Affine Objects

Mohamed Hashi

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1 Preliminaries

1.1 Basic Category Theory

Definition 1 (Presheaf category)**.** Let C be a category. Let $a \in \mathsf{C}$. Let $f : a' \to a$ We define

$$\hat{\mathsf{C}} := [\mathsf{C}^{\mathsf{op}}, \mathsf{Set}],$$

this should also be defined, even if just with "the category of presheaves on $\mathcal{C}$."

and the functor $h : \mathsf{C} \to \hat{\mathsf{C}}$ as follows

$$a \mapsto \mathrm{Hom}(-, a),$$

$$f \mapsto f \circ -.$$

This functor is fully faithful by the Yoneda lemma.

Definition 2 (Sections functor)**.** For any $a \in \mathsf{C}$ define the functor

$$\Gamma(a; -) : \hat{\mathsf{C}}(\mathsf{A}) \to \mathsf{A}$$

Is $\hat{\mathcal{C}}(A) = [\mathcal{C}^{op}, A]$? It's confusing (a little) to denote an object of $\mathcal{C}$ by a.

by

$$\mathfrak{F} \to \mathfrak{F}(a).$$

Let $L : I \to \mathsf{C}$ be diagram and assume that $\underline{\operatorname{colim}_{h(-)\circ L}\text{ exists in }\hat{\mathsf{C}}(\mathsf{A})}$. Define

is this because I could be a large category?

$$\Gamma(\underline{\operatorname*{colim}_{i \in I} L(i)}; -) : \hat{\mathsf{C}}(\mathsf{A}) \to \mathsf{A}$$

So this is a primitive symbol – $\operatorname*{colim}_{i \in I} L(i)$ need not exist?

by

$$\mathfrak{F} \to \mathrm{Hom}(\operatorname*{colim}_{i \in I} L(i), \mathfrak{F}) = \lim_{i \in I} \mathrm{Hom}(L(i), \mathfrak{F}).$$

By definition of a colimit these definitions coincide when a colimit exists in C.

~~You can sidestep the issues of colimit existence by putting~~

Remark. The category $\hat{C}$ is cocomplete so even if C does not have a terminal object, we can still compute the global sections. as $\Gamma(1;)$.

Definition 3 (Over/Under categories). Let C and C′ be categories. Let $F : C \to C'$ and $z \in C'$. Define the category C_z and C^z as

$$\mathrm{Obj}(C_z) := \{(a, w) \mid a \in C, w : F(a) \to z\},$$
$$\mathrm{Hom}((a, w), (b, v)) := \{f : a \to b \mid v \circ F(f) = w\},$$

and

$$\mathrm{Obj}(C^z) := \{(a, w) \mid a \in C, w : z \to F(a)\},$$
$$\mathrm{Hom}((a, w), (b, v)) := \{f : a \to b \mid F(f) \circ w = v\}.$$

We get faithful functors $C_z \to C : (a, w) \to a$ and $C^z \to C : (a, w) \to a$. We will call both functors localization functors and denote them by u. We will suppress the functor F where there can be no confusion. This is what you say when your notation includes F, but you sometimes choose not to write it. E.g. if you denoted the functor by $u_{F,z}$ and referred to it as u_z.

Definition 4 (Restriction). Let C be a category. ~~Let $a \in C$.~~ Let $\mathfrak{F} \in \hat{C}$.

The restriction of $\mathfrak{F}$ along a functor $\alpha : C \to D$ is defined to be $\alpha_! \mathfrak{F} : D \to \mathrm{Set}$.
This is not clear.

Definition 5 (direct image). Let $f : C \to D$. Define the direct image functor $f_* : \hat{D} \to \hat{C}$ as

$$f_* = - \circ f.$$

Definition 6 (inverse image). REDO

1.2 Topology

Definition 7 (Sieve). Let C be a category and $a \in C$. A sieve S on a is a subpresheaf of $h(a)$. Explicitly, for each $c \in C$, $S(c)$ is a subset of $\mathrm{Hom}(c, a)$ such that $fg \in S(\mathrm{Dom}(g))$ for all $f \in S(c)$ and for all $g \in h(c)$.

The maximal sieve on a, which is $h(a)$, will be denoted by $\max(a)$.

Definition 8 (Sieve category). Let C be a category and $a \in C$. The sieve category $\mathsf{Sieves}(a)$ is the subobject poset of the presheaf $h(a)$. Is this just a fancy way of saying that there's a notion of morphism $S \to S'$ given by containment?

Definition 9 (Pullback of sieve). Let C be a category and $a, b \in C$. Let S be a sieve on a. Let $f : b \to a$.

This makes it sound like f^*S depends on which c you take

For any $c \in C$ the sieve f^*S on b is given by $f^*S(c) = \{g \in \mathrm{Hom}(c, b) : fg \in S(c)\}$.

To show that this is actually a subpresheaf of $h(b)$, let $k : c \to c'$ and $h \in f^*S(c')$. Hence $fh \in S(c')$ and so $fhk \in S(c)$. Conclude that $hk \in f^*S(c')$.

This defines a functor $f^* : \mathsf{Sieves}(a) \to \mathsf{Sieves}(b)$.

Definition 10 (Sieve functors). Let C be a category. Let $a, b \in C$. Let $f : b \to a \in C_a$. Let $w : c \to a$. Let $g : w \to f \in C_a$. ← Is g a morphism $c \to b$ over a or just any morphism $c \to b$?

I think you can just remove this.

For every sieve $S \in \mathsf{Sieves}(f)$ define the sieve S' on b by $S'(c) = \bigcup_{g \in \mathrm{Hom}(c,b)} S(g)$. Let $h \in S'(c)$ and $k : c \to b$. Note that $hk \in S(gk)$ since S is a sieve on f, hence $hk \in S'(c)$. This shows that S' is a subpresheaf of $h(b)$. Let $S \xrightarrow{\alpha} R$ be a map of sieves. Define $\alpha' : S' \to R'$ to be

(don't h, k both go $c \to b$?

the argument of S should be an object of C_a. Perhaps you mean $S(f \circ g)$?

$$(\alpha')_c = \bigcup_{g \in \mathrm{Hom}(c,b)} \alpha_g.$$

For every sieve $R \in \mathsf{Sieves}(b)$ define the sieve $R^f \subset h(f)$ as follows. For each $g : c \to a \in C_a$,

$$R^f(g) = \{p : c \to b \in R(c) \mid g = f \circ p\}.$$

do you use "$\circ$" or not?

