

Affine Objects

Mohamed Hashi

June 16, 2016

1 Preliminaries

This chapter introduces all the basic notions that are needed but not assumed to be known to the reader. We will start with a discussion of some purely categorical notions like slice categories and presheaves. Secondly, we will introduce a notion of a topology on a category and look at some constructions that are relevant for us. Then we will introduce modules on ringed sites. Lastly, the notion of a scheme is introduced.

1.1 Basic Category Theory

Some categorical notions like presheaves and slice categories will be introduced in this section. See [awodayCat] and [sheavGeomLogic].

Definition 1 (Presheaf category). Let C be a category. Let $a \in C$. Let $f : a' \rightarrow a$. We define the category of presheaves on C as the category of contravariant functors to the category of sets Set . We will denote it by \hat{C} .

Define the functor $h : C \rightarrow \hat{C}$ as follows

$$a \mapsto \text{Hom}(-, a),$$

$$f \mapsto f \circ -.$$

This functor is fully faithful by the Yoneda lemma.

Notation 2. Let I, C be categories. Let $L : I \rightarrow C$ be a functor. The limit over this functor will be denoted by $\lim_{i \in I} L(i)$. The colimit will be denoted by $\text{colim}_{i \in I} L(i)$.

Definition 3 (Sections functor). For any $a \in C$ define the functor

$$\Gamma(a; -) : \hat{C} \rightarrow \text{Set}$$

1 Preliminaries

by

$$\mathfrak{F} \rightarrow \mathfrak{F}(\mathfrak{a}).$$

For any presheaf \mathfrak{G} , we define

$$\Gamma(\mathfrak{G}; -) : \hat{\mathcal{C}} \rightarrow \text{Set},$$

$$\mathfrak{F} \rightarrow \text{Hom}(\mathfrak{G}, \mathfrak{F}).$$

We will mostly use this for the terminal presheaf, which will allow us to compute the global sections.

Definition 4 (Over/Under categories). Let \mathcal{C} and \mathcal{C}' be categories. Let $F : \mathcal{C} \rightarrow \mathcal{C}'$ and $z \in \mathcal{C}'$. Define the category \mathcal{C}_z and \mathcal{C}^z as

$$\text{Obj}(\mathcal{C}_z) := \{(\mathfrak{a}, w) \mid \mathfrak{a} \in \mathcal{C}, w : F(\mathfrak{a}) \rightarrow z\},$$

$$\text{Hom}((\mathfrak{a}, w), (\mathfrak{b}, v)) := \{f : \mathfrak{a} \rightarrow \mathfrak{b} \mid v \circ F(f) = w\},$$

and

$$\text{Obj}(\mathcal{C}^z) := \{(\mathfrak{a}, w) \mid \mathfrak{a} \in \mathcal{C}, w : z \rightarrow F(\mathfrak{a})\},$$

$$\text{Hom}((\mathfrak{a}, w), (\mathfrak{b}, v)) := \{f : \mathfrak{a} \rightarrow \mathfrak{b} \mid F(f) \circ w = v\}.$$

We get faithful functors $\mathcal{C}_z \rightarrow \mathcal{C} : (\mathfrak{a}, w) \rightarrow \mathfrak{a}$ and $\mathcal{C}^z \rightarrow \mathcal{C} : (\mathfrak{a}, w) \rightarrow \mathfrak{a}$. We will call both functors localization functors and denote them by u .

Definition 5 (direct image). Let $f : \mathcal{C} \rightarrow \mathcal{D}$. Define the direct image functor $f_* : \hat{\mathcal{D}} \rightarrow \hat{\mathcal{C}}$ as

$$f_* = - \circ f.$$

We call $f_*\mathfrak{F}$ the restriction of \mathfrak{F} to \mathcal{C} . When f is the induced functor $\mathcal{C}_b \rightarrow \mathcal{C}_a$ coming from a morphism $g : b \rightarrow a$ then we sometimes denote it by $\mathfrak{F}|_b$.

Definition 6 (Restriction). Let \mathcal{C}, \mathcal{D} be categories. Let $\mathfrak{F} \in \hat{\mathcal{D}}$. Let $\alpha : \mathcal{C} \rightarrow \mathcal{D}$ be a functor. The restriction of \mathfrak{F} to \mathcal{C} along α is defined to be $\alpha_*\mathfrak{F}$.

1.2 Topology

In this section we will define a notion of a topology on a category and look at the related notions of sheaves, sites and restriction of sites.

See [sheavGeomLogic] for more details.

Definition 7 (Sieve). Let C be a category and $a \in C$. Define the maximal sieve $\max(a)$ on a to be the collection of all morphisms to a . In formula,

$$\max(a) = \{f \in C \mid \text{Codom}(f) = a\}.$$

A sieve S on a is a subcollection of $\max(a)$ such that $gf \in S$ for any $f \in S$ and any g .

Definition 8 (Sieve category). Let C be a category and $a \in C$. The sieve category $\text{Sieves}(a)$ consists of all the sieves on a as objects and inclusions of sieves as morphisms.

Definition 9 (Pullback of sieve). Let C be a category and $a, b \in C$. Let S be a sieve on a . Let $f : b \rightarrow a$.

The sieve f^*S on b is given by $f^*S = \{g \in \max b : fg \in S\}$ for any $c \in C$.

To show that this is actually a sieve on b , let $k : c \rightarrow c'$ and $h \in f^*S$. Hence $fh \in S$ and so $fhk \in S$. Conclude that $hk \in f^*S$.

This defines a functor $f^* : \text{Sieves}(a) \rightarrow \text{Sieves}(b)$.

Definition 10 (Grothendieck Topology). A Grothendieck topology \mathcal{T} is a family $\mathcal{T}(a)$ of ‘covering’ sieves for every $a \in C$ with the following conditions:

1. $\max(a) \in \mathcal{T}(a)$
2. $f^*R \in \mathcal{T}(a')$ if $R \in \mathcal{T}(a)$ for all $f : a' \rightarrow a$
3. if $f^*R \in \mathcal{T}(a')$ for all $f \in S$ with $S \in \mathcal{T}(a)$ then $R \in \mathcal{T}(a)$

Definition 11 (Basis). Let C be a category with pullbacks. A Grothendieck pretopology, or basis, \mathcal{B} is a collection $\mathcal{B}(a)$ of ‘covering’ families $\{f_i : a_i \rightarrow a\}$ of morphisms for every $a \in C$ with the following conditions.

