Amirkabir University of Technology
(Tehran Polytechnic)

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Modern control project
Report

Nonlinear Control of the Inverted Pendulum

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In the first phase of the project, we model the system and evaluate the stability, controllability and visibility features of the system.

1- Introduction of the generalities of physical system and examples of industrial applications

A cart inverted pendulum system has been served as a general model for robotic systems. The cart pendulum system is a non-linear, under-actuated system with unstable zero dynamics and must be controlled such that the position is at its unstable equilibrium. it is clear that the inverted pendulum is a system with many variations that render it a fundamental control problem. Apart from its variations, its analysis and control can be studied using a plethora of different techniques. The system can be controlled using various optimization techniques. Predictive control techniques can be applied, which aim at determining a series of future control actions that will balance the system. Adaptive control techniques can also be used. Such methods are useful when certain parameters of the system change during its simulation. Such methods are used for example when the cart goes through a different terrain and the friction coefficient changes or when an object is suddenly placed on the end of the rod, changing its center of gravity. Overall, the inverted pendulum is a system that helps engineers test the efficacy of new control methods and for that matter it works as a bridge between theoretical approaches and their application to real life problems. A cart inverted pendulum system has been served as a general model for robotic systems. Another example is driving a car, where the control is applied through the driver who is the one affecting the system's inputs, which are the speed and direction, aiming for a safe drive. Also, in every industrial facility each part of the production line functions under the supervision of digital controllers that ensure that each engine works properly according to specific control and design specifications.

2. Nonlinear system model and its parameters

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linearized equations of the pendulum (where the mass is concentrated at the top) are studied in the form of:

$$\dot{x} = Ax + Bu$$

Equation 1- state space form

If matrices A and B form a controllable pair, then the local optimal feedback control u is given by:

$$u = -R^{-1}B^TPx$$

Equation 2

where P satisfies the algebraic Riccati equation and R is a weighting matrix. the global pendulum model in the form of

$$\dot{x} = A(x)x + B(x)u$$

Equation 3

and the corresponding controllability of (A(x), B(x)) for almost all x (i.e. controllable, apart possibly from a set of measure zero) are analyzed. The x in A(x) and B(x) is fixed for each step, therefore the nonlinear model resembles the linear case for one step. This method also demonstrates effective robust control because of the update at each step, so any inaccuracy of A and B is adjusted automatically. Furthermore, the global control u is shown to be

$$u = -R^{-1}B(x)^TP(x)x$$

Equation 4

where P(x) satisfies the pointwise algebraic Riccati equation:

$$A^T(x)P(x) + P(x)A(x) + Q - P(x)B(x)R^{-1}B(x)^TP(x) = 0$$

Equation 5

where Q and R are weighting matrices. It is shown that the single pendulum installed on a cart can be controlled from a large range of initial positions, including the rest position where the pendulum hangs downwards. The double-link pendulum and car system can also be controlled from many initial positions.

Modelling of the Cart-Pendulum Systems

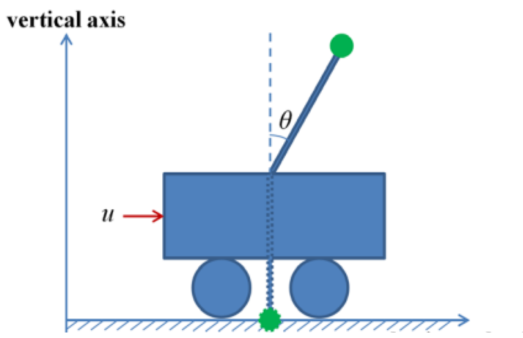

Figure 1- the single inverted pendulum and cart diagram

Often inverted pendulums are considered in combination with moving carts. The system of a single pendulum installed on a cart is drawn in Figure 1.

The dynamical model of the cart and the pendulum are equations of motions often obtained by applying force analysis using free body diagrams and Newton's second law F = ma. However, there are other methods available for achieving a system's dynamical model; for example, the Lagrangian approach which calculates the difference between total kinetic energy T and the total potential energy V of the system:

$$L = T - V$$

Equation 6

and the Hamiltonian equation which calculates the sum of the two types of energy:

$$H = T + V$$

Equation 7

It is usually easier to use the Lagrangian method than the one based on force analysis because all is required are the generalized kinetic and potential energy terms, so resolving of the forces (which is often complicated) is not needed. We adopt the Lagrangian approach for its simplicity, where the Lagrange's equations are given by:

$$\frac{d}{dt}\frac{\partial \mathrm{L}}{\partial \dot{\mathrm{x}}_i} - \frac{\partial \mathrm{L}}{\partial x_i} = f_i \qquad 1 \le i \le n$$

Equation 8

where xi represents the ith generalized coordinate and fii the ith generalized force applied on the object. These Lagrange's equations are equivalent to Newton's laws.

