Affine Objects

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1 Introduction

Hoi, dit is de introductie.

2 Preliminaries

2.1 Basic Category Theory

Definition 1 (Presheaf category)**.** Let C be a category. Let $a \in \mathsf{C}$. Let $f : a' \to a$ We define

$$\hat{\mathsf{C}} := [\mathsf{C}^{\mathsf{op}}, \mathsf{Set}],$$

and the functor $h : \mathsf{C} \to \hat{\mathsf{C}}$ as follows

$$a \mapsto \mathrm{Hom}(-, a),$$

$$f \mapsto f \circ -.$$

This functor is fully faithful by the Yoneda lemma.

Definition 2 (Sections functor)**.** For any $a \in \mathsf{C}$ define the functor

$$\Gamma(a; -) : \hat{\mathsf{C}}(\mathsf{A}) \to \mathsf{A}$$

by

$$\mathfrak{F} \to \mathfrak{F}(a).$$

Let $L : I \to \mathsf{C}$ be diagram and assume that $\underset{h(-)\circ L}{\mathrm{colim}}$ exists in $\hat{\mathsf{C}}(\mathsf{A})$. Define

$$\Gamma(\underset{i\in I}{\mathrm{colim}} L(i); -) : \hat{\mathsf{C}}(\mathsf{A}) \to \mathsf{A}$$

by

$$\mathfrak{F} \to \mathrm{Hom}(\underset{i\in I}{\mathrm{colim}} L(i), \mathfrak{F}) = \underset{i\in I}{\lim} \mathrm{Hom}(L(i), \mathfrak{F}).$$

By definition of a colimit these definitions coincide when a colimit exists in C.

Remark. The category $\hat{C}$ is cocomplete so even if C does not have a terminal object, we can still compute the global sections.

Definition 3 (Over/Under categories). Let C and C' be categories. Let $F : C \to C'$ and $z \in C'$. Define the category C_z and C^z as

$$\mathrm{Obj}(C_z) := \{(a, w) \mid a \in C, w : F(a) \to z\},$$
$$\mathrm{Hom}((a, w), (b, v)) := \{f : a \to b \mid v \circ F(f) = w\},$$

and

$$\mathrm{Obj}(C^z) := \{(a, w) \mid a \in C, w : z \to F(a)\},$$
$$\mathrm{Hom}((a, w), (b, v)) := \{f : a \to b \mid F(f) \circ w = v\}.$$

We get faithful functors $C_z \to C : (a, w) \to a$ and $C^z \to C : (a, w) \to a$. We will call both functors localization functors and denote them by u. We will suppress the functor F where there can be no confusion.

Definition 4 (Restriction).

Definition 5 (direct image). Let $f : C \to D$. Define the direct image functor $f_* : \hat{D} \to \hat{C}$ as

$$f_* = - \circ f.$$

Definition 6 (inverse image). Let C, D be a categories. Let $f : C \to D$ be a functor. Define the inverse image functor $f^* : \hat{C} \to \hat{D}$ as follows. Let $\mathfrak{F} \in \hat{C}$. For any $d \in D$

$$f^*(F)(d) = \operatorname*{colim}_{D_d} \mathfrak{F}u.$$

2.2 Topology

Definition 7 (Sieve). Let C be a category and $a \in C$. A sieve S on a is a subpresheaf of $h(a)$. Explicitly, for each $c \in C$, $S(c)$ is a subset of $\mathrm{Hom}(c, a)$ such that $fg \in S(\mathrm{Dom}(g))$ for all $f \in S(c)$ and for all $g \in h(c)$.

The maximal sieve on a, which is $h(a)$, will be denoted by $\max(a)$.

Definition 8 (Sieve category)**.** Let C be a category and $a \in C$. The sieve category $\mathsf{Sieves}(a)$ is the subobject poset of the presheaf $h(a)$.

Definition 9 (Pullback of sieve)**.** Let C be a category and $a, b \in C$. Let S be a sieve on a. Let $f : b \to a$.

For any $c \in C$ the sieve f^*S on b is given by $f^*S(c) = \{g \in \mathrm{Hom}(c, b) : fg \in S(c)\}$.

