Affine Objects

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1 Preliminaries

This chapter introduces all the basic notions that are needed but not assumed to be known to the reader. We will start with a discussion of some purely categorical notions like slice categories and presheaves. Secondly, we will introduce a notion of a topology on a category and look at some constructions that are relevant for us. Then we will introduce modules on ringed sites. Lastly, the notion of a scheme is introduced.

1.1 Basic Category Theory

Some categorical notions like presheaves and slice categories will be introduced in this section. See [1] and [3].

Definition 1.1.1 (Presheaf category)**.** Let C be a category. Let $a \in \mathsf{C}$. Let $f : a' \to a$ We define the category of presheaves on C as the category of contravariant functors to the category of sets Set. We will denote it by $\hat{\mathsf{C}}$.

Define the functor $h : \mathsf{C} \to \hat{\mathsf{C}}$ as follows

$$a \mapsto \mathrm{Hom}(-, a),$$

$$f \mapsto f \circ -.$$

This functor is fully faithful by the Yoneda lemma.

Notation 1.1.2. Let I, C be categories. Let $L : \mathsf{I} \to \mathsf{C}$ be a functor. The limit over this functor will be denoted by $\lim_{i \in \mathsf{I}} L(i)$. The colimit will be denoted by $\operatorname{colim}_{i \in \mathsf{I}} L(i)$.

Definition 1.1.3 (Sections functor)**.** For any $a \in \mathsf{C}$ define the functor

$$\Gamma(a; -) : \hat{\mathsf{C}} \to \mathsf{Set}$$

by

$$\mathfrak{F} \to \mathfrak{F}(a).$$

For any presheaf $\mathfrak{G}$, we define

$$\Gamma(\mathfrak{G}; -) : \hat{C} \to \mathsf{Set},$$
$$\mathfrak{F} \to \mathrm{Hom}(\mathfrak{G}, \mathfrak{F}).$$

We will mostly use this for the terminal presheaf, which will allow us to compute the global sections.

Definition 1.1.4 (Over/Under categories)**.** Let C and C' be categories. Let $F : C \to C'$ and $z \in C'$. Define the category C_z and C^z as

$$\mathrm{Obj}(C_z) := \{(a, w) \mid a \in C, w : F(a) \to z\},$$
$$\mathrm{Hom}((a, w), (b, v)) := \{f : a \to b \mid v \circ F(f) = w\},$$

and

$$\mathrm{Obj}(C^z) := \{(a, w) \mid a \in C, w : z \to F(a)\},$$
$$\mathrm{Hom}((a, w), (b, v)) := \{f : a \to b \mid F(f) \circ w = v\}.$$

We get faithful functors $C_z \to C : (a, w) \to a$ and $C^z \to C : (a, w) \to a$. We will call both functors localization functors and denote them by u.

Definition 1.1.5 (direct image)**.** Let $f : C \to D$. Define the direct image functor $f_* : \hat{D} \to \hat{C}$ as

$$f_* = - \circ f.$$

When u is the forgetful functor $C_a \to C$ then we denote u_* sometimes by $\mathfrak{F}\big|_a$.

Definition 1.1.6 (inverse image of presheaves)**.** Let C, D be a categories. Let $f : C \to D$ be a functor. Define the inverse image functor $f^{-1} : \hat{C} \to \hat{D}$ as follows. Let $\mathfrak{F} \in \hat{C}$. For any $d \in D$

$$f^{-1}(F)(d) = \operatorname*{colim}_{D_d} \mathfrak{F}u.$$

Lemma 1.1.7. *The functor* f_* *is left adjoint to every incarnation of* f^{-1}*.*

Corollary 1.1.8. *The functor* f_* *commutes with arbitrary colimits.*

1.2 Topology

In this section we will define a notion of a topology on a category and look at the related notions of sheaves, sites and restriction of sites.

See [3] for more details.

1.2.1 Basic

Definition 1.2.1 (Sieve)**.** Let C be a category and $a \in C$. Define the maximal sieve $\mathsf{max}(a)$ on a to be the collection of all morphisms to a. In formula,

$$\mathsf{max}(a) = \{f \in C \mid \mathsf{Codom}(f) = a\}.$$

A sieve S on a is a subcollection of $\mathsf{max}(a)$ such that $gf \in S$ for any $f \in S$ and any g.

Definition 1.2.2 (Sieve category)**.** Let C be a category and $a \in C$. The sieve category $\mathsf{Sieves}(a)$ consists of all the sieves on a as objects and inclusions of sieves as morphisms.

Definition 1.2.3 (Pullback of sieve)**.** Let C be a category and $a, b \in C$. Let S be a sieve on a. Let $f : b \to a$. The sieve f^*S on b is given by $f^*S = \{g \in \max b : fg \in S\}$ for any $c \in C$. To show that this is actually a sieve on b, let $k : c \to c'$ and $h \in f^*S$. Hence $fh \in S$ and so $fhk \in S$. Conclude that $hk \in f^*S$. This defines a functor $f^* : \mathsf{Sieves}(a) \to \mathsf{Sieves}(b)$.

Definition 1.2.4 (Grothendieck Topology)**.** A Grothendieck topology $\mathcal{T}$ is a family $\mathcal{T}(a)$ of 'covering' sieves for every $a \in C$ with the following conditions:

1. $\mathsf{max}(a) \in \mathcal{T}(a)$
2. $f^*R \in \mathcal{T}(a')$ if $R \in \mathcal{T}(a)$ for all $f : a' \to a$
3. if $f^*R \in \mathcal{T}(a')$ for all $f \in S$ with $S \in \mathcal{T}(a)$ then $R \in \mathcal{T}(a)$

Definition 1.2.5 (Basis)**.** Let C be a category with pullbacks. A Grothendieck pretopology, or basis, $\mathcal{B}$ is a collection $\mathcal{B}(a)$ of 'covering' families $\{f_i : a_i \to a\}$ of morphisms for every $a \in C$ with the following conditions.

1. every isomorphism is a covering singleton family,

2. (Stability) The pullback of a covering family is a covering family. If $\{f_i : a_i \to a\}$ is covering and $g : b \to a$, then $\{f'_i : a_i \times_a b \to b\}$ is covering.
3. (Transitivity) If $\{f_i : a_i \to a\}$ is a covering family and $\{f_{ij} : a_{ij} \to a_i\}$ for every i, then $\{f_{ij} : a_{ij} \to a\}$ is a covering family.

Generating a topology from a basis: take any sieve containing a covering family to be a covering sieve.

Remark 1.2.6. Sometimes you have a set of families S you would like to generate the topology. So one can take the smallest basis and the topology containing these families. This will be called 'the topology generated by S'.

1.2.2 Sites

Definition 1.2.7 (continuous functor)**.** Let $G : C \to D$ be a functor between sites. Let $c \in C$ and let R be a covering sieve on c. The functor G is said to preserves covers if $G(R)$ generates a covering sieve. It is enough to check that G sends covering families to covering families, if the topology of the sites is defined by a basis. If G also preserves pullbacks then we call it continuous. See [5, Tag 00WV].

Definition 1.2.8 (cocontinuous functor)**.** The functor G is said to lift covers or be cocontinuous if for every $R \in \mathrm{Cov}G(c)$ there is some $S \in \mathrm{Cov}c$ such that $G(S) \subset R$. See [5, Tag 00XJ].

Definition 1.2.9 (Site)**.** A site $(C, \mathcal{T})$ is a category C with a Grothendieck topology $\mathcal{T}$. A morphism of sites is a functor that preserves pullbacks and covers.

The category Sites has as objects sites and morphisms of sites as just defined. When no confusion can arise then we will use C to denote the whole site $(C, \mathcal{T})$.

Remark 1.2.10. In most resources a morphism of sites is defined to preserve all finite limits. To get the results that we want we only need preservation of pullbacks and we need the forgetful functor $C_a \to C$ to be a morphism of sites, hence this slightly weaker notion than usual.

