Human Capital Accumulation and Earnings Dynamics over the Life Cycle: Lessons from the Ben-Porath Model

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1 Introduction

Human capital investment and accumulation are essential components for many studies on a wide range of topics in Economics, including the economic growth and development of nations, the gender and black-white wage gaps, and the rate of return to schooling. Becker (1962) initiates the formal analysis of human capital studies and offers the first unified and comprehensive framework to study human capital investment with the standard Economic tools.[1]

Following Becker's work, Ben-Porath (1967) proposes a dynamic model that relates human capital accumulation to life-cycle earnings. Currently, this is the workhorse model when it comes to analyze the relation between human capital accumulation decisions and life-cycle outcomes. The vast majority of papers that model human capital use some variation of the Ben-Porath model. These variations lead to very different time profiles of investment in human capital, human capital accumulation, and earnings.

In this paper, we analyze the Ben-Porath model and several of its variations in order to inform researchers on the consequences of their modeling decisions. We proceed as follows: Section 2 presents the baseline specification. Many implications can be derived in closed form and the intuition can be easily explained with straightforward graphical analysis based on the assumptions of no depreciation and finite horizon. Section 3 analyzes the Haley-Rosen specification, which enables for finite horizon but still keeps tractability. Section 4 studies the model in a more general formulation by relaxing the neutrality assumption. Section 5 studies an specification that generates a Bang-Bang equilibrium. Section 6 allows for depreciation and develops conditions under which investment in human capital happens in different episodes over the life-cycle. Finally, Section 7 offers some final comments.

2 Basic Ben-Porath Model

The baseline Ben-Porath model studies how a single representative agent makes optimal life-cycle decisions on human capital investment to maximize her total lifetime disposable earnings. At each point of time, the agent's current stock of human capital, $H(t)$, and the rental rate of human capital, R, determine the amount of her potential earnings: $Y(t) = RH(t)$. The agent chooses two type of inputs in order to produce human capital: (i) a fraction of her current stock of human capital, $I(t)H(t)$, with $I(t) \in [0, 1]$; (ii) market goods, $D(t)$. Therefore, the cost of human capital investments includes both foregone earnings, $RI(t)H(t)$, and costs of the purchased market goods, $P_D D(t)$, where P_D is the price of the market goods.

Then, the agent's disposable earnings in period t, $E(t)$, are equivalent to her potential earnings in period t, $Y(t)$, less the total costs:

$$E(t) = RH(t) - RI(t)H(t) - P_D D(t) \tag{1}$$

The observed earning that the agent makes from her work in the labor market is $R(1-I(t))H(t)$, which are higher than her disposable earning and lower than her potential earnings. Subtracting this by the costs of purchased market goods, $P_D D(t)$, gives the level of her disposable earnings, $E(t)$.

The agent produces human capital through a production function that takes two inputs.

[1] Becker (2009) collates the work of the author on the subject which covers schooling, learning-by-doing, and on-the-job training.

Assumption 2.1 *(Strict Concavity of the Production Function)* $\forall\, t \in [0,T]$ $F(\cdot,\cdot)$ *is strictly concave in both of its arguments.*

The change in human capital stock at time t, which is summarized by the law of motion for $H(t)$, is defined as:

Definition 2.1 *(Law of Motion for Human Capital Stock in the Basic Ben-Porath Specification)*

$$\dot{H(t)} = F\left(I(t)H(t), D(t)\right) - \sigma H(t). \tag{2}$$

The law of motion for human capital stock embeds a neutrality assumption. Namely, the current stock of human capital at time t, $H(t)$, and the investment time at time t, $I(t)$, appear as a single argument in a multiplicative form in the flow production of human capital stock. This assumption simplifies our calculations by neutralizing the effect of $H(t)$ on the optimal decision of time investment. In particular, a higher level of $H(t)$ increases the marginal return of $I(t)$ in producing human capital and the marginal cost of $I(t)$ in foregone earnings, both in a multiplicative pattern. As a result, $H(t)$ cancels out in the first order condition for $I(t)$ as we show later.

The agent's life-cycle problem is to choose $I(t)$ and $D(t)$ over time to maximize her total disposable earnings subject to the law of motion for human capital. Given an initial condition of human capital, $H(0) = H_0$, the agent's problem is as follows.

Problem 2.1 *(Life-cycle Individual's Problem in the Basic Ben-Porath Model)*

$$\max_{I_t, D_t} \int_0^T \exp^{-rt} RH(t)(1 - I(t))dt$$

s.t.

$$\begin{aligned} H(0) &= H_0 \\ \dot{H(t)} &= F\left(I(t)H(t), D(t)\right) - \sigma H(t) \end{aligned}$$

Therefore, the current value Hamiltonian associated to the agent's maximization problem is

$$\mathcal{H}(\cdot) = \exp^{-rt}\left[RH(t) - RI(t)H(t) - P_D D(t)\right] + \mu(t)\dot{H(t)} \tag{3}$$

where $\mu(t)$ defines the shadow price of the human capital stock. Thus, the following conditions must be satisfied for the interior solution.

Condition 2.1 *(Optimality Conditions for the Life-cycle Individual's Problem in the Basic Ben-Porath Model)*

$$\frac{\partial \mathcal{H}(\cdot)}{\partial I(t)} = 0 \;\Leftrightarrow\; \exp^{-rt} R = \mu(t) F_1\left(I(t)H(t), D(t)\right) \tag{4}$$

$$\frac{\partial \mathcal{H}(\cdot)}{\partial D(t)} = 0 \;\Leftrightarrow\; \exp^{-rt} P_D = \mu(t) F_2\left(I(t)H(t), D(t)\right) \tag{5}$$

$$\frac{\partial \mathcal{H}(\cdot)}{\partial H(t)} = -\dot{\mu(t)} \;\Leftrightarrow\; \exp^{-rt} R\left(1 - I(t)\right) + \mu(t)\left(F_1\left(I(t)H(t), D(t)\right) I(t) - \sigma\right) = -\dot{\mu(t)} \tag{6}$$

$$\frac{\partial \mathcal{H}(\cdot)}{\partial \mu(t)} = \dot{H(t)} \;\Leftrightarrow\; \dot{H(t)} = F\left(I(t)H(t), D(t)\right) - \sigma H(t) \tag{7}$$

$$\textit{Transversality} \;:\; \lim_{t \to T} \mu(t)H(t) = 0 \tag{8}$$

where F_j is the first order derivative of the production function F with respect to argument j.

To simplify notation, combine the two terms with intertemporal meaning in the life-cycle decision problem into one term through $g(t) \equiv \exp^{rt} \mu(t)$. Then, combine (4) and (6) to get

$$\dot{\mu(t)} = -\exp^{-rt} R + \mu(t)\sigma \tag{9}$$

and note that $\dot{g(t)} = \dot{\mu(t)} \exp^{rt} + r\mu(t)e^{rt}$. Use (9) to obtain

$$\dot{g(t)} = (\sigma + r)g(t) - R. \tag{10}$$

Equation (8) implies that $\mu(T) = 0$, and therefore $g(T) = 0$ provided that $H(T) = 0$ has no economic sense. It is possible, thus, to solve (10) and obtain

$$g(t) = \frac{R}{\sigma + r}\left[1 - \exp^{(\sigma+r)(t-T)}\right], \tag{11}$$

which leads to $\dot{g(t)} < 0$. To wrap up the discussion note that the optimality conditions for the interior solution are:

$$\begin{aligned} g(t)F_1(I(t)H(t), D(t)) &= R \\ g(t)F_2(I(t)H(t), D(t)) &= P_D. \end{aligned} \tag{12}$$

The system in (12) consists of two equations and two unknowns that solve for the Marshallian demand for $I(t)H(t)$ and $D(t)$. Note that if the optimal solutions are interior and the cross-partial derivative of the production function with respect to its two arguments is assumed to be zero, i.e. $\frac{\partial^2 F}{\partial IH \partial D} = 0$, strict concavity of the production function together with $\dot{g(t)} < 0$ imply that both Marshallian demands are decreasing overtime. This is intuitive because the agent faces a finite horizon problem and the amount of time left to capture the returns of human capital investment decreases over time.

2.1 Earnings Dynamics

One of the fundamental questions that this basic model enables to ask is how earnings evolve over the life-cycle. Consider both the slope and curvature of the earnings dynamics in the case with no $D(t)$, i.e. $F_{D(t)} = 0$ so that the production function takes the single argument $I(t)H(t)$. Without loss of generality, also assume that $R \equiv 1$.

2.1.1 The Slope of Earnings Dynamics

Claim 2.1 *(Earnings over Time with no Depreciation) Let $\sigma = 0$. Then, when the optimal solution for $I(t)$ is interior, $\dot{E(t)} > 0$.*

Proof: Differentiate (1) and use (2) to write

$$\begin{aligned} \dot{E(t)} &= \dot{H(t)} - I(t)\dot{H}(t) \\ &= F\left(I(t)H(t)\right) - I(t)\dot{H}(t) \\ &> 0 \end{aligned} \tag{13}$$

where the inequality follows because the Marshallian demands for $I(t)H(t)$ is decreasing over time. $\boxtimes$

Claim 2.2 *(Earnings over Time with Depreciation) Let $\sigma > 0$. Then, $\dot{E(t)} \lesseqgtr 0$.*

Proof: Follow the same steps as in the proof of Claim 2.1 and note that the term $\sigma H(t)$ appears in the expression for $\dot{E(t)}$. This term could be $\lesseqgtr F\left(I(t)H(t)\right) - I(t)\dot{H}(t)$. $\boxtimes$

Claim 2.1 follows because with a positive amount of investment (interior solution) and no depreciation, human capital stock is accumulated over time. Moreover, investment on human capital declines over time, and thus disposable earnings increase overtime. Claim 2.2 follows because the accumulated stock of human capital may be driven down by a relatively high rate of depreciation.

