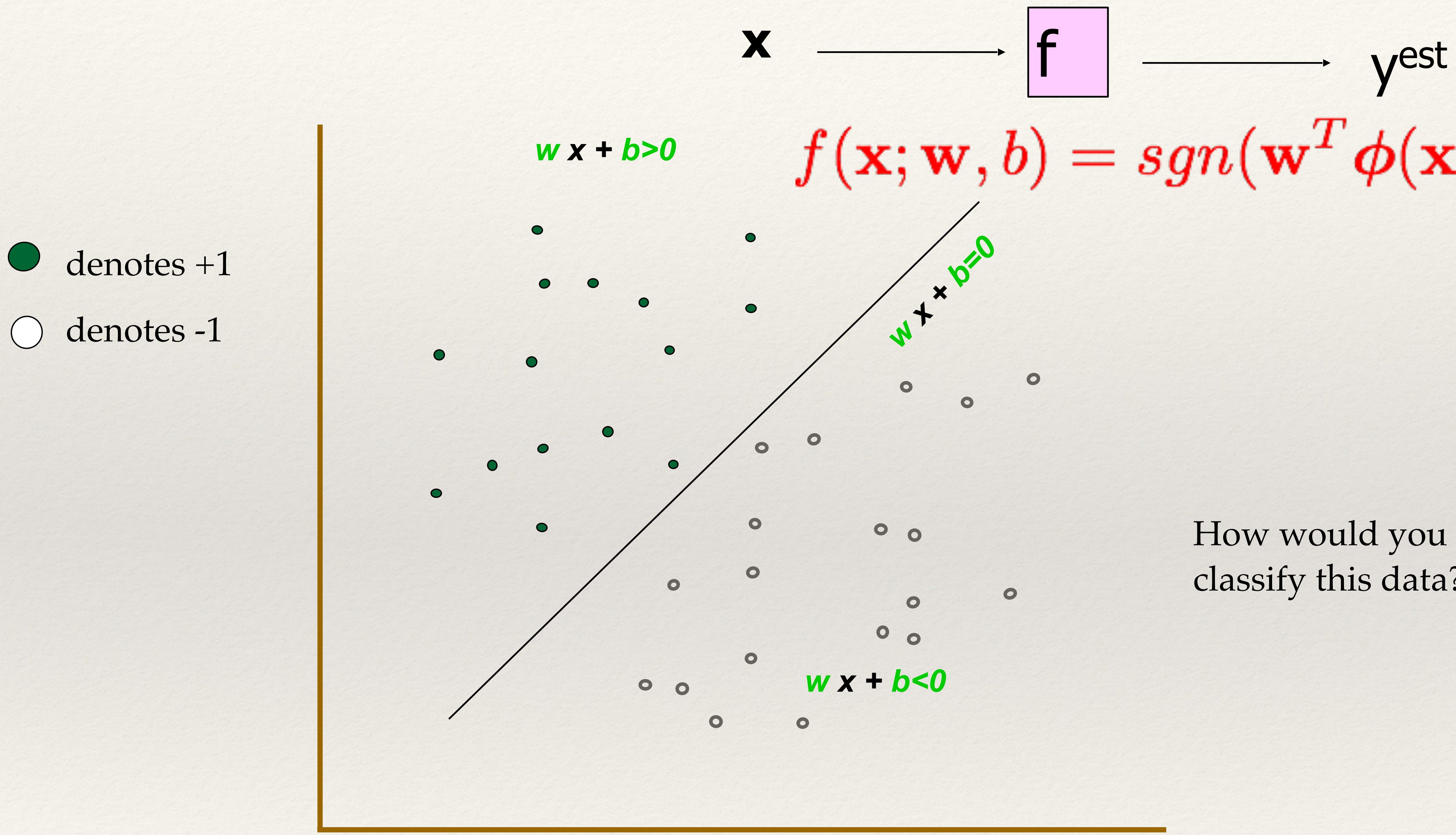
E9 205 Machine Learning for Sensory Signals

Support Vector Machines

16-03-2017

Linear Classifiers



$$f(\mathbf{x}; \mathbf{w}, b) = sgn(\mathbf{w}^T \phi(\mathbf{x}) + b)$$

How would you classify this data?

