INTRODUCTION

Set:-

↳ A well-defined collection of objects or elements.

↳ symbollically written as:

$$A = \{a, b, c, d\}$$

eg:- $\{a, b, c\}$, $\{a, c, b\}$, $\{b, c, a\}$ .... etc.

- **Set of sets**:- set containing other sets as its elements

  eg: $S = \{\{a,b\}, \{c, d, e\}\}$ etc.

- **Finite Sets**: set with finite no. of elements

  eg:. $S = \{1, 2, 3, 4, 5\}$; No. of elements = 5

- **Infinite Set**:- Set with infinite no. of elements

  eg:- set of all positive ~~numbers~~ integers, $N = \{1, 2, ...$

- **Singleton set**:- set with only one element

  eg: $S = \{a\}$

- **Null set**:- set with no element

  - denoted by $\{\}$ or ϕ

- **Universal set**:- Set of all elements under consideration.

  - denoted by U.

- **Equal sets**:- (Improper ~~sets)~~ subset) :-

  → Two sets are said to be equal if they have same elements.

  $S = \{a, b\}$, $T = \{b, a\}$; $S = T$ ⟹ order doesn't matter

  eg:- if $S = \{a, b\}$, $T = \{a, b\}$ then, $S = T$

- **Equinumerous Sets**:-

  → Two sets are said to be equinumerous if they have same cardinality (if same no. of elements)

  eg: If $S = \{a, b, c\}$, $T = \{1, 2, 3\}$ then

  S and T are equinumerous.

  Cardinality of $S(|S|)$ = Cardinality of $T(|T|) = 3$.

• **Subset**:-

→ Set S is subset of set T if all elements of S also belongs to T, written as $S \subseteq T$.

eg:- If $S = \{a, b, c\}$ and $T = \{a, b, c, d\}$ then, $S \subseteq T$.

| $S = \{a, b, c, d\}$ $U = \{a, b, c, d\}$
| $T = \{b, c\}$ $S \subseteq U$; $U \subseteq S$; $S = U$

• **Proper subset**:-

→ If S is subset of T but S does not contain all elements of T then S is called proper subset of T, written as $S \subset T$.

eg:- If $S \subseteq T$ and $S \neq T$, then S is proper subset of T $(S \subset T)$

$S = \{1, 2\}$; $T = \{3, 2, 1\}$ then $S \subset T$.

* If $S \subseteq T$ and $T \subseteq S$ then $S = T$.

• **Power set**:-

→ A set formed by all the subsets of given set S is called power set of S. (including set itself & null/empty set)

→ represented by $P(S)$ or 2^S.

eg: If $S = \{a, b\}$ then

power set of S,

$2^S = P(S) = \{\phi, \{a\}, \{b\}, \{a, b\}\}$

→ The no. of elements in power set of set S is $2^{|S|}$, where $|S|$ is the cardinality of S.

$|2^S| = 2^{|S|}$

Operations on set :-

(i) **Union $(\cup)$** of two or more sets is a set with elements belonging to atleast one of the sets.

eg: If $S = \{a, b, c\}$, $T = \{b, c, d\}$, then,

$S \cup T = \{a, b, c, d\}$

(ii) **Intersection $(\cap)$** of two or more sets is a set with elements that are common to all the sets.

eg: If $S = \{a, b, c\}$, $T = \{b, c, d\}$ then $S \cap T = \{b, c\}$

(iii) Difference of two sets $(S-T)$ is set of elements of S that are not elements of T.

eg: If $S = \{a, b, c, d\}$, $T = \{c, d, e, f\}$

then,

$$S - T = \{a, b\}$$

(iv) Complement of set S is the set of elements that doesnot belong belonging to universal set (U) to S.

eg: $U = \{a, b, c, d, e, f\}$

and $S = \{a, b, c, d\}$

then,

$$S^c \text{ or } \bar{S} = \{e, f\}$$

$\rightarrow S \cup \bar{S} = U$ and $S \cap \bar{S} = \phi$

$U = \{a, b, c, d, e, f, g, h\}$
$A = \{a, b, c, d\}$
$B = \{c, d, e, f\}$

$A \cup B = \{a, b, c, d, e, f\}$
$A \cap B = \{c, d\}$
$A - B = \{a, b\}$
$B - A = \{e, f\}$
$\bar{A} = \{e, f, g, h\}$

$A \cup \bar{A} = U$
$A \cap \bar{A} = \phi$

Set Laws :-

If A, B & C are sets then,

- Idempotency :- $A \cup A = A$
  $A \cap A = A$
- Commutativity :- $A \cup B = B \cup A$
  $A \cap B = B \cap A$
- Associativity :- $(A \cup B) \cup C = A \cup (B \cup C)$
  $(A \cap B) \cap C = A \cap (B \cap C)$
- Distributivity :- $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$
  $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$
- Absorption : $A \cap (A \cup B) = A$
  $A \cup (A \cap B) = A$

  $S \cap (S \cup T) = S$
  $S \cup (S \cap T) = S$

- De-Morgan's law:-

  For sets A & B,

  (i) $\overline{A \cup B} = \bar{A} \cap \bar{B}$

  (ii) $\overline{A \cap B} = \bar{A} \cup \bar{B}$

Cartesion Product :-

→ Cartesian product of a set S and T denoted by $S \times T$ is defined as the set of all ordered pairs (s,t) such that $s \in S$ and $t \in T$.

$(a,b) \neq (b,a)$

$S \times T \neq T \times S$

eg:- if $S = \{a,b\}$ and $T = \{c,d,e\}$

Then,

$S \times T = \{(a,c), (a,d), (a,e), (b,c), (b,d), (b,e)\}$

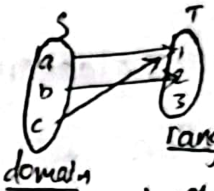
$S \times S = \{(a,a), (a,b), (b,a), (b,b)\}$

If no. of elements then no. of elements in $A \times B = m \times n$

Relation :- let A & B are 2 sets, then every relation from A to B is subset of $A \times B$.

Relation 'R' from a set S to set T is subset of $S \times T$ and is written as,

$_sR_t$ or R i.e. $R \subseteq S \times T$



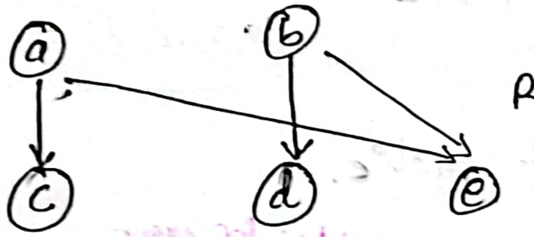
$R = \{(a,1), (b,2), (c,1)\}$

if $S = \{a,b\}$ and $T = \{c,d,e\}$

. Let Relation, $R = \{(a,c), (a,e), (b,d), (b,e)\}$

when x & y are in R, we write xRy.