In the case of a single pendulum-cart system, there are two x variables shown in Figure 2.1, namely the horizontal distance $x1$ (m) travelled by cart from the left reference, and the angle θ (rad) between the pendulum rod and the vertical axis. $\dot{x}_1$ and θ represent velocity of the cart along the horizontal axis and angular velocity of the rod around the rod-cart connection point, respectively. Here,

$$f1 = u \; \& \; f2 = 0$$

Equation 9

The total kinetic energy of the pendulum-cart system can be written as:

$$\begin{aligned} T &= \frac{1}{2}m_1\dot{x}_1^2 + \frac{1}{2}m_2\left[\frac{d}{dt}(x_1 + r\sin\theta)\right]^2 + \frac{1}{2}m_2\left[\frac{d}{dt}(r\cos\theta)\right]^2 \\ &= \frac{1}{2}m_1\dot{x}_1^2 + \frac{1}{2}m_2(\dot{x}_1 + r\dot{\theta}\cos\theta)^2 + \frac{1}{2}m_2(-r\dot{\theta}\sin\theta)^2 \\ &= \frac{1}{2}(m_1+m_2)\dot{x}_1^2 + m_2 r\dot{x}_1\dot{\theta}\cos\theta + \frac{1}{2}m_2 r^2\dot{\theta}^2 \,, \end{aligned}$$

Equation 10

where $m1$ and $m2$ are the masses of cart and of pendulum respectively, r denotes the length of the pendulum and g is acceleration due to gravity. The total potential energy of the system, using the bottom of the pendulum rest position as the vertical reference point, can be written as:

$$V = m_2 g(r + r\cos\theta)$$

Equation 11

Therefore, the Lagrangian equation is given by

$$\begin{aligned} L &= T - V \\ &= \frac{1}{2}(m_1+m_2)\dot{x}_1^2 + m_2 r\dot{x}_1\dot{\theta}\cos\theta + \frac{1}{2}m_2 r^2\dot{\theta}^2 - m_2 g r(1+\cos\theta)\,. \end{aligned}$$

Equation 12

$$\begin{aligned} \ddot{x}_1 &= \frac{m_2 r\dot{\theta}^2\sin\theta - m_2 g\sin\theta\cos\theta + u}{m_1 + m_2\sin^2\theta} \\ \ddot{\theta} &= \frac{-m_2 r\dot{\theta}^2\sin\theta\cos\theta + m_2 g\sin\theta + m_1 g\sin\theta - u\cos\theta}{r(m_1 + m_2\sin^2\theta)} \end{aligned}$$

Equation 13

which satisfies Newtown's second Law automatically. Note the two equations in above both have 2nd derivatives on the left-hand-side and are not yet in the standard state space model form. A state- space representation of the system can be written as

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= \frac{m_2 r x_4^2\sin x_3 - m_2 g\sin x_3\cos x_3 + u}{m_1 + m_2\sin^2 x_3} \\ \dot{x}_3 &= x_4 \\ \dot{x}_4 &= \frac{-m_2 r x_4^2\sin x_3\cos x_3 + m_2 g\sin x_3 + m_1 g\sin x_3 - u\cos x_3}{r(m_1 + m_2\sin^2 x_3)} \end{aligned}$$

Equation ~~14~~

by introducing three new variables, x_2, x_3 & x_4, i.e. $x_2 = \dot{x}_1$, $x_3 = \theta$ & $x4 = \dot{x}3$ and splitting each of the equations into two equations. This translates into the state-space matrix form as:

$$\begin{pmatrix}\dot{x}_1\\ \dot{x}_2\\ \dot{x}_3\\ \dot{x}_4\end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & \dfrac{-m_2 g \sin x_3 \cos x_3}{(m_1 + m_2 \sin^2 x_3)x_3} & \dfrac{m_2 r x_4 \sin x_3}{m_1 + m_2 \sin^2 x_3} \\ 0 & 0 & 0 & 1 \\ 0 & 0 & \dfrac{(m_1 + m_2) g \sin x_3}{r(m_1 + m_2 \sin^2 x_3)x_3} & \dfrac{-m_2 r x_4 \sin x_3 \cos x_3}{r(m_1 + m_2 \sin^2 x_3)} \end{pmatrix} \begin{pmatrix}x_1\\ x_2\\ x_3\\ x_4\end{pmatrix} + \begin{pmatrix} 0 \\ \dfrac{1}{m_1 + m_2 \sin^2 x_3} \\ 0 \\ \dfrac{-\cos x_3}{r(m_1 + m_2 \sin^2 x_3)} \end{pmatrix} \begin{pmatrix}u_1\\ u_2\\ u_3\\ u_4\end{pmatrix}.$$

Equation 15

One particular advantage of this method is that it can be easily extended to a general multi-link pendulum case, without the need to perform complex force analysis on the new and previous pendulum objects. For example, a double pendulum and cart system is illustrated in Figure 2.2, where $\theta 1$ and $\theta 2$ represent the angles between the 1st and the 2nd pendulum rods and the vertical axis.

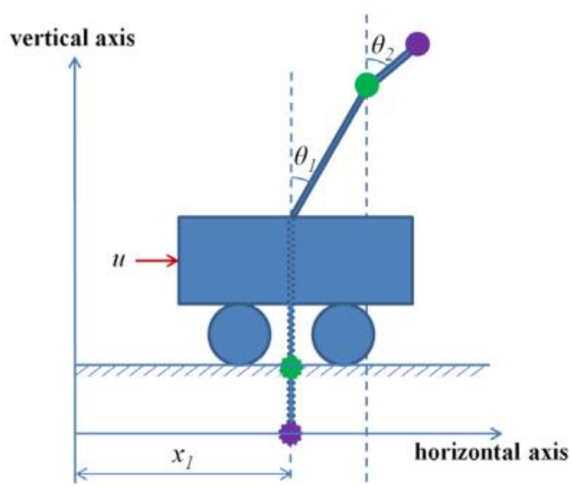

Figure 2- the double-link inverted pendulum and cart diagram

A similar energy analysis can be performed as before. Using the new vertical and horizontal references as indicated in Figure 2.2, the total kinetic energy and potential energy of the new system can be modified to
A similar energy analysis can be performed as before. Using the new vertical and horizontal references as indicated in Figure 2.2, the total kinetic energy and potential energy of the new system can be modified to:

$$
\begin{aligned}
T_2 &= T + \frac{1}{2}m_3\left[\frac{d}{dt}(x_1 + r_1\sin\theta_1 + r_2\sin\theta_2)\right]^2 + \frac{1}{2}m_3\left[\frac{d}{dt}(r_1\cos\theta_1 + r_2\cos\theta_2)\right]^2 \\
&= \frac{1}{2}(m_1+m_2+m_3)\dot{x}_1^2 + \frac{1}{2}(m_2+m_3)r_1^2\dot{\theta}_1^2 + \frac{1}{2}m_3r_2^2\dot{\theta}_2^2 + (m_2+m_3)r_1\dot{x}_1\dot{\theta}_1\cos\theta_1 \\
&\quad + m_3r_2\dot{x}_1\dot{\theta}_2\cos\theta_2 + m_3r_1r_2\dot{\theta}_1\dot{\theta}_2\cos(\theta_1-\theta_2),
\end{aligned}
$$

Equation 16

$$
\begin{aligned}
V_2 &= V + m_3g(r_1 + r_2 + r_1\cos\theta_1 + r_2\cos\theta_2) \\
&= (m_2+m_3)gr_1(1+\cos\theta_1) + m_3gr_2(1+\cos\theta_2),
\end{aligned}
$$

Equation 17

Which then leads to

$$
\begin{aligned}
L_2 &= T_2 - V_2 \\
&= \frac{1}{2}(m_1+m_2+m_3)\dot{x}_1^2 + \frac{1}{2}(m_2+m_3)r_1^2\dot{\theta}_1^2 + \frac{1}{2}m_3r_2^2\dot{\theta}_2^2 + (m_2+m_3)r_1\dot{x}_1\dot{\theta}_1\cos\theta_1 \\
&\quad + m_3r_2\dot{x}_1\dot{\theta}_2\cos\theta_2 + m_3r_1r_2\dot{\theta}_1\dot{\theta}_2\cos(\theta_1-\theta_2) - (m_2+m_3)gr_1(1+\cos\theta_1) \\
&\quad - m_3gr_2(1+\cos\theta_2),
\end{aligned}
$$

