

# Ben-Porath Notes

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Econ 350  
This draft, December 30, 2013

## Notes on Ben-Porath Human Capital Model

- Perfect Capital Markets
- No Nonmarket Benefits of Human Capital
- Fixed Labor Supply
- $H$  is human capital
- $I \in [0, 1]$  is investment time
- $D$  is goods input
- $F$  is a strictly concave function in two normal inputs

# Human Capital Production Function

- $\dot{H}(t) = F(I(t), H(t), D(t)) - \sigma H(t)$
- $F(I(t), H(t), D(t)) = F(I(t)H(t), D(t))$  (neutrality)
- $R$  is rental rate of human capital.
- Potential earnings:  $Y(t) = RH(t)$ .
- Observed earnings:

$$E(t) = RH(t) - \underbrace{RI(t)H(t)}_{\substack{\text{earnings} \\ \text{foregone}}} - \underbrace{P_D D(t)}_{\substack{\text{direct goods} \\ \text{costs}}}$$

- Consumer problem (max with respect to  $I(t), D(t)$ ):

$$\int_0^T e^{-rt} E(t) dt \quad \text{given } H(0) = H_0$$

- Formal solution (Hamiltonian): Flow of value from the optimal lifetime program

$$\underbrace{e^{-rt}[RH(t) - RI(t)H(t) - P_D D(t)]}_{\text{current flow}} + \underbrace{\mu(t)[\dot{H}]}_{\text{shadow price of human capital}}$$

- FOC Conditions (for interior solution):

$$I(t) : \quad Re^{-rt}H(t) = \mu(t)F_1H(t)$$

$$D(t) : \quad e^{-rt}P_D = \mu(t)F_2$$

$$\dot{\mu}(t) = -e^{-rt}[R - RI(t)] - \mu(t)F_1I(t) + \mu(t)\sigma$$

- Use FOC for investment to obtain:

$$\dot{\mu}(t) = -e^{-rt}R + \mu(t)\sigma.$$

Define  $g(t) = \mu(t)e^{+rt}$

$$\dot{g}(t) = \dot{\mu}e^{+rt} + r\mu(t)e^{+rt}$$

$$\dot{g}(t) = (\sigma + r)g(t) - R.$$

- Transversality:  $\lim_{t \rightarrow T} \mu(t)H(t) = 0$

$$\therefore \mu(T) = 0 \implies g(T) = 0$$


$$g(t) = \frac{R(1 - e^{(\sigma+r)(t-T)})}{\sigma + r}.$$

- Note that  $g(t)$  is a discount factor that adjusts for exponential depreciation of gross investment.
- $\dot{H}(t) + \sigma H(t) = F(IH(t), D(t)).$

- $0 < I(t) < 1$ , we can set up the problem in a “myopic” way.
- Gross “output” is  $F(I(t)H(t), D(t))$ .
- Returns on gross output:  $g(t)$ .
- Costs:  $P_D D(t) + R I(t) H(t)$ .
- Note: these are costs and returns as of period  $t$ .



- The agent's problem is:

$$\max_{I(t), D(t)} [g(t)F(I(t)H(t), D(t)) - P_D D(t) - R I(t)H(t)]$$


FOC:

- $g(t)F_1(I(t)H(t), D(t))H(t) = RH(t)$
- $g(t)F_2(I(t)H(t), D(t)) - P_D = 0.$

Demand functions are inverted first order conditions:

- $I(t)H(t) = I(t)H\left(\frac{R}{g(t)}, \frac{P_D}{g(t)}\right)$
- $D(t) = D\left(\frac{R}{g(t)}, \frac{P_D}{g(t)}\right)$

From normality of inputs, since  $\dot{g}(t) < 0$ , we have:

- $\dot{I}H(t) < 0, \dot{D}(t) < 0.$

- Then, if  $\sigma = 0$ ,  

$$\dot{E} = RF(I(t)H(t), D(t)) - R I \dot{H}(t) - P_D \dot{D}(t) > 0.$$
- Otherwise earnings can rise and then fall over the life cycle.  
 $(\sigma \neq 0).$
- What about  $\ddot{E}(t)$ ? Ben Porath chose a Cobb-Douglas form for  $F(I(t)H(t), D(t))$  and proves  $\ddot{E}(t) < 0$ .
- $\therefore$  Earnings increase at a decreasing rate over the life cycle.
- To simplify derivations, let  $F_2 \equiv 0$  (i.e. ignore  $D(t)$ ).

- First order condition for investment is:

$$g(t)F'(IH) = R.$$

$$\dot{g} = (\sigma + r)g(t) - R$$

- Differentiate the first order condition for investment.
- Set  $R = 1$  (for convenience)

$$\left(\text{Note that } \frac{\dot{g}}{g} = \sigma + r - \frac{1}{g}\right)$$

$$\dot{g}(t)F'(I(t)H(t)) + g(t)F''(I(t)H(t))I(t)\dot{H}(t) = 0.$$

$$\text{Thus } IH\dot{H}(t) = - \left( \frac{\dot{g}(t)}{g(t)} \right) \left[ \frac{F'}{F''} \right].$$

- To simplify notation, drop “t” argument for  $I(t), H(t), g(t)$  unless it clarifies matters to keep it explicit
- Then  $\ddot{IH} = - \left[ \frac{\ddot{g}}{g} - \left( \frac{\dot{g}}{g} \right)^2 \right] \frac{F'}{F''} - \frac{\dot{g}}{g} \left[ \dot{IH} - \frac{F'F'''}{(F'')^2} \dot{IH} \right]$ .
- Note that  $\ddot{g} = (\sigma + r)\dot{g}$ .
- $\therefore \frac{\ddot{g}}{\dot{g}} = (\sigma + r)$  and  $\frac{\ddot{g}}{g} = (\sigma + r)\frac{\dot{g}}{g}$  ( $\dot{g} \neq 0$ ).
- Thus, substituting for  $\dot{IH}$  we have

$$\ddot{IH} = - \left[ \frac{\ddot{g}}{g} - \left( \frac{\dot{g}}{g} \right)^2 \right] \frac{F'}{F''} + \left( \frac{\dot{g}}{g} \right)^2 \left[ 1 - \frac{F'F'''}{(F'')^2} \right] \left[ \frac{F'}{F''} \right].$$

- Earnings growth is given by (recall  $R = 1$ )

- $\dot{E} = F(IH) - I\dot{H} - \sigma H$

- $\ddot{E} = F'(IH)I\dot{H} - I\ddot{H} - \sigma\dot{H}$

- Since  $F' = \frac{1}{g}$  we have that

$$\ddot{E} = \frac{1}{g}I\dot{H} - I\ddot{H} - \sigma\dot{H}$$

- Set  $\sigma = 0$  for the moment and use the expression for  $I\dot{H}$  given above (including  $I\ddot{H}$ ).

Thus

- $$\ddot{E} = \dot{IH} \left[ \frac{1}{g} + \frac{\dot{g}}{g} \left( 1 - \frac{F' F'''}{(F'')^2} \right) \right] + \left( \frac{\ddot{g}}{g} - \left( \frac{\dot{g}}{g} \right)^2 \right) \frac{F'}{F''}.$$
- Use  $\dot{IH} = -\frac{\dot{g}}{g} \frac{F'}{F''}$  and  $\frac{\ddot{g}}{g} = (\sigma + r) \frac{\dot{g}}{g}$  to conclude that

$$\begin{aligned}
 \ddot{E} &= -\frac{\dot{g}}{g} \left[ \frac{F'}{F''} \right] \left\{ \frac{1}{g} + \frac{\dot{g}}{g} \left( 1 - \frac{F'F''}{(F'')^2} \right) \right\} \\
 &\quad + \left( (\sigma + r) \frac{\dot{g}}{g} - \left( \frac{\dot{g}}{g} \right)^2 \right) \frac{F'}{F''} \\
 &= -\frac{\dot{g}}{g} \left( \frac{F'}{F''} \right) \left\{ \frac{\frac{1}{g} + \frac{\dot{g}}{g} \left( 1 - \frac{F'F''}{(F'')^2} \right)}{-\frac{g(\sigma + r) - \dot{g}}{g}} \right\}
 \end{aligned}$$

$$\text{but } \dot{g} = (\sigma + r)g - 1 \quad (\sigma + r)g - \dot{g} = 1.$$

Thus

$$\begin{aligned}
 \ddot{E} &= \left( -\frac{\dot{g}}{g} \frac{F'}{F''} \right) \left( \frac{\dot{g}}{g} \right) \left( 1 - \frac{F' F'''}{(F'')^2} \right) \\
 &= \underbrace{- \left( \frac{\dot{g}}{g} \right)^2 \frac{F'}{F''}}_{+ \text{ (by concavity)}} \cdot \underbrace{\left( 1 - \frac{F' F'''}{(F'')^2} \right)}_{\text{Term depends on the sign of } F'''}
 \end{aligned}$$



- Define  $\eta = 1 - \frac{F'F'''}{(F'')^2}$ .
- Necessary condition for concavity of earnings profiles with age is  $F''' > 0$ ;
- Stronger condition is  $-\eta > 0$ .



