

1. exercise 3 (pg. 66)

3 Suppose v_1, \dots, v_m is a list of vectors in V . Define $T \in \mathcal{L}(\mathbb{F}^m, V)$ by

$$T(z_1, \dots, z_m) = z_1 v_1 + \dots + z_m v_m.$$

- (a) What property of T corresponds to v_1, \dots, v_m spanning V ?
 (b) What property of T corresponds to the list v_1, \dots, v_m being linearly independent?

(a) let's suppose v_1, \dots, v_m spans V . This means that for every vector $v \in V$, there exists scalars z_1, \dots, z_m such that $v = z_1 v_1 + \dots + z_m v_m$

now let's consider the map $T: \mathbb{F}^m \rightarrow V$ defined by

$$T(z_1, \dots, z_m) = z_1 v_1 + \dots + z_m v_m$$

given any $v \in V$, take the scalars z_1, \dots, z_m guaranteed by the spanning property, well by construction, $T(z_1, \dots, z_m) = v$

thus, every $v \in V$ lies in the range of T . Thus T is surjective

(b) let's suppose v_1, \dots, v_m is linearly independent

then the equation $z_1 v_1 + \dots + z_m v_m = 0$ only has one solution, $z_1 = z_2 = \dots = z_m = 0$

This means that the null space of T is just the zero vector

$$T(z_1, \dots, z_m) = 0 \text{ only happens when } (z_1, \dots, z_m) = (0, \dots, 0)$$

This means T is injective

2. exercise 5 (pg. 66)

5 Give an example of $T \in \mathcal{L}(\mathbb{R}^4)$ such that $\text{range } T = \text{null } T$.

↳ the set of all possible outputs if T equals the set of all vectors that get mapped to zero by T

we can define T as $T(x_1, x_2, x_3, x_4) = (0, 0, x_1, x_2)$

↳ take any vector in \mathbb{R}^4 and ignores the last two coordinates

the output is always of the form $(0, 0, a, b)$ so the range is $\{(0, 0, a, b) : a, b \in \mathbb{R}\}$

the null space: $(0, 0, x_1, x_2) = (0, 0, 0, 0)$ therefore $x_1 = 0$ and $x_2 = 0$

and thus the null space is all vectors $(0, 0, x_3, x_4)$

$$\text{for Range } T = \text{Null } T = \{(0, 0, a, b) : a, b \in \mathbb{R}\}$$

3. exercise 16 (pg. 67)

16 Suppose V and W are both finite-dimensional. Prove that there exists an injective linear map from V to W if and only if $\dim V \leq \dim W$.

① proving if there exists an injective linear map from V to W then $\dim V \leq \dim W$

let's say the linear map is T

well if T is injective, then we know $\dim \text{null } T = 0$

and we know $\dim V = \cancel{\dim \text{null } T} + \dim \text{range } T = \dim \text{range } T$

and the range of T is a subspace of W , so $\dim(\text{range } T) \leq \dim W$

and putting these together, $\dim V \leq \dim W$

② now, let's prove that if $\dim V \leq \dim W$, then there exists an injective linear map

we can let $\dim V = n$ and $\dim W = m$ and $n \leq m$

we can then let v_1, \dots, v_n be a basis for V and w_1, \dots, w_m be a basis for W

let's define a map $T \in \mathcal{L}(V, W): T(a_1 v_1 + \dots + a_n v_n) = a_1 w_1 + \dots + a_n w_n$

well if $T(v) = 0$ then that means $a_1 w_1 + \dots + a_n w_n = 0$

and since w_1, \dots, w_n are lin. ind., all $a_1 = \dots = a_n = 0$ so $v = 0$

thus T is injective

4. exercise 21 (pg. #67)

21 Suppose V is finite-dimensional, $T \in \mathcal{L}(V, W)$, and U is a subspace of W . Prove that $\{v \in V : Tv \in U\}$ is a subspace of V and

$$\dim\{v \in V : Tv \in U\} = \dim \text{null } T + \dim(U \cap \text{range } T).$$

let's say $S = \{v \in V : Tv \in U\}$

1. let's let $v_1, v_2 \in S$ which means $Tv_1 \in U$ and $Tv_2 \in U$

for some a, b $T(av_1 + bv_2) = aTv_1 + bTv_2$

both Tv_1 and Tv_2 are in U and U is a subspace so $aTv_1 + bTv_2 \in U$

thus $av_1 + bv_2 \in S$

↳ the zero vector $T(0) = 0 \in U$ so $0 \in S$ so S is a subspace of V

2. we can define $\tilde{T}: S \rightarrow U \cap \text{range } T$ by $\tilde{T}(v) = Tv$ for each $v \in S$