2.1.2 The Curvature of Earnings Dynamics

We now analyze the curvature of the earnings function for the case in which there is no depreciation.[2].

Claim 2.3 *(Concavity of the Earnings Function with no Depreciation) Assume that $\eta \equiv \left(1 - \frac{F'F'''}{F''^2}\right) < 0$. Then, the earnings function is strictly concave.*

Proof: First note that $\dot{E(t)} > 0$ by Claim 2.1. Since $F_{D(t)} = 0$ we can write the first order condition for investment as

$$g(t)F'(I(t)H(t)) = 1 \tag{14}$$

and differentiate it with respect to t to get

$$\begin{aligned} \dot{g(t)}F'(I(t)H(t)) + g(t)F''(I(t)H(t))I(t)\dot{H}(t) &= 0 \\ &\Leftrightarrow \\ I(t)\dot{H}(t) &= -\left(\frac{\dot{g(t)}}{g(t)}\right)\left[\frac{F'}{F''}\right]. \end{aligned} \tag{15}$$

Moreover, drop the argument t to shorten notation, and note that

$$I\ddot{H} = -\left[\frac{\ddot{g}}{g} - \left(\frac{\dot{g}}{g}\right)^2\right]\frac{F'}{F''} + \left(\frac{\dot{g}}{g}\right)^2\left[1 - \frac{F'F'''}{F''^2}\right]\left[\frac{F'}{F''}\right] \tag{16}$$

where we substitute in (15). Further, note that

$$\begin{aligned} \dot{E} &= F(IH) - I\dot{H} - \sigma H \\ \ddot{E} &= F'(IH)I\dot{H} - I\ddot{H} - \sigma\dot{H} \\ &= \frac{1}{g}I\dot{H} - I\ddot{H} - \sigma\dot{H}. \end{aligned} \tag{17}$$

[2] A similar analysis follows when $\sigma > 0$ for the cases in which either $\dot{E(t)} > 0$ or $\dot{E(t)} < 0$

and from (10) obtain $\frac{\ddot{g}}{g} = r\frac{\dot{g}}{g}$. Thus,

$$\begin{aligned}
\ddot{E} &= -\frac{\dot{g}}{g}\frac{F'}{F''}\left[\frac{1}{g} + \frac{\dot{g}}{g}\left(1 - \frac{F'F'''}{F''^2}\right)\right] + \left[r\frac{\dot{g}}{g} - \left(\frac{\dot{g}}{g}\right)^2\right]\frac{F'}{F''} \\
&= -\frac{\dot{g}}{g}\frac{F'}{F''}\left[\frac{1}{g} + \frac{\dot{g}}{g}\left(1 - \frac{F'F'''}{F''^2}\right) - \frac{gr - \dot{g}}{g}\right] \\
&= -\frac{\dot{g}}{g}\frac{F'}{F''}\left[\frac{1}{g} + \frac{\dot{g}}{g}\left(1 - \frac{F'F'''}{F''^2}\right) - \frac{1}{g}\right] \\
&= -\left(\frac{\dot{g}}{g}\right)^2\frac{F'}{F''}\left(1 - \frac{F'F'''}{F''^2}\right) \qquad (18)
\end{aligned}$$

where the third equality uses (10), i.e. $gr - \dot{g} = 1$. F is strictly concave and therefore $-\left(\frac{\dot{g}}{g}\right)^2\frac{F'}{F''} > 0$. Since $\left(1 - \frac{F'F'''}{F''^2}\right) < 0$ the claim follows. ☒

Therefore, $E(t)$ is concave if and only if $\eta < 0$, which implies a necessary condition for concavity: $F''' > 0$.

Example 2.1 *(Human Capital Production Functions and Earnings Concavity)*

- *Power Production Function 1 : consider the case of* $F(x) = \frac{Ax^\alpha}{\alpha}$ *for* $-\infty < \alpha < 1, A > 0$. *Then,* $\eta = \frac{1}{\alpha-1} < 0$. *Under this specification the earnings function is strictly concave with respect to time.*

- *Power Production Function 2 : consider the case of* $F(x) = a - bx^{-\alpha}$ *for* $-1 < \alpha < \infty, a, b, c > 0$. *Then,* $\eta = \frac{-1}{\alpha+1} < 0$. *Under this specification the earnings function is strictly concave with respect to time.*

- *Power Production Function 3: consider the case of* $F(x) = a - b\exp^{-cx}$ *with* $b, c > 0$. *Then,* $\eta = 0$.

- *Quadratic Production Function: any quadratic production function has* $F''' = 0$ *and does not induce concavity of earnings with respect to time.*

Importantly, all of these examples consider no depreciation of human capital, $\sigma = 0$.

2.2 The Specialization Period

Specialization happens when the agent devotes her complete human capital to produce human capital stock, i.e. when $I(t) = 1$ for $t \in [\underline{t}, \bar{t}]$. In order to analyze some of the properties of specialization periods we assume away $D(t)$ so that $F_{D(t)} = 0$ and rule out depreciation.

Recall that we can interpret $g(t)$ as the return to investment in human capital and that we show above that $\dot{g} < 0$. Then, there is at most one period of specialization at the beginning of the time horizon, if it happens. We denote this by $[0, t^*]$. Ben-Porath (1967) calls this the schooling period and it happens under the conditions that follow.

Condition 2.2 *(Conditions for the Existence of a Period of Specialization in the Basic Ben-Porath Model with no Depreciation)*

$$\begin{aligned}
F'(H(t))g(t) &> R \quad \forall\, t \in [0, t^*) \\
F'(H(t^*))g(t^*) &= R \\
I(t) &= 1 \quad \forall\, t \in [0, t^*] \\
H(t^*) &= \int_0^{t^*} F(H(\tau))d\tau + H_0
\end{aligned} \tag{19}$$

where $H(t^*)$ is the human capital stock accumulated up to time t^*.

Given that R is fixed, any decrease in the function $g(t)$ for a given t lowers t^* because it lowers the return to gross investment in human capital. For example, relatively high r implies relatively low t^* because the individual is relatively present oriented. Also, from the system (19), note that a high value of H_0 implies a lower value for t^* because it takes less time to obtain $H(t^*)$. If $\sigma > 0$, similar conditions characterize the specialization period. However, there may be more than one specialization period because, under some scenarios, a high value of σ may knock off capital such that various investment episodes are optimal. We defer that case for Section 6.2.

Case 2.1 *(No Depreciation and the Cobb-Douglas Production Function for Human Capital: Initial Level of Human Capital) In this case $\dot{H} = A\,(IH)^{\alpha}$ where $0 < \alpha < 1, A > 0$. As argued above, if it exists, specialization happens in the period $[0, t^*]$. Thus*

$$\begin{aligned}
\alpha A\,(H(0))^{\alpha-1}\, g(0) &> R \\
&\Leftrightarrow \\
H(0) &< \left[\frac{R}{g(0)\alpha A}\right]^{\frac{1}{\alpha-1}}.
\end{aligned} \tag{20}$$

As the conditions in (19) establish, the time spent in specialization is a decreasing function of $H(0)$. In this example, actually, the initial human capital needs to be below certain threshold in order for the individual to specialize during one period.

Case 2.2 *(No Depreciation and the Cobb-Douglas Production Function for Human Capital with Infinite Horizon: Initial Level of Human Capital) In the setting of Case 2.1 and if the horizon of the problem is infinite: $H(0) < \left(\frac{\alpha A}{r}\right)^{\frac{1}{1-\alpha}}$ because $g(t) = \frac{R}{r}$.*

Case 2.3 *(No Depreciation and the Cobb-Douglas Production Function for Human Capital: the Specialization Period) In the period of specialization $I(t) = 1$. Then,*

$$\dot{H} = A\,(H)^{\alpha}. \tag{21}$$

The general solution for (21) is

$$H(t) = [(1-\alpha)(At + K)]^{\frac{1}{1-\alpha}} \tag{22}$$

for some constant K. Given an initial condition $H(0) = H_0$, $K = \frac{H_0^{1-\alpha}}{1-\alpha}$ and

$$H(t) = \left[(1-\alpha)At + H_0^{1-\alpha}\right]^{\frac{1}{1-\alpha}}. \tag{23}$$

At the end of the specialization period, as established in (19):

$$\alpha g(t^*) A \left(H(t^*)\right)^{\alpha-1} = R. \tag{24}$$

If $T \to \infty$, $g(t) = \frac{R}{r}$ and

$$t^* = -\frac{H_0^{1-\alpha}}{A(1-\alpha)} + \frac{\alpha}{1-\alpha}\frac{1}{r}. \tag{25}$$

