Affine Objects

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1 Introduction

Hoi, dit is de introductie.

2 Preliminaries

2.1 Basic Category Theory

Definition 1 (Over/Under categories)**.** Let C and C' be categories. Let $F : \mathsf{C} \to \mathsf{C}'$ and $z \in \mathsf{C}'$. Define the category C_z and C^z as

$$\mathrm{Obj}(\mathsf{C}_z) := \{(a, w) \mid a \in \mathsf{C}, w : F(a) \to z\},$$
$$\mathrm{Hom}((a, w), (b, v)) := \{f : a \to b \mid v \circ F(f) = w\},$$

and

$$\mathrm{Obj}(\mathsf{C}^z) := \{(a, w) \mid a \in \mathsf{C}, w : z \to F(a)\},$$
$$\mathrm{Hom}((a, w), (b, v)) := \{f : a \to b \mid F(f) \circ w = v\}.$$

We get faithfull functors $\mathsf{C}_z \to \mathsf{C} : (a, w) \to a$ and $\mathsf{C}^z \to \mathsf{C} : (a, w) \to a$. We will call both functors localization functors and denote them by u. We will suppress the functor F where there can be no confusion.

Definition 2 (Presheaf category)**.** Let C be a category. Let $a \in \mathsf{C}$. Let $f : a' \to a$ We define

$$\hat{\mathsf{C}} := [\mathsf{C}^{\mathsf{op}}, \mathsf{Set}],$$

and the functor $h : \mathsf{C} \to \hat{\mathsf{C}}$ as follows

$$a \mapsto \mathrm{Hom}(-, a),$$

$$f \mapsto f \circ -.$$

This functor is fully faithfull by the Yoneda lemma.

2.2 Topology

Definition 3 (Sieve)**.** Let C be a category and $a \in C$. A sieve S on a is a subpresheaf of $h(a)$. Explicitely,

$$S(c) \subset \mathrm{Hom}(c, a)$$

such that

$$fg \in S(\mathrm{Dom}(g)), \forall f \in S(c), \forall g \in h(c).$$

The maximal sieve on a, which is $h(a)$ itself, will be denoted by $\max(a)$.

Definition 4 (Grothendieck Topology)**.** A Grothendieck topology $\mathcal{T}$ is a family $\mathcal{T}(a)$ of 'covering' sieves for every $a \in C$ with the following conditions:

- $\max(a) \in \mathcal{T}(a)$
- $f^*R \in \mathcal{T}(a')$ if $R \in \mathcal{T}(a)$ for any $f : a' \to a$
- if $f^*R \in \mathcal{T}(a')$ for all $f \in S$ with $S \in \mathcal{T}(a)$ then $R \in \mathcal{T}(a)$

Remark. Note that if $f \in R$ then $f^*R = \max(a')$. So if $R \subset S$ and R is covering then S is covering. Also $R \cap S$ is covering if and only if R and S are covering.

Definition 5 (Basis)**.** Let C be a category with pullbacks. A Grothendieck pretopology $\mathcal{B}$ is a collection $\mathcal{B}(a)$ of families $\{f_i : a_i \to a\}$ of 'covering' morphisms for every $a \in C$ with the following conditions.

- every isomorphism is a covering singleton family,
- (Stability) The pullback of a covering family is covering. If $\{f_i : a_i \to a\}$ is covering and $g : b \to a$, then $\{f_i' : a_i \times_a b \to b\}$ is covering.
- (Transitivity) If $\{f_i : a_i \to a\}$ is covering and $\{f_{ij} : a_{ij} \to a_i\}$ for every i, then $\{f_{ij} : a_{ij} \to a\}$ is covering.

Generating a topology from a basis: take any sieve containing a covering family to be a covering sieve.

2.2.1 Sheaves

Definition 6 (Matching family)**.** Let C be a category. Let $\mathfrak{F}$ be a presheaf on on C. Let $a \in C$ be an object. Let R be a sieve on a.

A set $\{x_i\}_{i \in R}$ with $x_i \in \Gamma(\mathrm{Dom}(i); \mathfrak{F})$ indexed by a sieve R and such that $x_{g \circ i} = \mathfrak{F}(g)(x_i)$ for any $g : b \to \mathrm{Dom}(i)$ and $b \in C$ is called a 'matching family'.

Definition 7 (Matching family/Morphisms)**.** Let C be a category. Let $\mathfrak{F}$ be a presheaf on on C. Let $a \in C$ be an object. Let R be a sieve on a.

