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HW #3

#1

8 Suppose $v_1, \ldots, v_m$ is linearly independent in V and $w \in V$. Prove that

$$\dim \operatorname{span}(v_1 + w, \ldots, v_m + w) \geq m - 1.$$

we can let $S = \{v_1 + w, \ldots, v_m + w\}$

for each $i = 1, \ldots, m-1$ we can consider

$$(v_{i+1} + w) - (v_i + w) = v_{i+1} - v_i$$

well this means that the vectors $v_2 - v_1, v_3 - v_2, \ldots, v_m - v_{m-1}$ are all in span (S)

next, we can suppose some scalars $a_1, \ldots, a_{m-1}$ exist such that

$$a_1(v_1 - v_2) + a_2(v_3 - v_2) + \ldots + a_{m-1}(v_m - v_{m-1}) = 0$$

we get $(-a_1)v_1 + (a_1 - a_2)v_2 + (a_2 - a_3)v_3 + \ldots + (a_{m-2} - a_{m-1})v_{m-1} + a_{m-1}v_m = 0$

well since $v_1, \ldots, v_m$ are linearly independent, all the coefficients must be 0

$-a_1 = 0 \Rightarrow a_1 = 0$

$a_1 - a_2 = 0 \rightarrow a_2 = 0$

$a_2 - a_3 = 0 \rightarrow a_3 = 0$ ...... $a_{m-2} - a_{m-1} = 0 \rightarrow a_{m-1} = 0$

and $a_{m-1} = 0$ and thus so all $a_i = 0$

therefore, the set $\{v_2 - v_1, \ldots, v_m - v_{m-1}\}$ is lin. independent

and thus, we've now found $m-1$ lin. ind. vectors in span (S) so $\dim(v_1 + w, \ldots, v_m + w) \geq m - 1$

#2

11 Suppose U and W are both four-dimensional subspaces of $\mathbf{C}^6$. Prove that there exist two vectors in $U \cap W$ such that neither of these vectors is a scalar multiple of the other.

we know that U and W are 4-dim. subspaces of $\mathbb{C}^6$.

well $\dim(U + W) = \dim(U) + \dim(W) - \dim(U \cap W)$

and $\dim(U) = 4$ and $\dim(W) = 4$

well, the sum can't get bigger than the whole space

$U + W$ is a subspace of $\mathbb{C}^6$, so its dimension is at most 6 so $\dim(U + W) \leq 6$

with this, we get $\dim(U + W) = 8 - \dim(U \cap W)$

since $\dim(U + W) \leq 6$ $\quad 8 - \dim(U \cap W) \leq 6 \Rightarrow \dim(U \cap W) \geq 2$

this means that $U \cap W$ has dim. at least 2 and any basis $U \cap W$ must have at least 2 vectors

any 2 basis vectors are lin. ind. by definition (not scalar multiples), so there exists two of these vectors in the intersection

#3

15 Suppose V is finite-dimensional and V_1, V_2, V_3 are subspaces of V with $\dim V_1 + \dim V_2 + \dim V_3 > 2 \dim V$. Prove that $V_1 \cap V_2 \cap V_3 \neq \{0\}$.

well we know for any two subspaces A and B, $\dim(A \cap B) = \dim A + \dim B - \dim(A+B)$

and with that $\dim(V_1 \cap V_2 \cap V_3) \geq \dim V_1 + \dim V_2 + \dim V_3 - 2\dim V$

we're given $\dim V_1 + \dim V_2 + \dim V_3 > 2\dim V$

so $\dim(V_1 \cap V_2 \cap V_3) > 0$

and if the dim. of the intersection is at least 1, it contains a nonzero vector and thus

$V_1 \cap V_2 \cap V_3 \neq \{0\}$

#4

3 Suppose that $T \in \mathcal{L}(\mathbf{F}^n, \mathbf{F}^m)$. Show that there exist scalars $A_{j,k} \in \mathbf{F}$ for $j = 1, \dots, m$ and $k = 1, \dots, n$ such that

$$T(x_1, \dots, x_n) = (A_{1,1}x_1 + \cdots + A_{1,n}x_n, \dots, A_{m,1}x_1 + \cdots + A_{m,n}x_n)$$

for every $(x_1, \dots, x_n) \in \mathbf{F}^n$.