To show that this is actually a subpresheaf of $h(b)$, let $k : c \to c'$ and $h \in f^*S(c')$. Hence $fh \in S(c')$ and so $fhk \in S(c)$. Conclude that $hk \in f^*S(c')$.

This defines a functor $f^* : \mathsf{Sieves}(a) \to \mathsf{Sieves}(b)$.

Definition 10 (Grothendieck Topology)**.** A Grothendieck topology $\mathcal{T}$ is a family $\mathcal{T}(a)$ of 'covering' sieves for every $a \in C$ with the following conditions:

- $\max(a) \in \mathcal{T}(a)$
- $f^*R \in \mathcal{T}(a')$ if $R \in \mathcal{T}(a)$ for all $f : a' \to a$
- if $f^*R \in \mathcal{T}(a')$ for all $f \in S$ with $S \in \mathcal{T}(a)$ then $R \in \mathcal{T}(a)$

Definition 11 (Basis)**.** Let C be a category with pullbacks. A Grothendieck pretopology $\mathcal{B}$ is a collection $\mathcal{B}(a)$ of families $\{f_i : a_i \to a\}$ of 'covering' morphisms for every $a \in C$ with the following conditions.

- every isomorphism is a covering singleton family,
- (Stability) The pullback of a covering family is a covering family. If $\{f_i : a_i \to a\}$ is covering and $g : b \to a$, then $\{f_i' : a_i \times_a b \to b\}$ is covering.
- (Transitivity) If $\{f_i : a_i \to a\}$ is a covering family and $\{f_{ij} : a_{ij} \to a_i\}$ for every i, then $\{f_{ij} : a_{ij} \to a\}$ is a covering family.

Generating a topology from a basis: take any sieve containing a covering family to be a covering sieve.

2.2.1 Sheaves

Definition 12 (Matching family)**.** Let C be a category. Let $\mathfrak{F}$ be a presheaf on on C. Let $a \in C$ be an object. Let R be a sieve on a. A set $\{x_i\}_{i \in R}$ with $x_i \in \Gamma(\mathrm{Dom}(i); \mathfrak{F})$

indexed by a sieve R and such that $x_{g \circ i} = \mathfrak{F}(g)(x_i)$ for any $g : b \to \mathrm{Dom}(i)$ and $b \in C$ is called a 'matching family'.

Definition 13 (Matching family/Morphisms)**.** Let C be a category. Let $\mathfrak{F}$ be a presheaf on on C. Let $a \in C$ be an object. Let R be a sieve on a. Define $\Gamma(R; \mathfrak{F}) = \mathrm{Hom}(R, \mathfrak{F})$. An element $\varphi \in \Gamma(R; \mathfrak{F})$ is uniquely identified by the matching family $\{\varphi(i)\}_{i \in R}$ of images. Conversely, any matching family $\{x_i\}_{i \in R}$, with $x_i \in \Gamma(\mathrm{Dom}(i); \mathfrak{F})$ indexed by R and such that $x_{g \circ i} = \mathfrak{F}(g)(x_i)$ for any $g : b \to \mathrm{Dom}(i)$ and $b \in C$, uniquely identifies a map $\varphi : R \to \mathfrak{F}$. Namely, take $\varphi_a(y) = x_y$.

Definition 14 (Amalgamation)**.** An amalgamation of a matching family $\{x_i\}_R$ is an element $x \in \Gamma(1; \mathfrak{F})$ such that $\mathfrak{F}(i)(x) = x_i$.

When you consider the matching family as a morphism φ, an amalgamation is a morphism $\phi : h(a) \to \mathfrak{F}$ that extends φ.

Definition 15 (Sheaves)**.** Let $(C, \mathcal{T})$ be a site. Let $\mathfrak{F} \in \hat{C}$.

A presheaf that admits a unique amalgamation of every matching family is called a sheaf. The category $\mathrm{Shv}(C)$ is the full subcategory in $\hat{C}$ all sheaves. Let i be the inclusion functor $\mathrm{Shv}(C) \to \hat{C}$.