(25) *provides some intuitive results: (i) an individual with relatively high initial human capital specializes during a relatively shorter period: $\frac{\partial t^*}{\partial H_0} < 0$; (ii) a relatively abler individual specializes during relatively longer period: $\frac{\partial t^*}{\partial A} > 0$; (iii) a relatively impatient individual specializes for a relatively shorter period: $\frac{\partial t^*}{\partial r} < 0$.*

Case 2.4 *(No Depreciation and the Cobb-Douglas Production for Human Capital: Post-school Earnings) Let $\tau = t - t^*$ define the post-school work experience and write post-school earnings as follows:*

$$E(\tau) = R\int_0^\tau \dot{H}(l + t^*)dl + RH(t^*) - RIH(\tau + t^*). \tag{26}$$

Now, from (19) the following equality holds:

$$\begin{aligned} \alpha g(t) A \left(IH(t)\right)^{\alpha-1} &= R \\ &\Leftrightarrow \\ IH(t) &= \left[\frac{\alpha g(t) A}{R}\right]^{\frac{1}{1-\alpha}} \end{aligned} \tag{27}$$

Combining (27) and the law of motion for human capital:

$$\dot{H} = A\left[\frac{\alpha g(t) A}{R}\right]^{\frac{\alpha}{1-\alpha}}. \tag{28}$$

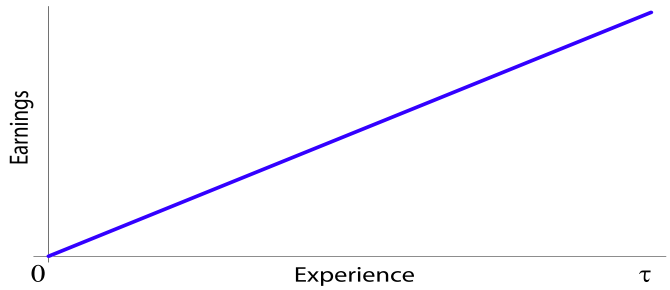
Then,

$$E(\tau) = R\int_0^\tau A\left[\frac{\alpha g(l+t^*)A}{R}\right]^{\frac{\alpha}{1-\alpha}} dl + RH(t^*) - R\left[\frac{\alpha g\left(\tau + t^*\right)A}{R}\right]^{\frac{1}{1-\alpha}} \tag{29}$$

and if $T \to \infty$

$$E(\tau) = RA\left[\frac{\alpha A}{r}\right]^{\frac{\alpha}{1-\alpha}}\tau. \tag{30}$$

Figure 1: Earnings and Experience, Cobb Douglas Technology and No Depreciation



When the time horizon is infinite, there is no concern with the reduction in time left for capturing returns to human capital investment and thus $g(t)$ is fixed overtime. Then when the solution is interior, the optimal choice on $I(t)H(t)$ is constant overtime, which implies the increase in $H(t)$ overtime is also a constant. This is why $E(t)$ increases at a constant rate as well. However, with a finite time horizon, the Cobb-Douglas production function with no depreciation implies a strictly concave earning function $E(t)$.

2.3 The Baseline Model Dynamics under the Cobb-Douglas Specification: a Summary

This section summarizes the dynamics of the main variables in the baseline model when there is no depreciation, market goods are ruled out, and the production function for human capital investment is Cobb-Douglas. We assume that the horizon is infinite to simplify the algebra but it is important to remark that the qualitative properties of the results remain unchanged under finite horizon. To wrap up the section we show simulations that illustrate how the variables of interest behave under various parameterizations (in all of them we set $R = 1$).

2.3.1 Human Capital

- At $t = 0$ an initial condition is given.
- At $0 < t < t^*$ the system (19) provides the conditions that human capital satisfies and its expression is given by (23).
- At $t = t^*$ (23) is still a valid expression for human capital. To obtain the exact quantity it suffices to substitute the expression for t^*, (25), into (23).
- At $t > t^*$ (19) and the expression for $\dot{H}$, (28), provide the expression for human capital.

Then,

$$H(t) = \begin{cases} H_0 & t = 0 \\ \left[(1-\alpha)At + H_0^{1-\alpha}\right]^{\frac{1}{1-\alpha}}, & 0 < t < t^* \\ \left[\frac{\alpha A}{r}\right]^{\frac{1}{1-\alpha}}, & t = t^* \\ A\left[\frac{\alpha A}{r}\right]^{\frac{\alpha}{1-\alpha}}(t-t^*) + \left[\frac{\alpha A}{r}\right]^{\frac{1}{1-\alpha}}, & t > t^*. \end{cases} \tag{31}$$

2.3.2 Investment

We focus on the case in which there is an specialization period, i.e. the case in which (20) holds. The combination of (27) and (31) gives the following

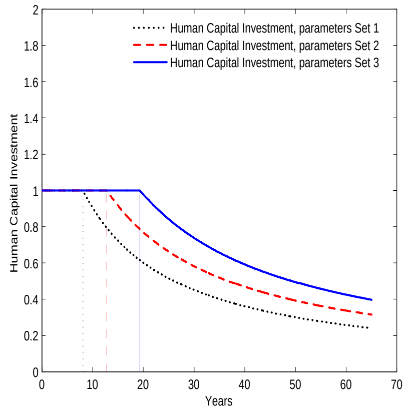
$$I(t) = \begin{cases} 1, & t = 0 \\ 1, & 0 < t < t^* \\ 1, & t = t^* \\ \frac{\left[\frac{\alpha A}{r}\right]^{\frac{1}{1-\alpha}}}{A\left[\frac{\alpha A}{r}\right]^{\frac{\alpha}{1-\alpha}}(t-t^*)+\left[\frac{\alpha A}{r}\right]^{\frac{1}{1-\alpha}}}, & t > t^*. \end{cases} \tag{32}$$

2.3.3 Earnings

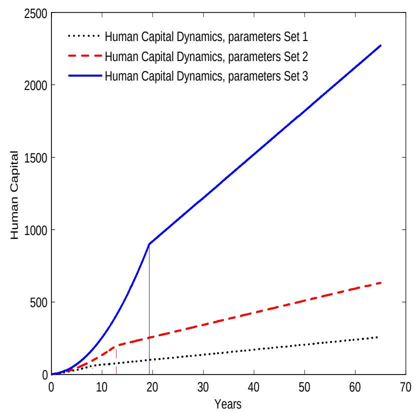
For the case of earnings we also focus on the case in which there is an specialization period, i.e. the case in which (20) holds. Thus, (1), (31), (32) define earnings as follows

$$E(t) = \begin{cases} 0, & t = 0 \\ 0, & 0 < t < t^* \\ 0, & t = t^* \\ RA\left[\frac{\alpha A}{r}\right]^{\frac{\alpha}{1-\alpha}}(t-t^*), & t > t^*. \end{cases} \tag{33}$$

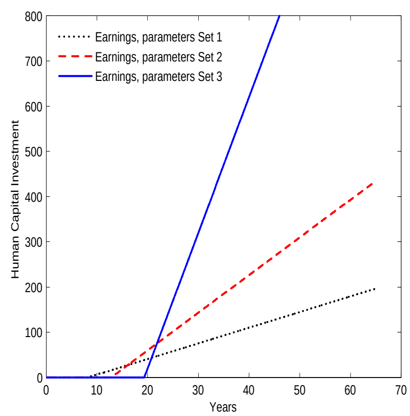
Figure 2: Dynamics with Variations in a Production Technology Parameter
$\alpha = .3$ (dotted); $\alpha = .4$ (dashed); $\alpha = .5$ (solid)
for $A = 3, r = .05, H_0 = 1$



(a) Human Capital Investment

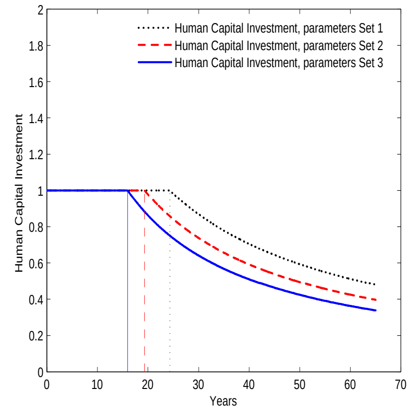


(b) Human Capital Stock

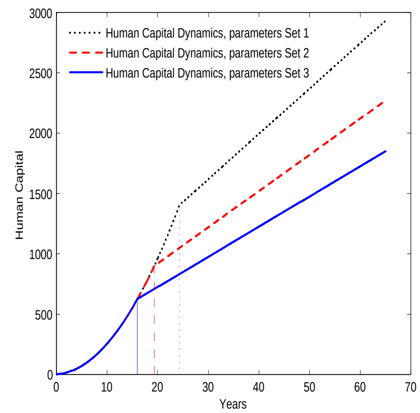


(c) Earnings

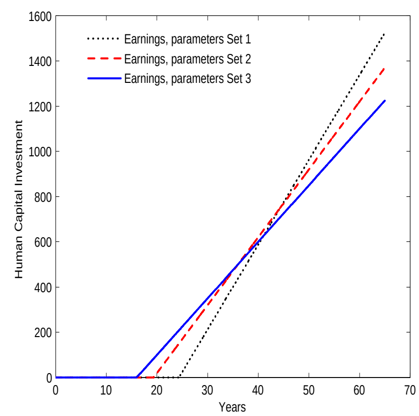
Figure 3: Dynamics with Variations in the Discounting Factor
$r = .04$ (dotted); $r = .05$ (dashed); $r = .06$ (solid)
for $A = 3, \alpha = .5, H_0 = 1$