1. every isomorphism is a covering singleton family,

1 Preliminaries

2. (Stability) The pullback of a covering family is a covering family. If $\{f_i : a_i \rightarrow a\}$ is covering and $g : b \rightarrow a$, then $\{f'_i : a_i \times_a b \rightarrow b\}$ is covering.
3. (Transitivity) If $\{f_i : a_i \rightarrow a\}$ is a covering family and $\{f_{ij} : a_{ij} \rightarrow a_i\}$ for every i , then $\{f_{ij} : a_{ij} \rightarrow a\}$ is a covering family.

Generating a topology from a basis: take any sieve containing a covering family to be a covering sieve.

Remark. Sometimes you have a set of families S you would like to generate the topology. So one can take the smallest basis and subsequently topology containing these families. This will be called ‘the topology generated by S ’.

Definition 12. Let $G : C \rightarrow D$ be a functor between sites. Let $c \in C$ and let R be a covering sieve on c . G induces a map $\text{Sieves}(c) \rightarrow \text{Sieves}(G(c))$ by sending a sieve R to the sieve generated by $G(R)$. The functor G is said to preserves covers or called cover-preserving if the induced map restricts to a map $\text{Cov}c \rightarrow \text{Cov}G(c)$.

Definition 13 (Site). A site (C, \mathcal{T}) is a category C with a Grothendieck topology \mathcal{T} . A morphism of sites is a functor that preserves finite limits and covers.

The category Sites has as objects sites and morphisms of sites as just defined. When no confusion can arise then we will use C to denote the whole site (C, \mathcal{T}) .

1.2.1 Sheaves

We will introduce the very important notion of a sheaf. See [sheavGeomLogic] for a more detailed treatment.

Definition 14 (Matching family). Let C be a category. Let \mathfrak{F} be a presheaf on C . Let $a \in C$ be an object. Let R be a sieve on a . A family $\{x_i\}_{i \in R}$ with $x_i \in \Gamma(\text{Dom}(i); \mathfrak{F})$ indexed by a sieve R and such that $x_{g \circ i} = \mathfrak{F}(g)(x_i)$ for any $g : b \rightarrow \text{Dom}(i)$ and $b \in C$ is called a ‘matching family’.

Definition 15 (Amalgamation). An amalgamation of a matching family $\{x_i\}_R$ is an element $x \in \Gamma(a; \mathfrak{F})$ such that $\mathfrak{F}(i)(x) = x_i$.

Definition 16 (Sheaf). Let (C, \mathcal{T}) be a site. Let $\mathfrak{F} \in \hat{C}$.

1 Preliminaries

A presheaf that admits a unique amalgamation for every matching family is called a sheaf. The category $\text{Shv}(\mathcal{C})$ is the full subcategory in $\hat{\mathcal{C}}$ of all sheaves.

Definition 17 (Plus construction). Let $(\mathcal{C}, \mathcal{T})$ be a site. Let $\alpha, \alpha' \in \mathcal{C}$ and $f : \alpha \rightarrow \alpha'$. Let $\mathfrak{F} \in \hat{\mathcal{C}}$. Define the functor $(-)^+ : \hat{\mathcal{C}} \rightarrow \hat{\mathcal{C}}$ as follows.

For all $\alpha \in \mathcal{C}$,

$$F^+(\alpha) = \frac{\{(R, \varphi) \mid R \in \mathcal{T}(\alpha), \varphi \in \Gamma(R; \mathfrak{F})\}}{\sim},$$

for all morphisms $f \in \mathcal{C}$,

$$F^+(f)([(R, \varphi)]) = [(f^*R, \varphi h(f))].$$

The equivalence relation is defined as $(R, \varphi) \sim (S, \phi)$ if $\varphi = \phi$ on some covering sieve $Q \subset R \cap S$.

Let $L : \mathfrak{F} \rightarrow \mathfrak{F}'$. Then

$$(L^+)_\alpha([(R, \varphi)]) = [(R, L \circ \varphi)]$$

This functor comes with a natural transformation $\omega : \text{Id} \rightarrow (-)^+$ defined by

$$\omega_{\mathfrak{F}, \alpha}(x) = [(\max(\alpha), y)]$$

where

$$y(i) = \mathfrak{F}(i)(x).$$

Lemma 18. *Let \mathfrak{F} be a presheaf, \mathfrak{G} a sheaf and $g : \mathfrak{F} \rightarrow \mathfrak{G}$ a morphism in $\hat{\mathcal{C}}$. Then g factors through $\omega_{\mathfrak{F}}$ via a unique g' .*

Proof. See [toposUU]. ■

Lemma 19. *For every presheaf \mathfrak{F} , F^+ is separated.*

Proof. See [toposUU]. ■

Lemma 20. *If \mathfrak{F} is separated, then F^+ is a sheaf.*

Proof. See [toposUU]. ■

1 Preliminaries

Definition 21. Define $\text{sh} = (-)^+ \circ (-)^+$.

Lemma 22 (Sheafification adjunction). *Let (C, \mathcal{T}) be a site. The functor sh is left adjoint to the inclusion $\hat{C} \rightarrow \text{Shv}(C)$ with unit*

$$\omega_{\mathcal{T}}^2 = \omega_{\mathcal{T}^+} \circ \omega_{\mathcal{T}}.$$

Lemma 23. *Let $F : (C, \mathcal{T}) \rightarrow (D, \mathcal{S})$ be a morphism of sites, that is a bijection of covers. Then $F_*(-)^+ = (-)^+F_*$.*

Proof. ■

Remark. It follows that $F_*\text{sh} = \text{sh}F_*$.

1.2.2 Relative topology

We will look at what the induced topology on a slice category looks like and what this implies for restriction of sheaves. See [stacks-project] for a more detailed treatment.

