfrak{O}(-1)$.

Lemma 39. *The composition of two restrictive functors is restrictive. If the composition* gf *is restrictive, then* g *is restrictive*

Proof. ∎

4 Caffine objects

4.0.1 introduction

Let $(C, \mathcal{T}, \mathfrak{O})$ be a ringed site.

Definition 40 (Caffine). Let $a \in C$ be an object. We call a *caffine* if the adjunction $\Gamma(1; -) \dashv \Lambda(-)$ is an equivalence of categories. Or equivalently that the unit η and co-unit ϵ of this adjunction are natural isomorphisms.

Example 41 (Examples of caffine objects). The main example to keep in mind is $\mathsf{Spec}\ R \in \mathsf{Sch}$.

Let $(*, \mathfrak{R})$ be a ringed space. This space is always caffine, because all presheaves are sheaves. If R is non-local, then this space is not a scheme. This is an example of a non-scheme caffine ringed space

4.0.2 Restrictive maps between caffine objects

Lemma 42 (Morphism between caffines is restrictive). *Let $b \xrightarrow{f} a \in C$ be a morphism between caffine objects, then f is restrictive.*

Proof. Let $\mathfrak{F}$ be a quasi-coherent module on C_a. We have to show that $\Gamma(a; \mathfrak{F}) \otimes_{\Gamma(a;\mathfrak{O})} \Gamma(b; \mathfrak{O}) \xrightarrow{k} \Gamma(b; \mathfrak{F})$ is an isomorphism. The map k is the adjunct of $\mathfrak{F}(f)$ with respect to the adjunction between restricting scalars and extending scalars along the map $\Gamma(a; \mathfrak{O}) \to \Gamma(b; \mathfrak{O})$. More concretely, this map is

$$k : x \otimes m \mapsto \mathfrak{F}(f)(x)m.$$

The argument will go as follows. First we observe that the morphism $\epsilon_{\mathfrak{F}} : \mathfrak{F} \to \Lambda(\Gamma(1;\mathfrak{F}))$ is an isomorphism because a is caffine. Second $i_a : \Gamma(1;\mathfrak{F})(a) \to \Lambda(\Gamma(1;\mathfrak{F}))(a)$ is an isomorphism by lemma .. . This holds for any caffine objects, so also for b. The consequence is that $\Gamma(1;\mathfrak{F})(f) = i_b^{-1}\mathfrak{F}(f) \circ i_a$, by naturality of the transformation $i : \Gamma(1;\mathfrak{F}) \to \mathfrak{F}$. Third, show that $\Gamma(1;\mathfrak{F})(f)$ has an isomorphism as adjunct along the same extension/restriction adjunction. Call this adjunct k'. Fourth, use naturality of adjunction bijections to conclude that k must also be an isomorphism.

Since a is caffine, $\mathfrak{F} = \Lambda(\Gamma(1;\mathfrak{F}))$. Since $\Lambda(-) = a(-) \circ -$, we know that $\mathfrak{F}$ is the sheafification of the presheaf

$$\Gamma(1;\mathfrak{F}) = c \to \Gamma(a;\mathfrak{F}) \otimes_{\Gamma(a;\mathfrak{O})} \Gamma(c;\mathfrak{O}).$$

Set $M = \Gamma(a;\mathfrak{F})$.

Define $k' : \Gamma(a;\mathfrak{M}) \otimes_{\Gamma(a;\mathfrak{O})} \Gamma(b;\mathfrak{O}) \to \Gamma(b;\mathfrak{M})$ By $k' : x \otimes m \mapsto M(f)(x)m$. If you unfold the constructions, it follows that

$$k'(x \otimes m) = x \otimes m \in \Gamma(b;\mathfrak{M}) = M \otimes \Gamma(b;\mathfrak{O})$$

is actually the identity.

We will prove that $A) The component at a caffine object of the unversal sheafification morphism i is an iso$ $\epsilon_{T,a}$. Hence when a is caffine then $\Gamma(1;i)$ on is an iso. Note that $\Gamma(1;i) : C/x \to \mathfrak{O}(x)$-Mod is equal to $\Gamma(x;i) : Cy \to$

B) Adjunction bijection respects composition with isos We have now that $M(f) = i_b^{-1} \circ \Lambda(F)(f) \circ i_a$. Let F be the bijection from the adjunction $\Gamma(1;-) \dashv \Lambda(-)$. Then

C) Hence k is also an iso.

