Affine Objects

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Contents

+ Introduction. What is the point of the preliminaries? What are you generalizing/proving?

1 Preliminaries

Every section and subsection should also start with at least a sentence or two of introduction, like you did on this page.

This chapter introduces all the basic notions that are needed but not assumed to be known to the reader. We will start with a discussion of some purely categorical notions like over categories and presheaves. Secondly, we will introduce a notion of a topology on a category and look at some constructions that are relevant for us. Then we will introduce modules on ringed sites. Lastly, the notion of a scheme is introduced.

1.1 Basic Category Theory

Some categorical notions like presheaves and over categories will be introduced in this section. See [1] and [4].

Definition 1.1.1 (Presheaf category). Let C be a category. Let $a \in C$. Let $f : a' \to a$ We define the category of presheaves on C as the category of contravariant functors to the category of sets Set. We will denote it by $\hat{C}$.

Define the functor $h : C \to \hat{C}$ as follows

$$a \mapsto \mathrm{Hom}(-, a),$$

$$f \mapsto f \circ -.$$

This functor is fully faithful by the Yoneda lemma.

Notation 1.1.2. Let I, C be categories. Let $L : I \to C$ be a functor. The limit over this functor will be denoted by $\lim_{i \in I} L(i)$. The colimit will be denoted by $\operatorname{colim}_{i \in I} L(i)$.

Definition 1.1.3 (Sections functor). For any $a \in C$ define the functor

$$\Gamma(a; -) : \hat{C} \to \mathsf{Set}$$

by

$$\mathfrak{F} \to \mathfrak{F}(a).$$

For any presheaf $\mathfrak{G}$, we define

$$\Gamma(\mathfrak{G}; -) : \hat{C} \to \mathrm{Set},$$
$$\mathfrak{F} \to \mathrm{Hom}(\mathfrak{G}, \mathfrak{F}).$$

We will mostly use this for the terminal presheaf, which will allow us to compute the global sections.

Definition 1.1.4 (Over/Under categories). Let C and C' be categories. Let $F : C \to C'$ and $z \in C'$. Define the category C_z and C^z as

$$\mathrm{Obj}(C_z) := \{(a, w) \mid a \in C, w : F(a) \to z\},$$
$$\mathrm{Hom}((a, w), (b, v)) := \{f : a \to b \mid v \circ F(f) = w\},$$

and

$$\mathrm{Obj}(C^z) := \{(a, w) \mid a \in C, w : z \to F(a)\},$$
$$\mathrm{Hom}((a, w), (b, v)) := \{f : a \to b \mid F(f) \circ w = v\}.$$

We get faithful functors $C_z \to C : (a, w) \to a$ and $C^z \to C : (a, w) \to a$. We will denote them both by u.

Definition 1.1.5 (direct image). Let $f : C \to D$. Define the direct image functor $f_* : \hat{D} \to \hat{C}$ as

$$f_* = - \circ f.$$

When u is the forgetful functor $C_a \to C$ then we denote $u_* \mathfrak{F}$ sometimes by $\mathfrak{F}|_a$.

Definition 1.1.6 (inverse image of presheaves). Let C, D be a categories. Let $f : C \to D$ be a functor. Define the inverse image functor $f^{-1} : \hat{C} \to \hat{D}$ as follows. Let $\mathfrak{F} \in \hat{C}$. For any $d \in D$

$$f^{-1}(\overset{\mathfrak{F}}{F})(d) = \operatorname*{colim}_{u \in D_d\,?} \mathfrak{F}u.$$

What is u??

Do you need D to be small for the colimit to exist?

Lemma 1.1.7. *The functor f_* is left adjoint to every incarnation of f^{-1}.*

Corollary 1.1.8. *The functor f_* commutes with arbitrary colimits.*

1.2 Topology

In this section we will define a notion of a topology on a category and look at the related notions of sheaves, sites and restriction of sites.

See [4] for more details.

1.2.1 Basic

Definition 1.2.1 (Sieve). Let C be a category and $a \in C$. Define the maximal sieve $\max(a)$ on a to be the collection of all morphisms to a. In formula,

$$\max(a) = \{f \in C \mid \mathrm{Codom}(f) = a\}.$$

Does gf mean f∘g?

A sieve S on a is a subcollection of $\max(a)$ such that $gf \in S$ for any $f \in S$ and any g.

Definition 1.2.2 (Sieve category). Let C be a category and $a \in C$. The sieve category $\mathrm{Sieves}(a)$ consists of all the sieves on a as objects and inclusions of sieves as morphisms.

Definition 1.2.3 (Pullback of sieve). Let C be a category and $a, b \in C$. Let S be a sieve on a. Let $f : b \to a$. The sieve f^*S on b is given by $f^*S = \{g \in \max b : fg \in S\}$ for any $c \in C$. To show that this is actually a sieve on b, let $k : c \to c'$ and $h \in f^*S$. Hence $fh \in S$ and so $fhk \in S$. Conclude that $hk \in f^*S$. This defines a functor $f^* : \mathrm{Sieves}(a) \to \mathrm{Sieves}(b)$.

Definition 1.2.4 (Grothendieck Topology). A Grothendieck topology $\mathcal{T}$ is a family $\mathcal{T}(a)$ of 'covering' sieves for every $a \in C$ with the following conditions:

1. $\max(a) \in \mathcal{T}(a)$
2. $f^*R \in \mathcal{T}(a')$ if $R \in \mathcal{T}(a)$ for all $f : a' \to a$
3. if $f^*R \in \mathcal{T}(a')$ for all $f \in S$ with $S \in \mathcal{T}(a)$ then $R \in \mathcal{T}(a)$

Definition 1.2.5 (Basis). Let C be a category with pullbacks. A Grothendieck pretopology, or basis, $\mathcal{B}$ is a collection $\mathcal{B}(a)$ of 'covering' families $\{f_i : a_i \to a\}$ of morphisms for every $a \in C$ with the following conditions.

1. every isomorphism is a covering singleton family,

2. (Stability) The pullback of a covering family is a covering family. If $\{f_i : a_i \to a\}$ is covering and $g : b \to a$, then $\{f_i' : a_i \times_a b \to b\}$ is covering.
3. (Transitivity) If $\{f_i : a_i \to a\}$ is a covering family and $\{f_{ij} : a_{ij} \to a_i\}$ for every i, then $\{f_{ij} : a_{ij} \to a\}$ is a covering family.

Generating a topology from a basis: take any sieve containing a covering family to be a covering sieve.

Remark 1.2.6. Sometimes you have a set of families S you would like to generate the topology. So one can take the smallest basis and the topology containing these families. This will be called 'the topology generated by S'.

1.2.2 Sites

def. of sites is later

Definition 1.2.7 (continuous functor). Let $G : C \to D$ be a functor ~~between sites~~. Let $c \in C$ and let R be a covering sieve on c. The functor G is said to preserves covers if $G(R)$ generates a covering sieve. It is enough to check that G sends ~~covering families~~ to covering families, if the topology of the sites is defined by a basis. If G also preserves pullbacks then we call it continuous. See [6, Tag 00WV].

Definition 1.2.8 (cocontinuous functor). The functor G is said to lift covers or be cocontinuous if for every $R \in \mathrm{Cov}G(c)$ there is some $S \in \mathrm{Cov}c$ such that $G(S) \subset R$. See [6, Tag 00XJ].

Definition 1.2.9 (Site). A site $(C, \mathcal{T})$ is a category C with a Grothendieck topology $\mathcal{T}$. A morphism of sites is a functor that preserves pullbacks and covers.

The category Sites has as objects sites and morphisms of sites as just defined. When no confusion can arise then we will use C to denote the whole site $(C, \mathcal{T})$.