"SVM and applications", Mingyue Tan. Univ of British Columbia

Linear Classifiers

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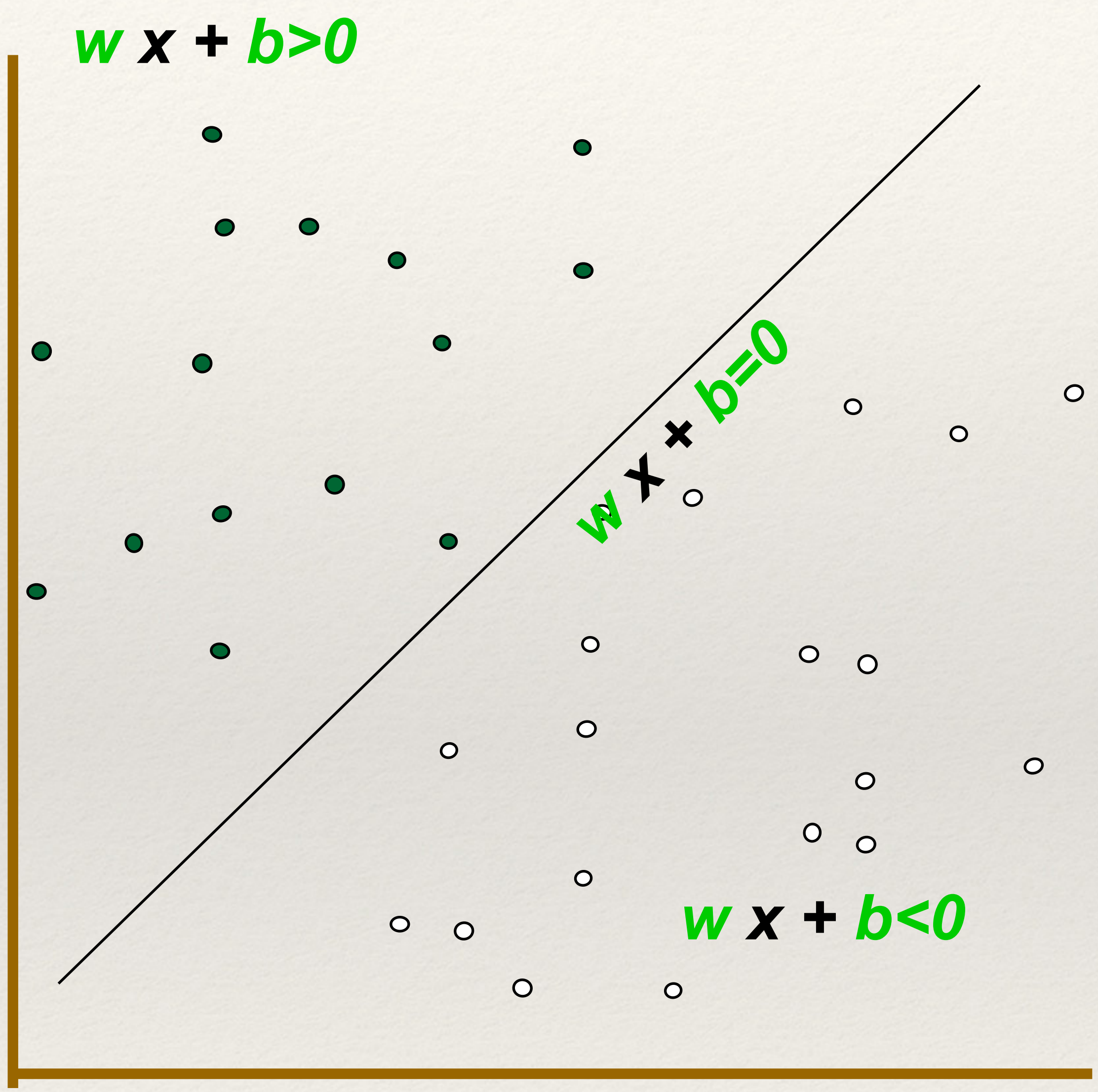
- • denotes +1
- ∘ denotes -1



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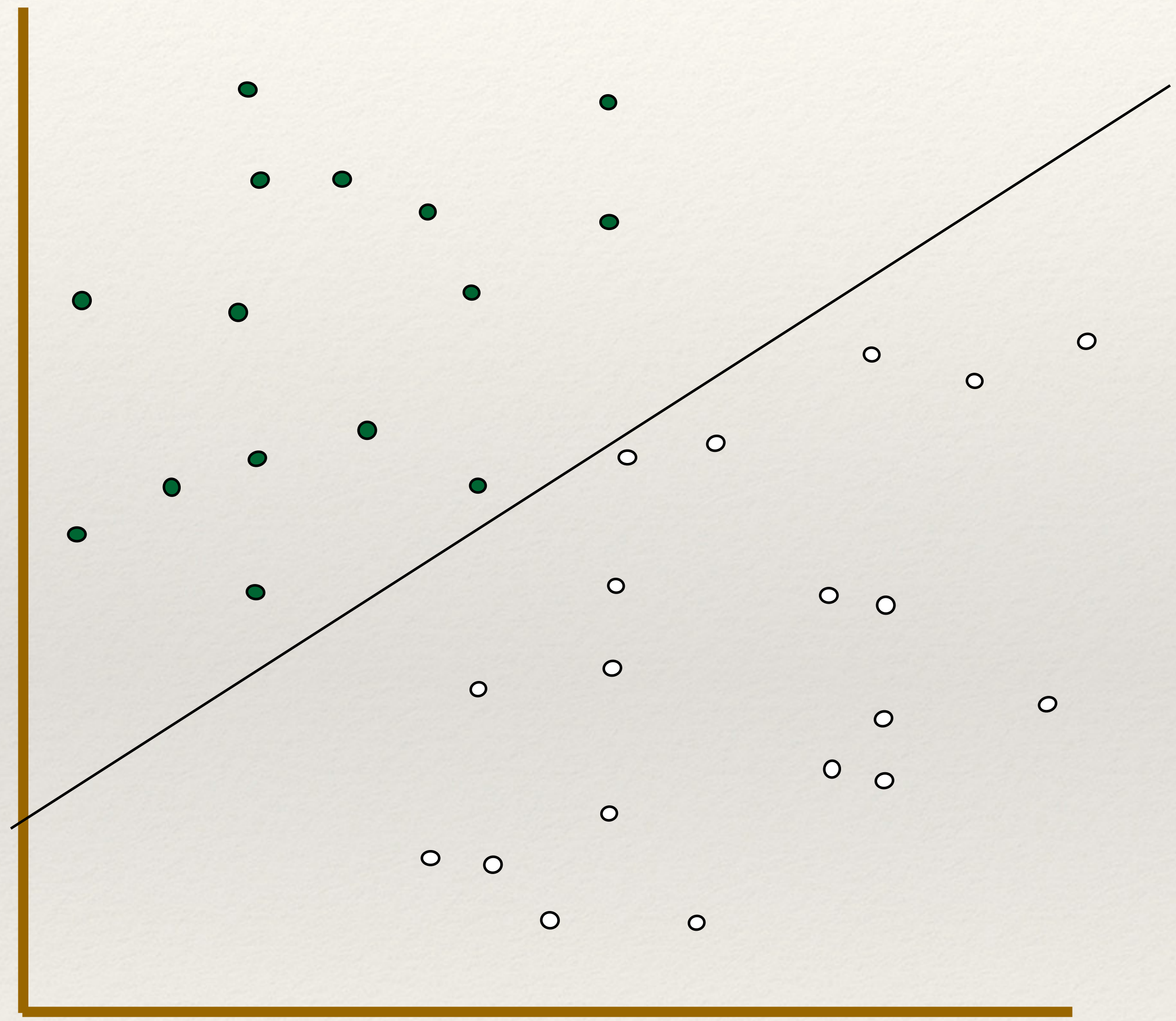