Equation 18

where $m3$ denotes the mass of the newly added pendulum, and $r1$ and $r2$ are the lengths of the original rod and the new rod respectively. The energy equations appear complicated; however, the analysis performed above is relatively straightforward in the sense that effect only comes from the new pendulum and corresponding energy terms can simply be added to the original equations. The process of obtaining a state-space model for the double-link pendulum cart system is also similar as the one is the single pendulum case. By solving the Langrage's equation and splitting each differential equation with a 2nd derivative into two equations containing only 1st derivatives (i.e. let $x2 = \dot{x}\ 1$, $x3 = \theta 1$, $x4 = \dot{x}\ 3$, $x5 = \theta 2$ and $x6 = \dot{x}\ 5$), we obtain standard state space model of the double-link pendulum cart system in matrix form:

$$\begin{pmatrix}\dot{x}_1\\ \dot{x}_2\\ \dot{x}_3\\ \dot{x}_4\\ \dot{x}_5\\ \dot{x}_6\end{pmatrix}=$$

$$\begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{m_2 ga\sin x_3\cos x_3}{-dx_3} & \frac{m_2 r_1 a x_4 \sin x_3}{d} & 0 & \frac{m_2 m_3 r_2 x_6[\sin(x_5-2x_3)-\sin x_5]}{-2d} \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & \frac{ge\sin x_3}{2r_1 dx_3} & \frac{fx_4}{-2d} & \frac{m_1 m_3 gb\sin x_5}{-r_1 dx_5} & \frac{r_2 hx_6}{-r_1 d} \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & \frac{m_1 gab\sin x_3}{-r_2 dx_3} & \frac{m_1 r_1 acx_4}{r_2 d} & \frac{m_1 ga\cos^2 x_3\sin x_5}{r_2 dx_5} & \frac{m_1 m_3 x_6\sin(2x_5-2x_3)}{-2d} \end{pmatrix}\times$$

$$\begin{pmatrix}x_1\\ x_2\\ x_3\\ x_4\\ x_5\\ x_6\end{pmatrix}+\begin{pmatrix}0\\ \frac{m_2+m_3c^2}{d}\\ 0\\ \frac{m_2\cos x_3-m_3c\sin x_5}{-r_1 d}\\ 0\\ \frac{ac\sin x_3}{-r_2 d}\end{pmatrix}\begin{pmatrix}u_1\\ u_2\\ u_3\\ u_4\\ u_5\\ u_6\end{pmatrix},$$

Equation 19

Where a, b, c, d, e , f, and h have been defined as the following:

$a = m_2+m_3$
$b = \cos x_3 \cos x_5$
$c = \sin(x_3 - x_5)$
$d = m_2 a \sin^2 x_3 + m_1 m_3 c^2 + m_1 m_2$
$e = 2m^2{}_2 + m_1 m_3 + 2m_2 m_3 + 2m_1 m_2 + m_1 m3 \cos(2x_5)$;
$f = m_2 a \sin(2x_3) – m_1 m_3 \sin(2x_5 – 2x_3)$;
$h = m_2 m_3 \sin x_3 (\cos x_5) + m_1 m_3 c$

3- Linearization

We used Lagrangian method to determine the dynamic model of the system which is given in the previous section. Now it is time to linearize the system. The general state space model of the system is

$$\dot{x} = Ax + Bu$$

Equation 20

where x , is the state variable, u is the control vector and (A ,B are controllable parameters. By quadratic infinite-time cost function, the linear optimal feedback control solution is:

$$u = -R^{-1}B(x)^T P(x)x$$

Equation 21

By implementing the above equation in the state space model we will have:

$$-\frac{d\boldsymbol{P}}{dt} = \boldsymbol{A}^T\boldsymbol{P} + \boldsymbol{P}\boldsymbol{A} + \boldsymbol{Q} - \boldsymbol{P}\boldsymbol{B}\boldsymbol{R}^{-1}\boldsymbol{B}^T\boldsymbol{P} = \boldsymbol{0}\,.$$

Equation 22

This equation is stable controlled system.