Example 1.2.11 (small site)**.** Let X be a topological space. Let the category $\mathsf{Open}(X)$ consist of the opens of X as objects and inclusion as the morphisms. Define a basis on this site consisting of the families $\{U_i \to U\}$ such that $\bigcup_i U_i = U_i$. A continouous map $f : X \to Y$ induces a morphism of sites $\mathsf{Open}(Y) \to \mathsf{Open}(X)$ by sending $U \mapsto f^{-1}(U)$. In this way you can embed the category of topological spaces into the category of sites.

If X is a ringed space we can turn the site $\mathsf{Open}(X)$ into a ringed site by setting the sheaf of rings of X to be the sheaf of rings of $\mathsf{Open}(X)$.

1.2.3 Sheaves

We will introduce the very important notion of a sheaf. See [3] for a more detailed treatment.

Definition 1.2.12 (Matching family). Let C be a category. Let $\mathfrak{F}$ be a presheaf on on C. Let $a \in C$ be an object. Let R be a sieve on a. A family $\{x_i\}_{i\in R}$ with $x_i \in \Gamma(\mathrm{Dom}(i); \mathfrak{F})$ indexed by a sieve R and such that $x_{g\circ i} = \mathfrak{F}(g)(x_i)$ for any $g : b \to \mathrm{Dom}(i)$ and $b \in C$ is called a 'matching family'.

Definition 1.2.13 (Amalgamation). An amalgamation of a matching family $\{x_i\}_R$ is an element $x \in \Gamma(a; \mathfrak{F})$ such that $\mathfrak{F}(i)(x) = x_i$.

Definition 1.2.14 (Sheaf). Let $(C, \mathcal{T})$ be a site. Let $\mathfrak{F} \in \hat{C}$.

A presheaf that admits a unique amalgamation for every matching family is called a sheaf. The category $\mathsf{Shv}(C)$ is the full subcategory in $\hat{C}$ of all sheaves.

Definition 1.2.15 (Plus construction). Let $(C, \mathcal{T})$ be a site. Let $a, a' \in C$ and $f : a \to a'$. Let $\mathfrak{F} \in \hat{C}$. Define the functor $(-)^+ : \hat{C} \to \hat{C}$ as follows.

For all $a \in C$,

$$F^+(a) = \frac{\{(R, \varphi) \mid R \in \mathcal{T}(a), \varphi \in \Gamma(R; \mathfrak{F})\}}{\sim},$$

for all morphisms $f \in C$,

$$F^+(f)([(R, \varphi)]) = [(f^*R, \varphi h(f))].$$

The equivalence relation is defined as $(R, \varphi) \sim (S, \phi)$ if $\varphi = \phi$ on some covering sieve $Q \subset R \cap S$.

Let $L : \mathfrak{F} \to \mathfrak{F}'$. Then

$$(L^+)_a([(R, \varphi)]) = [(R, L \circ \varphi)]$$

This functor comes with a natural transformation $\omega : \mathrm{Id} \to (-)^+$ defined by

$$\omega_{\mathfrak{F},a}(x) = [(\mathsf{max}(a), y]$$

where

$$y(i) = \mathfrak{F}(i)(x).$$

Lemma 1.2.16. *Let $\mathfrak{F}$ be a presheaf, $\mathfrak{G}$ a sheaf and $g : \mathfrak{F} \to \mathfrak{G}$ a morphism in $\hat{C}$. Then g factors through ω_F via a unique g'.*

Proof. See [4, p. 2.10]. ∎

Lemma 1.2.17. *For every presheaf $\mathfrak{F}$, F^+ is separated.*

Proof. See [4, p. 2.11]. ∎

Lemma 1.2.18. *If $\mathfrak{F}$ is separated, then F^+ is a sheaf.*

Proof. See [4, p. 2.12]. ∎

Definition 1.2.19. Define $sh = (-)^+ \circ (-)^+$. This functor is left adjoint to the inclusion $\hat{C} \to \mathsf{Shv}(C)$ with unit

$$\omega^2_{\mathfrak{F}} = \omega_{\mathfrak{F}^+} \circ \omega_{\mathfrak{F}}.$$

1.2.4 Relative topology

We will look at what the induced topology on a slice category looks like and what this implies for restriction of sheaves. See [5, Tag 03A4] for a more detailed treatement.

Remark 1.2.20. Let C be a category and $a, b \in C$. Let $f : b \to a \in C_a$. The map $\mathsf{max}(f) \to \mathsf{max}(b)$ sending a morphism to f to its underlying morphism in C is a bijection. Moreover composition in C and C_a are the same, so this identification respects pullback of sieves. This observation yields us that $\mathsf{Sieves}(b) = \mathsf{Sieves}(f)$.

Whenever R is a sieve on b, we will denote the corresponding sieve on f by R_f.

Definition 1.2.21 (Relative topology)**.** Let $(C, \mathcal{T})$ be a site. Let $a \in C$.

Define the induced topology $\mathcal{T}_a$ on C_a by, for each $f \in C_a$

$$\mathcal{T}_a(f) = \mathcal{T}(\mathrm{Dom}(f)).$$

The identification from Remark 1.2.20 implies that $\mathcal{T}_a$ is a Grothendieck topology.

Definition 1.2.22 (Oversite)**.** Let $Y = (C, \mathcal{T})$ be a site. Let $a \in C$. Define the site Y_a to be the category C_a with the induced topology T_a. We will denote it by just C_a.

Lemma 1.2.23. *The functor* $u : C_a \to C$ *is a morphism of sites that is cover lifting.*

Proof. Clearly u preserves pullbacks. By Remark 1.2.20 it is also immediate that u preserves covers and lifts covers. ∎

Definition 1.2.24 (Over ringed site)**.** Let $Y = (C, \mathcal{T}, \mathfrak{O})$ be a site. Let $a \in C$. Define the ringed site Y_a to be the site C_a with the structure sheaf $u_*\mathfrak{O}$ where $u : C_a \to C$ the forgetful functor. This makes sense because u is a morphism of sites, see Lemma 1.2.23.

Whenever u is used as a morphism of ringed sites it stands for (u, Id)

1.3 Modules

Presheaf modules and sheaf modules on a ringed site will be introduced in this section. Next we will introduce two main functors λ and Λ. The functors λ and Λ introduced here will be used extensively. See [5, Tag 03A4] for more detail.

Definition 1.3.1 (Presheaf modules)**.** Let $(C, \mathcal{T}, \mathfrak{O})$ be a ringed site.

A presheaf module on this ringed site is a presheaf of sets $\mathfrak{F}$ on C together with a map of presheaves

$$\mathfrak{O} \times \mathfrak{F} \to \mathfrak{F}$$

such that for every object $a \in C$ the map $\Gamma(a; \mathfrak{O}) \times \Gamma(a; \mathfrak{F}) \to \Gamma(a; \mathfrak{F})$ defines a $\Gamma(a; \mathfrak{O})$-module structure on $\Gamma(a; \mathfrak{F})$.

A morphism

$$\mathfrak{F} \to \mathfrak{G}$$

is a morphism of presheaf modules if

$$\begin{tikzcd}
\mathfrak{O} \times \mathfrak{F} \arrow[r] \arrow[d] & \mathfrak{F} \arrow[d] \\
\mathfrak{O} \times \mathfrak{G} \arrow[r] & \mathfrak{G}
\end{tikzcd}$$

commutes. The category of presheaf modules on C will be denoted $\mathsf{PMod}(\mathfrak{O})$.

Definition 1.3.2. Let $\mathfrak{F}$ be a sheaf of modules on $(C, \mathcal{T}, \mathfrak{O})$. It is called quasi-coherent if the following holds. For any object $a \in C$ there exists a covering sieve S such that for any map $f : b \to a$ in S there exists a presentation

$$\mathfrak{O}\big|_b^{\oplus I} \to \mathfrak{O}\big|_b^{\oplus J} \to \mathfrak{F}\big|_b \to 0$$

It is enough to have presentation for a generating set of S.

Quasi-coherent modules form a full subcategory of the category of sheafs of modules over $(C, \mathcal{T}, \mathfrak{O})$ which are denoted by $\mathsf{Qcoh}(\mathfrak{O})$.