→ when x & y are not in R, we write $x \not R y$.



$R \subseteq S \times T$

Total no. of relations possible $= 2^{m \times n}$

Similarly,

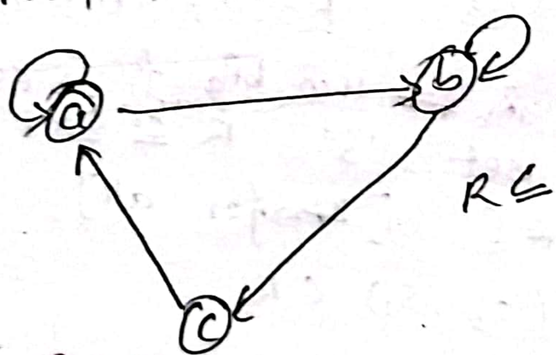
A relation R on set S [from S to S] is subset of $S \times S$.

; $R \subseteq S \times S$

eg:- if $S = \{a, b, c\}$

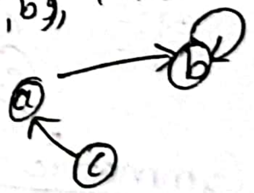
Let $R = \{(a,a), (a,b), (b,b), (b,c), (c,a)\}$

i.e.



$R \subseteq S \times S$

$R = \{(a,b), (b,b), (c,a)\}$



- Inverse Relation :-

A relation R^{-1} is inverse of a relation R if,

for all $s_1, s_2 \in S$, if $(s_1, s_2) \in R$ then, $(s_2, s_1) \in R^{-1}$

सब के लिए

$(\forall s_1, s_2) \in S$

($\forall$ → for all

$\exists$ → Exist करता

↳ There exist (at least one))

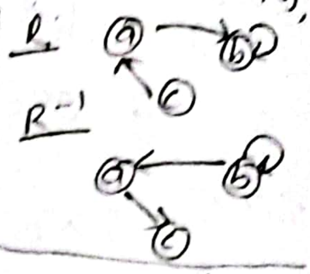
eg:- if $S = \{a, b, c\}$

$S \times S = \{(a,a), (a,b), (a,c), (b,a), (b,b), (b,c), (c,a), (c,b), (c,c)\}$

Let $R = \{(a,b), (a,c), (b,c), (c,c)\}$

then, $R^{-1} = \{(b,a), (c,a), (c,b), (c,c)\}$

$R = \{(a,b), (b,b), (c,a)\}$
$R^{-1} = \{(b,a), (b,b), (a,c)\}$



* Chain :-

→ chain in a binary relation 'R' is a sequence $(s_1, s_2, s_3, \ldots, s_N)$; $N \geq 1$, such that $(s_i, s_{i+1}) \in R$, $i = \{1, 2, \ldots, N-1\}$

~~(a,b,c,d) chain if~~

* Cyclic Relation :-

→ Relation 'R' is said to be cyclic relation on set S if there exist finite sequence of distinct elements. $(s_1, s_2, s_3, \ldots, s_N)$; $N \geq 1$; such that $(s_i, s_{i+1}) \in R$, $i \in \{1, 2, \ldots, N-1\}$ and $(s_N, s_1) \in R$

chain $\boxed{i = 1, 2, 3, \ldots N-1}$

(a, b, c, d) ⇒ chain if
$(a, b) \in R$
$(b, c) \in R$
$(c, d) \in R$

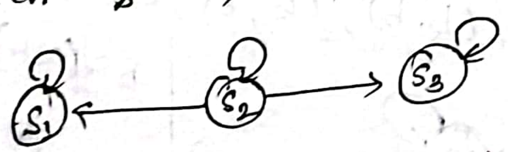
* Acyclic Relation → Not cyclic.

* Reflexive Relation :- If aR_a exists : for every $a \in S$.

A relation 'R' on set 'S', $R \subseteq S \times S$, is reflexive if, for each $s \in S$, $(s, s) \in R$.

$\boxed{\forall s \in S, (s, s) \in R}$

eg:-



$S = \{a, b, c\}$
$R = \{(a,a), (a,b), (b,b), (c,a)\}$ → Not reflexive
(c,c) missing

* Symmetric Relation :- If aR_b then bR_a for every $a, b \in S$.

A relation 'R' on set 'S', $R \subseteq S \times S$, is symmetric if $\forall s_1, s_2 \in S$ [∀ → for all] if $(s_1, s_2) \in R$ then, $(s_2, s_1) \in R$.

$R = R^{-1}$ for symmetric

eg:-

$R = \{(a,b), (b,a), (a,a), (a,c), (c\ldots$

* Transitive Relation:- if aRb & bRc exist, then aRc for every $a,b,c \in S$.

A relation 'R' on set 'S', $R \subseteq S \times S$, is transitive if $\forall s_1, s_2, s_3 \in S$.

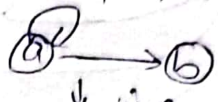
if $(s_1, s_2) \in R$, $(s_2, s_3) \in R$ then, $(s_1, s_3) \in R$.

$R = \{(a,b)\}$ — also transitive

$R = \{(a,b), (b,c), (a,a), (a,c)\}$

eg:-

$R = \{(a,b), (a,a)\}$



$(a,a) \in R$
$(a,b) \in R$
$(a,b) \in R$

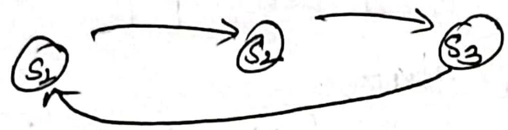
* Asymmetric Relation:-

A relation 'R' on set 'S', $R \subseteq S \times S$ is asymmetric if $\forall s_1, s_2 \in S$.

if $(s_1, s_2) \in R \Rightarrow (s_2, s_1) \notin R$.

"<"
if $a < b$ then $b \nless a$

eg:-



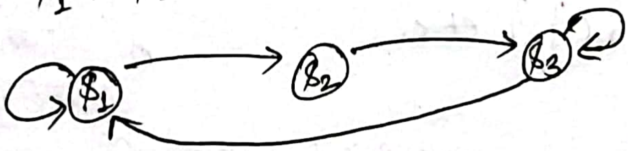
* Anti-Symmetric Relation:-

A relation 'R' on set 'S', $R \subseteq S \times S$, is antisymmetric if $\forall s_1, s_2 \in S$, if $(s_1, s_2) \in R$ and $(s_2, s_1) \in R$.

then, $s_1 = s_2$

"≤"
if $a \leq b$ & $b \leq a$ then, $a = b$

eg:-



	<	≤
Not reflexive	$(a < a)$ ✗	$a \leq a$ ✓ → Reflexive
Not symmetric ✗	$a < b \Rightarrow b \nless a$	$(a \leq b \Rightarrow b \leq a)$ ✗ not sy
Transitive	$a < b, b < c$ $a < c$	$a \leq b$, $b \leq c$, $\Rightarrow a \leq c$ transitive

* Equivalence Relation:-

A relation on set S, $R \subseteq S \times S$, is an equivalence relation if it is reflexive, symmetric and transitive.

i.e. $\forall s \in S$; $(s, s) \in R$

$\forall s_1, s_2 \in S$, if $(s_1, s_2) \in R \Rightarrow (s_2, s_1) \in R$

$\forall s_1, s_2, s_3 \in S$, if $(s_1, s_2) \in R$; $(s_2, s_3) \in R$
$\Rightarrow (s_1, s_3) \in R$.