This exercise shows that the linear map T has the form promised in the second to last item of Example 3.3.

we can let $e_1, \dots, e_n$ be the standard basis of F^n

and then $Te_n = (A_{1,i}, \dots, A_{m,i}) \in F^m$

then if we take some $(x_1, \dots, x_n) \in F^n$, we get that

$Tx_ie_i = x_i(A_{1,i}, \dots, A_{m,i})$

and therefore $T(x_1, \dots, x_n) = (A_{1,1}x_1 + \dots + A_{1,n}x_n, \dots, A_{m,1}x_1 + \dots + A_{m,n}x_n)$

#5

4 Suppose $T \in \mathcal{L}(V,W)$ and $v_1, \dots, v_m$ is a list of vectors in V such that $Tv_1, \dots, Tv_m$ is a linearly independent list in W. Prove that $v_1, \dots, v_m$ is linearly independent.

well T is a linear map from V to W, and $T(v_1), \dots, T(v_m)$ is lin. independent in W

let's suppose that $v_1, \dots, v_m$ is not lin. independent

this means that there are some scalars $a_1, \dots, a_m$, not all zero, such that

$$a_1 v_1 + a_2 v_2 + \dots + a_m v_m = 0$$

well since T is linear, we can do

$$T(a_1 v_1 + a_2 v_2 + \dots + a_m v_m) = T(0) = 0$$

and because T is linear

$$a_1 T(v_1) + a_2 T(v_2) + \dots + a_m T(v_m) = 0$$

well we said $a_1, \dots, a_m$ are not all $\emptyset$ but the equation says that a nontrivial lin. combination of $T(v_1), \dots, T(v_m)$ is zero

this means that $T(v_1), \dots, T(v_m)$ are lin. dependent which contradicts our assumption that they are lin. ind.

thus our assumption is wrong, and so $v_1, \dots, v_m$ must be lin. independent

#6

13 Suppose V is finite-dimensional. Prove that every linear map on a subspace of V can be extended to a linear map on V. In other words, show that if U is a subspace of V and $S \in \mathcal{L}(U,W)$, then there exists $T \in \mathcal{L}(V,W)$ such that $Tu = Su$ for all $u \in U$.

The result in this exercise is used in the proof of 3.125.

let's let $U \subseteq V$ be a subspace and let $S: U \to W$ be a linear map, where W is another vector space over F

well since U is a subspace of V and V is finite dimensional, there exists some subspace $M \subseteq V$ such that $V = U \oplus M$. This means that every vector $v \in V$ can be written as $v = u + m$ for some $u \in U$, $m \in M$

well let's say $T: V \to W$

for $v = u + m$ with $u \in U$ and $m \in M$, $T(v) = S(u)$

and for any $u \in U$, $T(u) = S(u)$, so T extends S

let's show additivity: let $v_1 = u_1 + m_1$ and $v_2 = u_2 + m_2$ with $u_1, u_2 \in U$ and $m_1, m_2 \in M$

then $v_1 + v_2 = (u_1 + m_1) + (u_2 + m_2) = (u_1 + u_2) + (m_1 + m_2)$ where $u_1 + u_2 \in U$ and $m_1 + m_2 \in M$

thus, $T(v_1 + v_2) = S(u_1 + u_2) = S(u_1) + S(u_2) = T(v_1) + T(v_2)$

and now for homogeneity: for $\lambda \in F$ and $v = u + m$, $\lambda v = \lambda(u + m) = (\lambda u) + (\lambda m)$ where $\lambda u \in U$, $\lambda m \in M$

then $T(\lambda v) = S(\lambda u) = \lambda S(u) = \lambda T(v)$

and therefore T is a linear map that extends S

$\therefore$ every lin. map from the subspace U can be extended to a linear map on V

