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HW#6

1. **4** Prove or give a counterexample: If $V$ is finite-dimensional and $U$ is a subspace of $V$ that is invariant under every operator on $V$, then $U = \{0\}$ or $U = V$.

when we say invariant under every operator, we mean that for every $T: V \to V$, $T(U) \subseteq U$

let's suppose $U \neq \{0\}$ and $U \neq V$ which means we can choose some $u \in U$ with $u \neq 0$ and there is some vector $v \in V$ with $v \notin U$

using Thm 3.10 $\left\{ \begin{array}{l} \text{now if we build some operator } T: V \to V \text{ such that } T(u) = v \\ \quad \text{for all other basis vectors, we can let } T \text{ send it to } 0 \end{array} \right.$

since $u \in U$ but $T(u) = v \notin U$, using $T$ moves a vector from $U$ to outside of $U$

but if $U$ is invariant under every operator, it should be the case that $T(u) \in U$ for all $T$ which leads to a contradiction

thus no nonzero subspace can be invariant under every operator, only $U = \{0\}$ and $U = V$ works

2.

7 Define $T \in \mathcal{L}(\mathbf{F}^3)$ by $T(z_1, z_2, z_3) = (2z_2, 0, 5z_3)$. Find all eigenvalues and eigenvectors of $T$.

$T(z_1, z_2, z_3) = \lambda(z_1, z_2, z_3)$

we can say $2z_2 = \lambda z_1$ $\quad 0 = \lambda z_2$ $\quad 5z_3 = \lambda z_3$

case 1: $\lambda = 5$ $\quad$ from $0 = \lambda z_2$ with $\lambda \neq 0$, we get $z_2 = 0$

from $2z_2 = \lambda z_1$ we get $0 = 5z_1$ thus $z_1 = 0$

$5z_3 = 5z_3 \Rightarrow z_3$ is free and must be nonzero for an eigenvector

thus $\{(0,0,z_3) : z_3 \in F\} = \text{span}\{(0,0,1)\}$

case 2: $\lambda = 0$ $\quad$ we get $2z_2 = 0$ $\quad 0 = 0$ $\quad 5z_3 = 0$ $\quad$ which forces $z_2 = 0$ and $z_3 = 0$

thus $\{(z_1, 0, 0) : z_1 \in F\} = \text{span}\{(1,0,0)\}$

and final eigenvalues: $\lambda = 0, 5$

3. **15** Suppose $V$ is finite-dimensional, $T \in \mathcal{L}(V)$, and $\lambda \in \mathbf{F}$. Show that $\lambda$ is an eigenvalue of $T$ if and only if $\lambda$ is an eigenvalue of the dual operator $T' \in \mathcal{L}(V')$.

we're assuming that $V$ is fin. dim., $T \in \mathcal{L}(V)$ and $\lambda \in F$

we're also letting $T' \in \mathcal{L}(V')$ be the dual operator and we define it as $T'(\varphi) = \varphi \circ T$ for all $\varphi \in V'$

we know these defenitions:

$\lambda$ is an eigenvalue of $T$ if there exists a nonzero vector $v \in V$ such that $T(v) = \lambda v$

$\lambda$ is an eigenvalue of $T'$ if there exists a nonzero $\varphi \in V'$ such that $T'(\varphi) = \lambda\varphi$ which means for all $v \in V$,

$T'(\varphi)(v) = \lambda\varphi(v)$ but since $T'(\varphi) = \varphi \circ T$, we're basically saying $\varphi(Tv) = \lambda\varphi(v)$ for all $v \in V$

let's first prove $(\Rightarrow)$

we're assuming $\lambda$ is an eigenvalue of $T$

if $\lambda$ is an eigenvalue of $T$, then there exilts a nonzero $v \in V$ s.t. $T(v) = \lambda v$

now, if we take any $\varphi \in V'$, we know that $T'(\varphi) = \varphi \circ T$

then for any $v \in V$, we have $T'(\varphi)(v) = (\varphi \circ T)(v) = \varphi(Tv)$

and specefically for our eigenvector $v$,

$$T'(\varphi)(v) = \varphi(Tv) = \varphi(\lambda v) = \lambda\varphi(v)$$

This shows that $T'$ on any $\varphi$ at $v$ scales $\varphi(v)$ by $\lambda$

and thus $\lambda$ is an eigenvalue of $T'$ corresponding to $\varphi$

now $(\Leftarrow)$ we're assuming $\lambda$ is an eigenvalue of $T'$

then, there exists some <u>nonzero</u> $\varphi \in V'$ such that $T'(\varphi) = \lambda\varphi$

by def. of $T$, this means that for every $v \in V$,

$$T'(\varphi)(v) = \varphi(Tv) = \lambda\varphi(v)$$

and we can say $\varphi(Tv - \lambda v) = 0$ for all $v \in V$

this means that every vector of form $Tv - \lambda v$ is sent to 0 by $\varphi$ → null space of $\varphi$

↳ if we say that $\lambda$ is not an eigenvalue of $T$ that would mean for every $w \in V$, there is a $v$ such that $w = Tv - \lambda v$, so every vector in $V$ is in the null space of $\varphi$

but if a functional is zero on all $V$, then it must be the zero functional

which leads to a contradiction because we assumed $\varphi$ is nonzero

thus $\lambda$ must be an eigenvalue of $T$

4.