Lemma 1.3.3. *Let $f : C \to D$ be a morphism of sites. Let $\mathfrak{F}$ be a quasi-coherent sheaf of modules on D. Then $f_*\mathfrak{F}$ is quasi-coherent.*

Proof. ■

Definition 1.3.4. Let $(C, \mathcal{T}, \mathfrak{O})$ be a ringed site. Let $R = \Gamma(1; \mathfrak{O})$. Let M, N be an R-module.

Define

$$\lambda : R\text{-}\mathsf{Mod} \to \mathsf{PMod}(\mathfrak{O})$$

by for all $a \in C$,

$$\lambda(M)(a) = M \otimes_R \Gamma(a; \mathfrak{O}),$$

for all $f : b \to a \in \mathsf{C}$,

$$\lambda(M)(f) = \mathrm{Id} \otimes \mathfrak{O}(f),$$

for all $g : M \to N \in \mathsf{R\text{-}Mod}$,

$$\lambda(g) = (a : g \otimes \mathrm{Id}).$$

Lemma 1.3.5. *Let* $(\mathsf{C}, \mathfrak{T}, \mathfrak{O})$ *be a ringed site. Let* $R = \Gamma(1; \mathfrak{O})$*. The functor* λ *is left adjoint to*

$$\Gamma(1; -) : \mathsf{PMod}(\mathfrak{O}) \to \mathsf{R\text{-}Mod}.$$

Proof. Let M, N be R-modules. Let $\mathfrak{F}, \mathfrak{G} \in \mathsf{PMod}(\mathfrak{O})$ be presheaf modules.

Let $\varphi : \lambda(M) \to \mathfrak{G}$ be a morphism of presheaf modules. Let $\phi : M \to \Gamma(1; \mathfrak{G})$ be a morphism of R-modules.

Define

$$\begin{aligned} \alpha = H_{M,\mathfrak{G}} :& \mathrm{Hom}(\lambda(M), \mathfrak{G}) \to \mathrm{Hom}(M, \Gamma(1; \mathfrak{G})) \\ :& \varphi \mapsto \varphi_1 \end{aligned}$$

where φ_1 is the component of φ on the global sections.

Define

$$\beta = L_{M,\mathfrak{G}} : \mathrm{Hom}(M, \Gamma(1; \mathfrak{G})) \to \mathrm{Hom}(\lambda(M), \mathfrak{G})$$

by, for each $a \in \mathsf{C}$

$$\beta(\phi)_a = \phi \otimes_R \Gamma(a; \mathfrak{O}).$$

We will show that β and α are mutually inverse.

Let $\varphi \in \mathrm{Hom}(\lambda(M), \mathfrak{G})$ and $a \in \mathsf{C}$. Let $d = \beta(\alpha(\varphi))$. Let $m \otimes g \in M \otimes_R \Gamma(a; \mathfrak{O})$. Let $p : \lambda(M)(1) \to \lambda(M)(a)$ be the projection map. Let $q : \mathfrak{G}(1) \to \mathfrak{G}(a)$ be the projection map. Then $d_a(m \otimes g) = \varphi_1(m) \otimes g$ and

$$\begin{aligned}\varphi_a(m \otimes g) &= g\varphi_a(m \otimes 1) \text{ by linearity}\\ &= g\varphi_a(p(m))\\ &= gq(\varphi_1(m)) \text{ by naturality of } \varphi\\ &= g(\varphi_1(m) \otimes 1)\\ &= \varphi_1(m) \otimes g.\end{aligned}$$

Hence $d = \varphi$. In words, the natural transformations from presheaves of the from $\lambda(M)$ are uniquely determined by their global sections component.

Let $d = \alpha(\beta(\phi))$. Let $m \in M$. Then $d(m) = (\phi \otimes_R R)(m) = \phi(m)$. Hence $d = \phi$, which makes α and β mutual inverses. Naturality is straightforward to check and is omitted. ∎

Definition 1.3.6. Let $(C, \mathcal{T}, \mathfrak{O})$ be a ringed site. Define

$$\Lambda : \mathsf{R\text{-}Mod} \to \mathsf{Mod}(\mathfrak{O})$$

by $sh \circ \lambda$.

It follows that we have the adjunction $\Lambda \dashv \Gamma(1; -)$.

This functor is the generalisation of [5, Tag 01BH] to general sites, so Λ on the site corresponding to a ringed space(we will define this later) will coincide with the construction defined in the Stacks Project.

1.4 -

Lemma 1.4.1. *Let* $f : (D, \mathcal{S}, \mathfrak{U}) \to (C, \mathcal{T}, \mathfrak{O})$ *be a morphism of ringed site. Let* $f^{\#}$ *be an isomorphism. We get a natural isomorphism* $f_* \circ \lambda \Rightarrow \lambda \circ (- \otimes_{\Gamma(1;\mathfrak{O})} \Gamma(1; \mathfrak{U}))$.

Proof. Define the natural transformation $t : \lambda \circ (- \otimes_{\Gamma(1;\mathfrak{O})} \Gamma(1; \mathfrak{U})) \Rightarrow u_* \circ \lambda$, by for each $\Gamma(1; \mathfrak{O})$-module M and for each $a \in D$,

$$t_{M,a} : M \otimes_{\Gamma(1;\mathfrak{O})} \Gamma(1; \mathfrak{U}) \otimes_{\Gamma(1;\mathfrak{U})} \Gamma(a; \mathfrak{U}) \to M \otimes_{\Gamma(1;\mathfrak{O})} \Gamma(a; \mathfrak{O}),$$

$$m \otimes r \otimes s \mapsto m \otimes r f^{\#,-1}(s).$$

Every component $t_{M,f}$ is an isomorphism by basic commutative algebra and the fact that $f^{\#}$ is an isomorphism ∎

Example 1.4.2. The morphism of ringed sites u is an example where Lemma 1.4.1 holds.

Lemma 1.4.3. *Let* $f : (D, \mathcal{S}, \mathfrak{U}) \to (C, \mathcal{T}, \mathfrak{O})$ *be a morphism of ringed site that is cover lifting. Let* $f^{\#}$ *be an isomorphism. Let* $\mathfrak{F}$ *be a quasi-coherent module on* C*. Let* $M = \Gamma(1; \mathfrak{F})$*. Consider* $\epsilon : \Lambda(M) \to \mathfrak{F}$*. Then* $f_*\epsilon : \Lambda(M \otimes_{\Gamma(1;\mathfrak{O})} \Gamma(1; \mathfrak{U}))) \to f_*\mathfrak{F}$ *is the counit of the adjunction* $\Lambda \dashv \Gamma(1; -)$ *on* D*.*

Proof. Let $a \in D$. We will show that $f_*\epsilon$ corresponds to the same morphism $\lambda(M \otimes_{\Gamma(1;\mathfrak{O})}) \to f_*\mathfrak{F}$ as the counit. By the universal property of the sheafification, this implies that $f_*\epsilon$ is the counit.

We have $(f_*\epsilon)_a(\omega^2(m \otimes r)) = rm$, which is the same as for the counit. ∎

Lemma 1.4.4 (Λ commutes with restriction)**.** *Let* $(C, \mathcal{T}, \mathfrak{O})$ *be a ringed site. Let* $f : (C, \mathcal{T}) \to (D, \mathcal{S})$ *be a morphism of sites that lift covers. Let* $f^{\#}$ *be an isomorphism.*

We have a natural isomorphism

$$f_* \circ \Lambda \Rightarrow \Lambda \circ (- \otimes_{\Gamma(1;\mathfrak{O})} \Gamma(1; \mathfrak{U})).$$

Proof. Follows from Lemma 1.4.1 and ??. ∎

1.5 Schemes

We will recap some definitions in scheme theory that we use later. See [6, 2] for thorough treatments of scheme theory.

Definition 1.5.1 (Spectrum of a ring)**.** Let R be a ring. The spectrum $\operatorname{Spec} R$ of R is the locally ringed space defined as follows. The underlying set is the set of prime ideals of R. The (Zariski) topology is generated by the basis of distinguised opens $D(f) = \{\mathfrak{p} \subset R | f \notin \mathfrak{p}\}$. The sheaf of rings is given on this basis by

$$D(f) \mapsto R_f.$$

A distinguised open $D(f)$ of Spec(R) viewed as locally ringed space is isomorphic to Spec(R_f), where the inclusion $\mathrm{Spec}(R_f) \to \mathrm{Spec}(R)$ corresponds to the canonical map $R \to R_f$.

