1.3 Modules

Presheaf modules and sheaf modules on a ringed site will be introduced in this section. The functors λ and Λ introduced here will be used extensively. See **[stacks-project]** for more detail.

Definition 30 (Presheaf modules)**.** Let $(C, \mathcal{T}, \mathfrak{O})$ be a ringed site. Let $R = \Gamma(1; \mathfrak{O})$.

A presheaf module on this ringed site is a presheaf of sets $\mathfrak{F}$ on C together with a map of presheaves

$$\mathfrak{O} \times \mathfrak{F} \to \mathfrak{F}$$

such that for every object $a \in C$ the map $\Gamma(a; \mathfrak{O}) \times \Gamma(a; \mathfrak{F}) \to \Gamma(a; \mathfrak{F})$ defines a $\Gamma(a; \mathfrak{O})$-module structure on $\Gamma(a; \mathfrak{F})$.

A morphism

$$\mathfrak{F} \to \mathfrak{G}$$

is a morphism of presheaf modules if

$$\begin{tikzcd} \mathfrak{O} \times \mathfrak{F} \arrow[r] \arrow[d] & \mathfrak{F} \arrow[d] \\ \mathfrak{O} \times \mathfrak{G} \arrow[r] & \mathfrak{G} \end{tikzcd}$$

commutes. The category of presheaf modules on C will be denoted $\mathsf{PMod}(\mathfrak{O})$.

category of sheaf modules?

Definition 31. Let $\mathfrak{F}$ be a sheaf of modules on $(C, \mathcal{T}, \mathfrak{O})$. It is called quasi-coherent if the following holds. For any object $a \in C$ there exists a covering sieve S such that for any map $f : b \to a$ in S there exists a presentation

$$\oplus_I b|_{\mathfrak{O}} \to \oplus_J b|_{\mathfrak{O}} \to b|_{\mathfrak{F}} \to 0$$

Quasi-coherent modules form a full subcategory of the category of sheafs of modules over $(C, \mathcal{T}, \mathfrak{O})$ which are denoted by $\mathsf{Qcoh}(\mathfrak{O})$.

Definition 32. Let $(\mathsf{C}, \mathcal{T}, \mathfrak{O})$ be a ringed site. Let $\mathsf{R} = \Gamma(1; \mathfrak{O})$. Let M, N be an R-module.

Define

$$\lambda : \mathsf{R}\text{-}\mathsf{Mod} \to \mathsf{PMod}(\mathfrak{O})$$

by for all $a \in \mathsf{C}$,

$$\lambda(\mathsf{M})(a) = \mathsf{M} \otimes_{\mathsf{R}} \Gamma(a; \mathfrak{O}),$$

for all $f : b \to a \in \mathsf{C}$,

$$\lambda(\mathsf{M})(f) = \mathrm{Id} \otimes \mathfrak{O}(f),$$

for all $g : \mathsf{M} \to \mathsf{N} \in \mathsf{R}\text{-}\mathsf{Mod}$,

$$\lambda(g) = (a : g \otimes \mathrm{Id}).$$

Lemma 33. *Let* $Y = (\mathsf{C}, \mathcal{T}, \mathfrak{O})$ *be a ringed site. The functor* λ *is left adjoint to*

$$\Gamma(1; -) : \mathsf{PMod}(\mathfrak{O}) \to \mathsf{R}\text{-}\mathsf{Mod}.$$

Proof. Let a be an object of C. Let M, N be R-modules. Let $\mathfrak{F}, \mathfrak{G} \in \mathsf{PMod}(\mathfrak{O})$ be presheaf modules.

Let $\varphi : \lambda(\mathsf{M}) \to \mathfrak{G}$ be a morphism of presheaf modules. Let $\phi : \mathsf{M} \to \Gamma(1; \mathfrak{G})$ be a morphism of presheaf modules.

Define

$$\begin{aligned} \alpha = H_{\mathsf{M},\mathfrak{G}} &: \mathrm{Hom}(\lambda(\mathsf{M}), \mathfrak{G}) \to \mathrm{Hom}(\mathsf{M}, \Gamma(1; \mathfrak{G})) && (1.1) \\ &: \varphi \mapsto \varphi_1 && (1.2) \end{aligned}$$

where φ_1 is the component of φ on the global sections.

