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Homework #5 : STAT 5113

Answer A1 : Let, $\mathbf{x} = (x_1, x_2, \dots, x_k)$, $\mathbf{p} = (p_1, p_2, \dots, p_k)$ and we know that $\sum_{j=1}^{k} x_j = n$

$$f(\mathbf{x}|\mathbf{p}) = \frac{n!}{x_1!x_2!\dots x_k!} \prod_{j=1}^{k} p_j^{x_j} = \frac{n!}{x_1!x_2!\dots x_k!} \cdot e^{\sum_{j=1}^{k}(x_j \ln p_j)} = h(\mathbf{x}) \cdot g(\mathbf{p}) \cdot e^{<t(\mathbf{x})\cdot\eta(\mathbf{p})>}$$

$$\text{Here,} \quad h(\mathbf{x}) = \frac{n!}{x_1!x_2!\dots x_k!}; \quad g(\mathbf{p}) = \sum_{j=1}^{k} p_j = 1; \quad t(\mathbf{x}) = \mathbf{x}; \quad \eta(\mathbf{p}) = (\ln p_1, \ln p_2, \dots, \ln p_k)$$

This k-nomial distribution belongs to exponential family with k parameters as it can be written as

$$f(\mathbf{x}|\mathbf{p}) = h(\mathbf{x}) \cdot g(\mathbf{p}) \cdot e^{<t(\mathbf{x})\cdot\eta(\mathbf{p})>}$$

Answer A2(a): For Beta distribution $\alpha > 0$ and $\beta > 0$, with pdf

$$f(x|\alpha,\beta) = \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} x^{\alpha-1}(1-x)^{\beta-1} = \frac{1}{x(1-x)} \cdot \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} \cdot e^{\alpha\ln(x)+\beta\ln(1-x)} = h(x) \cdot g(\alpha,\beta) \cdot e^{<t(x)\cdot\eta(\alpha,\beta)>}$$

$$\text{Here,} \quad h(x) = \frac{1}{x(1-x)}; \quad g(\alpha,\beta) = \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)}; \quad t(x) = (\ln x, \ln(1-x)); \quad \eta(\alpha,\beta) = (\alpha,\beta)$$

Hence, Beta distribution belongs to exponential family. Let, $\psi(\alpha,\beta) = -\ln g(\alpha,\beta)$ then,

$$f(x|\alpha,\beta) = h(x) \cdot e^{<t(x)\cdot\eta(\alpha,\beta)-\psi(\alpha,\beta)>}$$

Then, natural sufficient statistics $T = t(x) = (\ln x, \ln(1-x))$ and natural parameters are $\eta(\alpha,\beta) = (\alpha,\beta)$

Answer A2(b): For Rayleigh distribution $x > 0, \theta > 0$ and has pdf

$$f(x|\theta) = \frac{2}{\theta} x \exp(-x^2/\theta) = h(x) \cdot g(\theta) \cdot e^{<t(x)\cdot\eta(\theta)>}$$

$$\text{Here,} \quad h(x) = 2x; \quad g(\theta) = \frac{1}{\theta}; \quad t(x) = -x^2; \quad \eta(\theta) = 1/\theta$$

Hence, Rayleigh distribution belongs to exponential family. Let, $\psi(\theta) = -\ln g(\theta)$ then,

$$f(x|\theta) = h(x) \cdot e^{<t(x)\cdot\eta(\theta)-\psi(\theta)>}$$

Then, natural sufficient statistics $T = t(x) = -x^2$ and natural parameter is $\eta(\theta) = 1/\theta$

Answer A2(c): For Weibull distribution $x > 0, \alpha > 0, \beta > 0$ and has pdf

$$f(x;\alpha,\beta) = \frac{\beta}{\alpha} x^{\beta-1} \exp(-x^\beta/\alpha) = \frac{1}{x} \cdot \frac{\beta}{\alpha} \cdot e^{(\beta\ln x - x^\beta/\alpha)}$$