#7

16 Suppose V is finite-dimensional with $\dim V > 1$. Prove that there exist $S, T \in \mathcal{L}(V)$ such that $ST \neq TS$.

well let's suppose V is a finite-dimensional vector space with $\dim V > 1$

we can let $v_1, v_2, \ldots, v_m$ be a basis for V, where $m \geq 2$ because $\dim V > 1$

for any vector $v \in V$, we can say

$$v = a_1 v_1 + a_2 v_2 + \ldots + a_m v_m$$

then we can have two linear maps

$S \in \mathcal{L}(v)$ that swaps the coefficients of v_1 and v_2

$$S(v) = a_2 v_1 + a_1 v_2 + a_3 v_3 + \ldots + a_m v_m$$

and $T \in \mathcal{L}(v)$ that projects onto the v_1-direction

$T(v) = a_1 v_1$ so T sends v to just its v_1 component

well let's take an input $x = v_1 + 2v_2$

then $ST(x)$: $T(x) = T(v_1 + 2v_2) = T(1 \cdot v_1 + 2 \cdot v_2) = 1 \cdot v_1 = v_1$

$S(T(x)) = S(v_1) = 0 \cdot v_1 + 1 \cdot v_2 = v_2$

then $TS(x)$: $S(x) = S(v_1 + 2v_2) = 2v_1 + 1v_2$

$T(S(x)) = T(2v_1 + 1v_2) = 2v_1$

and we get $ST(x) = v_2$ $TS(x) = 2v_1$ and since $v_2 \neq 2v_1$, $ST \neq TS$

#8

17 Suppose V is finite-dimensional. Show that the only two-sided ideals of $\mathcal{L}(V)$ are $\{0\}$ and $\mathcal{L}(V)$.

*A subspace $\mathcal{E}$ of $\mathcal{L}(V)$ is called a **two-sided ideal** of $\mathcal{L}(V)$ if $TE \in \mathcal{E}$ and $ET \in \mathcal{E}$ for all $E \in \mathcal{E}$ and all $T \in \mathcal{L}(V)$.*

let's first check this case: $\{0\}$ is a two sided ideal

$\mathcal{L}(V)$ is also a two sided ideal because it is the whole space

now we can assume there is a nonzero, two sided ideal $\mathcal{E}$ in $\mathcal{L}(V)$

let's let $v_1, v_2, \ldots, v_n$ be a basis of V. well because $\mathcal{E} \neq \{0\}$, there is some nonzero map $T \in \mathcal{E}$

then for some i, $T(v_i) \neq 0$. we can say $T(v_i) = \sum_{j=i}^{n} a_j v_j$ with at least one $a_j \neq 0$

for each (k, j) we can say $S_{kj} \in \mathcal{L}(V)$ by $S_{kj}(v_m) = \begin{cases} v_k & m = j \\ 0 & m \neq j \end{cases}$ that is that S_{kj} sends v_j to v_k and everything else to $\emptyset$

we can do any building that we want

well then $A = \frac{1}{a} S_{kj}$ because $\mathcal{E}$ is a two sided ideal $AT \in \mathcal{E}$

well now $(AT)(v_i) = A(T(v_i)) = A\left(\sum_{m=1}^{n} a_m v_m\right) = a_j (A(v_j)) = a_j v_k$ and if we do $\frac{1}{a_j}$ then we get v_k

thus for any basis vector v_k, there is some value in $\mathcal{E}$ which sends v_i to v_k

now if we take all combinations of these maps, we can make anything in $\mathcal{L}(V)$

therefore $\mathcal{E}$ must contain every map in $\mathcal{E} = \mathcal{L}(V)$ and thus the only 2-sided ideals in $\mathcal{L}(V)$ are $\{0\}$ and $\mathcal{L}(V)$ itself