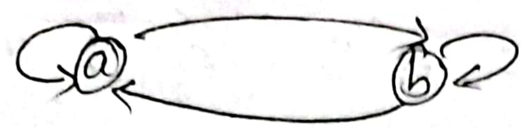
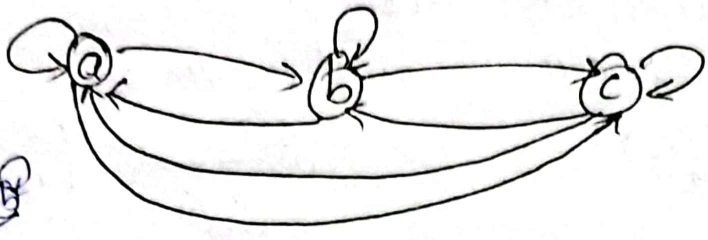
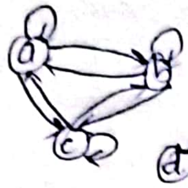
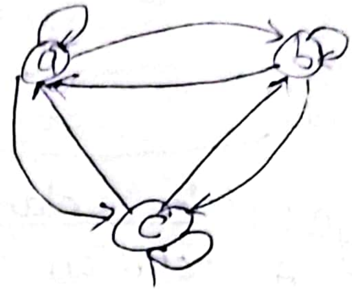
eg: 

fig: equivalence relation







(d) — d should be either completely outside or completely involved (2 way relation should be) → to be equivalence relation

Partition (π):-

Partition, π, of a set S, is a set of pairwise disjoint subset of the set S, such that,

- Union of all the elements of π, $\cup\pi = S$
- Each elements of π is non-empty

eg: If $S = \{a, b, c\}$, its partition sets may be,

$\pi_1 = \{\{a, b\}, \{c\}\}$

$\pi_2 = \{\{a\}, \{b\}, \{c\}\}$ etc.

$A = \{\{a,b\}, \{c,d\}, \{d,e,f\}\}$ — intersection so not partition

→ every pair must be disjoint.

Disjoint Set:-

Two sets are disjoint if they have no elements in common i.e.

A and B are disjoint if $A \cap B = \phi$

Equivalence class:

$Z = \{\dots, -2, -1, 0, 1, 2, 3 \dots\}$ → set of integers

$R = \{(s_1, s_2) : s_1, s_2 \in Z, (s_1 - s_2)$ is divisible by 5.

✓ Reflexive: $(a-a)$ is divisible by 5.
$= 0$ is divisible by 5. - true

✓ Symmetric: if $(a-b)$ is divisible by 5 ⇒ $(b-a)$ is also divisible by 5.

Transitive: if $(a-b)$ " " " " , $(b-c) \dots 5 \Rightarrow (a-c) \dots 5$

Equivalence class:-

Equivalence class is represented by $[s]$ is set of elements fulfilling the equivalence relation requirement, R.

i.e. $[s_1] = \{ s_2 : (s_2, s_1) \in R \}$

- Equivalence class is non-empty set.
- Equivalence classes are disjoint.
- The partition formed by equivalence class, π, is such that $\cup \pi = S$.

To show equivalence classes are disjoint,

$[s_1] = \{ s_2 : (s_2, s_1) \in R \}$, $[t_1] = \{ t_2 : (t_2, t_1) \in R \}$

$\pi = \{ [s_1], [s_2] \dots [s_N] \}$, $\cup \pi = S$

Let us assume, $[s_1]$ and $[t_1]$ are not ~~distin~~ disjoint.

i.e. $(t_2, s_2) \in R \Rightarrow (s_2, t_2) \in R$

again, $(s_2, s_1) \in R \Rightarrow (t_2, t_1) \in R$

$\Rightarrow (t_2, s_1) \in R \quad \Rightarrow (s_2, t_1) \in R$

$\Rightarrow t_2 \in [s_1] \quad s_2 \in [t_1]$

$[s_1] \subseteq [t_1]$

$[t_1] \subseteq [s_1]$

i.e, $[t_1] = [s_1]$ or if $[s_1]$ & $[t_1]$ are not disjoint, they are equal.

$[50] \cap [51] = \phi$

$[50] = \{ 50, 0, -5, 5, 10, \dots \} = [0]$

$[51] = \{ 51, 1, 6, 11, -4, -9, \dots \} = [1]$

$[52] = \{ 52, 2, 7, 12, -3, -8 \dots \} = [2]$

$[53] = \{ 53, 3, 8, 13, -2, -7 \dots \} = [3]$

$[54] = [54, 4, 9, 14, -1, -6, \dots \} = [4]$

$[55] = [50]$ | $[0] \cup [1] \cup [2] \cup [3] \cup [4]$
$= \mathbb{Z}$

Theorem:-

For any equivalence relation, R on set S, the corresponding equivalence forms the partition of the set.

eg:- of equivalence class.

Let $Z = \{ \ldots, -12, -11, \ldots, -1, 0, 1, 2, \ldots, 11, 12, \ldots \}$
= set of all integers.

Equivalence Relation, R can be defined as,

$$R = \{ (s_1, s_2) : s_1, s_2 \in Z, (s_1 - s_2) \text{ is divisible by } 5 \}$$

Then, equivalence classes.

$[0] = \{ \ldots, -5, 0, 5, 10, \ldots \}$

$[1] = \{ \ldots, -4, 1, 6, 11, \ldots \}$

$[2] = \{ \ldots, -3, 2, 7, 12, \ldots \}$

$[3] = \{ \ldots, -2, 3, 8, 13, \ldots \}$

$[4] = \{ \ldots, -1, 4, 9, 14, \ldots \}$

$\pi = \{ [0], [1], [2], [3], [4] \}$
$= \{ [-12], [-11], [-10], [-9], [-8] \}$

$\cup_\pi = Z$

If R is an equivalence relation on S then R^{-1} is also equivalence relation on S.