Note that restricting and sheafification commute.We can first restrict our presheaf to C/b and then sheafify. The global sections component of the universal sheafification morphism will be

$$M \otimes \Gamma(b;\mathfrak{O}) \to \Gamma(b;\mathfrak{F}),$$

$$m \otimes r \mapsto mr$$

because the triangle

$$\begin{tikzcd} \Gamma(a;\mathfrak{F}) \arrow[r, "\mathrm{restr}"] \arrow[d] & \Gamma(b;\mathfrak{F}) \\ \Gamma(a;\mathfrak{F}) \otimes \Gamma(b;\mathfrak{O}) \arrow[ur] & \end{tikzcd}$$

must commute by naturality. This is exactly the component of the unit η of the $\Gamma(1;-) \dashv \Lambda(-)$ on $(C, \mathcal{T})O/b$ for $\Gamma(a;\mathfrak{F}) \otimes \Gamma(b;\mathfrak{O})$. Since b is caffine, this is an isomorphism by assumption.

Let $(C, \mathcal{T}, \mathfrak{O})$ be a ringed site. Let $R = \Gamma(1;\mathfrak{O})$. Let $a \in C$ and $f : x \to y \in C$. ∎

Lemma 43 (Co-unit is iso when locally iso). *Let $a \in C$ be an object. Let $\{b_i \to a\}$ be a restrictive caffine cover, then ϵ is a natural isomorphism.*

Proof. Assume we have a restrictive caffine cover $\{b_i \to a\}$. Let $\mathfrak{F}$ be a quasi-coherent sheaf module. Set $M = \Gamma(a;\mathfrak{F})$. Set $M_i = \Gamma(b_i;\mathfrak{F})$ on C/a.

Consider the co-unit at $\mathfrak{F}$

$$\epsilon_a(\mathfrak{F}) : \Lambda(\Gamma(a;\mathfrak{F})) \to \mathfrak{F}.$$

This morphism restricted gives

$$\epsilon_{a,b_i}(\mathfrak{F}) : \Lambda(\Gamma(a;\mathfrak{F}))\big|_{b_i} \to \mathfrak{F}\big|_{b_i},$$

which is the same map as $\epsilon_b(\mathfrak{F}\big|_b)$. We only need to establish that $\Lambda(\Gamma(a;\mathfrak{F}))\big|_{b_i}$.

Because b_i is caffine, the canonical morphism given by sheafification

$$M \otimes_{\Gamma(a;\mathfrak{O})} \Gamma(b_i;\mathfrak{O}) \to M_i$$

is an isomorphism.

Hence the component

$$\epsilon_{b_i}(\mathfrak{F}\big|_{b_i}) : \Lambda(\Gamma(b_i;\mathfrak{F})) \to \mathfrak{F}\big|_{b_i}.$$

is an isomorphism, because ϵ_{b_i} is a natural isomorphism because b_i is caffine. The subscript b_i signifies that we are working in C/b_i.

But over an caffine object, a map is an isomorphism if and only if it is an isomorphism on global sections. In this case, using naturality of ϵ,

$$\epsilon_{b_i}(b_i) : \Gamma(1;\mathfrak{F}) \otimes_{\Gamma(a;\mathfrak{O})} \Gamma(b_i;\mathfrak{O}) \to \Gamma(b_i;\mathfrak{F}),$$

$$m \otimes r \to mr.$$

By restrictiveness of the map $b_i \to a$, this map is an isomorphism. A local isomorphism between sheaves is an isomorphism.

■

4.0.3 Caffine = Affine for schemes

Let $(X, \mathfrak{O})$ be a caffine scheme. Let $X \xrightarrow{F} \text{Spec } \Gamma(1;\mathfrak{O}) = Y$ be the adjunct of the identity map via the adjunction $(\text{Spec } -, \Gamma(1;-))$. Equivalently, let F be the component at X of the unit from this adjunction.

Let's introduce our variables. Let $x, y \in X$. Let $\mathfrak{p}, I, J \subset \mathfrak{O}(X)$ be ideals with $\mathfrak{p}$ prime. Let $a, b \in \mathfrak{O}(X)$ be global sections.

Definition 44. Define

$$\ker(x) = \ker(\mathfrak{O}(X) \to \kappa(x)),$$

$$D_X(a) = \{x \in X \mid a \notin \ker(x)\},$$

Define

$$V_X(I) = \text{Supp}(\Lambda_X(\frac{\mathfrak{O}(X)}{I})).$$

Remark. Recall that $F(x) = \ker(x)$ for $x \in X$. I will use $D_Y(a)$ for the distinguished open defined by a in the affine Y. Note that $D_X(ab) = D_X(a) \cap D_X(b)$ since $\ker(x)$ is a prime ideal.

Remark. If the support of a sheaf $\mathfrak{G}$ is empty, then locally all sections are zero. Hence all sections are equal to the zero section and $\mathfrak{G} = 0$.

