Affine Objects

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June 2, 2016

1 Preliminaries

This chapter introduces all the basic notions that are needed but not assumed to be known to the reader. We will start with a discussion of some purely categorical notions like slice categories and presheaves. Secondly, we will introduce a notion of a topology on a category and look at some constructions that are relevant for us. Then we will introduce modules on ringed sites. Lastly, the notion of a scheme is introduced.

1.1 Basic Category Theory

Some categorical notions like presheaves and slice categories will be introduced in this section. See [A10] and [MM92].

Definition 1 (Presheaf category)**.** Let C be a category. Let $a \in C$. Let $f : a' \to a$ We define the category of presheaves on C as the category of contravariant functors to the category of sets Set. We will denote it by $\hat{C}$.

Define the functor $h : C \to \hat{C}$ as follows

$$a \mapsto \mathrm{Hom}(-, a),$$

$$f \mapsto f \circ -.$$

This functor is fully faithful by the Yoneda lemma.

Notation 2. Let I, C be categories. Let $L : I \to C$ be a functor. The limit over this functor will be denoted by $\lim\limits_{i \in I} L(i)$. The colimit will be denoted by $\operatorname*{colim}\limits_{i \in I} L(i)$.

Definition 3 (Sections functor)**.** For any $a \in C$ define the functor

$$\Gamma(c; -) : \hat{C} \to \mathsf{Set}$$

by

$$\mathfrak{F} \to \Gamma(a; \mathfrak{F}).$$

Let $L : I \to C$ be a small diagram. Define

$$\Gamma(\operatorname*{colim}_{i \in I} L(i); -) : \hat{C} \to \mathsf{Set}$$

by

$$\mathfrak{F} \to \mathrm{Hom}(\operatorname*{colim}_{i \in I} L(i), \mathfrak{F}) = \lim_{i \in I} \mathrm{Hom}(L(i), \mathfrak{F}).$$

By definition of a colimit these definitions coincide when a colimit exists in C.

Remark. The category $\hat{C}$ is cocomplete so even if C does not have a terminal object, we can still compute the global sections as $\Gamma(1; -)$

Definition 4 (Over/Under categories)**.** Let C and C' be categories. Let $F : C \to C'$ and $z \in C'$. Define the category C_z and C^z as

$$\mathrm{Obj}(C_z) := \{(a, w) \mid a \in C, w : F(a) \to z\},$$
$$\mathrm{Hom}((a, w), (b, v)) := \{f : a \to b \mid v \circ F(f) = w\},$$

and

$$\mathrm{Obj}(C^z) := \{(a, w) \mid a \in C, w : z \to F(a)\},$$
$$\mathrm{Hom}((a, w), (b, v)) := \{f : a \to b \mid F(f) \circ w = v\}.$$

We get faithful functors $C_z \to C : (a, w) \to a$ and $C^z \to C : (a, w) \to a$. We will call both functors localization functors and denote them by u.

Definition 5 (direct image)**.** Let $f : C \to D$. Define the direct image functor $f_* : \hat{D} \to \hat{C}$ as

$$f_* = - \circ f.$$

Definition 6 (Restriction)**.** Let C, D be categories. Let $\mathfrak{F} \in \hat{D}$. Let $\alpha : C \to D$ be a functor. The restriction of $\mathfrak{F}$ to C along α is defined to be $\alpha_* \mathfrak{F}$.

1.2 Topology

In this section we will define a notion of a topology on a category and look at the related notions of sheaves, sites and restriction of sites.

See [MM92] for more details.

Definition 7 (Sieve)**.** Let C be a category and $a \in \mathsf{C}$. Define the maximal sieve $\mathsf{max}(a)$ on a to be the set of all morphisms to a. In formula,

$$\mathsf{max}(a) = \{f \in \mathsf{C} \mid \mathsf{Codom}(f) = a\}.$$

A sieve S is a subset of $\mathsf{max}(a)$ such that $gf \in S$ for any $f \in S$ and any g.

Remark. Let C be a category and $a, b \in \mathsf{C}$. Let $f : b \to a \in \mathsf{C}_a$.

Any morphism to b is also a morphism to f and vice versa. This observation yields us that $\mathsf{Sieves}(b) = \mathsf{Sieves}(f)$. Moreover composition in C and C_a are the same, so this identification respects pullback of sieves.