Proof:-

For R to be equivalence relation on S,

$\forall s \in S ; (s, s) \in R$

$\forall s_1, s_2 \in S ; (s_1, s_2) \in R \Rightarrow (s_2, s_1) \in R$

$\forall s_1, s_2, s_3 \in S ; (s_1, s_2) \in R, (s_2, s_3) \in R \Rightarrow (s_1, s_3) \in R.$

Then, we know,

$$R^{-1} = \{ (s_2, s_1) : (s_1, s_2) \in R \}$$

Now,

$$\left.\begin{array}{l} \forall s \in S, \ (s, s) \in R \\ \Rightarrow (s, s) \in R^{-1} \end{array}\right\} \text{Reflexive}$$

$$\forall s_1, s_2 \in S$$

$$\left.\begin{array}{l} \text{if } (s_1, s_2) \in R \Rightarrow (s_2, s_1) \in R \\ \text{then, } (s_2, s_1) \in R^{-1} \Rightarrow (s_1, s_2) \in R^{-1} \end{array}\right\} \text{symmetric}$$

$$\forall s_1, s_2, s_3 \in S$$

$$\left.\begin{array}{lll} \text{if } (s_2, s_1) \in R & \text{then} & (s_1, s_2) \in R^{-1} \\ (s_3, s_2) \in R & \text{then} & (s_2, s_3) \in R^{-1} \\ (s_3, s_1) \in R & \Rightarrow & (s_1, s_3) \in R^{-1} \end{array}\right\} \text{Transitive}$$

Since, R^{-1} is reflexive, symmetric and transitive,
$\therefore R^{-1}$ is also equivalence relation on S.

Partial Order :- (Non- strict) :- $\leq, \geq, \subseteq$

→ A relation on set S which is reflexive, antisymmetric and transitive is called a partial ordering on S.

→ A set on which there is a partial ordering defined is called a partial order set or poset.

denoted by $[S; R]$

i.e.

$$\forall s \in S ; (s, s) \in R$$

$$\forall s_1, s_2 \in S ; (s_1, s_2) \in R \ \& \ (s_2, s_1) \in R \Rightarrow s_1 = s_2.$$

$$\forall s_1, s_2, s_3 \in S ; (s_1, s_2) \in R, (s_2, s_3) \in R \Rightarrow (s_1, s_3) \in R$$

eg:

$S = \{1, 2, 3\}$

$R_1 = \{(1,1), (2,2), (3,3)\}$ → smallest poset on S.

$R_2 = \{(1,1), (2,2), (3,3), (1,2), (2,3), (1,3)\}$ ↳ largest poset on S.

* If 'A' is any set of real numbers, then $[A; \leq]$ is a poset.

$1.5 \leq 1.5$ - Reflexive

$(1 \leq 2)$ $(\cancel{2 \leq 1})$ → Antisymmetric

$(1 \leq 2)$ $(2 \leq 3)$ ⇒ $(1 \leq 3)$ → Transitive

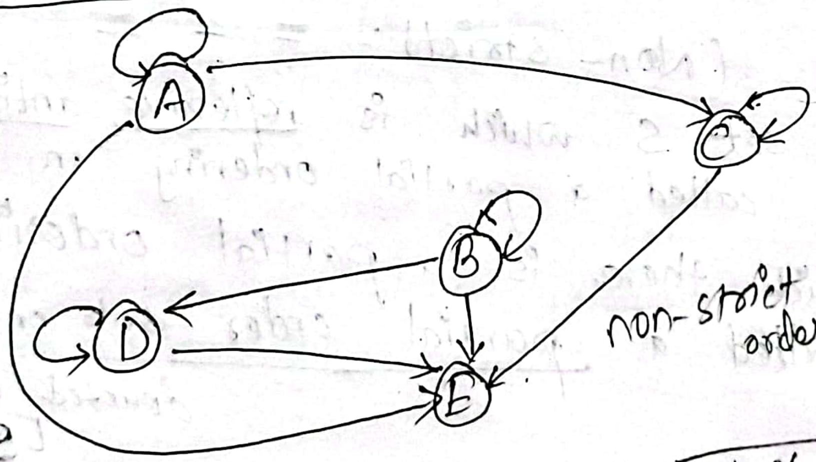
Strict Partial Order:- irreflexive, asymmetric and transitive.
$(<, >, \subset)$

Eg.- A subset is partial order relation (non-strict)
[proper subset is strict partial order].

$S = \{\{a\}, \{b\}, \{a, c\}, \{b, d, e\}, \{a, b, c, d, e\}\}$ (with A, B, C, D, E written above the five elements)

Subset Relation:

→ strict का उल्टा same relation hen there



non-strict partial order

Closure Property of Set:- [Set of integers is closed under subtraction operation]

→ A set is said to be closed under an operation if the result of operation on set elements also belong to same set.

$N = \{0, 1, 2, 3, 4, 5, 6 \ldots\}$
$a, b \Rightarrow \boxed{a + b = c}$ → set of natural no. closed under addition.

eg: Set of natural numbers is closed under addition operation but not under subtraction operation

* A set $T \subseteq S$ is said to be closed under some n-ary relation $R \subseteq S^n$ if

then, $(t_1, t_2, \dots, t_{n-1} \in T, (t_1, t_2, \dots t_{n-1}, t_n) \in R$

$t_n \in T$

eg: $S = \{1, 2, 3, 4, 5, 6, 7\}$

$T = \{1, 2, 3, 4, 6\}$; $T \subseteq S$

$R = \{ s_1, s_2, s_3 : s_3 = s_2 \text{ times } s_1 \}$

$= \{(1,1,1), (1,1,2), (1,3,3), (1,4,4), (1,5,5), (1,6,6), (1,7,7), (2,1,2), (2,2,4), (3,1,3), (3,2,6), (4,1,4), (5,1,5), (6,1,6), (7,1,7)\}$

Here,

$1, 2 \in T$, $(1, 2, 2) \in R \Rightarrow 2 \in T$

$1, 3, \in T$, $(2, 3, 6) \in R \Rightarrow 6 \in T$

and so on.

i.e. $T \subseteq S$ is closed under the relation $R \subseteq S^n$.

Theorem:-

For relations $R_i \subseteq S^n$, $n > 0$ and set $T \subseteq S$, there is a unique minimal set V having closure property P of set S under R_i and containing set T.

$\Rightarrow$ Consider $W_j \subseteq S$ such that and $T \subseteq W_j$

, W_j is closed under R_i $\forall i, j$

let, $V = \bigcap_j W_j$

As $\forall_{i,j}$, W_j is closed under R_i.

$s_1(i)\, s_2(i) \dots s_{n-1}(i) \in W_j$ and

$(s_1(i), s_2(i), \dots s_n(i)) \in R_i \Rightarrow s_n(i) \in W_j$

Since, V is intersection of W_j, $s_n(i) \in V$.

i.e. V is also closed under R_i and contains T.

Let us suppose, $V' \subset V$ is also closed under R_i & contains T. Then, V' must be equal to some W_j such that $V \subseteq V'$ ($\because V \subseteq W_j$). This is the contradiction.