This is a sieve because if $p \in R^f(g)$ and $h : g' \to g$ arbitrary, then $gh = fph$ so $ph \in R^f(gh)$. Let $S, R \in \mathsf{Sieves}(b)$. Let $\alpha : S \to R$. Define $\alpha^f : S^f \to R^f$ by setting for each $g : c \to a \in C_a$

$$(\alpha^f)_g = \alpha_b|_{S^f(g)}$$

Define functors

$$L^f : \mathsf{Sieves}(f) \to \mathsf{Sieves}(b),$$
$$Q^f : \mathsf{Sieves}(b) \to \mathsf{Sieves}(f).$$

By, for every sieve $S \in \mathsf{Sieves}(f)$

$$L^f(S) = S',$$

This is all obscuring the fact that S and S', or R and R^f, are the same collections of morphisms.

for every $h : S \to R \in \mathrm{Sieves}(f)$.

$$L^f(h) = h',$$

For every sieve $R \in \mathrm{Sieves}(b)$

$$Q^f(R) = R^f$$

For every sieve $k : S \to R \in \mathrm{Sieves}(b)$.

$$Q^f(k) = k^f.$$

Notation 11 (Sections over a sieve)**.** Let R be a sieve on a. Let $\Gamma(R;-) = \mathrm{Hom}(R,)$.

You repeat this in def. 20. I'd remove it here.

Lemma 12. *Let C be a category. Let $a, b \in C$. Let $f : b \to a \in C_a$. We have the equalities $L^fQ^f = Id$ and $Q^fL^f = Id$.*

Proof. Let $w : c \to a$. Let $g : w \to f \in C_a$.

Let $S \in \mathrm{Sieves}(f)$. Let $h \in Q^fL^f(S)(g)$. Hence $g = fh$ and $h \in L^f(S)(c)$. This implies $h \in S(fh) = S(g)$. Let $h \in S(g)$. So $g = fh$ and $h \in L^f(S)(\mathrm{Dom}(g)) = L^f(S)(c)$. This implies $h \in Q^fL^f(S)(g)$. Therefore $Q^fL^f(S)$ and S are the same sieve.

Let $h : S \to R \in \mathrm{Sieves}(f)$. Let $p \in S(g)$. Then by construction $L^fQ^f(h)_g(p) = Q^f(h)'_c(p) = h_c(p)$.

Let $R \in \mathrm{Sieves}(b)$. Let $h \in L^fQ^f(R)(c)$. Hence $h \in Q^f(R)(g)$ for some $g : c \to a$. So $g = hf$ and $h \in R(c)$. Let $h \in R(c)$. Hence $h \in Q^f(R)(hf)$ and since $\mathrm{Dom}(hf) = c$ we get $h \in L^fQ^f(R)(c)$. Therefore L^fQ^f and R are the same sieve.

Let $h : S \to R \in \mathrm{Sieves}(b)$. Let $p \in S(c)$. Then by construction $Q^fL^f(h)_c(p) = L^f(h)_{pf}(p) = h_c(p)$.

So $L^fQ^f = \mathrm{Id}$ and $Q^fL^f = \mathrm{Id}$. ■

what is w? Should this be Q^g?

should these be p's?

Lemma 13 (equivalence respects pullback)**.** *Let C be a category. Let $a, b, c \in C$. Let $f : b \to a$ and $g : c \to a$. Let $p : g \to f \in C_a$. Let $R \in \mathrm{Sieves}(f)$. Then $L^w u(g)^* = g^*Q^f$*

not mentioned here

Proof. Let $h \in p^*R^f(t)$ for some $t \in C_a$. Then $ph \in R^f(t)$, so $ph \in R(\mathrm{Dom}(t))$ and $t = fph$. This implies $h \in p^*R(\mathrm{Dom}(t))$ and since $g = fp$ also that $t = gh$. Hence $h \in (u(p)^*R)_g(t)$.

Let $h \in (u(p)^*R)_g(t)$ for some $t \in C_a$. Then $h \in u(p)^*R(\mathrm{Dom}(t))$ and $t = gh$. So we get $ph \in R(\mathrm{Dom}(t))$ and $t = fph$, so $ph \in R^f(t)$. Hence also $h \in p^*R^f(t)$. ■

One way to make this easier on yourself would be to define a sieve on a as a collection of morphisms to a, i.e. a sub"set" of $\max(a) = \{g \mid \mathrm{cod}(g) = a\}$. This has the bonus of giving ~~max~~ $\max(a)$ and $h(a)$ distinct meanings. Right now, it's confusing that you use them interchangeably.

$$\begin{tikzcd} \mathrm{Sieves}(b) \arrow[r, "Q^f"] \arrow[d, "u(g)^*"'] & \mathrm{Sieves}(f) \arrow[d, "g^*"] \\ \mathrm{Sieves}(c) \arrow[r, "Q^g"'] & \mathrm{Sieves}(a) \end{tikzcd}$$

Definition 14 (Grothendieck Topology). A Grothendieck topology $\mathcal{T}$ is a family $\mathcal{T}(a)$ of 'covering' sieves for every $a \in C$ with the following conditions:

1. $\mathsf{max}(a) \in \mathcal{T}(a)$
2. $f^*R \in \mathcal{T}(a')$ if $R \in \mathcal{T}(a)$ for all $f : a' \to a$
3. if $f^*R \in \mathcal{T}(a')$ for all $f \in S$ with $S \in \mathcal{T}(a)$ then $R \in \mathcal{T}(a)$

Definition 15 (Basis). Let C be a category with pullbacks. A Grothendieck pretopology $\mathcal{B}$ is a collection $\mathcal{B}(a)$ of families $\{f_i : a_i \to a\}$ of 'covering' morphisms for every $a \in C$ with the following conditions.

correct placement. Good job!

the morphisms themselves are not covering morphisms

1. every isomorphism is a covering singleton family,
2. (Stability) The pullback of a covering family is a covering family. If $\{f_i : a_i \to a\}$ is covering and $g : b \to a$, then $\{f_i' : a_i \times_a b \to b\}$ is covering.
3. (Transitivity) If $\{f_i : a_i \to a\}$ is a covering family and $\{f_{ij} : a_{ij} \to a_i\}$ for every i, then $\{f_{ij} : a_{ij} \to a\}$ is a covering family.

Generating a topology from a basis: take any sieve containing a covering family to be a covering sieve.

Definition 16 (Site). A site $(C, \mathcal{T})$ is a category C with a Grothendieck topology $\mathcal{T}$. A morphism of sites $G : (C, \mathcal{T}) \to (D$~~, S) is a functor~~ $G' : C \to D$ such that

~~**Definition 17** (Cover-preserving functor). A functor $G' : C \to D$ is called cover-preserving if~~ for every covering sieve R, the family $\{G'(f) | f \in R\}$ generates a S-covering sieve.

