Generalized Ben-Porath Model

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Table 1. Estimates of the human capital production function (males)[a].

Source	α	β	γ	A	r	σ	Restricted schooling and OJT model?	Labor supply	Synthetic cohorts?
Heckman (1976) two models	0.99 (.003)	−6.69 (.043)	–	45.49 (3.034)	0.10 (imposed)	0.0016 (0.00025)	No	Yes	Yes
US Bureau of the Census (1960) males	0.67 (0.052)	0 (imposed)	–	0.14×10^{-2} (0.04×10^{-2})	0.10 (imposed)	0 (constrained)	No	Yes	Yes
Heckman (1976)	0.812 (0.0225)	α[b] (restricted)	–	1.53 (1.62)	0.176 (0.275)	0.089 (0.068)	No	No	Yes
US Bureau of the Census (1960) males	0.52 (0.07)	α[b] (restricted)	–	17.3 (25.2)	0.196 (0.613)	0.037 (0.90)	No	Yes	Yes
Haley (1976) CPS (1956–1966) aggregates	0.578 (0.012)	α[b] (restricted)	–	0.019–0.04	0.04–0.069 (0.004) (0.003)	0.005–0.04 (0.014) (0.008)	No[c]	No	Yes
Brown (1976)[d] NLS young men	0.56–0.89	α[b] (restricted)	–	[f]	0.33–0.15	0 (imposed)	No[e]	No	No
Rosen (1976) US Census 1960 and 1970	0.5	1 (imposed)	–	$r+\varepsilon$ ($\varepsilon>0$) (see next column)	0.0725 (highschool) 0.0875 (college)	176 (0.275)	No	No	Yes

[a] $H_{t+1} = (1-\sigma)H_t + AI_t^{\alpha}H_t^{\beta}D_t^{\gamma}$.
r, interest rate; standard errors are given in parentheses.
[b] $\alpha = \beta$.
[c] All schooling groups.
[d] Brown makes alternative assumptions about the rate of growth of the price of labor services. See also Rosen.
[e] Only highschool graduates.
[f] Not reported.

Source: Browning, Hansen and Heckman (1999)

Table 2 (references cited)

- Heckman, J.J. (1976), "A life-cycle model of earnings, learning, and consumption", *Journal of Political Economy* 84(4, pt. 2): S11–S44.
- US Bureau of the Census (1960), 1960 Census Public Use Sample (United States Government Printing Office, Washington, DC).
- Haley, W.J. (1976), "Estimation of the earnings profile from optimal human capital accumulation", *Econometrica* 44: 1223–38.
- Brown, C. (1976), "A model of optimal human-capital accumulation and the wages of young high school graduates", *Journal of Political Economy* 84(2): 299–316.
- Rosen, S. (1976), "A theory of life earnings", *Journal of Political Economy* 84(Suppl.): 345–382.

Table 2. Estimated parameters for human capital production function.

Parameter	Estimated value			
	Males		Females	
	Highschool ($S=1$)	College ($S=2$)	Highschool ($S=1$)	College ($S=2$)
α	0.945(0.017)	0.939(0.026)	0.967	0.968
β	0.832(0.253)	0.871(0.343)	0.810	1.000
$A(1)$	0.081(0.045)	0.081(0.072)	0.079	0.057
$H_0(1)^*$	9.530(0.309)	13.622(0.977)	6.696	8.347
$A(2)$	0.085(0.053)	0.082(0.074)	0.082	0.057
$H_0(2)^*$	12.074(0.403)	14.759(0.931)	7.806	9.453
$A(3)$	0.087(0.056)	0.082(0.077)	0.084	0.058
$H_0(3)^*$	13.525(0.477)	15.614(0.909)	8.777	11.563
$A(4)$	0.086(0.054)	0.084(0.083)	0.086	0.058
$H_0(4)^*$	12.650(0.534)	18.429(1.095)	9.689	13.061

Source: Heckman, Lochner and Taber (1998)

- Human capital production function:
  $H_{a+1}^{S} = A^{S}(\theta)(I_a^S)^{\alpha_S}(H_a^S)^{\beta_S} + (1-\sigma)(H_a^S)^{\beta_S}$, with $S = 1, 2$.
  Standard errors are given in parentheses.
- Heckman, Lochner and Taber (1999) do not report the standard errors for females.
- Initial human capital for person of ability quantile using ability levels for NLSY.