The functor

$$\mathsf{Spec} : \mathsf{Rng} \to \mathsf{LRSpaces}$$

is left adjoint to

$$\Gamma(1; -) : \mathsf{LRSpaces} \to \mathsf{Rng},$$

see [5, Tag 01I1].

Definition 1.5.2 (Scheme)**.** We call the locally ringed space Spec(R) an affine scheme.

A scheme S is a locally ringed space that admits a covering of affine schemes. A morphism of schemes is a morphism of locally ringed spaces. The category of schemes we will denote by Sch.

Definition 1.5.3 (Sheaf of algebras)**.** A sheaf of algebras $\mathfrak{F}$ on a ringed site $(\mathcal{C}, \mathcal{T}, \mathfrak{O})$ is a sheaf of rings that comes with a (structure) morphism of sheaf of rings $\mathfrak{O} \to \mathfrak{F}$.

Definition 1.5.4 (Relative spec)**.** Let X be a scheme. Let $\mathfrak{S}$ be a sheaf of algebras on X that is quasi-coherent as a sheaf of modules.

Define the relative spectrum of $\mathfrak{S}$ over X to be the scheme

$$\mathrm{Rspec}\ \mathfrak{S} \to X$$

that you get by glueing the spectra $\mathrm{Spec}(\Gamma(V; \mathfrak{S})) \to V \subset X$ for every affine open V. See [5, Tag 01LW].

Definition 1.5.5 (Tilde functor)**.** Let Spec(R) be an affine scheme. Let M be a R-module. Define $\widetilde{M}$ to be the unique sheaf on Spec(R) with

$$\widetilde{M} : D(f) \mapsto M_f.$$

See [5, Tag 01HR].

Remark 1.5.6. This functor(with this notation) is commonly used in algebraic geometry texts. By [5, Tag 01I7] and uniqueness of left adjoints Λ and $\widetilde{-}$ are canonically isomorphic on spaces, hence on small Zariski sites. We will only use Λ in subsequent sections.

1.6 Small Zariski site

Lemma 1.6.1. *Let* R *be a ring. Let* M *be a* R*-module. Consider* $\lambda(M), \Lambda(M)$ *as sheaf modules on* $\mathsf{Open}(\mathsf{Spec}(R))$*. Then*

$$\omega^2_{\lambda(M),\mathsf{Spec}(R)} : \Gamma(\mathsf{Spec}(R); \lambda(M)) \to \Gamma(\mathsf{Spec}(R); \Lambda(M))$$

is an isomorphism.

Proof. Let $\mathfrak{p} \subset R$ be a prime ideal. As stated in Definition 1.3.6, we may use results from [5, Tag 01BH] in this setting. We will use that $\Lambda(M)_\mathfrak{p} = M \otimes_R R_\mathfrak{p}$. By naturality, localized at $\mathfrak{p}$, the map ω^2 sends m to $m \otimes 1 \in M \otimes_R R_\mathfrak{p}$, hence is the inverse of the multiplication map which is an isomorphism. Hence globally ω^2 is an isomorphism. ∎

Corollary 1.6.2. *Let* X *be a scheme. Consider* $\lambda(M), \Lambda(M)$ *as sheaf modules on* $\mathsf{Open}(X)$*. Let* $\mathsf{Spec}(R)$ *be an open subset of* X*. Let* M *be a* $\Gamma(X; \mathfrak{O})$*-module. Then* $\omega^2_{\lambda(M),\mathsf{Spec}(R)} : \Gamma(\mathsf{Spec}(R); \lambda(M)) \to \Gamma(\mathsf{Spec}(R); \Lambda(M))$ *is an isomorphism.*

Note that η_M is equal to $\omega^2_{\lambda(M),\mathsf{Spec}(R)}$ so we get the following.

Corollary 1.6.3. *Consider the adjunction* $\Lambda(-) \dashv \Gamma(1; -)$ *on* $\mathsf{Open}(\mathsf{Spec}(R))$*. The unit* $\eta_M : M \to \Gamma(\mathsf{Spec}(R); \Lambda(M))$ *is an isomorphism.*

Lemma 1.6.4. *Let* $\mathfrak{F}$ *be a sheaf of modules on scheme* X*.* $\mathfrak{F}$ *is quasi-coherent on* X *if and only if for any open* $\mathsf{Spec}(R) \subset X$ *the sheaf* $\mathfrak{F}|_{\mathsf{Spec}(R)}$ *is isomorphic to* $\Lambda(M)$ *with* $M = \Gamma(\mathsf{Spec}(R); \mathfrak{F})$*.*

Proof. $\Rightarrow$: By assumption we get local presentations indexed by a covering. Let $\bigcup_{i \in I} U_i = X$ be this covering. Assume without loss of generality that it is an affine open covering. Let $U_i = \mathsf{Spec}(R_i)$. Let $\mathfrak{O}_{U_i}^{\bigoplus K} \to \mathfrak{O}_{U_i}^{\bigoplus J} \to \mathfrak{F}|_{U_i} \to 0$ be one of the given presentations. Taking global sections gives us an exact sequence

$$R_i^{\bigoplus K} \to R_i^{\bigoplus J} \to \Gamma(U_i; \mathfrak{F}) \to 0.$$

Tensoring it with the localisation $R_{i,f}$ for any $f \in R_i$ yields

$$R_{i,f}^{\bigoplus K} \to R_{i,f}^{\bigoplus J} \to \Gamma(U_i; \mathfrak{F}) \otimes R_{i,f} \to 0.$$

Taking sections at $D(f)$ from the sheaf sequence yields

$$R_{i,f}^{\bigoplus K} \to R_{i,f}^{\bigoplus J} \to \Gamma(D(f); \mathfrak{F}) \to 0.$$

Hence $\mathfrak{F}|_{U_i}$ is the unique sheaf with $D(f) \mapsto \Gamma(U_i; \mathfrak{F})_f$, which we defined to be $\Lambda(\Gamma(U_i; \mathfrak{F}))$. By the affine communication lemma, this property holds for any affine and not just for the affines in this covering.

$\Leftarrow$: Let $M = \Gamma(X; \mathfrak{F})$. Take a presentation $R^{\bigoplus I} \to R^{\bigoplus J} \to M$ and apply $\Lambda(-)$. Then note that $\Lambda(-)$ commutes with arbitrary colimits since it is a left adjoint, see lemma ?. We have $\Lambda(R) = \mathfrak{O}_{\mathrm{Spec}(R)}$ so we get a presentation $\mathfrak{O}_{\mathrm{Spec}(R)}^{\bigoplus I} \to \mathfrak{O}_{\mathrm{Spec}(R)}^{\bigoplus J} \to \mathfrak{F}$ on every affine open subset $\mathrm{Spec}(R) \subset X$, hence $\mathfrak{F}$ is quasi-coherent. ∎

Corollary 1.6.5. *Consider the adjunction $\Lambda(-) \dashv \Gamma(1; -)$. For quasi-coherent sheaf $\mathfrak{F}$ the counit $\epsilon_{\mathfrak{F}} : \Lambda(M) \to \mathfrak{F}$ is an isomorphism.*

1.7 Big Zariski Site

In this section we will introduce the big Zariski ringed site and look at how quasi-coherence and Λ behave on this site.

Definition 1.7.1 (Big Zariski site). Define the big Zariski site to be $(\mathsf{Sch}, \mathcal{T}, \mathfrak{O})$ with the following components. The underlying category is Sch. The topology T is generated by the basis cosisting of the covering families $\{X_i \xrightarrow{f_i} X\}$ where f_i is an open immersion and $\bigcup_i f_i(X_i) = X$. The sheaf of rings $\mathfrak{O}$ sends $(U, \mathfrak{Q}) \to (X, \mathfrak{O})$ to $\Gamma(U; \mathfrak{Q})$.

We will mostly be interested in the sliced site Sch_X for a scheme X.

Definition 1.7.2 (k). Let X be a scheme. Define the functor $k : \mathsf{Open}(X) \to \mathsf{Sch}_X$ by $U \mapsto ((U, \mathfrak{O}_U), i)$ where $i : U \to X$ is the inclusion of the open subscheme into X.