Remark. Let C be a category and $a, b \in C$. Let $f : b \rightarrow a \in C_a$. The map $\max(f) \rightarrow \max(b)$ sending a morphism to f to its underlying morphism in C is a bijection. Moreover composition in C and C_a are the same, so this identification respects pullback of sieves. This observation yields us that $\text{Sieves}(b) = \text{Sieves}(f)$.

Whenever R is a sieve on b , we will denote the corresponding sieve on f by R_f .

Definition 24 (Relative topology). Let (C, \mathcal{T}) be a site. Let $a \in C$.

Define the induced topology \mathcal{T}_a on C_a by, for each $f \in C_a$

$$\mathcal{T}_a(f) = \mathcal{T}(\text{Dom}(f)).$$

The identification from Remark 1.2.2 implies that \mathcal{T}_a is a Grothendieck topology.

Definition 25 (Oversite). Let $Y = (C, \mathcal{T})$ be a site. Let $a \in C$. Define the site Y_a to be the category C_a with the induced topology \mathcal{T}_a .

Definition 26 (Natural transformation s). Let (C, \mathcal{T}) be a site. Let $a, b \in C$ and $f : b \rightarrow a$. Let $u : C_a \rightarrow C$.

1 Preliminaries

Let $\{\chi_i \mid i \in R\}$ be a compatible family indexed by a sieve R on b . The same set $\{\chi_i \mid i \in R\}$ is a compatible family on f indexed by the same sieve R . This yields a natural isomorphism

$$s : u_* \circ (-)^+ \rightarrow (-)^+ \circ u_*,$$

by

$$\begin{aligned} s_{\mathfrak{F}} : u_* \mathfrak{F}^+ &\rightarrow (u_* \mathfrak{F})^+ \\ s_{\mathfrak{F},f}([\chi_i \mid i \in R]) &= [\chi_i \mid i \in R_f]. \end{aligned}$$

We will treat s as an identification.

Lemma 27 (s and ω commute). *Let \mathfrak{F} be a presheaf on (C, \mathcal{T}) . Let $f : b \rightarrow a \in C$. Let $u : C_a \rightarrow C$ be the localisation morphism. Then $\omega_{u_* \mathfrak{F}} = s_{\mathfrak{F}} \circ u_* \omega_{\mathfrak{F}}$.*

Proof. For any section $\chi \in \Gamma(b; \mathfrak{F})$. Let $\chi_i = \mathfrak{F}(i)(\chi)$ for any morphism $i \in C$. Note that $\max(f) = \max(b)$. This implies that the compatible family $\{\chi_i\}$ indexed by the maximal sieve on f is sent by s to the same set $\{\chi_i\}$ indexed by the maximal sieve on b . In diagram form, that

$$\begin{array}{ccc} u_* \mathfrak{F} & & \\ \downarrow u_* \omega_{\mathfrak{F}} & \searrow \omega_{u_* \mathfrak{F}} & \\ u_* \mathfrak{F}^+ & \xrightarrow{s} & (u_* \mathfrak{F})^+ \end{array}$$

commutes. ■

Definition 28 (s^2). Define

$$s^2 : u_* \circ (-)^+ \circ (-)^+ \rightarrow (-)^+ \circ (-)^+ \circ u_*$$

as

$$s_{\mathfrak{F}}^2 = s_{\mathfrak{F}}^+ \circ s_{\mathfrak{F}^+}.$$

This is a natural isomorphism because s is one. Hence sheafifying and restricting commute via this iso

$$s^2 : u_* \circ sh_a \rightarrow sh_b \circ u_*.$$

Corollary 29. *Let \mathfrak{F} be a presheaf on (C, \mathcal{T}) . Let $f : b \rightarrow a \in C$. Let \mathfrak{F} be a presheaf on C . Then $\omega_{u_* \mathfrak{F}}^2 = s_{\mathfrak{F}}^2 \circ (u_* \omega_{\mathfrak{F}}^2)$.*

1.3 Modules

Presheaf modules and sheaf modules on a ringed site will be introduced in this section. The functors λ and Λ introduced here will be used extensively. See [stacks-project] for more detail.

Definition 30 (Presheaf modules). Let $(C, \mathcal{T}, \mathcal{O})$ be a ringed site. Let $R = \Gamma(1; \mathcal{O})$.

A presheaf module on this ringed site is a presheaf of sets \mathfrak{F} on C together with a map of presheaves

$$\mathcal{O} \times \mathfrak{F} \rightarrow \mathfrak{F}$$

such that for every object $u \in C$ the map $\Gamma(u; \mathcal{O}) \times \Gamma(u; \mathfrak{F}) \rightarrow \Gamma(u; \mathfrak{F})$ defines a $\Gamma(u; \mathcal{O})$ -module structure on $\Gamma(u; \mathfrak{F})$.

A morphism

$$\mathfrak{F} \rightarrow \mathfrak{G}$$

is a morphism of presheaf modules if

$$\begin{array}{ccc} \mathcal{O} \times \mathfrak{F} & \longrightarrow & \mathfrak{F} \\ \downarrow & & \downarrow \\ \mathcal{O} \times \mathfrak{G} & \longrightarrow & \mathfrak{G} \end{array}$$

commutes. The category of presheaf modules on C will be denoted $\text{PMod}(\mathcal{O})$.

Definition 31. Let \mathfrak{F} be a sheaf of modules on $(C, \mathcal{T}, \mathcal{O})$. It is called quasi-coherent if the following holds. For any object $u \in C$ there exists a covering sieve S such that for any map $f : b \rightarrow u$ in S there exists a presentation

$$\oplus_I b|_{\mathcal{O}} \rightarrow \oplus_J b|_{\mathcal{O}} \rightarrow b|_{\mathfrak{F}} \rightarrow 0$$

Quasi-coherent modules form a full subcategory of the category of sheafs of modules over $(C, \mathcal{T}, \mathcal{O})$ which are denoted by $\text{Qcoh}(\mathcal{O})$.

1 Preliminaries

Definition 32. Let $(C, \mathcal{T}, \mathfrak{D})$ be a ringed site. Let $R = \Gamma(1; \mathfrak{D})$. Let M, N be an R -module.