(basically \tilde{T} just applies T to vectors in S , but restricts the target to vectors in both U and $\text{range } T$)

the null space of all the $v \in S$ with $Tv = 0$ but by the way we defined S , this is just the set of vectors in V that are sent to zero by T

and every element in the colspace of \tilde{T} is something both in U and that can be written as Tv for some v

if we use $\dim S = \dim(\text{null } \tilde{T}) + \dim(\text{col } \tilde{T})$

and as we just saw $\dim S = \dim \text{null } T + \dim(U \cap \text{range } T)$

5. exercise 30 pg. 68

30 Suppose $\varphi \in \mathcal{L}(V, F)$ and $\varphi \neq 0$. Suppose $u \in V$ is not in $\text{null } \varphi$. Prove that

$$V = \text{null } \varphi \oplus \{au : a \in F\}.$$

↳ basically: every vector in V can uniquely be written as something in $\text{null } \varphi$ plus a scalar multiple of u

1. if we take any $v \in \text{null } \varphi \cap \{au : a \in F\}$

$$\hookrightarrow v = au \text{ for some } a \in F \text{ and } \varphi(v) = 0$$

and $\varphi(v) = \varphi(au) = a\varphi(u)$ since $u \notin \text{null } \varphi$, $\varphi(u) \neq 0$

$$\text{so } 0 = \varphi(v) = a\varphi(u) \Rightarrow a = 0$$

so $v = 0$, and the two subspaces intersect only at $\{0\}$

2. we can let $v \in V$ be any vector and set $b = \varphi(u)$ which is not zero and let $c = \varphi(v)$

we can see that $\exists a \in F$ so that $c = ab$ and $a = c/b$

$$\text{and now } v - au, \text{ we have } \varphi(v - au) = \varphi(v) - a\varphi(u) = c - ab = 0$$

$$\text{so } v - au \in \text{null } \varphi$$

$$\text{or } v = (v - au) + au \text{ with } v - au \in \text{null } \varphi \text{ and } au \text{ in } \{au : a \in F\}$$

\therefore every vector in V is in the sum of the two subspaces

3. if we say $v = w_1 + a_1u = w_2 + a_2u$, where $w_1, w_2 \in \text{null } \varphi, a_1, a_2 \in F$

$$\text{then } (w_1 - w_2) + (a_1 - a_2)u = 0$$

thus, $w_2 - w_1 = (a_1 - a_2)u$ is in both subspaces. but, the intersection is only $\{0\}$, so

$$a_1 - a_2 = 0 \Rightarrow a_1 = a_2 \text{ and thus } w_1 = w_2$$

6. exercise 5 (pg. 79)

- 5 Suppose V and W are finite-dimensional and $T \in \mathcal{L}(V, W)$. Prove that there exist a basis of V and a basis of W such that with respect to these bases, all entries of $M(T)$ are 0 except that the entries in row k , column k , equal 1 if $1 \leq k \leq \dim \text{range } T$.

we can let $\dim V = n$ and $\dim W = m$ and $\dim(\text{range } T) = r$

we can pick a basis w_1, \dots, w_r for $\text{range } T$

and then extend this to a basis of W , so we have w_1, \dots, w_m

then for each $k = 1, \dots, r$, since $w_k \in \text{range } T$, there is some vector $v_k \in V$ such that $Tv_k = w_k$

then $\{v_1, \dots, v_r\}$ is lin. ind. if not, we could say $0 = a_1 v_1 + \dots + a_r v_r$, which would give us $0 = T(a_1 v_1 + \dots + a_r v_r) = a_1 w_1 + \dots + a_r w_r$ which contradicts the independence of w_k

now we can let u_1, \dots, u_{n-r} be a basis of $\text{null } T$

then $\{u_1, \dots, u_{n-r}, v_1, \dots, v_r\}$ forms a basis for V

then for the matrix

$T(u_j) = 0$ for all u_j since these are null vectors

$T(v_k) = w_k$

and thus in the matrix that expresses T in these bases:

the first $n-r$ columns are all 0 's

for $k = 1, \dots, r$, the k th column has a 1 in the k th row and zero everywhere else