We will show that it preserves limits and covers. The terminal $X \in \mathsf{Open}(X)$ is send to the terminal $X \to X$. Let $U \to V$ and $W \to V$ be two morphism in $\mathsf{Open}(X)$. We have $k(U \cap W) = U \cap W \to X$ which is the pullback of $k(U) \to k(V)$ and $k(W) \to k(V)$.

Let $S = \{D(f_i) \to \mathrm{Spec}(R)\}$ be one of the generating family in $\mathrm{Open}(X)$. Note that $k(D(f_i))$ is isomorphic to the object $\mathrm{Spec}(R_{f_i}) \to \mathrm{Spec}(R)$. Hence $k(S)$ generates a covering sieve on Sch and hence on Sch_X.

So k is a morphism of sites as defined in Definition 1.2.9.

Lemma 1.7.3. *Let* X *be a scheme. Consider* $k : \mathrm{Open}(X) \to \mathrm{Sch}_X$. *Then* $k_*\Lambda = \Lambda$

Proof. ∎

Lemma 1.7.4.

Lemma 1.7.5. *Let* X *be a scheme. Let* $\mathfrak{F}$ *be a quasi-coherent sheaf on* Sch_X. *Then* $k_*\mathfrak{F}$ *is quasi-coherent.*

Proof. ∎

Lemma 1.7.6. *Let* $X = \mathrm{Spec}(R)$ *be a scheme. Let* $\mathfrak{F}$ *be a quasi-coherent sheaf on* Sch_X. *Let* $M = \Gamma(X;\mathfrak{F})$. *Then* $\epsilon_{\mathfrak{F}} : \Lambda(M) \to \mathfrak{F}$ *is an isomorphism.*

Proof. By lemma ?, k_* sends quasi-coherent sheafs to quasi-coherent sheafs. so $k_*\mathfrak{F} = \Lambda(M)$, where $k : \mathrm{Open}(X) \to \mathrm{Sch}_X$. Hence $\Gamma(D(f);\mathfrak{F}) = \Gamma(D(f);\Lambda\mathfrak{M}) = M_f$ for any $f \in R$.

Let $D(f_i)\big|_{\mathfrak{O}}^{\bigoplus J} \xrightarrow{\alpha_i} D(f_i)\big|_{\mathfrak{O}}^{\bigoplus K} \to D(f_i)\big|_{\mathfrak{F}} \to 0$ be a presentation with $(f_i) = (1)$. Note that the presheaf cokernel $\mathrm{Coker}(\alpha_i)$ of the sheaf morphism is $\lambda(\mathrm{Coker}(\alpha_{i,D(f_i)}))$ where $\alpha_{i,D(f_i)}$ is the component at $D(f_i)$ of α_i. So $D(f_i)\big|_{\mathfrak{F}} = \Lambda(M_f)$ since ω^2 is iso for affines.

Hence

$$\begin{aligned}\Gamma(D(f) \times_X \mathrm{Spec}(S);\mathfrak{F}) &= \Gamma(D(f) \times_X \mathrm{Spec}(S);\Lambda\mathfrak{F})\\ &= \Gamma(\mathrm{Spec}(S_f);\Lambda\mathfrak{M}_f)\\ &= M_f \otimes S_f\\ &= M \otimes S_f.\end{aligned}$$

By the sheaf property it follows that $\Gamma(\mathrm{Spec}(S);\mathfrak{F}) = \Gamma(X;\mathfrak{F}) \otimes S$, hence the counit $\Lambda(M) \to \mathfrak{F}$ is an isomorphism. ∎

Lemma 1.7.7. *Let* $X = \mathrm{Spec}(R)$ *be a scheme. Let* M *be a* R*-module. We work on the ringed site* Sch_X. *The counit* $\epsilon : M \to \Gamma(X;\Lambda(M))$ *is an isomorphism*

Proof. Let $\Lambda_a(M)$ be $\Lambda(M)$ over $\mathrm{Open}(X)$. By Lemma 1.7.3, we have $\Lambda_a(M) = k_*\Lambda(M)$. Hence $\Gamma(X; \Lambda(M)) = \Gamma(X; \Lambda_a(M))$. Moreover $\Gamma(X; \Lambda(M)) = M$ by Corollary 1.6.3. ∎

2 Restrictive

This section will introduce the notion of a restrictive morphism We will see some examples, non-examples and results in the category of schemes and see that this notion is closely related to affinenes.

For some of the examples and results see the chapter on quasi-coherent modules in [6].

Definition 2.0.1 (Restrictive morphism)**.** Let $(\mathsf{C}, \mathcal{T}, \mathfrak{O})$. A morphism $f : a \to b \in \mathsf{C}$ is called restrictive if for every quasi-coherent module $\mathfrak{G}$ on C_b the morphism

$$\widehat{f} : \Gamma(b; \mathfrak{G}) \otimes_{\Gamma(b;\mathfrak{O})} \Gamma(a; \mathfrak{O}) \to \Gamma(a; \mathfrak{G}) \tag{2.1}$$

is an isomorphism.

Remark 2.0.2. Assume we are in the context of Definition 2.0.1. Assume $\mathfrak{G} = \Lambda(M)$ for some $\Gamma(b; \mathfrak{O})$-module, then $\widehat{f} = \omega^2_a$ for the sheafification transformation $\omega^2_{\lambda(M)} : \lambda(M) \to \Lambda(M)$.

Example 2.0.3. In $\mathsf{Sch}_{\mathsf{Spec}(A)}$ the morphism $\mathsf{Spec}(A_f) \to \mathsf{Spec}(A)$ is restrictive. Let $\mathfrak{G}$ be a quasi-coherent sheaf on $\mathsf{Sch}_{\mathsf{Spec}(A)}$. This implies that $\mathfrak{G} = \Lambda(\Gamma(\mathsf{Spec}(A); \mathfrak{G}))$, see ?. The morphism

$$\Gamma(\mathsf{Spec}(A); \mathfrak{G}) \otimes_{\Gamma(1;\mathfrak{O})} \Gamma(\mathsf{Spec}(A_f); \mathfrak{O}) \to \Gamma(\mathsf{Spec}(A_f); \mathfrak{G}) = \Gamma(A; \mathfrak{G})_f,$$

$$m \otimes r \to rm$$

is an isomorphism by basic commutative algebra.

Example 2.0.4. Let R be a ring. Consider the open immersion $U = \mathsf{Spec}(R[x, y]) \setminus \{(x, y)\} \to \mathsf{Spec}(R[x, y])$ and the quasi-coherent sheaf $\mathfrak{G} = \Lambda(\frac{R[x,y]}{xy})$. The global sections of this sheaf are $\frac{R[x,y]}{xy}$, as shown in

Define $U_1 = D(x) \to U$ and $U_2 = D(y) \to U$. Note that these cover U together. We have $\Gamma(U_1;\mathfrak{G}) = 0$ and $\Gamma(U_1;\mathfrak{G}) = 0$, since $\frac{R[x,y]}{xy}_x = 0$ and $\frac{R[x,y]}{xy}_y = 0$. Hence since $\mathfrak{G}$ is a sheaf, we get $\Gamma(U;\mathfrak{G}) = 0$.

The sections over U of $\Lambda(R[x,y])$ are (also) $R[x,y]$. See [6, p. 4.4.1]. We conclude that $\Gamma(1;\mathfrak{G}) \otimes_{\Gamma(1;\mathfrak{O})} \Gamma(U;\mathfrak{O}) \to \Gamma(U;\mathfrak{G})$ is not an isomorphism.

Lemma 2.0.5. *Let $(C,\mathcal{T},\mathfrak{O})$ be a ringed site. Let $f : b \to a \in C$ and $g : c \to b \in C$ be morphisms.*

1. *If fg and f are restrictive, then g is.*
2. *if f and g are restrictive, then fg.*

If $\Gamma(b;\mathfrak{O}) \to \Gamma(c;\mathfrak{O})$ is faithfully flat then fg and g restrictive implies f is restrictive.