Lemma 45. *The set $V_X(I)$ is closed.*

Proof. Let $z \in X$ and M a $\mathfrak{O}$-module. Assume z is in the support of M, then $g \neq 0$ for any generating element $g \in M_z$.

Consider the exact sequence

$$\mathfrak{O}(X) \to \frac{\mathfrak{O}(X)}{I} \to 0.$$

The functor Λ_X is a left adjoint hence right exact so

$$\mathfrak{O} \xrightarrow{f} \Lambda_X(\frac{\mathfrak{O}(X)}{I}) \to 0$$

is exact. Hence the sequence

$$\mathfrak{O}_x \xrightarrow{f_x} \Lambda_X(\frac{\mathfrak{O}(X)}{I})_x \to 0$$

is exact. The global section $f(1)$ must generate $\Lambda_X(\frac{\mathfrak{O}(X)}{I})$ as a module by surjectivity of f. Similarly $f_x(1_x)$ generates $\Lambda_X(\frac{\mathfrak{O}(X)}{I})_x$.

Note that $f_x(1_x) = f(1)_x$ by definition of f_x, hence $f(1)_x$ is a generating element. Hence $\Lambda_X(\frac{\mathfrak{O}(X)}{I})_x \neq 0$ if and only if $f(1)_x \neq 0$.

This implies $V_X(I) = \mathrm{Supp}(f(1))$ which makes $V_X(I)$ closed as the support of a global section. ∎

The functor Λ_X is exact, so it commutes with quotients. So

$$\Lambda_X(\frac{\mathfrak{O}(X)}{I}) = \frac{\mathfrak{O}}{\Lambda_X(I)}$$

and

$$\Lambda_X(\frac{\mathfrak{O}(X)}{I})_x = \frac{\mathfrak{O}_x}{\Lambda_X(I)_x} = \frac{\mathfrak{O}_x}{I \otimes \mathfrak{O}_x}$$

$\frac{\mathfrak{O}_x}{\Lambda_X(I)_x} \neq 0$, which is the same as saying that $\Lambda_X(I)_x$ is a proper ideal of $\mathfrak{O}_x$. The sheaf $\Lambda_X(I)_x$ is the sheafification of the presheaf $(U \mapsto I \otimes \mathfrak{O}(U))$, hence the stalk at x of the sheaf is $\operatorname*{colim}_{x \in U} I \otimes \mathfrak{O}(U)$. The functor $I \otimes -$ is a left adjoint, hence commutes with colimits. So the stalk is isomorphic to $I \otimes \operatorname*{colim}_{x \in U} \mathfrak{O}(U) = I \otimes \mathfrak{O}_x$. See Stacks[01BH].

Lemma 46. *For $x \in X$ TFAE:*

1. $x \in V_X(I)$
2. $I\mathfrak{O}_x \neq \mathfrak{O}_x$
3. $I \subset \ker(x)$.

Proof. $1 \Rightarrow 2$:

Assume $x \in V_X(I)$. Then $\Lambda_X(\frac{\mathfrak{O}(X)}{I})_x = \frac{\mathfrak{O}_x}{I\mathfrak{O}_x} \neq 0$. Hence $I\mathfrak{O}_x \neq \mathfrak{O}_x$.

$2 \Rightarrow 3$:

Assume $I\mathfrak{O}_x \neq \mathfrak{O}_x$. Then $I\mathfrak{O}_x$ is proper hence contained in the unique maximal ideal of the local ring $\mathfrak{O}_x$, therefore $I \mapsto 0$ in $k(x)$ or equivalently $I \subset \ker(x)$.

$3 \Rightarrow 1$:

Assume $I \subset \ker(x)$. Then I maps into $\mathfrak{m}_x$, hence $I\mathfrak{O}_x \subset \mathfrak{m}_x$. Therefore

$$\frac{\mathfrak{O}_x}{\Lambda_X(I)_x} = \frac{\mathfrak{O}_x}{I\mathfrak{O}_x} \neq 0.$$

■

Corollary 47. *If* $y \in I$ *then* $D_X(y) \cap V_X(I) = \emptyset$

Proof. Assume $y \in I$. Let $z \in V_X(I)$, then $y \in \ker(z)$ by the previous lemma. This implies $z \notin D_X(y)$ ■

Corollary 48. $V_X(I) \cup V_X(J) = V_X(IJ)$

Proof. Let $z \in V_X(I) \cup V_X(J)$. Then $I \subset \ker(z)$ and $J \subset \ker(z)$ by the lemma, hence $IJ \subset \ker(z)$. Apply the lemma again to get $z \in V_X(IJ)$. Let $z \in V_X(IJ)$. Then $IJ \subset \ker(z)$ by the lemma. The ideal $\ker(z)$ is prime, so $I \subset \ker(z)$ or $J \subset \ker(z)$. Invoke the lemma again to get $z \in V_X(I) \cup V_X(J)$. ■

Lemma 49. *Every closed set* W *can be written as* $V_X(I)$ *for some ideal* I.

