Prelims

Let $Y = (X, \mathcal{T}, \mathfrak{O})$ be a ringed site. Let $R = \Gamma(1; \mathfrak{O})$. Let $a, b, c, c' \in X$. Let $\mathfrak{F}$ be a quasi-coherent module on Y_a. Let $M = \Gamma(a; \mathfrak{F}) = \Gamma(1; \mathfrak{F})$. Let $f : b \to a$.

Some basic definitions and constructions.

Definition 1 (Over/Under categories)**.** Let C and C' be categories. Let $F : C \to C'$ and $Z \in C'$. Define the category C_Z and C^Z as follows

$$\mathrm{Obj}(C_Z) := \{(X, u) \mid X \in C, u : F(X) \to Z\},$$

$$\mathrm{Hom}((X, u), (Y, v)) := \{f : X \to Y \mid v \circ F(f) = u\},$$

and

$$\mathrm{Obj}(C^Z) := \{(X, u) \mid X \in C, u : Z \to F(X)\},$$

$$\mathrm{Hom}((X, u), (Y, v)) := \{f : X \to Y \mid F(f)u = v\}.$$

We get faithfull functors $C_Z \to C : (X, u) \to X$ and $C^Z \to C : (X, u) \to X$. We will call both functors u and suppress the functor where there can be no confusion

Definition 2. Let M, N be an R-module. Let $g : M \to N$. Define

$$\lambda : R\text{-Mod} \to \mathrm{PMod}(Y)$$

by

$$\lambda(M)(a) = M \otimes_R \Gamma(a; \mathfrak{O}),$$

$$\lambda(M)(f) : \mathrm{Id} \otimes \mathfrak{O}(f),$$

$$\lambda(g) = (a : g \otimes \mathrm{Id}).$$

Definition 3. Define

$$\Lambda : R\text{-Mod} \to \mathrm{Mod}(Y)$$

by $sh \circ \lambda$.

This functor is left adjoing to the global sections functor, which I will prove in the next episode.

Lemma 4. *Let S be a subset of Hom$(-, f)$. Then $u(S)$ is a sieve on $u(f) = b$ if and only if S is a sieve on f.*

Proof. =>: Let $h : d \to b \in S$ and $k : e \to d$ be arbitrary. By assumption $u(hk) \in u(S)$. The functor u is faithfull, so $hk \in S$.

<=: Let $h : d \to b \in u(S)$ and $k : e \to d$ be arbitrary. By assumption $hk \in S$, hence $u(hk) \in u(S)$. ∎

We will define the induced topology $\mathcal{S}$ on C_a. That u considered as a map on sieves commutes with the pullback of sieves is used and will not be proved.

Definition 5. Let $\mathcal{T}(u(f))$ be the set of covering sieves on $u(f) \in X$. By the previous lemma sieves on $u(f)$ are sieves on f. Let $\mathcal{S}(f) = \{R \mid u(R) \in \mathcal{T}(u(f))$ be the induced topology. So $u(R)$ is covering on $u(f)$ if and only if R is covering on f.

a) Since u commutes with pullback of sieves, we have $\max(u(f)) = u(\max(f)) = \max(f)$, hence $\max(f) \in \mathcal{S}(f)$.

b) Let R be a covering sieve on f. Let $h : b' \to a$ and $p : b' \to b$ with $fp = h$. Commutativity of u and pulling back implies that $u(p)^*u(R) = u(p^*R)$. Hence p^*R is covering since $u(p^*R)$ is.

c) Let R be a covering sieve on f and Q be a sieve on f. Let $h : b' \to a$ and $p : b' \to b \in R$, hence with $fp = h$. Assume p^*Q is covering for every such p. Then $u(p^*Q) = u(p)^*u(Q)$ is covering for every p. We know that $u(R)$ is covering hence $u(Q)$ must be, which implies that Q is covering.

We proved that $\mathcal{S}$ is indeed a Grothendieck topology.

Main

Lemma 6. *Let a be caffine. The global component of the sheafification morphism is equal to the unit of $\Lambda \dashv \Gamma(1; -)$ in C_a.*

Proof. Let M be a $\Gamma(a; \mathfrak{O})$-module. Consider the following maps, which you get by repeatedly calling on an adjunction bijection. Let i be the universal sheafification morphism.

$$\Lambda(M) \to \Lambda(M)$$
$$i : \lambda(M) \to \Lambda(M) \text{ use sheafification adjunction}$$
$$M \to \Gamma(a; \Lambda(\mathfrak{M})) \text{ use } \lambda \dashv \Gamma(a; -)$$

If you compose the two adjunction bijections used, you get the bijection of $\Lambda \dashv \Gamma(a; -)$ by definition, so the last map is actually η_M. Hence $i_a = \eta_M$, which is an iso by assumption. ∎

Lemma 7. *Sheafifying and restricting commute. In formula form*

$$sh_b \circ *\big|_b \cong *\big|_b \circ sh_a.$$

Proof. I will prove that we have a natural isomorphism

$$s : sh_b \circ *\big|_b \to *\big|_b \circ sh_a.$$

Let $\mathfrak{F}$ be a presheaf on Y_a. Let $\mathfrak{H} = sh(F\big|_b)$ and $\mathfrak{K} = sh(F)\big|_b$. Let T be a covering sieve on g in Y_b and $j \in T$. Let S_j be a covering sieve on $\mathrm{Dom}(j)$ in Y_b and $i \in S_j$.