Define

$$\lambda : R\text{-Mod} \rightarrow \text{PMod}(\mathfrak{D})$$

by for all $\alpha \in C$,

$$\lambda(M)(\alpha) = M \otimes_R \Gamma(\alpha; \mathfrak{D}),$$

for all $f : b \rightarrow \alpha \in C$,

$$\lambda(M)(f) = \text{Id} \otimes \mathfrak{D}(f),$$

for all $g : M \rightarrow N \in R\text{-Mod}$,

$$\lambda(g) = (\alpha : g \otimes \text{Id}).$$

Lemma 33. Let $Y = (C, \mathcal{T}, \mathfrak{D})$ be a ringed site. The functor λ is left adjoint to

$$\Gamma(1; -) : \text{PMod}(\mathfrak{D}) \rightarrow R\text{-Mod}.$$

Proof. Let α be an object of C . Let M, N be R -modules. Let $\mathfrak{F}, \mathfrak{G} \in \text{PMod}(\mathfrak{D})$ be presheaf modules.

Let $\varphi : \lambda(M) \rightarrow \mathfrak{G}$ be a morphism of presheaf modules. Let $\phi : M \rightarrow \Gamma(1; \mathfrak{G})$ be a morphism of presheaf modules.

Define

$$\alpha = H_{M, \mathfrak{G}} : \text{Hom}(\lambda(M), \mathfrak{G}) \rightarrow \text{Hom}(M, \Gamma(1; \mathfrak{G})) \quad (1.1)$$

$$:\varphi \mapsto \varphi_1 \quad (1.2)$$

where φ_1 is the component of φ on the global sections.

Define

$$\beta = L_{M, \mathfrak{G}} : \text{Hom}(M, \Gamma(1; \mathfrak{G})) \rightarrow \text{Hom}(\lambda(M), \mathfrak{G})$$

by

$$\beta(\phi)_\alpha = \phi \otimes_R \Gamma(\alpha; \mathfrak{D}).$$

We will show that β and α are mutually inverse.

1 Preliminaries

Let $d = \beta(\alpha(\varphi))$. Let $m \otimes g \in M \otimes_R \Gamma(a; \mathfrak{D})$. Let $p : \lambda(M)(1) \rightarrow \lambda(M)(a)$ be the projection map. Let $q : \mathfrak{G}(1) \rightarrow \mathfrak{G}(a)$ be the projection map. Then $d_a(m \otimes g) = \varphi_1(m) \otimes g$ and

$$\begin{aligned} \varphi_a(m \otimes g) &= g\varphi_a(m \otimes 1) \text{ by linearity} \\ &= g\varphi_a(p(m)) \\ &= gq(\varphi_1(m)) \text{ by naturality of } \varphi \\ &= g(\varphi_1(m) \otimes 1) \\ &= \varphi_1(m) \otimes g. \end{aligned}$$

Hence $d = \varphi$. In words, the natural transformations from presheaves of the form $\lambda(M)$ are uniquely determined by their global sections component.

Let $d = \alpha(\beta(\phi))$. Let $m \in M$. Then $d(m) = (\phi \otimes_R R)(m) = \phi(m)$. Hence $d = \phi$, which makes H and L mutual inverses.

Naturality in M and \mathfrak{G}

Let $g : N \rightarrow M$ and $h : \mathfrak{F} \rightarrow \mathfrak{G}$. Let $\rho \in \text{Hom}(\lambda(N), \mathfrak{F})$. Let $k = H_{M, \mathfrak{G}}(h \circ \rho \circ \lambda(f))$. Let $l = h_1 \circ H_{N, \mathfrak{F}}(\rho) \circ f$.

Unfolding the definition for H shows that $k = h_1 \rho_1 f$ and $l = h_1 \rho_1 f$ as well. This proves naturality in M and \mathfrak{G} and the adjunction between λ and $\Gamma(1; -)$ between $R\text{-Mod}$ and $\text{PMod}(\mathfrak{D})$. ■

Definition 34. Let $(C, \mathcal{T}, \mathfrak{D})$ Define

$$\Lambda : R\text{-Mod} \rightarrow \text{Mod}(O)$$

by $\text{sh} \circ \lambda$.

It follows that we have the adjunction $\Lambda \dashv \Gamma(1; -)$. This functor is generalisation of [stacks-project]

Remark. By lemma ?, any morphism of sites that is a bijection of covers commutes with Λ . In the sense that if F is a bijection of covers, then $F_*\Lambda = \text{sh}F_*\lambda$.

Lemma 35 (λ commutes with restriction). *Let $(C, \mathcal{T}, \mathfrak{D})$ be a ringed site. Let $a \in C$. We have a natural isomorphism $t : u_* \circ \lambda \Rightarrow \lambda \circ (- \otimes_{\Gamma(1; \mathfrak{D})} \Gamma(a; \mathfrak{D}))$.*

1 Preliminaries

Proof. Define the natural transformation $t : \lambda \circ (- \otimes_{\Gamma(1; \mathfrak{D})} \Gamma(a; \mathfrak{D})) \Rightarrow u_* \circ \lambda$, by for each $\Gamma(1; \mathfrak{D})$ -module M and for each $f : b \rightarrow a \in C_a$,

$$t_{M,f} : M \otimes_{\Gamma(1; \mathfrak{D})} \Gamma(a; \mathfrak{D}) \otimes_{\Gamma(a; \mathfrak{D})} \Gamma(b; \mathfrak{D}) \rightarrow M \otimes_{\Gamma(1; \mathfrak{D})} \Gamma(b; \mathfrak{D}),$$

$$m \otimes r \otimes s \mapsto m \otimes rs.$$

Every component $t_{M,f}$ is an isomorphism by basic commutative algebra. ■

Let C be a category. Let $a \in C$. Let ϵ be the counit of the adjunction $\lambda \dashv \Gamma(1; -)$ on C . Let ϵ_a be the counit of the adjunction $\lambda_a \dashv \Gamma(a; -)$ on C_a .