Define

$$\beta = L_{\mathsf{M},\mathfrak{G}} : \mathrm{Hom}(\mathsf{M}, \Gamma(1; \mathfrak{G})) \to \mathrm{Hom}(\lambda(\mathsf{M}), \mathfrak{G})$$

by

$$\beta(\phi)_a = \phi \otimes_{\mathsf{R}} \Gamma(a; \mathfrak{O}).$$

We will show that β and α are mutually inverse.

Let $d = \beta(\alpha(\varphi))$. Let $m \otimes g \in M \otimes_R \Gamma(a; \mathfrak{O})$. Let $p : \lambda(M)(1) \to \lambda(M)(a)$ be the projection map. Let $q : \mathfrak{G}(1) \to \mathfrak{G}(a)$ be the projection map. Then $d_a(m \otimes g) = \varphi_1(m) \otimes g$ and

$$\begin{aligned}
\varphi_a(m \otimes g) &= g\varphi_a(m \otimes 1) \text{ by linearity} \\
&= g\varphi_a(p(m)) \\
&= gq(\varphi_1(m)) \text{ by naturality of } \varphi \\
&= g(\varphi_1(m) \otimes 1) \\
&= \varphi_1(m) \otimes g.
\end{aligned}$$

Hence $d = \varphi$. In words, the natural transformations from presheaves of the from $\lambda(M)$ are uniquely determined by their global sections component.

Let $d = \alpha(\beta(\phi))$. Let $m \in M$. Then $d(m) = (\phi \otimes_R R)(m) = \phi(m)$. Hence $d = \phi$, which makes H and L mutual inverses.

Naturality in M *and* $\mathfrak{G}$

Let $g : N \to M$ and $h : \mathfrak{F} \to \mathfrak{G}$. Let $\rho \in \mathrm{Hom}(\lambda(N), \mathfrak{F})$. Let $k = H_{M,\mathfrak{G}}(h \circ \rho \circ \lambda(f))$. Let $l = h_1 \circ H_{N,\mathfrak{F}}(\rho) \circ f$.

Unfolding the definition for H shows that $k = h_1\rho_1 f$ and $l = h_1\rho_1 f$ as well. This proves naturality in M and $\mathfrak{G}$ and the adjunction between λ and $\Gamma(1; -)$ between R-Mod and $\mathrm{PMod}(\mathfrak{O})$. ∎

Definition 34. Let $(C, \mathcal{T}, \mathfrak{O})$ Define

wrong $\mathcal{O}$

$$\Lambda : \text{R-Mod} \to \mathrm{Mod}(O)$$

by $sh \circ \lambda$.

It follows that we have the adjunction $\Lambda \dashv \Gamma(1; -)$. This functor is generalisation of **[stacks-project]**

?

Remark. By lemma ?, any morphism of sites that is a bijection of covers commutes with Λ. In the sense that if F is a bijection of covers, then $F_*\Lambda = shF_*\lambda$.

Lemma 35 (λ commutes with restriction). *Let* $(C, \mathcal{T}, \mathfrak{O})$ *be a ringed site. Let* $a \in C$. *We have a natural isomorphism* $t : u_* \circ \lambda \Rightarrow \lambda \circ (- \otimes_{\Gamma(1;\mathfrak{O})} \Gamma(a; \mathfrak{O}))$.

Proof. Define the natural transformation $t : \lambda \circ (- \otimes_{\Gamma(1;\mathcal{O})} \Gamma(a;\mathcal{O})) \Rightarrow u_* \circ \lambda$, by for each $\Gamma(1;\mathcal{O})$-module M and for each $f : b \to a \in C_a$,

$$t_{M,f} : M \otimes_{\Gamma(1;\mathcal{O})} \Gamma(a;\mathcal{O}) \otimes_{\Gamma(a;\mathcal{O})} \Gamma(b;\mathcal{O}) \to M \otimes_{\Gamma(1;\mathcal{O})} \Gamma(b;\mathcal{O}),$$

$$m \otimes r \otimes s \mapsto m \otimes rs.$$

Every component $t_{M,f}$ is an isomorphism by basic commutative algebra. ∎

Let C be a category. Let $a \in C$. Let ϵ be the counit of the adjunction $\lambda \dashv \Gamma(1;-)$ on C. Let ϵ_a be the counit of the adjunction $\lambda_a \dashv \Gamma(a;-)$ on C_a.