Proof. Let $\mathfrak{I}$ be some ideal sheaf inducing a closed subscheme structure on W. Let $\mathfrak{O}_W$ be the structure sheaf of this closed subscheme. By construction $V_X(I)$ is the support of the push-forward of $\mathfrak{O}_W$, hence $V_X(I) = W$. ■

Lemma 50. *The sets* $D_X(a)$ *form a basis for the topology of* X.

Proof. Let $U \subset X$ be any open. Let $x \in U$. By the previous lemma we get I such that $V_X(I) = U^c$. It follows that $x \notin V_X(I)$ and $I \not\subset \ker(x)$. So we get a $g \in I$ with $g \notin \ker(x)$. We get $x \in D_X(g)$ and by corollary .. $D_X(g) \subset U$. As stated earlier, $D_X(ab) = D_X(a) \cap D_X(b)$ since $\ker(x)$ is a prime ideal. So $D_X(a)$ form a basis. ∎

Lemma 51. *The map* F *is surjective.*

Proof. Let $\mathfrak{p} \in Y$ be a point in the target of F. Then $\Lambda_X(\kappa(\mathfrak{p}))$ is a quasi-coherent sheaf of modules. In fact $\kappa(\mathfrak{p}) \otimes_{\Gamma}(\mathfrak{O})\mathfrak{O}(U)$ is a $\mathfrak{O}(U)$ algebra, hence $\Lambda_X(\kappa(\mathfrak{p}))$ is a quasi-coherent sheaf of algebras. Hence we can compute the relative spec Rspec $(\Lambda_X(\kappa(\mathfrak{p}))) \to X$. The adjunct of the map

$$\text{Rspec } (\Lambda_X(\kappa(\mathfrak{p}))) \to Y$$

is the canonical morphism $g : R \to \kappa(\mathfrak{p})$. This morphism is also the adjunct of the composition

$$\text{Rspec } (\Lambda_X(\kappa(\mathfrak{p}))) \to \text{Spec } \kappa(\mathfrak{p}) \to X,$$

so both maps must be equal. This gives us a commutative square

$$\begin{tikzcd}
X \arrow[r, "F"] & Y \\
\text{Rspec } (\Lambda_X(\kappa(\mathfrak{p}))) \arrow[u] \arrow[r] & \text{Spec } \kappa(\mathfrak{p}) \arrow[u]
\end{tikzcd}$$

By lemma .., we know that $\Lambda_X(\kappa(\mathfrak{p}))$ is not the zero sheaf hence the structure sheaf of Rspec $(\Lambda_X(\kappa(\mathfrak{p})))$ non-zero. This implies that the scheme is not the empty scheme. Therefore the point $\mathfrak{p}$ is in the image of F. ∎

Lemma 52. *The closed set* $V_X(\mathfrak{p})$ *is irreducible. This implies that* F *is injective.*

Proof. Let $F(z) = \mathfrak{p}$ for some $z \in X$. By lemma .. this is possible. Let $y \in V_X(\mathfrak{p})$. Then $\ker(z) \subset \ker(y)$, hence if $y \in D_X(a)$ then $x \in D_X(a)$. Therefore y specialises to z, which thus must be $V_X(\mathfrak{p})$. This shows that it is irreducible. Uniqueness of generic points of closed irreducible subsets of schemes implies injectivity of F. ∎

Lemma 53. *The function F is open, hence a homeomorphism.*

Proof. Note that $F(D_X(a)) = \{F(x) \mid a \notin F(x)\} = F(X) \cap D_Y(a) = D_Y(a)$. Our map F is continuous and open, so a homeomorphism. ∎

Lemma 54. *If F is a homeomorphism, then X is affine.*

Proof. Let Spec $A_i = U_i \subset X$ be open and let $\bigcup_i U_i = X$. Assume it is a finite affine cover. Using our base, we get a cover of $U_i = \bigcup_j D_X(a_{ij})$ with a_{ij} global sections. Observe that $D_X(a_{ij}) \subset U_i$, hence $D(a_{ij}|_{U_i}) = D_X(a_{ij})$ which makes them affine. Continuing like this, we get a finite cover of affines $D_X(a_{ij})$ of X. since $F(X) = F(\bigcup_{ij} D_X(a_{ij})) = \bigcup_{ij} D_Y(a_{ij}) = Y$, we have $(a_{ij}) = (1)$. Affine-ness satisfies the two requirements for the affine communication lemma[HAG II Ex.2.17], hence X is affine. ∎

4.0.4 P1

Quasi-coherent modules on a scheme X can be defined for affine schemes first and as module sheaves that are $\tilde{M}$ for some $f^*\mathfrak{O}$-module M pulled back to every affine Spec $R \xrightarrow{f} X$. Let's consider the following conjecture.