**16** Suppose $v_1, \ldots, v_n$ is a basis of $V$ and $T \in \mathcal{L}(V)$. Prove that if $\lambda$ is an eigenvalue of $T$, then

$$|\lambda| \leq n \max\{|\mathcal{M}(T)_{j,k}| : 1 \leq j, k \leq n\},$$

where $\mathcal{M}(T)_{j,k}$ denotes the entry in row $j$, column $k$ of the matrix of $T$ with respect to the basis $v_1, \ldots, v_n$.

*See Exercise 19 in Section 6A for a different bound on $|\lambda|$.*

we're assuming that $v_1, \ldots, v_n$ is a basis of $V$ and $T \in \mathcal{L}(V)$ and $\lambda$ is an eigenvalue of $T$

we can let $M(T)$ be the $n \times n$ matrix of $T$ with respect to the basis $v_1, \ldots, v_n$

the entries of this are $M(T)_{j,k}$

this means that $T(v_k) = \sum_{j=1}^{n} M(T)_{j,k} v_j$

if we say $\lambda$ is an eigenvalue of $T$, then there exists a nonzero vector $v \in V$ such that $T(v) = \lambda v$

since $v_1, \ldots, v_n$ is a basis, we can write $v$ as a linear combination of the basis vectors: $v = c_1 v_1 + \ldots + c_n v_n$ where not all $c_k$ are $0$

then if we do $T(v) = T\left(\sum_{k=1}^{n} c_k v_k\right) = \sum_{k=1}^{n} c_k T(v_k)$

and if we substitute $T(v_k)$ with the matrix entries

$$T(v) = \sum_{k=1}^{n} c_k \left(\sum_{j=1}^{n} M(T)_{j,k} v_j\right) = \sum_{j=1}^{n} \left(\sum_{k=1}^{n} M(T)_{j,k} c_k\right) v_j$$

we also can say

$$T(v) = \lambda v = \lambda \sum_{j=1}^{n} c_j v_j = \sum_{j=1}^{n} (\lambda c_j) v_j$$

since $v_1, \ldots, v_n$ is a basis, the coefficients of the lin. combinations must be equal

comparing $v_j$ in both, we get $\lambda c_j = \sum_{k=1}^{n} M(T)_{j,k} c_k$ for each $j = 1, \ldots, n$

well this is the eigenvalue equation in matrix form: $M(T)c = \lambda c$ where $c$ is the column vector of the coefficients $\begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix}$

if we let $c_p$ be a coefficient with the largest abs. value among all the coefficients of $v$, so basically $|c_p| = \max\{|c_1|, \ldots, |c_n|\}$ and since $v$ is a non zero vector $|c_p| > 0$

so if we look at equation for row $p$: $\lambda c_p = \sum_{k=1}^{n} M(T)_{p,k} c_k$ and take abs. value of both sides and use Triangle Inequality, we get

$$|\lambda c_p| = \left|\sum_{k=1}^{n} M(T)_{p,k} c_k\right| \leq \sum_{k=1}^{n} |M(T)_{p,k} c_k| = \sum_{k=1}^{n} |M(T)_{p,k}| \, |c_k|$$

then if we let $M = \max \{ |M(T)_{j,k}| : 1 \le j, k \le n \}$

we can use the fact that $|c_k| \le |c_p|$ for all $k$, and $|M(T)_{p,k}| \le M$ for all $k$, and say

$$\sum_{k=1}^{n} |M(T)_{p,k}||c_k| \le \sum_{k=1}^{n} M|c_p| = M|c_p| \sum_{k=1}^{n} 1 = nM|c_p|$$

so we get $|\lambda c_p| \le nM|c_p|$

since $|c_p| > 0$, we can divide both sides by $|c_p|$ to get $|\lambda| \le nM$

and if we substitude to def. of $M$, we get $|\lambda| \le n \max\{ |M(T)_{j,k}| : 1 \le j, k \le n \}$

5. **19** Show that the forward shift operator $T \in \mathcal{L}(\mathbf{F}^\infty)$ defined by

$$T(z_1, z_2, \dots) = (0, z_1, z_2, \dots)$$

has no eigenvalues.