Remark 1.2.10. In most resources a morphism of sites is defined to preserve all finite limits. To get the results that we want we only need preservation of pullbacks and we need the forgetful functor $C_a \to C$ to be a morphism of sites, hence this slightly weaker notion than usual. Good

Example 1.2.11 (small site). Let X be a topological space. Let the category Open(X) consist of the opens of X as objects and inclusion as the morphisms. Define a basis on this site consisting of the families $\{U_i \to U\}$ such that $\bigcup_i U_i = U_i$. A continouous map $f : X \to Y$ induces a morphism of sites $\mathsf{Open}(Y) \to \mathsf{Open}(X)$ by sending $U \mapsto f^{-1}(U)$. In this way you can embed the category of topological spaces into the category of sites.

If X is a ringed space we can turn the site $\mathsf{Open}(X)$ into a ringed site by setting the sheaf of rings of X to be the sheaf of rings of $\mathsf{Open}(X)$.

What is a ringed site?

1.2.3 Sheaves

We will introduce the very important notion of a sheaf. See [4] for a more detailed treatment.

Definition 1.2.12 (Matching family)**.** Let C be a category. Let $\mathfrak{F}$ be a presheaf on on C. Let $a \in C$ be an object. Let R be a sieve on a. A family $\{x_i\}_{i \in R}$ with $x_i \in \Gamma(\mathrm{Dom}(i); \mathfrak{F})$ indexed by a sieve R and such that $x_{g \circ i} = \mathfrak{F}(g)(x_i)$ for any $g : b \to \mathrm{Dom}(i)$ and $b \in C$ is called a 'matching family'.

Definition 1.2.13 (Amalgamation)**.** An amalgamation of a matching family $\{x_i\}_R$ is an element $x \in \Gamma(a; \mathfrak{F})$ such that $\mathfrak{F}(i)(x) = x_i$.

Definition 1.2.14 (Sheaf)**.** Let $(C, \mathcal{T})$ be a site. Let $\mathfrak{F} \in \hat{C}$.

A presheaf that admits a unique amalgamation for every matching family is called a sheaf. The category $\mathsf{Shv}(C)$ is the full subcategory in $\hat{C}$ of all sheaves.

Definition 1.2.15 (Plus construction)**.** Let $(C, \mathcal{T})$ be a site. Let $a, a' \in C$ and $f : a \to a'$. Let $\mathfrak{F} \in \hat{C}$. Define the functor $(-)^+ : \hat{C} \to \hat{C}$ as follows.

For all $a \in C$,

$$F^+(a) = \frac{\{(R, \varphi) \mid R \in \mathcal{T}(a), \varphi \in \Gamma(R; \mathfrak{F})\}}{\sim},$$

for all morphisms $f \in C$,

$$F^+(f)([(R, \varphi)]) = [(f^* R, \varphi h(f))].$$

The equivalence relation is defined as $(R, \varphi) \sim (S, \phi)$ if $\varphi = \phi$ on some covering sieve $Q \subset R \cap S$.

Let $\mathsf{L} : \mathfrak{F} \to \mathfrak{F}'$. Then

$$(\mathsf{L}^+)_a([(\mathsf{R}, \varphi)]) = [(\mathsf{R}, \mathsf{L} \circ \varphi)]$$

This functor comes with a natural transformation $\omega : \mathrm{Id} \to (-)^+$ defined by

$$\omega_{\mathfrak{F},a}(x) = [(\max(a), y)]$$

where

$$y(i) = \mathfrak{F}(i)(x).$$

Lemma 1.2.16. *Let $\mathfrak{F}$ be a presheaf, $\mathfrak{G}$ a sheaf and $g : \mathfrak{F} \to \mathfrak{G}$ a morphism in $\hat{\mathsf{C}}$. Then g factors through ω_{F} via a unique g'.*

Proof. See [5, p. 2.10]. ∎

combine?

Lemma 1.2.17. *For every presheaf $\mathfrak{F}$, F^+ is separated.*

Proof. See [5, p. 2.11]. ∎

Lemma 1.2.18. *If $\mathfrak{F}$ is separated, then F^+ is a sheaf.*

Proof. See [5, p. 2.12]. ∎

Definition 1.2.19. Define $\mathsf{sh} = (-)^+ \circ (-)^+$. This functor is left adjoint to the inclusion $\hat{\mathsf{C}} \to \mathsf{Shv}(\mathsf{C})$ with unit

$$\omega^2_{\mathfrak{F}} = \omega_{\mathfrak{F}^+} \circ \omega_{\mathfrak{F}}.$$

1.2.4 Relative topology

We will look at what the induced topology on a over category looks like and what this implies for restriction of sheaves. See [6, Tag 03A4] for a more detailed treatement.

Remark 1.2.20. Let C be a category and $a, b \in \mathsf{C}$. Let $f : b \to a \in \mathsf{C}_a$. The map $\max(f) \to \max(b)$ sending a morphism to f to its underlying morphism in C is a bijection. Moreover composition in C and C_a are the same, so this identification respects pullback of sieves. This observation yields us that $\mathsf{Sieves}(b) = \mathsf{Sieves}(f)$.

Whenever R is a sieve on b, we will denote the corresponding sieve on f by R_f.

Definition 1.2.21 (Relative topology). Let $(\mathsf{C}, \mathcal{T})$ be a site. Let $a \in \mathsf{C}$.

Define the induced topology $\mathcal{T}_a$ on C_a by, for each $f \in \mathsf{C}_a$

$$\mathcal{T}_a(f) = \mathcal{T}(\mathrm{Dom}(f)).$$

The identification from Remark 1.2.20 implies that $\mathcal{T}_a$ is a Grothendieck topology.

Definition 1.2.22 (Oversite). Let $Y = (\mathsf{C}, \mathcal{T})$ be a site. Let $a \in \mathsf{C}$. Define the site Y_a to be the category C_a with the induced topology T_a. We will denote it by just C_a.

Lemma 1.2.23. *The functor* $u : \mathsf{C}_a \to \mathsf{C}$ *is a morphism of sites that is cover lifting.*

Proof. Clearly u preserves pullbacks. By Remark 1.2.20 it is also immediate that u preserves covers and lifts covers. ∎

ringed?

Definition 1.2.24 (Over ringed site). Let $Y = (\mathsf{C}, \mathcal{T}, \mathfrak{O})$ be a site. Let $a \in \mathsf{C}$. Define the ringed site Y_a to be the site C_a with the structure sheaf $u_*\mathfrak{O}$ where $u : \mathsf{C}_a \to \mathsf{C}$ the forgetful functor. This makes sense because u is a morphism of sites, see Lemma 1.2.23.

Whenever u is used as a morphism of ringed sites it stands for (u, Id)

?

1.3 Modules

Presheaf modules and sheaf modules on a ringed site will be introduced in this section. Next we will introduce two main functors λ and Λ. The functors λ and Λ introduced here will be used extensively. See [6, Tag 03A4] for more detail.

Definition 1.3.1 (Presheaf modules). Let $(\mathsf{C}, \mathcal{T}, \mathfrak{O})$ be a ringed site.

A presheaf module on this ringed site is a presheaf of sets $\mathfrak{F}$ on C together with a map of presheaves

$$\mathfrak{O} \times \mathfrak{F} \to \mathfrak{F}$$

such that for every object $a \in \mathsf{C}$ the map $\Gamma(a; \mathfrak{O}) \times \Gamma(a; \mathfrak{F}) \to \Gamma(a; \mathfrak{F})$ defines a $\Gamma(a; \mathfrak{O})$-module structure on $\Gamma(a; \mathfrak{F})$.

A morphism

$$\mathfrak{F} \to \mathfrak{G}$$

is a morphism of presheaf modules if

$$\begin{tikzcd}
\mathfrak{O} \times \mathfrak{F} \arrow[r] \arrow[d] & \mathfrak{F} \arrow[d] \\
\mathfrak{O} \times \mathfrak{G} \arrow[r] & \mathfrak{G}
\end{tikzcd}$$

commutes. The category of presheaf modules on C will be denoted $\mathsf{PMod}(\mathfrak{O})$. How about $\mathsf{Mod}(\mathfrak{O})$?