It's okay to combine these into a single definition

Definition 18 (Category of sites). The category Sites has as objects sites and as morphisms cover-preserving functors.

1.2.1 Sheaves

Definition 19 (Matching family). Let C be a category. Let $\mathfrak{F}$ be a presheaf on on C. Let $a \in C$ be an object. Let R be a sieve on a. A set $\{x_i\}_{i \in R}$ with $x_i \in \Gamma(\mathrm{Dom}(i); \mathfrak{F})$ indexed by a sieve R and such that $x_{g \circ i} = \mathfrak{F}(g)(x_i)$ for any $g : b \to \mathrm{Dom}(i)$ and $b \in C$ is called a 'matching family'.

Here, you're already implicitly thinking of R as a collection of morphisms to a.

Definition 20 (Matching family/Morphisms)**.** Let C be a category. Let $\mathfrak{F}$ be a presheaf on on C. Let $a \in C$ be an object. Let R be a sieve on a. Define $\Gamma(R;\mathfrak{F}) = \mathrm{Hom}(R,\mathfrak{F})$. An element $\varphi \in \Gamma(R;\mathfrak{F})$ is uniquely identified by the matching family $\{\varphi(i)\}_{i\in R}$ of images. Conversely, any matching family $\{x_i\}_{i\in R}$, with $x_i \in \Gamma(\mathrm{Dom}(i);\mathfrak{F})$ indexed by R and such that $x_{g\circ i} = \mathfrak{F}(g)(x_i)$ for any $g : b \to \mathrm{Dom}(i)$ and $b \in C$, uniquely identifies a map $\varphi : R \to \mathfrak{F}$. Namely, take $\varphi_a(y) = x_y$.

Definition 21 (Amalgamation)**.** An amalgamation of a matching family $\{x_i\}_R$ is an element $x \in \Gamma(a;\mathfrak{F})$ such that $\mathfrak{F}(i)(x) = x_i$.

When you consider the matching family as a morphism φ, an amalgamation is a morphism $\phi : h(a) \to \mathfrak{F}$ that extends φ.

Definition 22 (Separated presheaf)**.** A presheaf $\mathfrak{F}$ is separated if any matching family has at most one amalgamation.

Definition 23 (Sheaves)**.** Let $(C,\mathcal{T})$ be a site. Let $\mathfrak{F} \in \hat{C}$.

A presheaf that admits a unique amalgamation for every matching family is called a sheaf. The category $\mathsf{Shv}(C)$ is the full subcategory in $\hat{C}$ of all sheaves.

In other words, we call $\mathfrak{F}$ a sheaf if for each $a \in C$ and $R \in \mathcal{T}(a)$ the map

$$\begin{aligned}\Gamma(a;\mathfrak{F}) &\to \Gamma(R;\mathfrak{F})\\ x &\mapsto \{\mathfrak{F}(i)(x)\cancel{)} \mid i \in R\}\end{aligned}$$

is an isomorphism.

Definition 24 (Plus construction)**.** Let $(C,\mathcal{T})$ be a site. Let $a, a' \in C$ and $f : a \to a'$. Let $\mathfrak{F} \in \hat{C}$. Define the functor $(-)^+ : \hat{C} \to \hat{C}$ as follows.

For all $a \in C$,

$$\mathfrak{F}^+(a) = \frac{\{(R,\varphi) \mid R \in \mathcal{T}(a), \varphi \in \Gamma(R;\mathfrak{F})\}}{\sim},$$

and for all morphisms $f \in C$,

$$\mathfrak{F}^+(f)([(R,\varphi)]) = [(f^*R, \varphi h(f))].$$

The equivalence relation is defined as:

$$(R,\varphi) \sim (S,\phi)$$

does not need to be on separate lines.

if $\varphi = \phi$ on some covering sieve $Q \subset R \cap S$.

Let $L : \mathfrak{F} \to \mathfrak{F}'$. Then

$$(L^+)_a([(R,\varphi)]) = [(R, L \circ \varphi)]$$

This functor comes with a natural transformation $\omega : \mathrm{Id} \to (-)^+$ defined by

$$\omega_{\mathfrak{F},a}(x) = [(\max(a), y)]$$

where $y(i) = \mathfrak{F}(i)(x)$. this is earlier denoted y_i.

Lemma 25 (2.10 [1]). *Let $\mathfrak{F}$ be a presheaf, $\mathfrak{G}$ a sheaf and $g : \mathfrak{F} \to \mathfrak{G}$ a morphism in $\hat{C}$. Then g factors through $\omega_{\mathfrak{F}}$ via a unique g'.*

Lemma 26 (2.11 [1]). *For every presheaf $\mathfrak{F}$, $\mathfrak{F}^+$ is separated.*

Lemma 27 (2.12 [1]). *If $\mathfrak{F}$ is separated, then $\mathfrak{F}^+$ is a sheaf.*

Definition 28. Define $sh = (-)^+ \circ (-)^+$.

Lemma 29 (Sheafification adjunction). *Let $Y = (C, \mathcal{T})$ be a site. The functor sh is left adjoint to the inclusion $\hat{Y} \to \mathrm{Shv}(C)$ with unit*

$$\omega^2_{\mathfrak{F}} = \omega_{\mathfrak{F}^+} \circ \omega_{\mathfrak{F}}.$$

should be $\hat{C} \to \mathrm{Shv}(Y)$. Also there are still italicization problems.

1.2.2 Relative topology

Definition 30 (Relative topology). Let $(C, \mathcal{T})$ be a site. Let $a \in C$.

Set $\mathcal{T}_a(f) = \{R^f : R \in \mathcal{T}(b)\}$. Define the induced topology $\mathcal{T}_a$ on C_a by, for each $f \in C_a$

what is b?

$$\mathcal{T}_a(f) = Q^f(\mathcal{T}(\mathrm{Dom}(f))).$$

This is the same definition twice.

Lemma 31. *$\mathcal{T}_a$ defines a Grothendieck topology*

Proof. Axiom 1: Q^f is an equivalence of posets. So the terminal object is send to the terminal object. Hence $\max(f) \in \mathcal{T}_a(f)$.

Axiom 2 & 3 are consequences of: Q^f is an equivalence and Q^f commutes with sieve pullback. ∎

Definition 32 (Oversite). Let $Y = (C, \mathcal{T})$ be a site. Let $a \in C$. Define the site Y_a to be the category C_a with the induced topology T_a.

Definition 33 (Natural transformation s). Let $(C, \mathcal{T})$ be a site. Let $a, b \in C$ and $f : b \to a$.