Proof. Consider the following diagram

$$\begin{tikzcd}
\Gamma(a;\mathfrak{G}) \otimes_{\Gamma(a;\mathfrak{O})} \Gamma(c;\mathfrak{O}) \arrow[r, "\widehat{f}\otimes_{\Gamma(b;\mathfrak{O})}\Gamma(c;\mathfrak{O})"] \arrow[d, equal] & \Gamma(b;\mathfrak{G}) \otimes_{\Gamma(b;\mathfrak{O})} \Gamma(c;\mathfrak{O}) \arrow[d, "\widehat{g}"] \\
\Gamma(a;\mathfrak{G}) \otimes_{\Gamma(a;\mathfrak{O})} \Gamma(c;\mathfrak{O}) \arrow[r, "\widehat{fg}"] & \Gamma(c;\mathfrak{G}),
\end{tikzcd}$$

where $\widehat{-}$ is as in lemma ?.

This diagram commutes: going either direction sends $g \otimes r$ to rg. The results follows from commutativity. ∎

Lemma 2.0.6 (coproduct)**.** *Let X_1, X_2, Y be a schemes. $X_1 \xrightarrow{f_1} Y$ and $X_2 \xrightarrow{f_2} Y$ are restrictive morphisms if and only if the corresponding morphism $X_1 \sqcup X_2 \xrightarrow{f_1 \sqcup f_2} Y$ is restrictive.*

Proof. $\Rightarrow$:

Note that $\Gamma(X_1 \sqcup X_2; -) = \Gamma(X_1; -) \times \Gamma(X_2; -)$ by the sheaf property. We will show that

$$\widehat{f_1 \sqcup f_2} : \Gamma(Y; \mathfrak{G}) \otimes_{\Gamma(Y;\mathfrak{O})} \Gamma(X_1 \sqcup X_2; \mathfrak{O}) \to \Gamma(X_1 \sqcup X_2; \mathfrak{G})$$

is an isomorphism. Tensor commutes over products, so this becomes

$$\widehat{f_1} \times \widehat{f_2} : (\Gamma(Y; \mathfrak{G}) \otimes_{\Gamma(Y;\mathfrak{O})} \Gamma(X_1; \mathfrak{O})) \times (\Gamma(Y; \mathfrak{G}) \otimes_{\Gamma(Y;\mathfrak{O})} \Gamma(X_2; \mathfrak{O})) \to \Gamma(X_1; \mathfrak{G}) \times \Gamma(X_2; \mathfrak{G}).$$

By assumption $\widehat{f_1}$ and $\widehat{f_2}$ are isos, so their product is.

$\Leftarrow$: By Lemma 2.0.7, the canonical morphism $X_i \to X_1 \sqcup X_2$ is restrictive. By assumption $X_1 \sqcup X_2 \to Y$ is restrictive. Composing these morphisms yields f_i, by Lemma 2.0.5 this yields a restrictive morphism. ∎

Lemma 2.0.7. *Let* X, Y *be a schemes. The canonical morphism* $X \to X \sqcup Y$ *is restrictive.*

Proof. Let $\mathfrak{G}$ be a quasi-coherent sheaf on $X \sqcup Y$. By the sheaf property $\Gamma(X \sqcup Y; \mathfrak{G}) = \Gamma(X; \mathfrak{G}) \times \Gamma(Y; \mathfrak{G})$. The same holds also for $\mathfrak{O}$.

We are considering the morphism

$$\Gamma(X \sqcup Y; \mathfrak{G}) \otimes_{\Gamma(X \sqcup Y;\mathfrak{O})} \Gamma(X; \mathfrak{O}) \to \Gamma(X; \mathfrak{G}).$$

By the previous remark about disjoint unions and the sheaf property and some basic commutative algebra one sees that this becomes

$$(\Gamma(X; \mathfrak{G}) \otimes_{\Gamma(X \sqcup Y;\mathfrak{O})} \Gamma(X; \mathfrak{O})) \times (\Gamma(Y; \mathfrak{G}) \otimes_{\Gamma(X \sqcup Y;\mathfrak{O})} \Gamma(X; \mathfrak{O})) \to \Gamma(X; \mathfrak{G}).$$

Since $\Gamma(Y; \mathfrak{G}) \otimes_{\Gamma(X \sqcup Y;\mathfrak{O})} \Gamma(X; \mathfrak{O}) = 0$, we are left with

$$(\Gamma(X; \mathfrak{G}) \otimes_{\Gamma(X \sqcup Y;\mathfrak{O})} \Gamma(X; \mathfrak{O})) \to \Gamma(X; \mathfrak{G})$$
$$g \otimes r \to rg.$$

Note that $\Gamma(X; \mathfrak{G})$ already is an $\Gamma(X; \mathfrak{O})$-module and conclude that hence this morphism is an isomorphism. ∎

Lemma 2.0.8 (Restrictive to affines)**.** *If* $f : X \to \mathrm{Spec}(R)$ *is a restrictive open immersion, then* X *is affine.*

Proof. Since X is an open in $\mathrm{Spec}(R)$, we get a distinguised covering $\bigcup_i D(f_i) = X$ with $f_i \in R$ and $i \in I$. We will prove that $(f_i|_X)_{i \in I} = (1)$ in $S = \Gamma(X; \mathfrak{O})$.

Then we invoke the result in [2, Ex. 2.1.7] that states the following. For a scheme Y let Y_f be the support of $f \in \Gamma(Y; \mathfrak{O})$ as in lemma ?. if Y_{g_j} are affine and $(g_j)_{j \in J} = (1)$ then Y is affine.

Note that $D(f_i) = X_{f_i|_X}$. Consider $M = \frac{R}{(f_i)}$ as an R-module and look at $\Lambda(M)$. By restrictiveness we get $M \otimes_R S = \Lambda(M)(S)$ and by $M \otimes_R R_{f_i} = \Lambda(M)(D(f_i)) = M_{f_i} = 0$. Hence $\Lambda(M)(S) = 0$ by the sheaf axiom. This implies that $(f_i|_X)_{i \in I} = (1)$ in S. So X is affine. ∎

Lemma 2.0.9. *Any morphism* $\mathrm{Spec}(S) \xrightarrow{f} \mathrm{Spec}(R) \in \mathrm{Sch}_{\mathrm{Spec}(R)}$ *between affine schemes is restrictive.*

Proof. Let $\mathfrak{G}$ be a quasi-coherent module on $\mathrm{Sch}_{\mathrm{Spec}(R)}$. Set $M = \Gamma(\mathrm{Spec}(R); \mathfrak{G})$. We want to prove that

$$\widehat{f} : M \otimes_R S \to \Gamma(\mathrm{Spec}(S); \mathfrak{G})$$

is an isomorphism.

By Lemma 1.7.6, we get $\mathfrak{G} = \Lambda(M)$. As said in Remark 2.0.2, in this case $\widehat{f} = \omega^2_{\mathrm{Spec}(S)}$. By Lemma 1.6.1, we know that ω^2 is an isomorphism at affine schemes. ∎

Example 2.0.10 (Affine non-restrictive map)**.** One might expect(or want) that any property of all maps between affine schemes also hold for affine maps between any schemes. This is not the case for restrictiveness, so it is not local on the target.

Consider the canonical inclusions $\mathbb{A}^1 \to \mathbb{P}^1$ and the shifted quasi-coherent module $\mathfrak{O}(-1)$. This module is locally free of degree 1, this is often called an invertible module.

The global sections of the module $\mathfrak{O}(-1)$ are the elements of degree -1 in the global sections of $\mathfrak{O}$. There are no such elements, hence the global sections are the zero module.

On $\mathbb{A}^1$ all invertible modules are isomorphic to the structure sheaf. See [6, p. 14.2.8].

Similary any inclusion $\mathrm{Spec}(\kappa(\mathfrak{p})) \to \mathbb{P}^1$ of a point is not restrictive which can be shown with the same argument.

This is a (more opaque) way of saying that on projective space not every quasi-coherent sheaf is generated by global sections.

3 Caffine objects

3.1 Caffine objects

In this section we will introduce caffine objects, see some examples and non-examples and prove some properties. Most of these results will be generalisations of their counterpart for affine schemes.