Definition 1.3.2. Let $\mathfrak{F}$ be a sheaf of modules on $(\mathsf{C}, \mathfrak{T}, \mathfrak{O})$. It is called quasi-coherent if the following holds. For any object $a \in \mathsf{C}$ there exists a covering sieve S such that for any map $f : b \to a$ in S there exists a presentation

$$\mathfrak{O}|_b{}^{\oplus I} \to \mathfrak{O}|_b{}^{\oplus J} \to \mathfrak{F}|_b \to 0$$

It is enough to have presentation for a generating set of S.

Quasi-coherent modules form a full subcategory of the category of sheafs of modules over $(\mathsf{C}, \mathfrak{T}, \mathfrak{O})$ which are denoted by $\mathsf{Qcoh}(\mathfrak{O})$.

Lemma 1.3.3. *Let $f : \mathsf{C} \to \mathsf{D}$ be a morphism of* ringed? *sites. Let $\mathfrak{F}$ be a quasi-coherent sheaf of modules on D. Then $f_*\mathfrak{F}$ is quasi-coherent.*

Proof. ∎

Definition 1.3.4. Let $(\mathsf{C}, \mathfrak{T}, \mathfrak{O})$ be a ringed site. Let $R = \Gamma(1; \mathfrak{O})$. Let M, N be an R-module.

Define

$$\lambda : R\text{-}\mathsf{Mod} \to \mathsf{PMod}(\mathfrak{O})$$

by for all $a \in \mathsf{C}$,

$$\lambda(M)(a) = M \otimes_R \Gamma(a; \mathfrak{O}),$$

for all $f : b \to a \in C$,

$$\lambda(M)(f) = \mathrm{Id} \otimes \mathfrak{O}(f),$$

for all $g : M \to N \in$ R-Mod,

$$\lambda(g) = (a : g \otimes \mathrm{Id}).$$

Lemma 1.3.5. *Let* $(C, \mathcal{T}, \mathfrak{O})$ *be a ringed site. Let* $R = \Gamma(1; \mathfrak{O})$. *The functor* λ *is left adjoint to*

$$\Gamma(1; -) : \mathrm{PMod}(\mathfrak{O}) \to R\text{-Mod}.$$

Proof. Let M, N be R-modules. Let $\mathfrak{F}, \mathfrak{G} \in \mathrm{PMod}(\mathfrak{O})$ be presheaf modules.

Let $\varphi : \lambda(M) \to \mathfrak{G}$ be a morphism of presheaf modules. Let $\phi : M \to \Gamma(1; \mathfrak{G})$ be a morphism of R-modules.

Define

$$\alpha = H_{M,\mathfrak{G}} : \mathrm{Hom}(\lambda(M), \mathfrak{G}) \to \mathrm{Hom}(M, \Gamma(1; \mathfrak{G}))$$
$$: \varphi \mapsto \varphi_1$$

never used

where φ_1 is the component of φ on the global sections.

Define

$$\beta = L_{M,\mathfrak{G}} : \mathrm{Hom}(M, \Gamma(1; \mathfrak{G})) \to \mathrm{Hom}(\lambda(M), \mathfrak{G})$$

by, for each $a \in C$

$$\beta(\phi)_a = \phi \otimes_R \Gamma(a; \mathfrak{O}).$$

We will show that β and α are mutually inverse.

Let $\varphi \in \mathrm{Hom}(\lambda(M), \mathfrak{G})$ and $a \in C$. Let $d = \beta(\alpha(\varphi))$. Let $m \otimes g \in M \otimes_R \Gamma(a; \mathfrak{O})$. Let $p : \lambda(M)(1) \to \lambda(M)(a)$ be the projection map. Let $q : \mathfrak{G}(1) \to \mathfrak{G}(a)$ be the projection map. Then $d_a(m \otimes g) = \varphi_1(m) \otimes g$ and

$$\begin{aligned}\varphi_a(m \otimes g) &= g\varphi_a(m \otimes 1) \text{ by linearity}\\ &= g\varphi_a(p(m))\\ &= gq(\varphi_1(m)) \text{ by naturality of } \varphi\\ &= g(\varphi_1(m) \otimes 1)\\ &= \varphi_1(m) \otimes g.\end{aligned}$$

Hence $d = \varphi$. In words, the natural transformations from presheaves of the from $\lambda(M)$ are uniquely determined by their global sections component.

Let $d = \alpha(\beta(\phi))$. Let $m \in M$. Then $d(m) = (\phi \otimes_R R)(m) = \phi(m)$. Hence $d = \phi$, which makes α and β mutual inverses. Naturality is straightforward to check and is omitted. ∎

Definition 1.3.6. Let $(\mathsf{C}, \mathcal{T}, \mathfrak{O})$ be a ringed site. Define

$$\Lambda : \mathsf{R\text{-}Mod} \to \mathsf{Mod}(\mathfrak{O}) \leftarrow \text{undefined}$$

by $\mathrm{sh} \circ \lambda$.

It follows by that we have the adjunction $\Lambda \dashv \Gamma(1; -)$.

This functor is the generalisation of [6, Tag 01BH] to general sites, so Λ on the site corresponding to a ringed space(we will define this later) will coincide with the construction defined in the Stacks Project.

missing space

you've already used it on p. 7.

1.4 Restricting and module functors

The "let" means you are introducing new data. If $f^\#$ is part of the data of a morph. of ringed sites already, you should say "Suppose that $f^\#$ is..."

Even better would be to remind me where $f^\#$ goes.

Lemma 1.4.1. *Let $f : (\mathsf{D}, \mathcal{S}, \mathfrak{U}) \to (\mathsf{C}, \mathcal{T}, \mathfrak{O})$ be a morphism of ringed site. Let $f^\#$ be an isomorphism. We get a natural isomorphism $f_* \circ \lambda \Rightarrow \lambda \circ (- \otimes_{\Gamma(1;\mathfrak{O})} \Gamma(1; \mathfrak{U}))$.*

Proof. Define the natural transformation $t : \lambda \circ (- \otimes_{\Gamma(1;\mathfrak{O})} \Gamma(1; \mathfrak{U})) \Rightarrow u_* \circ \lambda$, by for each $\Gamma(1; \mathfrak{O})$-module M and for each $a \in \mathsf{D}$,

$$t_{M,a} : M \otimes_{\Gamma(1;\mathfrak{O})} \Gamma(1; \mathfrak{U}) \otimes_{\Gamma(1;\mathfrak{U})} \Gamma(a; \mathfrak{U}) \to M \otimes_{\Gamma(1;\mathfrak{O})} \Gamma(a; \mathfrak{O}),$$

$$m \otimes r \otimes s \mapsto m \otimes rf^{\#,-1}(s).$$

Every component $t_{M,f}$ is an isomorphism by basic commutative algebra and the fact that $f^\#$ is an isomorphism ∎

Example 1.4.2. The morphism of ringed sites u is an example where Lemma 1.4.1 holds.

Remind me what u is. Or cite an earlier example. Or both.

Lemma 1.4.3. *Let* $F : (C, \mathcal{T}) \to (D, \mathcal{S})$ *be a morphism of sites that lift covers. Then* $F_*(-)^+ = (-)^+F_*$ *and hence also* $F_* \circ sh = sh \circ F_*$. ← This is a very clear statement.

Proof. Let $c \in C$. Let $R \in Covc$. Let $\mathfrak{G}$ be a presheaf on D. Let $s = \{s_i \mid i \in R\}$ be a matching family in $F_*\mathfrak{G}$ on c indexed by R. So $s_i \in \Gamma(F(Dom(i)); \mathfrak{G})$. By the cover preserving assumption we know $F(R)$ generates a covering sieve S on $F(c)$. Define the natural transformation $T : F_*(-)^+ \to (-)^+F_*$ by

$$s \mapsto \{t_j \mid j \in S\}$$

where $t_j = F(h)s_i$ for some factorisation $j = F(i)h$. Such a factorisation always exists since S is generated by $F(R)$. This assignment is well-defined since F preserves pullbacks, see proof of [3, Lemma 2.3.3].