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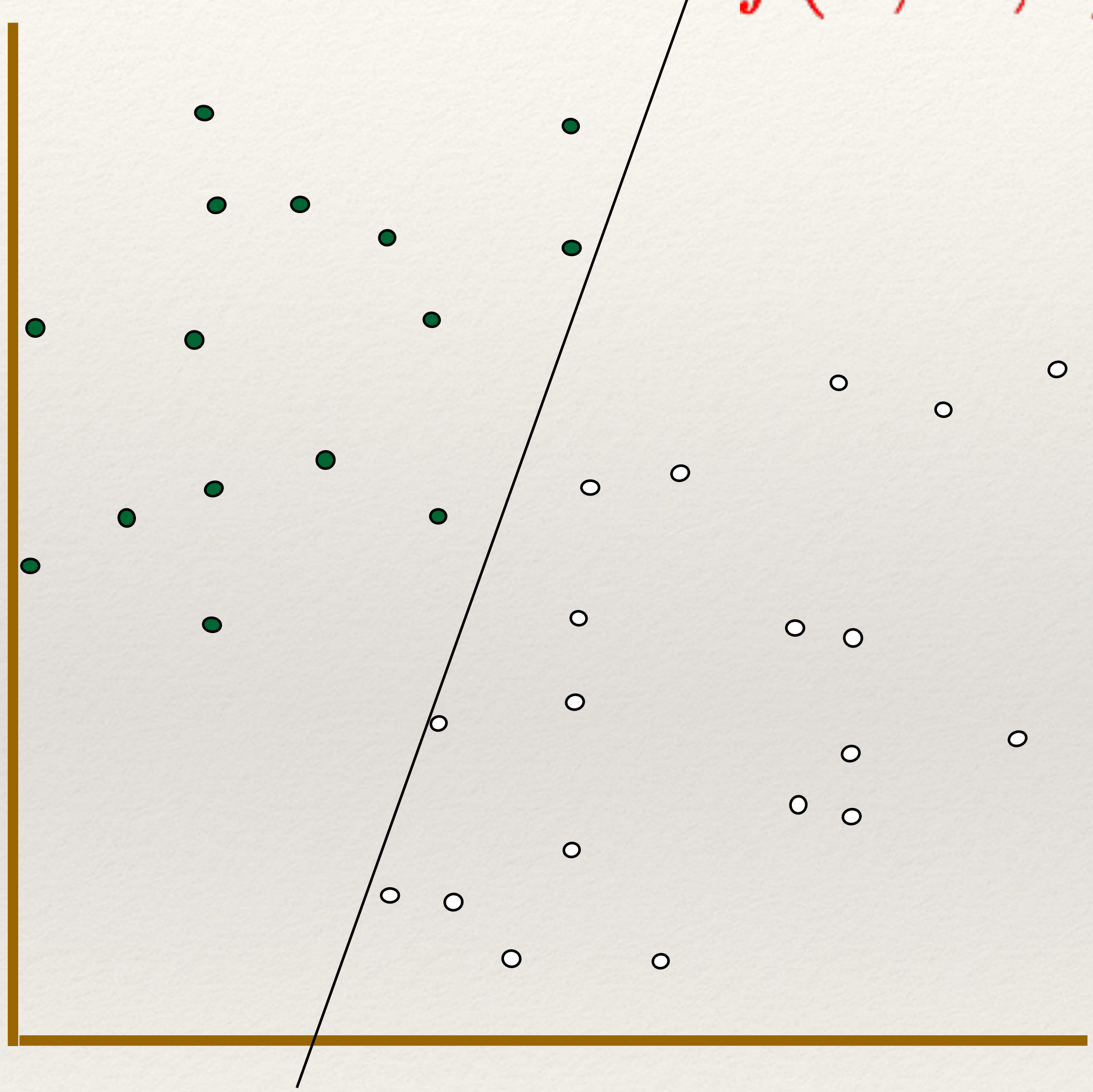