Definition 8 (Sieve category)**.** Let C be a category and $a \in \mathsf{C}$. The sieve category $\mathsf{Sieves}(a)$ consists of all the sieves on a as objects and inclusions of sieves as morphisms.

Definition 9 (Pullback of sieve)**.** Let C be a category and $a, b \in \mathsf{C}$. Let S be a sieve on a. Let $f : b \to a$.

The sieve f^*S on b is given by $f^*S(c) = \{g \in \mathrm{Hom}(c, b) : fg \in S(c)\}$ for any $c \in \mathsf{C}$.

To show that this is actually a subpresheaf of $h(b)$, let $k : c \to c'$ and $h \in f^*S(c')$. Hence $fh \in S(c')$ and so $fhk \in S(c)$. Conclude that $hk \in f^*S(c')$.

This defines a functor $f^* : \mathsf{Sieves}(a) \to \mathsf{Sieves}(b)$.

Definition 10 (Grothendieck Topology)**.** A Grothendieck topology $\mathcal{T}$ is a family $\mathcal{T}(a)$ of 'covering' sieves for every $a \in \mathsf{C}$ with the following conditions:

1. $\mathsf{max}(a) \in \mathcal{T}(a)$
2. $f^*R \in \mathcal{T}(a')$ if $R \in \mathcal{T}(a)$ for all $f : a' \to a$
3. if $f^*R \in \mathcal{T}(a')$ for all $f \in S$ with $S \in \mathcal{T}(a)$ then $R \in \mathcal{T}(a)$

Definition 11 (Basis)**.** Let C be a category with pullbacks. A Grothendieck pretopology $\mathcal{B}$ is a collection $\mathcal{B}(a)$ of 'covering' families $\{f_i : a_i \to a\}$ of morphisms for every $a \in C$ with the following conditions.

1. every isomorphism is a covering singleton family,
2. (Stability) The pullback of a covering family is a covering family. If $\{f_i : a_i \to a\}$ is covering and $g : b \to a$, then $\{f_i' : a_i \times_a b \to b\}$ is covering.
3. (Transitivity) If $\{f_i : a_i \to a\}$ is a covering family and $\{f_{ij} : a_{ij} \to a_i\}$ for every i, then $\{f_{ij} : a_{ij} \to a\}$ is a covering family.

Generating a topology from a basis: take any sieve containing a covering family to be a covering sieve.

Definition 12 (Site)**.** A site $(C, \mathcal{T})$ is a category C with a Grothendieck topology $\mathcal{T}$. A morphism of sites $G : (C, \mathcal{T}) \to (D, \mathcal{S})$ is a functor $G' : C \to D$ such that

A functor $G' : C \to D$ is called cover-preserving if for every covering sieve R, the family $\{G'(f) | f \in R\}$ generates a S-covering sieve.

The category Sites has as objects sites and as morphisms cover-preserving functors. When no confusion can arise then we will use C to denote the whole site $(C, \mathcal{T})$.

1.2.1 Sheaves

We will introduce the very important notion of a sheaf. See [MM92] for a more detailed treatment.

Definition 13 (Matching family)**.** Let C be a category. Let $\mathfrak{F}$ be a presheaf on on C. Let $a \in C$ be an object. Let R be a sieve on a. A set $\{x_i\}_{i \in R}$ with $x_i \in \Gamma(\mathrm{Dom}(i); \mathfrak{F})$ indexed by a sieve R and such that $x_{g \circ i} = \mathfrak{F}(g)(x_i)$ for any $g : b \to \mathrm{Dom}(i)$ and $b \in C$ is called a 'matching family'.

Definition 14 (Matching family/Morphisms)**.** Let C be a category. Let $\mathfrak{F}$ be a presheaf on on C. Let $a \in C$ be an object. Let R be a sieve on a. Define $\Gamma(R; \mathfrak{F}) = \mathrm{Hom}(R, F)$. An element $\varphi \in \Gamma(R; \mathfrak{F})$ is uniquely identified by the matching family $\{\varphi(i)\}_{i \in R}$ of images. Conversely, any matching family $\{x_i\}_{i \in R}$, with $x_i \in \Gamma(\mathrm{Dom}(i); \mathfrak{F})$ indexed by R and such that $x_{g \circ i} = \mathfrak{F}(g)(x_i)$ for any $g : b \to \mathrm{Dom}(i)$ and $b \in C$, uniquely identifies a map $\varphi : R \to \mathfrak{F}$. Namely, take $\varphi_a(y) = x_y$.