(a) Human Capital Investment

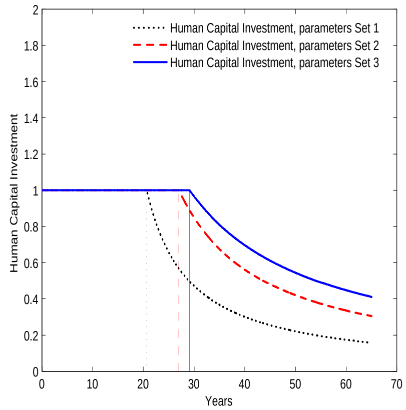


(b) Human Capital Stock

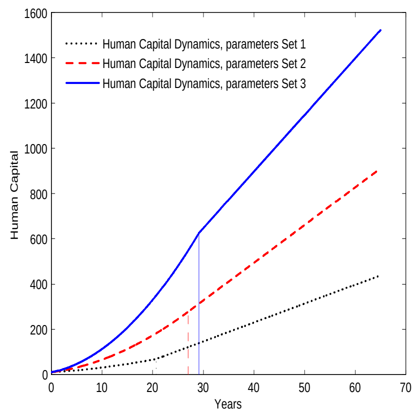


(c) Earnings

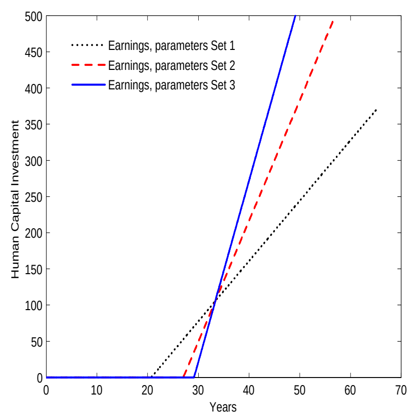
Figure 4: Dynamics with Variations in a Production Technology Parameter
$A = .5$ (dotted); $A = 1.0$ (dashed); $A = 1.5$ (solid)
for $r = .03, \alpha = .5, H_0 = 10$



(a) Human Capital Investment

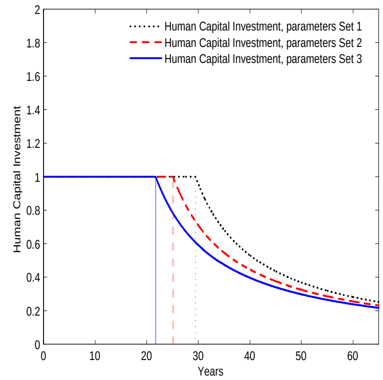


(b) Human Capital Stock

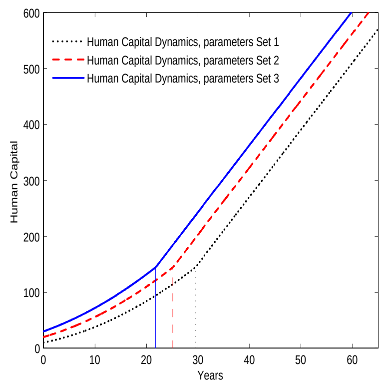


(c) Earnings

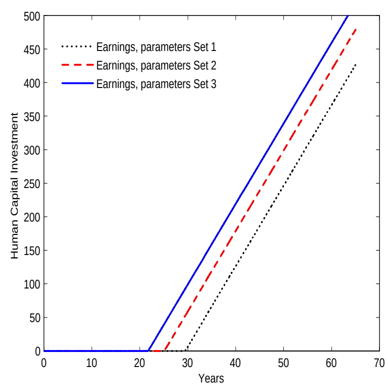
Figure 5: Dynamics with Variations in the Initial Level of Human Capital
$H_0 = 10$ (dotted); $H_0 = 20$ (dashed); $H_0 = 30$ (solid)
for $r = .025, \alpha = .5, A = .6,$



(a) Human Capital Investment



(b) Human Capital Stock



(c) Earnings

2.4 Rates of Return under the Cobb-Douglas Specification

We use this model to analyze returns both to schooling and post-schooling. In order to simplify the expressions we let $t \to \infty$. Similar implications hold for the finite time horizon problem.

2.4.1 Return to Schooling

We call schooling the period of specialization in which the individual devotes his complete human capital stock to produce new human capital. To define the return to schooling consider two scenarios: (i) the individual does not invest either in schooling or in post-schooling. Then in each period t she earns RH_0; (ii) the individual invests in schooling and does not make any post-schooling investments after that. Then in each after-schooling period τ she earns $R\left[\frac{\alpha A}{r}\right]^{\frac{1}{1-\alpha}}$. We define the (internal) rate of return of schooling as the discount rate at which the present values of the disposable income streams in the two scenarios are equal.

Definition 2.2 *("Internal" Rate of Return to Schooling) φ is the (internal) rate of return to schooling and solves the equation*

$$\begin{aligned}
\int_{t^*}^{\infty} \exp^{-\varphi t} R\left[\frac{\alpha A}{r}\right]^{\frac{1}{1-\alpha}} dt &= \int_0^{\infty} \exp^{-\varphi t} RH_0 dt \\
&\Rightarrow \\
\varphi &= \frac{\ln\left[\frac{\alpha A}{r}\right]^{\frac{1}{1-\alpha}} - \ln H_0}{-\frac{H_0^{1-\alpha}}{A(1-\alpha)} + \frac{\alpha}{1-\alpha}\frac{1}{r}}. \qquad (34)
\end{aligned}$$

2.4.2 Return to Post-schooling

Let $E(\tau)^{NPS}$ and $E(\tau)^{PS}$ denote earnings without and with post-schooling investment, respectively. By (33) we can write

$$\begin{aligned}
E(\tau)^{NPS} &= RH(t^*) \qquad (35)\\
&= R\left(\frac{\alpha A}{r}\right)^{\frac{1}{1-\alpha}} \\
E(\tau)^{PS} &= RA\left[\frac{\alpha A}{r}\right]^{\frac{\alpha}{1-\alpha}} \tau
\end{aligned}$$

so that the increment in earnings due to post-schooling at τ is

$$\Delta^{E(\tau)} \equiv E(\tau)^{PS} - E(\tau)^{NPS}. \qquad (36)$$

We can interpret $\Delta^{E(\tau)}$ as "returns less costs" from post-schooling, with $E(\tau)^{NPS}$ as the costs (i.e. foregone earnings) of post-schooling investments. Then, we define the (internal) rate of return to post-schooling as follows.

Definition 2.3 *("Internal" Rate of Return to Post-schooling) ϕ is the (internal) rate of return to schooling and solves the equation*

$$\int_0^{\infty} \exp^{-\phi\tau}\left[RA\left[\frac{\alpha A}{r}\right]^{\frac{\alpha}{1-\alpha}} \tau - R\left(\frac{\alpha A}{r}\right)^{\frac{1}{1-\alpha}}\right] d\tau = 0 \qquad (37)$$

Using the Laplace transform, (37) *implies*

$$\begin{aligned}\frac{1}{\phi^2}RA\left[\frac{\alpha A}{r}\right]^{\frac{\alpha}{1-\alpha}}-\frac{R}{\phi}A\left[\frac{\alpha A}{r}\right]^{\frac{1}{1-\alpha}} &= 0 \\ &\Rightarrow \\ \phi &= \frac{r}{\alpha}. \end{aligned} \tag{38}$$

The (internal) rate of return to post-schooling investment is a decreasing function of α. Recall that the internal rate of return of post-schooling measures the desirability of the investment opportunity. And with a higher α, individuals' level of human capital is relatively high even without any post-schooling investment, and thus the relative differences (measured by the ratios) between the amount of disposable earnings in the case with investment and in the case without investment are smaller. Therefore, investment in post-schooling is less desirable for more productive individuals. In addition, relatively patient individuals (who have relatively low r) require a smaller discount rate ϕ to equalize the scenario with post-schooling investment and the scenario without.

2.5 Earnings Growth and Patience in Finite Horizon

In this section we want to ask, in the same framework, how earnings growth depend on what defines relative patience in this model, the discount rate r. To do that, we investigate $\frac{\partial \dot{E(\tau)}}{\partial r}$.

Claim 2.4 *Assume that* $1-\frac{F'(\cdot)F'''(\cdot)}{F''^2}<0$ *(recall from Claim 2.3 that this is a sufficient condition for* $\ddot{E(t)}<0$ *in the current context). Then,* $\frac{\partial \dot{E(\tau)}}{\partial r}<0$.