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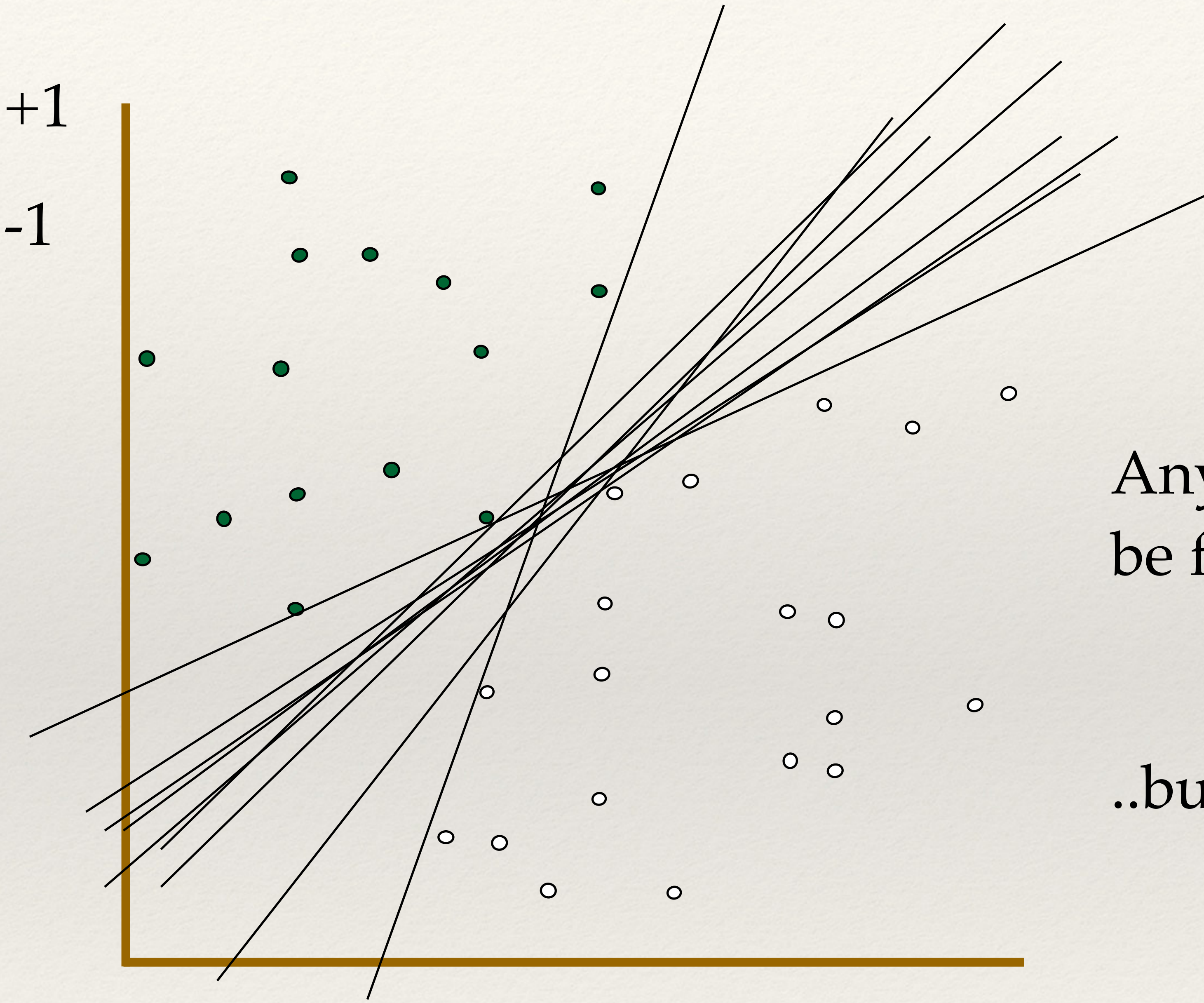

Any of these would be fine..

..but which is best?

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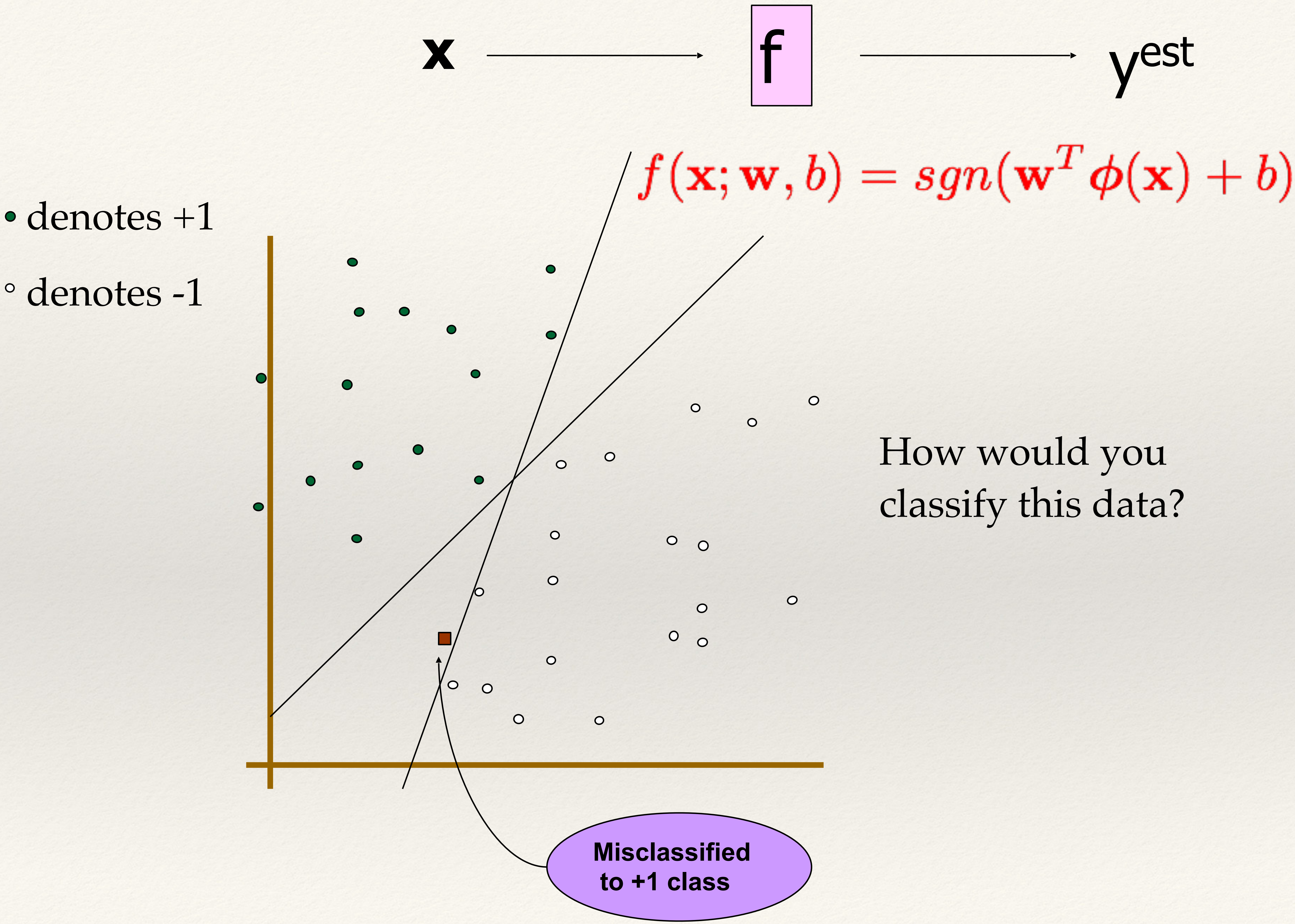
Linear Classifiers