If β is unknown then the term x^β is not a statistics and thus $f(x|\alpha,\beta)$ does not belong to exponential family. But if β is known then the term x^β is a statistics and thus we can write

$$f(x|\alpha,\beta) = h(x) \cdot g(\alpha,\beta) \cdot e^{<t(x)\cdot\eta(\alpha,\beta)>}$$

$$\text{Here,} \quad h(x) = \frac{1}{x}; \quad g(\alpha,\beta) = \frac{\beta}{\alpha}; \quad t(x) = (\ln x, x^\beta); \quad \eta(\alpha,\beta) = (\beta, -1/\alpha)$$

Hence, Weibull distribution belongs to exponential family. Let, $\psi(\alpha,\beta) = -\ln g(\alpha,\beta)$ then,

$$f(x|\alpha,\beta) = h(x) \cdot e^{<t(x)\cdot\eta(\alpha,\beta)-\psi(\alpha,\beta)>}$$

Then, natural sufficient statistics $T = t(x) = (\ln x, x^\beta)$ and natural parameters are $\eta(\alpha,\beta) = (\beta, -1/\alpha)$

Answer B3(a): Let, $X_1, X_2, \cdots, X_n \overset{iid}{\sim} \text{Unif}(0,\theta)$, where $\theta \sim \text{Pa}(\alpha,\beta)$; a Pareto random variable with pdf

$$f(\theta) = \frac{\alpha\beta^\alpha}{\theta^{\alpha+1}}; \quad \theta \geq \beta > 0; \quad \alpha > 0$$

$$\text{Let,} \quad X = (X_1, X_2, \cdots, X_n) \quad \text{and} \quad m = \max(X_1, X_2, \cdots, X_n)$$

$$\text{Likelihood,} \quad L(\theta) = f(X|\theta) = \frac{1}{\theta^n}; \quad \theta \geq m$$

$$\text{Joint distribution,} \quad f(\theta, X) = f(X|\theta)\cdot f(\theta) = \frac{1}{\theta^n}\frac{\alpha\beta^\alpha}{\theta^{\alpha+1}} = \frac{\alpha\beta^\alpha}{\theta^{\alpha+n+1}}$$

$$f(X) = \int_t^\infty p(\theta, X)d\theta = \int_t^\infty \frac{\alpha\beta^\alpha}{\theta^{\alpha+n+1}}d\theta = \alpha\beta^\alpha \frac{\theta^{-\alpha-n}}{-\alpha-n}\Bigg|_t^\infty = \frac{\alpha\beta^\alpha}{\alpha+n}\frac{1}{\theta^{\alpha+n}}\Bigg|_\infty^t = \frac{\alpha\beta^\alpha}{\alpha+n}\frac{1}{t^{\alpha+n}}$$

Now, if $m > \beta$ then $t = m$ thus,

$$f(X) = \frac{\alpha\beta^\alpha}{\alpha+n}\frac{1}{m^{\alpha+n}}$$

$$f(\theta|X) = \frac{f(\theta,X)}{f(X)} = \frac{(\alpha+n)m^{\alpha+n}}{\theta^{\alpha+n+1}}$$

Now, if $m < \beta$ then $t = \beta$ thus,

$$f(X) = \frac{\alpha\beta^\alpha}{\alpha+n}\frac{1}{\beta^{\alpha+n}} = \frac{\alpha}{\alpha+n}\frac{1}{\beta^n}$$

$$f(\theta|X) = \frac{f(\theta,X)}{f(X)} = \frac{(\alpha+n)\beta^{\alpha+n}}{\theta^{\alpha+n+1}}$$

In conclusion,

$$\theta|X \sim \text{Pa}(\alpha+n, \max(m,\beta)) \quad \text{where,} \quad m = \max(X_1, X_2, \cdots, X_n)$$