Proof: Without loss of generality, assume that R =1 and note that

$$\frac{\partial \dot{E(\tau)}}{\partial r}=F'(\cdot)\frac{\partial IH}{\partial r}-\frac{\partial}{\partial r}\dot{IH}. \tag{39}$$

From (12) we know that the first order condition of the agent's problem is

$$g(t)F'(\cdot)=1 \tag{40}$$

which by the implicit function theorem yields

$$\begin{aligned}\frac{\partial IH}{\partial r} &= \frac{\frac{\partial g(t)}{\partial r}F'(\cdot)}{2g(t)F''(\cdot)} \\ &< 0\end{aligned} \tag{41}$$

where the inequality follows from strict concavity of $F(\cdot)$ and $g(t)>0, \frac{\partial g(t)}{\partial r}<0$ (see (44)). Thus, the first term in (39) is negative. If we show that the second term is negative then we can sign (39) and provide a meaning for this results. In order to do that we need $\frac{\partial \dot{IH}}{\partial r}>0$. From (15) note that

$$\frac{\partial \dot{IH}}{\partial r}=-\frac{\dot{g}}{g}\left[1-\frac{F'(\cdot)F'''(\cdot)}{F''(\cdot)^2}\right]\frac{\partial IH}{\partial r}+\frac{F'(\cdot)}{F''(\cdot)}\frac{\partial}{\partial r}\left[-\frac{\dot{g}}{g}\right] \tag{42}$$

We know that $1-\frac{F'(\cdot)F'''(\cdot)}{F''^2}<0$ and $\dot{g}, \frac{\partial IH}{\partial r}<0$ the first term in (42) is positive. To sign the second term note that $\dot{g}=rg-1, -\frac{\dot{g}}{g}=\frac{1}{g}-r$. Then,

$$\frac{\partial}{\partial r}\left[-\frac{\dot{g}}{g}\right]=-\frac{1}{g^2}\frac{\partial g}{\partial r}-1. \tag{43}$$

To sign (43) note that

$$\begin{aligned}\frac{\partial g}{\partial r} &= \frac{\exp^{r(t-T)}\left(1-r(t-T)\right)-1}{r^2}\\ &< 0\end{aligned} \tag{44}$$

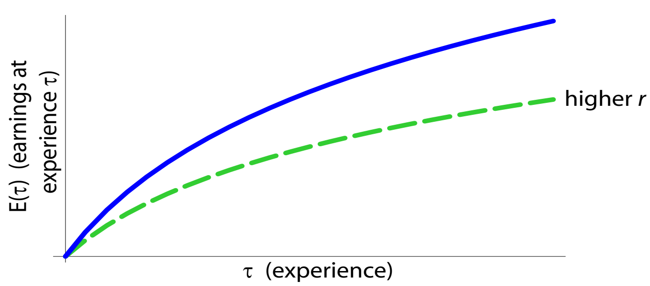
and

$$\begin{aligned}-\frac{\partial g}{g^2\partial r}-1 &= \frac{1}{r^2g^2}\exp^{r(t-T)}\left(1+r(t-T)-\exp^{r(t-T)}\right).\\ &< 0\end{aligned} \tag{45}$$

which implies that $\frac{\partial \dot{E}}{\partial r}<0$. ⊠

The graphical representation of Claim 2.4 is in Figure 6. It implies that the earnings function is relatively "less concave" for relatively impatient individuals (relatively high r). This is a consequence of their investment decisions: they spend less time in the schooling period and accumulate less human capital.

Figure 6: Earnings Profiles in Finite Horizon for Different Values of r



3 The Haley-Rosen Specification: Finite Horizon and the Autoregression Form

We analyze the finite horizon case under the specification that Haley (1976) and Rosen (1976) use. Specifically, we assume that $\dot{H}=A\left(IH\right)^{\alpha}, \alpha=\frac{1}{2}, \sigma=0$ and the exact same setting as in Section

2. Actually, in Section 2 we rely on infinite horizon to derive a set of closed form solutions to the individual's problem. In this section we relax the infinite horizon assumption and rely on the assumption $\alpha = \frac{1}{2}$ to gain the tractabiliy.

We focus on the dynamics of post-schooling earnings because one of the less credible consequence of the infinite horizon is the linearity of earnings on experience. From (29) we can write

$$\begin{aligned} E(\tau) &= RH(t^*) + R\int_0^\tau A\left[\frac{1}{2}\frac{g(t^*+l)A}{R}\right]dl - R\left[\frac{1}{2}\frac{g(t^*+\tau)A}{R}\right]^2 \\ &\Rightarrow \\ \dot{E(\tau)} &= \frac{g(t^*+\tau)A^2}{2R}\left(2R - rg(t^*+\tau)\right) \\ &\Rightarrow \\ \ddot{E(\tau)} &= -\frac{A^2}{R}g(t^*\dot{+}\tau)^2 \end{aligned} \tag{46}$$

where the second and third equalities use (10). Combining (10) and (46) we obtain a second order ODE with constant coefficients:

$$\ddot{E(\tau)} = 2r\dot{E(\tau)} - A^2R \tag{47}$$

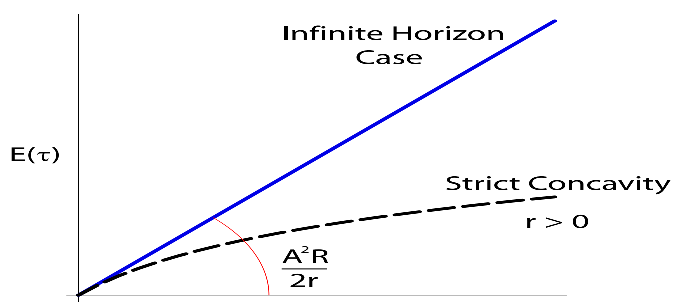
where the natural initial and terminal conditions that we impose are $E(0) = 0$ and $\dot{E(T)} = 0$ and then we guess and verify that the general solution to (47) is as the following.

$$E(\tau) = c_0 + c_1 \exp^{-2r\tau} + c_2\tau \tag{48}$$

So $E(0) = 0$ implies $c_1 + c_0 = 0$ and $\dot{E(T)} = 0$ implies $2rc_1 \exp^{2rT} + c_2 = 0$. Together with (47), we can solve for $c_0 = \frac{A^2R}{4r^2\exp^{2rT}}$, $c_1 = -c_0$, and $c_2 = \frac{A^2R}{2r}$. Therefore,

$$E(\tau) = \frac{A^2R}{4r^2}\exp^{-2rT}\left(1 - \exp^{2r\tau}\right) + \frac{A^2R}{2r}\tau. \tag{49}$$

Figure 7: Post-school Earnings in the Haley-Rosen Specification



Note:

From (46), we know that in the finite horizon case, the earnings function is strictly concave unless $t = T$. The intuition behind the linearity of the earnings function in the infinite horizon case is provided in Section 2.2. In contrast with the infinite case, the return to investment is decreasing over time as the agent is approaching to the final period of the time horizon, which implies that $I(t)H(t)$ is decreasing overtime. Thus the stock of human capital is increasing at a decreasing rate, which leads to the concavity of the earnings dynamics.

3.1 The Autoregression

From (49) it is possible to write

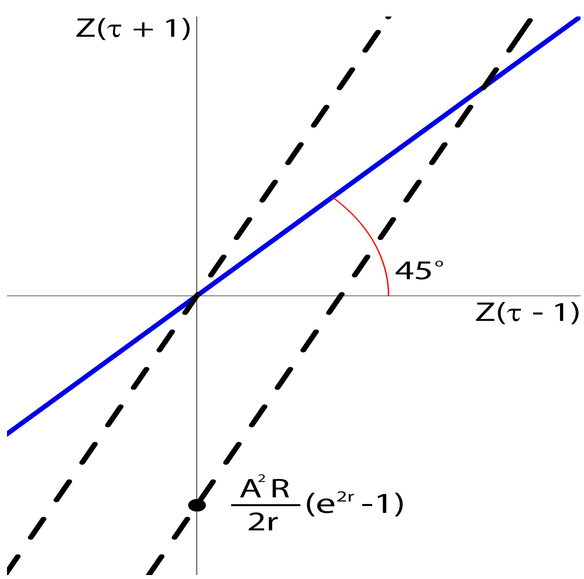
$$E(\tau+1) - E(\tau) = \frac{A^2R}{2r} + \frac{A^2R}{4r^2}\exp^{-2rT}\left(\exp^{2r\tau} - \exp^{2r(\tau+1)}\right) \tag{50}$$

which implies that

$$z(\tau+1) = \exp^{2r} z(\tau) + \frac{A^2R}{2r}\left(1 - \exp^{2r}\right) \tag{51}$$

where $z(\tau) \equiv E(\tau+1) - E(\tau)$ and we can analyze the growth dynamics of earnings. Consider a visual representation of (51)

Figure 8: Earnings Growth in the Haley-Rosen Representation



Note:

Apparently, the dynamics of the earnings growth are explosive. However, note that

$$\begin{aligned}\frac{\partial\left[E(\tau) - E(\tau-1)\right]}{\partial\tau} &= \frac{A^2R}{2r}\exp 2r(\tau - T)\left[\exp^{-2r} - 1\right] \\ &< 0\end{aligned} \tag{52}$$

so that even when the growth dynamics of earnings is explosive, the earnings dynamics, $E(t)$, can converge over time.