- Convention: $H(t), I(t)$ written as I, H unless it clarifies matters not doing so.
- Consider a more general Ben-Porath Model

$$\dot{H} = AI^{\alpha}H^{\beta} - \sigma H$$

Neutrality: $\alpha = \beta$.

- For simplicity assume no discounting ($r = 0$)
- No depreciation $\sigma = 0$
- Finite life $= T$
- Rental rate $= R$ (efficiency units, price of human capital)
- Initial endowment $= H_0$

- Problem ($0 \leq I \leq 1$; $0 < \alpha < 1$ for smooth problems):

$$\max \int_0^T [RH(t) - RI(t)H(t)]dt$$

such that $\dot{H} = AI^\alpha H^\beta$ and $H(0) = H_0$.

- Hamiltonian for problem:
Maximized Hamilton must be concave in state variable:

$$\mathcal{H} = RH(t)(1 - I(t)) + \mu(AI^\alpha H^\beta)$$

$\beta \leq 1$ needed for Mangasarian sufficient conditions.

- FOC:
$$\mu A \alpha I^{\alpha-1} H^{\beta} \geq RH \tag{$*$}$$
- Let "$\dot{}$" denote time rate of change.
$$\dot{\mu} = -\frac{\partial \mathcal{H}}{\partial H} = -R(1-I) - \beta \mu A I^{\alpha} H^{\beta-1}$$
- Rate of change of the shadow value of human capital declines with increases in the human capital stock.
- $\mu(T)H(T) = 0$ (transversality)
$$\mu(t) = \int_t^T \left[R(1 - I(u)) + \beta(\mu(u)) A I^{\alpha-1}(u) H^{\beta-1}(u) \right] \, du$$

- Now for the case with strict inequality in $(*)$, we have $I = 1$ (period of specialization associated with schooling or no earnings).

$$\begin{aligned} \alpha\mu A H^{\beta} &> RH \\ H^{\beta-1} &> \frac{R}{\alpha\mu A} \end{aligned}$$

- If $\beta > 1$, we get specialization in investment (no work) if

$$H > \left[\frac{R}{\alpha\mu A}\right]^{\frac{1}{\beta-1}}.$$

- Specialization at $t = 0$ requires

$$H_0 > \left[\frac{R}{\alpha \mu(0) A} \right]^{\frac{1}{\beta - 1}} = \left(\frac{R}{\alpha A} \right)^{\frac{1}{\beta - 1}} (\mu_0)^{\frac{1}{1-\beta}}$$

- If $\beta < 1$, specialization at $t = 0$ requires

$$H_0 < \left[\frac{R}{\alpha \mu(0) A} \right]^{\frac{1}{\beta - 1}} = \left(\frac{R}{\alpha A} \right)^{\frac{1}{\beta - 1}} (\mu_0)^{\frac{1}{1-\beta}} .$$

- When $\beta = 1$, specialization at $t = 0$ requires $\mu_0 > \dfrac{R}{\alpha A}$.

- Person just specializing ($I = 1$ is the interior solution) if

$$\alpha\mu A H^{\beta} = RH \qquad (I = 1)$$
$$\mu = \left(\frac{R}{\alpha A}\right) H^{1-\beta}$$

