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An Analytical Formula for Optimal Tuning of the State Feedback Controller Gains for the Cart-Inverted Pendulum System

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Abstract: This paper resolves the long-standing problem of state-feedback gain tuning for Cart-Inverted-Pendulum system by identifying a natural choice for the closed loop poles that ensures speed of response as well as robust stabilization. It first identifies the robustness issue that arises due to gain variations in the input and the output signals of this single-input-two output system. It then proposes a robustness measure for such a system and obtains the controller gains that maximize the same. Simulations, as well as experimental results, confirm the superiority of this controller over existing ones.

Keywords: Cart-Inverted-Pendulum system, Stabilization, Multi-Loop approach

1. INTRODUCTION

Balancing an inverted pendulum on a cart moving on a finite rail (see Fig.1) is a benchmark control problem. The system here has one input (the force, u, applied to the cart) and two outputs (the pendulum angle, θ, w.r.t. the upward vertical, and the cart position, x, on the rail). A standard approach for solving this problem consists of (i) defining the states of the system to be $\theta, \dot{\theta}, x, \dot{x}$, (ii) obtaining θ and x from measurements, and deriving $\dot{\theta}$ and $\dot{x}$ therefrom, and (iii) then providing state-feedback type control by choosing appropriate gains corresponding to these states. These choices should, naturally, be such as to i) increase the speed of response, ii) tolerate more variations in the gains of the physical signals that link the system to the controller, and iii) reduce the control input. To see how these objectives are met in the considerable number of works available in literature for the last three decades (most of which are based on the LQR [D. Chatterjee and Joglekar (2002)], [Z.-M. Wang (2003)], [Harrison (2003)], [Muskinja and Tovornik (2006)] and some on the H-infinity approach [Ouyang et al. (2012)], [Linden and Lambrechts (1993)]), a review of the same has been undertaken. This review reveals the following:

(1) While the methods applied and/or the tuning processes (the choice of Q, R in LQR, or, weights in H_∞) used are all different, all the designs are found to lead to closed-loop poles with a dominant-pole structure, i.e.,
$$p_{1,2} = -\alpha \pm j\alpha\sqrt{1/\zeta_1^2 - 1}; \quad p_{3,4} = -\beta \pm j\beta\sqrt{1/\zeta_2^2 - 1};$$
where $\zeta_1, \zeta_2 \leq 1$; $\alpha, \beta > 0$ and $\beta \geq 5\alpha$.
(2) While a larger α yields a faster response, the compensation requires greater control effort. A large value of the ratio β/α accentuates the control effort requirement.
(3) Regarding the gain margins, GM_u, GM_θ, GM_x, associated respectively with the signals u, θ, x, however, it is found that (a) while the GM_u is guaranteed to be with in $[1/2 - \infty]$[Anderson and Moore (1989)], (b) the GM_θ and GM_x are much poorer. More important, none of the design approaches used appears to have made any effort to improve these although it is known in literature ([Yang and Kabamba (1994)], [Das and Paul (2011)]) that robustness w.r.t. u does not automatically guarantee the same w.r.t. θ and x.

Several questions arise from the above. Is dominant pole structure the right choice for good control here? Is the robustness associated with the θ or x signal necessarily going to be much poorer compared to the same of u, or is it possible that there exists some suitable choice of the closed-loop poles that may improve the overall robustness satisfying the response requirement as well?

These questions prompt the authors to consider the overall robustness as $min\{GM_u, GM_\theta, GM_x\}$ and to look for an optimal closed-loop pole configuration that would maximize this measure and yet maintain good speed of response. (So far as the control input required is concerned, it is assumed to remain within the given bound.) To this end, it is first noted that the plant-controller interconnection is based on *three* signals, the u, θ and x and the loops involved in the compensated system may be viewed from the point of view of any one of these three. Therefore, the gain margins corresponding to the three signals will, in general, be different, though the closed-loop characteristic equation of the system yielded by them is the same. This paper identifies the most restrictive loop from the robustness point of view and finds out the closed-loop pole locations that maximizes the robustness of the same. In section 2, the system transfer functions and the characteristic equation corresponding to state-feedback control is obtained. In section 3, the poles and zeros of the u-, x- and θ-loops are derived and the loop that is most restrictive in terms of robustness is identified, the closed loop choice of poles that maximizes the system robustness is then found, which leads to a choice of the controller gains in closed form. Simulation and experimental results presented in section 4 establish the superiority of the controller proposed over the existing ones.

