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HW #7 Math 405

1.

we're assuming V is finite dimensional and $T \in \mathcal{L}(V)$ has minimal polynomial $\longrightarrow m(T)$

$$4 + 5z - 6z^2 - 7z^3 + 2z^4 + z^5 = m_T(z)$$

by defenition of a minimal polynomial, T satisfies this polynomial, so $m_T(T) = 0$

so $\quad T^5 + 2T^4 - 7T^3 - 6T^2 + 5T + 4I = 0$

we also know by Thm 5.32 that T is not invertible $\Leftrightarrow$ constant term of minimal polynomial of T is $\emptyset$

well since $4 \neq 0$, 0 is not an eigenvalue which mean that T is invertible and T^{-1} exists

we can manipulate the equation to find T^{-1}, if we multiply the entire equation by $(T^{-1})^5$

$$(T^{-1})^5 (T^5 + 2T^4 - 7T^3 - 6T^2 + 5T + 4I) = (T^{-1})^5 (0)$$

then we get $\quad I + 2T^{-1} - 7(T^{-1})^2 - 6(T^{-1})^3 + 5(T^{-1})^4 + 4(T^{-1})^5 = 0$

$\quad\quad$ ↳ this shows that T^{-1} is a root of $g(z)$

$$g(z) = 4z^5 + 5z^4 - 6z^3 - 7z^2 + 2z + 1$$

since $g(T^{-1}) = 0$, the minimal polynomial of T^{-1} must divide $g(z)$

well by def. a minimal polynomial must be monic. The leading coeff of $g(z) = 4$ so we can

divide the entire polynomial

$$m_{T^{-1}}(z) = \frac{1}{4} g(z) = \frac{1}{4}(4z^5 + 5z^4 - 6z^3 - 7z^2 + 2z + 1)$$

$$m_{T^{-1}}(z) = z^5 + \frac{5}{4}z^4 - \frac{3}{2}z^3 - \frac{7}{4}z^2 + \frac{1}{2}z + \frac{1}{4}$$

2.

we're letting $T \in \mathcal{L}(V)$ and T^2 have an upper-triangular matrix with respect to some basis of V

by lemma 36, we know that if T as an upper-triangular matrix wrt some basis of V then the min. polynomial equals $(z - \lambda_1) \dots (z - \lambda_m)$ for some $\lambda_1, \dots, \lambda_m \in F$

thus, the min. polynomial for T^2 can be written as $(T^2 - \lambda_1 I) \dots (T^2 - \lambda_m I)$

$\underrightarrow{\text{adding } I \text{ so we can construct}}$

$\xrightarrow{\text{by factoring out}}$

this can then be written as $(T - \sqrt{\lambda_1} I)(T + \sqrt{\lambda_1} I) \dots (T - \sqrt{\lambda_m} I)(T + \sqrt{\lambda_m} I) = 0$ $\quad$ (*)

$\hookrightarrow$ these square roots would need to exist in F

we know lemma 33 says that suppose $T \in \mathcal{L}(V)$ and V has a basis wrt which T has an upper-triangular matrix with diagonal matrix with diagonal entries $\lambda_1, \dots, \lambda_n$ then $(T - \lambda_1 I) \dots (T - \lambda_n I) = 0$

(*) shows that there is a polynomial with only linear factors in T that when applied to T gives 0

by lemma 33, (*) existing means that T itself also has an upper-triangular matrix wrt some basis of V

3.

we can let some $T: F^2 \to F^2$ by something like $T(w, z) = (w + z, w + z)$

and the standard basis would be $((1, 0), (0, 1))$ on F^2

in other words $M(T) = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$

$\hookrightarrow$ nonzero numbers on the diagonal

T is not invertible because the determinant $(1)(1) - (1)(1) = 0$

12 Suppose V is finite-dimensional, $T \in \mathcal{L}(V)$ has an upper-triangular matrix with respect to some basis of V, and U is a subspace of V that is invariant under T.

   (a) Prove that $T|_U$ has an upper-triangular matrix with respect to some basis of U.

   (b) Prove that the quotient operator T/U has an upper-triangular matrix with respect to some basis of V/U.

The quotient operator T/U was defined in Exercise 38 in Section 5A.