In other words, we call $\mathfrak{F}$ a sheaf if the map

$$\mathfrak{F}(a) \to \mathfrak{F}(R)$$

$$a : x \mapsto \{\mathfrak{F}(i)(x)\}_{i \in R}$$

is an isomorphism.

Definition 16 (Plus construction)**.** Let $(C, \mathcal{T})$ be a site. Let $a, a' \in C$ and $f : a \to a'$. Let $\mathfrak{F} \in \hat{C}$. Define the functor $(-)^+ : \hat{C} \to \hat{C}$ as follows

On objects:

$$\mathfrak{F}^+(a) = \frac{\{(R, \varphi) \mid R \in \mathcal{T}(a), \varphi \in \Gamma(R; \mathfrak{F})\}}{\sim},$$

$$\mathfrak{F}^+(f)([(R, \varphi)]) = [(f^*R, \varphi h(f))].$$

The equivalence relation is defined as:

$$(R, \varphi) \sim (S, \phi)$$

if $\varphi = \phi$ on some $Q \subset R \cap S$

Let $L : \mathfrak{F} \to \mathfrak{F}'$. Then

$$(L^+)_a([(R, \varphi)]) = [(R, L \circ \varphi)]$$

This functor comes with a natural transformation $\omega : \mathrm{Id} \to (-)^+$ defined by

$$\omega_{\mathfrak{F},a}(x) = [(\max(a), y]$$

$$y(i) = \mathfrak{F}(i)(x).$$

Definition 17. Define $\mathrm{sh} = (-)^+ \circ (-)^+$.

Lemma 18. *Let $Y = (C, \mathcal{T})$ be a site. The functor* sh *is left adjoint to the inclusion Shv*$(Y) \to$ *Shv*(C) *with unit*

$$\omega^2 : \mathit{Id} \xrightarrow{\omega} (-)^+ \xrightarrow{\omega} \mathrm{sh}$$

Proof. ∎

2.2.2 Relative topology

Definition 19 (Sieve functors)**.** Let C be a category. Let $a, b \in C$. Let $f : b \to a \in C_a$. Let $w : c \to a$. Let $g : w \to f \in C_a$.

For every sieve $S \in \mathsf{Sieves}(f)$ define the sieve S' by $S'(c) = \bigcup_{g \in \mathrm{Hom}(c,b)} S(g)$.

Let $h \in S'(c)$ and $k : c \to b$. Note that $hk \in S(gk)$ since S is a sieve on f, hence $hk \in S'(c)$. This shows that S' is a subpresheaf of $h(b)$.

Let $S \in \mathsf{Sieves}(f)$. Let $h : S \to \mathfrak{F} \in \hat{C_a}$.

Define $h' : S' \to u^*\mathfrak{F}$ to be

$$(h')_c = \bigcup_{g \in \mathrm{Hom}(c,b)} h_g.$$

For every sieve $R \in \mathsf{Sieves}(b)$ define the sieve $R^f \subset h(f)$ as follows. For each $g : c \to a \in C_a$,

$$R^f(g) = \{p : c \to b \in R(c) \mid g = f \circ p\}.$$

This is a sieve because if $p \in R^f(g)$ and $h : g' \to g$ arbitrary, then $gh = fph$ so $ph \in R^f(gh)$.

Let $S \in \mathsf{Sieves}(b)$. Let $h : S \to \mathfrak{G} \in \hat{C}$. Define $h^f : S^f \to \mathfrak{G}u$ by setting for each $g : c \to a \in C_a$

$$(h^f)_g = h_b|_{S^f(g)}$$

Define functors

$$L^f : \mathsf{Sieves}(f) \to \mathsf{Sieves}(b),$$
$$Q^f : \mathsf{Sieves}(b) \to \mathsf{Sieves}(f).$$

By, for every sieve $S \in \mathsf{Sieves}(f)$

$$L^f(S) = S',$$

for every $h : S \to R \in \mathsf{Sieves}(f)$.

$$L^f(h) = h',$$

For every sieve $R \in \mathsf{Sieves}(b)$

$$Q^f(R) = R^f$$

For every sieve $k : S \to R \in \mathsf{Sieves}(b)$.

$$Q^f(k) = k^f.$$

(Necessary to proof the functor axioms?)