Let $\{x_i\}$ be a compatible family indexed by a sieve R on b. The same set $\{x_i\}$ is a compatible family on f indexed by $Q^f(R)$. Define the natural transformation

$$s : u_* \circ (-)^+ \to (-)^+ \circ u_*,$$

what is u here? $\mathcal{C}_a \to \mathcal{C}$?

by

$$s_{\mathfrak{F}} : u_*\mathfrak{F}^+ \to (u_*\mathfrak{F})^+$$
$$s_{\mathfrak{F},f}([\{x_i \mid i \in R\}]) = [\{x_i \mid i \in Q^f(R)\}].$$

Lemma 34 (Restriction commutes with plus). *The transformation $s_{\mathfrak{F}}$ is an isomorphism of functors.*

f ?

Proof. We use the variables from definition ?. The morphism $s_{\mathfrak{F},f}$ has an inverse, again sending the set to itself and applying Q^f on the indexing sieve. These are mutual inverses, so $s_{\mathfrak{F},f}$ is an isomorphism for each f. ■

Lemma 35 (s and ω commute). *Let $\mathfrak{F}$ be a presheaf on $Y = (C, \mathcal{T})$. Let $f : b \to a \in C$. Let $u : Y_a \to Y$ be the localisation morphism. Then $\omega_{u_*\mathfrak{F}} = s_{\mathfrak{F}} \circ (u_*\omega_{\mathfrak{F}})$.*

Proof. For any section $x \in \Gamma(b; \mathfrak{F})$. Let $x_i = \mathfrak{F}(i)(x)$ for any morphism $i \in C$. Note that $Q^f(\max(f)) = \max(b)$. This implies that the compatible family $\{x_i\}$ indexed by the maximal sieve on f is sent by s to the same set $\{x_i\}$ indexed by the maximal sieve on b. In diagram form, that

$$\begin{tikzcd} u_*\mathfrak{F} \arrow[d, "u_*\omega_{\mathfrak{F}}"] \arrow[dr, "\omega_{u_*\mathfrak{F}}"] & \\ u_*\mathfrak{F}^+ \arrow[r, "s"] & (u_*\mathfrak{F})^+ \end{tikzcd}$$

commutes. ■

what does it mean for two morphisms to be isomorphic?

Corollary 36. *Let $(\mathsf{C}, \mathfrak{T}, \mathfrak{O})$ be a ringed site. Let $b \in \mathsf{C}$. Let M be a $\Gamma(1; \mathfrak{O})$-module. The morphism $\omega^2_{\lambda(M),1,b} : \lambda(M)(b) \to \Lambda(M)(b)$ is isomorphic to*

$$\omega^2_{\lambda(M \otimes \mathfrak{O}(b)),b,Id_b} : \lambda(\lambda(M \otimes \mathfrak{O}(b)) \to \Lambda(\lambda(M \otimes \mathfrak{O}(b)).$$

These are not yet defined.

How do you apply λ and Λ more than once?

Definition 37. Define

$$s^2 : u_* \circ (-)^+ \circ (-)^+ \to (-)^+ \circ (-)^+ \circ u_*$$

as

$$s^2_{\mathfrak{F}} = (-)^+(s_{\mathfrak{F}}) \circ s_{(-)^+(\mathfrak{F})}.$$

$\mathfrak{F}^+$?

Lemma 38. *Let $\mathfrak{F}$ be a presheaf on $(\mathsf{C}, \mathfrak{T})$. Let $f : b \to a \in \mathsf{C}$. Let $\mathfrak{F}$ be a presheaf on C. Then $\omega^2_{u_*\mathfrak{F}} = s^2_{\mathfrak{F}} \circ (u_*\omega^2_{\mathfrak{F}})$.*

Proof. Let $a \in \mathsf{C}$. We have

$$\omega^2_{u_*\mathfrak{F}} = \omega_{u_*\mathfrak{F}^+} \circ \omega_{u_*\mathfrak{F}},$$

this is different (and nicer)

$$s^2_{\mathfrak{F}} = \boxed{s_{\mathfrak{F}}{}^+ \circ s_{\mathfrak{F}^+}},$$

$$u_*\omega^2_{\mathfrak{F}} = u_*(\omega_{\mathfrak{F}^+} \circ \omega_{\mathfrak{F}}).$$

Doesn't this lemma follow immediately from #35?

~~$s^2_{\mathfrak{F}} \circ (u_*\omega^2_{\mathfrak{F}})$~~

~~$= \underbrace{s_{\mathfrak{F}}^+ \circ s_{\mathfrak{F}^+} \circ u_*\omega_{\mathfrak{F}^+}} \circ u_*\omega_{\mathfrak{F}}$~~

~~$= s_{\mathfrak{F}}^+ \circ \omega_{u_*\mathfrak{F}^+} \circ u_*\omega_{\mathfrak{F}}$~~

yuck, nevermind

different u

Let $x \in \Gamma(b; u_*\mathfrak{F})$.

Then

$$\omega^2_{u_*\mathfrak{F},f}(x) = \{\{x_i \mid i \in \max(\mathrm{Dom}(j))\}_j \mid j \in \max(f)\}$$

with $x_i = \mathfrak{F}i(x)$. We also have

$$u_*\omega^2_{\mathfrak{F},f}(x) = \{\{x_i \mid i \in \max(\mathrm{Dom}(j))\}_j \mid j \in \max(b)\},$$

How is $j \in \max(b)$ and $\mathrm{Dom}(j) \in \mathsf{C}_a$?

where $\mathrm{Dom}(j) \in \mathsf{C}_a$. Apply $s_{\mathfrak{F}^+}$ on this to get

$$\{\{x_i \mid i \in \max(\mathrm{Dom}(j))\}_j \mid j \in \max(f)\},$$

where $\mathrm{Dom}(j) \in \mathsf{C}$. Lastly, apply $s_{\mathfrak{F}^+}{}^+$ to get

why is this here?

$$\{\{x_i \mid i \in \max(\mathrm{Dom}(j))\}_j \mid j \in \max(f)\}$$

where $\mathrm{Dom}(j) \in \mathsf{C}_a$. The statement of the lemma is established. ∎

Yes, it follows:

$$\omega^2_{u_*\mathfrak{F}} = \omega_{(u_*\mathfrak{F})^+} \circ \omega_{u_*\mathfrak{F}} = \omega_{(u_*\mathfrak{F})^+} \circ s_{\mathfrak{F}} \circ u_*\omega_{\mathfrak{F}}$$

$$= s_{\mathfrak{F}}^+ \circ (u_*\omega_{\mathfrak{F}})^+ \circ \omega_{u_*\mathfrak{F}} \quad \text{(naturality of } \omega)$$

$$= s_{\mathfrak{F}}^+ \circ (u_*\omega_{\mathfrak{F}})^+ \circ s_{\mathfrak{F}} \circ u_*\omega_{\mathfrak{F}}$$

$$= s_{\mathfrak{F}}^+ \circ s_{\mathfrak{F}^+} \circ u_*\omega_{\mathfrak{F}^+} \circ u_*\omega_{\mathfrak{F}} \quad \text{(naturality of } s \circ u_*\omega) = s^2 \circ u_*\omega^2.$$