- Note: if  $F(x) = \frac{Ax^\alpha}{\alpha}$ ,  $-\infty < \alpha < 1$ ,  $A > 0$ ,  $F'(x) = Ax^{\alpha-1}$
- $F''(x) = (\alpha - 1)Ax^{\alpha-2}$
- $F'''(x) = (\alpha - 1)(\alpha - 2)Ax^{\alpha-3}$
- $\eta = \frac{\alpha-2}{\alpha-1} < 0$ . Thus  $\ddot{E}$  is negative (concavity).
- If  $F(x) = a - be^{-cx}$ , for  $b, c > 0$ ,  $\eta = 0$  and  $\ddot{E}$  negative.
- Obviously fails with quadratic technologies.

## Period of Specialization

- Period of specialization is associated with full time investment.
- Assume  $F_2 \equiv 0$  (ignore D).
- Suppose that at time  $t$

$$F'(H_0)g(t) > R.$$

- Then it pays to specialize.
- How to solve? Initially assume  $\sigma = 0$ .
- Note that marginal returns to investment decline with capital stock growth ( $F' \downarrow$ ) and with time  $\dot{g} < 0$ .

- Then there is at *most* one period of specialization:  $[0, t^*]$ .
- This is “schooling” in the Ben-Porath model.
- $t^*$  is characterized by

$$F'(H(t^*))g(t^*) = R$$

$$I(t^*) = 1 \text{ (at the endpoint of the interval)}$$

$$H(t^*) = \int_0^{t^*} F(H(\tau)) d\tau + H_0.$$

- Note that anything that lowers  $g(t)$  (and not  $R$ ) lowers  $t^*$ .
- Thus the higher  $r$ , the lower  $t^*$ .
- Note, also, that the higher  $H_0$ , the lower  $t^*$ , since it takes less time to acquire  $H(t^*)$ .

- Now to get  $H(\tau)$ , notice that  $\dot{H} = F(H)$  in the period of specialization.
- Solve jointly to get  $t^*$ .
- Now, if  $\sigma > 0$ , we get the same condition for specialization but could get cycling in the model. (Initially, high  $\sigma$  knocks off capital makes specialization in investment productive again.)
- Let  $\sigma = 0$ , thus no cycling possible in the model.

Cobb-Douglas example:

$$\dot{H} = A(IH)^\alpha - \sigma H, \quad 0 < \alpha < 1, \quad A > 0$$

A period of specialization arises if

$$g(t_0)^\alpha A(H_0)^{\alpha-1} > R.$$

Then if

$$(H_0)^{\alpha-1} > \left[ \frac{R}{g(t_0)^\alpha A} \right]$$

$$\text{or } H_0 < \left[ \frac{R}{g(t_0)^\alpha A} \right]^{\frac{1}{\alpha-1}},$$

the agent will specialize.

If  $T \rightarrow \infty$ , the condition simplifies to

$$H_0 < \left( \frac{r}{\alpha A} \right)^{\frac{1}{\alpha-1}} = \left( \frac{\alpha A}{r} \right)^{\frac{1}{1-\alpha}}$$

$$\text{since } g(t) = \frac{R}{r} \quad \text{🗨️}$$

If the condition required for specialization is satisfied, we obtain:

$$\begin{aligned}\dot{H} &= A(IH)^\alpha \\ \therefore \frac{\dot{H}}{H^\alpha} &= A\end{aligned}$$

$$\begin{aligned}H(t)^{1-\alpha} &= (1-\alpha)At + (1-\alpha)K_0 \\ H(t) &= [(1-\alpha)At + (1-\alpha)K_0]^{\frac{1}{1-\alpha}} \\ [K_0(1-\alpha)]^{\frac{1}{1-\alpha}} &= H_0 \\ K_0(1-\alpha) &= H_0^{1-\alpha} \\ K_0 &= \frac{H_0^{1-\alpha}}{(1-\alpha)}\end{aligned}$$



Therefore, we have that during the period of specialization (schooling) human capital is accumulating via the following growth process:

$$\begin{aligned} H(t) &= [A(1 - \alpha)t + K_0(1 - \alpha)]^{\frac{1}{1-\alpha}} \\ &= [A(1 - \alpha)t + H_0^{1-\alpha}]^{\frac{1}{1-\alpha}}. \end{aligned}$$

At the end of the period of specialization we have that

$$\alpha g(t^*) A (H(t^*))^{\alpha-1} = R.$$

Let  $T \rightarrow \infty$ , then  $g(t^*) = R/r$  and  $t^*$  is defined by solving

$$\alpha \frac{R}{r} A (A(1 - \alpha)t^* + H_0^{1-\alpha})^{-1} = R.$$

Thus,

$$\left(\frac{\alpha A}{r}\right) = A(1 - \alpha) t^* + H_0^{1-\alpha}$$

$$\text{Schooling: } t^* = -\frac{H_0^{1-\alpha}}{A(1 - \alpha)} + \left(\frac{\alpha}{1 - \alpha}\right) \frac{1}{r}$$

Higher  $A$ , higher  $t^*$  “ability to learn.”

Higher  $H_0$ , lower  $t^*$  “ability to earn.”

Define post school work experience as  $\tau = t - t^*$ . Then

$$E(\tau) = R \int_0^{\tau} \dot{H}(\ell + t^*) d\ell + RH(t^*) - RIH(\tau + t^*).$$

At school leaving age and beyond we have

$$\alpha g(t) A(IH(t))^{\alpha-1} = R.$$

Thus, we have

$$\begin{aligned} [IH(t)]^{\alpha-1} &= \frac{R}{\alpha g(t) A} \\ IH(t) &= \left[ \frac{\alpha g(t) A}{R} \right]^{\frac{1}{1-\alpha}}. \end{aligned}$$

Thus,

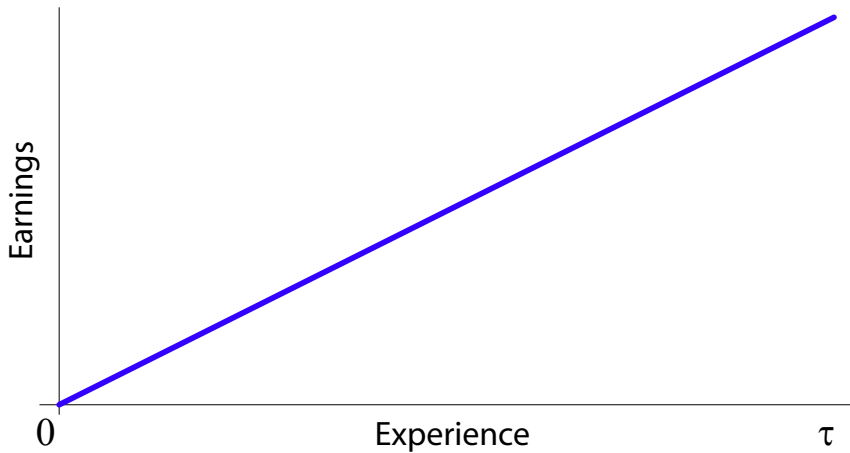
$$\dot{H} = A \left[ \frac{\alpha g(t)A}{R} \right]^{\frac{\alpha}{1-\alpha}}.$$