Lemma 20. *Let* C *be a category. Let* $a, b \in C$. *Let* $f : b \to a \in C_a$. *We have the equalities* $L^f Q^f = Id$ *and* $Q^f L^f = Id$.

Proof. Let $w : c \to a$. Let $g : w \to f \in C_a$.

Let $S \in \mathsf{Sieves}(f)$. Let $h \in Q^f L^f(S)(g)$. Hence $g = fh$ and $h \in L^f(S)(c)$. This implies $h \in S(fh) = S(g)$. Let $h \in S(g)$. So $g = fh$ and $h \in L^f(S)(\mathrm{Dom}(g)) = L^f(S)(c)$. This implies $h \in Q^f L^f(S)(g)$. Therefore $Q^f L^f(S)$ and S are the same sieve.

Let $h : S \to R \in \mathsf{Sieves}(f)$. Let $p \in S(g)$. Then by construction $L^f Q^f(h)_g(p) = Q^f(h)'_c(p) = h_c(p)$.

Let $R \in \mathrm{Sieves}(b)$. Let $h \in L^f Q^f(R)(c)$. Hence $h \in Q^f(R)(g)$ for some $g : c \to a$. So $g = hf$ and $h \in R(c)$. Let $h \in R(c)$. Hence $h \in Q^f(R)(hf)$ and since $\mathrm{Dom}(hf) = c$ we get $h \in L^f Q^f(R)(c)$. Therefore $L^f Q^f$ and R are the same sieve.

Let $h : S \to R \in \mathrm{Sieves}(b)$. Let $p \in S(c)$. Then by construction $Q^f L^f(h)_c(p) = L^f(h)_{pf}(p) = h_c(p)$.

So $L^f Q^f = \mathrm{Id}$ and $Q^f L^f = \mathrm{Id}$. ∎

Definition 21 (Relative topology)**.** Let $(C, \mathcal{T})$ be a site. Let $a \in C$.

Set $\mathcal{T}_a(f) = \{R^f : R \in \mathcal{T}(b)\}$. Define the induced topology $\mathcal{T}_a$ on C_a by, for each $f \in C_a$

$$\mathcal{T}_a(f) = Q^f(\mathcal{T}(\mathrm{Dom}(f))).$$

Lemma 22. *$\mathcal{T}_a$ defines a Grothendieck topology*

Proof. Axiom 1: Q^f is an equivalence of posets. So the terminal object is send to the terminal object. Hence $\max(f) \in \mathcal{T}_a(f)$.

Axiom 2 & 3 are consequences of: Q^f is an equivalence and Q^f commutes with sieve pullback. ∎

Lemma 23. *Let C be a category. Let $a, b \in C$. Let $(x_i)_T$ be a matching family for some presheaf $\mathfrak{F}$ on b indexed by sieve T. For any $f : b \to a$ the family $(x_i)_{T^f}$ is matching again.*

Proof. Let $u : C_a \to C$ be the localization functor. Only when a domain has an 'a' as subscript, is it taken in C_a.

We have $x_i \in \Gamma(\mathrm{Dom}(i); \mathfrak{F})$. Hence also $x_i \in \Gamma(fi; \mathfrak{F}u) = \Gamma(\mathrm{Dom}(i)_a; \mathfrak{F}u)$, where now i is considered as a morphism in C_a. Note that

$$(\mathfrak{F}u)(p)(x_i) = \mathfrak{F}(p)(x_i) = x_{iop}$$

for any $p : c \to \mathrm{Dom}(i)$, since $(x_i)_T$ is a matching family in C. ∎

Lemma 24. *Let $Y = (C, \mathcal{T})$. Let $a, b \in C$. Let $f : b \to a$. Sheafifying and restricting commute. We will exhibit an natural isomorphism*

$$s : sh_b \circ *\big|_b \to *\big|_b \circ sh_a.$$

Proof. We will construct the natural transformation

$$s : sh_b \circ *|_b \to *|_b \circ sh_a$$

and prove it is an isomorphism.

Considering the matching families as morphisms, s is given by

$$(\varphi, R) \mapsto (\varphi', R').$$

Let $g : c \to b \in Y_b$.

