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Math 405 HW #1

1. Suppose S is a nonempty set. Let V^S denote the set of functions from S to V. Define a natural addition and scalar multiplication on V^S, and show that V^S is a vector space with these definitions.

defining a natural addition and subtraction

suppose V is any vector space over F → a vector space is always defined over some field F

for all $f, g \in V^S$, $\forall a \in F$, we can define — F is the set of scalars that you're allowed to multiply vectors by

$$(f+g)(x) = f(x) + g(x), \quad \forall x \in S$$

$$(af)(x) = a(f(x)), \quad \forall x \in S$$

f and g are functions that take inputs from S and give outputs in V

since $f, g \in V^S: S \to V$ so $f(x) + g(x) \in V$ and $a(f(x)) \in V$ meaning $f+g \in V^S$ and $af \in V^S$

adding two functions in V^S stays in V^S and multiplying a function by a scalar stays in V^S

showing that V^S is a vector space over F:

1. commutativity

$$(f+g)(x) = f(x) + g(x) = g(x) + f(x) = (g+f)(x)$$

2. associativity:

$$((f+g)+h)(x) = (f+g)(x) + h(x) = f(x) + g(x) + h(x) = f(x) + (g(x) + h(x))$$

$$= f(x) + (g+h)(x) = (f + (g+h))(x)$$

$$((ab)f)(x) = (ab)f(x) = a(b f(x)) = (a(bf))(x)$$

3. additive identity:

additive identity is $0(x) = 0$

$$(f+0)(x) = f(x) + 0(x) = f(x) + 0 = f(x)$$

4. additive inverse:

for all $f \in V^S$, define the inverse of f to be $-f: S \to V$ such that for all $x \in S$ (define a new function that does this)

$$(-f)(x) = -f(x)$$

since $f(x) \in V$, its additive inverse $-f(x)$ is also in V and their sum is 0 so

$$f(x) + (-f)(x) = 0$$

5. multiplicative identity:

$$(1f)(x) = 1(f(x)) = f(x) = (f)(x)$$

6. distributive properties

For all $a \in F$ and $x \in S$ and $f, g \in V^S$ since $f(x), g(x) \in V$ and vector addition in V is distributive,

$$a(f(x) + g(x)) = a(f(x)) + a(g(x)) = (af)(x) + (ag)(x) = (af + ag)(x)$$

and for all $a, b \in F$, $x \in S$ and $f \in V^S$

$$(a+b)(f(x)) = a(f(x)) + b(f(x)) = (af)(x) + (bf)(x)$$

$\therefore$ we've shown that V^S is a vector space over F

2. For each of the following subsets of $\mathbf{F}^3$, determine whether it is a subspace of $\mathbf{F}^3$.

(a) $\{(x_1, x_2, x_3) \in \mathbf{F}^3 : x_1 + 2x_2 + 3x_3 = 0\} = U$

(b) $\{(x_1, x_2, x_3) \in \mathbf{F}^3 : x_1 + 2x_2 + 3x_3 = 4\} = U$

(c) $\{(x_1, x_2, x_3) \in \mathbf{F}^3 : x_1 x_2 x_3 = 0\} = U$

(d) $\{(x_1, x_2, x_3) \in \mathbf{F}^3 : x_1 = 5x_3\} = U$

(a) ① additive identity $0 + 2\cdot 0 + 3\cdot 0 = 0$ thus $(0,0,0) \in U$

② closed under addition

$(x_1 + x_2, x_3) + (y_1, y_2, y_3) = (x_1 + y_1, x_2 + y_2, x_3 + y_3)$

$(x_1 + y_1) + 2(x_2 + y_2) + 3(x_3 + y_3) = (x_1 + 2x_2 + 3x_3) + (y_1 + 2y_2 + 3y_3) = 0 + 0 = 0 \in U$ ✓

③ closed under scalar multiplication:

$a(x_1, x_2, x_3) = (ax_1, ax_2, ax_3)$

$ax_1 + 2ax_2 + 3ax_3 = a(x_1 + 2x_2 + 3x_3) = a \cdot 0 = 0$ ✓

∴ (a) is a subspace

(b) ① additive identity $0 + 2\cdot 0 + 3\cdot 0 \neq 4$ and thus $0 = (0,0,0) \notin U$