Corollary 39. *Let $Y = (C, \mathcal{T})$. Let $a, b \in C$. Let $f : b \to a \in C$. Sheafifying and restricting commute via the iso*

this is what you had been calling u_*?

$$s^2 : sh_b \circ *|_b \to *|_b \circ sh_a.$$

Lemma 40 (λ commutes with restriction)**.** *Let $(C, \mathcal{T}, \mathfrak{O})$ be a ringed site. Let $a \in C$. We have a natural isomorphism $t : u_* \circ \lambda \Rightarrow \lambda \circ (- \otimes_{\Gamma(1;\mathfrak{O})} \Gamma(a; \mathfrak{O}))$.*

Proof. Define the natural transformation $t : \lambda \circ (- \otimes_{\Gamma(1;\mathfrak{O})} \Gamma(a; \mathfrak{O})) \Rightarrow u_* \circ \lambda$, by for each $\Gamma(1; \mathfrak{O})$-module M and for each $f : b \to a \in C_a$,

$$t_{M,f} : M \otimes_{\Gamma(1;\mathfrak{O})} \Gamma(a; \mathfrak{O}) \otimes_{\Gamma(a;\mathfrak{O})} \Gamma(b; \mathfrak{O}) \to M \otimes_{\Gamma(1;\mathfrak{O})} \Gamma(b; \mathfrak{O}),$$

$$m \otimes r \otimes s \mapsto m \otimes rs.$$

Every component $t_{M,f}$ is an isomorphism by basic commutative algebra. ∎

Let C be a category. Let $a \in C$. Let ϵ be the counit of the adjunction $\lambda \dashv \Gamma(1; -)$ on C. Let ϵ_a be the counit of the adjunction $\lambda_a \dashv \Gamma(a; -)$ on C_a.

Lemma 41 (λ counit commute with restriction)**.** *We have $\epsilon|_a \cong \epsilon_a$ on presheaves of the form $\lambda(M \otimes \Gamma(b; \mathfrak{O}))$ via*

I'm not sure how to read these.

$$t^{-1} \epsilon_{\lambda(M)}|_{a\lambda(M)}|_a \, t = \epsilon_a|_{\lambda(M \otimes \Gamma(b;\mathfrak{O}))}.$$

Proof. Both maps are the identity map if you unfold them. ∎

Lemma 42 (Λ commutes with restriction)**.** *Let $(C, \mathcal{T}, \mathfrak{O})$ be a ringed site. Let $f : b \to a \in C$. We have a natural isomorphism*

What does f do here?

$$q : u_* \circ \Lambda \to \Lambda \circ (- \otimes_{\Gamma(1;\mathfrak{O})} \Gamma(a; \mathfrak{O})).$$

Proof. Define q to be the composition

$$u_* \circ sh \circ \lambda \overset{s^2_\lambda}{\Rightarrow} sh \circ u_* \circ \lambda \text{ by lemma ?}$$
$$\overset{sh(t)}{\Rightarrow} sh \circ \lambda \circ u_* \text{ by lemma ?}$$

From lemma ? we get that q is an isomorphism. ∎

Is $\epsilon|_a$ the same as ϵ_a?

Lemma 43 (Λ counit commute with restriction). *Let $(\mathsf{C}, \mathfrak{T}, \mathfrak{O})$ be a ringed site. Let $a \in \mathsf{C}$. Let ϵ be the counit of the adjunction $\Lambda \dashv \Gamma(1; -)$ on C. Let ϵ_a be the counit of the adjunction $\Lambda_a \dashv \Gamma(1; -)$ on C_a. Let M be a $\Gamma(a; \mathfrak{O})$-module. We have $\epsilon|_a \cong \epsilon_a$ on modules of the form $\Lambda(M)$.*

Proof. Fix $\Lambda(M)$. Let ϵ_λ be the counit of the adjunction $\lambda \dashv \Gamma(1; -)$. By lemma ?, restricting this counit, yields the counit $\epsilon_{a,\lambda}$ of the adjunction $\lambda_a \dashv \Gamma(a; -)$. We have the commuting diagram

$$\begin{tikzcd} \lambda(M) \arrow[d, "\omega^2"'] \arrow[dr, "\epsilon_\lambda"] & \\ \Lambda(M) \arrow[r, "\epsilon"] & \mathfrak{F} \end{tikzcd}$$

After restriction and applying some natural isomorphisms, we have the commuting diagram

$$\begin{tikzcd} \lambda(M \otimes_{\Gamma(1;\mathfrak{O})} \Gamma(a; \mathfrak{O})) \arrow[d, "\omega_a^2"'] \arrow[dr, "\epsilon_{a,\lambda}"] & \\ \Lambda(M \otimes_{\Gamma(1;\mathfrak{O})} \Gamma(a; \mathfrak{O})) \arrow[r, shift left, "q^{-1}\epsilon_a q"] \arrow[r, shift right, "\epsilon|_a"'] & \mathfrak{F}|_a \end{tikzcd}$$

By the universal property of ω_a^2, it follows that $q^{-1}\epsilon_a q = \epsilon|_a$. ∎

Lemma 44. *Let $(\mathsf{C}, \mathfrak{T})$ be a* ringed? *site. Let $a \in \mathsf{C}$. Consider the conunit ϵ of $\Lambda(-) \dashv \Gamma(1; -)$ on C and the counit ϵ_a of the same adjunction on C_a. Then $\epsilon|_a = \epsilon_a$.*

Proof. Let $\mathfrak{F}$ be a sheaf module on C. Let $f : b \to a \in \mathsf{C}$. We want to show that $\epsilon_{\mathfrak{F},a,f} : \Lambda(\Gamma(1; \mathfrak{F})) \to \mathfrak{F}$ is the same map as $\epsilon_\mathfrak{F}|_{af} : \Lambda(\Gamma(1; \mathfrak{F})) \to \mathfrak{F}$

$$\alpha : \lambda(\Gamma(1; \mathfrak{F})) \to \mathfrak{F},$$

$$m \otimes r \mapsto mr$$

composed with $\omega^2_{\lambda(\Gamma(1;\mathfrak{F}))}$. ∎

Is this the same as Lemma 43 but more general? Then you can prove it all in one go, first for $\Lambda(M)$ modules and then in general.

