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HW #5

#1 ex. 9 pg. 94

9 Suppose V is finite-dimensional and $T: V \to W$ is a surjective linear map of V onto W. Prove that there is a subspace U of V such that $T|_U$ is an isomorphism of U onto W.

Here $T|_U$ means the function T restricted to U. Thus $T|_U$ is the function whose domain is U, with $T|_U$ defined by $T|_U(u) = Tu$ for every $u \in U$.

we're assuming $T: V \to W$ is surjective

if T is surjective, we know that range $T = W$ and

$$\dim V = \dim(\text{null } T) + \dim W$$

let's choose a basis $\{v_1, \ldots, v_k\}$ for null T and extend this to a basis of V by adding vectors $\{u_1, \ldots, u_m\}$ where $m = \dim W$

then we get $\{v_1, \ldots, v_k, u_1, \ldots, u_m\}$ which is a basis for V

then if we let $U = \text{span}\{u_1, \ldots, u_m\}$ we get $\dim U = m = \dim W$

now $T|_u$ is injective:

if $T(a_1u_1 + \ldots + a_mu_m) = 0$, then $a_1u_1 + \ldots + a_mu_m \in$ null T but the basis vectors $u_1, \ldots, u_m$ are lin. ind. from null T, so all $a_i = 0$

$T|_u$ is surjective:

since T is surjective, $T(u_1), \ldots, T(u_m)$ form a basis for W and any $w \in W$ can be written as a linear combination of the above

thus $T|_u : U \to W$ is an isomorphism

#2 ex. 15 pg. 94

15 Suppose $T \in \mathcal{L}(V)$ and $v_1, \ldots, v_m$ is a list in V such that $Tv_1, \ldots, Tv_m$ spans V. Prove that $v_1, \ldots, v_m$ spans V.

let's let $T \in \mathcal{L}(V)$ and $v_1, \ldots, v_m$ be a list in V s.t. $Tv_1, \ldots, Tv_m$ spans V

we want to show that any vector in V can be written as a lin. combination of $v_1, \ldots, v_m$

① because $Tv_1, \ldots, Tv_m$ spans V, this means that for every vector $w \in V$, there are scalars s.t. $w = a_1Tv_1 + \ldots + a_mTv_m$

each $T(v_1), \ldots, T(v_m)$ itself is in V and any vector in V can be written as a lin. comb. of them

each $T(v_i)$ depends linearly on v_i and these vectors lie in the same space V

② if we suppose that $v_1, \ldots, v_m$ do not span V, then their span is a subspace of $U \subseteq V$ and there is some $w \in V$ that is not a lin. combination of $v_1, \ldots, v_m$

③ but if we consider $T(w)$

there exists $b_1, \ldots, b_m$ s.t. $T(w) = b_1T(v_1) + \ldots + b_mT(v_m) = T(b_1v_1 + \ldots + b_mv_m)$

which gives us $T(w - (b_1v_1 + \ldots + b_mv_m)) = 0$ which is in null T

w can be written as a vector in span $v_1, \ldots, v_m$ plus something in the null space $\rightarrow$ which means it can be written as a combination of v_i $\Rightarrow$ which leads to a contradiction thus, the only way for $Tv_1, \ldots, Tv_m$ to span V is if $v_1, \ldots, v_m$ already spans V

#3 ex: 1 pg. 103

1 Suppose T is a function from V to W. The *graph* of T is the subset of $V \times W$ defined by

$$\text{graph of } T = \{(v, Tv) \in V \times W : v \in V\}.$$

Prove that T is a linear map if and only if the graph of T is a subspace of $V \times W$.

Formally, a function T from V to W is a subset T of $V \times W$ such that for each $v \in V$, there exists exactly one element $(v, w) \in T$. In other words, formally a function is what is called above its graph. We do not usually think of functions in this formal manner. However, if we do become formal, then this exercise could be rephrased as follows: Prove that a function T from V to W is a linear map if and only if T is a subspace of $V \times W$.

we're saying $T: V \to W$ and graph of $T = \{(v, Tv) \in V \times W : v \in V\}$

① let's first suppose T is linear, then its graph is a subspace

to check about a subspace:

① zero element $(0, T0) = (0, 0)$ is in the graph

② closed under addition let $v_1, v_2 \in V$ then $(v_1, Tv_1) + (v_2, Tv_2) = (v_1 + v_2, Tv_1 + Tv_2) = (v_1 + v_2, T(v_1 + v_2))$

③ closed under scalar mult. let $\lambda \in \mathbb{F}$ then $\lambda(v, Tv) = (\lambda v, \lambda Tv) = (\lambda v, T(\lambda v))$

② next, let's suppose the graph is a subspace, and we'll prove that T is linear

to prove linearity:

① addition:

let's let $v_1, v_2 \in V$

we know (v_1, Tv_1) and (v_2, Tv_2) are in the graph

then $(v_1 + v_2, Tv_1 + Tv_2)$

↳ for this to be in the graph, there must be some w such that $w = T(v_1 + v_2)$ with $(v_1 + v_2, w)$

therefore, $(v_1 + v_2, Tv_1 + Tv_2) = (v_1 + v_2, T(v_1 + v_2))$

thus, $T(v_1 + v_2) = Tv_1 + Tv_2$

② scalar multiplication

we know (v, Tv) is in the graph

then $\lambda(v, Tv)$ must also be in the graph

so $(\lambda v, T(\lambda v)) = (\lambda v, \lambda Tv)$, which means $T(\lambda v) = \lambda Tv$

$\therefore$ T is a linear map

#4 ex. 4 pg. 103

3 Suppose $V_1, \ldots, V_m$ are vector spaces. Prove that $\mathcal{L}(V_1 \times \cdots \times V_m, W)$ and $\mathcal{L}(V_1, W) \times \cdots \times \mathcal{L}(V_m, W)$ are isomorphic vector spaces.