$\boldsymbol{R}$ is a weighting matrix and $\boldsymbol{P}$ is a positive-definite Hermitian

To linearize the system we have to define a quiescent point. In the case of two inverted pendulum and cart system the quiescent point is when the inverted pendulums are kept in a small neighborhood of the vertical upright position. Therefore, $x_{3,}$ x_4 $x_{5,}$ and x_6 are small. Thus:

$$\sin x_3 \approx x_3, \sin x_5 \approx x_5, \cos x_3 \approx \cos x_5 \approx 1, \sin^2 x_3 \approx \sin^2 x_5 \approx 0, \cos^2 x_3 \approx \cos^2 x_5 \approx 1$$

$$\sin^2(x_3 - x_5) \approx 0 \text{ and } x_4^2 \sin x_3 \approx x_4^2 \sin x_5 \approx x_6^2 \sin x_3 \approx x_6^2 \sin x_5 \approx 0$$

Equation 23

Then the linearized state-space model can be simplified as:

$$\boldsymbol{A_2} = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -\frac{(m_2+m_3)g}{m_1} & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & \frac{Mg}{2r_1 m_1 m_2} & 0 & -\frac{m_3 g}{r_1 m_2} & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & -\frac{(m_2+m_3)g}{r_2 m_2} & 0 & \frac{(m_2+m_3)g}{r_2 m_2} & 0 \end{pmatrix} \text{ and } \boldsymbol{B_2} = \begin{pmatrix} 0 \\ \frac{1}{m_1} \\ 0 \\ -\frac{1}{r m_1} \\ 0 \\ 0 \end{pmatrix},$$

Equation 24

And M=2$m^2{}_2$ + m_1m_3 + 2m_2m_3 + 2m_1m_2 + m_1m3

The linear models represent the systems adequately when the starting positions of the inverted pendulums are near the vertical upright positions. Because the linear optimal feedback controlled systems are stable at the equilibriums where $\boldsymbol{x}$ = 0, the linear representations of the systems will always be valid as long as the initial positions are within a small neighborhood of the vertical upright position.

4-Jordan block and diagonal form

First we obtained the state space form of the system by the use of ss function in Matlab. With state space variable the transfer function matrix can be derived. For the Jordan normal form we used Jordan function in Matlab and the result are shown below.

```
V =

    0.0045         0   -0.0002   -0.0000    0.0002    0.0000
         0    0.0045   -0.0019   -0.0004   -0.0019   -0.0004
         0         0    0.0153    0.0070   -0.0153   -0.0070
         0         0    0.1206    0.1294    0.1206    0.1294
         0         0    0.0224   -0.0096   -0.0224    0.0096
         0         0    0.1767   -0.1767    0.1767   -0.1767


j =

         0    1.0000         0         0         0         0
         0         0         0         0         0         0
         0         0    7.8880         0         0         0
         0         0         0   18.4277         0         0
         0         0         0         0   -7.8880         0
         0         0         0         0         0  -18.4277
```

Figure 3

The transfer function matrix is:

$$V = \begin{pmatrix} 0.0045 & 0 & -0.0002 & 0 & 0.0002 & 0 \\ 0 & 0.0045 & -0.0019 & -0.0004 & -0.0019 & -0.0004 \\ 0 & 0 & 0.0153 & 0.007 & -0.0153 & -0.007 \\ 0 & 0 & 0.1206 & 0.1294 & 0.1206 & 0.1294 \\ 0 & 0 & 0.0224 & -0.0096 & -0.0224 & -0.0096 \\ 0 & 0 & 0.1767 & -0.1767 & 0.1767 & -0.1767 \end{pmatrix}$$

With the Jordan normal form, new state space variables are obtained.

```
sysj =

  A =
            x1      x2      x3      x4      x5      x6
   x1        0       1       0       0       0       0
   x2        0       0       0       0       0       0
   x3        0       0   7.888       0       0       0
   x4        0       0       0   18.43       0       0
   x5        0       0       0       0  -7.888       0
   x6        0       0       0       0       0  -18.43

  B =
                u1
   x1    1.982e-16
   x2          100
   x3          -10
   x4          -10
   x5          -10
   x6          -10

  C =
                u1          x2          x3          x4          x5          x6
   y1            0           0     0.01529    0.007023    -0.01529   -0.007023
   y2            0           0      0.0224   -0.009587     -0.0224    0.009587

  D =
        u1
   y1    0
   y2    0

Continuous-time state-space model.
```

Figure 4-state space form

Matlab Code:

```
%% Part 4 : Jordan
[V,j]=jordan(A)
%%J = ss2ss(sys,inv(V))
A_J = jordan(A)
B_J = inv(V)*B
C_J = C*V
D_J = D
sysj = ss(A_J,B_J,C_J,D_J)
```

5-transform function

```
G =

  From input to output...
                     -5 s^2 + 1.51e-13 s + 980
   1:  ---------------------------------------------------------
       s^4 - 5.684e-14 s^3 - 401.8 s^2 + 1.182e-11 s + 2.113e04

                                980
   2:  ---------------------------------------------------------
       s^4 - 5.684e-14 s^3 - 401.8 s^2 + 1.182e-11 s + 2.113e04

Continuous-time transfer function.
```

Figure 5

The pole zero map is as the figure 6.