Lemma 36 (λ counit commute with restriction). *We have $u_*\epsilon \cong \epsilon_a$ on presheaves of the form $\lambda_a(M \otimes \Gamma(b;\mathcal{O}))$ with M a $\Gamma(1;\mathcal{O})$-module via*

$$t \circ u_*\epsilon \circ t^{-1} = \epsilon_a.$$

Proof. Both maps are the identity map if you unfold them. ∎

Lemma 37 (Λ commutes with restriction). *Let $(C, \mathcal{T}, \mathcal{O})$ be a ringed site. We have a natural isomorphism*

$$q : u_* \circ \Lambda \to \Lambda \circ (- \otimes_{\Gamma(1;\mathcal{O})} \Gamma(a;\mathcal{O})).$$

Proof. Define q to be the composition

$$u_* \circ sh \circ \lambda \overset{s^2\lambda}{\Rightarrow} sh \circ u_* \circ \lambda$$
$$\overset{sh(t)}{\Rightarrow} sh \circ \lambda \circ - \otimes_{\Gamma(1;\mathcal{O})} \Gamma(a;\mathcal{O}))$$

By Definition 26 and lemma 35, t and s^2 are isomorphisms so q is an isomorphism as well. ∎

1.4 Schemes

We will recap the parts of scheme theory here that we use. See [**vakil**, **HAG**] for thorough treatments of scheme theory.

Definition 38 (Spectrum of a ring). Let R be a ring. The spectrum Spec$R$ of R is the locally ringed space defined as follows. The underlying set is the set of prime ideals of R. The (Zariski) topology is generated by the basis of distinguised opens $D(f) = \{\mathfrak{p} \subset R | f \notin \mathfrak{p}\}$. The sheaf of rings is given on this basis by

$$D(f) \mapsto R_f.$$

A distinguised open $D(f)$ of $\mathrm{Spec}(R)$ viewed as locally ringed space is isomorphic to $\mathrm{Spec}(R_f)$, where the inclusion $D(f) \to \mathrm{Spec}(R)$ corresponds to the canonical map $R \to R_f$.

Definition 39 (Locus of a point). Let $(X, \mathfrak{O})$ be a scheme. Define the locus of point $x \in X$ to be

$$\ker(x) = \ker(\Gamma(X; \mathfrak{O}) \to \kappa(x)).$$

Note that $\ker(x)$ is a prime ideal of $\Gamma(X; \mathfrak{O})$

Lemma 40. *For any* $X \in \mathsf{LRSpaces}$ *and* $R \in \mathsf{Rng}$ *we have an isomorphism*

$$\underline{Hom}_{\mathsf{LRSpaces}}(X, \underline{Spec}(R)) \to \underline{Hom}_{\mathsf{Rng}}(R, \Gamma(X; \mathfrak{O}))$$

that is natural in X *and* R. *In short* fix italiciization

$$\underline{Spec}: \mathsf{Rng} \to \mathsf{LRSpaces}$$

is adjoint to

$$\Gamma(1; -) : \mathsf{LRSpaces} \to \mathsf{Rng}.$$

Proof. Sending a morphism of locally ringed spaces $X \to \mathrm{Spec}\,R$ to its global component $R \to \Gamma(X; \mathfrak{O})$ will turn out to be an isomorphism. The inverse is the following map.