Earnings are given by

$$\begin{aligned} E(\tau) = & R \int_0^{\tau} A \left[ \frac{\alpha g(\ell + t^*)A}{R} \right]^{\frac{\alpha}{1-\alpha}} d\ell + RH(t^*) \\ & - R \left[ \frac{\alpha g(\tau + t^*)A}{r} \right]^{\frac{1}{1-\alpha}}. \end{aligned}$$

Let  $T \rightarrow \infty$ , then  $g(t) = \frac{R}{r}$

$$\begin{aligned} E(\tau) &= RA \left[ \frac{\alpha A}{r} \right]^{\frac{\alpha}{1-\alpha}} \tau + R \left[ \frac{\alpha A}{r} \right]^{\frac{1}{1-\alpha}} - R \left[ \frac{\alpha A}{r} \right]^{\frac{1}{1-\alpha}} \\ &= RA \left[ \frac{\alpha A}{r} \right]^{\frac{\alpha}{1-\alpha}} \tau. \end{aligned}$$



# Human Capital Dynamics

$$t_0 < t < T, \quad T \rightarrow \infty, \quad t^* = \left( \frac{\alpha}{1-\alpha} \right) \frac{1}{r} - \frac{H_0^{1-\alpha}}{A(1-\alpha)}$$

$$t = t_0 \Rightarrow H(t) = H_0$$

$$t_0 < t < t^* \Rightarrow H(t) = (A(1-\alpha)t + H_0^{1-\alpha})^{\frac{1}{1-\alpha}}$$

$$t = t^* \Rightarrow H(t) = \left( \frac{\alpha A}{r} \right)^{\frac{1}{1-\alpha}}$$

$$t^* < t \Rightarrow H(t) = \left( \frac{\alpha A}{r} \right)^{\frac{\alpha}{1-\alpha}} (t - t^*) + H(t^*)$$

# Investment Dynamics

$$t_0 < t < T, \quad T \rightarrow \infty, \quad t^* = \left( \frac{\alpha}{1-\alpha} \right) \frac{1}{r} - \frac{H_0^{1-\alpha}}{A(1-\alpha)}$$

$$t = t_0 \Rightarrow I(t) = 1 \quad \text{if} \quad F'(H_0)g(t) > R$$

$$t_0 < t \leq t^* \Rightarrow I(t) = 1$$

$$t^* < t \Rightarrow I(t) = \frac{\left( \frac{\alpha A}{r} \right)^{\frac{1}{1-\alpha}}}{\left( \frac{\alpha A}{r} \right)^{\frac{\alpha}{1-\alpha}} (t - t^*) + H(t^*)}$$

$$I(t) = \left( \left( \frac{\alpha A}{r} \right)^{-1} (t - t^*) + 1 \right)^{-1}$$



# Earnings Dynamics

$$t_0 < t < T, \quad T \rightarrow \infty, \quad t^* = \left( \frac{\alpha}{1-\alpha} \right) \frac{1}{r} - \frac{H_0^{1-\alpha}}{A(1-\alpha)}$$

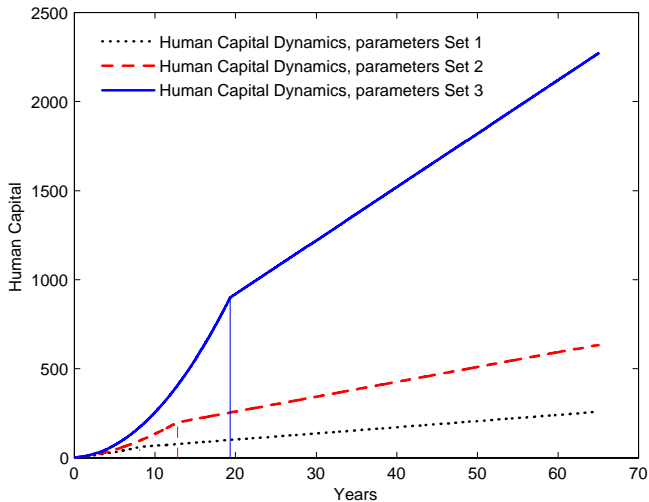
$$E(t) = RH(t) \cdot (1 - I(t)), \text{ so}$$

$$t_0 < t \leq t^* \Rightarrow I(t) = 1 \Rightarrow E(t) = 0$$

$$\begin{aligned} t^* < t \Rightarrow E(t) &= RH(t) - RH(t)I(t) \\ &= RH(t) - \left( \frac{\alpha A}{r} \right)^{\frac{1}{1-\alpha}} \\ &= R(A(1-\alpha)t + H_0^{1-\alpha})^{\frac{1}{1-\alpha}} - \left( \frac{\alpha A}{r} \right)^{\frac{1}{1-\alpha}} \end{aligned}$$

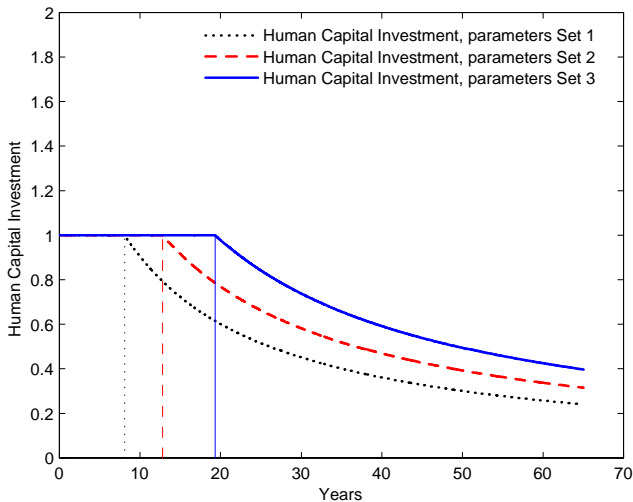
# Human capital dynamics, varying $\alpha$ ( $A = 3$ , $r = 0.05$ , $H_0 = 1$ )

$\alpha = 0.3$  (dotted line),  $\alpha = 0.4$  (dashed line),  $\alpha = 0.5$  (solid line)



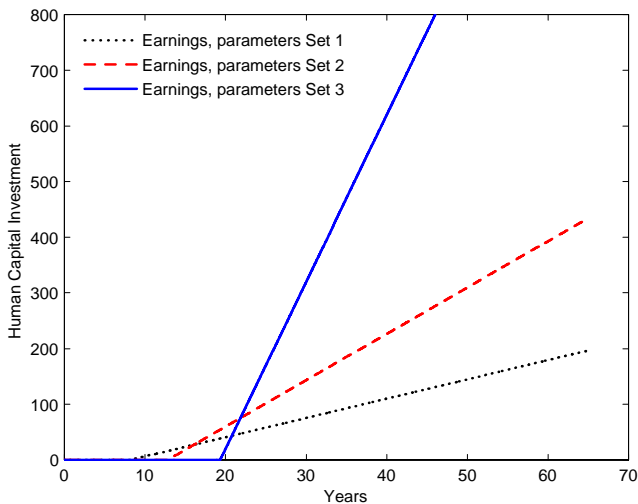
# Human investment dynamics, varying $\alpha$ ( $A = 3$ , $r = 0.05$ , $H_0 = 1$ , $R = 1$ )

$\alpha = 0.3$  (dotted line),  $\alpha = 0.4$  (dashed line),  $\alpha = 0.5$  (solid line)



