STAT 5113: Statistical Inference

Homework 5 – Due March 12, 2018

A. **One or more of the following problems *might* be graded.**

1. Does the Multinomial distribution, having pmf

$$f(x_1, \ldots, x_k) = \binom{n}{x_1 \ldots x_k} \prod_{j=1}^{k} p_j^{x_j}$$

belong to an exponential family? Is the number of parameters k or $k-1$?

2. Verify whether or not the following distributions belong to an exponential family. If they do, specify the natural sufficient statistic and the natural parameter.

   a. The $\mathcal{B}eta(\alpha, \beta)$ distribution.

   b. The Rayleigh distribution, having pdf

$$f(x) = \frac{2}{\theta}\, x \exp\left\{-x^2/\theta\right\}, \qquad x > 0,\ \theta > 0.$$

   c. The Weibull distribution, having pdf

$$f(x) = \frac{\beta}{\alpha}\, x^{\beta-1} \exp\left\{-x^{\beta}/\alpha\right\}, \qquad x > 0,\ \alpha > 0,\ \beta > 0.$$

   Consider separately the case when β is considered known and the case when it is not.

B. **The following problems *will* be graded.**

3. Consider a sample of size n from the $\mathcal{U}nif(0, \theta)$ distribution. Assume a Pareto prior distribution for θ, having pdf

$$p(\theta) = \frac{\alpha \beta^{\alpha}}{\theta^{\alpha+1}}, \qquad \theta \geq \beta,$$

where α and β are positive constants (sometimes called *hyper*-parameters).

   a. Show that the posterior distribution of θ has again a Pareto distribution, specifying the formulas for updating the parameters α and β based on the observations.

   b. Verify that the posterior distribution of θ based on the full data set is the same as the posterior based on the distribution of the sufficient statistic $T = \max\{X_i :\ i = 1, \ldots, n\}$.

4. Let $X_1, \ldots, X_n \overset{iid}{\sim} \mathcal{N}(\mu, 1)$.

   a. Find a one-dimensional sufficient statistic T and find its distribution.

   b. Let $I(\mu)$ be Fisher information for the original model, and let $I_T(\mu)$ be Fisher information for the reduced model determined by T. Show that $I(\mu) = I_T(\mu)$.