There is no assumption in the exercise above or in the two following exercises that the vector spaces are finite-dimensional.

let's define a map S such that $\mathcal{L}(V_1 \times \ldots \times V_m, W) \to \mathcal{L}(V_1, W) \times \ldots \times \mathcal{L}_m(V_m, W)$

↳ this means that for each linear map T, we're doing $V_1 \times \ldots \times V_m \to W$ to $(T_1, \ldots, T_m)$ where each T_i is $T_i(v_i) = T(0, \ldots, v_i, \ldots 0)$

we must show injective and subjective

injective: if T gives all zero maps T_i, then T must send every basis vector to 0, so T is the zero map

subjective: given some linear maps $(T_1, \ldots, T_m)$ we can say $T: V_1 \times \ldots \times V_m \to W$ by

$$T(v_1, \ldots, v_m) = T_1(v_1) + \ldots + T_m(v_m)$$

this returns T_i for each of the corresponding spots

to show linearity: $S(aT + bX) = aS(T) + bS(X)$

thus these are isomorphic as vector spaces

#5 ex 10 pg. 103

10 Suppose $A_1 = v + U_1$ and $A_2 = w + U_2$ for some $v, w \in V$ and some subspaces U_1, U_2 of V. Prove that the intersection $A_1 \cap A_2$ is either a translate of some subspace of V or is the empty set.

we're saying U_1 and U_2 are subspaces of V

let's say some x is in $A_1 \cap A_2$ iff $x = v + u_1 = w + u_2$ for some $u_1 \in U_1$, $u_2 \in U_2$

we can say $v + u_1 = w + u_2$

$u_1 - u_2 = w - v$

$u_1 = w - v + u_2$

so any $x \in A_1 \cap A_2$ is of the form $x = v + u_1 = v + (w - v + u_2) = w + u_2$ for some $u_2 \in U_2$

such that $w - v + u_2 \in U_1$

we can define $S = \{u_2 \in U_2 : w - v + u_2 \in U_1\}$

then $A_1 \cap A_2 = \{w + u_2 : u_2 \in S\}$

now we just have to show S is a translate or something empty

let's say $s_0 \in S$ then $u_2 \in S$, $w - v + u_2 \in U_1$

well this means $w - v + u_2 \in U_1 \Rightarrow u_2 \in (U_1 - (w - v)) \cap U_2$

or $S = (U_1 - (w - v)) \cap U_2$

if S is empty, $A_1 \cap A_2 = \emptyset$

if not S is a translate of $U_1 \cap U_2$

if $s_0 \in S$ then every other $u_2 \in S$ can be written as

$u_2 = s_0 + t \quad t \in U_1 \cap U_2$

so $S = s_0 + (U_1 \cap U_2)$ and thus $A_1 \cap A_2 = w + s_0 + (U_1 \cap U_2)$

which is a translate of subspace $U_1 \cap U_2$

#6 ex. 14 pg #104

14 Suppose U and W are subspaces of V and $V = U \oplus W$. Suppose $w_1, \dots, w_m$ is a basis of W. Prove that $w_1 + U, \dots, w_m + U$ is a basis of V/U.

we're saying U and W are subspaces of V and $V = U \oplus W$

and $w_1, \dots, w_m$ is a basis of W

to show that $w_1 + U, \dots, w_m + U$ is a basis of V/U we must show linear independence and spanning

for the list to be lin. ind., $\lambda_1(w_1 + U) + \dots + \lambda_m(w_2 + U) + U = 0 + U = (\lambda_1 w_1 + \dots + \lambda_m w_m) + U$

$\lambda_1 w_1 + \dots + \lambda_m w_m - 0 \in U$

this means that $\lambda_1 = \dots = \lambda_m$ must be 0 because $U \cap W = \{0\}$

thus, $w_1 + U, \dots, w_m + U$ is linearly independent

next, showing $w_1 + U, \dots, w_m + U$ spans V/U

for all $v + U \in V/U$, we can say that $v = u + \lambda_1 w_1 + \dots + \lambda_m w_m$ where $u \in U$

this means $v - (\lambda_1 w_1 + \dots + \lambda_m w_m) = u \in U$

well then $v + U = (\lambda_1 w_1 + \dots + \lambda_m w_m) + U$ and this

gives us $\lambda_1(w_1 + U) + \dots + \lambda_m(w_m + U)$

which tells us that $w_1 + U, \dots, w_m + U$ spans V/U