Define $\Gamma(R; \mathfrak{F}) = \mathrm{Hom}(R, \mathfrak{F})$. An element $\varphi \in \Gamma(R; \mathfrak{F})$ is uniquely identified by the matching family $\{\varphi(i)\}_{i \in R}$ of images. Conversely, any matching family $\{x_i\}_{i \in R}$, with $x_i \in \Gamma(\mathrm{Dom}(i); \mathfrak{F})$ indexed by R and such that $x_{g \circ i} = \mathfrak{F}(g)(x_i)$ for any $g : b \to \mathrm{Dom}(i)$ and $b \in C$, uniquely identifies a map $\varphi : R \to \mathfrak{F}$. Namely, take $\varphi_a(y) = x_y$.

Definition 8 (Amalgamation)**.** An amalgamation of a matching family $\{x_i\}_R$ is an element $x \in \Gamma(1; \mathfrak{F})$ such that $\mathfrak{F}(i)(x) = x_i$.

When you consider the matching family as a morphism φ, an amalgamation is a morphism $\phi : h(a) \to \mathfrak{F}$ that extends φ.

Definition 9 (Sheaves)**.** Let $(C, \mathcal{T})$ be a site. Let $\mathfrak{F} \in \hat{C}$.

A presheaf that admits a unique amalgamation of every matching family is called a sheaf. The category $\mathrm{Shv}(C)$ is the full subcategory in $\hat{C}$ all sheaves. Let i be the inclusion functor $\mathrm{Shv}(C) \to \hat{C}$.

Definition 10 (Sheaves #2)**.** Let $(C, \mathcal{T})$ be a site. Let $\mathfrak{F} \in \hat{C}$. Let R be a sieve on $a \in C$.

We call $\mathfrak{F}$ a sheaf if the map

$$\mathfrak{F}(a) \to \mathfrak{F}(R)$$
$$a : x \mapsto \{\mathfrak{F}(i)(x)\}_{i \in R}$$

is an isomorphism.

Definition 11 (Plus construction)**.** Let $(C, \mathcal{T})$ be a site. Let $a, a' \in C$ and $f : a \to a'$. Let $\mathfrak{F} \in \hat{C}$. Define the functor $(-)^+ : \hat{C} \to \hat{C}$ as follows

On objects:

$$\mathfrak{F}^+(a) = \frac{\{(R, \varphi) \mid R \in \mathcal{T}(a), \varphi \in \Gamma(R; \mathfrak{F})\}}{\sim},$$

$$\mathfrak{F}^+(f)([[(R,\varphi)]]) = [(f^*R, h(f)\varphi)].$$

The eq. relation is defined as:

$$(R,\varphi) \sim (S,\phi)$$

if $\varphi = \phi$ on some $Q \subset R \cap S$

Let $L : \mathfrak{F} \to \mathfrak{F}'$. Then

$$(L^+)_a([(R,\varphi)]) = [(R, L \circ \varphi)]$$

This functor comes with a natural transformation $\omega : \mathrm{Id} \to (-)^+$ defined by

$$\omega_{\mathfrak{F},a}(x) = [(\max(a), y]\ , y(i) = \mathfrak{F}(i)(x).$$

Definition 12. Define $\mathrm{sh} = (-)^+ \circ (-)^+$.

Lemma 13. *Let $Y = (C, \mathcal{T})$ be a site. The functor* sh *is left adjoint to the inclusion* *Shv*$(Y) \to$ *Shv*(C) *with unit*

$$\omega^2 : \mathit{Id} \xrightarrow{\omega} (-)^+ \xrightarrow{\omega} \mathrm{sh}$$

Proof. ∎

2.2.2 Relative topology

Definition 14 (Relative topology)**.** Let $(C, \mathcal{T})$ be a site. Let $a \in C$.

The topology $\mathcal{T}$ induces a topology $\mathcal{T}_a$ on C_a as follows. Let $f : b \to a \in C_a$. Let $R \in \mathcal{T}(b)$. Define the sieve $R_f \subset h(f)$ as follows. Let $g : b' \to a \in C_a$

$$R_f(g) = \{p : b' \to b \in R(b') \mid g = f \circ p\}.$$

This is a sieve because if $p \in R_f(g)$ and $h : g' \to g$ arbitrary, then $gh = fph$ so $ph \in R_f(gh)$.

Set $\mathcal{T}_a(f) = \{R_f :\ R \in \mathcal{T}(b)\}$.