3.2 From the Haley-Rosen Specification to the Mincer Equation

The earnings function in the Haley-Rosen specification actually lead to the Mincer equation. To see that take logs of (49) and obtain

$$\ln E(\tau) = \ln\left(\frac{A^2 R}{2r}\right) + \ln\tau + \ln\left[1 + \frac{\exp^{-2rT} - \exp^{2r(\tau-T)}}{2r\tau}\right]. \tag{53}$$

We can approximate around τ_0 the second and third terms in (53) to obtain

$$\begin{aligned}
\ln(\tau) &\approx \ln(\tau_0) + \frac{1}{\tau_0}(\tau-\tau_0) - \frac{1}{\tau_0^2}\frac{(\tau-\tau_0)^2}{2!} \\
\ln\left[1 + \frac{\exp^{-2rT} - \exp^{2r(\tau-T)}}{2r\tau}\right] &\approx \xi_0 + \xi_1(\tau-\tau_0) + \xi_2\frac{(\tau-\tau_0)^2}{2!}
\end{aligned} \tag{54}$$

for the adequate ξ_0, ξ_1, ξ_2. Thus,

$$\ln(\tau) + \ln\left[1 + \frac{\exp^{-2rT} - \exp^{2r(\tau-T)}}{2r\tau}\right] \approx \alpha_0 + \alpha_1(\tau-\tau_0) + \alpha_2(\tau-\tau_0)^2 \tag{55}$$

with $\alpha_0 \equiv \ln(\tau_0) + \xi_0, \alpha_1 \equiv \frac{1}{\tau_0} + \xi_1, \alpha_2 \equiv \frac{-\frac{1}{\tau_0^2}+\xi_2}{2}$. This leads to the so called Mincer equation (see Mincer, 1974):

$$\ln E(\tau) = k_0 + k_1\tau + k_2\tau^2 \tag{56}$$

where $k_0 = \alpha_0 - \tau_0\alpha_1 + \alpha_2\tau_0^2, k_2 = \alpha_2$. Also, it provides a baseline to compare "Ben-Porath" with "Mincer" coefficients. Table 1 provides different combinations of the parameters r, τ_0, T that lead to different values of k_1, k_2 that are close to the estimates that Mincer (1974) obtains.

Table 1: The Ben-Porath and the Mincer Coefficients

Parameters			Ben Porath Coefficients	
r	τ_0	T	k_1	k_2
0.0225	29.54	41.43	0.081	−0.0010
0.05	25	60	0.0808	−0.0008
0.05	20	65	0.1002	−0.0013
0.0675	24.70	74.77	0.081	−0.0008
Mincer Coefficients			0.081	−0.0012

Note: the Mincer model or Mincer equation is $\ln(\mathrm{E}) = k_0 + k_1\tau + k_2\tau^2$, where τ is experience.

Now, if $rT \approx 0$ then $\exp^{-rT} \approx 1$ and (53) becomes

$$\ln E(\tau) \approx \ln\left(\frac{A^2 R}{2r}\right) + \ln\tau + \ln\left[1 + \frac{1 - \exp^{2r\tau}}{2r\tau}\right] \tag{57}$$

which leads to various observations. The Haley-Rosen specification of the Ben-Porath model implies no economic content for the Mincerian rate of return on post-school investment. Put differently, an extension of (56) which includes post-school investment does not have a structural counterpart. Actually, this model implies that the entire economic content is in the intercept (see (57).(57) implies that, *caeteris paribus*, schooling has no effect on earnings. Mincer (1974) finds that the contrary holds. However, we claim that his finding does not necessarily argue against the Ben-Porath model. It could simply be the case that Mincer (1974) does not include ability measures in his estimations, which appear in (57), and therefore finds a positive coefficient on schooling.

4 Generalized Ben-Porath Model

We now generalize the model in Section 2 by relaxing the neutrality assumption so that the production function of human capital is more general. We focus our analysis on specialization because the analysis of other conditions is very similar to that of Section 2.

In particular, the law of motion for human capital stock in the generalized Ben-Porath model is

$$\dot{H} = AI^{\alpha}H^{\beta} - \sigma H. \tag{58}$$

So the model in Section 2 is a particular case of this general formulation when $\alpha = \beta$. To simplify the analysis of the implications of this model we assume that there is neither discounting nor depreciation, i.e. $r = \sigma = 0$. To ease notation we neglect the argument t when possible. We analyze this model in finite horizon.

The Hamiltonian of the problem is

$$\mathcal{H} = RH(t)\left(1 - I(t)\right) + \mu(t)\left(AI(t)^{\alpha}H(t)^{\beta}\right) \tag{59}$$

where $\mu(t)$ defines the shadow price of human capital. The following condition must be satisfied for interior solutions.

Condition 4.1 *(Optimality Conditions for the Life-Cycle Individual's Problem in the Generalized Ben-Porath Model)*

$$\frac{\partial \mathcal{H}(\cdot)}{\partial I(t)} = 0 \quad \Leftrightarrow \quad \mu(t)A\alpha I(t)^{\alpha-1}H(t)^{\beta} = RH(t) \tag{60}$$

$$\frac{\partial \mathcal{H}(\cdot)}{\partial H(t)} = -\dot{\mu(t)} \quad \Leftrightarrow \quad -R(1 - I(t)) - \beta\mu(t)AI(t)^{\alpha}H(t)^{\beta-1} = \dot{\mu(t)} \tag{61}$$

$$\frac{\partial \mathcal{H}(\cdot)}{\partial \mu(t)} = \dot{H} \quad \Leftrightarrow \quad \dot{H(t)} = AI(t)^{\alpha}H(t)^{\beta} \tag{62}$$

$$\textit{Transversality} \quad : \quad \lim_{t \to T} \mu(t)H(t) = 0 \tag{63}$$

Condition 4.1 is equivalent to the Mangasarian sufficient conditions for a global optimum if $\beta \leq 1$ (see Mangasarian, 1966).

4.1 Specialization

If $\frac{\partial \mathcal{H}(\cdot)}{\partial I(t)} > 0$ with $I(t) = 1$, the agent would specialize. Thus the condition that guarantees specialization is as follows.

Condition 4.2 *(Conditions for Specialization in the Generalized Ben-Porath Model)*

$$\textit{Conditions for Specialization} : \begin{cases} H > \left[\frac{R}{\alpha A\mu}\right]^{\frac{1}{\beta-1}}, & \beta > 1 \\ 1 > \left[\frac{R}{\alpha A\mu}\right], & \beta = 1 \\ H < \left[\frac{R}{\alpha A\mu}\right]^{\frac{1}{\beta-1}}, & \beta < 1. \end{cases} \tag{64}$$

During the period(s) of specialization (61), (62) become

$$\dot{\mu} = -\beta\mu A H^{\beta-1} \tag{65}$$
$$\dot{H} = AH^{\beta} \tag{66}$$

and we can solve for the dynamics of human capital stock in this region

$$H(t) = \begin{cases} c_0 \exp^{At}, & \beta = 1 \\ (At + c_1)^{\frac{1}{1-\beta}} (1-\beta)^{\frac{1}{1-\beta}}, & \beta \neq 1. \end{cases} \tag{67}$$

The initial condition for the human capital stock leads to $c_0 = H_0$ and $c_1 = \frac{H_0^{1-\beta}}{1-\beta}$ which implies that

$$H(t) = \begin{cases} H_0 \exp^{At-1}, & \beta = 1 \\ \left(At + \frac{H_0^{\frac{1}{1-\beta}}}{1-\beta} \right)^{\frac{1}{1-\beta}} (1-\beta)^{\frac{1}{1-\beta}}, & \beta \neq 1. \end{cases} \tag{68}$$

Also, we can solve (65) and find that

$$\mu(t) = \begin{cases} k_0 \exp^{-At}, & \beta = 1 \\ \frac{k_1}{(At+c_1)^{\frac{\beta}{1-\beta}}}, & \beta \neq 1 \end{cases} \tag{69}$$

for which there is an exact solution given an initial condition $\mu(0) = \mu_0$. This is, we can find k_0, k_1 in (69) provided $\mu_0 > 0$ (it is a price). In particular, note that $k_0 = \mu_0 > 0$ and $k_1 = \mu_0 c_1^{\frac{\beta}{1-\beta}} > 0$ for $0 < \beta < 1$.

Let t^* denote the time when specialization ends. It must be true that, then, (60) holds with strict equality

$$\mu(t^*) A\alpha H(t^*)^{\beta} = RH(t^*) \tag{70}$$

which implies that

$$t^* = \frac{1}{A}\left(\ln\left[\frac{A\alpha}{R} + \ln k_0\right]\right) \tag{71}$$

for $\beta = 1$. For $\beta \neq 1$, t^* solves

$$\frac{k_1}{(At^* + c_0)^{\frac{\beta}{\beta-1}}} \frac{A\alpha}{R} = \left[At^* (1-\beta)^{\frac{1}{1-\beta}} + H_0^{1-\beta} (1-\beta)^{\frac{\beta}{1-\beta}} \right]^{1-\beta}. \tag{72}$$

To wrap up the discussion we ask if the period of specialization is unique for some particular cases.