Well-definedness:

Suppose $x = (\varphi, V)$ and $y = (\phi, W)$ are equivalent sections, or matching families, over g. Let $L^g(R)$ be the covering sieve on which they are the same. Then $\varphi' = \phi'$ on $Q^g L^g(R) = R$. So $s_g(x) = s_g(y)$ Hence this map is well-defined.

Injectivity: Let $s_g(\varphi, V) = s_g(\phi, W)$. Then there is some covering sieve $R \subset V'^f \cap W'^f$ on c on which they agree. Hence φ and ϕ coincide on R'_g by the equivalence.

Surjectivity: Let $y = (\varphi, V)$ be an element of $\Gamma(g; \mathfrak{K})$. Then $s_g(\varphi'^g, V'^g) = y$. Hence s_g is surjective.

Naturality: Let $h : d \to b$ and $t : d \to c$, such that $gt = h$. We will show that $s_h \mathfrak{H}(t) = \mathfrak{K}(t) s_g$. See below diagram.

$$\begin{tikzcd}
\Gamma(h; \mathfrak{H}) \arrow[r, "s_h"] & \Gamma(h; \mathfrak{K}) \\
\Gamma(g; \mathfrak{H}) \arrow[u, "\mathfrak{H}(t)"] \arrow[r, "s_g"] & \Gamma(g; \mathfrak{K}) \arrow[u, "\mathfrak{K}(t)"']
\end{tikzcd}$$

Let $x = (\varphi, V) \in \Gamma(g; \mathfrak{H})$. Then $\mathfrak{K}(t)(s_g(x)) = (\varphi' h(u(f)), t^*(V'))$ and $s_h(\mathfrak{H}(t)(x)) = ((\varphi h(f))', (t^* V)') = (\varphi' h(u(f)), (t^* V)')$. Hence s is natural. ■

2.3 Modules

Definition 25 (Presheaf modules)**.** Let $Y = (C, \mathcal{T}, \mathfrak{O})$ be a ringed site. Let $R = \Gamma(1; \mathfrak{O})$.

A presheaf module on Y is a presheaf of sets $\mathfrak{F}$ on C together with a map of presheaves

$$\mathfrak{O} \times \mathfrak{F} \to \mathfrak{F}$$

such that for every object $a \in C$ the map $\Gamma(a;\mathfrak{O}) \times \Gamma(a;\mathfrak{F}) \to \Gamma(a;\mathfrak{F})$ defines a $\Gamma(a;\mathfrak{O})$-module structure on $\Gamma(a;\mathfrak{F})$.

A morphism

$$\mathfrak{F} \to \mathfrak{G}$$

is a morphism of presheaf modules if

$$\begin{tikzcd} \mathfrak{O} \times \mathfrak{F} \arrow[r] \arrow[d] & \mathfrak{F} \arrow[d] \\ \mathfrak{O} \times \mathfrak{G} \arrow[r] & \mathfrak{G} \end{tikzcd}$$

commutes. The category of presheaf modules on Y will be denoted $\mathrm{PMod}(Y)$.

Definition 26. Let M, N be an R-module.

Define

$$\lambda : \text{R-Mod} \to \mathrm{PMod}(Y)$$

by for all $a \in C$,

$$\lambda(M)(a) = M \otimes_R \Gamma(a;\mathfrak{O}),$$

for all $f : b \to a \in C$,

$$\lambda(M)(f) : \mathrm{Id} \otimes \mathfrak{O}(f),$$

for all $g : M \to N \in \text{R-Mod}$,

$$\lambda(g) = (a : g \otimes \mathrm{Id}).$$

Lemma 27. *Let $Y = (X, \mathfrak{T}, \mathfrak{O})$ be a ringed site. The functor λ is left adjoint to*

$$\Gamma(1;-) : PMod(Y) \to \text{R-Mod}$$

.

Proof. Let a be an object of C. Let M, N be R-modules. Let $\mathfrak{F}, \mathfrak{G} \in \mathrm{PMod}(Y)$ be presheaf modules.

Let $\varphi : \lambda(M) \to \mathfrak{G}$ be a morphism of presheaf modules. Let $\phi : M \to \Gamma(1; \mathfrak{G})$ be a morphism of presheaf modules.