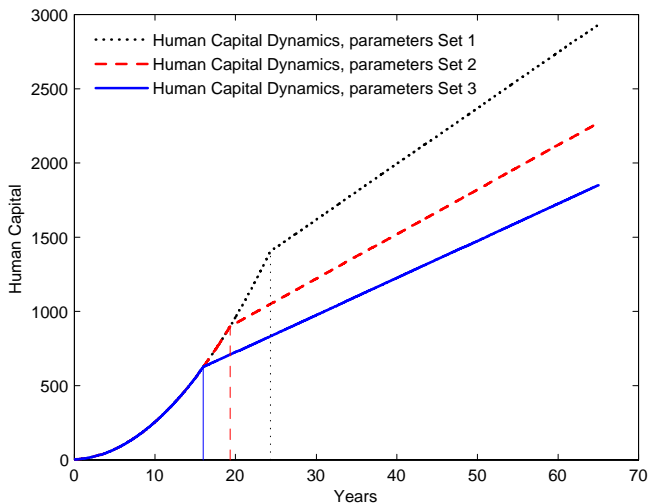
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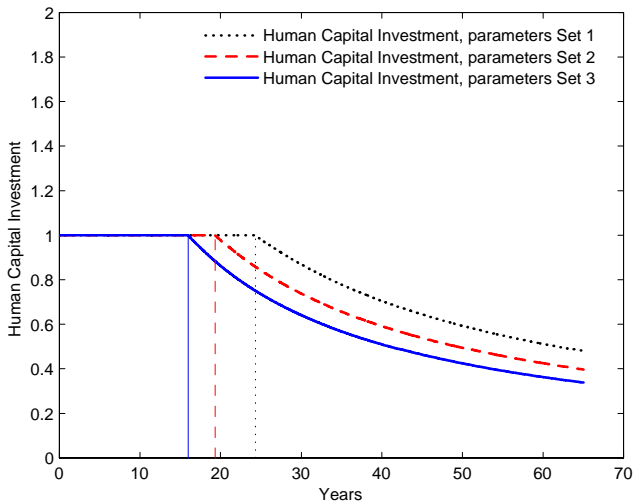
# Human capital dynamics, varying $r$ ( $A = 3$ , $H_0 = 1$ , $\alpha = 0.5$ )

$r = 0.04$  (dotted line),  $r = 0.05$  (dashed line),  $r = 0.06$  (solid line)



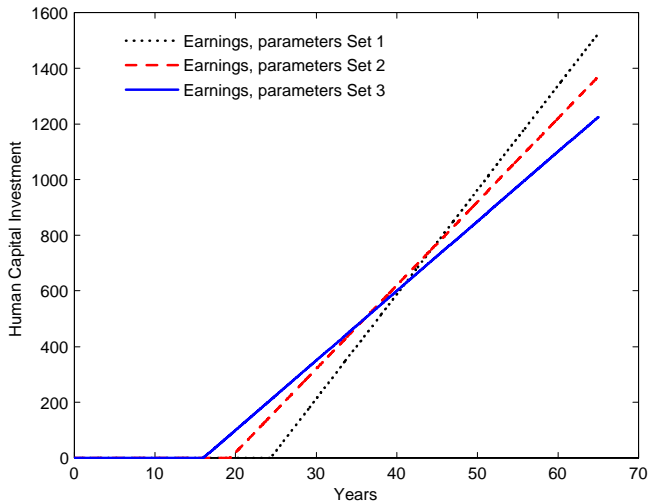
# Human investment dynamics, varying $r$ ( $A = 3$ , $H_0 = 1$ , $\alpha = 0.5$ )

$r = 0.04$  (dotted line),  $r = 0.05$  (dashed line),  $r = 0.06$  (solid line)



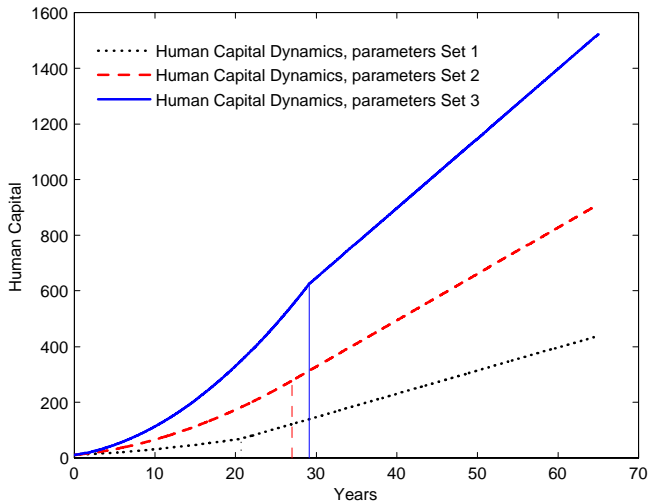
# Earnings dynamics, varying $r$ ( $A = 3$ , $H_0 = 1$ , $\alpha = 0.5$ , $R = 1$ )

$r = 0.04$  (dotted line),  $r = 0.05$  (dashed line),  $r = 0.06$  (solid line)



# Human capital dynamics, varying $A$ ( $r = 0.03$ , $H_0 = 10$ , $\alpha = 0.5$ )

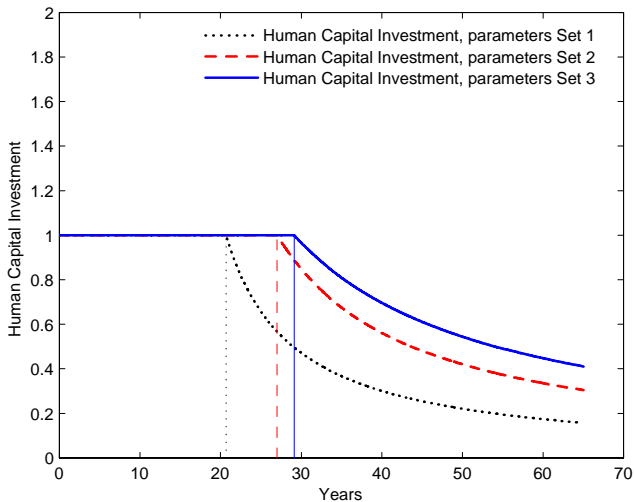
$A = 0.5$  (dotted line),  $A = 1.0$  (dashed line),  $A = 1.5$  (solid line)





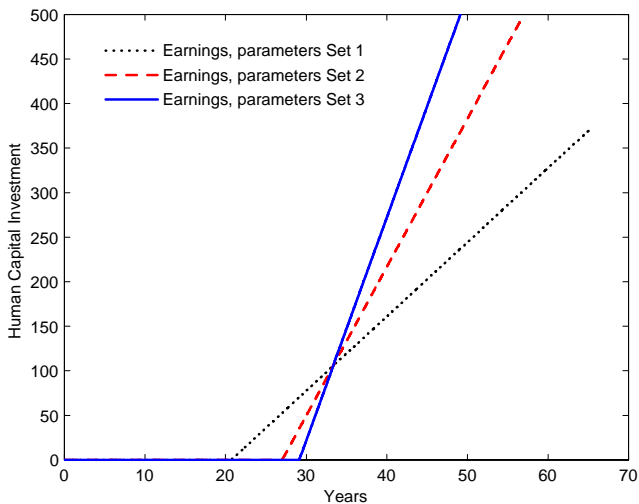
# Human investment dynamics, varying $A$ ( $r = 0.03$ , $H_0 = 10$ , $\alpha = 0.5$ )

$A = 0.5$  (dotted line),  $A = 1.0$  (dashed line),  $A = 1.5$  (solid line)



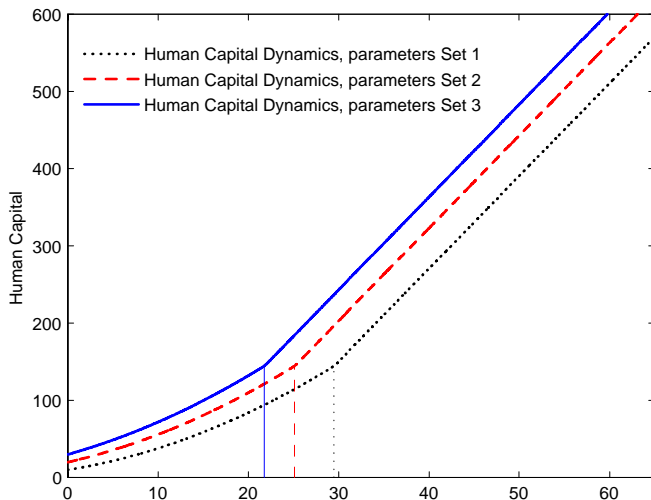
# Earnings dynamics, varying $A$ ( $r = 0.03$ , $H_0 = 10$ , $\alpha = 0.5$ )

