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HW #8

1. ex. 14 pg. 173

14 (a) Give an example of a finite-dimensional complex vector space and an operator T on that vector space such that T^2 is diagonalizable but T is not diagonalizable.

(b) Suppose $\mathbf{F} = \mathbf{C}$, k is a positive integer, and $T \in \mathcal{L}(V)$ is invertible. Prove that T is diagonalizable if and only if T^k is diagonalizable.

(a) we're looking at V finite-dimensional complex vector space and T on this vector space

we can let $T = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$

T is not diagonalizable; the only eigenvalue is 0

↳ the min. polynomial is x^2 which has a repeated root so T is not diagonalizable

T^2 is $\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}^2 = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$

the zero operator is diagonalizable thus, we've satisfied the requirement

(b) we're saying $F = \mathbb{C}$, k is a positive integer, and $T \in \mathcal{L}(V)$ is invertible

① let's first assume T is diagonalizable

if T is diagonalizable, there exists an invertible matrix P and a diagonal matrix D such that $T = PDP^{-1}$ then $T^k = (PDP^{-1})^k = PDP^{-1}PDP^{-1} \dots PDP^{-1} = PD^kP^{-1}$

since D is a diagonal matrix, D^k is also a diagonal matrix with the diagonal entries raised to the power of k thus T^k is similar to a diagonal matrix D^k which means T^k is diagonalizable.

② let's assume T^k is diagonalizable. Its minimal polynomial shouldn't have any repeated roots. we can let these values be $\mu_1, \mu_2, \dots \mu_m$. Then the minimal polynomial of T^k is $m_{T^k}(x) = (x-\mu_1)(x-\mu_2) \dots (x-\mu_m)$. The eigenvalues of T^k are the k^{th} powers of the eigenvalues of T. we can let $\lambda_1, \dots, \lambda_n$ be the eigenvalues of T. Since T is invertible, none of its eigenvalues are 0. Thus, all $\lambda_i^k = \mu_j$ are non-zero. The min. poly. of T divides a polynomial whose roots are simple — T satisfies a polynomial $p(x)$ such that $p(T^k) = 0$ where $p(x) = m_{T^k}(x)$

$$p(T) = (T^k - \mu_1 I)(T^k - \mu_2 I) \dots (T^k - \mu_m I)$$

for each μ_j, we can look at $z^k = \mu_j$. In $\mathbb{C}$, this has k distinct non-zero roots — $\omega_{j,1}, \dots, \omega_{j,k}$. we can factor each term in the polynomial $x^k - \mu_j = (x - \omega_{j,1}) \dots (x - \omega_{j,k})$. Thus $p(x)$ factors into linear terms with roots $\omega_{j,i}$ for all j, i. Since all μ_j are distinct and non-zero, all the roots can be picked to be distinct. So the polynomial $p(x) = \prod_{j=1}^{m} (x^k - \mu_j)$ is a polynomial that T satisfies, and all of its roots are distinct. Since T is a root of a polynomial with distinct roots, the min. poly. must also have distinct roots and thus T is diagonalizable.

2 ex. 16 pg. 173

16 Suppose that $T \in \mathcal{L}(V)$ is diagonalizable. Let $\lambda_1, \ldots, \lambda_m$ denote the distinct eigenvalues of T. Prove that a subspace U of V is invariant under T if and only if there exist subspaces $U_1, \ldots, U_m$ of V such that $U_k \subseteq E(\lambda_k, T)$ for each k and $U = U_1 \oplus \cdots \oplus U_m$.

we're assuming $T \in \mathcal{L}(V)$ is diagonalizable and we're letting $\lambda_1, \ldots, \lambda_m$ denote the distinct eigenvalues of T

① let's first assume that a subspace U of V is invariant under T

well because T is diagonalizable, we can write $V = E(\lambda_1, T) \oplus \ldots \oplus E(\lambda_m, T)$ by Thm. 45

so every $v \in V$ can be uniquely written as $v = v_1 + \ldots + v_m$ where $v_k \in E(\lambda_k, T)$

we can let $U_k = U \cap E(\lambda_k, T)$ → the part of U in $E(\lambda_k, T)$

↳ ok so suppose $u \in U$. we can write $u = v_1 + \ldots + v_m$ with $v_k \in E(\lambda_k, T)$

because U is T-invariant, $T(u) \in U$ and since each eigenspace itself is T-invariant,

if we do repeated applications, we can show that $v_k \in U$ → $v_k \in U \cap E(\lambda_k, T) = U_k$