∴ (b) is not a subspace

(c) ① additive identity $0 \cdot 0 \cdot 0 = 0$ thus $(0,0,0) \in U$

② closed under addition

ex $(0,1,1) \in U$ and $(1,0,1) \in U$

$(0,1,1) + (1,0,1) = (0+1, 1+0, 1+1) = (1,1,2) \notin U$

∴ (c) is not a subspace

(d) ① additive identity $0 = 5 \cdot 0$ thus $(0,0,0) \in U$ ✓

② closed under addition

$(x_1, x_2, x_3) + (y_1, y_2, y_3) = (x_1 + y_1, x_2 + y_2, x_3 + y_3)$

$x_1 + y_1 = 5x_3 + 5y_3 = 5(x_3 + y_3)$ ✓

③ closed under scalar multiplication

$a(x_1, x_2, x_3) = (ax_1, ax_2, ax_3)$

$ax_1 = a(5x_3) = 5(ax_3)$ ✓

∴ (d) is a subspace

3.

3 Show that the set of differentiable real-valued functions f on the interval $(-4,4)$ such that $f'(-1) = 3f(2)$ is a subspace of $\mathbf{R}^{(-4,4)}$.

→ the vector space of all real valued functions defined on $(-4,4)$

we can let U be the set of differentiable real-valued functions f on the interval $(-4,4)$ such that $f^{-1}(-1) = 3f(2)$ is a subspace of $\mathbb{R}^{(-4,4)}$.

① additive itentity

the zero function $f(x) = 0$ $\quad f'(x) = 0$ for all x

$$\left.\begin{array}{ll} \text{at } x = -1 & f'(-1) = 0 \\ \text{at } x = 2 & f(2) = 0 \end{array}\right\} \; 0 = 3 \cdot 0 \checkmark$$

② closed under addition

let $f, g \in U$ then we have $f'(-1) = 3f(2)$ and $g'(-1) = 3g(2)$

$$\begin{aligned} (f+g)'(-1) &= (f'+g')(-1) \\ &= f'(-1) + g'(-1) \\ &= 3f(2) + 3g(2) \\ &= 3(f(2) + g(2)) \\ &= 3(f+g)(2) \end{aligned}$$

and thus $f+g \in U$ ✓

③ closed under scalar multiplication

let $\lambda \in F$ and $f \in U$ then $f'(-1) = 3f(2)$

$$\begin{aligned} (\lambda f)'(-1) &= (\lambda f')(-1) \\ &= \lambda f'(-1) \\ &= \lambda(3f(2)) \\ &= 3\lambda f(2) = 3(\lambda f)(2) \end{aligned}$$

and thus $\lambda f \in U$ ✓

$\therefore$ U is a subspace

4.

7 Prove or give a counterexample: If U is a nonempty subset of $\mathbf{R}^2$ such that U is closed under addition and under taking additive inverses (meaning $-u \in U$ whenever $u \in U$), then U is a subspace of $\mathbf{R}^2$.

we can give the following counter example

$U = \{ (x_1, x_2) \in \mathbb{R}^2 : x_1, x_2 \in \mathbb{Z} \}$

the above is not closed under scalar multiplication if $a = \frac{1}{2} \in \mathbb{R}^2$ and $(x_1, x_2) = (1,1)$

$a(x_1, x_2) = \frac{1}{2}(1,1) \notin \mathbb{Z}$

$\therefore$ U is not a subspace over $\mathbb{R}^2$

5.

9 A function $f\colon \mathbf{R} \to \mathbf{R}$ is called *periodic* if there exists a positive number p such that $f(x) = f(x+p)$ for all $x \in \mathbf{R}$. Is the set of periodic functions from $\mathbf{R}$ to $\mathbf{R}$ a subspace of $\mathbf{R}^{\mathbf{R}}$? Explain.

① additive identity $\quad f(x) = f(x+p)$ $\quad$ the zero function outputs zero everywhere

the zero function $f(x) = 0$ is 0 no matter what x is $\quad f(x+p) = 0$ and $f(x) = 0$ ✓

② closed under addition

let's say f has period p $\quad f(x) = f(x+p)$

and g has period q $\quad g(x) = g(x+q)$

$(f+g)(x) = f(x) + g(x)$ $\quad$ for $f+g$ to be periodic, it has to repeat after some period l

then

$(f+g)(x) = (f+g)(x+l)$ which is $f(x) + g(x) = f(x+l) + g(x+l)$

but the period p of f and period q of g might be different

there might be no single l such that both f and g repeat after l

we cannot guarantee $f+g$ is periodic

$\therefore$ not a subspace over $\mathbb{R}^{\mathbb{R}}$

6.