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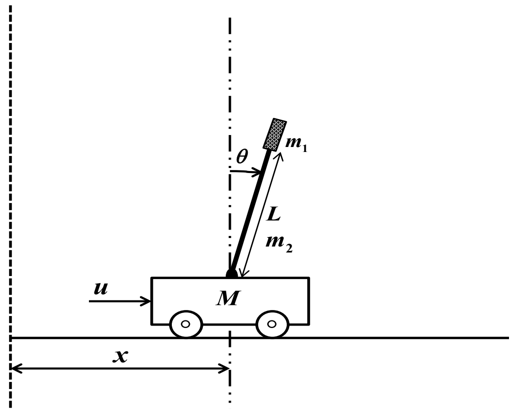

Fig. 1. Schematic of Cart-Inverted Pendulum System.

2. SYSTEM DESCRIPTION

Referring to a cart- inverted-pendulum (CIP) system as shown in Fig.1, let M be the cart mass, m_1 be the bob mass, m_2 the (uniformly distributed) pole mass and L the pole length. Then the two relevant transfer functions, linearized about the upper equilibrium point and simplified assuming zero air and rail friction, can be obtained (following standard procedure [Williams (2011)]) to be

$$\begin{bmatrix}\theta(s)/U(s)\\ X(s)/U(s)\end{bmatrix} = \begin{bmatrix}\dfrac{k_u G_\theta}{(s^2-p^2)}\\ \dfrac{k_u G_x(s^2-z^2)}{s^2(s^2-p^2)}\end{bmatrix} \triangleq k_u G(s) \tag{1}$$

with

$$G_\theta = \frac{-ml}{[(M+m)J+Mml^2]};\; G_x = \frac{J+ml^2}{[(M+m)J+Mml^2]};$$
$$p^2 = \frac{(M+m)mgl}{[(M+m)J+Mml^2]};\; z^2 = \frac{mgl}{J+ml^2};$$
$$J = \frac{(8m_1+5m_2)m_2L^2}{6(m_1+m_2)};\quad l = \frac{(2m_1+m_2)L}{2(m_1+m_2)};\quad m = m_1+m_2; \tag{2}$$

(Note that l is the distance of the pendulum CG from the pivot, g the acceleration due to gravity and J the moment of inertia of the pendulum about its CG). Here, k_u, nominally unity, stands for the variations in u.

This system, obviously, is unstable. State feedback is used to stabilize the same. This paper denotesthe gains corresponding to the states $\theta, \dot{\theta}, x, \dot{x}$ as $k_\theta\alpha_\theta, k_\theta, k_x\alpha_x, k_x$ respectively, and the control input becomes

$$U(s) = -\left[k_\theta(s+\alpha_\theta)\;\; k_x(s+\alpha_x)\right]\begin{bmatrix}\theta(s)\\ X(s)\end{bmatrix} \triangleq C(s)\begin{bmatrix}\theta(s)\\ X(s)\end{bmatrix} \tag{3}$$

Then the system characteristic equation becomes

$$I + C(s)G(s) = 0 \tag{4}$$

$$\Leftrightarrow s^2(s^2-p^2) + G_\theta k_u k_\theta s^2(s+\alpha_\theta) + G_x k_u k_x (s^2-z^2)(s+\alpha_x) = 0 \tag{5}$$

3. CONTROLLER DESIGN

While eq.(5) is enough for the purpose of pole placement, in order to understand the robustness of the system w.r.t. the u, x and θ signal gain variations it would be helpful to rewrite the same in the following forms:

$$s^2(s^2-p^2) + \mathbf{k_u}[G_\theta k_\theta s^2(s+\alpha_\theta) + G_x k_x (s^2-z^2)(s+\alpha_x)] = 0 \tag{6}$$

$$s^2[s^2-p^2+k_u G_\theta k_\theta(s+\alpha_\theta)] + \mathbf{k_x}[G_x k_u (s^2-z^2)(s+\alpha_x)] = 0 \tag{7}$$

$$s^2(s^2-p^2) + k_u G_x k_x (s^2-z^2)(s+\alpha_x) + \mathbf{k_\theta}[k_u G_\theta s^2(s+\alpha_\theta)] = 0 \tag{8}$$