Define
$$\alpha = H_{M,\mathfrak{G}} : \mathrm{Hom}(\lambda(M), \mathfrak{G}) \to \mathrm{Hom}(M, \Gamma(1; \mathfrak{G}))$$
by
$$\alpha(\varphi) = \varphi_1,$$
where φ_1 is the component of φ on the global sections.

Define
$$\beta = L_{M,\mathfrak{G}} : \mathrm{Hom}(M, \Gamma(1; \mathfrak{G})) \to \mathrm{Hom}(\lambda(M), \mathfrak{G})$$
by
$$\beta(\phi)_a = \phi \otimes_R \Gamma(a; \mathfrak{O}).$$

We will show that β and α are mutually inverse.

Let $d = \beta(\alpha(\varphi))$. Let $m \otimes g \in M \otimes_R \Gamma(a; \mathfrak{O})$. Let $p : \lambda(M)(1) \to \lambda(M)(a)$ be the projection map. Let $q : \mathfrak{G}(1) \to \mathfrak{G}(a)$ be the projection map. Then $d_a(m \otimes g) = \varphi_1(m) \otimes g$ and

$$\begin{aligned} \varphi_a(m \otimes g) &= g\varphi_a(m \otimes 1) \text{ by linearity} \\ &= g\varphi_a(p(m)) \\ &= gq(\varphi_1(m)) \text{ by naturality of } \varphi \\ &= g(\varphi_1(m) \otimes 1) \\ &= \varphi_1(m) \otimes g. \end{aligned}$$

Hence $d = \varphi$. In words, the natural transformations from presheaves of the from $\lambda(M)$ are uniquely determined by their global sections component.

Let $d = \alpha(\beta(\phi))$. Let $m \in M$. Then $d(m) = (\phi \otimes_R R)(m) = \phi(m)$. Hence $d = \phi$, which makes H and L mutual inverses.

Naturality in M *and* $\mathfrak{G}$

Let $g : N \to M$ and $h : \mathfrak{F} \to \mathfrak{G}$. Let $\rho \in \mathrm{Hom}(\lambda(N), \mathfrak{F})$. Let $k = H_{M,\mathfrak{G}}(h \circ \rho \circ \lambda(f))$. Let $l = h_1 \circ H_{N,\mathfrak{F}}(\rho) \circ f$.

Unfolding the definition for H shows that $k = h_1 \rho_1 f$ and $l = h_1 \rho_1 f$ as well. This proves naturality in M and $\mathfrak{G}$ and the adjunction between λ and $\Gamma(1; -)$. ■

Definition 28. Define

$$\Lambda : \mathsf{R\text{-}Mod} \to \mathsf{Mod}(Y)$$

by $sh \circ \lambda$.

It follows from lemma .. that we have the adjunction $\Lambda \dashv \Gamma(1; -)$.

3 Caffine objects

3.1 introduction

Definition 29 (Caffine object)**.** Let $Y = (C, \mathcal{T}, \mathfrak{O})$ be a ringed site. Let $a \in C$ be an object. We call a caffine if the unit η and co-unit ϵ of the adjunction $\Gamma(1; -) \dashv \Lambda(-)$ on Y_a are natural isomorphisms.

Example 30 (Examples of caffine objects)**.** The main example to keep in mind is $\mathsf{Spec}\ R \in \mathsf{Sch}$.

Example 31. Let $(*, \mathfrak{R})$ be a ringed space. This space is always caffine, because all presheaves are sheaves. If R is non-local, then this space is not a scheme. This is an example of a non-scheme caffine ringed space.