Let $\varphi : R \to \Gamma(X; \mathfrak{O})$ be given. We need to construct a morphism of locally ringed spaces $(f, f^{\#})$. Define $f(x) = \ker x$. For distinguised open $D(f) \subset \mathrm{Spec}(R)$, define $f^{\#}_{D(f)}(\frac{s}{f}) = \frac{s}{f}$. This makes sense because $f \in \Gamma(X; \mathfrak{O})$ is invertible in $D_X(f)$

These maps are mutually inverses and we have naturality. add reference ∎

Definition 41 (Scheme). We call the locally ringed space $\mathrm{Spec}(R)$ an affine scheme.

A scheme S is a locally ringed space that admits a covering of affine schemes. A morphism of schemes is a morphism of locally ringed spaces. The category of schemes we will denote by Sch.

When we define a sheaf $\mathfrak{F}$ 'on' a scheme S, then this means we define this sheaf on the small Zariski site corresponding to S. not defined until §1.5

Definition 42 (Tilde functor). Let Spec(R) be an affine scheme. Let M be a R-module. Define $\widetilde{M}$ to be the unique sheaf with on Spec R

$$\widetilde{M} : D(f) \mapsto M_f.$$

See [**stacks-project**].

Remark. This functor(with this notation) is commonly used in algebraic geometry texts. We will show that it is equal to Λ on the small Zariski site and then only use Λ.

Remark. The next result will prove that our definition of quasi-coherence in the sense of Definition 31 coincides with the usual definition for schemes using $\widetilde{-}$. See [**vakil**] for the usual definition.

Lemma 43. *Let $\mathfrak{F}$ be a sheaf of modules on scheme X. $\mathfrak{F}$ is quasi-coherent on X if and only if for any open Spec(R) $\subset$ X the sheaf $\mathfrak{F}|_{Spec(R)}$ is isomorphic to $\Gamma(\widetilde{Spec(R)}; \mathfrak{F})$.* ← rephrase: it's confusing that the ~ doesn't cover Γ(Spec R; 𝔉). perhaps "$\widetilde{M}$, where M is the R-module Γ(Spec R; 𝔉)."

use ⊕K or 𝒪(K) or coproduct for

Proof. $\Rightarrow$: By assumption we get local presentations indexed by a covering. Let $\bigcup_{i \in I} U_i = X$ be this covering. Assume without loss of generality that it is an affine open covering. Let $U_i = \mathrm{Spec}(R_i)$. Let $\mathfrak{O}^K_{U_i} \to \mathfrak{O}^J_{U_i} \to \mathfrak{F}|_{U_i} \to 0$ be one of the given presentations. Taking global sections gives us an exact sequence

$$R_i^K \to R_i^J \to \Gamma(U_i; \mathfrak{F}) \to 0.$$

Tensoring it with the localisation $R_{i,f}$ for any $f \in R_i$ yields

$$R_{i,f}^K \to R_{i,f}^J \to \Gamma(U_i; \mathfrak{F}) \otimes R_{i,f} \to 0.$$

Taking sections at D(f) from the sheaf sequence yields

$$R_{i,f}^K \to R_{i,f}^J \to \Gamma(D(f); \mathfrak{F}) \to 0.$$

Hence $\mathfrak{F}|_{U_i}$ is the unique sheaf with $D(f) \mapsto \Gamma(U_i; \mathfrak{F})_f$, which we defined to be $\Gamma(\widetilde{U_i; \mathfrak{F}})$. By the affine communication lemma, this property holds for any affine and not just for the affines in this covering.

arbitrary colimits?

what is M?

$\Leftarrow$: Take a presentation $R^I \to R^J \to M$ and apply $\widetilde{-}$. Then note that $\widetilde{-}$ commutes with infinite coproducts and $\widetilde{R} = \mathfrak{O}_{Spec(R)}$. So we get a presentation $\mathfrak{O}^I_{Spec(R)} \to \mathfrak{O}^J_{Spec(R)} \to \mathfrak{F}$ on every affine open subset $Spec(R) \subset X$, hence $\mathfrak{F}$ is quasi-coherent. ■

$\oplus I$ or (I)

Lemma 44. *Let $\mathfrak{F}$ be a sheaf of modules on scheme X. Then $\mathfrak{F}$ is quasi-coherent if and only if for any $D(f) = Spec(S_f) \to Spec(S)$ opens in X with $f \in S$ the induced map*

$$\Gamma(Spec(S); \mathfrak{F}) \otimes_S S_f \to \Gamma(D(f); \mathfrak{F})$$

is an isomorphism.