Eq. (6), may be visualized to represent the u-loop of the system. Of the four poles of this loop, one is in the RHP at p, two at origin and one in the LHP at $-p$. The three (finite) loop zeros, however, are all *assignable* using the state-feedback gains. Since on feedback the loop poles tend to the loop zeros, the stabilization/pole-placement problem as seen from the view point of this loop does not seem to be a critical one (except for the control input limitation).

Eq. (7) refers to the x-loop of the system. It is seen to have a double pole at origin and two more poles arising out of the controller, and two *fixed* zeros at $\pm z$ and an *assignable* one at $-\alpha_x$. It is evident from a root-locus point of view that in order that the open-loop poles at origin become a closed-loop pair that corresponds to sufficiently high damping and fast settling, $-\alpha_x$ must be of the same order of magnitude as z. This, in turn, implies that the above pair of closed-loop poles must lie in the vicinity of these zeros so that the remaining two poles may be placed sufficiently far in the LHP thereby ensuring sufficient GM w.r.t. k_x in spite of the presence of the RHP zero at z. Unlike the u-loop, therefore, this loop shows that the response of the system may not be made arbitrarily fast (even in the absence of control input limitation).

Eq. (8) refers to the θ-loop. Since $p \approx z$, it is easy to see that the two of the loop poles are now at $\sim \pm z$, the third in LHP to the right of $-\alpha_x$ and the fourth in RHP at a distance more than that of the third from origin. In effect, then, the mean distance from origin of the poles in RHP are more than the same of the ones in LHP. (Note that the last fact is valid even if $p \not\approx z$.) Again, two of the zeros of the loop are at origin (*fixed*), the other at α_θ being *assignable*. This α_θ is chosen in such a way that both the RHP and LHP pairs of poles become, in closed-loop, such that they correspond to sufficiently high damping and fast settling. Now none of these pairs can be made to be far in the LHP since then the other pair would tend to origin. Clearly, the best choice of α_θ would be the one that yields the closed-loop poles as a repeated complex pair having a damping factor of ζ and frequency ω. Fig. 2 shows the root-loci of this loop for such an optimum α_θ and the loci that correspond to perturbations to it.

Clearly, the θ-loop, by the virtue of the two unstable poles and two zeros at origin it has, is the most restrictive of the three. So, the closed-loop pole configuration dictated by it is chosen. Following the study of its root locus as made above one may chose the system characteristic equation as

$$(s^2 + 2\zeta\omega s + \omega^2)^2 = 0 \tag{9}$$

Where $0 \le \zeta \le 1$. Now comparing (9) with (5) it is routine to obtain

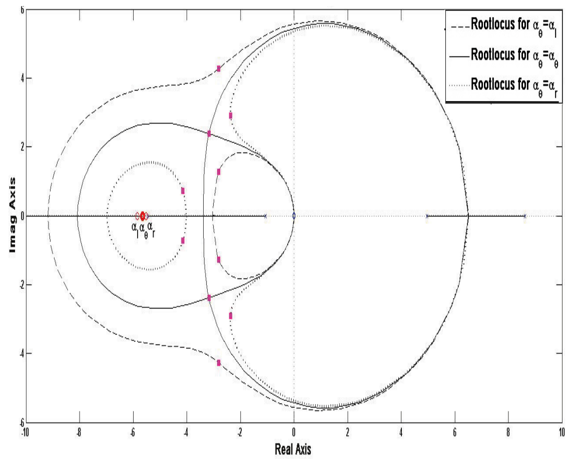

Fig. 2. Root locus plot of θ- loop for α_θ variation.