Definition 15 (Amalgamation)**.** An amalgamation of a matching family $\{x_i\}_R$ is an element $x \in \Gamma(a; \mathfrak{F})$ such that $\mathfrak{F}(i)(x) = x_i$.

When you consider the matching family as a morphism φ, an amalgamation is a morphism $\phi : h(a) \to \mathfrak{F}$ that extends φ.

Definition 16 (Separated presheaf)**.** A presheaf $\mathfrak{F}$ is separated if any matching family has at most one amalgamation.

Definition 17 (Sheaves)**.** Let $(C, \mathcal{T})$ be a site. Let $\mathfrak{F} \in \hat{C}$.

A presheaf that admits a unique amalgamation for every matching family is called a sheaf. The category $\mathsf{Shv}(C)$ is the full subcategory in $\hat{C}$ of all sheaves.

In other words, we call $\mathfrak{F}$ a sheaf if for each $a \in C$ and $R \in \mathcal{T}(a)$ the map

$$\begin{aligned} \Gamma(a; \mathfrak{F}) &\to \Gamma(R; \mathfrak{F}) \\ x &\mapsto \{x_i \mid i \in R\} \end{aligned}$$

where $x_i = \mathfrak{F}(i)(x)$ is an isomorphism.

Definition 18 (Plus construction)**.** Let $(C, \mathcal{T})$ be a site. Let $a, a' \in C$ and $f : a \to a'$. Let $\mathfrak{F} \in \hat{C}$. Define the functor $(-)^+ : \hat{C} \to \hat{C}$ as follows.

For all $a \in C$,

$$F^+(a) = \frac{\{(R, \varphi) \mid R \in \mathcal{T}(a), \varphi \in \Gamma(R; \mathfrak{F})\}}{\sim},$$

for all morphisms $f \in C$,

$$F^+(f)([(R, \varphi)]) = [(f^*R, \varphi h(f))].$$

The equivalence relation is defined as $(R, \varphi) \sim (S, \phi)$ if $\varphi = \phi$ on some covering sieve $Q \subset R \cap S$.

Let $L : \mathfrak{F} \to \mathfrak{F}'$. Then

$$(L^+)_a([(R, \varphi)]) = [(R, L \circ \varphi)]$$

This functor comes with a natural transformation $\omega : \mathrm{Id} \to (-)^+$ defined by

$$\omega_{\mathfrak{F},a}(x) = [(\mathsf{max}(a), y]$$

where

$$y(i) = \mathfrak{F}(i)(x).$$

Lemma 19 (2.10 [ToposNotes])**.** *Let $\mathfrak{F}$ be a presheaf, $\mathfrak{G}$ a sheaf and $g : \mathfrak{F} \to \mathfrak{G}$ a morphism in $\hat{C}$. Then g factors through ω_F via a unique g'.*

Lemma 20 (2.11 [ToposNotes])**.** *For every presheaf $\mathfrak{F}$, F^+ is separated.*

Lemma 21 (2.12 [ToposNotes])**.** *If $\mathfrak{F}$ is separated, then F^+ is a sheaf.*

Definition 22. Define $sh = (-)^+ \circ (-)^+$.

Lemma 23 (Sheafification adjunction)**.** *Let $(C, \mathcal{T})$ be a site. The functor sh is left adjoint to the inclusion $\hat{C} \to \mathsf{Shv}(C)$ with unit*

$$\omega^2_{\mathfrak{F}} = \omega_{\mathfrak{F}^+} \circ \omega_{\mathfrak{F}}.$$

1.2.2 Relative topology

We will look at what the induced topology on a slice category looks like and what this implies for restriction of sheaves. See [Stacks, Tag 03A4] for a more detailed treatement.

Definition 24 (Relative topology)**.** Let $(C, \mathcal{T})$ be a site. Let $a \in C$.

Define the induced topology $\mathcal{T}_a$ on C_a by, for each $f \in C_a$

$$\mathcal{T}_a(f) = \mathcal{T}(\mathrm{Dom}(f)).$$

The identification from Remark 1.2 implies that $\mathcal{T}_a$ is a Grothendieck topology.