Remark. Every sieve S on $f : b \to a$ can be considered as a sieve S' on b by defining $S'(b') = \bigcup_g S(g)$ with $g : b' \to b$. This is clearly a sieve. We have $S'_f = S$ and $(S_f)' = S$. Hence $S \in \mathcal{T}(b)$ if and only if $S_f \in \mathcal{T}_a(b)$.

Lemma 15. *$\mathcal{T}_a$ defines a Grothendieck topology*

Proof. We will prove the axioms one by one.

- Axiom 1

Let $p : b' \to b \in h(f)(g)$ with $g : b' \to a$. Then $g = f \circ p$ hence $p \in \max(b)_f(g)$ so $\max(b)_f = h(f) = \max(f)$. This proves that $\max(f) \in \mathcal{T}_a(f)$.

- Axiom 2

Let $p : b' \to b \in h(f)g$ with $g : b' \to a$. Let $R_f \in \mathcal{T}_a(f)$. We have to show that $p^*R_f \in \mathcal{T}_a(g)$. We will prove $p^*R_f = (p^*R)_g$, which implies the desired result.

Let $h \in p^*R_f(t)$ for some $t \in C_a$. Then $ph \in R_f(t)$, so $ph \in R(\mathrm{Dom}(t))$ and $t = fph$. This implies $h \in p^*R(\mathrm{Dom}(t))$ and since $g = fp$ also that $t = gh$. Hence $h \in (p^*R)_g(t)$.

Let $h \in (p^*R)_g(t)$ for some $t \in C_a$. Then $h \in p^*R(\mathrm{Dom}(t))$ and $t = gh$. So we get $ph \in R(\mathrm{Dom}(t))$ and $t = fph$, so $ph \in R_f(t)$. Hence also $h \in p^*R_f(t)$.

- Axiom 3

Let $R_f \in \mathcal{T}_a(f)$. Let S_f be any sieve on f for some sieve S on b. Any sieve on f can be written as S_f for some S by the above remark Assume that $p^*S_f \in \mathcal{T}_a(g)$ for any $p : b' \to b \in R_f(g)$ with $g : b' \to a$.

By assumption $p^*S_f = (p^*S)_g$ is covering for any $p \in R_f(g)$, hence p^*S is covering for any $p \in R(b')$, hence S is covering so S_f is. ■

Lemma 16. *Let C be a category. Let $a, b \in C$. Let $(x_i)_T$ be a matching family for some presheaf $\mathfrak{F}$ on b indexed by sieve T. For any $f : b \to a$ the family $(x_i)_{T_f}$ is matching again.*

Proof. Let $u : C_a \to C$ be the localization functor. Only when a domain has an 'a' as subscript, is it taken in C_a.

We have $x_i \in \Gamma(\mathrm{Dom}(i); \mathfrak{F})$. Hence also $x_i \in \Gamma(fi; \mathfrak{F}u) = \Gamma(\mathrm{Dom}(i)_a; \mathfrak{F}u)$, where now i is considered as a morphism in C_a. Note that

$$(\mathfrak{F}u)(p)(x_i) = \mathfrak{F}(p)(x_i) = x_{iop}$$

for any $p : c \to \mathrm{Dom}(i)$, since $(x_i)_T$ is a matching family in C. ∎

Lemma 17. *Let* $Y = (C, \mathfrak{T})$. *Let* $a, b \in C$. *Let* $f : b \to a$. *Sheafifying and restricting commute. In formula form*

$$\mathrm{sh}_b \circ *\big|_b \cong *\big|_b \circ \mathrm{sh}_a.$$

Proof. <!– This proof is not improved from last time, left it out for now. Will rewrite it using new parts about relative topology. –>

∎

2.3 Modules

Definition 18 (Presheaf modules)**.** Let $Y = (C, \mathfrak{T}, \mathfrak{O})$ be a ringed site. Let $R = \Gamma(1; \mathfrak{O})$.

A presheaf module on Y is a presheaf of sets $\mathfrak{F}$ on C together with a map of presheaves

$$\mathfrak{O} \times \mathfrak{F} \to \mathfrak{F}$$

such that for every object $a \in C$ the map $\Gamma(a; \mathfrak{O}) \times \Gamma(a; \mathfrak{F}) \to \Gamma(a; \mathfrak{F})$ defines a $\Gamma(a; \mathfrak{O})$-module structure on $\Gamma(a; \mathfrak{F})$.