Proof. $\Rightarrow$:

Assume $\mathfrak{F}$ is quasi-coherent. Let $Spec(S) \subset X$ be a open affine subset. By Lemma 43 we have $\mathfrak{F}|_{Spec(S)} = \widetilde{M}$ for some ~~R~~ S-module M. By construction $\Gamma(D(f); \mathfrak{F}) = M_f$ and hence the induced map is an isomorphism by basic commutative algebra.

$\Leftarrow$:

Assume that the induced map is an isomorphism for every $f \in S$ with $Spec(S) \subset X$ open affine subset. Choose collection f_i such that $(f_i) = (1)$. Then $\Gamma(D(f_i); \mathfrak{F}) = \Gamma(Spec(S); \mathfrak{F}) \otimes_S S_{f_i}$ by assumption. Hence $F|_{Spec(S)} = \Gamma(\widetilde{Spec(S)}; \mathfrak{F})$. By Lemma 43 this implies that $\mathfrak{F}$ is quasi-coherent. ■

Shouldn't this be "choose any $f \in S$?"

Definition 45 (Sheaf of algebras). A sheaf of algebras $\mathfrak{F}$ on a ringed site $(C, \mathcal{T}, \mathfrak{O})$ is a sheaf of rings that comes with a (structure) morphism of sheaf of rings $\mathfrak{O} \to \mathfrak{F}$.

Definition 46 (Relative spec). Let X be a scheme. Let $\mathfrak{S}$ be a sheaf of algebras on X that is quasi-coherent as a sheaf of modules.

Define the relative spectrum of $\mathfrak{S}$ over X to be the scheme

wrong S

$$Rspec(S) \to X$$

that you get by glueing the spectra $Spec(\Gamma(V; \mathfrak{S})) \to V \subset X$ for every affine open V. See [**stacks-project**].

1.5 Scheme sites

In this section we will introduce the big and small Zariski ringed site and look at how quasi-coherence and Λ behave on these sites.

Definition 47 (Small Zariski site)**.** Let X be a scheme. Define the small Zariski ringed site of X to be $(\mathsf{Open}(X), \mathcal{T}, \mathfrak{O})$ with the following components. The underlying category is $\mathsf{Open}(X)$. The topology $\mathcal{T}$ is generated by the families $\{D(f_i) \to \mathsf{Spec}(R) \cancel{=} U\}$ where $(f_i) = R$ with $f_i \in R$ for any open affine $\mathsf{Spec}(R) \subset X$. The sheaf of rings is just the sheaf of rings coming from X. This sheaf is already defined on $\mathsf{Open}(X)$.

Do you want an "over X" version?

I think you want arbitrary open covers

Definition 48 (Big Zariski site)**.** Define the big Zariski site to be $(\mathsf{Sch}, \mathcal{T}, \mathfrak{O})$ with the following components. The underlying category is Sch. ~~The topology T is generated~~ by the families $\{\mathsf{Spec}(R_{f_i}) \to \mathsf{Spec}(R)\}$ where $(f_i) = R$ with $f_i \in R$ for any open affine $\mathsf{Spec}(R) \subset X$. The sheaf of rings $\mathfrak{O}$ sends $(U, \mathfrak{Q}) \to (X, \mathfrak{O})$ to $\mathfrak{Q}U$.

there's no X here

Don't you usually say $\Gamma(U, Q)$?

Definition 49 (k)**.** Let X be a scheme. Define the functor $k : \mathsf{Open}(X) \to \mathsf{Sch}_X$ by $U \mapsto ((U, \mathfrak{O}_U), i)$ where $i : U \to X$ is the inclusion of the open subscheme into X.