Definition 25 (Oversite)**.** Let $Y = (C, \mathcal{T})$ be a site. Let $a \in C$. Define the site Y_a to be the category C_a with the induced topology T_a.

Definition 26 (Natural transformation s)**.** Let $(C, \mathcal{T})$ be a site. Let $a, b \in C$ and $f : b \to a$. Let $u : C_a \to C$.

Let $\{x_i \mid i \in R\}$ be a compatible family indexed by a sieve R on b. The same set $\{x_i \mid i \in R\}$ is a compatible family on f indexed by the same sieve R. This yields a natural isomorphism

$$s : u_* \circ (-)^+ \to (-)^+ \circ u_*,$$

by

$$s_{\mathfrak{F}} : u_*\mathfrak{F}^+ \to (u_*\mathfrak{F})^+$$
$$s_{\mathfrak{F},f}([\{x_i \mid i \in R\}]) = [\{x_i \mid i \in R\}].$$

We will treat s as an identification.

Lemma 27 (s and ω commute)**.** *Let $\mathfrak{F}$ be a presheaf on $(C, \mathcal{T})$. Let $f : b \to a \in C$. Let $u : C_a \to C$ be the localisation morphism. Then $\omega_{u_*\mathfrak{F}} = s_{\mathfrak{F}} \circ u_*\omega_{\mathfrak{F}}$.*

Proof. For any section $x \in \Gamma(b; \mathfrak{F})$. Let $x_i = \mathfrak{F}(i)(x)$ for any morphism $i \in C$. Note that $\max(f) = \max(b)$. This implies that the compatible family $\{x_i\}$ indexed by the maximal sieve on f is sent by s to the same set $\{x_i\}$ indexed by the maximal sieve on b. In diagram form, that

$$\begin{tikzcd}
u_*\mathfrak{F} \arrow[d, "u_*\omega_{\mathfrak{F}}"'] \arrow[dr, "\omega_{u_*\mathfrak{F}}"] & \\
u_*\mathfrak{F}^+ \arrow[r, "s"] & (u_*\mathfrak{F})^+
\end{tikzcd}$$

commutes. ∎

Definition 28 (s^2)**.** Define

$$s^2 : u_* \circ (-)^+ \circ (-)^+ \to (-)^+ \circ (-)^+ \circ u_*$$

as

$$s^2_{\mathfrak{F}} = {s_{\mathfrak{F}}}^+ \circ s_{\mathfrak{F}^+}.$$

Lemma 29. *Let $\mathfrak{F}$ be a presheaf on $(C, \mathcal{T})$. Let $f : b \to a \in C$. Let $\mathfrak{F}$ be a presheaf on C. Then $\omega^2_{u_*\mathfrak{F}} = s^2_{\mathfrak{F}} \circ (u_*\omega^2_{\mathfrak{F}})$.*

Proof. Let $a \in C$. We have the following identities.

$$\begin{aligned}
\omega^2_{u_*\mathfrak{F}} &= \omega_{u_*\mathfrak{F}^+} \circ \omega_{u_*\mathfrak{F}} \text{ by definition} \\
&= \omega_{u_*\mathfrak{F}^+} \circ s_{\mathfrak{F}} \circ u_*\omega_{\mathfrak{F}} \text{ by applying Lemma 27} \\
&= {s_{\mathfrak{F}}}^+ \circ u_*{\omega_{\mathfrak{F}}}^+ \circ u_*\omega_{\mathfrak{F}} \text{ by naturality of } \omega \\
&= {s_{\mathfrak{F}}}^+ \circ u_*{\omega_{\mathfrak{F}}}^+ \circ s_{\mathfrak{F}} \circ u_*\omega_{\mathfrak{F}} \text{ by applying Lemma 27} \\
&= {s_{\mathfrak{F}}}^+ \circ s_{\mathfrak{F}^+} \circ u_*\omega_{\mathfrak{F}^+} \circ u_*\omega_{\mathfrak{F}} \text{ by naturality of } s \circ u_*\omega \\
&= s^2_{\mathfrak{F}} \circ u_*\omega^2_{\mathfrak{F}}
\end{aligned}$$