We will prove that T is injective. Let $s' = \{s'_i \mid i \in R'\}$. Assume $T(s) = T(s')$ then $\{t_j \mid j \in S\} = \{t'_{j'} \mid j' \in S'\}$, hence there exists covering sieve $Q \subset S \cap S'$ such that $t_k = t'_k$ for every $k \in Q$. Since F lifts covers, there exists covering sieve $P \subset R \cap R'$ on c such that $F(P)$ generates a sieve $P' \subset Q$. Hence $t_k = F(h)s_k = F(h)s'_k = t'_k$ with $k \in P'$ and $k = F(k)h$. Take $h = Id$ to conclude that $s = s'$ on P, hence they are in the same equivalence class.

Now comes surjectivity. Let $t = \{t_j \mid j \in S\} \in ..$ on $F(c)$. We get a covering sieve R such that $F(R) \subset S$. Consider this matching family now on the covering sieve generated by $F(R)$, which amounts to taking a different representing element from the equivalence class. So now $F(R)$ generates S. For any $j \in S$ we get a factorisation $j = F(i)h$ for some $i \in R$. We want to have $s_i = t_{F(i)}$. This produces a matching family because restriction in $F_*\mathfrak{G}$ is exactly the same as in $\mathfrak{G}$. Note that now $T(s) = t$. Hence we got surjectivity. ∎

Example 1.4.4. The morphism of ringed sites u satisfies the conditions of Lemma 1.4.3 and so commutes with sheafification.

Maybe combine with Example 1.4.2, or even move to the beginning of the subsection as motivation.

Corollary 1.4.5. *We have* $F_*\omega = \omega$ *and hence also* $F_*\omega^2 = \omega^2$

Lemma 1.4.6 (Λ commutes with restriction). *Let* $(C, \mathcal{T}, \mathfrak{O})$ *be a ringed site. Let* $f : (C, \mathcal{T}) \to (D, \mathcal{S})$ *be a morphism of sites that lift covers. Let* $f^{\#}$ *be an isomorphism.*

We have a natural isomorphism

$$f_* \circ \Lambda \Rightarrow \Lambda \circ (- \otimes_{\Gamma(1;\mathfrak{O})} \Gamma(1; \mathfrak{U})).$$

Proof. Follows from Lemmas 1.4.1 and 1.4.3. ■

Again, "Suppose"

Lemma 1.4.7. *Let* $f : (D, \mathcal{S}, \mathfrak{U}) \to (C, \mathcal{T}, \mathfrak{O})$ *be a morphism of ringed sites that is cover lifting. Let* $f^{\#}$ *be an isomorphism. Let* $\mathfrak{F}$ *be a quasi-coherent module on* C. *Let* $M = \Gamma(1; \mathfrak{F})$. *Consider* $\epsilon : \Lambda(M) \to \mathfrak{F}$. *Then* $f_*\epsilon : \Lambda(M \otimes_{\Gamma(1;\mathfrak{O})} \Gamma(1; \mathfrak{U}))) \to f_*\mathfrak{F}$ *is the counit of the adjunction* $\Lambda \dashv \Gamma(1; -)$ *on* D.

Proof. Let $a \in D$. We will show that $f_*\epsilon$ corresponds to the same morphism $\lambda(M \otimes_{\Gamma(1;\mathfrak{O})}) \to f_*\mathfrak{F}$ as the counit. By the universal property of the sheafification, this implies that $f_*\epsilon$ is the counit.

We have $(f_*\epsilon)_a(\omega^2(m \otimes r)) = rm$, which is the same as for the counit. ■

1.5 Schemes

We will recap some definitions in scheme theory that we use later. See [7, 2] for thorough treatments of scheme theory.

Definition 1.5.1 (Spectrum of a ring). Let R be a ring. The spectrum $\operatorname{Spec} R$ of R is the locally ringed space defined as follows. The underlying set is the set of prime ideals of R. The (Zariski) topology is generated by the basis of distinguised opens $D(f) = \{\mathfrak{p} \subset R | f \notin \mathfrak{p}\}$. The sheaf of rings is given on this basis by

$$D(f) \mapsto R_f.$$

A distinguised open $D(f)$ of $\operatorname{Spec}(R)$ viewed as locally ringed space is isomorphic to $\operatorname{Spec}(R_f)$, where the inclusion $\operatorname{Spec}(R_f) \to \operatorname{Spec}(R)$ corresponds to the canonical map $R \to R_f$.

The functor

$$\mathsf{Spec} : \mathsf{Rng} \to \mathsf{LRSpaces}$$

is left adjoint to

$$\Gamma(1; -) : \mathsf{LRSpaces} \to \mathsf{Rng},$$

see [6, Tag 01I1].

Definition 1.5.2 (Scheme)**.** We call the locally ringed space $\mathrm{Spec}(R)$ an affine scheme.

A scheme S is a locally ringed space that admits a covering of affine schemes. A morphism of schemes is a morphism of locally ringed spaces. The category of schemes we will denote by Sch.

Definition 1.5.3 (Sheaf of algebras)**.** A sheaf of algebras $\mathfrak{F}$ on a ringed site $(\mathcal{C}, \mathcal{T}, \mathfrak{O})$ is a sheaf of rings that comes with a (structure) morphism of sheaf of rings $\mathfrak{O} \to \mathfrak{F}$.
sheaves

Definition 1.5.4 (Relative spec)**.** Let X be a scheme. Let $\mathfrak{S}$ be a sheaf of algebras on X that is quasi-coherent as a sheaf of modules.

Define the relative spectrum of $\mathfrak{S}$ over X to be the scheme

$$\mathrm{Rspec}\,\mathfrak{S} \to X$$

that you get by glueing the spectra $\mathrm{Spec}(\Gamma(V; \mathfrak{S})) \to V \subset X$ for every affine open V. See [6, Tag 01LW].

Definition 1.5.5 (Tilde functor)**.** Let $\mathrm{Spec}(R)$ be an affine scheme. Let M be a R-module. Define $\widetilde{M}$ to be the unique sheaf on $\mathrm{Spec}(R)$ with

$$\widetilde{M} : D(f) \mapsto M_f.$$

See [6, Tag 01HR].

Remark 1.5.6. This functor(with this notation) is commonly used in algebraic geometry texts. By [6, Tag 01I7] and uniqueness of left adjoints Λ and $\widetilde{-}$ are canonically isomorphic on spaces, hence on small Zariski sites. We will only use Λ in subsequent sections.

1.6 Small Zariski site

Lemma 1.6.1. *Let* R *be a ring. Let* M *be a* R*-module. Consider* $\lambda(M), \Lambda(M)$ *as sheaf modules on* $\mathrm{Open}(\mathrm{Spec}(R))$*. Then*

$$\omega^2_{\lambda(M),\mathrm{Spec}(R)} : \Gamma(\mathrm{Spec}(R); \lambda(M)) \to \Gamma(\mathrm{Spec}(R); \Lambda(M))$$

is an isomorphism.

Proof. Let $\mathfrak{p} \subset R$ be a prime ideal. As stated in Definition 1.3.6, we may use results from [6, Tag 01BH] in this setting. We will use that $\Lambda(M)_\mathfrak{p} = M \otimes_R R_\mathfrak{p}$. By naturality, localized at $\mathfrak{p}$, the map ω^2 sends m to $m \otimes 1 \in M \otimes_R R_\mathfrak{p}$, hence is the inverse of the multiplication map which is an isomorphism. Hence globally ω^2 is an isomorphism. ∎

Corollary 1.6.2. *Let* X *be a scheme. Consider* $\lambda(M), \Lambda(M)$ *as sheaf modules on* $\mathrm{Open}(X)$*. Let* $\mathrm{Spec}(R)$ *be an open subset of* X*. Let* M *be a* $\Gamma(X; \mathfrak{O})$*-module. Then* $\omega^2_{\lambda(M),\mathrm{Spec}(R)} : \Gamma(\mathrm{Spec}(R); \lambda(M)) \to \Gamma(\mathrm{Spec}(R); \Lambda(M))$ *is an isomorphism.*

?