(a) we're assuming V is fin.dim. and $T \in \mathcal{L}(V)$ has an upper-triangular matrix with respect to some basis of V

and U is a subspace of V that is invariant under T

we know that a matrix is upper triangular if all entries below the diagonal are 0 $a_{ij} = 0$ for $i > j$

let's let the basis for V be $(v_1, v_2, ..., v_n)$ so that T is upper triangular with respect to this basis and V is fin.dim.

well since U is a subspace of V, there exists a basis $(u_1, ..., u_k)$ of U where each u_j is a linear combination of the original basis vectors $v_1, ..., v_n$

because U is invariant under T, for each u_j, $T(u_j) \in U$

we can write $T(u_j)$ as a linear combination of $(u_1, ..., u_k)$. $T(u_j) = \sum_{i=1}^{k} a_{ij} u_i$ → some constant

and since the matrix of T w.r.t. $(v_1, ..., v_n)$ is upper-triangular, the coefficients satisfy $T(v_j) \in \text{span}(v_j, v_{j+1}, ..., v_n)$

each u_j, being a combination of the v_i will have the same "upper-triangular" relation relative to the specified order

Thus, w.r.t. the basis $(u_1, ..., u_k)$ of U, the matrix (a_{ij}) of the $T|_U$ will be upper triangular

and thus, $T|_U$ has an upper-triangular matrix w.r.t. some basis of U

(b) let's look at the basis of V/U which by definition is $\{ v_{m+1} + U, ..., v_n + U \}$

well we also know that there exists a basis $v_1, ..., v_n$ such that $T v_k \in \text{span}(v_1, ..., v_k)$ for each k

well we know $T(v_k + U) = T(v_k) + U$ because of the map induced by T on V/U

and then we can do $T(v_k + U) \in \text{span}(v_{m+1} + U, ..., v_k + U)$ for some $k > m$

and thus T/U has an upper triangular matrix

5.

1 Suppose V is a finite-dimensional complex vector space and $T \in \mathcal{L}(V)$.

(a) Prove that if $T^4 = I$, then T is diagonalizable.
(b) Prove that if $T^4 = T$, then T is diagonalizable.
(c) Give an example of an operator $T \in \mathcal{L}(\mathbf{C}^2)$ such that $T^4 = T^2$ and T is not diagonalizable.

(a) we're assuming V is fin. dim. complex vector space and $T \in \mathcal{L}(V)$

let's assume that $T^4 = I$

we can let $p(x) = x^4 - 1$ and since $T^4 = I$, we know that $p(T) = 0$

this means that the minimal polynomial of T, or $m(x)$, must divide $p(x)$

$p(x) = x^4 - 1$ can be $(x^2 - 1)(x^2 + 1) = (x-1)(x+1)(x-i)(x+i)$

the roots of this are $1, -1, i, -i$ which are all distinct

well since the min. poly. divides $p(x)$ and all roots are distinct, the roots of $m(x)$ must also be distinct. This is because any root of $m(x)$ is also a root of $p(x)$. Since the roots of $m(x)$ are distinct, T is diagonalizable.

(b). we're saying $T^4 = T$ we can then let $p(x) = x^4 - x$ well then since $T^4 = T$, we can say $p(T) = T^4 - T = 0$

this means that $m(x)$ must divide $p(x)$

$p(x) = x^4 - x$ can be written as $x^4 - x = x(x^3 - 1) = x(x-1)(x^2 + x + 1)$

$$x = \frac{-1 \pm \sqrt{1^2 - 4(1)(1)}}{2} = \frac{-1 \pm \sqrt{-3}}{2} = \frac{-1 \pm i\sqrt{3}}{2}$$

so $p(x) = x(x-1)\left(x - \frac{-1 + i\sqrt{3}}{2}\right)\left(x - \frac{-1 - i\sqrt{3}}{2}\right)$

so the roots of $p(x)$ are $0, 1, \frac{-1 + i\sqrt{3}}{2}, \frac{-1 - i\sqrt{3}}{2}$ which are all distinct

thus $m(x)$ divides $p(x)$ and since the roots of $p(x)$ are distinct, the roots of $m(x)$ must also be distinct thus T is diagonalizable

(c) we want $T^4 = T^2$ but T is not diagonalizable

$T^4 = T^2 \Rightarrow T^2(T^2 - I) = 0$ so the minimal polynomial divides $x^2(x-1)(x+1)$

$\hookrightarrow$ the roots of this are 0 (multiplicity 2), 1 and -1

a non-diagonalizable operator must have minimal polynomial with at least one repeated root

since $p(x)$ has a repeated root at 0, the minimal polynomial could be $x^2, x^2(x-1), x^2(x+1)$ or $x^2(x-1)(x+1)$

if we take $T = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ and then $T^2 = 0$ and $T^4 = T^2 = 0$ so $T^4 = T^2$

the min. poly. is x^2 the only eigenvalue is 0, with multiplicity 2
$\hookrightarrow$ thus T is not diagonalizable

6.