Claim 4.1 *(Uniqueness of the Specialization Period) If the period of specialization exists, it is unique when either when $\beta = 1$ or when $\beta \in [0, 1]$.*

Proof: In both cases (69) implies that $\dot{\mu(t)} < 0$. Importantly, $\mu(t)$ is the shadow price or value of human capital. Thus, $\dot{I(t)} < 0$ and, if it exists, the period of specialization is unique. ☒

5 The Basic Sheshinski Specification: Bang-Bang Equilibria

The Basic Sheshinski specification is a particular case of the Generalized Ben-Porath model in Section 4 in which $\alpha = \beta = 1$.

Definition 5.1 *(Law of Motion for Human Capital Stock in the Basic Sheshinski Specification)*

$$\dot{H(t)} = AI(t)H(t) - \sigma H(t). \tag{73}$$

Proceeding as in Section 2 and Section 4 we can write down the current value Hamiltonian and obtain the following optimality conditions for the interior solution.

Condition 5.1 *(Optimality Conditions for the Life-cycle Individual's Problem in the Basic Sheshinski Specification)*

$$\frac{\partial \mathcal{H}(\cdot)}{\partial I(t)} = 0 \quad \Leftrightarrow \quad \mu(t) \exp^{rt} = \frac{R}{A} \tag{74}$$

$$\frac{\partial \mathcal{H}(\cdot)}{\partial H(t)} = -\dot{\mu(t)} \quad \Leftrightarrow \quad -\exp^{-rt} R(1 - I(t)) - \mu(t) \left(AI(t) - \sigma \right) = \dot{\mu(t)} \tag{75}$$

$$\frac{\partial \mathcal{H}(\cdot)}{\partial \mu(t)} = \dot{H(t)} \quad \Leftrightarrow \quad \dot{H(t)} = AI(t)H(t) - \sigma H(t) \tag{76}$$

$$\textit{Transversality} \quad : \quad \lim_{t \to T} \mu(t)H(t) = 0 \tag{77}$$

Claim 5.1 *(Bang-Bang in the Sheshinski Specification) Assume that $\sigma + r < A$ and that there is an initial period of specialization.*[3] *Then, the solution to the problem is Bang-Bang, i.e. either $I = 0$ or $I = 1$.*

[3]Note that $\sigma + r > A$ implies that $\dot{g(t)} > 0$ and this violates the transversality condition.

Proof: Define $g(t) = \mu(t)\exp^{rt}$ and use (75), (77) to obtain

$$\begin{aligned} \dot{g} &= -R + (R - Ag)I + (\sigma + r)g && (78) \\ g(T) &= 0. && (79) \end{aligned}$$

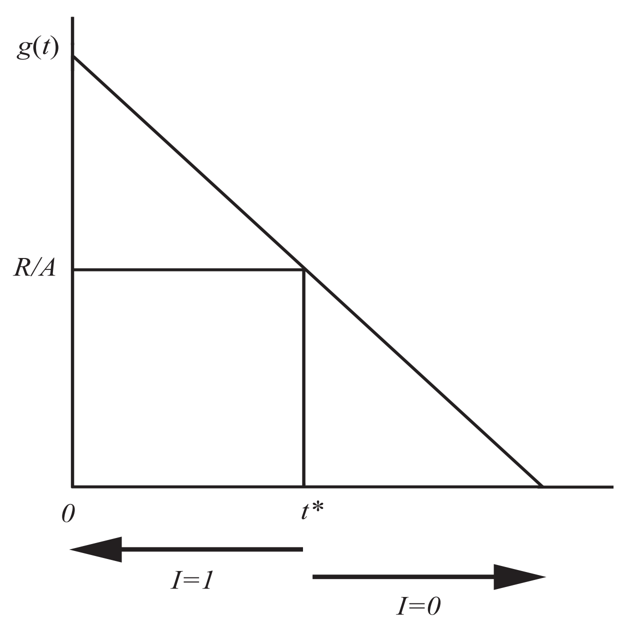
In the specialization period $I(t) = 1$. If $\sigma + r < A$, $\dot{g(t)} < 0$. Actually, by (74), as $g(t)$ decreases to $\frac{R}{A}$, $I(t)$ switches from it's upper bound 1 to it lower bound 0. Then, with $I(t) = 0$ we can use $g(T) = 0$ and write

$$\begin{aligned} \dot{g(t)} &= (\sigma + r)g(t) - R \\ &\Rightarrow \\ g(t) &= \frac{R}{\sigma + r}\left[1 - \exp^{(\sigma+r)(t-T)}\right]. && (80) \end{aligned}$$

for which $\dot{g(t)} < 0$ as well. Therefore, once $I(t)$ reaches zero it never goes back again to a positive value. This formulation has a Bang-Bang solution. ⊠

It follows that the schooling period, if it exists, is unique and at the beginning of the investment cycle. If it does not exist the individual does not invest at all in human capital. Figure 9 is a graphical representation of Claim 5.1.

Figure 9: Bang-Bang Equilibrium in the Basic Sheshinski Specification



Note:

We can actually solve for t^*, the length of the schooling period, using the fact that $g(t^*) = \frac{R}{A}$ by (74) and $g(t^*) = \frac{R}{\sigma+r}\left[1 - \exp^{(\sigma+r)(t^*-T)}\right]$ by (80):

$$\begin{aligned}
\frac{R}{A} &= \frac{R}{\sigma+r}\left[1 - \exp^{(\sigma+r)(t^*-T)}\right] \\
&\Leftrightarrow \\
t^* &= \frac{1}{\sigma+r}\ln\frac{A-(\sigma+r)}{A} + T.
\end{aligned} \tag{81}$$

Thus, (i) longer life horizons imply more schooling, $\frac{\partial t^*}{\partial T} > 0$; (ii) greater depreciation implies less schooling, $\frac{\partial t^*}{\partial \sigma} < 0$; (iii) higher relative impatience implies less schooling, $\frac{\partial t^*}{\partial r} < 0$; (iv) higher productivity implies more schooling, $\frac{\partial t^*}{\partial A} > 0$; (v) initial human capital does not affect schooling, $\frac{\partial t^*}{\partial H_0} = 0$.

5.1 From the Basic Sheshinski Specification to the Mincer Equation

Assume that there is a period of specialization. From (66) we know that in the period $[0, t^*]$

$$\begin{aligned}
\dot{H(t)} &= (A-\sigma)H(t) \\
&\Rightarrow \\
H(t) &= H_0 \exp^{(A-\sigma)t}.
\end{aligned} \tag{82}$$

At t^*, actually, $I(t) = 0$ so earnings are $Y(t) = RH(t^*)$ Then,

$$\ln Y(t^*) = \ln(RH_0) + (A-\sigma)t^*. \tag{83}$$

According to this model, the returns to schooling, $A - \sigma$, are given by the productivity of the human capital less the human capital depreciation rate.

6 The Modified Sheshinski Specification

The last variation to the Ben-Porath model that we consider allows for human capital investment cycles. This is, it allows for investment in human capital stock to happen in different episodes.

Definition 6.1 *(Law of Motion for Human Capital in the Modified Sheshinski Specification)*

$$\dot{H} = AI - \sigma H. \tag{84}$$

Note that the human capital production function does not depend on $H(t)$. The optimality conditions for an interior solution in this model are the following.

Condition 6.1 *(Optimality Condition for the Life-cycle Individual's Problem in the Modified Sheshinski Specification)*

$$\frac{\partial \mathcal{H}(\cdot)}{\partial I(t)} = 0 \quad \Leftrightarrow \quad \mu \exp^{rt} = \frac{RH}{A} \tag{85}$$

$$\frac{\partial \mathcal{H}(\cdot)}{\partial H(t)} = -\dot{\mu(t)} \quad \Leftrightarrow \quad \dot{\mu} = \mu\sigma - \exp^{-rt} R(1-I) \tag{86}$$

$$\frac{\partial \mathcal{H}(\cdot)}{\partial \mu(t)} = \dot{H(t)} \quad \Leftrightarrow \quad \dot{H(t)} = AI - \sigma H \tag{87}$$

$$\textit{Transversality} \quad : \quad \lim_{t \to T} \mu(t)H(t) = 0. \tag{88}$$

6.1 No Depreciation: a Schooling Model

Define $g(t) = \mu(t)\exp^{rt}$ and use (86) to obtain

$$\dot{g} = g(\sigma + r) - R(1 - I). \tag{89}$$

Let $\sigma = 0$. Then $\dot{g} = -R(1-I) + rg$. And $\dot{H} = A$ when $I = 1$. So the solution for the human capital trajectory when $I = 1$ is

$$H(t) = At + H_0. \tag{90}$$

At $t = 0$, $I = 1$ if $g(0) > \frac{R}{A}H_0$. Importantly, $I = 1$ implies that $\dot{g(t)} = rg(t) > 0$. As t grows, the return for gross investment grows because the payoff period gets closer. $I = 1$ cannot be a solution forever because the agent receives no earnings if she invests all of the time during the complete life-cycle.

Claim 6.1 *(Uniqueness of the Period of Specialization in the Modified Sheshinski Specification with no Depreciation) If the specialization period exists then it is unique.*

Proof: Based on (85), if a specialization period exists and if $g(t) - \frac{RH(t)}{A}$ is strictly decreasing overtime, then the specialization period must occur at the beginning of the life cycle and is unique. So in the following we show that $\dot{g(t)} - \frac{R}{A}\dot{H(t)} < 0$.