7. exercise 3 (pg. 93)

3 Suppose V is finite-dimensional and $T \in \mathcal{L}(V)$. Prove that the following are equivalent.

- (a) T is invertible.
- (b) TV_1, \dots, TV_n is a basis of V for every basis v_1, \dots, v_n of V .
- (c) TV_1, \dots, TV_n is a basis of V for some basis v_1, \dots, v_n of V .

let's first show (a) \Rightarrow (b)

if T is invertible, then for every basis v_1, \dots, v_n of V , the list TV_1, \dots, TV_n is a basis

let's take v_1, \dots, v_n of V

if we take a linear combination TV_1, \dots, TV_n

$$a_1 TV_1 + \dots + a_n TV_n = 0$$

$$\text{this is } T(a_1 v_1 + \dots + a_n v_n) = 0$$

and since T is invertible, this means $a_1 v_1 + \dots + a_n v_n = 0$

but since v_1, \dots, v_n are a basis, the only solution is $a_1 = \dots = a_n = 0$

thus TV_1, \dots, TV_n are lin. ind.

and since there are n of these, they also span V and thus form a basis

now (b) \Rightarrow (c)

if the statement is true for every basis then it's definitely true for some basis too

now (c) \Rightarrow (a)

if there's at least one basis v_1, \dots, v_n where TV_1, \dots, TV_n is a basis, then T is invertible

we can say we have a basis: TV_1, \dots, TV_n that forms a basis of V

if we make an inverse

there's a linear map S such that $S(Tv_k) = v_k$ for $k=1, \dots, n$

then for any vector $v \in V$, we can say $v = a_1 v_1 + \dots + a_n v_n$

$$(ST)(v) = S(T(a_1 v_1 + \dots + a_n v_n)) = S(a_1 TV_1 + \dots + a_n TV_n) = a_1 S(Tv_1) + \dots +$$

thus $ST = I$ which means T is invertible

8. exercise 6 (pg. 94)

6 Suppose that W is finite-dimensional and $S, T \in \mathcal{L}(V, W)$. Prove that $\text{null } S = \text{null } T$ if and only if there exists an invertible $E \in \mathcal{L}(W)$ such that $S = ET$.

we're supposing W is finite dimensional, $S, T \in \mathcal{L}(V, W)$

① let's show that if $S = ET$ with E invertible, then $\text{null } S = \text{null } T$

let's suppose that $S = ET$, where E is invertible

then for any $v \in V$, $S(v) = E(T(v))$

then $S(v) = 0 \Rightarrow E(T(v)) = 0$

since E is invertible, only zero maps to zero $E(w) = 0 \Rightarrow w = 0$

thus $E(T(v)) = 0 \Rightarrow T(v) = 0$

and thus $v \in \text{null } T$

now if $T(v) = 0 \Rightarrow E(T(v)) = E(0) = 0$ so $S(v) = 0$

and thus $v \in \text{null } S$

② next, if $\text{null } S = \text{null } T$, then there exists E which is invertible s.t. $S = ET$

well if $\text{null } S = \text{null } T$ then we can say $\text{null } T \subseteq \text{null } S$

whenever we have two maps with one's null space inside the other, we can say

$S = \tilde{E}T$ where \tilde{E} is a linear map on W

if we restrict \tilde{E} to only act on the colspace of T , and say this is E'

we can then see if E' is injective

if $E'(w) = 0$ for $w \in \text{range } T$

then $w = T(v)$ for some v

$E'(w) = S(v) = 0 \Rightarrow v \in \text{null } S$

but $\text{null } S = \text{null } T \Rightarrow T(v) = w = 0$

$\therefore E'(w) = 0 \Rightarrow w = 0$ so E' is injective

and since E' is injective and its range T is fin. dim., we can

extend E' to an invertible map E defined on all W

and now for all v , $S(v) = \tilde{E}T(v) = ET(v)$