$A = 0.5$  (dotted line),  $A = 1.0$  (dashed line),  $A = 1.5$  (solid line)



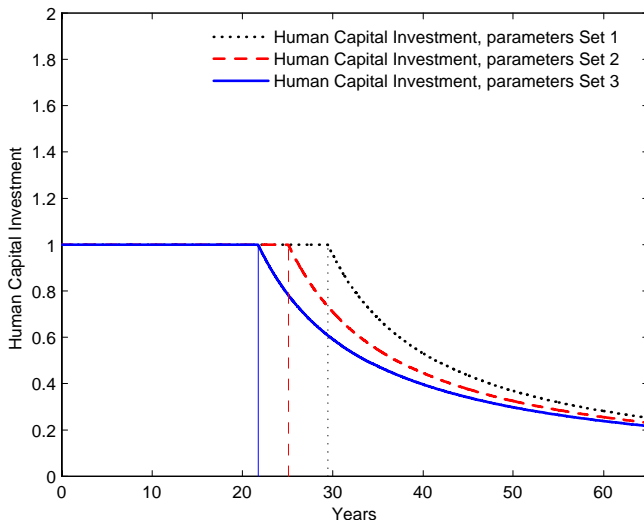
Human capital dynamics, varying  $H_0$  ( $A = 0.6$ ,  $r = 0.025$ ,  $\alpha = 0.5$ ,  $R = 1.0$ )

$H_0 = 10$  (dotted line),  $H_0 = 20$  (dashed line),  $H_0 = 30$  (solid line)



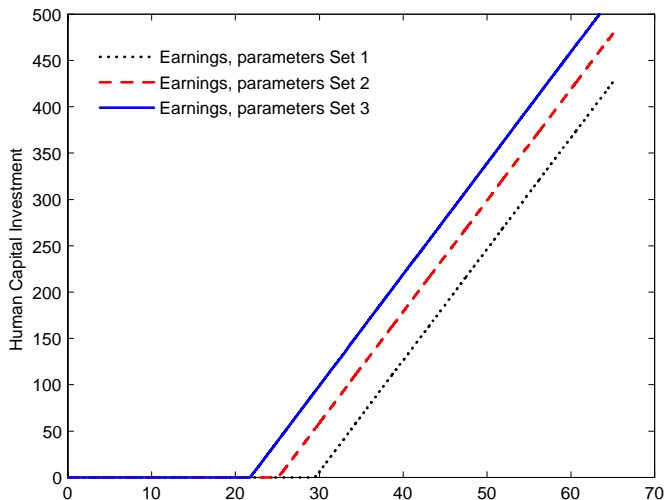
# Earnings dynamics, varying $H_0$ ( $A = 0.6$ , $r = 0.025$ , $\alpha = 0.5$ , $R = 1.0$ )

$H_0 = 10$  (dotted line),  $H_0 = 20$  (dashed line),  $H_0 = 30$  (solid line)



Human investment dynamics, varying  $H_0$  ( $A = 0.6$ ,  $r = 0.025$ ,  $\alpha = 0.5$ ,  $R = 1.0$ )

$H_0 = 10$  (dotted line),  $H_0 = 20$  (dashed line),  $H_0 = 30$  (solid line)



# Finite Horizon Ben Porath Model in Level and Autogressive Form ( $\alpha = 1/2$ )

- $\dot{H} = A(IH)^\alpha$
- $\alpha = 1/2$  (Haley, 1976; Rosen, 1976)
- $\sigma = 0$
- $R$  = rental rate

- $$\dot{E}(\tau) = \frac{A^2}{2}g(\tau + t^*) - 2R \left[ \frac{A}{2} \frac{g(\tau + t^*)}{R} \right] \left[ \frac{A}{2R} \dot{g}(\tau + t^*) \right]$$

$$\dot{g} = rg - R$$

- Thus,

$$\dot{E}(\tau) = \left[ \frac{A^2}{2R} \right] g[2R - rg]$$

$$\ddot{E}(\tau) = \frac{-A^2}{R} (\dot{g})^2.$$

- Using  $\dot{g} = rg - R$ ,

$$\begin{aligned}\dot{E}(\tau) &= \frac{A^2}{2R} \left( \frac{\dot{g} + R}{r} \right) \left( 2R - r \frac{(\dot{g} + R)}{r} \right) \\ &= \frac{A^2}{2Rr} (R^2 - (\dot{g})^2).\end{aligned}$$

- Thus,

$$\begin{aligned}\dot{E}(\tau) &= \frac{A^2}{2Rr} R^2 - \frac{1}{2r} \frac{A^2}{R} (\dot{g})^2 \\ &= \frac{A^2}{2Rr} R^2 + \frac{1}{2r} \ddot{E}(\tau).\end{aligned}$$

- Thus,

$$\ddot{E}(\tau) = 2r\dot{E}(\tau) - A^2R. \quad (1)$$



- This is a standard ordinary differential equation with constant coefficients. The solution is of the form

$$E(\tau) = c_1 e^{2r\tau} + c_2 \tau + c_0.$$

- We can pin this equation down knowing that

$$E(0) = 0 \quad (\text{so } c_1 + c_0 = 0)$$

$$\dot{E}(T) = 0 \quad (\text{so } 2rc_1 e^{2rT} + c_2 = 0).$$

- Finally, optimality produces (1) above to get  $c_0$ .
- Set

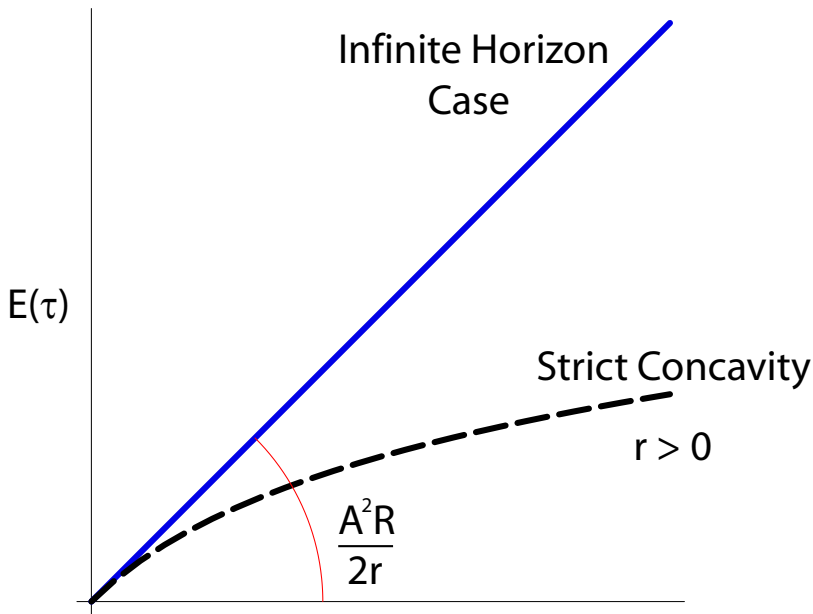
$$c_1 = -c_0$$

$$c_2 = \frac{A^2 R}{2r} e^{2rT},$$

using  $E(T) = 0$  and (1).

- Thus

$$E(\tau) = \frac{A^2 R}{(2r)^2} e^{-2rT} (1 - e^{2r\tau}) + \left( \frac{A^2 R}{2r} \right) \tau. \quad (2)$$



- This, in its essential form, is the equation that Brown (JPE, 1976) fits; from the  $\tau$  term, one can identify  $\frac{A^2 R}{2r}$ .
- From the exponential (in  $\tau$ ) one can pick up  $r$  and  $A^2 R$ , but his estimates are poor,  $r \rightarrow 0$ .
- But from Brown,  $T \rightarrow \infty$  is a good approximation. (His sample is young). Thus

$$E(\tau) \doteq \frac{RA^2}{2}\tau.$$

- Thus “ $r$ ” is not identified.