Given that $\dot{g(t)} = rg(t) - R(1 - I(t))$ and $g(T) = 0$, we have:

$$g(t) = R \int_t^T (1 - I(s)) \exp^{r(t-s)} ds \tag{91}$$

Then taking derivative with respect to time gives:

$$\dot{g(t)} = R\left[-1 + I(t) + r\int_t^T (1 - I(s)) \exp^{r(t-s)} ds\right] \tag{92}$$

Together with (84), we have:

$$
\begin{aligned}
\dot{g(t)} - \frac{R}{A}\dot{H(t)} &= -R + Rr\int_t^T (1 - I(s)) \exp^{r(t-s)} ds &(93)\\
&\leq -R + Rr\int_t^T \exp^{r(t-s)} ds &(94)\\
&= -R \exp^{r(t-T)} &(95)\\
&< 0, &(96)
\end{aligned}
$$

where the first inequality follows from setting $I(\tau) = 0$. ☒

Finally, to compute the optimal schooling length, t^*, note that (85) holds with strict equality at t^* and (90) is valid so that

$$g(t^*) = \frac{R}{A}\left(At^* + H_0\right). \tag{97}$$

(91) is also valid for t^*. Then,

$$\left(1 - \exp^{r(t^*-T)}\right) = \frac{r}{A}\left(At^* + H_0\right) \tag{98}$$

and thus $\frac{\partial t^*}{\partial H_0} < 0$, $\frac{\partial t^*}{\partial A} > 0$ and $\frac{\partial t^*}{\partial r} < 0$ as in the model of Section 2.

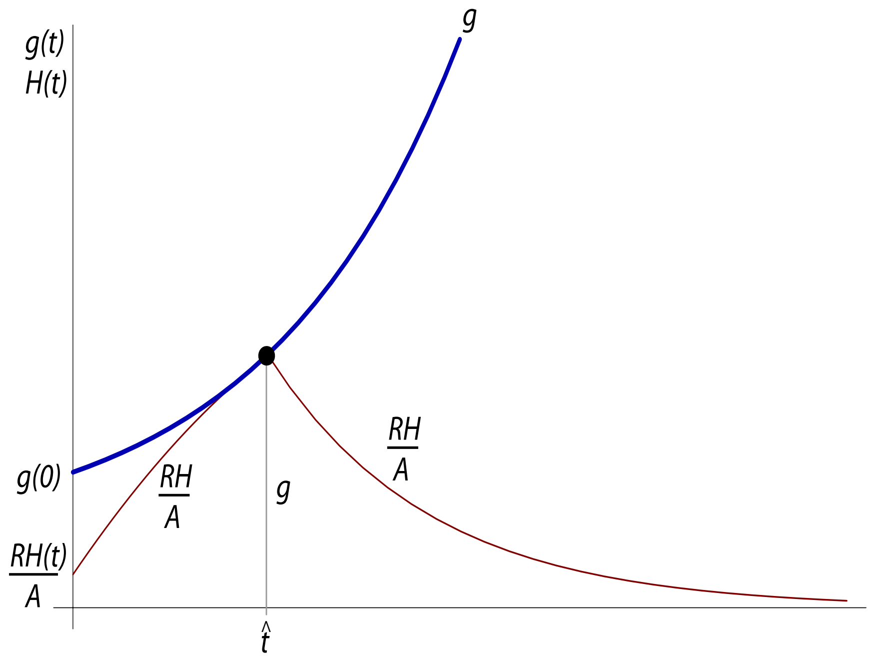
6.2 Depreciation

Let us give some conditions under which human capital investment would have different episodes over the life cycle. First assume that $g(0) > \frac{H_0 R}{A}$ so that there is a specialization period to begin with. We can solve (87) and (89) to obtain

$$
\begin{aligned}
H(t) &= \left[H_0 - \frac{A}{\sigma}\right] \exp^{-\sigma t} + \frac{A}{\sigma}\\
g(t) &= g_0 \exp^{(r+\sigma)t}
\end{aligned}
$$

with $g_0 > 0$. Once the solution becomes interior, $g(t) = \frac{R}{A}H(t)$ by (85). Assume that $\sigma < \frac{A}{H_0}$ so that $\dot{H(0)} > 0$. Then, graphically,

Figure 10: Return to Gross Investment in Human Capital in the Modified Sheshinski Specification



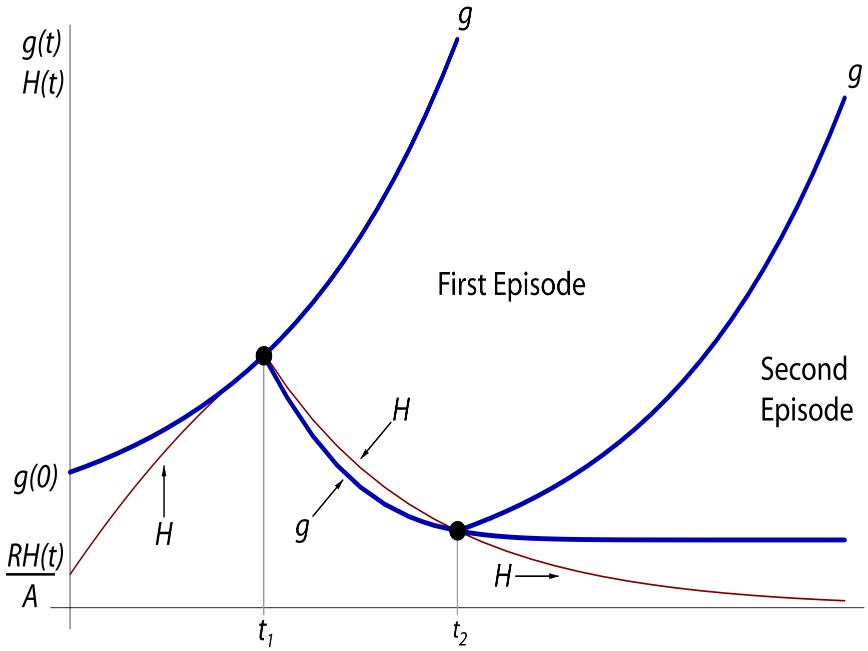
Note:

Let t_1 denote the time in which the first period of specialization finishes. If the solution "bangs-out" to $I = 0$ we can use (89) and (84) to get

$$\begin{aligned} \dot{g} &= (\sigma + r)g - R \\ H(t) &= H(t_1)\exp^{-\sigma(t-t_1)} \end{aligned} \tag{99}$$

for $t_1 < t < t_2$. Likewise, we can define a period t_2 in which the solution "bangs-in" again and so on. Graphically,

Figure 11: Human Capital Investment Episodes in the Modified Sheshinski Specification



Note:

In $t < t_1$, $I = 1$ implies $\dot{g} > 0$. g needs to decrease for the problem to respect the transversality condition. Thus, in the neighborhood of t_1 it has to be that $g(\dot{t_1}) < \frac{RH(\dot{t_1})}{A}$ (see Figure (11)). If we take the expression from the right of $g(t_1)$ this requires

$$\begin{aligned}
-R(\sigma + r)g(t_1) &< \frac{RH(\dot{t_1})}{A} \\
&= \frac{-\sigma RH(t_1)}{A} \\
&= -\sigma g(t_1) \\
&\Leftrightarrow \\
g(t_1) &< \frac{R}{r}. \qquad (100)
\end{aligned}$$

We can follow an analogous reasoning to construct conditions under which human capital investment happens in different episodes during the life-cycle.

To wrap up this section note that we have an initial period of specialization if $g_0 > \frac{RH_0}{A}$. At $t = 0$, however, it should be the case that the slope of $\frac{RH_0}{A}$ exceeds $\dot{g}$. Otherwise the expressions for g in the specialization period and the "interior" case do not intersect and the solution violates the transversality condition. This implies that $R\left[1 - \frac{\sigma H_0}{A}\right] > g_0(\sigma + r)$. High initial levels of human capital, low productivity, high discount, high depreciation, and low returns to human capital rule out an initial specialization period. Suppose the conditions described above hold so that specialization happens. We cannot show that $g(t_3) < g(t_1)$ so that is better to accumulate “all the human capital required for life” in the first period of specialization. This is what we call cycling in the investment on human capital and it is a consequence of depreciation.

References

Becker, G. S. (1962). Investment in Human Capital: A Theoretical Analysis. *The Journal of Political Economy 70*(5), 9–49.

Becker, G. S. (2009). *Human capital: A theoretical and empirical analysis, with special reference to education*. University of Chicago Press.

Ben-Porath, Y. (1967). The Production of Human Capital and the Life Cycle of Earnings. *The Journal of Political Economy 75*(4), 352–365.

Haley, W. J. (1976). Estimation of the Earnings Profile from Optimal Human Capital Accumulation. *Econometrica: Journal of the Econometric Society*, 1223–1238.

Mangasarian, O. L. (1966). Sufficient Conditions for the Optimal Control of Non-linear Systems. *SIAM Journal on Control 4*(1), 139–152.

Mincer, J. A. (1974). Schooling, Experience, and Earnings. In *Schooling, experience, and earnings*, pp. 41–63. Columbia University Press.

Rosen, S. (1976). A Theory of Life Earnings. *The Journal of Political Economy 84*(4), S45–S67.