Conjecture 55 (P1). Let $a \in C$. A sheaf module $\mathfrak{F}$ on $(C, \mathfrak{T}, \mathfrak{O})/a$ is quasi-coherent if and only if $\mathfrak{F}$ is quasi-coherent on $(C, \mathfrak{T}, \mathfrak{O})/b$ for any affine $b \to a$

Remark. The only if direction always holds.

Definition 56. We say that a ringed site $(C, \mathcal{O}, \mathfrak{T})$ has *enough affines* if any object has an affine covering $\{b_i \to a\}$.

Lemma 57 (P1 holds with enough affines). *P1 holds for any ringed site with enough affines.*

Proof. Quasi-coherence is a local property. So if every object admits a affine cover P1 holds. ■

Lemma 58 (Finite poset has enough affines). *Any finite poset has enough affines.*

NA-chain proof. Let $x_0 \in C$. If x_0 is covered by the maximal sieve only or the maximal sieve and the empty sieve, it is affine and we are done. Assume otherwise. Let $S = \{y_i \to x_0\}$ be a non-maximal, non-empty cover of x_0. Then S does not contain isomorphisms.

We can associate to any non-maximal non-empty covering sieve S of an element x_0, the set of all NA-chains $x_0 \leftarrow x_1 \leftarrow \ldots \leftarrow x_n$. An NA-chain, associated to R, is a chain of maps ending in x_0 such that $x_i \leftarrow x_{i+1}$ is contained in a non-maximal, non-empty cover of x_i, where $x_0 \leftarrow x_1$ is contained in R.

By finiteness of C, any chain of maps is bounded by the size of C or contains a cycle. If a chain contains a cycle, it contains isomorphisms. By construction, no isomorphism can be present in a NA-chain. Therefore the length of any NA-chain is bounded by $\|C\|$.

Let H be a NA-chain associated to S of maximal length m. Then the last map $\ldots \leftarrow h \leftarrow g$ in H has an affine object g as domain, because H cannot be increased and so g has no non-maximal, non-empty coverings which makes it affine. Also the non-maximal, non-empty covering of h where this map appears must be an affine covering by applying the same reasoning to the other objects occurring in it. Hence all objects occurring at the $(m-1)$th place in any NA-chain admits an affine cover. Let $i \leqslant m-1$. Assume all elements at the $(i-1)$th place admit an affine cover. Let b be a object occurring at the $(i-1)$th place in a chain. It is either affine or all objects in any non-maximal, non-empty cover occur at the ith place in some chain hence admit an affine cover. Therefore any non maximal, non empty cover on b can be refined to an affine cover. This provides us with an affine cover of b. By reversed induction, x_0 admits a affine cover. ■

Lemma 59 (Non-quasi-coherent sheaf). *Any category admits a non-quasi-coherent sheaf.*

Proof. Let C be a ringed site with no affines. Therefore no object can have the empty sieve as a covering sieve, because that would make all sheafs trivial localized at this object. Let $\mathfrak{O}$ be its structure sheaf.

Let a be an object of C. Let b be an object of C such that $\mathrm{Hom}(b, a) = \emptyset$.

The following situation, the commuting square with conditions on the maps and objects, will be called S1. Note that a, b are fixed and not variables in S1.

$$\begin{tikzcd} e \arrow[r, "L"] & b \\ d \arrow[u, "K"] \arrow[r, "G"] & c \arrow[u, "F"'] \end{tikzcd}$$

With $\mathfrak{O}(d) \neq 0$, $\mathrm{Hom}(e, a) = \emptyset$ and $\mathrm{Hom}(c, a) \neq \emptyset$.

Assume that for any S a covering sieve on b.

1) every map $F \in S$ as in S1, so F has codomain b and its domain maps to a, we can find maps G, K, L to complete to S1 with $L \in S$.

2) For every $L \in S$ as in S1, so L has codomain b and its domain does not map to a, we can complete the to get S1.

Consequences: Every non-empty covering sieve of b contains maps L and F that fit in S1. 'Objects under a get under every object under b'. Call this assumption $A1$ en $A2$.