■

Corollary 30. *Let* $(C, \mathcal{T})$*. Let* $a, b \in C$*. Let* $f : b \to a \in C$*. Sheafifying and restricting commute via the iso*

$$s^2 : sh_b \circ u_* \to u_* \circ sh_a.$$

Lemma 31 (λ commutes with restriction)**.** *Let* $(C, \mathcal{T}, \mathfrak{O})$ *be a ringed site. Let* $a \in C$*. We have a natural isomorphism* $t : u_* \circ \lambda \Rightarrow \lambda \circ (- \otimes_{\Gamma(1;\mathfrak{O})} \Gamma(a; \mathfrak{O}))$*.*

Proof. Define the natural transformation $t : \lambda \circ (- \otimes_{\Gamma(1;\mathfrak{O})} \Gamma(a; \mathfrak{O})) \Rightarrow u_* \circ \lambda$, by for each $\Gamma(1; \mathfrak{O})$-module M and for each $f : b \to a \in C_a$,

$$t_{M,f} : M \otimes_{\Gamma(1;\mathfrak{O})} \Gamma(a; \mathfrak{O}) \otimes_{\Gamma(a;\mathfrak{O})} \Gamma(b; \mathfrak{O}) \to M \otimes_{\Gamma(1;\mathfrak{O})} \Gamma(b; \mathfrak{O}),$$

$$m \otimes r \otimes s \mapsto m \otimes rs.$$

Every component $t_{M,f}$ is an isomorphism by basic commutative algebra. ■

Let C be a category. Let $a \in C$. Let ϵ be the counit of the adjunction $\lambda \dashv \Gamma(1; -)$ on C. Let ϵ_a be the counit of the adjunction $\lambda_a \dashv \Gamma(a; -)$ on C_a.

Lemma 32 (λ counit commute with restriction)**.** *We have* $u_*\epsilon \cong \epsilon_a$ *on presheaves of the form* $\lambda_a(M \otimes \Gamma(b; \mathfrak{O}))$ *with* M *a* $\Gamma(1; \mathfrak{O})$*-module via*

$$t \circ u_*\epsilon \circ t^{-1} = \epsilon_a.$$

Proof. Both maps are the identity map if you unfold them. ∎

Lemma 33 (Λ commutes with restriction)**.** *Let* $(C, \mathcal{T}, \mathfrak{O})$ *be a ringed site. We have a natural isomorphism*

$$q : u_* \circ \Lambda \to \Lambda \circ (- \otimes_{\Gamma(1;\mathfrak{O})} \Gamma(a;\mathfrak{O})).$$

Proof. Define q to be the composition

$$\begin{aligned} u_* \circ \mathrm{sh} \circ \lambda &\overset{s^2\lambda}{\Rightarrow} \mathrm{sh} \circ u_* \circ \lambda \\ &\overset{\mathrm{sh}(t)}{\Rightarrow} \mathrm{sh} \circ \lambda \circ - \otimes_{\Gamma(1;\mathfrak{O})} \Gamma(a;\mathfrak{O})) \end{aligned}$$

By Definition 26 and lemma 31, t and s^2 are isomorphisms so q is an isomorphism as well. ∎

1.3 Modules

Presheaf modules and sheaf modules on a ringed site will be introduced in this section. The functors λ and Λ introduced here will be used extensively. See [Stacks, Tag 03A4] for more detail.

Definition 34 (Presheaf modules)**.** Let $Y = (C, \mathcal{T}, \mathfrak{O})$ be a ringed site. Let $R = \Gamma(1;\mathfrak{O})$.

A presheaf module on Y is a presheaf of sets $\mathfrak{F}$ on C together with a map of presheaves

$$\mathfrak{O} \times \mathfrak{F} \to \mathfrak{F}$$

such that for every object $a \in C$ the map $\Gamma(a;\mathfrak{O}) \times \Gamma(a;\mathfrak{F}) \to \Gamma(a;\mathfrak{F})$ defines a $\Gamma(a;\mathfrak{O})$-module structure on $\Gamma(a;\mathfrak{F})$.

A morphism

$$\mathfrak{F} \to \mathfrak{G}$$

is a morphism of presheaf modules if

$$\begin{array}{ccc} \mathfrak{O} \times \mathfrak{F} & \longrightarrow & \mathfrak{F} \\ \downarrow & & \downarrow \\ \mathfrak{O} \times \mathfrak{G} & \longrightarrow & \mathfrak{G} \end{array}$$

commutes. The category of presheaf modules on Y will be denoted $\mathsf{PMod}(Y)$.