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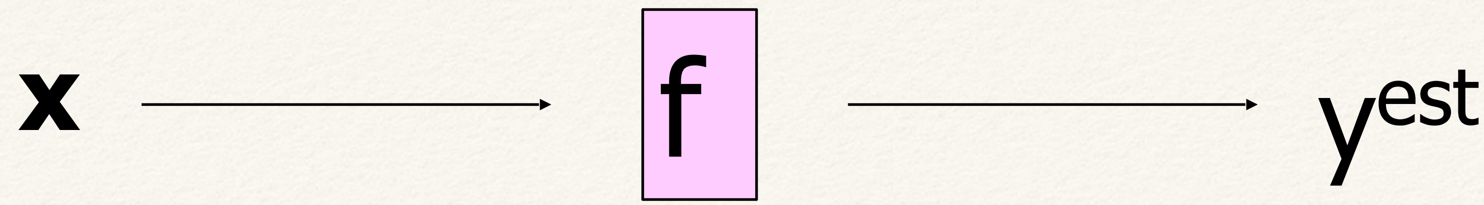

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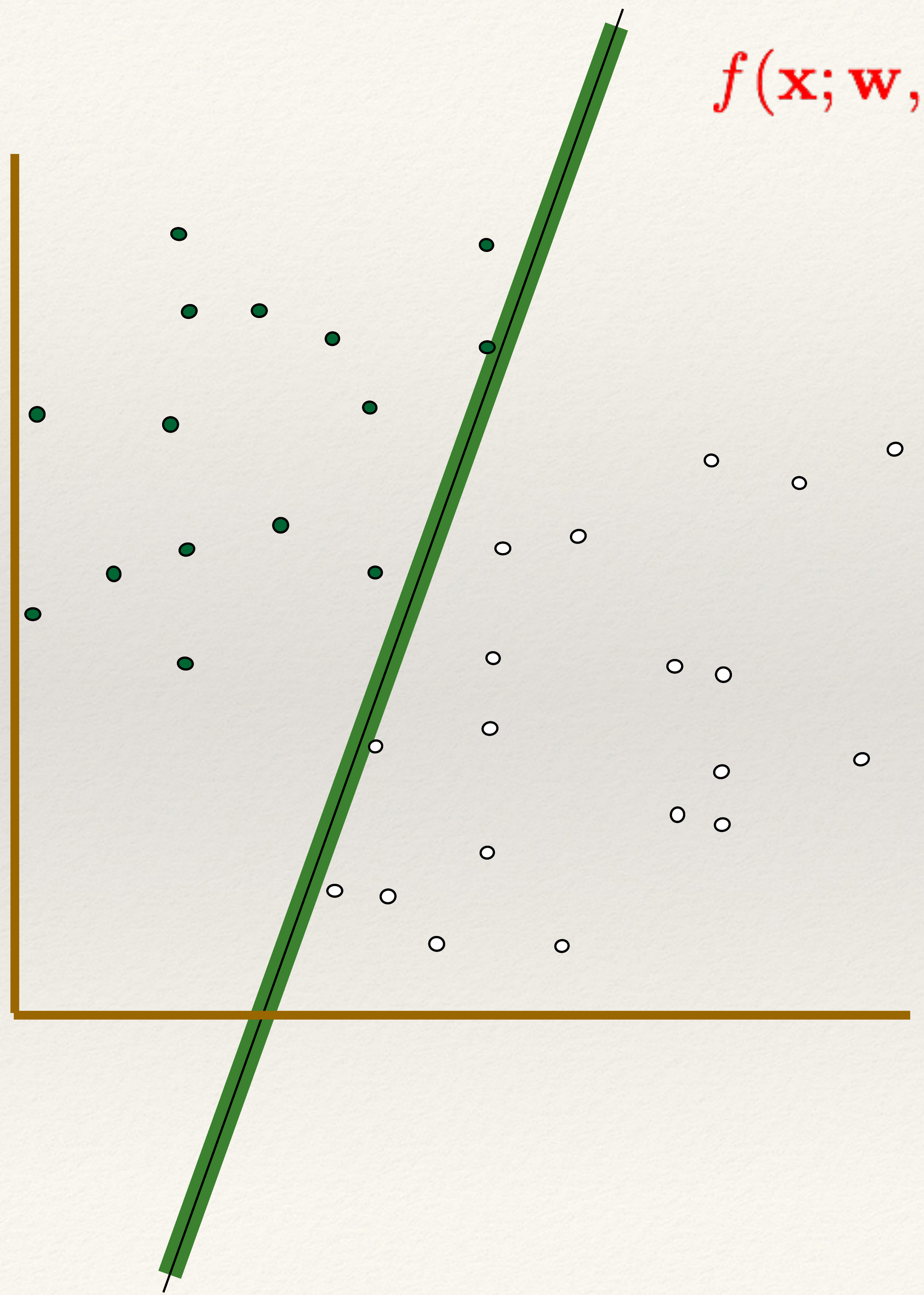


Define the **margin** of a linear classifier as the width that the boundary could be increased by before hitting a datapoint.