Define the presheaf $\mathfrak{F}$ as

$$x \mapsto \mathfrak{O}(x) \text{ if } \mathrm{Hom}(x, a) = \emptyset,$$
$$x \mapsto \mathfrak{O}(x)[y] \text{ otherwise },$$

$$u \xrightarrow{f} v \mapsto \mathfrak{O}(f) \text{ if } \mathrm{Hom}(x, a) = \emptyset,$$
$$(u \xrightarrow{f} v) \mapsto (\mathfrak{O}(u) \to \mathfrak{O}(u)[y]) \circ \mathfrak{O}(f) \text{ otherwise }.$$

Let $G = \mathfrak{F}^{++}$.

Let S be a covering sieve on b. If S is empty, then $G(b) = 0 = \mathfrak{O}(b)$. Assume otherwise. Let (x_f) be a matching family indexed by S. Let $x_f \in G(u) \setminus O(u)$ for $u \xrightarrow{f} b$ such that $\mathrm{Hom}(u, a) \neq \emptyset$, which is possible by $A2$. Set $F = f$ and complete to S1. Then

$\mathfrak{F}(G)(x_F) = \mathfrak{F}(K)(x_L) = x_{KL}$ since it is a matching family, which is impossible because $\mathfrak{F}(G)(x_F)$ is not in the image of $\mathfrak{F}(K)$. So all matching families (x_f) have components that are elements of $O(\mathrm{Dom}(f)) \subset G(\mathrm{Dom}(f))$, which already have unique amalgamations in $\mathfrak{O}(b)$. Hence $G(b) = \mathfrak{O}(b)$.

Let U be a global cover of the category. By A1, $U(b)$ contains an element $g : e \to b$ with $\mathrm{Hom}(e, a) = \emptyset$. Set $g = L$ and complete S1. The element $y \in G(d)$ is not generated locally by sections of $G(e)$. Hence G is not locally presentable.

Examples that satisfy A1&A2

- Open category of any irreducible space.
- Neighbourhood space of any point in any topological space.
- categories with pullbacks, terminal object and are irreducible.

■

Example 60 (Stacks 01BL example). Let $L = (\mathbb{R}, O_R)$ be the real line with the euclidean topology and the sheaf of continuous real valued functions as structure sheaf. Let

$$X = \frac{\bigcup_{i=0}^{\infty} L_i}{\sim}$$

with $[i, x] \sim [j, y]$ if and only if $i = j$ and $x = y$ or $y = x = 0$. The real lines are glued to each other at zero. Define the open $U_n \subset X$ as $U_n \cap L_i = (-\frac{1}{n}, \frac{1}{n})$. These opens form a basis of neighbourhoods of 0. Let $f : \mathbb{R} \to \mathbb{R}$ be any continuous function such that $f(x) = 0$ if $x \in (-1, 1)$ and $f(x) = 1$ if $x \in (-\infty, -2) \cup (2, \infty)$. Let $f_n(x) = f(nx)$.

Define the sheaf map

$$\bigoplus_i O_R \xrightarrow{\alpha} \bigoplus_{ij} O_R,$$

$$e_i \mapsto \sum_j f_i 1_{L_j} e_{ij}.$$

To proof that this is well-defined, we need to show that the sum $\sum_j f_i 1_{L_j} e_{ij}$ is locally finite for every i. Let $[k, y] \in X$. If $y \neq 0$, then

$$W_{[k,y]} = \{[k, z] \in X \mid z \in (y - \delta, y + \delta\} \subset L_k$$

is open in X and $\alpha_{W_{[k,y]}}(e_i) = f_i e_{ik}$ for any $\delta < |\frac{y}{2}|$. If $y = 0$, then $\alpha_{U_n}(e_i) = 0$ if $n > i$ because f_i is zero on U_n. Hence we found a cover on which our sum is locally finite, which makes α well-defined.

Let U be any open of any topological space X. Let $\mathfrak{F}$ be any presheaf. Consider the map of presheafs

$$\mathfrak{F}|_U{}^+ \xrightarrow{g} \mathfrak{F}^+|_U$$

defined by the components

$$g_V : \underset{S \in \mathrm{Cov}(V)}{\mathrm{colim}} \mathrm{Match}(S, F) \xrightarrow{\mathrm{id}} \underset{S \in \mathrm{Cov}(V)}{\mathrm{colim}} \mathrm{Match}(S, F).$$

Every component is a isomorphism, hence g is an isomorphism.

The adjunction $(\Lambda(-), \Gamma(X, -))$ implies that $\Lambda(-)$ commutes with arbitrary colimits. Moreover

$$O_X \cong \Lambda(\Gamma(X, O_X))$$

so

$$\bigoplus_i O_R \cong \Lambda(\bigoplus_i \Gamma(X, O_X)).$$

This shows that α is a morphism between associated sheafs. Let $\beta : \bigoplus_i \Gamma(U, O_X) \to \bigoplus_{ij} \Gamma(U, O_X)$ for some open U. Then $\Lambda(\beta)(e_i) = \sum_{j \in J_i} a_{ij} e_{ij}$ where J_i is finite for every i.