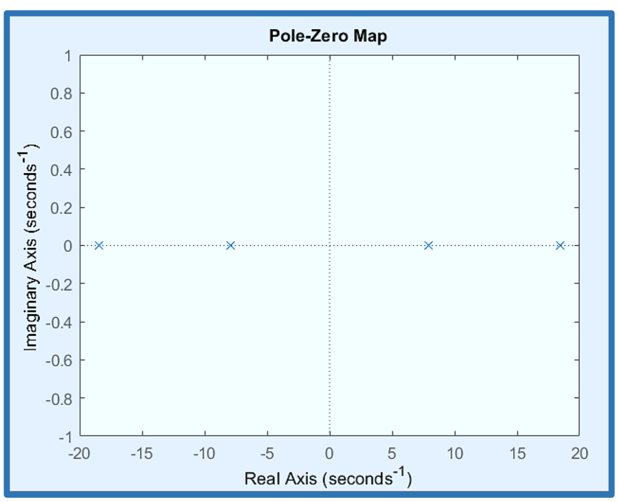

Figure 6- pole-zero map

Matlab Code:

```
s = tf('s')
%G = ss2tf(A,B,C,D)
G = C*inv(s*eye(6)-A)*B+D
P = pole(G)
Z = tzero(G)
pzmap(G)
```

Poles = 18.4277, -18.4277, 7.888, -7.888

No zero

6- stability

Inner stability will divide into two part:

1-Stability in the sense of Lyapunov : it's to be said if we have $\dot{x} = Ax$

Then if the eigenvalue of A has a none positive value which includes (0 and negative value) the stability will be stable in the sense of Lyapunov.

2- Asymptotically stability:

If the eigenvalue of A is strictly negative it will be stable in an as asymptotically way

With the Jordan form obtained in Part 4, we have the eigenvalues on the main diameter:

A =

	x1	x2	x3	x4	x5	x6
x1	0	1	0	0	0	0
x2	0	0	0	0	0	0
x3	0	0	7.888	0	0	0
x4	0	0	0	18.43	0	0
x5	0	0	0	0	-7.888	0
x6	0	0	0	0	0	-18.43

Figure 7

And also we obtained the eigenvalues by (command eig(A)) :

Eigen value : 0, 0, -18.4277, -7.888, 18.4277, 7.888

Lyapunov Stability: The real parts of eigenvalues are negative or zero, provided that the zeros are simple, means the maximum Jordan matrix order is 1.

Asymptotic stability: Eigenvalues should have negative real part
Internal stability: the system which is Internal stable should be asymptotic stable and also Lyapunov stable at the same time.
Since we have two special values with positive real part, the asymptotic stability condition is not established, so it is not internally stable.

Since the order of the Jordan block corresponds to zero eigenvalues equal to 2, it is not Lyapunov stable and so not boundary stable.

Matlab Code:

```
syms s
eig_A = eig(A)
w = isstable(sys)
```

7- BIBO stability

According to the LTI system, (multi-input-multi-output or single-output-single-input) with rational conversion function, is BIBO stable if and only if all its poles have a real negative part.

Poles = 18.4277, -18.4277, 7.888, -7.888

So the system is not input/output stable (BIBO) because 2 poles of the conversion function are at the right of the imaginary axis.

Matlab command: pole(G)

8-

We want to determine the system state transition matrix and the system response and output response of the system to a desired initial condition and the input of the single step obtained.

$$\mathbf{\Phi} = (s\mathbf{I} - \mathbf{A})^{-1}$$

Equation 25-state transition matrix

So we use the following code:

```
>>syms t s

fi = (inv(s*eye(6)-A))
fi_t =ilaplace(fi)
u = 1/s
x_0 = [1;0;0;0;0;0]
X = fi*x_0 + fi*B*u
Y = C*fi*x_0 + C*fi*B*u +D *u
X_t = ilaplace(X)
Y_t = ilaplace(Y)
```

*Note that another way to find the state transition matrix is expm(A*t) command.*

Due to the large size of the equations in the resulting matrices, the results are available in the attached MATLAB file.