We will show that it preserves limits and covers. The terminal $X \in \mathsf{Open}(X)$ is send to the terminal $X \to X$. Let $U \to V$ and $W \to V$ be two morphism in $\mathsf{Open}(X)$. We have $k(U \cap W) = U \cap W \to X$ which is the pullback of $k(U) \to k(V)$ and $k(W) \to k(V)$.

Let $S = \{D(f_i) \to \mathsf{Spec}(R)\}$ be one of the generating family in $\mathsf{Open}(X)$. Note that $k(D(f_i))$ is isomorphic to the object $\mathsf{Spec}(R_{f_i}) \to \mathsf{Spec}(R)$. Hence $k(S)$ generates a covering sieve on Sch and hence on Sch_X.

So k is morphism of sites.

Lemma 50. *Let R be a ring. Let M be a R-module. $\widetilde{M} = \Lambda(M)$ on the small Zariski site coming from $\mathsf{Spec}(R)$.*

Proof. Let $U \subset \mathsf{Spec}(R)$ be any open subset. Let $\bigcup_{i \in I} D(f_i) = U$ be a cover of U. Define $\lambda(M) \to \widetilde{M}$ at U to be $m \otimes u \mapsto mu$.

There is a unique morphism $\widetilde{M} \to \Lambda(M)$ such that

$$\begin{tikzcd} \widetilde{M} \arrow[r] & \Lambda(M) \\ \lambda(M) \arrow[u] \arrow[ur] & \end{tikzcd}$$

commutes.

We will give this morphism only on $D(f)$ for some $f \in R$. To make this diagram commute we must send $\frac{m}{f}$ to $\{m \otimes \frac{1}{f_j} \mid j \in \max(D(f))\}$. This defines a sheaf morphism and moreover this the unique way to make the diagram commute. By the universal property of the sheafification $\widetilde{M} = \Lambda(M)$. Note that the witnessing isomorphism is functorial in M.

??

∎

By the previous remark about disjoint unions and the sheaf property and some basic commutative algebra one sees that this becomes

$$(\Gamma(X;\mathfrak{G}) \otimes \Gamma(X;\mathfrak{O})) \times (\Gamma(Y;\mathfrak{G}) \otimes \Gamma(X;\mathfrak{O})) \to \Gamma(X;\mathfrak{G}).$$

Since $\Gamma(Y;\mathfrak{G}) \otimes \Gamma(X;\mathfrak{O}) = 0$, we are left with

over what?

$$(\Gamma(X;\mathfrak{G}) \otimes \Gamma(X;\mathfrak{O})) \to \Gamma(X;\mathfrak{G})$$
$$g \otimes r \to rg.$$

Note that $\Gamma(X;\mathfrak{G})$ already is an $\Gamma(X;\mathfrak{O})$-module and conclude that hence this morphism is an isomorphism. ■

Lemma 56. *If $X_i \to Y$ is restrictive for each $i \in I$, where I is an (possibly infinite) indexing set, Then $\bigsqcup_{i\in I} X_i \to Y$ is restrictive.*

Proof. ■

Lemma 57 (coproduct van restrictive). *Let X_1, X_2, Y be a schemes. $X_1 \to Y$ and $X_2 \to Y$ are restrictive morphisms if and only if the corresponding morphism $X_1 \sqcup X_2 \to Y$ is restrictive.*

($\Leftarrow$) *Proof.* Note that $\Gamma(X_1 \sqcup X_2; -) = \Gamma(X_1; -) \times \Gamma(X_2; -)$ by the sheaf property. We will show that

$$\Gamma(Y;\mathfrak{G}) \otimes_{\Gamma(Y;\mathfrak{O})} \Gamma(X_1 \sqcup X_2;\mathfrak{O}) \to \Gamma(X_1 \sqcup X_2;\mathfrak{G})$$

is an isomorphism.