Lemma 36 (λ counit commute with restriction). *We have $u_* \epsilon \cong \epsilon_a$ on presheaves of the form $\lambda_a(M \otimes \Gamma(b; \mathfrak{D}))$ with M a $\Gamma(1; \mathfrak{D})$ -module via*

$$t \circ u_* \epsilon \circ t^{-1} = \epsilon_a.$$

Proof. Both maps are the identity map if you unfold them. ■

Lemma 37 (Λ commutes with restriction). *Let $(C, \mathcal{T}, \mathfrak{D})$ be a ringed site. We have a natural isomorphism*

$$q : u_* \circ \Lambda \rightarrow \Lambda \circ (- \otimes_{\Gamma(1; \mathfrak{D})} \Gamma(a; \mathfrak{D})).$$

Proof. Define q to be the composition

$$u_* \circ sh \circ \lambda \xrightarrow{s^2 \lambda} sh \circ u_* \circ \lambda$$

$$\xrightarrow{sh(t)} sh \circ \lambda \circ - \otimes_{\Gamma(1; \mathfrak{D})} \Gamma(a; \mathfrak{D})$$

By Definition 26 and lemma 35, t and s^2 are isomorphisms so q is an isomorphism as well. ■

1.4 Schemes

We will recap the parts of scheme theory here that we use. See [vakil, HAG] for thorough treatments of scheme theory.

1 Preliminaries

Definition 38 (Spectrum of a ring). Let R be a ring. The spectrum $\text{Spec} R$ of R is the locally ringed space defined as follows. The underlying set is the set of prime ideals of R . The (Zariski) topology is generated by the basis of distinguished opens $D(f) = \{\mathfrak{p} \subset R \mid f \notin \mathfrak{p}\}$. The sheaf of rings is given on this basis by

$$D(f) \mapsto R_f.$$

A distinguished open $D(f)$ of $\text{Spec}(R)$ viewed as locally ringed space is isomorphic to $\text{Spec}(R_f)$, where the inclusion $D(f) \rightarrow \text{Spec}(R)$ corresponds to the canonical map $R \rightarrow R_f$.

Definition 39 (Locus of a point). Let (X, \mathfrak{O}) be a scheme. Define the locus of point $x \in X$ to be

$$\ker(x) = \ker(\Gamma(X; \mathfrak{O}) \rightarrow \kappa(x)).$$

Note that $\ker(x)$ is a prime ideal of $\Gamma(X; \mathfrak{O})$

Lemma 40. *For any $X \in \text{LRSpaces}$ and $R \in \text{Rng}$ we have an isomorphism*

$$\text{Hom}_{\text{LRSpaces}}(X, \text{Spec}(R)) \rightarrow \text{Hom}_{\text{Rng}}(R, \Gamma(X; \mathfrak{O}))$$

that is natural in X and R . In short

$$\text{Spec} : \text{Rng} \rightarrow \text{LRSpaces}$$

is adjoint to

$$\Gamma(1; -) : \text{LRSpaces} \rightarrow \text{Rng}.$$

Proof. Sending a morphism of locally ringed spaces to its global component will turn out to be an isomorphism. The inverse is the following map.

Let $\varphi : R \rightarrow \Gamma(X; \mathfrak{O})$ be given. We need to construct a morphism of locally ringed spaces $(f, f^\#)$. Define $f(x) = \ker x$. For distinguished open $D(f) \subset \text{Spec}(R)$, define $f^\#_{D(f)}(\frac{s}{f}) = \frac{s}{f}$. This makes sense because $f \in \Gamma(X; \mathfrak{O})$ is invertible in $D_X(f)$

These maps are mutually inverses and we have naturality. ■

Definition 41 (Scheme). We call the locally ringed space $\text{Spec}(R)$ an affine scheme.

A scheme S is a locally ringed space that admits a covering of affine schemes. A morphism of schemes is a morphism of locally ringed spaces. The category of schemes we will denote by Sch .

1 Preliminaries

When we define a sheaf \mathfrak{F} ‘on’ a scheme S , then this means we define this sheaf on the small Zariski site corresponding to S .

Definition 42 (Tilde functor). Let $\mathrm{Spec}(R)$ be an affine scheme. Let M be a R -module. Define \widetilde{M} to be the unique sheaf with

$$D(f) \mapsto M_f.$$

See [stacks-project].

Remark. This functor (with this notation) is commonly used in algebraic geometry texts. We will show that it is equal to Λ on the small Zariski site and then only use Λ .

Remark. The next result will prove that our definition of quasi-coherence in the sense of Definition 31 coincides with the usual definition for schemes using $\widetilde{}$. See [vakil] for the usual definition.

Lemma 43. *Let \mathfrak{F} be a sheaf of modules on scheme X . \mathfrak{F} is quasi-coherent on X if and only if for any open $\mathrm{Spec}(R) \subset X$ the sheaf $\mathfrak{F}|_{\mathrm{Spec}(R)}$ is isomorphic to $\Gamma(\widetilde{\mathrm{Spec}(R)}; \mathfrak{F})$.*

Proof. \Rightarrow : By assumption we get local presentations indexed by a covering. Let $\bigcup_{i \in I} U_i = X$ be this covering. Assume without loss of generality that it is an affine open covering. Let $U_i = \mathrm{Spec}(R_i)$. Let $\mathfrak{O}_{U_i}^K \rightarrow \mathfrak{O}_{U_i}^J \rightarrow \mathfrak{F}|_{U_i} \rightarrow 0$ be one of the given presentations. Taking global sections gives us an exact sequence

$$R_i^K \rightarrow R_i^J \rightarrow \Gamma(U_i; \mathfrak{F}) \rightarrow 0.$$

Tensoring it with the localisation $R_{i,f}$ for any $f \in R_i$ yields

$$R_{i,f}^K \rightarrow R_{i,f}^J \rightarrow \Gamma(U_i; \mathfrak{F}) \otimes R_{i,f} \rightarrow 0.$$

Taking sections at $D(f)$ from the sheaf sequence yields

$$R_{i,f}^K \rightarrow R_{i,f}^J \rightarrow \Gamma(D(f); \mathfrak{F}) \rightarrow 0.$$

Hence $\mathfrak{F}|_{U_i}$ is the unique sheaf with $D(f) \mapsto \Gamma(U_i; \mathfrak{F})_f$, which we defined to be $\Gamma(\widetilde{U_i}; \mathfrak{F})$. By the affine communication lemma, this property holds for any affine and not just for the affines in this covering.