$$
\begin{aligned}
k_x &= \frac{-4\zeta\omega^3}{G_x z^2}\\
\alpha_x &= \frac{\omega}{4\zeta}\\
k_\theta &= \frac{4\zeta\omega z^2+4\zeta\omega^3}{G_\theta z^2}\\
\alpha_\theta &= \frac{p^2z^2+2\omega^2z^2+4\zeta^2\omega^2z^2+\omega^4}{4\zeta\omega z^2+4\zeta\omega^3}
\end{aligned} \tag{10}
$$

So far as ω is concerned, a higher value implies a faster response but requires high control effort. This paper proposes to choose ζ and ω in order to maximize the minimum of the loop gain margins. In this regard u-loop is found to be the least critical one as results are available in literature [Anderson and Moore (1989)] that guarantee a gain-margin of $[1/2-\infty]$ for the input-side of a state-feedback controller. The same, however, is not true for the other loops [Yang and Kabamba (1994)] and are therefore considered for further investigation. To this end, to get an understanding about the gain margins of x and θ loop, this paper first finds out the critical loop gains k_{xi} and $k_{\theta i}$,respectively, associated with them. Here, k_{xi} is the x-loop gain at which the closed loop poles become marginally stable and any further increase in the same will make the system unstable. Similarly, $k_{\theta i}$ is the θ-loop gain at which the corresponding root locus crosses-in the imaginary axis and any further decrease in the gain makes the system unstable. This paper next defines the gain margins of these loops as

$$GM_x \triangleq k_{xi}/k_x; \quad GM_\theta \triangleq k_\theta/k_{\theta i}; \tag{11}$$

Hence, the system gain margin can be defined as

$$GM \triangleq min\{GM_\theta, GM_x\} \tag{12}$$

Next in this note the ζ and ω are found that maximizes this GM. To this end it is straight forward to find the critical gains related to different loops from the Routh's table of the system characteristic equation (given by eq.(5)). The critical gains are found as

$$k_{xi} = -\frac{G_\theta k_\theta}{G_x} + \frac{p^2}{G_x(\alpha_\theta-\alpha_x)} \tag{13}$$

$$k_{\theta i} = -\frac{G_x k_x}{G_\theta} + \frac{p^2}{G_\theta(\alpha_\theta-\alpha_x)} \tag{14}$$

In view of eq.(10), eq.(13) and eq.(14), eq.(11) becomes

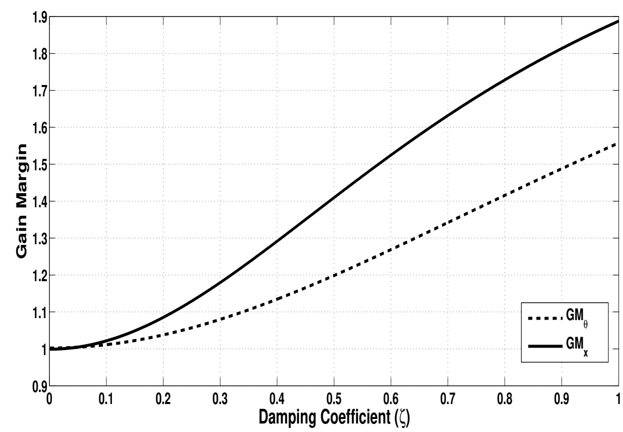

Fig. 3. GM_x and GM_θ plots for variation in ζ (with a given ω).

$$GM_x = \frac{(z^2+\omega^2)(1+4\zeta^2)}{p^2+\omega^2(1+4\zeta^2)} \tag{15}$$

$$GM_\theta = \frac{p^2z^2+\omega^2(p^2+z^2+4\zeta^2z^2)+\omega^4(1+4\zeta^2)}{p^2z^2+2\omega^2p^2+\omega^4(4\zeta^2+1)} \tag{16}$$

For any choice of ω, it is easy to see from eq.(15) and eq.(16) that GM_θ and GM_x increase for an increase in ζ. Thus the obvious choice of ζ comes out to be 1, for any given ω. Fig.3 shows how these gain margins vary with variation in ζ. Note that $\zeta = 1$ yields a good transient behavior. For the above mentioned choice eq.(15) and eq.(16) reduces to

$$GM_x(\omega) = 1+\frac{5z^2-p^2}{p^2+5\omega^2} \tag{17}$$

$$GM_\theta(\omega) = 1+\frac{\omega^2(5z^2-p^2)}{p^2z^2+2\omega^2p^2+5\omega^4} \tag{18}$$