- Write this as an autoregression:

$$E(\tau + 1) - E(\tau) = \frac{A^2 R}{(2r)^2} e^{-2rT} (e^{2r\tau} - e^{2r(\tau+1)}) + \frac{A^2 R}{2r}$$
$$E(\tau) - E(\tau - 1) = \frac{A^2 R}{(2r)^2} e^{-2rT} (e^{2r(\tau-1)} - e^{2r\tau}) + \frac{A^2 R}{2r}.$$

- Multiply second equation by  $e^{2r}$ :

$$\begin{aligned} e^{2r}[E(\tau) - E(\tau - 1)] &= \frac{A^2 R}{2r^2} e^{-2rT} (e^{2r\tau} - e^{2r(\tau+1)}) + e^{2r} \frac{(A^2 R)}{2r} \\ &= E(\tau + 1) - E(\tau) - (1 - e^{2r\tau}) \frac{A^2 R}{2r}. \end{aligned}$$

- Thus

$$E(\tau + 1) - E(\tau) = e^{2r}[E(\tau) - E(\tau - 1)] - (e^{2r} - 1) \frac{A^2 R}{2r}.$$

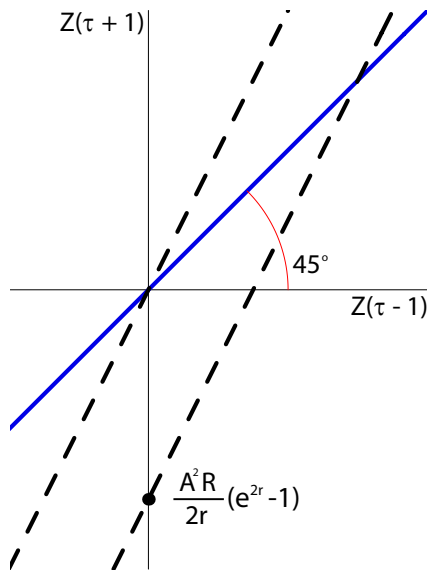
- Let

$$Z(\tau + 1) = E(\tau + 1) - E(\tau)$$

$$Z(\tau) = E(\tau) - E(\tau - 1)$$

$$Z(\tau + 1) = e^{2r} Z(\tau) - (e^{2r} - 1) \left( \frac{A^2 R}{2r} \right).$$





- Apparently explosive, it actually converges. Observe:

$$\begin{aligned} E(\tau) - E(\tau - 1) &= \frac{A^2 R}{(2r)^2} e^{-2rT} (e^{2r(\tau-1)} - e^{2r\tau}) + \frac{A^2 R}{2r} \\ &= \frac{A^2 R}{2r} \left[ 1 + \frac{e^{-2rT}}{2r} e^{2r\tau} (1 - e^{2r}) \right] \end{aligned}$$

- $$\frac{\partial [E(\tau) - E(\tau - 1)]}{\partial \tau} = \frac{A^2 R}{2r} (e^{-2rT} e^{2r\tau} (1 - e^{2r}) < 0$$

- Increments are actually decreasing.

- Let  $b = e^{2r}$ .

- $$c = - \left( \frac{e^{2r} - 1}{2r} \right) \frac{A^2 R}{2r} = \left( \frac{1 - e^{2r}}{2r} \right) A^2 R$$

$$Z(T) = \underbrace{(b)^T (Z_0)}_{\text{growing}} + c \underbrace{\sum_{j=0}^{T-1} b^j}_{\text{declining}},$$

but converges to a constant (even though autoregression is “explosive”).

## Deriving Mincer from Ben Porath

Using (2), we obtain

$$E(\tau) = \left( \frac{A^2 R}{2r} \right) \left[ \tau + \frac{e^{-2rT} - e^{2r(\tau-T)}}{2r} \right].$$

In logs,

$$\begin{aligned} \ln E(\tau) &= \ln \left( \frac{A^2 R}{2r} \right) + \ln \tau + \ln \left[ 1 + \frac{e^{-2rT} - e^{2r(\tau-T)}}{2r\tau} \right] \\ &= \ln \left[ \frac{A^2 R}{2r} \right] + \ln \tau + \ln \left[ 1 + \frac{e^{-rT}(1 - e^{2r\tau})}{2r\tau} \right]. \end{aligned}$$

# The Taylor Expansions

$$\ln(\tau) \doteq \ln(\tau_0) + \frac{1}{\tau_0}(\tau - \tau_0) - \frac{1}{\tau_0^2} \frac{(\tau - \tau_0)^2}{2!}$$

$$\ln\left(1 + \frac{e^{-2rT} - e^{2r(\tau-T)}}{2r\tau}\right) \doteq \xi_0 + \xi_1(\tau - \tau_0) + \xi_2 \frac{(\tau - \tau_0)^2}{2!}$$

$$\xi_0 \equiv \ln\left(1 + \frac{e^{-2rT} - e^{2r(\tau_0-T)}}{2r\tau_0}\right)$$

$$\xi_1 \equiv -\left(\frac{e^{-2rT} + e^{2r(\tau_0-T)}(2r\tau_0 - 1)}{\tau_0(2r\tau_0 + e^{-2rT} - e^{2r(\tau_0-T)})}\right)$$

$$\xi_2 \equiv \left[ \frac{(e^{-2rT} + e^{2r(\tau_0-T)}(2r\tau_0 - 1))}{(\tau_0(2r\tau_0 + e^{-2rT} - e^{2r(\tau_0-T)}))^2} (4r\tau_0 + e^{-2rT} - e^{2r(\tau_0-T)}(2r\tau_0 + 1)) \right. \\ \left. - \left( \frac{(2r)^2 \tau_0 e^{2r(\tau_0-T)}}{(\tau_0(2r\tau_0 + e^{-2rT} - e^{2r(\tau_0-T)}))^2} \right) \right]$$

Adding the terms together:

$$\ln(\tau) + \ln\left(1 + \frac{e^{-2rT} - e^{2r(\tau-T)}}{2r\tau}\right)$$

$$\doteq \alpha_0 + \alpha_1(\tau - \tau_0) + \alpha_2(\tau - \tau_0)^2$$

$$\alpha_0 \equiv \ln(\tau_0) + \xi_0$$

$$\alpha_1 \equiv \xi_1 + \frac{1}{\tau_0}$$

$$\alpha_2 \equiv \left(-\frac{1}{\tau_0^2} + \xi_2\right) / 2$$

To obtain Mincer Equations:

$$\ln(\tau) + \ln\left(1 + \frac{e^{-2rT} - e^{2r(\tau-T)}}{2r\tau}\right) \doteq k_0 + k_1\tau + k_2\tau^2$$

$$k_0 \equiv \alpha_0 - \tau_0\alpha_1 + \alpha_2\tau_0^2$$

$$k_1 \equiv \alpha_1 - 2\alpha_2\tau_0$$

$$k_2 \equiv \alpha_2$$

## Mincer Obtained:

- Mincer coefficients

$$\hat{k}_1 = 0.081$$

$$\hat{k}_2 = -0.0012$$

- Using  $r = 0.0225$ ,  $\tau_0 = 29.54$ ,  $T = 41.43$ ,

$$k_1 = 0.081$$

$$k_2 = -0.0010$$



Parameters			Ben Porath Coefficients	
$r$	$\tau_0$	$T$	$k_1$	$k_2$
0.0225	29.54	41.43	0.081	-0.0010
0.05	25	60	0.0808	-0.0008
0.05	20	65	0.1002	-0.0013
0.0675	24.70	74.77	0.081	-0.0008
Mincer Coefficients			0.081	-0.0012

$$\text{Model: } \ln(\text{Earnings}) = k_0 + k_1\tau + k_2\tau^2$$

- Suppose

$$rT \doteq 0 \quad \text{and} \quad e^{-rT} = 1.$$

- $$\ln E(\tau) \doteq \ln \left( \frac{A^2 R}{2r} \right) + \ln \tau + \ln \left[ 1 + \frac{1 - e^{2r\tau}}{2r\tau} \right]$$