Hence,

V cannot have proper subset which is closed under R_i and contains T.

Hence,

V is a unique minimal set having closure property of S under R_i, containing T.

[V is closure of T under P].

Function: special kind of relation

Function is a subset of $S \times T$ that assign, for each element of S, a unique element of T.

→ An association of each object of one kind with a unique object of other kinds.

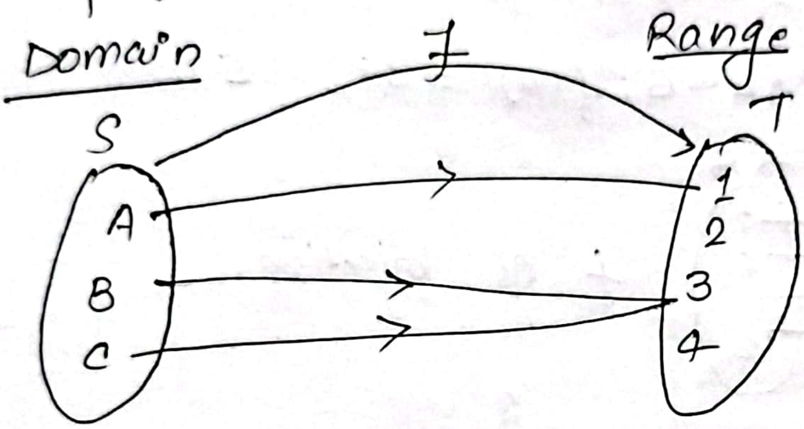
It is written as,

$$f : S \to T \text{ [function } f\text{, such that } S \text{ maps } T]$$

or,

$$t = f(s),\ t \in T,\ s \in S.$$

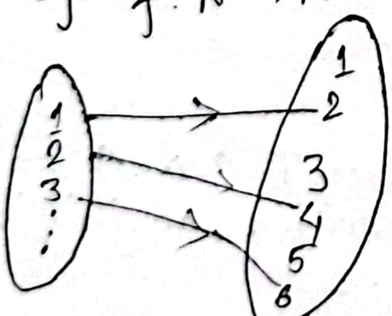
→ The elements of S forms the domain of definition of f and elements of T forms the range of f.



One-to-One function:

Distinct elements of domain of 'f' is mapped to distinct element of range of 'f' by this function.

eg: $f : N \to N$ such that $f(n) = 2n,\ n \in N,\ N = \{0, 1, 2 \ldots\}$ → is one-to-one and into



$$f(s_1) \neq f(s_2) \Rightarrow s_1 \neq s_2\ ;\ s_1, s_2 \in S.$$

Onto Function: → co.domain का euta element का [illegible]

A function $f : S \to T$ is onto function if each element $t \in T$ is an image of atleast one element $s \in S$.

eg:- $f : R \to R_p$, such that $f(x) = x$

$R = [-\infty, \infty]$ [R is set of all real numbers, R_p is set of all non-negative real numbers]

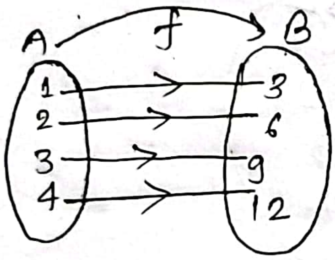
$R_p = [0, \infty]$

Bijection:-

A function $f : S \to T$ is bijection if it is both one-to-one and onto. Such a function f is also termed as <u>one-to-one correspondence</u>.

eg:- Let $A = \{1, 2, 3, 4\}$
$B = \{3, 6, 9, 12\}$

Then,
the function, $f : A \to B : f(a) = 3a ; a \in A$



f is bijection.

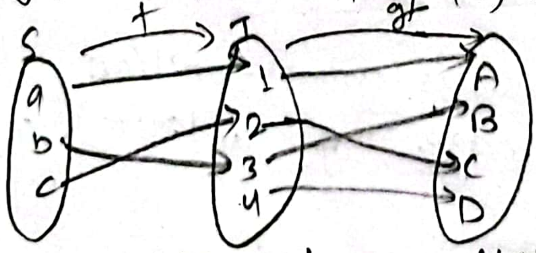
Inverse of a function → f^{-1} exist only if f is bijection fn.

Inverse of a bijection function $f : S \to T$, represented by f^{-1}, is a function $f^{-1} : T \to S$.

[Note: If f is not bijection then f^{-1} doesnot exist].

Composite function:-

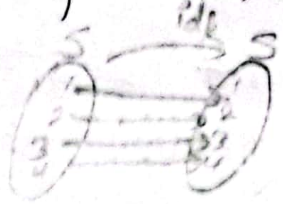
Composite function of $f : S \to T$ and $g : T \to V$ is a function $g \circ f : S \to V$, also written as $g(f(s)) = v$, $s \in S$, $v \in V$ where, $f(s) = t$, $t \in T$.



$g.f : S \to V$ or $g(f(s)) = v$ where, $f(s) = t$.

Identity function:- → is Bijection fun.

A function on set S is identity function, written as id_S if



$$id_S = \{ (s, s) : s \in S \}$$

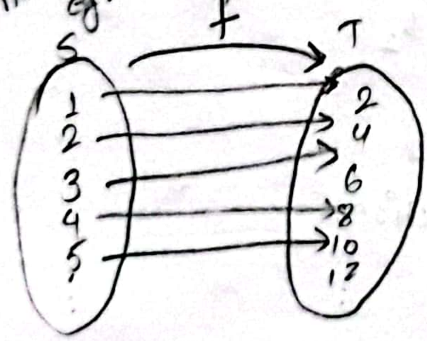
In other words,
for some $f: S \to S$, such that, $f(s) = s$; $s \in S$.

Q Is bijection an equivalence Relation?

countably infinite
↳ set of integers

uncountable
↳ set of real numbers

equinumerous
eg.



$f(n) = 2n$

$|S| = |T|$

S - set of all positive integer
T = set of all positive even integer

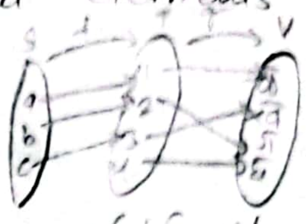
$S \subset T$

If f and g are functions then gof is also a function.

Let $f: S \to T$, $g: T \to V$ then $gof: S \to V$

i.e. $g(f(s)) = v$, $s \in S$, $v \in V$.

Suppose that $s_1 \in S$ maps two different elements $v_1, v_2 \in V$.