Answer B3(b): Here, sufficient statistic $T = \max\{X_i : i = 1, \cdots, n\}$, let's find the distribution of T

$$F_T(m) = f(T \leq m) = \prod_{i=1}^n f(X_i \leq m) = \Big(\frac{m}{\theta}\Big)^n$$

$$f_T(m) = \frac{d}{dt}F_T(m) = n\Big(\frac{m}{\theta}\Big)^{n-1}\cdot\frac{1}{\theta} = \frac{n}{\theta}\Big(\frac{m}{\theta}\Big)^{n-1}$$

$$\text{Joint distribution,} \quad f(\theta, T) = f(T|\theta)\cdot f(\theta) = \frac{n}{\theta}\Big(\frac{m}{\theta}\Big)^{n-1}\frac{\alpha\beta^\alpha}{\theta^{\alpha+1}} = \frac{\alpha\beta^\alpha}{\theta^{n+\alpha+1}}nm^{n-1}$$

$$f(T) = \int_t^\infty f(\theta,T)d\theta = \frac{\alpha\beta^\alpha nm^{n-1}}{(n+\alpha)t^{n+\alpha}}$$

$$f(\theta|T) = \frac{f(\theta,T)}{f(T)} = \frac{(\alpha+n)t^{\alpha+n}}{\theta^{\alpha+n+1}}$$

So, we end up in the same integration and the same conditions as before with $\theta \geq \beta > 0; \quad \theta \geq m$ and $m = \max(X_1, X_2, \cdots, X_n)$. if $m > \beta$ then $t = m$ and if $m < \beta$ then $t = \beta$.

$$\theta|X \sim \text{Pa}(\alpha+n, \max(m,\beta))$$

We end up having the same posterior distribution in both models.

Answer B4(a): Here, $X_1, X_2, \cdots, X_n \overset{iid}{\sim} \text{N}(\mu, 1)$ and $X = (X_1, X_2, \cdots, X_n)$

$$L(\mu) = f(X|\mu, \sigma^2 = 1) = \prod_{i=1}^n \frac{1}{\sqrt{2\pi}}\exp\Big[-\frac{(x_i-\mu)^2}{2}\Big] = \Big(\frac{1}{\sqrt{2\pi}}\Big)^n e^{-\frac{1}{2}\sum_{i=1}^n(x_i-\mu)^2} = \Big(\frac{1}{\sqrt{2\pi}}\Big)^n e^{-\frac{1}{2}\sum_{i=1}^n x_i^2 + \mu\sum_{i=1}^n x_i - \frac{n}{2}\mu^2}$$

$$\text{Likelihood,} \quad L(\mu) = \Big(\frac{1}{\sqrt{2\pi}}\Big)^n e^{-\frac{1}{2}\sum_{i=1}^n x_i^2}\cdot e^{\mu\sum_{i=1}^n x_i - \frac{n}{2}\mu^2} \propto e^{\mu\sum_{i=1}^n x_i - \frac{n}{2}\mu^2} = h(t(x), \mu)$$

$$\text{Sufficient statistics,} \quad T = t(x) = \sum_{i=1}^n x_i \sim N(n\mu, n)$$

$$\text{Log-likelihood,}\quad l(\mu) = \ln L(\mu) = -\frac{1}{2}\sum_{i=1}^{n} x_i^2 + \mu\sum_{i=1}^{n} x_i - \frac{n}{2}\mu^2 + C$$

$$l'(\mu) = \sum_{i=1}^{n} x_i - n\mu$$

$$l''(\mu) = -n$$

$$\text{Fisher Information,}\quad I(\mu) = \mathbb{E}(-l''(\mu)) = n$$

Answer B4(b): For the reduced model $T \sim N(n\mu, n)$

$$\text{Likelihood,}\quad L(\mu) = f(T = t|\mu) = \frac{1}{\sqrt{2\pi n}} e^{-\frac{(t-n\mu)^2}{2n}}$$

$$\text{Log-likelihood,}\quad l(\mu) = \ln L(\mu) \propto -\frac{(t - n\mu)^2}{2n}$$

$$l'(\mu) = -\frac{1}{2n} 2(t - n\mu)(-n) = t - n\mu$$

$$l''(\mu) = -n$$

$$I_T(\mu) = \mathbb{E}(-l''(\mu)) = \mathbb{E}(n) = n$$

$$\text{Hence,}\quad I(\mu) = I_T(\mu)$$