1 Preliminaries

\Leftarrow : Take a presentation $R^I \rightarrow R^J \rightarrow M$ and apply $\widetilde{}$. Then note that $\widetilde{}$ commutes with infinite coproducts and $\widetilde{R} = \mathfrak{O}_{\mathrm{Spec}(R)}$. So we get a presentation $\mathfrak{O}_{\mathrm{Spec}(R)}^I \rightarrow \mathfrak{O}_{\mathrm{Spec}(R)}^J \rightarrow \mathfrak{F}$ on every affine open subset $\mathrm{Spec}(R) \subset X$, hence \mathfrak{F} is quasi-coherent. ■

Lemma 44. *Let \mathfrak{F} be a sheaf of modules on scheme X . Then \mathfrak{F} is quasi-coherent if and only if for any $D(f) = \mathrm{Spec}(S_f) \rightarrow \mathrm{Spec}(S)$ opens in X with $f \in S$ the induced map*

$$\Gamma(\mathrm{Spec}(S); \mathfrak{F}) \otimes_S S_f \rightarrow \Gamma(D(f); \mathfrak{F})$$

is an isomorphism.

Proof. \Rightarrow :

Assume \mathfrak{F} is quasi-coherent. Let $\mathrm{Spec}(S) \subset X$ be a open affine subset. By Lemma 43 we have $\mathfrak{F}|_{\mathrm{Spec}(S)} = \widetilde{M}$ for some R -module M . By construction $\Gamma(D(f); \mathfrak{F}) = M_f$ and hence the induced map is an isomorphism by basic commutative algebra.

\Leftarrow :

Assume that the induced map is an isomorphism for every $f \in S$ with $\mathrm{Spec}(S) \subset X$ open affine subset. Choose collection f_i such that $(f_i) = (1)$. Then $\Gamma(D(f_i); \mathfrak{F}) = \Gamma(\mathrm{Spec}(S); \mathfrak{F}) \otimes_S S_{f_i}$ by assumption. Hence $\mathfrak{F}|_{\mathrm{Spec}(S)} = \Gamma(\mathrm{Spec}(S); \mathfrak{F})$. By Lemma 43 this implies that \mathfrak{F} is quasi-coherent. ■

Definition 45 (Sheaf of algebras). A sheaf of algebras \mathfrak{F} on a ringed site $(C, \mathcal{T}, \mathfrak{O})$ is a sheaf of rings that comes with a (structure) morphism of sheaf of rings $\mathfrak{O} \rightarrow \mathfrak{F}$.

Definition 46 (Relative spec). Let X be a scheme. Let \mathfrak{G} be a sheaf of algebras on X that is quasi-coherent as a sheaf of modules.

Define the relative spectrum of \mathfrak{G} over X to be the scheme

$$\mathrm{Rspec} \mathfrak{G} \rightarrow X$$

that you get by glueing the spectra $\mathrm{Spec}(\Gamma(V; \mathfrak{G})) \rightarrow V \subset X$ for every affine open V . See [stacks-project].

1.5 Scheme sites

In this section we will introduce the big and small Zariski ringed site and look at how quasi-coherence and \wedge behave on these sites.

Definition 47 (Small Zariski site). Let X be a scheme. Define the small Zariski ringed site of X to be $(\text{Open}(X), \mathcal{T}, \mathfrak{D})$ with the following components. The underlying category is $\text{Open}(X)$. The topology \mathcal{T} is generated by the families $\{D(f_i) \rightarrow \text{Spec}(R) = U\}$ where $(f_i) = R$ with $f_i \in R$ for any open affine $\text{Spec}(R) \subset X$. The sheaf of rings is just the sheaf of rings coming from X . This sheaf is already defined on $\text{Open}(X)$.

Definition 48 (Big Zariski site). Define the big Zariski site to be $(\text{Sch}, \mathcal{T}, \mathfrak{D})$ with the following components. The underlying category is Sch . The topology \mathcal{T} is generated by the families $\{\text{Spec}(R_{f_i}) \rightarrow \text{Spec}(R)\}$ where $(f_i) = R$ with $f_i \in R$ for any open affine $\text{Spec}(R) \subset X$. The sheaf of rings \mathfrak{D} sends $(U, \mathfrak{D}_U) \rightarrow (X, \mathfrak{D})$ to \mathfrak{D}_U .

Definition 49 (k). Let X be a scheme. Define the functor $k : \text{Open}(X) \rightarrow \text{Sch}_X$ by $U \mapsto ((U, \mathfrak{D}_U), i)$ where $i : U \rightarrow X$ is the inclusion of the open subscheme into X .

We will show that it preserves limits and covers. The terminal $X \in \text{Open}(X)$ is sent to the terminal $X \rightarrow X$. Let $U \rightarrow V$ and $W \rightarrow V$ be two morphism in $\text{Open}(X)$. We have $k(U \cap W) = U \cap W \rightarrow X$ which is the pullback of $k(U) \rightarrow k(V)$ and $k(W) \rightarrow k(V)$.

Let $S = \{D(f_i) \rightarrow \text{Spec}(R)\}$ be one of the generating family in $\text{Open}(X)$. Note that $k(D(f_i))$ is isomorphic to the object $\text{Spec}(R_{f_i}) \rightarrow \text{Spec}(R)$. Hence $k(S)$ generates a covering sieve on Sch and hence on Sch_X .

So k is morphism of sites.

Lemma 50. *Let R be a ring. Let M be a R -module. $\widetilde{M} = \wedge(M)$ on the small Zariski site coming from $\text{Spec}(R)$.*

Proof. Let $U \subset \text{Spec}(R)$ be any open subset. Let $\bigcup_{i \in I} D(f_i) = U$ be a cover of U . Define $\lambda(M) \rightarrow \widetilde{M}$ at U to be $m \otimes u \mapsto mu$.