5 Suppose V is a finite-dimensional complex vector space and $T \in \mathcal{L}(V)$.
Prove that T is diagonalizable if and only if

$$V = \text{null}(T - \lambda I) \oplus \text{range}(T - \lambda I)$$

for every $\lambda \in \mathbf{C}$.

we're assuming V is fin. dim. complex vect space and $T \in \mathcal{L}(V)$

let's first prove that if T is diagonalizable then $V = \text{null}(T - \lambda I) \oplus \text{range}(T - \lambda I)$

let's assume that T is diagonalizable

we can let $S = T - \lambda I$. Well since T is diagonalizable, S is also diagonalizable because it has the same

eigenvectors as T just with the eigenvalues shifted by $-\lambda$
$\longrightarrow$ by Thm. 5.55
well if S is diagonalizable, then there exists a basis of V consisting of eigenvectors of S

↳ let's let this basis be $v_1, \ldots, v_n$ so for every $j \in 1, 2, \ldots, n$, we have some

$\lambda \in \mathbb{C}$ such that $Tv_j = \lambda_j v_j$. we can do the following: where $m \in \{1, 2, \ldots, n\}$ such that

$\lambda_j = 0$ for $j = 1, 2, \ldots, m$ and $\lambda_j \neq 0$ for $j = m+1, \ldots, n$

so we have some $V = \text{span} \{v_1, \ldots, v_m\} \oplus \text{span} \{v_{m+1}, v_n\}$. Now we must show that

$\text{null}(s) = \text{span}\{v_1, \ldots, v_m\}$ and $\text{range}(s) = \text{span}\{v_{m+1}, \ldots, v_n\}$

well span $\{v_1, \ldots, v_m\}$ is the eigenspace corresponding to 0

Thus every element $v \in \text{span}\{v_1, \ldots, v_m\}$ satisfies $Sv = 0$ thus $v \in \text{null}(S)$

additionally, any $u \in \text{null}(s)$, we get $S \cdot u = 0$ thus u is an eigenvector so

$u \in \text{span} \{v_1, \ldots, v_m\}$ thus $\text{null}(s) = \text{span}\{v_1, \ldots, v_m\}$

now for any $j \in m+1, \ldots, n$ we get $S(\lambda_j^{-1} v_j) = v_j$ thus range S is a subset of $\{v_{m+1}, \ldots, v_n\}$

for every $y \in \text{range } S$ we get $y = Sx$ for some $x \in V$ and since $v_1, \ldots, v_n$ is a basis

of V we get $x = a_1 v_1 + \ldots + a_n v_n$ for some $a_1, \ldots, a_n$ thus $y = Sx = S(a_1 v_1 + \ldots + a_n v_n)$

$$= \lambda_{m+1} a_{m+1} v_{m+1} + \ldots + \lambda_n a_n v_n \in \text{span} \{v_{m+1}, \ldots, v_n\}$$

now we can get that range $S = \text{span}\{v_{m+1}, \ldots, v_n\}$. Therefore, $V = \text{null } S \oplus \text{range } S$

now let's prove in the opposite direction

let's assume that $V = \text{null}(T - \lambda I) \oplus \text{range}(T - \lambda I)$ for every $\lambda \in \mathbb{C}$

the min. polynomial of T splits completely over $\mathbb{C}$ so $m_T(z) = \prod_{i=1}^{m} (z - \lambda_i)^{n_i}$ for

distinct eigenvalues $\lambda_1, \ldots, \lambda_m$ and exponents $n_i \geq 1$

we can do a proof by contradiction that for some i, $n_i > 1$. Then T is not diagonalizable and there

is a nontrivial intersection: $\text{null}(T - \lambda_i I) \cap \text{range}(T - \lambda_i I) \neq \{0\}$

but by the assumption, the intersection is always $\{0\}$ so all of the exponents must be $n_i = 1$

well a min. polynomial with all $n_i = 1$ means T is diagonalizable.