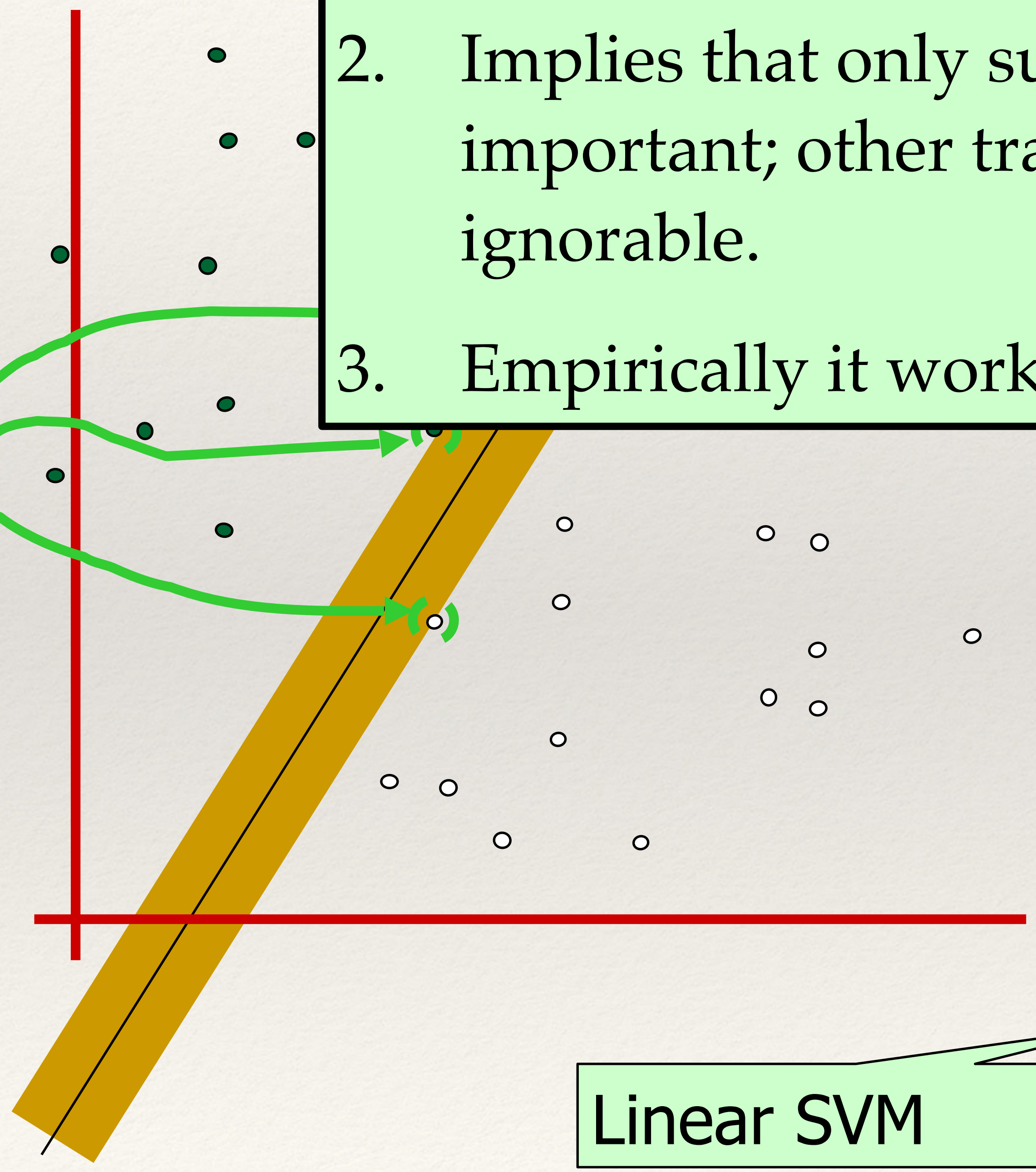
Maximum Margin

$\mathbf{x} \longrightarrow \boxed{f} \longrightarrow y^{est}$

1. Maximizing the margin is good according to intuition
2. Implies that only support vectors are important; other training examples are ignorable.
3. Empirically it works very very well.

• denotes +1

◦ denotes -1



Support Vectors are those data points that the margin pushes up against

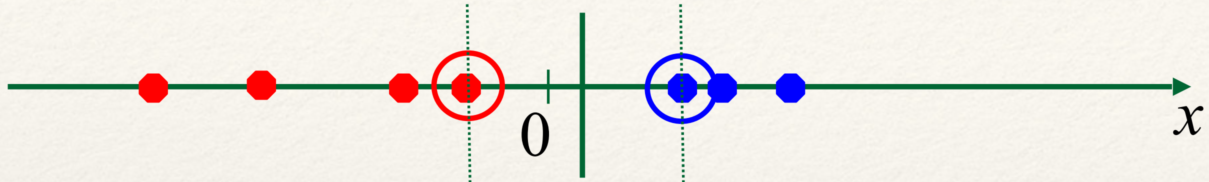
linear classifier with the, um, maximum margin.