Thus, $u \in U$ is in $U_1 \oplus \ldots \oplus U_m$

conversely, any sum of $u_k \in U_k$ is in U since each $U_k \subset U$

This is a direct sum because eigenvectors belonging to distinct eigenvalues are linearly independent, and the original breakdown of V is direct so $U = U_1 \oplus \ldots \oplus U_m$ with each $U_k \subseteq E(\lambda_k, T)$

② we're assuming there exists subspaces $U_k \subseteq E(\lambda_k, T)$ with $U = U_1 \oplus \ldots \oplus U_m$

each $E(\lambda_k, T)$ is T-invariant by def. of eigenspace ($Tv = \lambda_k v$ where $v \in E(\lambda_k, T)$) and thus any subspace $U_k \subset E(\lambda_k, T)$ is also T-invariant: $T(U_k) \subseteq E(\lambda_k, T)$ and U_k

Thus, for any $u = u_1 + \ldots + u_m$ with $u_k \in U_k$

$T(u) = T(u_1) + \ldots + T(u_m) = \lambda_1 u_1 + \ldots + \lambda_m u_m \in U_1 \oplus \ldots \oplus U_m = U$ so $T(u) \in U$ for all $u \in U$ thus U is T-invariant

3. ex. 3 pg. 180

3 Suppose $S, T \in \mathcal{L}(V)$ are such that $ST = TS$. Suppose $p \in \mathcal{P}(\mathbf{F})$.

(a) Prove that null $p(S)$ is invariant under T.

(b) Prove that range $p(S)$ is invariant under T.

See 5.18 for the special case $S = T$.

we're assuming $S, T \in \mathcal{L}(V)$ are such that $ST = TS$ and we're supposing $p \in \mathcal{P}(F)$

(a) a subspace U is invariant under T if for every vector $u \in U$, $T(u)$ is also in U

we can let $v \in \text{null } p(s)$

by the def. of null space, this means $p(s)v = 0$

Since S and T commute, any polynomial in S also commutes with T

$$p(S)T = Tp(s)$$

This can be seen by: $p(s) = a_n S^n + \ldots + a_1 S + a_0 I$

Then $p(S)T = (a_n S^n + \ldots + a_1 S + a_0 I)T = a_n S^n T + \ldots + a_1 ST + a_0 IT$

since $ST = TS$, we can say $S^k T = TS^k$ for any non-negative $\mathbb{Z}$ k so,

$$p(s)T = a_n TS^n + \ldots + a_1 TS + a_0 TI = T(a_n S^n + \ldots + a_1 S + a_0 I) = Tp(s)$$

now: $p(s)T(v) = Tp(S)(v)$

since $v \in \text{null } p(S)$, $p(s)v = 0$

$$p(s)T(v) = T(0) = 0$$

thus $p(s)T(v) = 0$ means that $T(v)$ is in the null space of $p(s)$

thus null $p(s)$ is invariant under T

(b) A subspace U is invariant under T if for every vector $u \in U$, $T(u)$ is also in U

we can let $u \in \text{range } p(s)$. By definition of the range, there exists some vector $v \in V$ such that $u = p(S)v$

we want to show that $T(u)$ is also in the range of $p(s)$

$$T(u) = T(p(s)v)$$

$p(s)$ and T commute so $(Tp(s) = p(s)T)$ we can say $T(u) = p(s)(T(v))$

Since $T(v)$ is a vector in V, $p(s)(T(v))$ is an element of the range of $p(s)$ by the definition of the range thus $T(u)$ is in the range of $p(s)$

and thus $p(s)$ is invariant under T

4.

4 Prove or give a counterexample: If A is a diagonal matrix and B is an upper-triangular matrix of the same size as A, then A and B commute.

we can try $A = \underbrace{\begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}}_{\text{diagonal matrix}}$ and $B = \underbrace{\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}}_{\text{upper triangular}}$

$$\left.\begin{aligned} AB &= \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \\ BA &= \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix} = \begin{pmatrix} 0 & 2 \\ 0 & 0 \end{pmatrix} \end{aligned}\right\} \text{don't commute!}$$