Definition 35. Let M, N be an R-module.

Define

$$\lambda : \mathsf{R\text{-}Mod} \to \mathsf{PMod}(Y)$$

by for all $a \in C$,

$$\lambda(M)(a) = M \otimes_R \Gamma(a; \mathfrak{O}),$$

for all $f : b \to a \in C$,

$$\lambda(M)(f) : \mathrm{Id} \otimes \mathfrak{O}(f),$$

for all $g : M \to N \in \mathsf{R\text{-}Mod}$,

$$\lambda(g) = (a : g \otimes \mathrm{Id}).$$

Lemma 36. *Let* $Y = (X, \mathfrak{T}, \mathfrak{O})$ *be a ringed site. The functor* λ *is left adjoint to*

$$\Gamma(1; -) : \mathsf{PMod}(Y) \to \mathsf{R\text{-}Mod}$$

.

Proof. Let a be an object of C. Let M, N be R-modules. Let $\mathfrak{F}, \mathfrak{G} \in \mathsf{PMod}(Y)$ be presheaf modules.

Let $\varphi : \lambda(M) \to \mathfrak{G}$ be a morphism of presheaf modules. Let $\phi : M \to \Gamma(1; \mathfrak{G})$ be a morphism of presheaf modules.

Define

$$\alpha = H_{M,\mathfrak{G}} : \mathrm{Hom}(\lambda(M), \mathfrak{G}) \to \mathrm{Hom}(M, \Gamma(1; \mathfrak{G}))$$

by

$$\alpha(\varphi) = \varphi_1,$$

where φ_1 is the component of φ on the global sections.

Define

$$\beta = L_{M,\mathfrak{G}} : \mathrm{Hom}(M, \Gamma(1;\mathfrak{G})) \to \mathrm{Hom}(\lambda(M), \mathfrak{G})$$

by

$$\beta(\phi)_a = \phi \otimes_R \Gamma(a;\mathfrak{O}).$$

We will show that β and α are mutually inverse.

Let $d = \beta(\alpha(\varphi))$. Let $m \otimes g \in M \otimes_R \Gamma(a;\mathfrak{O})$. Let $p : \lambda(M)(1) \to \lambda(M)(a)$ be the projection map. Let $q : \mathfrak{G}(1) \to \mathfrak{G}(a)$ be the projection map. Then $d_a(m \otimes g) = \varphi_1(m) \otimes g$ and

$$\begin{aligned}
\varphi_a(m \otimes g) &= g\varphi_a(m \otimes 1) \text{ by linearity} \\
&= g\varphi_a(p(m)) \\
&= gq(\varphi_1(m)) \text{ by naturality of } \varphi \\
&= g(\varphi_1(m) \otimes 1) \\
&= \varphi_1(m) \otimes g.
\end{aligned}$$

Hence $d = \varphi$. In words, the natural transformations from presheaves of the from $\lambda(M)$ are uniquely determined by their global sections component.

Let $d = \alpha(\beta(\phi))$. Let $m \in M$. Then $d(m) = (\phi \otimes_R R)(m) = \phi(m)$. Hence $d = \phi$, which makes H and L mutual inverses.

Naturality in M and $\mathfrak{G}$

Let $g : N \to M$ and $h : \mathfrak{F} \to \mathfrak{G}$. Let $\rho \in \mathrm{Hom}(\lambda(N), \mathfrak{F})$. Let $k = H_{M,\mathfrak{G}}(h \circ \rho \circ \lambda(f))$. Let $l = h_1 \circ H_{N,\mathfrak{F}}(\rho) \circ f$.

Unfolding the definition for H shows that $k = h_1\rho_1 f$ and $l = h_1\rho_1 f$ as well. This proves naturality in M and $\mathfrak{G}$ and the adjunction between λ and $\Gamma(1;-)$. ∎

Definition 37. Let $(\mathcal{C}, \mathcal{T}, \mathfrak{O})$ Define

$$\Lambda : \mathsf{R\text{-}Mod} \to \mathsf{Mod}(O)$$

by $\mathrm{sh} \circ \lambda$.