Assume that $\alpha = \Lambda(\beta)$ over some neighbourhood U of 0. Then there exists a m such that $U_m \subset U$. Let $k > 2m$. Then $f_k \neq 0$ on U_m, hence $f_k 1_{L_j} \neq 0$ on U_m for every j and so no coëfficients vanish of $\alpha_{U_m}(e_k) = \sum_j f_k 1_{L_j} e_{kj}$. This contradicts $\alpha = \Lambda(\beta)$.

Example 61 (Category without enough affines #1). Let X, f_j and U_n be as in the previous example. Define the full subcategory $N(y) \xrightarrow{i} \mathrm{Open}(()X)$ of all opens U that contain the point $y \in X$.

This category has all fibre products. Let $U \to V \leftarrow W$ be two morphisms. Then $U \leftarrow U \cap W \to W$ is the pullback.

On this category $N(y)$, let a family $\{f_i : U_i \to U\}$ be covering if $\bigcup_i f_i(U_i) = U$.

Let $V \xrightarrow{f} U$ be an isomorphism, then $f(V) = U$ so $\{f\}$ is a covering family.

Let $\{U_i \xrightarrow{f_i} U\}$ be a covering family. For every i, let $\{U_{ij} \xrightarrow{f_{ij}} U_i\}$ be a covering family. By definition this gives that $\bigcup_i f_i(U_i) = U$ and $\bigcup_j f_{ij}(U_i j) = U_i$ for every i. Hence

$$\bigcup_{i,j}(f_i \circ f_{ij})(U_{ij}) = \bigcup_i f_i(U_i) = U$$

and so the family $\{U_{ij} \xrightarrow{f_i \circ f_{ij}} U\}$ is covering.

Let $V \to U$ be a morphism in $N(y)$ and $\{U_i \xrightarrow{f_i} U\}$ be a covering family on U. This tells us that $\bigcup_i f_i(U_i) = U$, hence also $\bigcup_i g_i(U_i \cap V) = V$ where $g_i : U_i \cap V \to V$ is the pullback of f_i. Hence $\{U_i \cap V \xrightarrow{g_i} V\}$ is a covering family of V.

All criteria for a pretopology are established. Let τ be the generated Grothendieck topology.

Let $\mathfrak{F}$ be a sheaf on $\mathrm{Open}(X)$. Let $\hat{\hat{\mathfrak{F}}} = \mathfrak{F} \circ i$. Let $\{U_i \to V\}$ be a covering family on V in $N(y)$. Let $(x)_i$ be a matching family of $\hat{\hat{\mathfrak{F}}}$ indexed by $\{U_i \to V\}$, so $x_i \in \hat{\hat{\mathfrak{F}}}(U_i) = \mathfrak{F}(U_i)$. Note that $\{U_i \to V\}$ is also a covering family on V in $\mathrm{Open}(X)$, hence $(x)_i$ is also a matching family of $\mathfrak{F}$ on V. Since $\mathfrak{F}$ is a sheaf, there exists a unique amalgamation $x \in \mathfrak{F}(V) = \hat{\hat{\mathfrak{F}}}(V)$ such that $x = x_i$ in $\mathfrak{F}(U_i) = \hat{\hat{\mathfrak{F}}}(U_i)$. This shows that $\hat{\hat{\mathfrak{F}}}$ is a sheaf, hence i is continuous.

Let $\mathfrak{O}_{X,y} = \mathfrak{O}_X \circ \tau$. This is a sheaf of rings by the previous. We constructed a ringed site $(N(y), \tau, \mathfrak{O}_{X,y})$.

Let F be the inclusion functor from the category of sheafs to the category of presheafs. We have the adjunctions:

1. $(\Lambda(-), \Gamma(X, -))$,
2. $((-)^{++}, F)$.

The structure sheaf $O_{X,y}$ is, trivially, isomorphic to $\Lambda(\Gamma(X, O_{X,y}))$. By adjunction (2)

$$\bigoplus_i \mathfrak{O}_{X,y} \cong (\bigoplus_i \mathfrak{O}_{X,y})^{+s},$$

where the coproduct on the left hand side is in the category of sheafs and the coproduct on the right hand side in the category of presheafs.

By adjunction (1)

$$\bigoplus_i \Lambda(\Gamma(X, O_{X,y})) \cong \Lambda(\bigoplus_i \Gamma(X, O_{X,y})).$$

Combine these 3 observation to get

$$\bigoplus_i \mathfrak{O}_{X,y} \cong \Lambda(\bigoplus_i \Gamma(X, O_{X,y})),$$

which shows that $\bigoplus_i \mathfrak{O}_{X,y}$ is quasi-coherent.