$gof: S \to V$

i.e. Let $g(f(s_1)) = v_1$ & $g(f(s_2)) = v_2$

then,

$$f(s_1) = t_1 \quad \& \quad g(t_1) = v_1$$

and, $f(s_1) = t_2$ and $g(t_2) = v_2$

But, since f is a function, $f(s_1)$ ~~or~~ must be unique

i.e. $f(s_1) = t_1 = t_2$

$\Rightarrow g(t_1) = g(t_2)$ $\because g$ is a function & $t_1 = t_2$.

i.e. $v_1 = v_2 \Rightarrow gof$. is also a function.

If f and g are one-to-one functions then gof is also one-to-one function.

Let $f: S \to T$, $g: T \to V$ and $gof: S \to V$.

Let for $s_1, s_2 \in S$, $g(f(s_1)) = g(f(s_2))$

but as g is one-to-one.

$$g(f(s_1)) = g(f(s_2)) \Rightarrow f(s_1) = f(s_2)$$

Again, as f is also one-to-one.

$$\cancel{g(f(s_1))} \; f(s_1) = f(s_2) \Rightarrow s_1 = s_2$$

i.e. $g(f(s_1)) = g(f(s_2)) \Rightarrow s_1 = s_2$

$\therefore gof$ is one-to-one function.

If f is onto & g is onto then gof is also onto function.

$\Rightarrow$ Let $f: S \to T$, $g: T \to V$

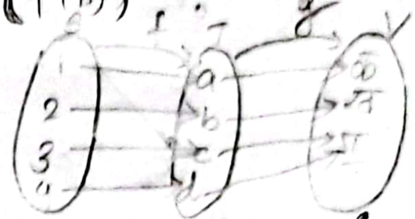
Then, $\forall_v \in V$, $\exists_t \in T$ such that $g(t) = v$

Similarly, $\forall_t \in T$, $\exists_s \in S$, such that $f(s) = t$.

Therefore, for any $s \in S$, $gof(s) = g(f(s)) = g(t) = v$.

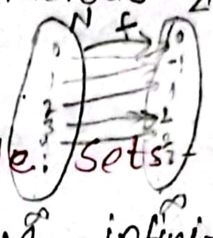
Also, $\forall_v \in V$, $\exists_s \in S$ such that $g(f(s)) = v$.

$\therefore$ gof is also onto function.



Equinumerous:

Two sets S and T are equinumerous if there is Bijection between set S and T.



$f(n) = \begin{cases} \frac{n}{2} & \text{if } n \text{ is even} \\ -\frac{n-1}{2} & \text{if } n \text{ is odd} \end{cases}$

Countably Infinite and Uncountable Sets:

(count गर्न सकिने infinite elements भएको set, position set गर्न सकिने)

A set $S = \{s_1, s_2, \dots\}$ is countably infinite if there is Bijection of S with the set of natural numbers.

i.e. $f: N \to S$.

If there is no such bijection, then set is <u>uncountable</u>.

Eg.. $f(n) = 2n$, $n \in N$, is <u>countably infinite</u>.

whereas, set of real numbers is <u>uncountable</u>.

Dovetailing

Dovetailing is a technique of interweaving the enumeration of several sets.

* If A, B, C are three countably infinite sets given as,

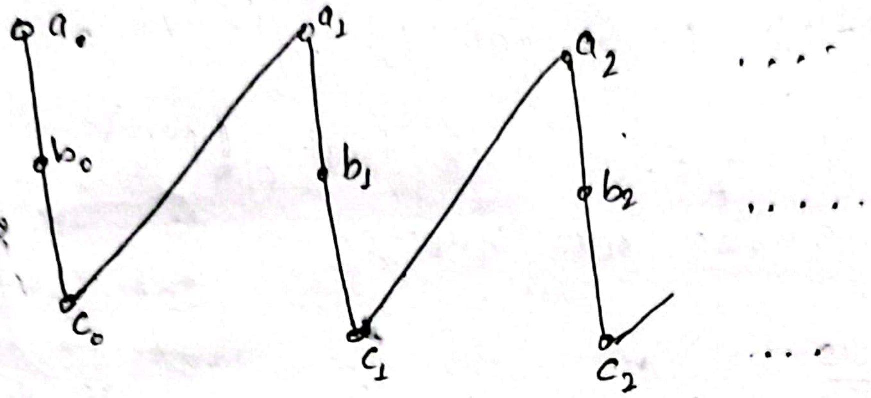
$A = \{a_0, a_1, a_2, \dots\}$
$B = \{b_0, b_1, b_2, \dots\}$
$C = \{c_0, c_1, c_2, \dots\}$

$A \cup B = \{a_0, a_1, a_2 \dots \infty, b_0, b_1 \dots \infty\}$
$= \{a_0, b_0, a_1, b_1, \dots\}$

Then, $A \cup B \cup C$ is also countably infinite.

Here, for $A \cup B \cup C$, if we try to list entire elements of 'A' first then, we will never reach to the elements of 'B' and 'C'.

Using the technique of ~~devo~~ dovetailing, i.e. visiting elements, altering between the sets as shown in the figure,



Dovetailing for $A \cup B \cup C$.

So,

$$A \cup B \cup C = \{a_0, b_0, c_0, a_1, b_1, c_1, \dots\}$$

Position of, $a_n \Rightarrow 3n$

$b_n \Rightarrow 3n+1$

$c_n \Rightarrow 3n+2$

i.e. $A \cup B \cup C$ is countably infinite.

~~Ex~~ Union of countably infinite collection of countably infinite set is countably infinite.

i.e. if N is countably infinite, then $N \times N$ (cartesian product) is also countably infinite.

$$\Rightarrow N \times N = \{0\} \times N \cup \{1\} \times N \cup \dots \infty$$