## Conclusion

- There may be no economic content in Mincer's "rate of return" on post-school investment.
- All of the economic content is in the intercept term.
- Note, however, holding experience constant, there should be no effect of schooling on the earnings function.
- Mincer finds an effect. This would seem to argue against the Ben-Porath model.
- Not necessarily. Look at equation

$$t^* = \frac{1}{r} - \frac{1}{2} \frac{H_0^{1/2}}{A} \quad \text{for } \alpha = 1/2 \text{ and } T \text{ "big."}$$

- Suppose  $A$  is randomly distributed in the population.
- Then, we have that if  $H_0$  is distributed independently of  $A$ , the coefficient on  $t^*$  (length of schooling) is

$$E \left[ \left( -\frac{1}{2} \frac{H_0^{1/2}}{A} \right) (2 \ln A) \right] > 0.$$

- Thus, the coefficient on schooling is

$$-E \left( H_0^{1/2} \right) E \left( \frac{\ln A}{A} \right).$$

If  $A$  is Pareto;

$$F(A) = \left( \frac{\alpha}{A_0} \right) \left( \frac{A_0}{A} \right)^{\alpha+1}, \quad A_0 > 0, \alpha > 0.$$

Integrate by parts to reach

$$\begin{aligned} E \left( \frac{\ln A}{A} \right) &= -\frac{(A_0)^{\alpha+1} \alpha}{A_0} (\ln A_0) A_0^{-(\alpha+1)} - \frac{1}{\alpha+1} \\ &= -\frac{\alpha \ln A_0}{A_0} - \frac{1}{\alpha+1} \end{aligned}$$

Therefore, the coefficient on schooling is

$$E(H_0)^{1/2} \left[ \frac{1}{\alpha + 1} + \frac{\alpha \ln A_0}{A_0} \right] > 0.$$

Since units of  $H_0$  are arbitrary, we are done.

Therefore, positive coefficient on schooling solely as a consequence of *not* including ability measures.

## Rate of Return to Post-School Investment

Let  $T \rightarrow \infty$ . Without post-school investment, person makes

$$R \left[ \frac{\alpha A}{r} \right]^{\frac{1}{1-\alpha}}.$$

Increment in earnings at post-school age  $\tau$  is simply

$$\underbrace{R A \left[ \frac{\alpha A}{r} \right]^{\frac{1}{1-\alpha}}}_{\text{Earnings (above school-  
ing earnings) at } \tau} \tau - \underbrace{R \left[ \frac{\alpha A}{r} \right]^{\frac{1}{1-\alpha}}}_{\text{Costs}}.$$

- $\phi$  is that rate that equates returns and costs. Thus, solve for  $\phi$ .

$$\int_0^{\infty} e^{-\phi\tau} \left[ RA \left[ \frac{\alpha A}{r} \right]^{\frac{1}{1-\alpha}} \tau - R \left[ \frac{\alpha A}{r} \right]^{\frac{1}{1-\alpha}} \right] d\tau = 0$$

- Use the Laplace transform.
- Then

$$\frac{1}{\phi^2} RA \left[ \frac{\alpha A}{r} \right]^{\frac{\alpha}{1-\alpha}} - \frac{1}{\phi} R \left[ \frac{\alpha A}{r} \right]^{\frac{1}{1-\alpha}} = 0$$

$$\phi = \frac{r}{\alpha}.$$



- Therefore the rate of return to post-schooling investment is  $r/\alpha$ .
- Smaller  $\alpha$ , higher  $\phi$ .
- Thus, the lower the productivity (i.e.,  $\alpha$ ), the higher  $\phi$ .

## Rate of Return to Schooling (Holding Post-School Investment Fixed)

Person without schooling can earn  $RH_0$ . With schooling can earn  $RA \left[ \frac{\alpha A}{r} \right]^{\frac{\alpha}{1-\alpha}}$ . (Assuming no post school investment.)

Recall that (for  $T \rightarrow \infty$ ), optimal schooling is given by

$$t^* = \frac{1}{r} - \frac{1}{2} \frac{H_0^{1/2}}{A}.$$

During this period (before  $t^*$ ), under our assumptions, there are no earnings.

Then the rate of return is given by comparing

$$\int_{t^*}^{\infty} e^{-\phi t} \left[ R \left( \frac{\alpha A}{r} \right)^{\frac{1}{1-\alpha}} \right] dt \text{ with } \int_0^{\infty} e^{-\phi t} R H_0 dt.$$

Solve for  $\phi$ :

$$\begin{aligned} \left[ \frac{\alpha A}{r} \right]^{\frac{1}{1-\alpha}} e^{-\phi t^*} &= H_0 \\ \ln \left[ \frac{\alpha A}{r} \right]^{\frac{1}{1-\alpha}} - \phi t^* &= \ln H_0 \\ \phi = \frac{\ln \left[ \frac{\alpha A}{r} \right]^{\frac{1}{1-\alpha}} - \ln H_0}{t^*} &= \frac{\ln \left[ \frac{\alpha A}{r} \right]^{\frac{1}{1-\alpha}} - \ln H_0}{\frac{1}{r} - \frac{1}{2} \frac{H_0^{1/2}}{A}} \end{aligned}$$

Has no simple relationship to the rate of return to investment.

## Growth of Earnings

- Keep time argument implicit unless being explicit helps.
- $E$ ,  $H$ ,  $I\dot{H}$  all depend on  $t$ .
- Growth of earnings:

$$\dot{E} = f(I\dot{H}) - (I\dot{H})$$

$$\frac{\partial \dot{E}}{\partial r} = ?$$

- FOC:

$$g(t) f'(I\dot{H}) = 1$$

$$g(t) = \frac{1 - e^{r(t-T)}}{r}$$

- Totally differentiate FOC with respect to  $t$ :

$$\dot{g}f'(IH) + gf''(IH)(I\dot{H}) = 0$$

$$- \left( \frac{\dot{g}}{g} \frac{f'}{f''} \right) = (I\dot{H})$$

- First note that

$$\frac{\partial \dot{E}}{\partial r} = f' \left( \frac{\partial IH}{\partial r} \right) - \frac{\partial}{\partial r} \left[ (I\dot{H}) \right].$$

- Now observe further that

$$\frac{\partial(IH)}{\partial r} < 0.$$

- Thus the first term is negative.
- Observe that we can show that

$$\frac{\partial(\dot{IH})}{\partial r} > 0$$

if concavity on earnings is satisfied ( $\ddot{E} < 0$ ).

- Intuition: the time rate of decrease in  $IH$  is slowed down ( $r \uparrow \Rightarrow IH \downarrow$ ; the function is “less concave”).
- If we can establish this, we know that the contribution of the second term is negative and

$$\frac{\partial \dot{E}}{\partial r} < 0.$$



- To show this, observe that

$$\frac{\partial[I\dot{H}]}{\partial r} = \left[ -\frac{\dot{g}}{g} \right] \left[ 1 - \frac{f' f'''}{(f'')^2} \right] \frac{\partial(IH)}{\partial r} + \left( \frac{f'}{f''} \right) \frac{\partial}{\partial r} \left[ -\frac{\dot{g}}{g} \right].$$

- From the earlier notes, concavity of earnings function in experience ( $\ddot{E} < 0$ )

$$\left[ 1 - \frac{f' f'''}{(f'')^2} \right] < 0.$$

- The first term is positive, since  $\dot{g} < 0$  and

$$\frac{\partial(IH)}{\partial r} < 0.$$

- To investigate the second term, we determine that

$$\dot{g} = rg - 1, \quad \frac{\dot{g}}{g} = r - \frac{1}{g}, \quad -\frac{\dot{g}}{g} = \frac{1}{g} - r.$$

- Now,

$$\frac{\partial}{\partial r} \left[ -\frac{\dot{g}}{g} \right] = -\frac{1}{g^2} \frac{\partial g}{\partial r} - 1.$$

- This term is negative. Why?