9-

In this section, we want to obtain the initial conditions in such a way that a certain frequency is not excited from the output.

The columns of matrix v_i are the right Eigen Vector

$$Av_i = \lambda_i v_i$$

Equation 26

The rows of matrix w_i^T are the left Eigen Vector

$$w_i^T A = \lambda_i w_i^T$$

Equation 27

$$\begin{cases} x(t) = Tz(t) = \sum_{i=1}^{n} v_i e^{\lambda_i t} z_i(0) \\ z(t) = T^{-1}x(t) \rightarrow z_i(0) = w_i^T x(0) = c_i \end{cases} \rightarrow x(t) = \sum_{i=1}^{n} v_i e^{\lambda_i t} c_i$$

Equation 28

In order not to excite a particular frequency from the output, the corresponding constant C must be zero. Therefore, our initial condition must be such that its internal product in the special left vector of all modes except our desired mode is zero.

$$\begin{cases} w_i^T x(0) \neq 0 & i \neq i_0 \\ w_i^T x(0) = 0 & i = i_0 \end{cases}$$

Equation 29

```
Wi_T =

  220.0000         0    2.0000   -0.0000    1.0000   -0.0000
         0  220.0000    0.0000    2.0000   -0.0000    1.0000
         0         0   15.7760    2.0000   11.5567    1.4651
         0         0   36.8554    2.0000  -25.1556   -1.3651
         0         0  -15.7760    2.0000  -11.5567    1.4651
         0         0  -36.8554    2.0000   25.1556   -1.3651
```

Figure 8

If we do not want to excite the first frequency, given that the fourth and sixth values of the matrix are zero and not zero in the other lines, we can define the initial conditions as follows:

$$x(0) = [0\ 0\ 0\ 1\ 0\ 0]' \ or \ \ x(0) = [0\ 0\ 0\ 0\ 0\ 1]' \ \ or \ \ x(0) = [0\ 0\ 0\ 1\ 0\ 1]'$$

In this case we have:

$$\begin{cases} w_i^T x(0) \neq 0 & i \neq 1 \\ w_i^T x(0) = 0 & i = 1 \end{cases}$$

Equation 30

If we do not want to excite the second frequency, given that the third and fifth values of the second row of matrix are zero and not zero in the other lines, we can define the initial conditions as follows:

$$if \quad x(0) = [0\ 0\ 1\ 0\ 0\ 0]' \ or \ \ x(0) = [0\ 0\ 0\ 0\ 1\ 0]' \ \ or \ \ x(0) = [0\ 0\ 1\ 0\ 1\ 0]'$$

$$\begin{matrix} w_i^T x(0) \neq 0 & i \neq 2 \\ w_i^T x(0) = 0 & i = 2 \end{matrix}$$

Equation 31

Matlab Code:

```
%% Part 9 :Specify the initial conditions
Vi = V ;
Wi_T = inv(V);
x0_1 = [0;0;0;1;0;0];
c_1 = Wi_T*x0_1;
x0_2 = [0;0;1;0;0;0];
c_2 = Wi_T*x0_2;
```

10-

In this section, we want to decompose the system using the Kalman method and obtain controllable or visible subsystems.

The system is controllable but unobservable because the rank of the visibility matrix is 4, so we select 4 independent rows of the matrix observability:

```
rref_obsv =

     0     0     1     0     0     0
     0     0     0     1     0     0
     0     0     0     0     1     0
     0     0     0     0     0     1
     0     0     0     0     0     0
     0     0     0     0     0     0
     0     0     0     0     0     0
     0     0     0     0     0     0
     0     0     0     0     0     0
     0     0     0     0     0     0
     0     0     0     0     0     0
     0     0     0     0     0     0
```

Figure 9-rref(observability)

We add two more independent rows to get the base vector:

$$r_5 = [1\ \ 1\ \ 0\ \ 0\ \ 0\ \ 0] \qquad r_6 = [0\ \ 1\ \ 0\ \ 0\ \ 0\ \ 0]$$

Figure 10-The added rows

So the unique conversion matrix is:

```
T_k =

     0     0     1     0     0     0
     0     0     0     1     0     0
     0     0     0     0     1     0
     0     0     0     0     0     1
     1     1     0     0     0     0
     0     1     0     0     0     0
```