It follows that we have the adjunction $\Lambda \dashv \Gamma(1; -)$.

Definition 38. Let $\mathfrak{F}$ be a sheaf of modules on $(C, \mathcal{T}, \mathfrak{O})$. It is called quasi-coherent if the following holds. For any object $a \in C$ there exists a covering sieve S such that for any map $f : b \to a$ in S there exists a presentation

$$\bigoplus_I \mathfrak{O}\big|_b \to \bigoplus_J \mathfrak{O}\big|_b \to \mathfrak{F}\big|_b \to 0$$

Quasi-coherent modules form a full subcategory of the category of sheafs of modules over $(C, \mathcal{T}, \mathfrak{O})$ which are denoted by $\mathsf{Qcoh}(\mathfrak{O})$.

1.4 Schemes

Definition 39 (Spectrum of a ring)**.** Let R be a ring. The spectrum $\mathsf{Spec}R$ of R is the ringed space defined as follows. The underlying set is the set of prime ideals of R. The (zariski) topology is generated by the basis $D(f) = \{\mathfrak{p} \subset R | f \notin \mathfrak{p}\}$. The sheaf of rings is given by

$$D(f) \mapsto R_f.$$

Definition 40 (Distinguised open)**.** Let $\mathsf{Spec}R$ be a affine scheme. The set

$$D(f) = \{\mathfrak{p} \subset R | f \notin \mathfrak{p}\}$$

for a global section f is called a distinguised open. The open $D(f)$ is isomorphic to $\mathsf{Spec}(R_f)$ as a locally ringed space.

Definition 41 (Locus of a point)**.** Let $(X, \mathfrak{O})$ be a scheme. Define the locus of a global section $x \in \Gamma(1; \mathfrak{O})$ to be

$$\ker(x) = \ker(\mathfrak{O}(X) \to \kappa(x)).$$

Lemma 42. *The functor*

$$\mathit{Spec} : \mathsf{Rng} \to \mathsf{LRSpaces}$$

is left adjoint to

$$\Gamma(1; -) : \mathsf{LRSpaces} \to \mathsf{Rng}.$$

With unit

$$F = \eta : (X, \mathfrak{O}) \to \mathit{Spec}(\Gamma(1; \mathfrak{O})).$$
$$x \mapsto \ker(x),$$

Proof. ∎

Definition 43 (Affine scheme)**.** We call the ringed space $\mathrm{Spec}(R)$ an affine scheme.

Definition 44 (Scheme)**.** A scheme S is a locally ringed space that admits a covering of affine schemes. A morphism of schemes is a morphism of locally ringed spaces. The category of schemes we will denote by Sch.

2 Restrictive

2.1 Restrictive

Definition 45 (Restrictive morphism)**.** Let $(\mathsf{C}, \mathfrak{T}, \mathfrak{O})$. A morphism $f : a \to b \in \mathsf{C}$ is called restrictive if for every quasi-coherent module $\mathfrak{G}$ on C the morphism

$$\Gamma(b; \mathfrak{G}) \otimes_{\Gamma(b;\mathfrak{O})} \Gamma(a; \mathfrak{O}) \to \Gamma(a; \mathfrak{G})$$

is an isomorphism.

Example 46. In $\mathsf{Sch}_{/\mathsf{Spec}(A)}$ the morphism $\mathrm{Spec}(A_f) \xrightarrow{f} \mathrm{Spec}(A)$ is restrictive. Let $\mathfrak{G}$ be a quasi-coherent sheaf on $\mathsf{Sch}_{/\mathsf{Spec}(A)}$. This implies that $\mathfrak{G} = \Gamma(\widetilde{\mathrm{Spec}(A)}; \mathfrak{G})$. The morphism

$$\Gamma(\mathrm{Spec}(A); \mathfrak{G}) \otimes_{\Gamma(1;\mathfrak{O})} \Gamma(\mathrm{Spec}(A_f); \mathfrak{O}) \to \Gamma(\mathrm{Spec}(A_f); \mathfrak{G}) = \Gamma(A; \mathfrak{G})_f,$$

$$m \otimes r \to rm$$

is an isomorphism by basic commutative algebra.