Let $x = (x_{i,j}) \in sh(F\big|_b)$ be indexed by S_j and T. We have $x_{i,j} \in \Gamma(\mathrm{Dom}(i); \mathfrak{F})$. Define $s_g(x) = (x_{u(i),u(j)})$ with indexing covering sieves $u(S_j), u(T)$.

Let $x \sim y$. Let R be the covering sieve on which they are the same. Then $s_g(x) \sim s_g(y)$ because they are the same on $u(R)$. Hence this map is well-defined.

Let $s_g(x) = s_g(y)$. Then there is some covering sieve R on fg on which they agree. Consider $u^*(R)$ as a covering sieve on g and its is clear that x and y must agree on it, hence the map is injective.

Let $y = (y_{k,l})$ be an element of $\Gamma(c; \mathfrak{K})$ which is indexed by V, W. Then $s_g(y') = y$ where y' has the same elements as y but is indexed by $u(V), u(W)$, so $y' \in \Gamma(c; \mathfrak{H})$. Hence s_g is surjective.

Let $h : c' \to b$ and $p : c' \to c$, such that $gp = h$. We will show that $s_h \mathfrak{H}(t) = \mathfrak{K}(t) s_g$. See below diagram. Let $x = (x_{i,j}) \in \Gamma(c; \mathfrak{H})$ with indexing covering sieves S_j and T. Then

$\mathfrak{K}(t)(s_g(x)) = (x_{k,l})$ with indexing covering sieves t^*S_l and t^*T. The other one becomes $s_h(\mathfrak{H}(t)(x)) = (x_{k,l})$ with indexing covering sieves t^*S_l and t^*T. Hence s is natural.

$$\begin{tikzcd}
\Gamma(h;\mathfrak{H}) \arrow[r, "s_h"] & \Gamma(h;\mathfrak{K}) \\
\Gamma(g;\mathfrak{H}) \arrow[r, "s_g"] \arrow[u, "\mathfrak{H}(t)"] & \Gamma(g;\mathfrak{K}) \arrow[u, "\mathfrak{K}(t)"']
\end{tikzcd}$$

∎

Proposition 8. *The adjunct of* f

$$\Gamma(a;\mathfrak{F}) \otimes_{\Gamma(a;\mathfrak{O})} \Gamma(b;\mathfrak{O}) \to \Gamma(b;\mathfrak{F})$$

is an isomorphism.

Consider

$$\begin{tikzcd}
\mathsf{PMod}(Y_a) \arrow[r, "sh_a"] \arrow[d, "*|_b"] & \mathsf{Mod}(Y_a) \arrow[d, "*|_b"] & \Gamma(a;\mathfrak{O})\text{-}\mathsf{Mod} \arrow[l, "\Lambda"'] \arrow[d, "-\otimes\Gamma(b;\mathfrak{O})"] \arrow[ll, bend right, "\lambda"'] \\
\mathsf{PMod}(Y_b) \arrow[r, "sh_a"] & \mathsf{Mod}(Y_b) & \Gamma(b;\mathfrak{O})\text{-}\mathsf{Mod} \arrow[l, "\Lambda"] \arrow[ll, bend left, "\lambda"]
\end{tikzcd}$$

By a previous lemma, the left square commutes. By definition the two ‘triangles’ commute too and the outer square commute, hence the right square also commutes. Therefore $M \otimes \Gamma(b;\mathfrak{O}) \cong \Gamma(b;\mathfrak{F})$. This is the proof you wrote down friday.

The requirement is not to find any isomorphism but a specific one. So I think this is not enough and we need to do some bookkeeping and see if the witnessing isomorphism is our map.

Consider

$$\begin{tikzcd}
\Lambda(M)\big|_b & & \Lambda(M \otimes \Gamma(b;\mathfrak{O})) \arrow[ll, "s"'] \\
\lambda(M \otimes \Gamma(b;\mathfrak{O})) \arrow[u, "i|_b"] \arrow[urr, "j"] & &
\end{tikzcd}$$

The natural transformation j is the universal sheafification morphism coming from sh_b. We have seen that $\Gamma(b;j)$ and s are isomorphisms

Let $g : c \to b$. Let $x = m \otimes r \in \lambda(M \otimes \Gamma(c;\mathfrak{O}))$. Then $j_g(x) = (x_i)$ indexed by the maximal sieve on g and $i_g(x) = i_{fg}(x) = (x_i)$ indexed by the maximal sieve on gf. Hence we get $s_g(j_g(x)) = i_g(x)$, so the triangle commutes. Evaluating everything on the terminal, in this case on b, shows that two out of three maps are isomorphisms, hence i_b is an isomorphism.