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HW #2 Math 405

1. exercise 2 p.37

Prove or give a counterexample: If v_1, v_2, v_3, v_4 spans V, then the list

$$v_1 - v_2, v_2 - v_3, v_3 - v_4, v_4$$

also spans V.

let's first show that every vector in V can be written using the new list

let's suppose v is any vector in V. if v_1, v_2, v_3, v_4 spans V, then for any v, there exists some a, b, c, d such that

$$v = av_1 + bv_2 + cv_3 + dv_4$$

we can say $w_1 = v_1 - v_2$ $\quad w_2 = v_2 - v_3$ $\quad w_3 = v_3 - v_4$ $\quad w_4 = v_4$

and thus

$w_1 = v_1 - v_2 \Rightarrow v_1 = w_1 + v_2$

$w_2 = v_2 - v_3 \Rightarrow v_2 = w_2 + v_3$

$w_3 = v_3 - v_4 \Rightarrow v_3 = w_3 + v_4$

$w_4 = v_4$

and then we get $v_3 = w_3 + w_4$ $\quad v_2 = w_2 + v_3 = w_2 + w_3 + w_4$ $\quad v_1 = w_1 + v_2 = w_1 + w_2 + w_3 + w_4$

and then we get $v = av_1 + bv_2 + cv_3 + dv_4 = a(w_1 + w_2 + w_3 + w_4) + b(w_2 + w_3 + w_4) + c(w_3 + w_4) + dw_4$

and then we can get $v = aw_1 + (a+b)w_2 + (a+b+c)w_3 + (a+b+c+d)w_4$

this shows that any vector v in V can be written as a combination of w_1, w_2, w_3, w_4 so the new list spans V

and in the other direction, any combination of the new list can also be written using the original basis, so the span is the same

2. exercise 3 pg. 37

3 Suppose $v_1, \ldots, v_m$ is a list of vectors in V. For $k \in \{1, \ldots, m\}$, let

$$w_k = v_1 + \cdots + v_k.$$

Show that $\operatorname{span}(v_1, \ldots, v_m) = \operatorname{span}(w_1, \ldots, w_m)$.

to show $\operatorname{span}(v_1, \ldots, v_m) = \operatorname{span}(w_1, \ldots, w_m)$, we can show $\operatorname{span}(v_1, \ldots, v_m) \subseteq \operatorname{span}(w_1, \ldots, w_m)$ and $\operatorname{span}(w_1, \ldots, w_m) \subseteq \operatorname{span}(v_1, \ldots, v_m)$

① let's first show $\operatorname{span}(w_1, \ldots, w_m) \subseteq \operatorname{span}(v_1, \ldots, v_m)$

each w_k is in $\operatorname{span}(v_1, \ldots, v_m)$, because w_k is the sum of the first k v_j's. And thus any linear combination of $w_1, \ldots, w_m$ is also a linear combination of the v_j's meaning $\operatorname{span}(w_1, \ldots, w_m) \subseteq \operatorname{span}(v_1, \ldots, v_m)$

② next we must show $\operatorname{span}(v_1, \ldots, v_m) \subseteq \operatorname{span}(w_1, \ldots, w_m)$

every v_k can be built from the w_j's

$v_1 = w_1$

and for $k \geq 2$, $v_k = w_k - w_{k-1}$ (basically subtracting the previous sum from the curr. sum to "isolate" v_k)

so any collection of $v_1, \ldots, v_m$ can be turned into a combination of $w_1, \ldots, w_m$. for example, if we want

$a_1 v_1 + a_2 v_2 + \ldots + a_m v_m$ we can do this:

$a_1 w_1 + a_2(w_2 - w_1) + a_3(w_3 - w_2) + \ldots + a_m(w_m - w_{m-1})$

and then we can get $(a_1 - a_2) w_1 + (a_2 - a_3) w_2 + \ldots + (a_{m-1} - a_m) w_{m-1} + a_m w_m$

so every combination of the v_j's is also a combination of the w_k's meaning

$\operatorname{span}(v_1, \ldots, v_m) \subseteq \operatorname{span}(w_1, \ldots, w_m)$

3. exercise 10 pg. 38

Prove or give a counterexample: If $v_1, v_2, \ldots, v_m$ is a linearly independent independent list of vectors in V and $\lambda \in \mathbf{F}$ with $\lambda \neq 0$, then $\lambda v_1, \lambda v_2, \ldots, \lambda v_m$ is linearly v_m is linearly independent.

let's say $a_1, a_2, \ldots, a_m \in F$ and these satisfy $a_1(\lambda v_1) + a_2(\lambda v_2) + \ldots + a_m(\lambda v_m) = \emptyset$

and since $v_1, v_2, \ldots, v_m$ are all linearly independent, then $a_1\lambda, a_2\lambda, \ldots, a_m\lambda$ must be $\emptyset$ but, we're assuming $\lambda \neq 0$, thus $a_1, a_2, \ldots, a_m$ must each equal $\emptyset$

and thus, $\lambda v_1, \lambda v_2, \lambda v_3, \ldots, \lambda v_m$ is linearly independent.