A morphism

$$\mathfrak{F} \to \mathfrak{G}$$

is a morphism of presheaf modules if

$$\begin{array}{ccc} \mathfrak{O} \times \mathfrak{F} & \longrightarrow & \mathfrak{F} \\ \downarrow & & \downarrow \\ \mathfrak{O} \times \mathfrak{G} & \longrightarrow & \mathfrak{G} \end{array}$$

commutes. The category of presheaf modules on Y will be denoted $\mathrm{PMod}(Y)$.

Definition 19. Let M, N be an R-module. Let $f : b \to a \in C$. Let $g : M \to N \in \mathsf{R\text{-}Mod}$. Define

$$\lambda : \mathsf{R\text{-}Mod} \to \mathrm{PMod}(Y)$$

by

$$\lambda(M)(a) = M \otimes_R \Gamma(a; \mathfrak{O}),$$

$$\lambda(M)(f) : \mathrm{Id} \otimes \mathfrak{O}(f),$$

$$\lambda(g) = (a : g \otimes \mathrm{Id}).$$

Lemma 20. *Let $Y = (X, \mathcal{T}, \mathfrak{O})$ be a site. The functor λ is left adjoint to*

$$\Gamma(1; -) : \mathit{PMod}(Y) \to \mathsf{R\text{-}Mod}$$

.

Proof. Let a be an object of C. Let M, N be R-modules. Let $\mathfrak{F}, \mathfrak{G} \in \mathrm{PMod}(Y)$ be presheaf modules.

Let $\varphi : \lambda(M) \to \mathfrak{G}$ be a morphism of presheaf modules. Let $\phi : M \to \Gamma(1; \mathfrak{G})$ be a morphism of presheaf modules.

Define

$$\alpha = H_{M,\mathfrak{G}} : \mathrm{Hom}(\lambda(M), \mathfrak{G}) \to \mathrm{Hom}(M, \Gamma(1; \mathfrak{G}))$$

by

$$\alpha(\varphi) = \varphi_1,$$

where φ_1 is the component of φ on the globalsections.

Define

$$\beta = L_{M,\mathfrak{G}} : \mathrm{Hom}(M, \Gamma(1;\mathfrak{G})) \to \mathrm{Hom}(\lambda(M), \mathfrak{G})$$

by

$$\beta(\phi)_a = \phi \otimes_R \Gamma(a;\mathfrak{O}).$$

We will show that β and α are mutually inverse.

Let $d = \beta(\alpha(\varphi))$. Let $m \otimes g \in M \otimes_R \Gamma(a;\mathfrak{O})$. Let $p : \lambda(M)(1) \to \lambda(M)(a)$ be the projection map. Let $q : \mathfrak{G}(1) \to \mathfrak{G}(a)$ be the projection map. Then $d_a(m \otimes g) = \varphi_1(m) \otimes g$ and

$$\begin{aligned}
\varphi_a(m \otimes g) &= g\varphi_a(m \otimes 1) \quad \text{linearity} \\
&= g\varphi_a(p(m)) \\
&= gq(\varphi_1(m)) \quad \text{naturality of } \varphi \\
&= g(\varphi_1(m) \otimes 1) \\
&= \varphi_1(m) \otimes g.
\end{aligned}$$

Hence $d = \varphi$. In words, the natural transformations from presheaves of the from $\lambda(M)$ are unique determined by their global sections component.

Let $d = \alpha(\beta(\phi))$. Let $m \in M$. Then $d(m) = (\phi \otimes_R R)(m) = \phi(m)$. Hence $d = \phi$, which makes H and L mutual inverses.

Naturality in M *and* $\mathfrak{G}$

Let $g : N \to M$ and $h : \mathfrak{F} \to \mathfrak{G}$. Let $\rho \in \mathrm{Hom}(\lambda(N), \mathfrak{F})$. Let $k = H_{M,\mathfrak{G}}(h \circ \rho \circ \lambda(f))$. Let $l = h_1 \circ H_{N,\mathfrak{F}}(\rho) \circ f$.

Unfolding the definition for H shows that $k = h_1\rho_1 f$ and $l = h_1\rho_1 f$ as well. This proves naturality in M and $\mathfrak{G}$ and the adjunction between λ and $\Gamma(1;-)$. ■

Definition 21. Define

$$\Lambda : \mathsf{R\text{-}Mod} \to \mathsf{Mod}(Y)$$

by $sh \circ \lambda$.

It follows from lemma .. that we have the adjunction $\Lambda \dashv \Gamma(1;-)$.