we're saying $T \in \mathcal{L}(F^\infty)$ and $T(z_1, z_2, z_3, \dots) = (0, z_1, z_2, z_3, \dots)$

let's assume that there exists a scalar $\lambda$ and a non-zero vector $z = (z_1, z_2, z_3, \dots)$ such that $Tz = \lambda z$

then using the eigenvalue equation:

$$T(z_1, z_2, z_3, \dots) = \lambda(z_1, z_2, z_3, \dots)$$

$$(0, z_1, z_2, z_3, \dots) = (\lambda z_1, \lambda z_2, \lambda z_3, \dots)$$

if we do this, we get
$$0 = \lambda z_1$$
$$z_1 = \lambda z_2$$
$$z_2 = \lambda z_3$$
$$\vdots$$

there are two cases:

case 1: $\lambda = 0$ — if $\lambda = 0$, then with the first equation, $0 = \lambda z_1$ is trivially satisfied

the rest of the equations become

$$z_1 = 0 \cdot z_2 = 0$$
$$z_2 = 0 \cdot z_3 = 0$$
$$\vdots$$
which implies that $z_1 = z_2 = z_3 = \dots = 0$

so the only eigenvector corresponding to $\lambda = 0$ is $z = (0, 0, \dots)$ but by def. an eigenvector must not be $\emptyset$ thus $\lambda = 0$ is not an eigenvalue

case 2: $\lambda \ne 0$

if $\lambda \ne 0$, then $0 = \lambda z_1 \Rightarrow$ it must be that $z_1 = 0$

then $z_1 = \lambda z_2 \Rightarrow 0 = \lambda z_2 \Rightarrow z_2 = 0$ .... and so

we can see that $z_n = 0$ for all $n \ge 1$ meaning the only eigenvector is $z = (0, 0, \dots)$

but an eigenvector must not be $\emptyset$

both cases lead to a contradiction, thus the forward shift operator has no eigenvalues

6.

**39** Suppose $V$ is finite-dimensional and $T \in \mathcal{L}(V)$. Prove that $T$ has an eigenvalue if and only if there exists a subspace of $V$ of dimension $\dim V - 1$ that is invariant under $T$.

let's first prove ($\Rightarrow$) if $T$ has an eigenvalue, then the subspace exists

let $\lambda \in F$ be an eigenvalue of $T$

then there exists a nonzero vector $v \in V$ such that $T(v) = \lambda v$

we can make subspace $U$ of $\dim V - 1$ that is invariant under $T$

since $v \neq 0$, we can extend $\{v\}$ to be a basis of $V$: $\{v, v_2, v_3, \dots, v_n\}$

then we can let $U = \operatorname{span}\{v_2, v_3, \dots, v_n\}$

then $\dim U = n-1 = \dim V - 1$

now we must show that $T(U) \subseteq U$

we can take $u \in U$

since $\{v_1, v_2, \dots, v_n\}$ is a basis, we can say $T(u) = a_1 v_1 + a_2 v_2 + \dots + a_n v_n$ for some $a_i$ scalars

we must show that $a_1 = 0$ meaning $T(u)$ has no component in the $v$ direction so it lies completely inside $U$

we can define $\varphi \in V'$ by $\varphi(v) = 1$ $\varphi(v_j) = 0$ for all $j \geq 2$

then for any $w = b_1 v + \dots + b_n v_n$, $\varphi(w) = b_1$

now: $(T'(\varphi))(v) = \varphi(Tv) = \varphi(\lambda v) = \lambda(\varphi v) = \lambda$

for any $u \in U$ $(T'\varphi)(u) = \varphi(Tu) = a_1$ but we can also do $T'\varphi = \lambda\varphi$ from $Tv = \lambda v$ which means

$(T'\varphi)(u) = \lambda\varphi(u) = \lambda \cdot 0 = 0$ and thus $a_1 = 0$

therefore $T(u)$ has no $v$-component, so $T(u) \in U$ and $U$ is invariant under $T$, and $\dim U = \dim V - 1$

proving ($\Leftarrow$)

let's assume there exists a subspace $U \subseteq V$ such that $\dim U = \underbrace{\dim V}_{n} - 1$ and $T(U) \subseteq U$

we want to show that $T$ has an eigenvalue

we can choose $\{w_1, \dots, w_{n-1}\}$ for $U$ and extend to a basis of $V$ $\{w_1, \dots, w_{n-1}, v\}$

now: $T(v) = u + cv$ where $u \in U$ and $c$ is a scalar

if we suppose $T(v) = u + cv$

↳ if $u \neq 0$ then $(T - cI)(v) = u$

if $c$ were not an eigenvalue, then $T - cI$ on $v$ would "enter" the subspace,

contradicting direct sum defenitions

thus we can show that $T$ has an eigenvalue

7. **10** Suppose $V$ is finite-dimensional, $T \in \mathcal{L}(V)$, and $v \in V$. Prove that

$$\operatorname{span}(v, Tv, \ldots, T^m v) = \operatorname{span}(v, Tv, \ldots, T^{\dim V - 1} v)$$

for all integers $m \geq \dim V - 1$.