4. exercise 13 pg. 38

13 Suppose $v_1, \ldots, v_m$ is linearly independent in V and $w \in V$. Show that

$$v_1, \ldots, v_m, w \text{ is linearly independent} \iff w \notin \operatorname{span}(v_1, \ldots, v_m).$$

let's suppose that $v_1, \ldots, v_m, w$ is linearly independent

proof by contradiction:

* let's suppose that $w \in \operatorname{span}(v_1, \ldots, v_m)$ so $w = a_1v_1 + \ldots + a_mv_m$ for some numbers $a_1, \ldots, a_m$

  this is the same as $a_1v_1 + \ldots + a_mv_m - w = \emptyset$

  ↳ in this equation, not all coefficients are $\emptyset$, the coefficient for w is -1

  thus, there exists $a_1, a_2, \ldots, a_m, -1$, not all $\emptyset$, such that $a_1v_1 + \ldots + a_mv_m + (-w) = \emptyset$

  by the def. of linear dependence, $v_1, \ldots, v_m, w$ is linearly dependent, which contradicts our initial assumption. thus $w \notin \operatorname{span}(v_1, \ldots, v_m)$

* let's suppose $w \notin \operatorname{span}(v_1, \ldots, v_m)$.

  if $v_1, \ldots, v_m, w$ is linearly dependent, then we have $v_j \in \operatorname{span}(v_1, \ldots, v_{j-1})$ for some j or $w \in \operatorname{span}(v_1, \ldots, v_m)$

  but since $v_1, \ldots, v_m$ is linearly independent, there is no $j \in \{1, \ldots, m\}$ such that $v_j \in \operatorname{span}(v_1, \ldots, v_{j-1})$

  and by assumption we know $w \notin \operatorname{span}(v_1, \ldots, v_m)$ thus $v_1, \ldots, v_m, w$ is linearly dependent

5. exercise 17 pg. 38

17 Prove that V is infinite-dimensional if and only if there is a sequence $v_1, v_2, \ldots$ of vectors in V such that $v_1, \ldots, v_m$ is linearly independent for every positive integer m.

1. first proving that if V is infinite dimensional, then the sequence exists

we can start with any nonzero vector v_1

since V is infinite-dimensional, $\{v_1\}$ doesn't span all of V. So there's a vector v_2 not in the span of v_1, which makes v_1, v_2 linearly independent

adding v_3, not in the span of v_1, v_2 — now v_1, v_2, v_3 are linearly independent

if we continue doing this, always finding a new vector not in the span of all the previous ones

and doing this, we get a sequence $v_1, v_2, \ldots$, where for any m, the first m are linearly independent

2. next if the sequence exists, then V is infinite dimensional

let's suppose we have the sequence $v_1, v_2, \ldots$ wher $v_1, \ldots, v_m$ is linearly independent for every m

this means, for any positive integer, there are at least that many linearly ind. vectors in V

but in a finite dim. space, the largest possible size for a lin. ind. set equals the dimension

and if we can always find a bigger ind. set, the space can't be finite dim. and thus V must be infinite dim.

6. exercise 4 pg. 43

4 (a) Let U be the subspace of $\mathbf{C}^5$ defined by

$$U = \{(z_1, z_2, z_3, z_4, z_5) \in \mathbf{C}^5 : 6z_1 = z_2 \text{ and } z_3 + 2z_4 + 3z_5 = 0\}.$$

Find a basis of U.

(b) Extend the basis in (a) to a basis of $\mathbf{C}^5$.

(c) Find a subspace W of $\mathbf{C}^5$ such that $\mathbf{C}^5 = U \oplus W$.