Lemma 47. *Let* $(\mathsf{C}, \mathfrak{T}, \mathfrak{O})$ *be a ringed site. If* $f : b \to a \in \mathsf{C}$ *and* $g : c \to b \in \mathsf{C}$ *are restrictive, then* fg *is.*

Proof. Let $\mathfrak{G}$ be a quasi-coherent sheaf on C. Note that the morphism

$$\mathfrak{G}(fg) \otimes_{\Gamma(a;\mathfrak{O})} \Gamma(c; \mathfrak{G}) : \Gamma(a; \mathfrak{G}) \otimes_{\Gamma(a;\mathfrak{O})} \Gamma(c; \mathfrak{G}) \to \Gamma(c; \mathfrak{G})$$

is equal to $(\mathfrak{G}(g)(\mathfrak{G}(f) \otimes_{\Gamma(a;\mathfrak{O})} \Gamma(b; \mathfrak{G}))) \otimes_{\Gamma(b;\mathfrak{O})} \Gamma(c; \mathfrak{G})$. By assumption

$$\mathfrak{G}(f) \otimes_{\Gamma(a;\mathfrak{O})} \Gamma(b; \mathfrak{G}) : \Gamma(a; \mathfrak{G}) \otimes_{\Gamma(a;\mathfrak{O})} \Gamma(b; \mathfrak{G}) \to \Gamma(b; \mathfrak{G})$$

and

$$\mathfrak{G}(g) \otimes_{\Gamma(b;\mathfrak{O})} \Gamma(c;\mathfrak{G})\Gamma(b;\mathfrak{G}) \otimes_{\Gamma(b;\mathfrak{O})} \Gamma(c;\mathfrak{G}) \to \Gamma(c;\mathfrak{G})$$

are isomorphisms, hence

$$(\mathfrak{G}(g)(\mathfrak{G}(f) \otimes_{\Gamma(a;\mathfrak{O})} \Gamma(b;\mathfrak{G}))) \otimes_{\Gamma(b;\mathfrak{O})} \Gamma(c;\mathfrak{G})$$

is an isomorphism. ■

Example 48. Let R be a ring. Consider the open immersion $U = \mathrm{Spec}(R[x,y]) \setminus \{(x,y)\} \to \mathrm{Spec}(R[x,y])$ and the quasi-coherent sheaf $\mathfrak{G} = \Lambda(\frac{R[x,y]}{xy})$. The global sections of this sheaf are $\frac{R[x,y]}{xy}$, because for affine schemes the counit is an isomorphism.

Define $U_1 = D(x) \to U$ and $U_2 = D(y) \to U$. Note that these cover U together. We have $\Gamma(U_1;\mathfrak{G}) = 0$ and $\Gamma(U_1;\mathfrak{G}) = 0$, since $\frac{R[x,y]}{xy}_x = 0$ and $\frac{R[x,y]}{xy}_y = 0$. Hence since $\mathfrak{G}$ is a sheaf, we get $\Gamma(U;\mathfrak{G}) = 0$.

The sections over U of $\Lambda(R[x,y])$ are (also) $R[x,y]$. We conclude that $\Gamma(1;\mathfrak{G}) \otimes_{\Gamma(1;\mathfrak{O})} \Gamma(U;\mathfrak{O}) \to \Gamma(U;\mathfrak{G})$ is not an isomorphism.

Example 49 (Affine non-restrictive map)**.** Consider the canonical inclusions $\mathbb{A}^1 \to \mathbb{P}^1$ and the shifted quasi-coherent module $\mathfrak{O}(-1)$. This module is even locally free of degree 1, this is often called an invertible module.

The global sections of the module $\mathfrak{O}(-1)$ are the elements of degree -1 in the global sections of $\mathfrak{O}$. There are no such elements, hence the global sections are the zero module.

On $\mathbb{A}^1$ all invertible modules are isomorphic to the structure sheaf. We conclude that the canonical inclusions cannot be restrictive.

Any inclusion $\mathrm{Spec}(\kappa(\mathfrak{p})) \to \mathbb{P}^1$ of a point is not restrictive which can be shown with the same argument.

This is another(more opaque) way of saying that on projective space not every quasi-coherent sheaf is generated by global sections.

Lemma 50 (Restrictive to affines)**.** *If $f : X \to Spec(R)$ is a restrictive open immersion. then X is affine.*

Proof. ■

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