Definition 3.1.1 (Caffine object)**.** Let $(\mathsf{C}, \mathcal{T}, \mathfrak{O})$ be a ringed site. Let $a \in \mathsf{C}$ be an object. We call a caffine if the unit η and co-unit ϵ of the adjunction $\Lambda \dashv \Gamma(a; -)$ on C_a are natural isomorphisms for any $\Gamma(a; \mathfrak{O})$-module M and any quasi-coherent $\mathfrak{F}$ on C_a

In other words, that $\Lambda : \Gamma(a; \mathfrak{O})\text{-Mod} \to \mathsf{Qcoh}(\mathfrak{O})$ is an equivalence of categories with $\Gamma(a; -)$ as pseudo-inverse and the unit and co-unit as witnessing natural isomorphisms.

Example 3.1.2 (Examples of caffine objects)**.** The main example to keep in mind is $\mathsf{Spec}(R) \in \mathsf{Sch}$. See Lemmas 1.7.6 and 1.7.7 for proofs that the unit and co-unit are isomorphisms.

Example 3.1.3. The scheme $\mathbb{P}^1$ is not caffine in Sch. The counit at the quasi-coherent sheaf of modules $\mathfrak{O}(-1)$ has signature $\epsilon : \Lambda(0) \to \mathfrak{O}(-1)$. Since $\mathfrak{O}(-1)$ is locally free of degree 1, this cannot be an isomorphism.

Definition 3.1.4 (caffine cover)**.** Let $(\mathsf{C}, \mathcal{T}, \mathfrak{O})$ be a ringed site. A family of maps $\{a_i \to a\}$ is called a caffine covering of a if every a_i is caffine and the family is a covering family.

Definition 3.1.5. We say that a ringed site $(\mathsf{C}, \mathcal{T}, \mathfrak{O})$ has enough affines if any object admits a caffine covering.

Lemma 3.1.6. *Let $(\mathsf{C}, \mathcal{T}, \mathfrak{O})$ be a ringed site. Let $a \in \mathsf{C}$. Let $\{b_i \to a\}$ be a caffine covering on a. Assume every map $b_i \xrightarrow{i} a$ is restrictive. Then the counit ϵ of the adjunction $\Lambda_a(-) \dashv \Gamma(a; -)$ is a natural isomorphism.*

Proof. Let $\mathfrak{F}$ be a quasi-coherent sheaf module. Set $M = \Gamma(a; \mathfrak{F})$. Set $M_i = \Gamma(b_i; \mathfrak{F})$. Set $\beta_i = i_* \epsilon_{\mathfrak{F},a}$. By Lemma 1.4.3, we have$\beta_i \cong \epsilon_{i_*\mathfrak{F},b_i}$. Since b_i is caffine β_i is an isomorphism. Hence ϵ is an isomorphism between sheafs, since it is locally an isomorphism. ∎

Lemma 3.1.7. *Let* $(C, \mathfrak{T}, \mathfrak{O})$ *be a ringed site. Let* $a \in C$. *Let* M *be a* $\Gamma(a; \mathfrak{O})$*-module. The component*

$$\omega^2_{\lambda(M),a} : \lambda(M)(a) \to \Lambda(M)(a)$$

at Id_a *of the sheafification morphism*

$$\omega^2_{\Lambda(M)} : \lambda(M) \to \Lambda(M)$$

is equal to the unit of $\Lambda \dashv \Gamma(1; -)$ *in* C_a.

Proof. Consider the following maps, which you get by repeatedly calling on an adjunction.

$$\mathrm{Id} : \Lambda(M) \to \Lambda(M)$$

$$\omega^2_{\Lambda(M)} : \lambda(M) \to \Lambda(M) \text{ use sheafification adjunction, see lemma ..}$$

$$\omega^2_{\lambda(M),a} M \to \Gamma(a; \Lambda(M)) \text{ take sections at } a$$

We took the adjunct of Id with respect to the sheafification adjunction as the first step. Then we took the adjunct of the result wrt the λ ajunction. Hence we get the adjunct of Id wrt the Λ adjunction. so the last map is actually the unit of the Λ adjunction. ∎

Corollary 3.1.8. *Let* $(C, \mathfrak{T}, \mathfrak{O})$ *be a ringed site. Let* $a \in C$ *be caffine. Then* $\omega^2_{\lambda(M),a}$ *is an isomorphism for any* $\mathfrak{O}(a)$*-module* M.

Theorem 3.1.9 (Morphism between caffines is restrictive)**.** *Let* $Y = (C, \mathfrak{T}, \mathfrak{O})$. *Let* $f : b \to a \in C$ *be a morphism between caffine objects, then* f *is restrictive.*

Proof. Let $\mathfrak{F}$ be a quasi-coherent module on Y_a. Let $M = \Gamma(a; \mathfrak{F})$. Since a is caffine, we have $\mathfrak{F} = \Lambda(M)$.

We have to show that the adjunct, along the extension of scalars adjunction, of $\mathfrak{F}(f)$

$$\Gamma(a; \mathfrak{F}) \otimes_{\Gamma(a;\mathfrak{O})} \Gamma(b; \mathfrak{O}) \to \Gamma(b; \mathfrak{F})$$

is an isomorphism.

This adjunct is the component at b of the natural transformation $\omega^2_{\lambda(\Gamma(1;\mathfrak{F}))}$. Since b is caffine, this component is an isomorphism. ∎

3.2 Caffine schemes

Let X be a caffine scheme. We will prove that that the the counit of $\mathrm{Spec} \dashv \Gamma(1;-)$, namely

$$\epsilon_X : X \to \mathrm{Spec}(\Gamma(1;\mathfrak{O})),$$

is an isomorphism. Set $R = \Gamma(1;\mathfrak{O})$.

Lemma 3.2.1. *The sets $D_X(a)$ form a basis for the topology of X, with a a global section.*

Proof. Let $U \subset X$ be any open. Let $x \in U$. By lemma ? we get I such that $V_X(I) = U^c$. It follows that $x \notin V_X(I)$ and $I \not\subset \ker(x)$. So we get a $g \in I$ with $g \notin \ker(x)$. We get $x \in D_X(g)$ and by corollary ?$D_X(g) \subset U$. As stated earlier, $D_X(ab) = D_X(a) \cap D_X(b)$ since $\ker(x)$ is a prime ideal. So $D_X(a)$ form a basis. ∎

Lemma 3.2.2. *Every closed set $W \subset X$ can be written as $V_X(I)$ for some ideal $I \subset \Gamma(1;\mathfrak{O})$.*

Proof. Let $\mathfrak{I}$ be some ideal sheaf inducing a closed subscheme structure on W. This is always a quasi-coherent module. Let $I = \Gamma(1;\mathfrak{I}) = \Lambda(I)$. Let $\mathfrak{O}_W$ be the structure sheaf of this closed subscheme. By construction $i_!\mathfrak{O}_W = i^{-1}\frac{\mathfrak{O}}{\mathfrak{I}}$ along the inclusion $W \xrightarrow{i} X$. Hence $V_X(I) = \mathrm{Supp}\ i_!\mathfrak{O}_W = W$ ∎

Lemma 3.2.3. *Let $I \subset \Gamma(1;\mathfrak{O})$ be an ideal. The set $V_X(I)$ is closed.*

Proof. Let $z \in X$ and M a $\mathfrak{O}$-module. Assume z is in the support of M, then $g \neq 0$ for any generating element $g \in M_z$.

Consider the exact sequence

$$\mathfrak{O}(X) \to \frac{\mathfrak{O}(X)}{I} \to 0.$$

The functor Λ_X is a left adjoint hence right exact so

$$\mathfrak{O} \xrightarrow{f} \Lambda_X(\frac{\mathfrak{O}(X)}{I}) \to 0$$

is exact. Hence the sequence

$$\mathfrak{O}_x \xrightarrow{f_x} \Lambda_X(\frac{\mathfrak{O}(X)}{I})_x \to 0$$

is exact. The global section $f(1)$ must generate $\Lambda_X(\frac{\mathfrak{O}(X)}{I})$ as a module by surjectivity of f. Similarly $f_x(1_x)$ generates $\Lambda_X(\frac{\mathfrak{O}(X)}{I})_x$.