This is the simplest kind of SVM (Called an LSVM)

Linear SVM

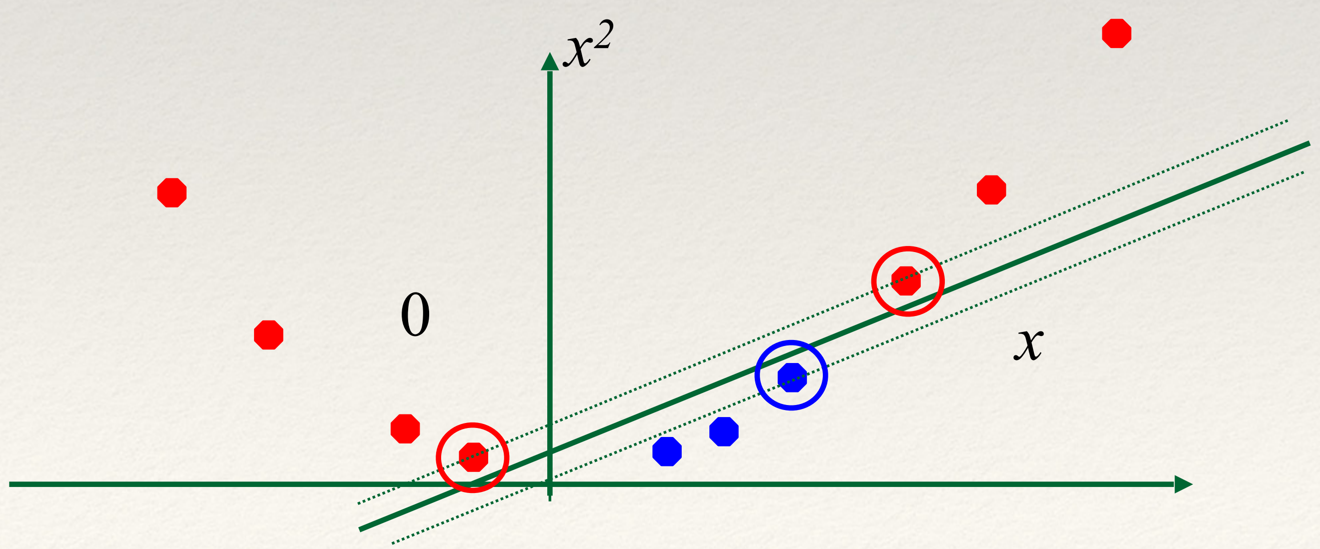
Non-linear SVMs

- Datasets that are linearly separable with some noise work out great:



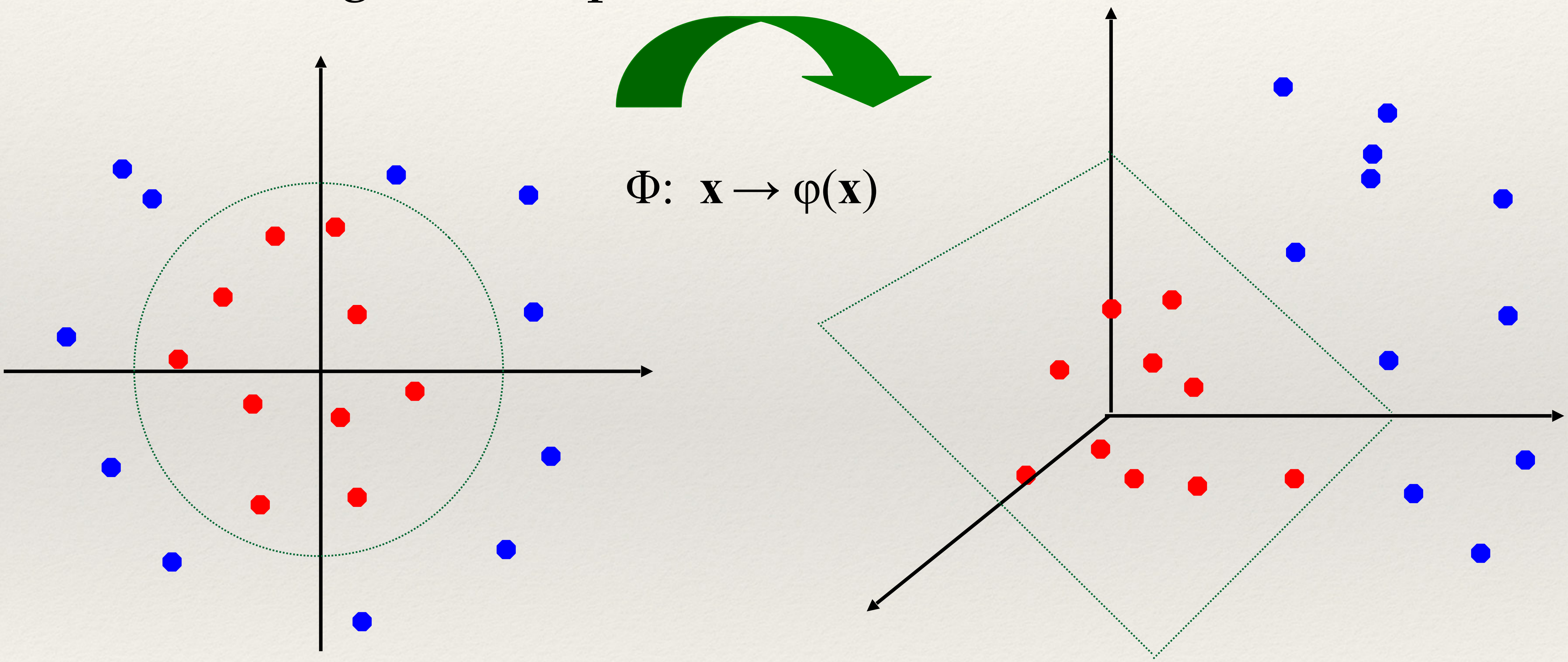
- But what are we going to do if the dataset is just too hard?