There is a unique morphism $\widetilde{M} \rightarrow \wedge(M)$ such that

1 Preliminaries

$$\begin{array}{ccc} \widetilde{M} & \longrightarrow & \Lambda(M) \\ \uparrow & \nearrow & \\ \lambda(M) & & \end{array}$$

commutes.

We will give this morphism only on $D(f)$ for some $f \in R$. To make this diagram commute we must send $\frac{m}{f}$ to $\{m \otimes \frac{1}{f_j} \mid j \in \max(D(f))\}$. This defines a sheaf morphism and moreover this the unique way to make the diagram commute. By the universal property of the sheafification $\widetilde{M} = \Lambda(M)$. Note that the witnessing isomorphism is functorial in M . ■

2 Restrictive

This section will introduce the notion of a restrictive morphism. We will see some examples, non-examples and results in the category of schemes and see that this notion is closely related to affinenes.

For some of the examples and results see the chapter on quasi-coherent modules in [vakil].

2.1 Restrictive

Definition 51 (Restrictive morphism). Let $(C, \mathcal{T}, \mathfrak{D})$. A morphism $f : a \rightarrow b \in C$ is called restrictive if for every quasi-coherent module \mathfrak{G} on C_b the morphism

$$\Gamma(b; \mathfrak{G}) \otimes_{\Gamma(b; \mathfrak{D})} \Gamma(a; \mathfrak{D}) \rightarrow \Gamma(a; \mathfrak{G})$$

is an isomorphism.

Example 52. In $\text{Sch}_{\text{Spec}(A)}$ the morphism $\text{Spec}(A_f) \rightarrow \text{Spec}(A)$ is restrictive. Let \mathfrak{G} be a quasi-coherent sheaf on $\text{Sch}_{\text{Spec}(A)}$. This implies that $\mathfrak{G} = \Lambda(\Gamma(\text{Spec}(A); \mathfrak{G}))$, see ???. The morphism

$$\Gamma(\text{Spec}(A); \mathfrak{G}) \otimes_{\Gamma(1; \mathfrak{D})} \Gamma(\text{Spec}(A_f); \mathfrak{D}) \rightarrow \Gamma(\text{Spec}(A_f); \mathfrak{G}) = \Gamma(A; \mathfrak{G})_f,$$

$$m \otimes r \rightarrow rm$$

is an isomorphism by basic commutative algebra.

Example 53. Let R be a ring. Consider the open immersion $U = \text{Spec}(R[x, y]) \setminus \{(x, y)\} \rightarrow \text{Spec}(R[x, y])$ and the quasi-coherent sheaf $\mathfrak{G} = \Lambda(\frac{R[x, y]}{xy})$. The global sections of this sheaf are $\frac{R[x, y]}{xy}$, as shown in

2 Restrictive

Define $U_1 = D(x) \rightarrow U$ and $U_2 = D(y) \rightarrow U$. Note that these cover U together. We have $\Gamma(U_1; \mathfrak{G}) = 0$ and $\Gamma(U_2; \mathfrak{G}) = 0$, since $\frac{R[x,y]}{xy} \big|_x = 0$ and $\frac{R[x,y]}{xy} \big|_y = 0$. Hence since \mathfrak{G} is a sheaf, we get $\Gamma(U; \mathfrak{G}) = 0$.

The sections over U of $\Lambda(R[x,y])$ are (also) $R[x,y]$. See [vakil]. We conclude that $\Gamma(1; \mathfrak{G}) \otimes_{\Gamma(1; \mathfrak{D})} \Gamma(U; \mathfrak{D}) \rightarrow \Gamma(U; \mathfrak{G})$ is not an isomorphism.

Lemma 54. *Let $(C, \mathcal{T}, \mathfrak{D})$ be a ringed site. Let $f : b \rightarrow a \in C$ and $g : c \rightarrow b \in C$ be morphisms.*

1. *If fg and f are restrictive, then g is.*
2. *if f and g are restrictive, then fg .*

If $\Gamma(b; \mathfrak{D}) \rightarrow \Gamma(c; \mathfrak{D})$ is faithfully flat then fg and g restrictive implies f is restrictive

Proof. Consider the following diagram

$$\begin{array}{ccc}
 \Gamma(a; \mathfrak{G}) \otimes_{\Gamma(a; \mathfrak{D})} \Gamma(c; \mathfrak{D}) & \xrightarrow{\hat{f} \otimes_{\Gamma(b; \mathfrak{D})} \Gamma(c; \mathfrak{D})} & \Gamma(b; \mathfrak{G}) \otimes_{\Gamma(b; \mathfrak{D})} \Gamma(c; \mathfrak{D}) \\
 \parallel & & \downarrow \hat{g} \\
 \Gamma(a; \mathfrak{G}) \otimes_{\Gamma(a; \mathfrak{D})} \Gamma(c; \mathfrak{D}) & \xrightarrow{\hat{f}g} & \Gamma(c; \mathfrak{G}),
 \end{array}$$

where $\hat{}$ is the adjoint map of $\mathfrak{G}(-)$ with respect to the extension/restriction of scalars adjunction.

This diagram commutes: going either direction sends $g \otimes r$ to rg . The results follows from commutativity. ■

Lemma 55. *Let X, Y be a schemes. The canonical morphism $X \rightarrow X \sqcup Y$ is restrictive.*

Proof. Let \mathfrak{G} be a quasi-coherent sheaf on $X \sqcup Y$. By the sheaf property $\Gamma(X \sqcup Y; \mathfrak{G}) = \Gamma(X; \mathfrak{G}) \times \Gamma(Y; \mathfrak{G})$. The same holds also for \mathfrak{D} .