3 Caffine objects

3.1 Restrictive maps between caffine objects

Lemma 22. *Let* $(C, \mathcal{T}, \mathfrak{O})$ *be a ringed site. Let* a *be caffine. Let* M *be a* $\Gamma(a; \mathfrak{O})$*-module. The component* $\omega^2\lambda(M), a$ *at* a *of the sheafification morphism* $\omega^2_{\Lambda(M)} : \lambda(M) \to \Lambda(M)$ *is equal to the unit of* $\Lambda \dashv \Gamma(1; -)$ *in* C_a.

Proof. Consider the following maps, which you get by repeatedly calling on an adjunction.

$$\mathrm{Id} : \Lambda(M) \to \Lambda(M)$$

$$\omega^2_{\Lambda(M)} : \lambda(M) \to \Lambda(M) \text{ use sheafification adjunction, see lemma ..}$$

$$\omega^2_{\lambda(M),a} M \to \Gamma(a; \Lambda(\mathfrak{M})) \text{ take sections at } a$$

We took the adjunct of Id wrt the sheafification adjunction as the first step. Then we took the adjunct of the result wrt the λ ajunction. Hence we get the adjunct of Id wrt the Λ adjunction. so the last map is actually the unit of the Λ adjunction. This map is an isomorphism because we assume a to be caffine. ∎

Theorem 23 (Morphism between caffines is restrictive)**.** *Let* $Y = (C, \mathcal{T}, \mathfrak{O})$. *Let* $f : b \to a \in C$ *be a morphism between caffine objects, then* f *is restrictive.*

Proof. Let $\mathfrak{F}$ be a quasi-coherent module on Y_a. Let $M = \Gamma(a; \mathfrak{F})$. Since a is caffine, we have $\mathfrak{F} = \Lambda(M)$.

We have to show that the adjunct of f

$$\Gamma(a; \mathfrak{F}) \otimes_{\Gamma(a;\mathfrak{O})} \Gamma(b; \mathfrak{O}) \to \Gamma(b; \mathfrak{F})$$

is an isomorphism.

Consider

$$\begin{tikzcd}
\mathrm{PMod}(Y_a) \arrow[r, "sh_a"] \arrow[d, "*|_b"] & \mathrm{Mod}(Y_a) \arrow[d, "*|_b"] & \Gamma(a;\mathfrak{O})\text{-Mod} \arrow[l, "\Lambda"'] \arrow[d, "-\otimes\Gamma(b;\mathfrak{O})"] \arrow[ll, bend right, "\lambda"'] \\
\mathrm{PMod}(Y_b) \arrow[r, "sh_a"] & \mathrm{Mod}(Y_b) & \Gamma(b;\mathfrak{O})\text{-Mod} \arrow[l, "\Lambda"'] \arrow[ll, bend left, "\lambda"]
\end{tikzcd}$$

By a previous lemma, the left square commutes. By definition the two 'triangles' commute too and the outer square commute, hence the right square also commutes. Therefore $M \otimes \Gamma(b;\mathfrak{O}) \cong \Gamma(b;\mathfrak{F})$. This is the proof you wrote down friday.

The requirement is not to find any isomorphism but a specific one. So I think this is not enough and we need to do some bookkeeping and see if the witnessing isomorphism is our map.

Let i be the morphism of presheaves at $\lambda(M)$ of the natural transformation ω^2 coming with sh_a as defined in lemma ?. Let j be the morphism at $\lambda(M \otimes \Gamma(b;\mathfrak{O}))$ of the natural transformation ω^2 coming with sh_a as defined in lemma ? .

Consider

$$\begin{tikzcd}
\Lambda(M)\big|_b & \Lambda(M \otimes \Gamma(b;\mathfrak{O})) \arrow[l, "s_{\lambda(M)}"'] \\
\lambda(M \otimes \Gamma(b;\mathfrak{O})) \arrow[u, "i|_b"] \arrow[ur, "j"] &
\end{tikzcd}$$

We have seen that the component j_b at b, the global component, is an isomorphism in lemma ?. since b is caffine and that $s_{\lambda(M)}$ is an isomorphism as constructed in lemma ?.

We will prove commutativity of the triangle. Let $g : c \to b \in Y_b$. Let $\mathfrak{M} = \lambda(M \otimes \Gamma(c;\mathfrak{O}))$. Let $x = m \otimes r \in \mathfrak{M}$.

- TODO

Evaluating everything on the terminal object, in this case on $\mathfrak{b}$, shows that two out of three maps are isomorphisms, hence $i_{\mathfrak{b}}$ is an isomorphism. ∎