11 Prove that the intersection of every collection of subspaces of V is a subspace of V.

the intersection $\cap V_k$ means we only keep the vectors that are inside every V_k

① additive inverse

every subspace contains 0 so 0 must be in all V_k and thus 0 is in $\cap V_k$

② closed under addition

let's say u, v are in $\cap U_k$ then they are in every V_k

and we know $\{V_k\}$ is closed under addition so $u + v \in V_k$ and thus $u+v \in \cap V_k$

③ closed under scalar multiplication

let's say $u \in \cap V_k$ and then u is in every V_k. Multiplying u by a scalar a is still in every V_k

and thus au is in the intersection.

7.

12 Prove that the union of two subspaces of V is a subspace of V if and only if one of the subspaces is contained in the other.

we can say that V_1 and V_2 are subspaces of V

1. we must show that if $V_1 \subseteq V_2$ then the union is a subspace of V

if $V_1 \subseteq V_2$ then $V_1 \cup V_2$ is just V_2 and we know V_2 is a subspace of V

$\therefore V_1 \cup V_2$ is a subspace of V

2. we must show that if $V_1 \cup V_2$ is a subspace of V then either $V_1 \subseteq V_2$ or $V_2 \subseteq V_1$

proof by contradiction

let's say $x \in V_1$ and $x \notin V_2$ and $y \notin V_1$ and $y \in V_2$

if $V_1 \cup V_2$ is a subspace of V then $x + y \in V_1 \cup V_2$

if $x + y \in V_1 \cup V_2$ then

$x+y \in V_1$ and $x+y \in V_2$

we know $(-x) \in V_1$

but $x+y = x + (-x) + y = y \in V_1$ and this leads to a contradiction

and $x+y = x+y(-y) = x \in V_2$ which also leads to a contradiction

$\therefore$ if $V_1 \cup V_2$ is a subspace of V then either $V_1 \subseteq V_2$ or $V_2 \subseteq V_1$

and thus proving both sides of the arrow, we've proven the statement

8.

13 Prove that the union of three subspaces of V is a subspace of V if and only if one of the subspaces contains the other two.

let's say V_1, V_2, and V_3 are subspaces of V

① we must show that if one of the subspaces contains the other two then $V_1 \cup V_2 \cup V_3$ is a subspace of V

well if one of V_1 V_2 V_3 contains the other two then $V_1 \cup V_2 \cup V_3$ is a subspace of V

② we must show that if the union of three subspaces of V is a subspace of V then one of the subspaces contains the other two

let's assume that $V_1 \cup V_2 \cup V_3$ is a subspace of V

then we can check the properties of subspaces

1. closure under addition: if we take $v_1 \in V_1$ and $v_2 \in V_2$. Since $V_1 \cup V_2 \cup V_3$ itself is a subspace, then we know that the sum $v_1 + v_2$ must also be in $V_1 \cup V_2 \cup V_3$ because we know subspaces satisfy closure under addition

and then if $v_1 + v_2$ is in $V_1 \cup V_2 \cup V_3$, it must be in either V_1, V_2, or V_3

then, we can formulate the following 3 cases:

case 1: let's suppose $v_1 \in V_1$ and $v_2 \in V_2$ and $v_1 + v_2 \in V_1$ then V_2 <u>must</u> be contained in V_1

we know this since V_1 is closed under addition and under additive inverses

and we know $v_1 + v_2 \in V_1$ and $v_1 \in V_1$ so their difference $(v_1 + v_2) - v_1$ is also in V_1

↳ the additive inverse

but $(v_1 + v_2) - v_1 = v_2$ which means $v_2 \in V_1$ so V_2 <u>must</u> be contained in V_1

case 2: let's suppose $v_1 \in V_1$ and $v_2 \in V_2$ and $v_1 + v_2 \in V_2$. Then V_1 must be contained in V_2

case 3: let's suppose $v_1 + v_2 \in V_3$ and by the same logic, either V_1 or V_2 must be contained in V_3

and by looking at each combination, we can see that if $V_1 \cup V_2 \cup V_3$ is a subspace, at least one of V_1, V_2, V_3 must contain the other two

and thus proving both sides of the arrow, we've proven the statement