1.3 Schemes

Definition 45 (Distinguised open). Let SpecR be a affine scheme. The set

$$D(f) = \{\mathfrak{p} \subset R | f \not\in \mathfrak{p}\}$$

for a global section f is called a distinguised open.

This is somehow out of order

Lemma 46. *A distinguised open* $D(f)$ *is affine. In particular,* $D(f) = \mathit{Spec} R_f$.

Proof. ∎

Definition 47 (Spectrum of a ring). Let R be a ring. The spectrum SpecR of R is the ringed space defined as follows. The underlying set is the set of prime ideals of R. The (zariski) topology is generated by the basis $D(f) = \{\mathfrak{p} \subset R | f \not\in \mathfrak{p}\}$. The sheaf of rings is given by

$$D(f) \mapsto R_f.$$

Definition 48 (Locus of a point). Let $(X, \mathfrak{O})$ be a scheme. Define the locus of a global section $x \in \Gamma(1; \mathfrak{O})$ to be

$$\ker(x) = \ker(\mathfrak{O}(X) \to \kappa(x)).$$

Lemma 49. *The functor*

$$\mathit{Spec} : \mathsf{Rng} \to \mathsf{LRSpaces}$$

is left adjoint to

$$\Gamma(1; -) : \mathsf{LRSpaces} \to \mathsf{Rng}.$$

With unit

$$F = \eta : (X, \mathfrak{O}) \to \mathit{Spec}(\Gamma(1; \mathfrak{O})).$$
$$x \mapsto \ker(x),$$

Proof. ∎

Definition 50 (Affine scheme). We call the ringed space SpecR an affine scheme.

Definition 51 (Scheme). A scheme S is a locally ringed space that admits a covering of affine schemes. A morphism of schemes is a morphism of locally ringed spaces. The category of schemes we will denote by Sch.

You don't need to define schemes.

2 Restrictive

2.1 Restrictive

What is such a functor?

Definition 52 (Restrictive functor). A functor $F : (C, \mathcal{T}, \mathfrak{O}) \to (D, \mathcal{S}, \mathfrak{U})$ between ringed sites is called restrictive if for every quasi-coherent module $\mathfrak{G}$ on $(D, \mathcal{S}, \mathfrak{U})$ the co-unit ϵ of $\Lambda(-) \dashv \Gamma(1; -)$ induces an isomorphism

If ε is the counit of $\Lambda \dashv \Gamma$, isn't $\varepsilon_G : \Lambda(\Gamma(G)) \to G$?

You have F, f, and $\mathfrak{f}$. What do they mean?

$$\epsilon_{\mathfrak{G}} : \mathfrak{G} \to f_* f^{-1} \mathfrak{G},$$

$$\epsilon_{\mathfrak{G},1} : \Gamma(1; \mathfrak{G}) \to \Gamma(1; \mathfrak{f}_* \mathfrak{f}^{-1} \mathfrak{G})$$

$$\epsilon_{\mathfrak{G},1} \otimes_{\Gamma(1;\mathfrak{O})} \Gamma(1; \mathfrak{U}) : \Gamma(1; \mathfrak{G}) \otimes_{\Gamma(1;\mathfrak{O})} \Gamma(1; \mathfrak{U}) \to \Gamma(1; \mathfrak{f}_* \mathfrak{f}^{-1} \mathfrak{G}).$$

Definition 53 (Restrictive morphism). A morphism $f : a \to b \in C$ is called restrictive if the induced functor

$$C_a \to C_b$$

is restrictive.

Isn't every Spec A → Spec B?

Example 54. In Sch, the morphism $\mathrm{Spec}(A_f) \to \mathrm{Spec}(A)$ is restrictive.

Lemma 55. *The composition of two restrictive functors is restrictive. If the composition* gf *is restrictive, then* g *is restrictive*

Proof. ■

Non-Example 56. The open immersion $\mathrm{Spec}(R^2) \setminus 0 \to \mathrm{Spec}(R^2)$ is not restrictive. The quasi-coherent sheaf $\Lambda(\frac{R[x,y]}{xy})$ fails to satisfy the condition from the definition.

Non-Example 57 (Affine non-restrictive map). Both canonical inclusions $\mathbb{A}^1 \to \mathbb{P}^1$ are not restrictive. Look at the quasi-coherent module $\mathfrak{O}(-1)$. There are no global sections but on every affine chart this invertible sheaf is trivial.

Non-Example 58. Any inclusion $\mathrm{Spec}(\kappa(\mathfrak{p})) \to \mathbb{P}^1$ is not restrictive. Look at $\mathcal{O}(-1)$.

Lemma 59 (Restrictive to affines). *If $f : X \to \mathit{Spec}(R)$ is a restrictive open immersion. then X is affine.*

Proof. ∎

3 Caffine objects

3.1 Caffine objects

only for quasicoherent $\mathcal{O}$-modules, right?

Definition 60 (Caffine object). Let $Y = (C, \mathcal{T}, \mathfrak{O})$ be a ringed site. Let $a \in C$ be an object. We call a caffine if the unit η and co-unit ϵ of the adjunction $\Lambda(-) \dashv \Gamma(1; -)$ on Y_a are natural isomorphisms. Another way to put this: Γ is an equivalence.

Example 61 (Examples of caffine objects). The main example to keep in mind is $\mathsf{Spec}(R) \in \mathsf{Sch}$.

Definition 62 (caffine cover). Let $(X, \mathcal{T}, \mathfrak{O})$ be a ringed site. A family of maps $\{a_i \to a\}$ is called a caffine covering of a if every a_i is caffine and the family is a covering family.

Definition 63. We say that a ringed site $(\mathcal{C}, \mathcal{O}, \mathfrak{T})$ has enough affines if any object admits a caffine covering.
different notation

Lemma 64. *Let $(C, \mathcal{T}, \mathfrak{O})$ be a ringed site. Let $a \in C$. Let $\{b_i \to a\}$ be a caffine covering on a. Assume every map $b_i \to a$ is restrictive. Then the counit ϵ of the adjunction $\Lambda(-) \dashv \Gamma(1; -)a$ is a natural isomorphism.*
?

Proof. Let $\mathfrak{F}$ be a quasi-coherent sheaf module. Set $M = \Gamma(a; \mathfrak{F})$. Set $M_i = \Gamma(b_i; \mathfrak{F})$. Set $\beta_i = \epsilon_{\mathfrak{F},a}|_{b_i}$. By lemma ? $\beta_i \cong \epsilon_{b_i}$, hence beta$a_i$ is an isomorphism.