- In a period of specialization, $I = 1$

$$\dot{\mu} = -\beta\mu A H^{\beta-1}$$
$$\dot{H} = A H^{\beta}$$

- Then,

$$\frac{\dot{H}}{H^{\beta}} = A \qquad \text{or} \qquad \frac{dH}{H^{\beta}} = A\,dt$$

$$\frac{[H(t)]^{1-\beta}}{1-\beta} = At + c_0 \quad , \quad \beta \neq 1$$

$$H(t) = (At + c_0)^{\frac{1}{1-\beta}} (1-\beta)^{\frac{1}{1-\beta}}$$

(making t dependence explicit)

$$H(0) = H_0 = c_0^{\frac{1}{1-\beta}} (1-\beta)^{\frac{1}{1-\beta}}$$

$$\left(\frac{H_0}{(1-\beta)^{\frac{1}{1-\beta}}} \right)^{1-\beta} = c(0).$$

- When $\beta = 1$,

$$
\begin{aligned}
\ln H(t) &= At + c_0 \\
H(t) &= e^{At+c_0} \\
H(0) = H_0 &= e^{c_0} \qquad\qquad \ln H_0 = c_0.
\end{aligned}
$$

- When $\beta \neq 1$,

$$\dot{\mu} = -\beta \mu A [H]^{\beta - 1} = -\beta \mu A \left[\frac{1}{(At + c_0)(1 - \beta)} \right]$$

$$\frac{\dot{\mu}}{\mu} = \frac{-\beta}{1 - \beta} \cdot \frac{A}{At + c_0} \qquad c_0 \geq 0$$

$$\ln \mu(t) = -\left(\frac{\beta}{1 - \beta} \right) \cdot \ln(At + c_0) + c_1$$

$$\mu(t) = e^{c_1} e^{-\frac{\beta}{1-\beta} \ln(At + c_0)} = \frac{e^{c_1}}{(At + c_0)^{\beta/1-\beta}}$$

- At $t = 0$,

$$\mu(0) = \frac{e^{c_1}}{c_0^{\beta/1-\beta}} = \frac{e^{c_1}}{\left(\frac{H(0)}{(1-\beta)^{\frac{1}{1-\beta}}}\right)} = \frac{e^{c_1}}{H(0)^{\beta}}(1-\beta)^{\beta/1-\beta}$$

$$\frac{\mu(0)H(0)^{\beta}}{(1-\beta)^{\beta/1-\beta}} = e^{c_1}$$

$$c_1 = \ln\left[\frac{\mu(0)[H(0)]^{\beta}}{(1-\beta)^{\beta/1-\beta}}\right].$$

- When $\beta = 1$,

$$\frac{\dot{\mu}}{\mu} = -A$$

$$\ln \mu(t) = -At + c_1^*$$

$$\mu(t) = e^{c_1^*} e^{-At}$$

$$\mu(0) = e^{c_1^*} \qquad \ln \mu(0) = c_1^*$$

- Now at end of period of specialization, we must have

$$\mu(t^*)A\alpha[H(t)]^\beta = RH(t).$$

- Thus for $\beta = 1$, specialization ends (schooling ends) when

$$\mu(t^*)A\alpha = R \qquad \mu(t^*) = \left(\frac{R}{A\alpha}\right)$$

$$e^{c_1^*}e^{-At^*} = \left(\frac{R}{A\alpha}\right) \qquad \frac{e^{c_1^*}A\alpha}{R} = e^{At^*}$$

$$c_1^* + \ln\left(\frac{A\alpha}{R}\right) = At^* \qquad \frac{1}{A}\left[c_1^* + \ln\left(\frac{A\alpha}{R}\right)\right] = t^*.$$

- When $\beta \neq 1$

$$\mu(t^*)\frac{A\alpha}{R} = [H(t^*)]^{1-\beta} \qquad H(t^*) = \left[\frac{\mu(t^*)A\alpha}{R}\right]^{\frac{1}{1-\beta}}.$$

- Substituting from above we find that we get

$$\frac{e^{c_1}}{(At+c_0)^{\beta/1-\beta}}\frac{A\alpha}{R} = (At+c_0)(1-\beta).$$

- The Ben Porath case is $\alpha = \beta$.
- Therefore, $\dot{\mu} = -R$ (trivial dynamics).

$$\mu(t) = -Rt + c_1, \quad \mu(T) = 0 \Rightarrow c_1 = RT$$
$$\mu(t) = -R(T - t)$$