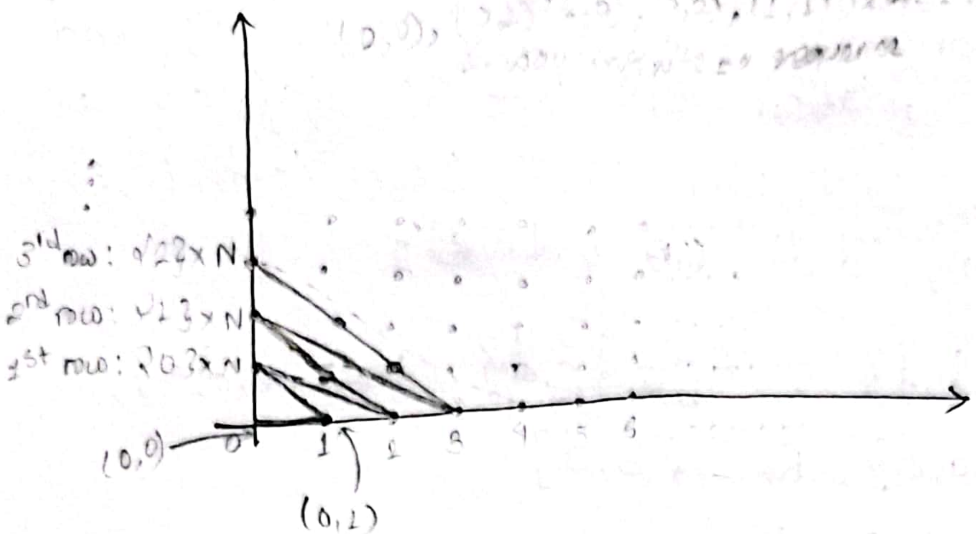
where,

$$\{0\} \times N = \{(0,0), (0,1), (0,2) \dots\}$$

$$\{1\} \times N = \{(1,0), (1,1), (1,2) \dots\}$$

$\vdots$

Let us arrange the numbers in grid as follows :-



For $N \times N$, in first step collect first element $(0,0)$ of the first row, then, in second step, collect second element at first row $(0,1)$ and first element of second row $(1,0)$. Processing in similar manner, in n^{th} step, collect n^{th} element of 1^{st} row, $(n-1)^{th}$ elements of second row, ... and lastly 1^{st} element of n^{th} row, such that any element with position index (i,j) can be mapped with unique position index m; $m = 0, 1, 2, \ldots$

as,

$$\overset{\text{position}}{m} = \frac{1}{2}\left[(i+j)^2 + 3i + j\right].$$

i.e. $f : N \times N \to N$

$\Rightarrow$ $N \times N$ is countably infinite.

Diagonalization Principle :- used to prove a set is uncountable.

For a binary relation, R, on set S, if $D = \{a : a \in S, \text{ and } (a,a) \notin R\}$ be diagonal set for R and

$\forall a \in S$, if $R_a = \{b : b \in S \text{ and } (a,b) \in R\}$

Then, D is distint from each R_a.

Proof:

Let us consider set $S = \{a, b, c, d, e\}$ and R be binary relation on set S defined as,

$$R = \{(a,b), (a,d), (b,c), (b,d), (c,c), (c,d), (c,e), (d,a), (d,d), (e,b), (e,d)\}$$

i.e.

$$R_a = \{b, d\}, \quad R_b = \{c, d\}, \quad R_c = \{c, d, e\},$$

$$R_d = \{a, d\}, \quad R_e = \{b, d\}$$

R may be pictorized as square array of 5 rows and 5 columns labeled with elements of S. Let s_i and s_j represent row and column of array;

$$s_i, s_j \in S.$$

Place 1 in the box corresponding to (s_i, s_j) if

$$(s_i, s_j) \in R$$

Place 0 in the box corresponding to (s_i, s_j) if

$$(s_i, s_j) \notin R.$$

i.e.

	a	b	c	d	e
a	0	1	0	1	0
b	0	0	1	1	0
c	0	0	1	1	1
d	1	0	0	1	0
e	0	1	0	1	0

The sequence of boxes along the diagonal is

0	0	1	1	0

and its complement is

1	1	0	0	1

i.e, Diagonal set, $D = \{ s : s \in S$ and $(s, s) \notin R \}$
$= \{a, b, e\}$

This D is different from each row of the array. By construction of D, it differs from first row in first position, second row in second position and so on.

For the same reason, this diagonalization principle holds for infinite set as well. The diagonal set D always differs from set Ra. at the position of ath row and a^{th} column.

* Set of real numbers in the interval [0, 1) is uncountable.

⇒ Suppose that the set of real number in the interval [0, 1) is countable and can be listed as $x_0, x_1, x_2, \ldots$

Let us write each x_i in its decimal expansion:

$$x_i = 0.d_{i0}\, d_{i1}\, d_{i2} \ldots$$

Let us define a number x such that its ~~i~~ i^{th} digit is $\begin{cases} 0 & \text{if } d_{ii} \neq 0 \text{ and} \\ 1 & \text{if } d_{ii} = 0 \end{cases}$

Then, $x \neq x_i$ for all i.

But x is a real number in [0, 1) and it must equal to same x_i. Hence the contradiction i.e, our assumption was wrong.

Thus, set of real numbers in [0, 1) is uncountable.

Proof Techniques:

* Proof By Induction:-

In this technique of proof, we show that the elements of an infinite set has certain property by using following steps of arguments.

- Basis step : The property is true for $n=0$ (or some other natural number)
- Induction step:- For any arbitrary natural number $k \geq 0$.

The property is true for $n=k$ is hypothesized. Then, based on this hypothesis, property is shown to be true for $n=k+1$.

Eg:- Prove that $n < 2^n$ for $n \geq 0$. non-negative

$(n \geq 1)$ n is positive integer

n is non-neg. integer $(n \geq 0)$

Proof:-

Basis: for $n=0$,

$0 < 2^0$ i.e. $0 < 1$ is true.

Induction Step: Let the statement be true for $n=k$ where k is arbitrary natural number.

i.e. $k < 2^k$ be true [Induction Hypothesis]

Then, for $n=k+1$,

We need to prove, $k+1 < 2^{k+1}$

We have, from induction hypothesis,

$k < 2^k$ ——— ①

→ Add 1 on b.s.

$k+1 < (2^k+1)$

Also, we know,

$1 \leq 2^k$ ——— ② $[\because k \geq 0]$

Adding 2^k on both side

$1 \leq 2^k$

$(2^k+1) \leq 2^k+2^k$

$\Rightarrow k+1 < 2^k+2^k$

From ① and ②,

$k+1 < 2^k+2^k$

$\Rightarrow k+1 < 2.2^k$

$\Rightarrow k+1 < 2^{k+1}$

i.e. statement is true for $n=k+1$

Hence, $n < 2^n$. proved.

Eg②:- Prove the following by induction.

$$1+2+3+\dots+n = \frac{n(n+1)}{2}$$

Ans:-

Basis:- In basis part, we have to prove for case $n=1$.

For $n=1$, LHS $=1$

RHS, $= \frac{1(1+1)}{2} = \frac{2}{2} = 1$

Induction:-

We have to prove:-

$$1+2+3+\dots+n = \frac{n(n+1)}{2}$$

Let us consider the statement is true for $n=k$.

$$1+2+3+\dots+k = \frac{k(k+1)}{2} \quad \text{--- (i)}$$

Then,

for $n=k+1$,

$$(1+2+3+\dots+k)+(k+1) = \frac{k(k+1)}{2} + (k+1)$$

$$= \cancel{\frac{(k+1)}{2}} \; (k+1)\left(\frac{k}{2}+1\right)$$

$$= \frac{(k+1)(k+2)}{2}$$

$$= \frac{(k+1)((k+1)+1)}{2} \quad \text{--- (ii)}$$

From eqⁿ (ii),

given statement is true for $n=k+1$.

" " all $n \in N$.