Defined here

Note that η_M is equal to $\omega^2_{\lambda(M),\mathrm{Spec}(R)}$ so we get the following.

Corollary 1.6.3. *Consider the adjunction* $\Lambda(-) \dashv \Gamma(1; -)$ *on* $\mathrm{Open}(\mathrm{Spec}(R))$*. The unit* $\eta_M : M \to \Gamma(\mathrm{Spec}(R); \Lambda(M))$ *is an isomorphism.*

Lemma 1.6.4. *Let* $\mathfrak{F}$ *be a sheaf of modules on scheme* X*.* $\mathfrak{F}$ *is quasi-coherent on* X *if and only if for any open* $\mathrm{Spec}(R) \subset X$ *the sheaf* $\mathfrak{F}|_{\mathrm{Spec}(R)}$ *is isomorphic to* $\Lambda(M)$ *with* $M = \Gamma(\mathrm{Spec}(R); \mathfrak{F})$*.*

Proof. $\Rightarrow$: By assumption we get local presentations indexed by a covering. Let $\bigcup_{i \in I} U_i = X$ be this covering. Assume without loss of generality that it is an affine open covering. Let $U_i = \mathrm{Spec}(R_i)$. Let $\mathfrak{O}_{U_i}^{\oplus K} \to \mathfrak{O}_{U_i}^{\oplus J} \to \mathfrak{F}|_{U_i} \to 0$ be one of the given presentations. Taking global sections gives us an exact sequence

$$R_i^{\oplus K} \to R_i^{\oplus J} \to \Gamma(U_i; \mathfrak{F}) \to 0.$$

Tensoring it with the localisation $R_{i,f}$ for any $f \in R_i$ yields

$$R_{i,f}^{\oplus K} \to R_{i,f}^{\oplus J} \to \Gamma(U_i; \mathfrak{F}) \otimes R_{i,f} \to 0.$$

Taking sections at $D(f)$ from the sheaf sequence yields

$$R_{i,f}^{\oplus K} \to R_{i,f}^{\oplus J} \to \Gamma(D(f);\mathfrak{F}) \to 0.$$

Hence $\mathfrak{F}|_{U_i}$ is the unique sheaf with $D(f) \mapsto \Gamma(U_i;\mathfrak{F})_f$, which we defined to be $\Lambda(\Gamma(U_i;\mathfrak{F}))$. By the affine communication lemma, this property holds for any affine and not just for the affines in this covering.

$\Leftarrow$: Let $M = \Gamma(X;\mathfrak{F})$. Take a presentation $R^{\oplus I} \to R^{\oplus J} \to M$ and apply $\Lambda(-)$. Then note that $\Lambda(-)$ commutes with arbitrary colimits since it is a left adjoint, see lemma ?. We have $\Lambda(R) = \mathfrak{O}_{\mathsf{Spec}(R)}$ so we get a presentation $\mathfrak{O}_{\mathsf{Spec}(R)}^{\oplus I} \to \mathfrak{O}_{\mathsf{Spec}(R)}^{\oplus J} \to \mathfrak{F}$ on every affine open subset $\mathsf{Spec}(R) \subset X$, hence $\mathfrak{F}$ is quasi-coherent. ∎

Corollary 1.6.5. *Consider the adjunction $\Lambda(-) \dashv \Gamma(1;-)$. For quasi-coherent sheaf $\mathfrak{F}$ the counit $\epsilon_{\mathfrak{F}} : \Lambda(M) \to \mathfrak{F}$ is an isomorphism.*

1.7 Big Zariski Site

In this section we will introduce the big Zariski ringed site and look at how quasi-coherence and Λ behave on this site.

Definition 1.7.1 (Big Zariski site). Define the big Zariski site to be $(\mathsf{Sch}, \mathcal{T}, \mathfrak{O})$ with the following components. The underlying category is Sch. The topology T is generated by the basis cosisting of the covering families $\{X_i \xrightarrow{f_i} X\}$ where f_i is an open immersion and $\bigcup_i f_i(X_i) = X$. The sheaf of rings $\mathfrak{O}$ sends $(U,\mathfrak{O}) \to (X,\mathfrak{O})$ to $\Gamma(U;\mathfrak{O})$.

We will mostly be interested in the site Sch_X for a scheme X.

Definition 1.7.2 (k). Let X be a scheme. Define the functor $k : \mathsf{Open}(X) \to \mathsf{Sch}_X$ by $U \mapsto ((U, \mathfrak{O}_U), i)$ where $i : U \to X$ is the inclusion of the open subscheme into X.

We will show that it preserves limits and covers. The terminal $X \in \mathsf{Open}(X)$ is send to the terminal $X \to X$. Let $U \to V$ and $W \to V$ be two morphism in $\mathsf{Open}(X)$. We have $k(U \cap W) = U \cap W \to X$ which is the pullback of $k(U) \to k(V)$ and $k(W) \to k(V)$.

Let $S = \{D(f_i) \to \mathsf{Spec}(R)\}$ be one of the generating family in $\mathsf{Open}(X)$. Note that $k(D(f_i))$ is isomorphic to the object $\mathsf{Spec}(R_{f_i}) \to \mathsf{Spec}(R)$. Hence $k(S)$ generates a covering sieve on Sch and hence on Sch_X.

So k is a morphism of sites as defined in Definition 1.2.9.

Lemma 1.7.3. *Let* X *be a scheme. Consider* $k : \mathsf{Open}(X) \to \mathsf{Sch}_X$. *Then* $k_*\Lambda = \Lambda$

Proof. ∎

Lemma 1.7.4.

Lemma 1.7.5. *Let* X *be a scheme. Let* $\mathfrak{F}$ *be a quasi-coherent sheaf on* Sch_X. *Then* $k_*\mathfrak{F}$ *is quasi-coherent.*

Proof. ∎

Lemma 1.7.6. *Let* $X = \mathrm{Spec}(R)$ *be a scheme. Let* $\mathfrak{F}$ *be a quasi-coherent sheaf on* Sch_X. *Let* $M = \Gamma(X;\mathfrak{F})$. *Then* $\epsilon_{\mathfrak{F}} : \Lambda(M) \to \mathfrak{F}$ *is an isomorphism.*

You're calling these both ε

Proof. By lemma ?, k_* sends quasi-coherent sheafs to quasi-coherent sheafs. so $k_*\mathfrak{F} = \Lambda(M)$, where $k : \mathsf{Open}(X) \to \mathsf{Sch}_X$. Hence $\Gamma(D(f);\mathfrak{F}) = \Gamma(D(f);\Lambda\mathfrak{M}) = M_f$ for any $f \in R$.

Let $D(f_i)|_{\mathfrak{O}}{}^{\oplus J} \xrightarrow{\alpha_i} D(f_i)|_{\mathfrak{O}}{}^{\oplus K} \to D(f_i)|_{\mathfrak{F}} \to 0$ be a presentation with $(f_i) = (1)$. Note that the presheaf cokernel $\mathrm{Coker}(\alpha_i)$ of, the sheaf morphism is $\lambda(\mathrm{Coker}(\alpha_{i,D(f_i)}))$ where $\alpha_{i,D(f_i)}$ is the component at $D(f_i)$ of α_i. So $D(f_i)|_{\mathfrak{F}} = \Lambda(M_f)$ since ω^2 is iso for affines.