We are considering the morphism

$$\Gamma(X \sqcup Y; \mathfrak{G}) \otimes_{\Gamma(X \sqcup Y; \mathfrak{D})} \Gamma(X; \mathfrak{D}) \rightarrow \Gamma(X; \mathfrak{G}).$$

2 Restrictive

By the previous remark about disjoint unions and the sheaf property and some basic commutative algebra one sees that this becomes

$$(\Gamma(X; \mathfrak{G}) \otimes \Gamma(X; \mathfrak{D})) \times (\Gamma(Y; \mathfrak{G}) \otimes \Gamma(X; \mathfrak{D})) \rightarrow \Gamma(X; \mathfrak{G}).$$

Since $\Gamma(Y; \mathfrak{G}) \otimes \Gamma(X; \mathfrak{D}) = 0$, we are left with

$$(\Gamma(X; \mathfrak{G}) \otimes \Gamma(X; \mathfrak{D})) \rightarrow \Gamma(X; \mathfrak{G})$$

$$g \otimes r \rightarrow rg.$$

Note that $\Gamma(X; \mathfrak{G})$ already is an $\Gamma(X; \mathfrak{D})$ -module and conclude that hence this morphism is an isomorphism. ■

Lemma 56. *If $X_i \rightarrow Y$ is restrictive for each $i \in I$, where I is an (possibly infinite) indexing set, Then $\bigsqcup_{i \in I} X_i \rightarrow Y$ is restrictive.*

Proof. ■

Lemma 57 (coproduct van restrictive). *Let X_1, X_2, Y be a schemes. $X_1 \rightarrow Y$ and $X_2 \rightarrow Y$ are restrictive morphisms if and only if the corresponding morphism $X_1 \sqcup X_2 \rightarrow Y$ is restrictive.*

Proof. Note that $\Gamma(X_1 \sqcup X_2; -) = \Gamma(X_1; -) \times \Gamma(X_2; -)$ by the sheaf property. We will show that

$$\Gamma(Y; \mathfrak{G}) \otimes_{\Gamma(Y; \mathfrak{D})} \Gamma(X_1 \sqcup X_2; \mathfrak{D}) \rightarrow \Gamma(X_1 \sqcup X_2; \mathfrak{G})$$

is an isomorphism.

Let $g \otimes (r, s) \in \Gamma(Y; \mathfrak{G}) \otimes_{\Gamma(Y; \mathfrak{D})} \Gamma(X_1 \sqcup X_2; \mathfrak{D})$. This map sends it to (rg, sg) . By assumption the maps

$$\Gamma(Y; \mathfrak{G}) \otimes_{\Gamma(Y; \mathfrak{D})} \Gamma(X_1; \mathfrak{D}) \rightarrow \Gamma(X_1; \mathfrak{G})$$

$$g \otimes r \mapsto rg$$

$$\Gamma(Y; \mathfrak{G}) \otimes_{\Gamma(Y; \mathfrak{D})} \Gamma(X_2; \mathfrak{D}) \rightarrow \Gamma(X_2; \mathfrak{G})$$

$$g \otimes s \mapsto sg$$

2 Restrictive

are isomorphisms. This implies directly that our map $g \otimes (r, s) \rightarrow (rg, sg)$ is an isomorphism. ■

Lemma 58 (Restrictive to affines). *If $f : X \rightarrow \text{Spec}(\mathbb{R})$ is a restrictive open immersion, then X is affine.*

Proof. Since X is an open in $\text{Spec}(\mathbb{R})$, we get a distinguished covering $\bigcup_i D(f_i) = X$ with $f_i \in \mathbb{R}$. We will prove that $(f_i) = (1)$ in $S = \Gamma(X; \mathfrak{O})$. Then we invoke the result in [harts]: if X_{f_i} are all affine and $(f_i) = (1)$ then X is affine.

Consider $M = \frac{\mathbb{R}}{(f_i)}$ as an \mathbb{R} -module and look at $\Lambda(M)$. By restrictiveness we get $M \otimes_{\mathbb{R}} S = \Lambda(M)(S)$ and by $M \otimes_{\mathbb{R}} \mathbb{R}_{f_i} = \Lambda(M)(D(f_i)) = M_{f_i} = 0$. Hence $\Lambda(M)(S) = 0$ by the sheaf axiom. This implies that $(f_i) = (1)$ in S . ■

Lemma 59. *Any morphism $\text{Spec}(S) \xrightarrow{f} \text{Spec}(\mathbb{R}) \in \text{Sch}_{\text{Spec}(\mathbb{R})}$ between affine schemes is restrictive.*

Proof. Let \mathfrak{G} be a quasi-coherent module on $\text{Sch}_{\text{Spec}(\mathbb{R})}$. Set $M = \Gamma(\text{Spec}(\mathbb{R}); \mathfrak{G})$. We want to prove that

$$M \otimes_{\mathbb{R}} S \rightarrow \Gamma(\text{Spec}(S); \mathfrak{G})$$

is an isomorphism.

Note that $\mathfrak{G} = \Lambda(M)$ since $\text{Spec}(\mathbb{R})$ is affine. See ???. For the same reason, we get $\text{Spec}(S)|_{\mathfrak{G}} = \Lambda(M \otimes \Gamma(\text{Spec}(S); \mathfrak{O}))$. By lemma ?, we know that ω^2 is an isomorphism at affine schemes.

By lemma ?, we know that for a module of the form $\Lambda(M)$ the ω^2 at the codomain of f is equal to \hat{f} if both domain and codomain are affine schemes. ■

Example 60 (Affine non-restrictive map). One might expect(or want) that any property of all maps between affine schemes also hold for affine maps between any schemes. This is not the case for restrictiveness.

Consider the canonical inclusions $\mathbb{A}^1 \rightarrow \mathbb{P}^1$ and the shifted quasi-coherent module $\mathfrak{O}(-1)$. This module is even locally free of degree 1, this is often called an invertible module.

2 Restrictive

The global sections of the module $\mathcal{O}(-1)$ are the elements of degree -1 in the global sections of \mathcal{O} . There are no such elements, hence the global sections are the zero module.

On \mathbb{A}^1 all invertible modules are isomorphic to the structure sheaf. See [vakil]. We conclude that the canonical inclusions cannot be restrictive.

Any inclusion $\mathrm{Spec}(\kappa(\mathfrak{p})) \rightarrow \mathbb{P}^1$ of a point is not restrictive which can be shown with the same argument.

This is a (more opaque) way of saying that on projective space not every quasi-coherent sheaf is generated by global sections.