3.2 Restrictive maps between caffine objects

Lemma 32. *Let* $(C, \mathcal{T}, \mathfrak{O})$ *be a ringed site. Let* a *be caffine. Let* M *be a* $\Gamma(a; \mathfrak{O})$*-module. The component* $\omega^2_{\lambda(M),a}$ *at* a *of the sheafification morphism* $\omega^2_{\Lambda(M)} : \lambda(M) \to \Lambda(M)$ *is equal to the unit of* $\Lambda \dashv \Gamma(1; -)$ *in* C_a*.*

Proof. Consider the following maps, which you get by repeatedly calling on an adjunction.

$$\mathrm{Id} : \Lambda(M) \to \Lambda(M)$$

$$\omega^2_{\Lambda(M)} : \lambda(M) \to \Lambda(M) \text{ use sheafification adjunction, see lemma ..}$$

$$\omega^2_{\lambda(M),a} M \to \Gamma(a; \Lambda(\mathfrak{M})) \text{ take sections at } a$$

We took the adjunct of Id with respect to the sheafification adjunction as the first step. Then we took the adjunct of the result wrt the λ ajunction. Hence we get the adjunct of Id wrt the Λ adjunction. so the last map is actually the unit of the Λ adjunction. This map is an isomorphism because we assume a to be caffine. ∎

Theorem 33 (Morphism between caffines is restrictive)**.** *Let* $Y = (C, \mathcal{T}, \mathfrak{O})$*. Let* $f : b \to a \in C$ *be a morphism between caffine objects, then* f *is restrictive.*

Proof. Let $\mathfrak{F}$ be a quasi-coherent module on Y_a. Let $M = \Gamma(a; \mathfrak{F})$. Since a is caffine, we have $\mathfrak{F} = \Lambda(M)$.

We have to show that the adjunct, along the extension of scalars adjunction, of $\mathfrak{F}(f)$

$$\Gamma(a; \mathfrak{F}) \otimes_{\Gamma(a;\mathfrak{O})} \Gamma(b; \mathfrak{O}) \to \Gamma(b; \mathfrak{F})$$

is an isomorphism.

Consider

$$\begin{tikzcd}
\mathsf{PMod}(Y_a) \arrow[r, "sh_a"] \arrow[d, "*|_b"] & \mathsf{Mod}(Y_a) \arrow[d, "*|_b"] & \Gamma(a;\mathfrak{O})\text{-}\mathsf{Mod} \arrow[l, "\Lambda"'] \arrow[d, "-\otimes\Gamma(b;\mathfrak{O})"] \arrow[ll, bend right, "\lambda"'] \\
\mathsf{PMod}(Y_b) \arrow[r, "sh_b"] & \mathsf{Mod}(Y_b) & \Gamma(b;\mathfrak{O})\text{-}\mathsf{Mod} \arrow[l, "\Lambda"] \arrow[ll, bend left, "\lambda"]
\end{tikzcd}$$

By a previous lemma, the left square commutes. By definition the two 'triangles' commute too and the outer square commute, hence the right square also commutes. Therefore $M \otimes \Gamma(b; \mathfrak{O}) \cong \Gamma(b; \Lambda(\mathfrak{M})) \cong \Gamma(b; \mathfrak{F})$.

The requirement is not to find any isomorphism but a specific one. So I think this is not enough and we need to do some bookkeeping and see if the witnessing isomorphism is our map.

Let i be the morphism of presheaves at $\lambda(M)$ of the natural transformation ω^2 coming with sh_a as defined in lemma ?. Let j be the morphism at $\lambda(M \otimes \Gamma(b; \mathfrak{O}))$ of the natural transformation ω^2 coming with sh_a as defined in lemma ? .

Consider

$$\begin{tikzcd}
& \Lambda(M)\big|_b & & \Lambda(M \otimes \Gamma(b; \mathfrak{O})) \arrow[ll, "s_{\lambda(M)}"'] \\
\lambda(M \otimes \Gamma(b; \mathfrak{O})) \arrow[ur, "i\big|_b"] \arrow[urrr, "j"] & & &
\end{tikzcd}$$

We have seen that the component j_b at b, the global component, is an isomorphism in lemma ?. since b is caffine. The map $s_{\lambda(M)}$ is an isomorphism as constructed in lemma ?.

We will prove commutativity of the triangle. Let $g : c \to b \in Y_b$. Let $\mathfrak{M} = \lambda(M \otimes \Gamma(c; \mathfrak{O}))$. Let $x = m \otimes r \in \mathfrak{M}$.

- TODO

Evaluating everything on the terminal object, in this case on b, shows that two out of three maps are isomorphisms, hence i_b is an isomorphism. $\blacksquare$