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Homework #4 : STAT 5113

Answer A1 : If λ is the 'kill rate' of Armadillo per day and the number of killed Armadillo by people $1, 2, \cdots, n$ is respectively $X_1, X_2, \cdots, X_n$ (i.i.d) then the likelihood $L(\lambda)$

$$L(\lambda) = \prod_{i=1}^{n} \frac{e^{-\lambda}\lambda^{x_i}}{x_i!} = \frac{e^{-n\lambda}\lambda^{\sum_{i=1}^{n} x_i}}{\prod_{i=1}^{n} x_i!} = Ce^{-n\lambda}\lambda^{n\bar{x}}, \quad \bar{x} \text{ is the sample mean}$$

$$\text{log-likelihood}, \quad l(\lambda) = \ln L = C - n\lambda + n\bar{x}\ln\lambda$$

$$l'(\lambda) = -n + \frac{n\bar{x}}{\lambda}$$

$$\text{for MLE} \quad l'(\lambda) = 0 \Rightarrow \hat{\lambda} = \bar{x}, \quad [n \neq 0]$$

Code in R:

```
x = c(rep(0,29),rep(1,8),rep(2,1))
n = length(x)
lambda_MLE = mean(x)

> lambda_MLE
[1] 0.263158
```

This result can be verified by using R

```
nll = function(dummy){
  -sum(dpois(x, lambda = dummy, log = TRUE))
}

require(stats4)
output = mle(nll, start = list(dummy = 0.1),
             nobs = n, method = "L-BFGS-B", lower = c(1e-8, 1e-8))
lambda_MLE_usingR = output@coef["dummy"]

> lambda_MLE_usingR
   dummy
0.263159
```

Answer A2 :

$$l''(\lambda) = -\frac{n\bar{x}}{\lambda^2}$$

$$\text{Fisher Information}, \quad I(\lambda) = \mathbb{E}(-l'') = \frac{n\bar{x}}{\lambda^2}$$

$$\text{by ML Asymptotics}, \quad \hat{\lambda} \;\; \dot{\sim} \;\; N(\lambda, \frac{\lambda^2}{n\bar{x}})$$

$$\text{Asymptotic Standard Error, } \mathrm{ASE}_\lambda(\hat{\lambda}) = \sqrt{\frac{\lambda^2}{n\bar{x}}}$$

$$\text{Estimate}, \quad \widehat{\mathrm{ASE}_\lambda}(\hat{\lambda}) = \sqrt{\frac{\hat{\lambda}^2}{n\bar{x}}} = \sqrt{\frac{\bar{x}}{n}}$$

Code in R:

```
ASE_estimate = sqrt(mean(x)/n)

> ASE_estimate
[1] 0.0832178
```

Answer A3 : The following code generates ASE estimate using parametric Bootstrap

```
B = 1e6
set.seed(1990)
lambda_bootstrap = rep(0,B)
for (i in 1:B){
  y = rpois(n, lambda_MLE)
  lambda_bootstrap[i] = mean(y)
}

> sd(lambda_bootstrap) # close to the value found in A2
[1] 0.0832253
```

Answer A4 : For any ith hunter the probability of killing no Armadillo is $P(X_i = 0) = \frac{e^{-\lambda}\lambda^0}{0!} = e^{-\lambda}$. So, for ith hunter the probability of killing at least one Armadillo is $\psi(\lambda) = 1 - e^{-\lambda}$. Using the estimate $\hat{\lambda}$, the probability of killing at least one Armadillo collectively $\psi(\hat{\lambda}) = 1 - e^{-\hat{\lambda}}$

```
kill = function(z) 1-exp(-z)
kp_MLE = kill(lambda_MLE)

> kp_MLE
[1] 0.231379
```

Answer A5 : For Delta method, if $\hat{\lambda}$ is MLE of λ, then for any differentiable function of λ namely $\psi(\lambda)$

$$\psi(\hat{\lambda}) \quad \dot{\sim} \quad N\Big(\psi(\lambda), [\psi'(\lambda)]^2[I(\lambda)]^{-1}\Big)$$

$$\text{here,} \quad \psi'(\lambda) = e^{-\lambda}$$

$$\psi(\hat{\lambda}) \quad \dot{\sim} \quad N\Big(1 - e^{-\lambda}, \frac{\lambda^2 e^{-2\lambda}}{n\bar{x}}\Big)$$

$$\text{ASE} = \sqrt{\frac{\lambda^2 e^{-2\lambda}}{n\bar{x}}}$$

$$\widehat{\text{ASE}} = \sqrt{\frac{\hat{\lambda} e^{-2\hat{\lambda}}}{n}}$$

Code in R:

```
ASE_delta = sqrt((lambda_MLE*exp(-2*lambda_MLE))/n)

> ASE_delta
[1] 0.06396293567
```

Answer A6 : ASE of $\psi(\lambda)$, the probability of at least one kill, using Bootstrap

```
kp_bootstrap = kill(lambda_bootstrap)
sd(kp_bootstrap)

> sd(kp_bootstrap) # close to the value found in A5
[1] 0.0634601
```

Answer B1 :

$$\text{Prior distribution,} \quad \lambda \sim \text{Gamma}(\alpha_0, \beta_0)$$

If $\beta_0 = 1.1$, and prior mean $= 0.25$ then, $\alpha_0 = 0.25\beta_0 = 0.275$

$$f(\lambda|\underline{x}) \propto L(\lambda)f(\lambda) = e^{-n\lambda}\lambda^{n\bar{x}}e^{-\beta_0\lambda}\lambda^{\alpha_0-1} = e^{-(n+\beta_0)\lambda}\lambda^{(n\bar{x}+\alpha_0)-1}$$

$$f(\lambda|\underline{x}) \sim \text{Gamma}(n\bar{x} + \alpha_0, n + \beta_0)$$

$$\text{Bayesian Estimate,} \quad \hat{\lambda} = \mathbb{E}(\lambda|\underline{x}) = \frac{n\bar{x} + \alpha_0}{n + \beta_0}$$

I am using Monte Carlo method to find out the Expectation

Code in R

```
beta_0 = 1.1
alpha_0 = 0.25*beta_0
alpha_1 = alpha_0 + sum(x)
beta_1 = n+beta_0
lambda_hat_bayes = alpha_1/beta_1

> lambda_hat_bayes # close to the value found in A4
[1] 0.262788

MC = 1e3 # number of Monte Carlo simulations
y = rep(0,MC)
kill_prob_bayes = rep(0,MC)
set.seed(1990)
for (i in 1:MC){
  temp = rpois(n,lambda_hat_bayes)
  kill_prob_bayes[i] = kill(mean(temp))
}
kp_bayes = mean(kill_prob_bayes)
CI = quantile(kill_prob_bayes,probs = c(0.025, 0.975))

> CI
     2.5%     97.5%
0.0999124 0.3606917
```

The ML estimate found in A4 is within this 95% CI limit.

Answer B2 : If the data contains all zeros then from frequency point of view, $\hat{\lambda} = \bar{x} = 0$ and from Bayesian point of view $\hat{\lambda} = \mathbb{E}(\lambda|x) = \frac{\alpha_0}{n+\beta_0}$. **Comment:** Observed data pulls the prior estimate of $\mathbb{E}(\lambda) = 0.25$ close to MLE estimate of $\hat{\lambda} = 0$

Code in R:

```
> post_estimate = alpha_0/(n + beta_0)
[1] 0.00703325
```