Let $g\otimes(r,s) \in \Gamma(Y;\mathfrak{G})\otimes_{\Gamma(Y;\mathfrak{O})}\Gamma(X_1\sqcup X_2;\mathfrak{O})$. This map sends it to (rg, sg). By assumption the maps

$$\Gamma(Y;\mathfrak{G}) \otimes_{\Gamma(Y;\mathfrak{O})} \Gamma(X_1;\mathfrak{O}) \to \Gamma(X_1;\mathfrak{G})$$
$$g \otimes r \mapsto rg$$
$$\Gamma(Y;\mathfrak{G}) \otimes_{\Gamma(Y;\mathfrak{O})} \Gamma(X_2;\mathfrak{O}) \to \Gamma(X_2;\mathfrak{G})$$
$$g \otimes s \mapsto sg$$

This is all just a long way of saying that tensor product distributes over $\oplus$.

are isomorphisms. This implies directly that our map $g \otimes (r, s) \to (rg, sg)$ is an isomorphism.

($\Rightarrow$) ? Follows from lemmas 54, 55. ■

Lemma 58 (Restrictive to affines). *If $f : X \to Spec(R)$ is a restrictive open immersion, then X is affine.*

Proof. Since X is an open in $\mathrm{Spec}(R)$, we get a distinguised covering $\bigcup_i D(f_i) = X$ with $f_i \in R$. We will prove that $(f_i|_X) = (1)$ in $S = \Gamma(X; \mathfrak{O})$. Then we invoke the result in **[harts]**: if X_{f_i} are all affine and $(f_i|_X) = (1)$ then X is affine.

Consider $M = \frac{R}{(f_i)}$ as an R-module and look at $\Lambda(M)$. By restrictiveness we get $M \otimes_R S = \Lambda(M)(S)$ and by $M \otimes_R R_{f_i} = \Lambda(M)(D(f_i)) = M_{f_i} = 0$. Hence $\Lambda(M)(S) = 0$ by the sheaf axiom. This implies that $(f_i) = (1)$ in S. You also need to at least ~~...~~ claim that $X_{f_i} = D(f_i)$. ■

Lemma 59. *Any morphism $Spec(S) \xrightarrow{f} Spec(R) \in \mathrm{Sch}_{Spec(R)}$ between affine schemes is restrictive.*

Proof. Let $\mathfrak{G}$ be a quasi-coherent module on $\mathrm{Sch}_{\mathrm{Spec}(R)}$. Set $M = \Gamma(\mathrm{Spec}(R); \mathfrak{G})$ We want to prove that

$$M \otimes_R S \to \Gamma(\mathrm{Spec}(S); \mathfrak{G})$$

is an isomorphism.

Why isn't this $G|_{\mathrm{Spec}\,S} = \Lambda(M)(S)$?

~~Note that $\mathfrak{G} = \Lambda(M)$ since $\mathrm{Spec}(R)$~~ is affine. See ??. For the same reason, we get $\mathrm{Spec}(S)|_{\mathfrak{G}} = \Lambda(M \otimes \Gamma(\mathrm{Spec}(S); \mathfrak{O}))$. By lemma ?, we know that ω^2 is an isomorphism ~~at affine schemes.~~

By lemma ?, we know that for a module of the form $\Lambda(M)$ the ω^2 at the codomain of f is equal to $\hat{f}$ if both domain and codomain are affine schemes. ■

Example 60 (Affine non-restrictive map). One might expect(or want) that any property of all maps between affine schemes also hold for affine maps between any schemes. This is not the case for restrictiveness. So restrictiveness is not local on the target.

Consider the canonical inclusions $\mathbb{A}^1 \to \mathbb{P}^1$ and the shifted quasi-coherent module $\mathfrak{O}(-1)$. This module is even locally free of degree 1, this is often called an invertible module.

The global sections of the module $\mathfrak{O}(-1)$ are the elements of degree -1 in the global sections of $\mathfrak{O}$. There are no such elements, hence the global sections are the zero module.

On $\mathbb{A}^1$ all invertible modules are isomorphic to the structure sheaf. See [**vakil**]. We conclude that the canonical inclusions cannot be restrictive.

Any inclusion $\operatorname{Spec}(\kappa(\mathfrak{p})) \to \mathbb{P}^1$ of a point is not restrictive which can be shown with the same argument.

This is a (more opaque) way of saying that on projective space not every quasi-coherent sheaf is generated by global sections.