we're letting $V$ be a fin. dim. vector space, $T \in \mathcal{L}(V)$, and $v \in V$

let's let $n = \dim V$

we can say $S = (v, Tv, T^2 v, \ldots, T^m v)$

$A = \operatorname{span}(v, \ldots, T^m v)$

$B = \operatorname{span}(v, \ldots, T^{n-1} v)$

we want to show that $A = B$ for $m \geq n-1$

proven earlier in textbook

in any n-dim. space, any set of more than $n$ vectors is linearly dependent

so in the list $v, Tv, \ldots, T^n v$, there must be a nontrivial linear comb. among the first $n+1$ vectors:

$a_0 v + a_1 Tv + \ldots + a_n T^n v = 0$ with not all $a_i$ zero

thus $T^n v$ can always be written as a lin comb. of $v, Tv, \ldots, T^{n-1} v$

and by the same logic:

$T^{n+1} v$ is a comb. of previous vectors

so all higher powers add nothing new to the span

now we must show $A \subseteq B$

any element in $A$ can be written as a comb. of $v, Tv, \ldots, T^m v$

but for $k \geq n$, each $T^k v$ can be written as a comb. of $v, Tv, \ldots, T^{n-1} v$ so these don't increase the span

thus $A \subseteq B$

now we must show $B \subseteq A$

since $m \geq n-1$, all basis vectors of $B$ $(v, \ldots, T^{n-1} v)$ are also in $A$. So, every element of $B$ is in $A$

thus $A = B$

8.

**11** Suppose $V$ is a two-dimensional vector space, $T \in \mathcal{L}(V)$, and the matrix of $T$ with respect to some basis of $V$ is $\begin{pmatrix} a & c \\ b & d \end{pmatrix}$.

(a) Show that $T^2 - (a+d)T + (ad - bc)I = 0$.

(b) Show that the minimal polynomial of $T$ equals

$$\begin{cases} z - a & \text{if } b = c = 0 \text{ and } a = d, \\ z^2 - (a+d)z + (ad - bc) & \text{otherwise.} \end{cases}$$

(a) let's first compute $T^2 = T \cdot T = \begin{pmatrix} a & c \\ b & d \end{pmatrix}\begin{pmatrix} a & c \\ b & d \end{pmatrix}$

$$T^2 = \begin{pmatrix} a^2+bc & c(a+d) \\ b(a+d) & d^2+bc \end{pmatrix}$$

now, we can compute $T^2-(a+d)T+(ad-bc)I$

$$(a+d)T = (a+d)\begin{pmatrix} a & c \\ b & d \end{pmatrix} = \begin{pmatrix} a(a+d) & c(a+d) \\ b(a+d) & d(a+d) \end{pmatrix}$$

$$(ad-bc)I = \begin{pmatrix} ad-bc & 0 \\ 0 & ad-bc \end{pmatrix}$$

now $T^2-(a+d)T+(ad-bc)I$

↳ and if we do this

first entry: $(a^2+bc)-(a^2+ad)+(ad-bc) = \emptyset$

second entry: $(ac+cd)-(ac+cd)+0=0$

bottom left: $(ba+db)-(ba+db)+0=0$

bottom right: $(d^2+bc)-(ad+d^2)+(ad-bc)=0$

every entry is $\emptyset$ thus $T^2-(a+d)T+(ad-bc)I=0$

(b) $m_T(z)$ is the monic polynomial of least degree such that $m_T(T)=0$

case 1: $b=c=0$ and $a=d$ then $T=aI$

then $T(v)=av$ for all $v$

if we try $p(z)=z-a$ then $p(T)(v)=(T-aI)(v) = T(v)-av = av-av=0$

this works for any vector $v$, and is of degree 1 and monic

there are no lower-degree polynomials that work so the minimal polynomial is $z-a$

case 2: try degree 1: $p(z)=z-r$ for any $r \in \mathbb{F}$

$p(T)=T-rI$ if $T \neq rI$ then there's at least one entry of $T-rI$ that is not zero, so

$p(T)$ is not the zero operator

try degree 2: characteristic polynomial $p(z)=z^2-(a+d)z+(ad-bc)$

$p(T)=T^2-(a+d)T+(ad-bc)I$

calculate $p(T)$ as an operator:

for any $v$, do $T(T(v))-(a+d)T(v)+(ad-bc)v$

when do all calculations (from part (a)), we get $\emptyset$

so the min. poly. must be degree 2 $m_T(z) = z^2-(a+d)z+(ad-bc)$