Hence

$$\begin{aligned}\Gamma(D(f) \times_X \mathrm{Spec}(S);\mathfrak{F}) &= \Gamma(D(f) \times_X \mathrm{Spec}(S);\Lambda\mathfrak{F})\\ &= \Gamma(\mathrm{Spec}(S_f);\Lambda\mathfrak{M}_f)\\ &= M_f \otimes S_f\\ &= M \otimes S_f.\end{aligned}$$

By the sheaf property it follows that $\Gamma(\mathrm{Spec}(S);\mathfrak{F}) = \Gamma(X;\mathfrak{F}) \otimes S$, hence the counit $\Lambda(M) \to \mathfrak{F}$ is an isomorphism. ∎

Lemma 1.7.7. *Let* $X = \mathrm{Spec}(R)$ *be a scheme. Let* M *be a* R*-module. We work on the ringed site* Sch_X. *The counit* $\epsilon_M : M \to \Gamma(X;\Lambda(M))$ *is an isomorphism*

Proof. Let $\Lambda_a(M)$ be $\Lambda(M)$ over $\mathsf{Open}(X)$. By Lemma 1.7.3, we have $\Lambda_a(M) = k_*\Lambda(M)$. Hence $\Gamma(X;\Lambda(M)) = \Gamma(X;\Lambda_a(M))$. Moreover $\Gamma(X;\Lambda(M)) = M$ by Corollary 1.6.3. ∎

The idea is to conclude that $\mathrm{Qcoh}(\mathrm{Sch}_X) \cong \Gamma(X,\mathcal{O})$-Mod, right? It would be helpful to say that somewhere.

2 Restrictive morphisms

This section will introduce the notion of a restrictive morphism We will see some examples, non-examples and results in the category of schemes and see that this notion is closely related to affinenes.

For some of the examples and results see the chapter on quasi-coherent modules in [7].

Definition 2.0.1 (Restrictive morphism). Let $(C, \mathcal{T}, \mathfrak{O})$. A morphism $f : a \to b \in C$ is called restrictive if for every quasi-coherent module $\mathfrak{G}$ on C_b the morphism

$$\widehat{f} : \Gamma(b; \mathfrak{G}) \otimes_{\Gamma(b;\mathfrak{O})} \Gamma(a; \mathfrak{O}) \to \Gamma(a; \mathfrak{G}) \tag{2.1}$$

is an isomorphism.

Remark 2.0.2. Let us be in the setting of Definition 2.0.1. Assume $\mathfrak{G} = \Lambda(M)$ where $M = \Gamma(1; \mathfrak{G})$. Consider $\omega^2_{\mathfrak{G},a} : \lambda(M)(a) \to \Lambda(M)(a)$. This is map is equal to $\widehat{f}$.

Remark 2.0.3. Assume we are in the context of Definition 2.0.1. Assume $\mathfrak{G} = \Lambda(M)$ for some $\Gamma(b; \mathfrak{O})$-module, then $\widehat{f} = \omega^2_a$ for the sheafification transformation $\omega^2_{\lambda(M)} : \lambda(M) \to \Lambda(M)$. Duplicate?

Example 2.0.4. In $\mathsf{Sch}_{\mathsf{Spec}(A)}$ the morphism $\mathrm{Spec}(A_f) \to \mathrm{Spec}(A)$ is restrictive. Let $\mathfrak{G}$ be a quasi-coherent sheaf on $\mathsf{Sch}_{\mathsf{Spec}(A)}$. This implies that $\mathfrak{G} = \Lambda(\Gamma(\mathrm{Spec}(A); \mathfrak{G}))$, see ?. The morphism

$$\Gamma(\mathrm{Spec}(A); \mathfrak{G}) \otimes_{\Gamma(1;\mathfrak{O})} \Gamma(\mathrm{Spec}(A_f); \mathfrak{O}) \to \Gamma(\mathrm{Spec}(A_f); \mathfrak{G}) = \Gamma(A; \mathfrak{G})_f,$$

$$m \otimes r \to rm$$

is an isomorphism by basic commutative algebra.

Example 2.0.5. Let R be a ring. Consider the open immersion $U = \mathrm{Spec}(R[x, y]) \setminus \{(x, y)\} \to \mathrm{Spec}(R[x, y])$ and the quasi-coherent sheaf $\mathfrak{G} = \Lambda(\frac{R[x,y]}{xy})$. The global sections of this sheaf are $\frac{R[x,y]}{xy}$, as shown in ?

3 Caffine objects

3.1 Caffine objects

In this section we will introduce caffine objects, see some examples and non-examples and prove some properties. Most of these results will be generalisations of their counterpart for affine schemes.

Definition 3.1.1 (Caffine object). Let $(\mathsf{C}, \mathcal{T}, \mathfrak{O})$ be a ringed site. Let $\mathfrak{a} \in \mathsf{C}$ be an object. We call $\mathfrak{a}$ caffine if the unit η and co-unit ϵ of the adjunction $\Lambda \dashv \Gamma(\mathfrak{a}; -)$ on $\mathsf{C}_{\mathfrak{a}}$ are natural isomorphisms for any $\Gamma(\mathfrak{a}; \mathfrak{O})$-module M and any quasi-coherent $\mathfrak{F}$ on $\mathsf{C}_{\mathfrak{a}}$

In other words, that $\Lambda : \Gamma(\mathfrak{a}; \mathfrak{O})\text{-Mod} \to \mathsf{Qcoh}(\mathfrak{O})$ is an equivalence of categories with $\Gamma(\mathfrak{a}; -)$ as pseudo-inverse and the unit and co-unit as witnessing natural isomorphisms.

Example 3.1.2 (Examples of caffine objects). The main example to keep in mind is $\mathsf{Spec}(R) \in \mathsf{Sch}$. See Lemmas 1.7.6 and 1.7.7 for proofs that the unit and co-unit are isomorphisms.

Example 3.1.3. The scheme $\mathbb{P}^1$ is not caffine in Sch. The counit at the quasi-coherent sheaf of modules $\mathfrak{O}(-1)$ has signature $\epsilon : \Lambda(0) \to \mathfrak{O}(-1)$. Since $\mathfrak{O}(-1)$ is locally free of degree 1, this cannot be an isomorphism.

Definition 3.1.4 (caffine cover). Let $(\mathsf{C}, \mathcal{T}, \mathfrak{O})$ be a ringed site. A family of maps $\{\mathfrak{a}_i \to \mathfrak{a}\}$ is called a caffine covering of $\mathfrak{a}$ if every $\mathfrak{a}_i$ is caffine and the family is a covering family.

Definition 3.1.5. We say that a ringed site $(\mathsf{C}, \mathcal{T}, \mathfrak{O})$ has enough affines if any object admits a caffine covering.

Lemma 3.1.6. *Let $(\mathsf{C}, \mathcal{T}, \mathfrak{O})$ be a ringed site. Let $\mathfrak{a} \in \mathsf{C}$. Let $\{\mathfrak{b}_i \to \mathfrak{a}\}$ be a caffine covering on $\mathfrak{a}$. Assume every map $\mathfrak{b}_i \xrightarrow{i} \mathfrak{a}$ is restrictive. Then the counit ϵ of the adjunction $\Lambda_{\mathfrak{a}}(-) \dashv \Gamma(\mathfrak{a}; -)$ is a natural isomorphism.*

This requires that the # of the morphism of ringed sites be an isomorphism. It would be helpful to mention there that this holds for slice categories.

Proof. Let $\mathfrak{F}$ be a quasi-coherent sheaf module. Set $M = \Gamma(a; \mathfrak{F})$. Set $M_i = \Gamma(b_i; \mathfrak{F})$. Set $\beta_i = i_* \epsilon_{\mathfrak{F},a}$. By Lemma 1.4.7, we have$\beta_i \cong \epsilon_{i_*\mathfrak{F},b_i}$. Since b_i is caffine β_i is an isomorphism. Hence ϵ is an isomorphism between sheafs, since it is locally an isomorphism. ■

Lemma 3.1.7. *Let $(C, \mathcal{T}, \mathfrak{O})$ be a ringed site. Let $a \in C$. Let M be a $\Gamma(a; \mathfrak{O})$-module. The component*

$$\omega^2_{\lambda(M),a} : \lambda(M)(a) \to \Lambda(M)(a)$$

at Id_a of the sheafification morphism

$$\omega^2_{\lambda(M)} : \lambda(M) \to \Lambda(M)$$

is equal to the unit of $\Lambda \dashv \Gamma(1; -)$ in C_a.