- How about… mapping data to a higher-dimensional space:



Non-linear SVMs: Feature spaces

- General idea: the original input space can always be mapped to some higher-dimensional feature space where the training set is separable:



The "Kernel Trick"

- The linear classifier relies on dot product between vectors $k(x_i,x_j)=\mathbf{x_i}^T\mathbf{x_j}$
- If every data point is mapped into high-dimensional space via some transformation Φ: $\mathbf{x} \rightarrow \phi(\mathbf{x})$, the dot product becomes:

$$k(\mathbf{x_i},\mathbf{x_j})= \phi(\mathbf{x_i})^T\phi(\mathbf{x_j})$$

- A *kernel function* is some function that corresponds to an inner product in some expanded feature space.
- Example:

  2-dimensional vectors x=[x_1 x_2]; let $k(\mathbf{x_i},\mathbf{x_j})=(1 + \mathbf{x_i}^T\mathbf{x_j})^2$,

  Need to show that $K(\mathbf{x_i},\mathbf{x_j})= \phi(\mathbf{x_i})^T\phi(\mathbf{x_j})$:

$$k(\mathbf{x_i},\mathbf{x_j})=(1 + \mathbf{x_i}^T\mathbf{x_j})^2,$$

$$= 1+ x_{i1}^2x_{j1}^2 + 2\, x_{i1}x_{j1}\, x_{i2}x_{j2}+ x_{i2}^2x_{j2}^2 + 2x_{i1}x_{j1} + 2x_{i2}x_{j2}$$

$$= [1\;\; x_{i1}^2\;\; \sqrt{2}\, x_{i1}x_{i2}\;\; x_{i2}^2\;\; \sqrt{2}x_{i1}\;\; \sqrt{2}x_{i2}]^T [1\;\; x_{j1}^2\;\; \sqrt{2}\, x_{j1}x_{j2}\;\; x_{j2}^2\;\; \sqrt{2}x_{j1}\;\; \sqrt{2}x_{j2}]$$

$= \phi(\mathbf{x_i})^T\phi(\mathbf{x_j})$, where $\phi(\mathbf{x}) = [1\;\; x_1^2\;\; \sqrt{2}\, x_1x_2\;\; x_2^2\;\; \sqrt{2}x_1\;\; \sqrt{2}x_2]$

What Functions are Kernels?

- For many functions $k(\mathbf{x}_i,\mathbf{x}_j)$ checking that

  $$k(\mathbf{x}_i,\mathbf{x}_j) = \boldsymbol{\phi}(\mathbf{x}_i)^T\boldsymbol{\phi}(\mathbf{x}_j)$$ can be cumbersome.

- Mercer's theorem: Every semi-positive definite symmetric function is a kernel
  - Semi-positive definite symmetric functions correspond to a semi-positive definite symmetric Gram matrix:

$$\mathbf{K} = \begin{bmatrix} k(\mathbf{x}_1,\mathbf{x}_1) & k(\mathbf{x}_1,\mathbf{x}_2) & k(\mathbf{x}_1,\mathbf{x}_3) & \dots & k(\mathbf{x}_1,\mathbf{x}_N) \\ k(\mathbf{x}_2,\mathbf{x}_1) & k(\mathbf{x}_2,\mathbf{x}_2) & k(\mathbf{x}_2,\mathbf{x}_3) & & k(\mathbf{x}_2,\mathbf{x}_N) \\ \dots & \dots & \dots & \dots & \dots \\ k(\mathbf{x}_N,\mathbf{x}_1) & k(\mathbf{x}_N,\mathbf{x}_2) & k(\mathbf{x}_N,\mathbf{x}_3) & \dots & k(\mathbf{x}_N,\mathbf{x}_N) \end{bmatrix}$$

Examples of Kernel Functions

- Linear: $k(\mathbf{x_i},\mathbf{x_j}) = \mathbf{x_i}^T\mathbf{x_j}$

- Polynomial of power p: $k(\mathbf{x_i},\mathbf{x_j}) = (1+ \mathbf{x_i}^T\mathbf{x_j})^p$

- Gaussian (radial-basis function network):

$$k(\mathbf{x}_i, \mathbf{x}_j) = \exp \frac{-||\mathbf{x}_i - \mathbf{x}_j||^2}{\sigma^2}$$

- Sigmoid: $k(\mathbf{x_i},\mathbf{x_j}) = \tanh(\beta_0\mathbf{x_i}^T\mathbf{x_j} + \beta_1)$

SVM Formulation

- Goal - **1) Correctly classify all training data**

$$\left.\begin{array}{ll} \mathbf{w}^T\phi(\mathbf{x}_n) + b \geq 1 & if \quad t_n = +1 \\ \mathbf{w}^T\phi(\mathbf{x}_n) + b \leq 1 & if \quad t_n = -1 \end{array}\right\}$$

$$t_n(\mathbf{w}^T\phi(\mathbf{x}_n) + b) \geq 1$$

2) Define the Margin

$$\frac{1}{||\mathbf{w}||} min_n \left[t_n(\mathbf{w}^T\phi(\mathbf{x}_n) + b)\right]$$

3) Maximize the Margin

$$argmax_{\mathbf{w},b}\left\{\frac{1}{||\mathbf{w}||} min_n \left[t_n(\mathbf{w}^T\phi(\mathbf{x}_n) + b)\right]\right\}$$

- Equivalently written as

$$argmin_{\mathbf{w},b} \frac{1}{2}||\mathbf{w}||^2 \text{ such that } t_n(\mathbf{w}^T\phi(\mathbf{x}_n) + b) \geq 1$$