Now comparing the numerator and denominator of the fractional term in eq.(17) and eq.(18), one may conclude $GM_x(\omega) < GM_\theta(\omega)$. Therefore from definition given in eq.(12) it is evident that $GM = GM_\theta(\omega)$ and the value of $\omega = \omega_{op}$ that maximizes $GM_\theta(\omega)$ will be the global choice for closed loop poles. Fig.4 gives a graphical solution for ω_{op}. The same can be found by differentiating eq.(18) w.r.t. ω and then equating the derivative to zero. The choice is as follows

$$\omega_{op}^4 = \frac{p^2z^2}{5} \tag{19}$$

The parameters given by eq.(10) corresponding to the choice $\zeta = 1$ and $\omega = \omega_{op}$ then allow one to tune the controller gains for *any* CIP system in the *simplest* fashion to ensure (as the next section would confirm) *significantly improved* performance and robustness. This choice may, then, be said to be a *canonical* one. Further, as is often the case, if $M \gg m$, then $p \approx z$, and these gains, in terms of the physical parameters of the CIP system, become

$$k_x = -1.2Mz; \quad \alpha_x = z/6; \quad k_\theta = -3Mg/z; \quad \alpha_\theta = z; \tag{20}$$

where z is as given in eq.(2).

4. RESULTS AND DISCUSSION

The performance of the controller proposed has been tested for a CIP system [Feedback (2007)] having $M = 2.4kg, m_1 = 0.21kg, m_2 = 0.05kg, L = 0.36m$ and subject to the limitations $|x_{max}| = 0.4m$ and $|u_{max}| = 20N$ via (i) simulation, and (ii)

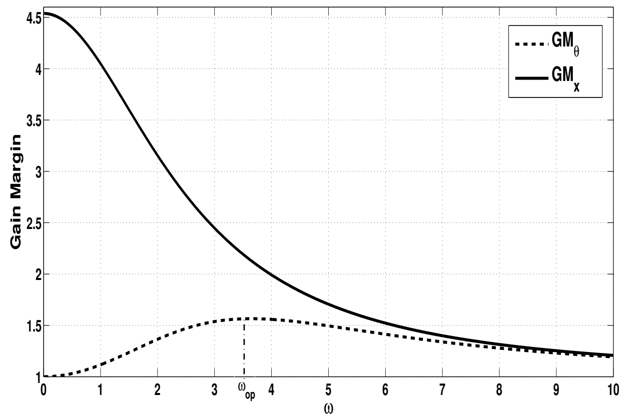

Fig. 4. GM_x and GM_θ plots for variation in ω (with $\zeta = 1$).

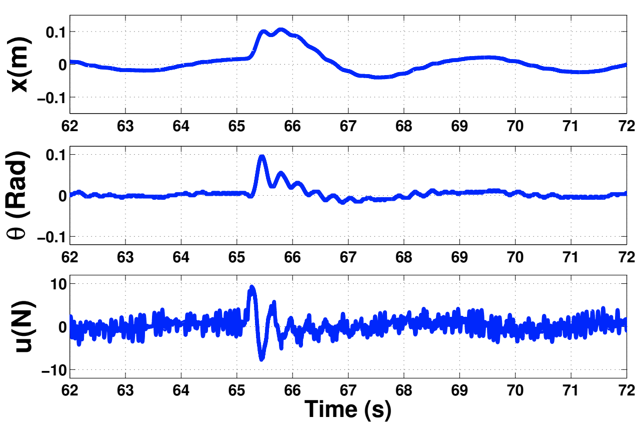

Fig. 5. Experimental plots of x,θ, and u.

physical implementation using a *MATLAB-Simulink* based real-time interface working with a sampling interval of 3ms. For comparison, performance of the same system has also been tested for controller gains employed in [D. Chatterjee and Joglekar (2002)] and [Das and Paul (2011)]. Some comparison has also been made with the performance of the controller proposed in [Muskinja and Tovornik (2006)] for a similar system.