∎

Lemma 65. *Let $(C, \mathcal{T}, \mathfrak{O})$ be a ringed site. Let $a \in C$. Let M be a $\Gamma(a; \mathfrak{O})$-module. The component*

$$\omega^2_{\lambda(M),a} : \lambda(M)(a) \to \Lambda(M)(a)$$

at Id_a of the sheafification morphism

$$\omega^2_{\Lambda(M)} : \lambda(M) \to \Lambda(M)$$

is equal to the unit of $\Lambda \dashv \Gamma(1;-)$ *in* C_a.

Proof. Consider the following maps, which you get by repeatedly calling on an adjunction.

$$\mathrm{Id} : \Lambda(M) \to \Lambda(M)$$

$$\omega^2_{\Lambda(M)} : \lambda(M) \to \Lambda(M) \text{ use sheafification adjunction, see lemma ..}$$

$$\omega^2_{\lambda(M),a} M \to \Gamma(a; \Lambda(\mathfrak{M})) \text{ take sections at } a$$

We took the adjunct of Id with respect to the sheafification adjunction as the first step. Then we took the adjunct of the result wrt the λ ajunction. Hence we get the adjunct of Id wrt the Λ adjunction. so the last map is actually the unit of the Λ adjunction. ∎

Corollary 66. *Let* $(C, \mathfrak{T}, \mathfrak{O})$ *be a ringed site. Let* $a \in C$ *be caffine. Then* $\omega^2_{\lambda(M),a}$ *is an isomorphism for any* $\mathfrak{O}(a)$*-module* M.

Theorem 67 (Morphism between caffines is restrictive). *Let* $Y = (C, \mathfrak{T}, \mathfrak{O})$. *Let* $f : b \to a \in C$ *be a morphism between caffine objects, then* f *is restrictive.*

Proof. Let $\mathfrak{F}$ be a quasi-coherent module on Y_a. Let $M = \Gamma(a; \mathfrak{F})$. Since a is caffine, we have $\mathfrak{F} = \Lambda(M)$.

We have to show that the adjunct, along the extension of scalars adjunction, of $\mathfrak{F}(f)$

$$\Gamma(a; \mathfrak{F}) \otimes_{\Gamma(a;\mathfrak{O})} \Gamma(b; \mathfrak{O}) \to \Gamma(b; \mathfrak{F})$$

is an isomorphism.

This adjunct is the component at b of the natural transformation $\omega^2_{\lambda(\Gamma(1;\mathfrak{F})),a}$. Since b is caffine, this component is an isomorphism.

∎

does this not belong here?

3.2 Caffine schemes

Let X be a caffine scheme. We will prove that that the the counit of Spec $\dashv \Gamma(1;-)$, namely

$$\epsilon_X : X \to \mathrm{Spec}(\Gamma(1;\mathfrak{O}))$$

, is an isomorphism. Set $R = \Gamma(1;\mathfrak{O})$.

Lemma 68. *The sets $D_X(a)$ form a basis for the topology of X, with a a global section.*

Proof. Let $U \subset X$ be any open. Let $x \in U$. By lemma ? we get I such that $V_X(I) = U^c$. It follows that $x \notin V_X(I)$ and $I \not\subset \ker(x)$. So we get a $g \in I$ with $g \notin \ker(x)$. We get $x \in D_X(g)$ and by corollary ?$D_X(g) \subset U$. As stated earlier, $D_X(ab) = D_X(a) \cap D_X(b)$ since $\ker(x)$ is a prime ideal. So $D_X(a)$ form a basis. ■

Lemma 69. *Every closed set $W \subset X$ can be written as $V_X(I)$ for some ideal $I \subset \Gamma(1;\mathfrak{O})$.*

Proof. Let $\mathfrak{I}$ be some ideal sheaf inducing a closed subscheme structure on W. This is always a quasi-coherent module. Let $I = \Gamma(1;\mathfrak{I}) = \Lambda(I)$. Let $\mathfrak{O}_W$ be the structure sheaf of this closed subscheme. By construction $i_!\mathfrak{O}_W = i^*\frac{\mathfrak{O}}{\mathfrak{I}}$ along the inclusion $W \xrightarrow{i} X$. Hence $V_X(I) = \mathrm{Supp}\ i_!\mathfrak{O}_W = W$ ■

Lemma 70. *Let $I \subset \Gamma(1;\mathfrak{O})$ be an ideal. The set $V_X(I)$ is closed.*

Proof. Let $z \in X$ and M a $\mathfrak{O}$-module. Assume z is in the support of M, then $g \neq 0$ for any generating element $g \in M_z$.

Consider the exact sequence

$$\mathfrak{O}(X) \to \frac{\mathfrak{O}(X)}{I} \to 0.$$

The functor Λ_X is a left adjoint hence right exact so

$$\mathfrak{O} \xrightarrow{f} \Lambda_X(\frac{\mathfrak{O}(X)}{I}) \to 0$$

is exact. Hence the sequence

$$\mathfrak{O}_x \xrightarrow{f_x} \Lambda_X(\frac{\mathfrak{O}(X)}{I})_x \to 0$$

is exact. The global section $f(1)$ must generate $\Lambda_X(\frac{\mathfrak{O}(X)}{I})$ as a module by surjectivity of f. Similarly $f_x(1_x)$ generates $\Lambda_X(\frac{\mathfrak{O}(X)}{I})_x$.

Note that $f_x(1_x) = f(1)_x$ by definition of f_x, hence $f(1)_x$ is a generating element. Hence $\Lambda_X(\frac{\mathfrak{O}(X)}{I})_x \neq 0$ if and only if $f(1)_x \neq 0$.

This implies $V_X(I) = \mathrm{Supp}(f(1))$ which makes $V_X(I)$ closed as the support of a global section. ∎

Lemma 71. *For* $x \in X$ *TFAE:*

1. $x \in V_X(I)$
2. $I\mathfrak{O}_x \neq \mathfrak{O}_x$
3. $I \subset \ker(x)$.

Proof. $1 \Rightarrow 2$:

Assume $x \in V_X(I)$. Then $\Lambda_X(\frac{\mathfrak{O}(X)}{I})_x = \frac{\mathfrak{O}_x}{I\mathfrak{O}_x} \neq 0$. Hence $I\mathfrak{O}_x \neq \mathfrak{O}_x$.

$2 \Rightarrow 3$:

Assume $I\mathfrak{O}_x \neq \mathfrak{O}_x$. Then $I\mathfrak{O}_x$ is proper hence contained in the unique maximal ideal of the local ring $\mathfrak{O}_x$, therefore $I \mapsto 0$ in $k(x)$ or equivalently $I \subset \ker(x)$.