- $$\begin{aligned}\frac{\partial g}{\partial r} &= \frac{-(t - T)e^{r(t-T)}}{r} - \frac{1 - e^{r(t-T)}}{r^2} \\ &= \frac{1}{r^2} [e^{r(t-T)} (1 - r(t - T)) - 1]\end{aligned}$$

- Now observe that

$$e^{r(T-t)} > 1 + r(T - t) \quad \text{for } T \geq t.$$

- Thus

$$\frac{\partial g}{\partial r} < 0.$$

- Consider next that

$$\begin{aligned}
 \frac{-\partial g}{g^2 \partial r} - 1 &= \frac{1}{r^2} \left[ \frac{1 - e^{r(t-T)} (1 - r(t-T))}{g^2} \right] - 1 \\
 &= \frac{1}{g^2 r^2} \left[ 1 - e^{r(t-T)} (1 - r(t-T)) - (1 - e^{r(t-T)})^2 \right] \\
 &= \frac{1}{(rg)^2} \left[ 1 - e^{r(t-T)} (1 - r(t-T)) - 1 + 2e^{r(t-T)} - e^{2r(t-T)} \right] \\
 &= \frac{1}{(rg)^2} \left[ e^{r(t-T)} \right] \left[ 1 + r(t-T) - e^{r(t-T)} \right].
 \end{aligned}$$

- This expression is clearly negative.
- Set  $x \equiv T - t$ :

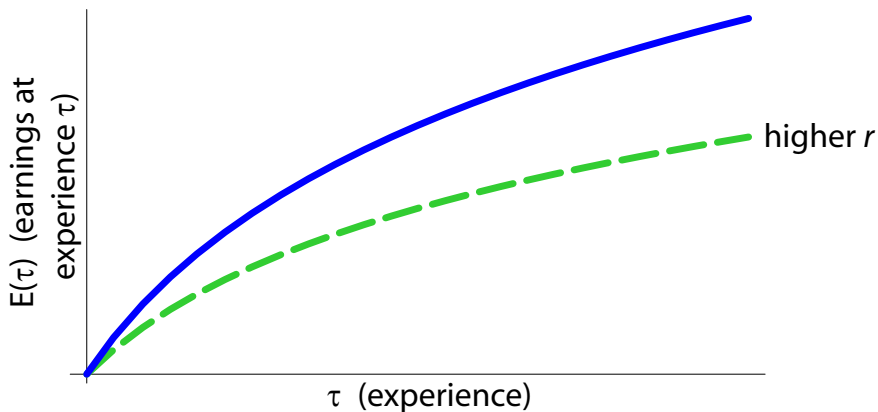
$$(1) \quad 1 - rx - e^{-rx} = 0 \quad \text{when } x = 0.$$

$$(2) \quad \frac{\partial}{\partial x} (1 - rx - e^{-rx}) = -r + re^{-rx} < 0.$$

- Thus from concavity ( $f'' < 0$ ),

$$\left( \frac{f'}{f''} \right) \frac{\partial}{\partial r} \left[ -\frac{\dot{g}}{g} \right] > 0.$$

- Now the proposition is proved for  $\sigma = 0$  with  $\ddot{E} < 0$  everywhere. *Q.E.D.*



**Appendix: Haley-Rosen:** Let  $\alpha = 1/2$ .

$$E(\tau) = RH(t^*) + R \int_0^\tau A \left( \frac{1}{2} \frac{g(t^* + \ell)A}{R} \right) d\ell - R \left[ \frac{1}{2} \frac{g(\tau + t^*)A}{R} \right]^2.$$

This can be written as a simple autoregression in earnings:

$$\begin{aligned} \dot{E}(\tau) &= R \left[ A \left( \frac{1}{2} \frac{g(t^* + \tau)A}{R} \right) - 2R \left[ \frac{1}{2} \frac{g(\tau + t^*)A}{R} \right] \frac{A}{2R} \dot{g}(\tau + t^*) \right] \\ &= \frac{1}{2} A^2 [g(t^* + \tau)(R - \dot{g}(t^* + \tau))]. \end{aligned}$$

$$\dot{g} = rg - R$$



Thus

$$\dot{E}(\tau) = \frac{A^2}{2R} [g(t^* + \tau) (R - \dot{g}(t^* + \tau))]$$

$$\dot{g} = rg - R$$

$$\ddot{g} = r\dot{g}.$$

**Haley-Rosen:**  $\alpha = \beta = 1/2$

$$E(\tau) = RH(t^*) + R \int_0^\tau A \left( \frac{1}{2} \frac{g(t^* + \ell)A}{R} \right) d\ell - R \left[ \frac{A}{2} \frac{g(\tau + t^*)}{R} \right]^2$$

$$\begin{aligned} \dot{E}(\tau) &= \frac{A^2}{2} g(\tau + \tau^*) - 2R \left[ \frac{A}{2} \frac{g(\tau + t^*)}{R} \right] \left[ \frac{A}{2R} \dot{g} \right] \\ &= \frac{A^2}{2} g(\tau + t^*) - \frac{1}{2} \frac{A^2}{R} g \dot{g} \\ &= \frac{1}{2} A^2 g \left[ 1 - \frac{\dot{g}}{R} \right] \quad \text{use: } \dot{g} = rg - R \\ &= \frac{1}{2} \frac{A^2}{R} g [R - \dot{g}] = \frac{A^2}{2R} g [R - rg + R] \\ &= \frac{A^2}{2R} g [2R - rg] \end{aligned}$$

$$\begin{aligned}\ddot{E}(\tau) &= \frac{A^2}{2R}[\dot{g}(2R - rg) + g(-r\dot{g})] \\ &= \frac{A^2}{2R}\dot{g}[2R - 2rg] = \frac{A^2}{R}\dot{g}(R - rg) \\ &= -\frac{A^2}{2}(\dot{g})^2.\end{aligned}$$

Notice that  $\dot{E}(\tau)$  can be written as

$$\begin{aligned}\dot{E}(\tau) &= \frac{A^2}{2R} \left( \frac{\dot{g} + R}{r} \right) \left( 2R - r \frac{(\dot{g} + R)}{r} \right) \\ &= \frac{A^2}{2R} \left( \frac{\dot{g} + R}{r} \right) (2R - \dot{g} - R) \\ &= \frac{A^2}{2R} \left( \frac{\dot{g} + R}{r} \right) (R - \dot{g}) = \frac{A^2}{2Rr} (R^2 - (\dot{g})^2).\end{aligned}$$

Thus we conclude that

$$\begin{aligned}\dot{E}(\tau) &= \frac{A^2}{2Rr}R^2 - \frac{1}{2r}\frac{A^2}{R}(\dot{g})^2 \\ &= \frac{A^2}{2Rr}R^2 + \frac{1}{2r}\ddot{E}\end{aligned}$$

so that

$$\ddot{E}(\tau) - 2r\dot{E}(\tau) + A^2R = 0.$$

Integrate once to reach

$$\dot{E}(\tau) - 2rE(\tau) + A^2R\tau + c_0 = 0$$

where  $c_0$  is a constant of integration.

Then “reduced equation” is

$$\dot{E}(\tau) = 2rE(\tau)$$

so that

$$E(\tau) = c_1 e^{2r\tau},$$

$c_1$  is constant of integration.

The general solution is thus:

$$E(\tau) = c_0 + c_2\tau + c_1 e^{2r\tau}.$$

For a period of specialization,  $E(0) = 0$  so that  $c_1 + c_0 = 0$ .

$$\dot{E}(\tau) = 2rc_1e^{2r\tau} + c_2$$

so that at  $\tau = 0$ ,

$$(2rc_1e^{2r\tau} + c_2) - 2r[c_1e^{2r\tau} + c_0 + c_2\tau] + A^2R\tau + c_0 = 0.$$

Thus we conclude that

$$c_2 = \frac{A^2R}{2r}$$

To this point, the equation looks like

$$E(\tau) = c_0(1 - e^{2r\tau}) + \frac{A^2 R}{2r} \tau.$$

Now there is no investment at the end of life.

$$\dot{E}(\tau) = 0.$$

Thus

$$\dot{E}(T) = 0 = -2rc_0e^{2rT} + \frac{A^2 R}{2r}$$

so  $c_0 = \frac{A^2 R}{(2r)^2} e^{-2rT}$ . Thus

$$E(\tau) = \frac{A^2 R}{(2r)^2} e^{-2rT} (1 - e^{2r\tau}) + \frac{A^2 R}{2r} \tau.$$