Set $y = 0 \in X$. Define the sheaf map

$$\bigoplus_i O_{X_y} \xrightarrow{\alpha} \bigoplus_{ij} O_{X_y},$$

$$e_i \mapsto \sum_j f_i 1_{L_j} e_{ij}.$$

Fix i. We will prove that $\alpha_X(e_i)$ is a well-defined global section. Let $m > i$. Let $V_k = L_k \cup U_m$ and $\{V \to =\}\{V_k\}$. By construction f_i is zero on U_m, hence $f_i 1_{L_j}$ is zero on V_k if $k \neq j$ and so $\sum_j f_i 1_{L_j} e_{ij} = f_i 1 L_k e_{ik}$ on V_k. This shows that $\alpha_X(e_i)$ is a well-defined section on any element of the cover $\{U_i \to V\}$ and this family is matching since the sections are functions and the 'restriction' maps are actual restriction.

Assume there exists $\beta : \bigoplus_i \Gamma(V, O_{X,y}) \to \bigoplus_{ij} \Gamma(V, O_{X,y})$ such that $\Lambda(\beta) = \alpha_V$. Then $\alpha_V(e_i) = \sum_j f_i 1_{L_j} e_{ij}$ is not just locally finite over some cover, but actually finite globally on V for all i. So almost all $f_i 1_{L_j}$ are zero on V. Note that $y \in V$, so $U_d \subset V$ for some d. Let $i > 2d$, then $f_i \neq 0$ on $(-\frac{1}{d}, \frac{1}{d})$ and so $f_i 1_{L_j} \neq 0$ on U_d for any j. Hence $\alpha_V(e_i) = \sum_j f_i 1_{L_j} e_{ij}$ is not a finite sum for $i > 2d$. This contradicts our assumption.

Let $U \subset X$ be $U \cap L_j = U_j$. Fix i. Then $f_i 1_{L_j} = 0$ if $i < j$, hence

$$\sum_j f_i 1_{L_j} e_{ij} = \sum_{j \leqslant i} f_i 1_{L_j} e_{ij}$$

is a finite sum.

The restriction of any quasi-coherent sheaf is quasi-coherent. Observe that α, and its restrictions, is a morphism between quasi-coherent sheafs but does not come from a map of modules. Therefore $\Lambda(-)_V : \Gamma(V, O_{X,y}) - \mathrm{Mod} \to \mathsf{Qcoh}(\mathfrak{Y})$ is not full for any V and no object V is affine in $N(y)$.

Example 62 (Category without enough affines #1).

Example 63 (General P1 does not hold). The category C is $\mathbb{Z} \times \mathbb{Z}$ with the usual ordering. An element (i,j) is only non-trivially covered by $\{(i,j-1) \to (i,j), (i-1,j) \to (i,j)\}$. Let k be any field. Let $R = k[x_{ij} | i,j \in \mathbb{Z}]$. Define the structure sheaf as $O(i,j) = R[x_{kl}^{-1} | i \leqslant k \;\&\; j \leqslant l]$.

Fix $(a,b) \in C$. Consider the over category $C \downarrow (a,b)$ at this point. Let $(i,j) \to (a,b)$ be an object of $C \downarrow (a,b)$. Define the presheaf of modules $F(i,j) = O(i,j)/(x_{a-1,b}x_{a,b-1})$ on X. Then $a > i$ or $b > j$ or ($i = a$ and $j = b$). If $a > i$ or $b > j$, then $x_{a-1,b}$ or $x_{a,b-1}$ is invertible in $O(i,j)$, hence $F(i,j) = 0$ in both cases. This presheaf is zero everywhere except at (a,b), hence sheafifies to the zero sheaf. In other words: $\Lambda(\frac{O(a,b)}{(x_{a-1,b}x_{a,b-1})}) = 0$, where Λ is the 'tilde' functor. Hence (a,b) is not affine, which shows that C has no affine objects.

Consider $G = O(i,j)[y_{kl} | k \leqslant i \;\&\; l \leqslant j]$. Let $\bigoplus_{k \in I} O \xrightarrow{\alpha} G$ be any sheaf map. Let $\alpha_{00}(e_k)$ be the image of the generators $e_k \in \bigoplus_{k \in I} O$ in the global sections. The section $y_{1,1} \in G(1,1)$ cannot be written as a finite sum $\sum_k \lambda_k \alpha_{00}(e_k)$ for scalars $\lambda_k \in O(i,j)$ for any (i,j). This shows that α is not surjective hence G is not quasi-coherent(locally presentable).

5 Cohomology