∴ By principle of mathematical induction,

$1+2+3+\dots+n = \frac{n(n+1)}{2}$ is true for all $n \in N$.

eg(3):- Prove by mathematical induction, for $n \geq 1$.

$$1^2 + 2^2 + \dots + n^2 = \frac{n(n+1)(2n+1)}{6}$$

Soln:-

Let $p(n) = 1^2 + 2^2 + \dots + n^2 = \frac{n(n+1)(2n+1)}{6}$

Basis:- For $n = 1$, $P(1) = 1 = LHS$.

$$RHS = P(1) = \frac{1(1+1)(2 \cdot 1+1)}{6} = \frac{1 \times 2 \times 3}{6} = 1$$

$\therefore P(1)$ is true.

Induction:- Let us assume, $(n = k) : k \geq 1$

$$1^2 + 2^2 + \dots + k^2 = \frac{k(k+1)(2k+1)}{6} \text{ is true}$$

Now, for $k+1$,

$$1^2 + 2^2 + 3^2 + \dots + k^2 + (k+1)^2$$

$$= \frac{k(k+1)(2k+1)}{6} + (k+1)^2$$

$$= \frac{k(k+1)(2k+1) + 6(k+1)^2}{6}$$

$$= \frac{(k+1)[k(2k+1) + 6(k+1)]}{6}$$

$$= \frac{(k+1)[2k^2 + 7k + 6]}{6}$$

$$= \frac{(k+1)(k+2)(2k+3)}{6}$$

$$= p(k+1) = RHS.$$

$\therefore$ By principle of mathematical induction,

$$1^2 + 2^2 + 3^2 + \dots + k^2 = \frac{n(n+1)(2n+1)}{6}$$

Proof by construction:-

prime no. betn 20 & 25

→ construct a prime no. 23 → lies betn 20 & 25.

Proof By contradiction:-

rational no:-

$\Rightarrow \frac{p}{q} ; q \neq 0$

smallest form

$\frac{20}{30} = 2/3$ $\frac{15}{9} = \frac{5}{3}$

2 & 3 are relatively prime

$\frac{9}{8}$ → relatively prime

• prove $\sqrt{2}$ is irrational.

→ let $\sqrt{2}$ be rational

$\sqrt{2} = \frac{a}{b}$; $b \neq 0$ and a & b are relatively prime

$\Rightarrow a = \sqrt{2}\, b$

Squaring on both sides,

$a^2 = 2b^2$ ——— ①.

$\Rightarrow a^2$ is even $\Rightarrow$ means a is even

then, a can be written as,

$a = 2k$, where k is some integer

Squaring,

$a^2 = (2k)^2$

$\Rightarrow 2b^2 = 4k^2$ [$\because$ from ① $a^2 = 2b^2$]

$\Rightarrow b^2 = 2k^2$

$\Rightarrow b^2$ is even $\Rightarrow b$ is even

As 'a' & 'b' both are even, they are not relatively prime

Hence, contradiction occurs

$\therefore \sqrt{2}$ is not rational no.

* Pigeonhole principle:-

Pigeonhole Principle:

"There is no one-to-one function from set S ~~on~~ to T, given S and T are finite and $|S| > |T|$.

i.e. if you have fewer pigeonholes then the pigeons and you need to put all pigeons in the holes then atleast two pigeons must share a single pigeonhole.

* If for a binary relation R on a finite set S, there is a path from s_1 to s_n ; $s_1, s_2 \dots s_n \in S$, then the length of shortest path from s_1 to s_n can be almost $|S|$.

Proof:- Let the path be $(s_1, s_2, \dots s_n)$ and let it be the shortest path from s_1 to s_n. Let the length of this path, $l > |S|$.

But, as well as the elements of this path,

$s_1, s_2 \dots s_n \in S$. & $|S| = n$

$\Rightarrow l > n$.

eg:- $L = \{w : w \in \Sigma^*, w \text{ contains } 101\}, \Sigma = \{0, 1\}$
Then, string '1011' $\in L$ but "0011" $\notin L$.

* Kleene Star of language, L is set of all strings obtained by concatenating zero or more strings from set L.

$$L^* = \{ w \in \Sigma^*, w = w_1 w_2 w_3 \dots w_n ; w_i \in L, i \in \{1,2, \dots\} \}$$

Note:-

$$(\Sigma^*)^* = \Sigma^*$$

$$\phi^* = \{e\}$$

$L = \{ab, aa\}$
$L^* = \{e, ab, aa, abab, abaa, aaab, aaaa, \dots\}$

L^* ⇒ patterns repeat of that language

Regular Expression:-

→ Regular expressions over a set 'Σ' is defined recursively as:-

represents set of strings

- A symbol, $\$ \in \Sigma$ is regular expression.
- Empty string, e is regular expression.
- Null set, ϕ is regular expression.
- Union of S_1 and S_2, $(S_1 \cup S_2)$ is regular expression if S_1 and S_2 are regular expressions.
- Concatenation of S_1 and S_2 $(S_1 S_2)$ is regular expression if S_1 and S_2 are regular expressions.
- Kleene star of S (S^*) is regular expression if S is regular expression.

eg:- for $\Sigma = \{a, b, c\}$, Regular expression can be: $\phi, a, e, ab, a^*b, a \cup b, a^* \cup b, aa^*b$ etc.

→ w contains abb
$(a \cup b)^* abb (a \cup b)^*$

→ w starts with abb
$abb(a \cup b)^*$

→ w contains atleast 2 b
$(a \cup b)^* ba^* b (a \cup b)^*$

→ length of w is atleast 2
$(a \cup b)^+$ or, $(a \cup b)(a \cup b)(a \cup b)^*$ or, $(a \cup b)^* (aa \cup ab \cup ba \cup bb)$

w contains atmost 2 a
$b^*(e \cup a \cup ab^*a)b^*$

* Prove by induction on length, w for any alphabets $w \in \Sigma^*$; $\{w\}$ is a regular language.

Proof:

Basis: for $|w| = 0$, $w = e \Rightarrow \{w\} = \{e\}$ is regular language

Induction:- for $|w| = n$, $\{w\}$ be regular language for length of string n.

for $|w| = n+1$, $w = \alpha_1 \alpha_2 \cdots \alpha_n \alpha_{n+1}$; $\alpha_i \in \Sigma$

$\Rightarrow w' \alpha_{n+1}$ where $w' = \alpha_1 \alpha_2 \cdot \alpha_n$

Since,

$|w'| = n \Rightarrow \{w'\}$ is regular language

also, as $\alpha_{n+1} \in \Sigma$ $\{\alpha_{n+1}\}$ is regular language.

Then,

concatenation of $\{w'\}$ and $\{\alpha_{n+1}\}$, $\{w' \alpha_{n+1}\}$ is also regular language.

i.e. for $|w| = n+1$, $\{w\}$ is regular language

Hence,

by principle of mathematical induction, $\{w\}$ for any alphabet Σ, $w \in \Sigma^*$, is a regular language.