$3 \Rightarrow 1$:

Assume $I \subset \ker(x)$. Then I maps into $\mathfrak{m}_x$, hence $I\mathfrak{O}_x \subset \mathfrak{m}_x$. Therefore

$$\frac{\mathfrak{O}_x}{\Lambda_X(I)_x} = \frac{\mathfrak{O}_x}{I\mathfrak{O}_x} \neq 0.$$

∎

Corollary 72. *If* $y \in I$ *then* $D_X(y) \cap V_X(I) = \emptyset$

Proof. Assume $y \in I$. Let $z \in V_X(I)$, then $y \in \ker(z)$ by the previous lemma. This implies $z \notin D_X(y)$ ∎

Corollary 73. $V_X(I) \cup V_X(J) = V_X(IJ)$

Proof. Let $z \in V_X(I) \cup V_X(J)$. Then $I \subset \ker(z)$ and $J \subset \ker(z)$ by the lemma, hence $IJ \subset \ker(z)$. Apply the lemma again to get $z \in V_X(IJ)$. Let $z \in V_X(IJ)$. Then $IJ \subset \ker(z)$ by the lemma. The ideal $\ker(z)$ is prime, so $I \subset \ker(z)$ or $J \subset \ker(z)$. Invoke the lemma again to get $z \in V_X(I) \cup V_X(J)$. ∎

Lemma 74 (Stalks). *Let* $I \subset \Gamma(1; \mathfrak{O})$. *Let* $x \in X$. *Then* $\Lambda(I)_x = I \otimes \mathfrak{O}_x$.

Proof. The functor Λ_X is exact, so it commutes with quotients. So

$$\Lambda_X(\frac{\mathfrak{O}(X)}{I}) = \frac{\mathfrak{O}}{\Lambda_X(I)}$$

and

$$\Lambda_X(\frac{\mathfrak{O}(X)}{I})_x = \frac{\mathfrak{O}_x}{\Lambda_X(I)_x} = \frac{\mathfrak{O}_x}{I \otimes \mathfrak{O}_x}$$

$\frac{\mathfrak{O}_x}{\Lambda_X(I)_x} \neq 0$, which is the same as saying that $\Lambda_X(I)_x$ is a proper ideal of $\mathfrak{O}_x$. The sheaf $\Lambda_X(I)_x$ is the sheafification of the presheaf $(U \mapsto I \otimes \mathfrak{O}(U))$, hence the stalk at x of the sheaf is $\operatorname{colim}_{x \in U} I \otimes \mathfrak{O}(U)$. The functor $I \otimes -$ is a left adjoint, hence commutes with colimits. So the stalk is isomorphic to $I \otimes \operatorname{colim}_{x \in U} \mathfrak{O}(U) = I \otimes \mathfrak{O}_x$. See Stacks[01BH]. ∎

Lemma 75. *If* η_X *is a homeomorphism, then* X *is affine.*

Proof. Let $\operatorname{Spec} A_i = U_i \subset X$ be open affines and let $\bigcup_i U_i = X$. Assume it is a finite affine cover. Using our base, we get a cover of $U_i = \bigcup_j D_X(a_{ij})$ with a_{ij} global sections. Observe that $D_X(a_{ij}) \subset U_i$, hence $D_{U_i}(a_{ij}|_{U_i}) = D_X(a_{ij})$ which makes them affine. Continuing like this, we get a finite cover of affines $D_X(a_{ij})$ of X. Since

$$F(X) = F(\bigcup_{ij} D_X(a_{ij})) = \bigcup_{ij} D_Y(a_{ij}) = \operatorname{Spec} R,$$

we have $(a_{ij}) = (1)$. Affine-ness satisfies the two requirements for the affine communication lemma[HAG II Ex.2.17], hence X is affine. ∎

Lemma 76. *The map* ϵ_X *is surjective.*

Proof. Let $\mathfrak{p} \in \mathrm{Spec}R$ be a point in the target of ϵ_X. Then $\Lambda_X(\kappa(\mathfrak{p}))$ is a quasi-coherent sheaf of modules. In fact $\kappa(\mathfrak{p}) \otimes_{\Gamma}(\mathfrak{O})\mathfrak{O}(U)$ is a $\mathfrak{O}(U)$ algebra, hence $\Lambda_X(\kappa(\mathfrak{p}))$ is a quasi-coherent sheaf of algebras. Hence we can compute the relative spec Rspec $(\Lambda_X(\kappa(\mathfrak{p}))) \to X$. The adjunct of the map

$$\mathrm{Rspec}\ (\Lambda_X(\kappa(\mathfrak{p}))) \to \mathrm{Spec}R$$

is the canonical morphism $g : R \to \kappa(\mathfrak{p})$. This morphism is also the adjunct of the composition

$$\mathrm{Rspec}\ (\Lambda_X(\kappa(\mathfrak{p}))) \to \mathrm{Spec}\kappa(\mathfrak{p}) \to X,$$

so both maps must be equal. This gives us a commutative square

$$\begin{tikzcd}
X \arrow[r, "\epsilon_X"] & \mathrm{Spec}R \\
\mathrm{Rspec}\ (\Lambda_X(\kappa(\mathfrak{p}))) \arrow[u] \arrow[r] & \mathrm{Spec}\kappa(\mathfrak{p}) \arrow[u]
\end{tikzcd}$$

By lemma .., we know that $\Lambda_X(\kappa(\mathfrak{p}))$ is not the zero sheaf hence the structure sheaf of Rspec $(\Lambda_X(\kappa(\mathfrak{p})))$ non-zero. This implies that the scheme is not the empty scheme. Therefore the point $\mathfrak{p}$ is in the image of ϵ_X. ∎

Lemma 77. *The closed set $V_X(\mathfrak{p})$ is irreducible. This implies that ϵ_X is injective.*

Proof. Let $F(z) = \mathfrak{p}$ for some $z \in X$. By lemma .. this is possible. Let $y \in V_X(\mathfrak{p})$. Then $\ker(z) \subset \ker(y)$, hence if $y \in D_X(a)$ then $x \in D_X(a)$. Therefore y specialises to z, which thus must be $V_X(\mathfrak{p})$. This shows that it is irreducible. Uniqueness of generic points of closed irreducible subsets of schemes implies injectivity of F. ∎

Lemma 78. *The counit ϵ_X is open, hence a homeomorphism.*

Proof. Note that $\epsilon_X(D_X(a)) = \{\epsilon_X(x) \mid a \notin \ker(x)\} = \epsilon_X(X) \cap D_{\mathrm{Spec}R}(a) = D_{\mathrm{Spec}R}(a)$. Our map ϵ_X is continuous and open, so a homeomorphism. ∎