Note that $f_x(1_x) = f(1)_x$ by definition of f_x, hence $f(1)_x$ is a generating element. Hence $\Lambda_X(\frac{\mathfrak{O}(X)}{I})_x \neq 0$ if and only if $f(1)_x \neq 0$.

This implies $V_X(I) = \mathrm{Supp}(f(1))$ which makes $V_X(I)$ closed as the support of a global section. ∎

Lemma 3.2.4. *For $x \in X$ TFAE:*

1. $x \in V_X(I)$
2. $I\mathfrak{O}_x \neq \mathfrak{O}_x$
3. $I \subset \ker(x)$.

Proof. $1 \Rightarrow 2$:

Assume $x \in V_X(I)$. Then $\Lambda_X(\frac{\mathfrak{O}(X)}{I})_x = \frac{\mathfrak{O}_x}{I\mathfrak{O}_x} \neq 0$. Hence $I\mathfrak{O}_x \neq \mathfrak{O}_x$.

$2 \Rightarrow 3$:

Assume $I\mathfrak{O}_x \neq \mathfrak{O}_x$. Then $I\mathfrak{O}_x$ is proper hence contained in the unique maximal ideal of the local ring $\mathfrak{O}_x$, therefore $I \mapsto 0$ in $k(x)$ or equivalently $I \subset \ker(x)$.

$3 \Rightarrow 1$:

Assume $I \subset \ker(x)$. Then I maps into $\mathfrak{m}_x$, hence $I\mathfrak{O}_x \subset \mathfrak{m}_x$. Therefore

$$\frac{\mathfrak{O}_x}{\Lambda_X(I)_x} = \frac{\mathfrak{O}_x}{I\mathfrak{O}_x} \neq 0.$$

∎

Corollary 3.2.5. *If* $y \in I$ *then* $D_X(y) \cap V_X(I) = \emptyset$

Proof. Assume $y \in I$. Let $z \in V_X(I)$, then $y \in \ker(z)$ by the previous lemma. This implies $z \notin D_X(y)$ ∎

Corollary 3.2.6. $V_X(I) \cup V_X(J) = V_X(IJ)$

Proof. Let $z \in V_X(I) \cup V_X(J)$. Then $I \subset \ker(z)$ and $J \subset \ker(z)$ by the lemma, hence $IJ \subset \ker(z)$. Apply the lemma again to get $z \in V_X(IJ)$. Let $z \in V_X(IJ)$. Then $IJ \subset \ker(z)$ by the lemma. The ideal $\ker(z)$ is prime, so $I \subset \ker(z)$ or $J \subset \ker(z)$. Invoke the lemma again to get $z \in V_X(I) \cup V_X(J)$. ∎

Lemma 3.2.7 (Stalks)**.** *Let* $I \subset \Gamma(1; \mathfrak{O})$. *Let* $x \in X$. *Then* $\Lambda(I)_x = I \otimes \mathfrak{O}_x$.

Proof. The functor Λ_X is exact, so it commutes with quotients. So

$$\Lambda_X(\frac{\mathfrak{O}(X)}{I}) = \frac{\mathfrak{O}}{\Lambda_X(I)}$$

and

$$\Lambda_X(\frac{\mathfrak{O}(X)}{I})_x = \frac{\mathfrak{O}_x}{\Lambda_X(I)_x} = \frac{\mathfrak{O}_x}{I \otimes \mathfrak{O}_x}$$

$\frac{\mathfrak{O}_x}{\Lambda_X(I)_x} \neq 0$, which is the same as saying that $\Lambda_X(I)_x$ is a proper ideal of $\mathfrak{O}_x$. The sheaf $\Lambda_X(I)_x$ is the sheafification of the presheaf $(U \mapsto I \otimes \mathfrak{O}(U))$, hence the stalk at x of the sheaf is $\operatorname{colim}_{x \in U} I \otimes \mathfrak{O}(U)$. The functor $I \otimes -$ is a left adjoint, hence commutes with colimits. So the stalk is isomorphic to $I \otimes \operatorname{colim}_{x \in U} \mathfrak{O}(U) = I \otimes \mathfrak{O}_x$. See Stacks[01BH]. ∎

Lemma 3.2.8. *If* η_X *is a homeomorphism, then* X *is affine.*

Proof. Let $\operatorname{Spec} A_i = U_i \subset X$ be open affines and let $\bigcup_i U_i = X$. Assume it is a finite affine cover. Using our base, we get a cover of $U_i = \bigcup_j D_X(a_{ij})$ with a_{ij} global sections. Observe that $D_X(a_{ij}) \subset U_i$, hence $D_{U_i}(U_i|_{a_{ij}}) = D_X(a_{ij})$ which makes them affine. Continuing like this, we get a finite cover of affines $D_X(a_{ij})$ of X. Since

$$F(X) = F(\bigcup_{ij} D_X(a_{ij})) = \bigcup_{ij} D_Y(a_{ij}) = \operatorname{Spec} R,$$

we have $(a_{ij}) = (1)$. Affine-ness satisfies the two requirements for the affine communication lemma [2, Ex.2.17], hence X is affine. ■

Lemma 3.2.9. *The map ϵ_X is surjective.*

Proof. Let $\mathfrak{p} \in \operatorname{Spec} R$ be a point in the target of ϵ_X. Then $\Lambda_X(\kappa(\mathfrak{p}))$ is a quasi-coherent sheaf of modules. In fact $\kappa(\mathfrak{p}) \otimes_{\Gamma}(\mathfrak{O})\mathfrak{O}(U)$ is a $\mathfrak{O}(U)$ algebra, hence $\Lambda_X(\kappa(\mathfrak{p}))$ is a quasi-coherent sheaf of algebras. Hence we can compute the relative spec Rspec $(\Lambda_X(\kappa(\mathfrak{p}))) \to X$. The adjunct of the map

$$\text{Rspec } (\Lambda_X(\kappa(\mathfrak{p}))) \to \operatorname{Spec} R$$

is the canonical morphism $g : R \to \kappa(\mathfrak{p})$. This morphism is also the adjunct of the composition

$$\text{Rspec } (\Lambda_X(\kappa(\mathfrak{p}))) \to \operatorname{Spec} \kappa(\mathfrak{p}) \to X,$$

so both maps must be equal. This gives us a commutative square

$$\begin{tikzcd} X \arrow[r, "\epsilon_X"] & \operatorname{Spec} R \\ \text{Rspec } (\Lambda_X(\kappa(\mathfrak{p}))) \arrow[u] \arrow[r] & \operatorname{Spec} \kappa(\mathfrak{p}) \arrow[u] \end{tikzcd}$$

By lemma .., we know that $\Lambda_X(\kappa(\mathfrak{p}))$ is not the zero sheaf hence the structure sheaf of Rspec $(\Lambda_X(\kappa(\mathfrak{p})))$ non-zero. This implies that the scheme is not the empty scheme. Therefore the point $\mathfrak{p}$ is in the image of ϵ_X. ■

Lemma 3.2.10. *The closed set $V_X(\mathfrak{p})$ is irreducible. This implies that ϵ_X is injective.*

Proof. Let $F(z) = \mathfrak{p}$ for some $z \in X$. By lemma .. this is possible. Let $y \in V_X(\mathfrak{p})$. Then $\ker(z) \subset \ker(y)$, hence if $y \in D_X(a)$ then $x \in D_X(a)$. Therefore y specialises to z, which thus must be $V_X(\mathfrak{p})$. This shows that it is irreducible. Uniqueness of generic points of closed irreducible subsets of schemes implies injectivity of F. ■

Lemma 3.2.11. *The counit ϵ_X is open, hence a homeomorphism.*

Proof. Note that $\epsilon_X(D_X(a)) = \{\epsilon_X(x) \mid a \notin \ker(x)\} = \epsilon_X(X) \cap D_{\operatorname{Spec} R}(a) = D_{\operatorname{Spec} R}(a)$. Our map ϵ_X is continuous and open, so a homeomorphism. ∎

Proposition 3.2.12. *The caffine scheme X is affine.*

Proof. By lemmas ?,?,? F is an homeo and so by lemma ? the other direction holds too. ∎

3.3 Enough caffines

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