Proof. Consider the following maps, which you get by repeatedly calling on an adjunction.

$$\mathrm{Id} : \Lambda(M) \to \Lambda(M)$$

$$\omega^2_{\Lambda(M)} : \lambda(M) \to \Lambda(M) \text{ use sheafification adjunction}$$

$$\omega^2_{\lambda(M),a} M \to \Gamma(a; \Lambda(M)) \text{ take sections at } a$$

We took the adjunct of Id with respect to the sheafification adjunction as the first step. Then we took the adjunct of the result wrt the λ ajunction. Hence we get the adjunct of Id wrt the Λ adjunction. so the last map is actually the unit of the Λ adjunction. ■

Corollary 3.1.8. *Let $(C, \mathcal{T}, \mathfrak{O})$ be a ringed site. Let $a \in C$ be caffine. Then $\omega^2_{\lambda(M),a}$ is an isomorphism for any $\mathfrak{O}(a)$-module M.*

Theorem 3.1.9 (Morphism between caffines is restrictive)**.** *Let $Y = (C, \mathcal{T}, \mathfrak{O})$. Let $f : b \to a \in C$ be a morphism between caffine objects, then f is restrictive.*

Proof. Let $\mathfrak{F}$ be a quasi-coherent module on Y_a. Let $M = \Gamma(a; \mathfrak{F})$. Since a is caffine, we have $\mathfrak{F} = \Lambda(M)$.

We have to show that

$$\widehat{f} : \Gamma(a; \mathfrak{F}) \otimes_{\Gamma(a;\mathfrak{O})} \Gamma(b; \mathfrak{O}) \to \Gamma(b; \mathfrak{F})$$

is an isomorphism, where $\widehat{f}$ as defined in Equation (2.0.1).

Note that $\widehat{f}$ is the component at b of the natural transformation $\omega^2_{\lambda(\Gamma(1;\mathfrak{F}))}$, see Remark 2.0.2. Since b is caffine, this component is an isomorphism by Corollary 3.1.8. ■

3.2 Caffine schemes

in which ringed site? Sch? Sch$_X$? Open(X)?

Let X be a caffine scheme. Set $R = \Gamma(X;\mathfrak{O})$. We will prove that the canonical morphism

$$\epsilon_X : X \longrightarrow \mathrm{Spec}(\Gamma(X;\mathfrak{O}))$$

is an isomorphism. This morphism sends $x \in X$ to $\ker x$ as defined later.

Definition 3.2.1 (Support of module). Let $\mathfrak{M}$ be a $\mathfrak{O}$-module. The support $\mathrm{Supp}(\mathfrak{M})$ of this module is the subspace

$$\{x \in X \mid \mathfrak{M}_x = 0\} \subset X.$$

Definition 3.2.2 (Set cut out by ideal). Define the set cut out by an ideal I of $\Gamma(X;\mathfrak{O})$ to be

$$V_X(I) = \mathrm{Supp}(\Lambda_X(\frac{\mathfrak{O}(X)}{I})).$$

These can be replaced by R if you want.

Definition 3.2.3 (Locus of a point). Let $(X,\mathfrak{O})$ be a scheme. Define the locus of point $x \in X$ to be

$$\ker(x) = \ker(\Gamma(X;\mathfrak{O}) \longrightarrow \kappa(x)).$$

Note that $\ker(x)$ is a prime ideal of $\Gamma(X;\mathfrak{O})$.

Definition 3.2.4. For a global section $a \in R$ define

$$D_X(a) = \{x \in X \mid a \notin \ker(x)\}.$$

Later you use $I\mathcal{O}_x$.

Lemma 3.2.5 (Stalks). *Let M be a $\Gamma(X;\mathfrak{O})$-module. Let $x \in X$. Then* $\Lambda(I)_x \neq I \otimes \mathfrak{O}_x$.

not used

what is I?

Proof. See [6, Tag 01BH]. ■

Lemma 3.2.6. *For $x \in X$ TFAE:*

We seem to have
$\Lambda_X(\mathcal{O}(X)/I) = \Lambda_X(\mathcal{O}(X))/\Lambda_X(I)$,
$\left(\Lambda_X(\mathcal{O}(X))/\Lambda_X(I)\right)_x = \frac{\Lambda_X(\mathcal{O}_X)_x}{\Lambda_X(I)_x}$
and $\Lambda_X(I)_x = I\mathcal{O}_{X,x}$

1. $x \in V_X(I)$
2. $I\mathfrak{O}_x \neq \mathfrak{O}_x$
3. $I \subset \ker(x)$.

Proof. $1 \Rightarrow 2$:

Assume $x \in V_X(I)$. Then $\overbrace{\Lambda_X(\frac{\mathfrak{O}(X)}{I})_x = \frac{\mathfrak{O}_x}{I\mathfrak{O}_x}}^{\text{why are these equal?}} \neq 0$. Hence $I\mathfrak{O}_x \neq \mathfrak{O}_x$.

$2 \Rightarrow 3$:

Assume $I\mathfrak{O}_x \neq \mathfrak{O}_x$. Then $I\mathfrak{O}_x$ is proper hence contained in the unique maximal ideal of the local ring $\mathfrak{O}_x$, therefore $I \mapsto 0$ in $k(x)$ or equivalently $I \subset \ker(x)$.

$3 \Rightarrow 1$:

Assume $I \subset \ker(x)$. Then I maps into $\mathfrak{m}_x$, hence $I\mathfrak{O}_x \subset \mathfrak{m}_x$. Therefore

$$\frac{\mathfrak{O}_x}{\Lambda_X(I)_x} = \frac{\mathfrak{O}_x}{I\mathfrak{O}_x} \neq 0.$$

earlier it was $I \otimes \mathcal{O}_x$.

∎

Corollary 3.2.7. *If* $y \in I$ *then* $D_X(y) \cap V_X(I) = \emptyset$.

Proof. Assume $y \in I$. Let $z \in V_X(I)$, then $y \in \ker(z)$ by the previous lemma. This implies $z \notin D_X(y)$. ∎

reference

Corollary 3.2.8. $V_X(I) \cup V_X(J) = V_X(IJ)$.

"Conversely,"

Proof. Let $z \in V_X(I) \cup V_X(J)$. Then $I \subset \ker(z)$ ~~and~~ or $J \subset \ker(z)$ by the lemma, hence $IJ \subset \ker(z)$. Apply the lemma again to get $z \in V_X(IJ)$. Let $z \in V_X(IJ)$. Then $IJ \subset \ker(z)$ by the lemma. The ideal $\ker(z)$ is prime, so $I \subset \ker(z)$ or $J \subset \ker(z)$. Invoke the lemma again to get $z \in V_X(I) \cup V_X(J)$. ∎

still true since $IJ \subseteq I$ and $IJ \subseteq J$.

Lemma 3.2.9. *Let* $I \subset \Gamma(X; \mathfrak{O})$ *be an ideal. The set* $V_X(I)$ *is closed.*

Proof. Consider the exact sequence

$$\mathfrak{O}(X) \to \frac{\mathfrak{O}(X)}{I} \to 0.$$

The functor Λ is a left adjoint hence right exact so

$$\mathfrak{O} \xrightarrow{f} \Lambda(\frac{\mathfrak{O}(X)}{I}) \to 0$$

is exact. Hence the sequence

$$\mathfrak{O}_x \xrightarrow{f_x} \Lambda(\frac{\mathfrak{O}(X)}{I})_x \to 0$$

is exact. (The global section $f(1)$ must generate $\Lambda(\frac{\mathfrak{O}(X)}{I})$ as a module by surjectivity of f.) Similarly $f_x(1_x)$ generates $\Lambda(\frac{\mathfrak{O}(X)}{I})_x$. ← This I agree with.

This is true locally, but I don't see why globally

Note that $f_x(1_x) = f(1)_x$ by definition of f_x, hence $f(1)_x$ is a generating element. Hence $\Lambda(\frac{\mathfrak{O}(X)}{I})_x \neq 0$ if and only if $f(1)_x \neq 0$.