The responses have been obtained in simulation for the cases of (i) an initial θ of 4°, and (ii) a step x-input of $0.1m$. The overshoots corresponding to both the cases, as well as the undershoot corresponding to the latter, as yielded by the different controllers noted above are tabulated in Table 1. The settling time for each case is also noted. The present design is, clearly, much superior to the others in all respects.

The steady state responses of x, θ and u as obtained experimentally using the proposed controller are plotted in Fig.5. It also shows how the compensated system reacts to a disturbance in θ at 65.2 s. Comparing this with the ones presented in [D. Chatterjee and Joglekar (2002)] and [Das and Paul (2011)] the superiority of the present controller in respect of the magnitudes of the persistent oscillations becomes evident.

The gain margins of the different loops involved in the system have also been obtained in simulation for all the designs noted above. The following observations may be made in this respect:

Table 1.

Performance Parameter			Chatterjee (2002)	Muskinja (2006)	Das (2011)	Proposed
Simulation	Overshoot	θ	30%	28.3%	23.3%	13.33 %
		x	19.4%	13.8%	22%	3.34%
	Undershoot	x	18%	3%	5.7%	0.9%
	Settling time (s)	θ	5.1	4.1	4.3	2.1
		x	5.9	4.5	5.2	1.7
	GM	θ	1.3	1.13	1.16	1.69
		x	1.38	1.21	1.35	2.22
		u	2.7	3.9	4.5	2.9
	PM	θ	53.7°	58.4°	57.6°	54.6°
		x	42.4°	38.3°	43.2°	37.4°
		u	54.3°	65.8°	73.4°	58.9°
	Delay Margin (s)	θ	0.016	0.003	0.017	0.025
		x	0.48	0.31	0.51	0.235
		u	0.084	0.012	0.029	0.0763
Exp	GM	θ	1.24	–	1.18	1.55
		x	1.28	–	1.4	2.1
		u	2.45	–	3.6	2.23

(1) As the three finite u-loop zeros lie in LHP, the limitation on control input is the only factor that decides what would be the maximum value of k_u that just causes instability.
(2) As the x-loop has two poles at origin, $k_x \to 0$ would cause low frequency, undamped, oscillations in x. Hence, the finite-limit on cart movement i.e. rail length decides the lower limit on k_x. Likewise, the presence of two zeros at origin in the θ-loop will cause rail-length to limit the maximum value of k_θ.
(3) The instabilities corresponding to minimum possible value of k_u and k_θ or maximum possible value of k_x are, however, independent of physical limits and may be directly obtained from the corresponding loop transfer functions.

Table 1 shows that while the gain margin for u-variation is about the same for all the designs, the ones for θ- and x-variations are (as they are expected to be) markedly better for the proposed design than for the other ones.

The GMs corresponding to u, x and θ loops have also been obtained experimentally for the system [Feedback (2007)] and are presented in Table 1. These results confirm the findings of the simulations.

The PMs and the corresponding delay-margins (DM), $DM \triangleq PM/\omega_g$, where ω_g is the gain-crossover frequency, as obtained for the different loops from the respective transfer functions are also given in Table 1. It is well known [Karl J. Astrom (1997)] that if a continuous-time controller is to be implemented digitally (as is universally done for the CIP system) then the sampling interval, T, must satisfy the condition $T < 2DM$. It is then seen from Table 1 that (i) it is the θ-loop that determines the maximum permissible T, and (ii) the present design allows for about 50% higher T to be used. It has been experimentally verified that while the nominal designs given in [D. Chatterjee and Joglekar (2002)] and [Das and Paul (2011)] work for $T \leq 10ms$, the one presented here allows up to $T = 15ms$.

5. CONCLUSION

Based on a unique, analytical choice of the closed-loop poles, this paper has presented a state feedback controller for the CIP system that maximizes the worst of the loop gain margins involved in the system attaining at the same time a speed of response that is significantly higher than what has so far been achieved in literature. Simulation as well as physical implementation results confirm its superiority in respects of robustness as well as response.It is felt that the multi-loop based design approach used in this paper can be extended to yield superior controllers for the several multi-link pendulum problems considered in literature both in continuous and discrete domain.

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