Figure 11- Unique Conversion Matrix

The Kalman decompose is as below:

$\bar{A}_o$ $\quad$ 0 $\quad$ $\bar{A}_{21}$ $\quad$ $\bar{A}_{\bar{o}}$

```
A_k =

         0    1.0000         0         0         0         0
  205.8000         0  -98.0000         0         0         0
         0         0         0    1.0000         0         0
 -196.0000         0  196.0000         0         0         0
   -0.9800         0         0         0         0    1.0000
   -0.9800         0         0         0         0         0
```

$\bar{B}o$ $\quad$ $\bar{B}_{\bar{o}}$

```
B_k =

         0
   -5.0000
         0
         0
    0.5000
    0.5000
```

$\bar{C}_o$ $\quad$ 0

```
C_k =

     1     0     0     0     0     0
     0     0     1     0     0     0
```

Matlab Code:

```
%% Part 10 : Kalman
 n = 6
 % Cheking Controlability & observability
 n1 = rank(ctrb(A,B))
 if(n1 == n)
    disp("System is controllable")
```

```
else
   disp("System is not fully controllable")
end
n2 = rank(obsv(A,C))
if(n2 == n)
   disp("System is observable")
else
   disp("System is not fully observable")
end
% Create Unique Conversion Matrix
rref_obsv = rref(obsv(A,C))
T_k = [0 0 1 0 0 0;0 0 0 1 0 0;0 0 0 0 1 0;0 0 0 0 0 1;1 1 0 0 0 0;0 1 0 0 0 0]
% Kalman Analysis
A_k = T_k*A*inv(T_k)
B_k = T_k*B
C_k = C*inv(T_k)
A_bar_o = A_k(1:n2,1:n2)
A_bar_o_bar = A_k(n2 + 1:n,n2 + 1:n)
A_bar_21 = A_k(n2 + 1:n,1:n2)
A_bar_0 = A_k(1:n2,n2 + 1:n)
B_bar_o = B_k(1:n2,:)
B_bar_o_bar = B_k(n2 + 1:n,:)
C_bar_o = C_k(:,1:n2)
```

11 –

In the last part, we want to check whether the realization of the state space is minimal or not and also to determine the of the system. To do this, we use the following code:

```
%% Part 11 : Minimal Realization
sys = ss(A,B,C,D)
sysr = ss(A_bar_o,B_bar_o,C_bar_o,D)
%sysr = minreal(sys)
if(rank(sys.A) == rank(sysr.A))
  disp("The Realization is minimal")
else
  disp("The Realization is not minimal")
end;
eig_o = eig(A_bar_o)       % observable poles
eig_o_bar = eig(A_bar_o_bar) % unobservable poles
```

The minimum realization will be as follows:

```
sys =

  A =
          x1      x2      x3      x4      x5      x6
   x1      0       1       0       0       0       0
   x2      0       0   -0.98       0       0       0
   x3      0       0       0       1       0       0
   x4      0       0   205.8       0     -98       0
   x5      0       0       0       0       0       1
   x6      0       0    -196       0     196       0

  B =
        u1
   x1    0
   x2  0.5
   x3    0
   x4   -5
   x5    0
   x6    0

  C =
       x1  x2  x3  x4  x5  x6
   y1   0   0   1   0   0   0
   y2   0   0   0   0   1   0

  D =
       u1
   y1   0
   y2   0

Continuous-time state-space model.
```

```
2 states removed.

sysr =

  A =
           x1     x2     x3     x4
   x1       0      1      0      0
   x2   205.8      0    -98      0
   x3       0      0      0      1
   x4    -196      0    196      0

  B =
        u1
   x1    0
   x2   -5
   x3    0
   x4    0

  C =
        x1  x2  x3  x4
   y1    1   0   0   0
   y2    0   0   1   0

  D =
        u1
   y1    0
   y2    0

Continuous-time state-space model.

The Realization is not minimal
```

To determine the stabilizability and detectablity of the system, it is necessary to determine the controllable and visible poles.

According to the figure 10:

Invisible poles = 0,0

Visible poles = -18.4277, -7.8880, 7.8880, 18.4277 *-All poles are controllable-*

Due to the fact that the system is controllable, it is also stabilizable, but due to the existence of zero conjugate states which are invisible, the system is detectable.