This implies $V_X(I) = \mathrm{Supp}(f(1))$ which makes $V_X(I)$ closed as the support of a global section. ■

This is the first time you need the caffine hypothesis. It would be good to indicate this somehow.

Lemma 3.2.10. *Every closed set $W \subset X$ can be written as $V_X(I)$ for some ideal $I \subset \Gamma(x; \mathfrak{O})$.*

by definition

Proof. Let $\mathfrak{I}$ be some ideal sheaf inducing a closed subscheme structure on W. This is always a quasi-coherent module. Let $I = \Gamma(x; \mathfrak{I})$. Since X is caffine, we get $\mathfrak{I} = \Lambda(I)$. Let $\mathfrak{O}_W$ be the structure sheaf of this closed subscheme. Consider the closed immersion $W \xrightarrow{i} X$. By construction $\mathfrak{I} \to \mathfrak{O} \to i_*\mathfrak{O}_W \to 0$ is exact, hence $i_*\mathfrak{O}_W = \frac{\mathfrak{O}}{\mathfrak{I}}$. Hence $V_X(I) = \mathrm{Supp}\ (i_*\mathfrak{O}_W) = W$. ■

Lemma 3.2.11. *The sets $D_X(a)$ form a basis for the topology of X, with $a \in R$.*

You also need to show these are open.

Proof. Let $U \subset X$ be any open. Let $x \in U$. By Lemma 3.2.10 we get I such that $V_X(I) = U^c$. It follows that $x \notin V_X(I)$ and $I \not\subset \ker(x)$. So we get a $g \in I$ with $g \notin \ker(x)$. We get $x \in D_X(g)$ and $D_X(g) \subset U$. Why is this? Because $D_X(g) \cap V_X(I) = \emptyset$ "U^c"

Let $a, b \in R$. Note that $D_X(ab) = D_X(a) \cap D_X(b)$ since $\ker(x)$ is a prime ideal. So the opens $D_X(a)$ form a basis. ■

For example, you could say at the beginning of the section 3.2 "First we make a few general constructions that work over any scheme" and then here "Now for the remainder of the section we assume X is caffine in Sch."

Lemma 3.2.12. *The map ϵ_X is surjective.* ← It is helpful to say again: $X \longrightarrow \mathrm{Spec}(\Gamma(\mathcal{O}; X))$.

Or put in each statement what assumptions you are making on X.

Proof. Let $\mathfrak{p} \in \operatorname{Spec} R$ be a point in the target of ϵ_X. Then $\Lambda_X(\kappa(\mathfrak{p}))$ is a quasi-coherent sheaf of modules. In fact $\kappa(\mathfrak{p}) \otimes_{\Gamma(X;\mathfrak{O})} \mathfrak{O}(U)$ is a $\mathfrak{O}(U)$ algebra for any open $U \subset X$, hence $\Lambda_X(\kappa(\mathfrak{p}))$ is a quasi-coherent sheaf of algebras. Hence we can compute the relative spec $\text{Rspec }(\Lambda_X(\kappa(\mathfrak{p}))) \xrightarrow{h} X$.

Since X is caffine we have $\Lambda_X(\kappa(\mathfrak{p}))(X) = \kappa(\mathfrak{p})$. So

$$\text{Rspec }(\Lambda_X(\kappa(\mathfrak{p}))) \xrightarrow{\epsilon_X \circ h} \operatorname{Spec}(R)$$

factors as

$$\text{Rspec }(\Lambda_X(\kappa(\mathfrak{p}))) \to \operatorname{Spec}(\kappa(\mathfrak{p})) \to \operatorname{Spec}(R).$$

This gives us a commutative square

$$\begin{tikzcd}
X \arrow[r, "\epsilon_X"] & \operatorname{Spec} R \\
\text{Rspec }(\Lambda_X(\kappa(\mathfrak{p}))) \arrow[u] \arrow[r] & \operatorname{Spec} \kappa(\mathfrak{p}) \arrow[u]
\end{tikzcd}$$

Since $\Lambda_X(\kappa(\mathfrak{p}))$ is not the zero sheaf hence the structure sheaf of $\text{Rspec }(\Lambda_X(\kappa(\mathfrak{p})))$ is non-zero. This implies that $\text{Rspec }(\Lambda_X(\kappa(\mathfrak{p})))$ is not the empty scheme. Therefore the point $\mathfrak{p}$ is in the image of $\text{Rspec }(\Lambda_X(\kappa(\mathfrak{p}))) \to \operatorname{Spec}(\kappa(\mathfrak{p})) \to \operatorname{Spec}(R)$, hence also in the image of ϵ_X. ∎

Lemma 3.2.13. *Let $\mathfrak{p} \subset R$ be a~~n~~ prime ideal. The closed set $V_X(\mathfrak{p})$ is irreducible. This implies that ϵ_X is injective.*

Proof. Let $\epsilon_X(z) = \mathfrak{p}$ for some $z \in X$. By lemma .. this is possible. So $\ker z = \mathfrak{p}$. Let $y \in V_X(\mathfrak{p})$. Then $\ker(z) \subset \ker(y)$, hence if $y \in D_X(a)$ then $z \in D_X(a)$. Therefore z is contained in any open subset of $V_X(\mathfrak{p})$, hence it is irreducible. Any element that is ~~send~~ sent to $\mathfrak{p}$ will be the generic point of $V_X(\mathfrak{p})$. Uniqueness of generic points implies injectivity of F.

why is that?

∎

extra line break

Lemma 3.2.14. *The counit ϵ_X is open, hence a homeomorphism.*

What is ker $\mathfrak{p}$ for $\mathfrak{p} \in \operatorname{Spec} R$? an ideal of R? Unclear.

What is a? $\notin$

Proof. Since ϵ_X is a morphism of locally ringed spaces, if a /in $\ker x$ then $a \notin \ker \epsilon_X(x)$. Hence $\epsilon_X(D_X(a)) = \{\epsilon_X(x) \mid a \notin \ker(x)\} = \epsilon_X(X) \cap D_{\operatorname{Spec} R}(a) = D_{\operatorname{Spec} R}(a)$. Our map ϵ_X is continuous and open, so a homeomorphism. ∎

Lemma 3.2.15. *If* η_X ? *is a homeomorphism, then* X *is affine.*

Proof. Let $\operatorname{Spec} A_i = U_i \subset X$ be open affines and ~~let~~ suppose $\bigcup_i U_i = X$. Assume it is a finite affine cover, which can be done since X is quasi-compact. Using our base, we get a cover of $U_i = \bigcup_j D_X(a_{ij})$ with a_{ij} global sections. Observe that $D_X(a_{ij}) \subset U_i$, hence $D_{U_i}(a_{ij}|_{U_i}) = D_X(a_{ij})$ which makes them affine. Continuing like this, we get a finite cover of affines $D_X(a_{ij})$ of X. Since

$$\epsilon_X(X) = \epsilon_X(\bigcup_{ij} D_X(a_{ij})) = \bigcup_{ij} D_{\operatorname{Spec}(R)}(a_{ij}) = \operatorname{Spec} R,$$

we have $(a_{ij}) = (1)$. Now both requirements of [2, Ex.2.17] are satisfied, hence X is affine. ∎

What are these requirements? A review at the beginning of the proof would help.

Proposition 3.2.16. *The caffine scheme* X *is affine.*

Proof. By Lemmas 3.2.12 to 3.2.14 F ? is ~~an~~ a homeomorphism and so by Lemma 3.2.15 ϵ_X is an isomorphism. ∎

Technically, you concluded that X is affine. It's easy to see, though, that this is equivalent to ϵ_X being an isomorphism.
You could even have a proof early on in the section that (1) X is affine
(2) $\Leftrightarrow$ ϵ_X is a homeo.
(3) $\Leftrightarrow$ ϵ_X is an iso.
Because (1) $\Rightarrow$ (2) $\Rightarrow$ (3) are the directions you prove (or are easy).
That would also motivate why you care so much about the topology of X.