(a) the first cond. gives $z_2 = 6z_1$

the second cond. gives $z_3 = -2z_4 - 3z_5$

so any vector in U can be written as $(z_1, 6z_1, -2z_4 - 3z_5, z_4, z_5)$

where z_1, z_4, and z_5 can be anything

we can say

$$= z_1(1,6,0,0,0) + z_4(0,0,-2,1,0) + z_5(0,0,-3,0,1)$$

and thus $\{(1,6,0,0,0), (0,0,-2,1,0), (0,0,-3,0,1)\}$ is a basis for U

(b) we can start with (a) and add more vectors — we want 5 lin. ind.

we can do $(0,1,0,0,0)$ and $(0,0,1,0,0)$ giving us

$$(1,6,0,0,0), (0,0,-2,1,0), (0,0,-3,0,1), (0,1,0,0,0), (0,0,1,0,0)$$

(c) the three from (a) span U

the two we added must span W

and thus W is the space spanned by $(0,1,0,0,0)$ and $(0,0,1,0,0)$

and a basis for W is $\{(0,1,0,0,0), (0,0,1,0,0)\}$

7.

7 Suppose v_1, v_2, v_3, v_4 is a basis of V. Prove that

$$v_1 + v_2, v_2 + v_3, v_3 + v_4, v_4$$

is also a basis of V.

1. first we can show lin. ind.

let's suppose $a_1(v_1+v_2) + a_2(v_2+v_3) + a_3(v_3+v_4) + a_4 v_4 = \emptyset$

$a_1 v_1 + (a_1+a_2)v_2 + (a_2+a_3)v_3 + (a_3+a_4)v_4 = \emptyset$

and since v_1, v_2, v_3, v_4 is a basis from our assumption, the coefficients must be $\emptyset$

$a_1 = 0$

and $a_1 + a_2 = 0$ means $a_2 = 0$ since $a_1 = 0$

and $a_2 + a_3 = 0$ means $a_3 = 0$ since $a_2 = 0$

and $a_3 + a_4 = 0$ means $a_4 = 0$ since $a_3 = 0$

so, all $a_i = 0$ and thus, it's lin. ind.

2. next we must show that the list spans V

well since V is 4-dimensional, and we have 4 lin. ind. vectors, they will automatically span V

we can also show that each v_i can be written using the new list

v_4 is directly in the list

$v_3 = (v_3+v_4) - v_4$

$v_2 = (v_2+v_3) - (v_3+v_4) + v_4$

$v_1 = (v_1+v_2) - [(v_2+v_3) - (v_3+v_4) + v_4]$

and thus the new list spans V

8. exercise 11 pg. 43 Suppose V is a real vector space. Show that if $v_1, \ldots, v_n$ is a basis of V (as a real vector space), then $v_1, \ldots, v_n$ is also a basis of the complexification $V_{\mathbf{C}}$ (as a complex vector space).

See Exercise 8 in Section 1B for the definition of the complexification $V_{\mathbf{C}}$.

this question is basically saying given a list of real vectors $v_1, \ldots, v_n$ that form a basis for a real space V

↳ we have to show that they also form a basis for the complexified space V_c

↳ complexification is using complex numbers as coefficients for vectors

in other words, we must show that $v_1, \ldots, v_n$ spans V_c

let's suppose V is a real vector space and let's let $v_1, \ldots, v_n$ be a basis of V over $\mathbb{R}$

↳ complexification V_c includes vectors of form $u + iv$, where $u, v \in V$ and i is the imaginary part

we can then take any vector in V_c, let's say $u + iv$ where $u, v \in V$

since $v_1, \ldots, v_n$ is a real basis, we can say

$$u = a_1 v_1 + \ldots + a_n v_n \quad \text{for some } a_i \text{ and } b_i \in \mathbb{R}$$

$$v = b_1 v_1 + \ldots + b_n v_n$$

then $u + iv = (a_1 v_1 + \ldots + a_n v_n) + i(b_1 v_1 + \ldots + b_n v_n) = (a_1 + ib_1) v_1 + \ldots + (a_n + ib_n) v_n$

where $a_i + ib_i$ are the complex numbers

and thus any vector in V_c can be written as a linear combination of $v_1, \ldots, v_n$

in terms of being linearly ind.:

let's suppose $c_1 v_1 + \ldots + c_n v_n = 0$ for some $c_i \in \mathbb{C}$

we can write the $c_i = a_i + ib_i$ where a_i and $b_i \in \mathbb{R}$

$$(a_1 + ib_1) v_1 + \ldots + (a_n + ib_n) v_n = 0$$

$$(a_1 v_1 + \ldots + a_n v_n) + i (b_1 v_1 + \ldots + b_n v_n) = 0$$

but in V_c, this means $a_1 v_1 + \ldots + a_n v_n = 0$

$b_1 v_1 + \ldots + b_n v_n = 0$

since $\{v_1, \ldots, v_n\}$ are lin. ind. over $\mathbb{R}$, $a_i = 0$ $b_i = 0$ and thus $c_i = 0$

thus $v_1, \ldots, v_n$ spans V_c and is lin. ind. over $\mathbb{C}$ and thus is a basis of the complexification