- General case:

$$\dot{\mu} = R\left[-1 + \underbrace{R^{\frac{1}{\alpha-1}}\left(\frac{1}{A}\right)^{\frac{1}{1-\alpha}}\alpha^{\frac{1}{1-\alpha}}\mu^{\frac{1}{1-\alpha}}H^{\frac{\beta-1}{1-\alpha}}}_{I}(1-\beta/\alpha)\right]$$

$$= R[-1 + I \underbrace{(1-\beta/\alpha)}_{\text{adjustment to } I}]$$

- Therefore, we have that if $\beta/\alpha > 1$, $\dot{\mu} < 0$.
- If $\beta < 0$, $\dot{\mu}$ might be > 0.
- Assume for the moment that $\beta \geq 0$. Then what do we have?
- $\dot{\mu} = -R$ during period of specialization.

- At the end of the period of specialization (if one occurs), we have that

$$I = 1 = \left(\frac{R}{A\alpha}\right)^{\frac{1}{\alpha-1}} \mu^{\frac{1}{1-\alpha}} H^{\frac{\beta-1}{1-\alpha}}. \qquad (**)$$

- Assume that $\dot{\mu} < 0$ for $0 < \alpha < 1$.
- As t increases, right hand side of $(**)$ decreases if $\dfrac{\beta - 1}{1 - \alpha} < 0$, i.e., $\beta < 1$.

$$\frac{\beta}{1-\beta}+1=\frac{\beta+1-\beta}{1-\beta}=\frac{1}{1-\beta}$$

$$\left[\frac{e^{c_1}}{1-\beta}\frac{A\alpha}{R}\right]=(At+c_0)^{\frac{1}{1-\beta}}$$

$$\frac{1}{A}\left[\frac{e^{c_1}}{1-\beta}\frac{A\alpha}{R}\right]^{1-\beta}-c_0=t^*$$

- End of first specialization period. (This is associated with schooling.)

- Question: Is there more than one period of specialization?
- Look ahead to interior segment. In the interior we get:

$$\mu A \alpha I^{\alpha-1} = R H^{1-\beta}$$

$$I^{\alpha-1} = \left(\frac{R}{\mu A \alpha}\right) H^{1-\beta}$$

$$I = \left(\frac{R}{\mu A \alpha}\right)^{\frac{1}{\alpha-1}} H^{\frac{1-\beta}{\alpha-1}} = \left(\frac{R}{A \alpha}\right)^{\frac{1}{\alpha-1}} \mu^{\frac{1}{1-\alpha}} H^{\frac{\beta-1}{1-\alpha}}$$

- Substitute into costate (shadow price) equation:

$$\dot{\mu} = -R(1 - I) - \beta\mu A I^{\alpha} H^{\beta-1}$$

$$= -(R) + R^{\frac{\alpha}{\alpha-1}} \left(\frac{1}{A}\right)^{\frac{1}{\alpha-1}} \mu^{\frac{1}{1-\alpha}} H^{\frac{\beta-1}{1-\alpha}} (\alpha)^{\frac{\alpha}{1-\alpha}} (\alpha - \beta)$$

  for $\beta > 0$, $\dot{\mu} < 0$.

- Then $I \downarrow$ monotonically over the life cycle when $\beta < 1$.
- $\beta = 1$, obviously $\mu(t) \downarrow \Rightarrow I(t) \downarrow$ monotonically.
- Therefore, we have at most one period of specialization, and it is early on (beginning of life).

- Take $\beta \neq 1$. For a person who specializes, the lifecycle is as follows:
    - $[0, t^*]$ school
    - $[t^*, T]$ work
- Then we solve from t^* on

$$\dot{\mu} = -R + R^{\alpha/\alpha-1}\left(\frac{1}{A}\right)^{\frac{1}{\alpha-1}} \mu^{\frac{1}{1-\alpha}} H^{\beta-1/1-\alpha}(\alpha)^{\alpha/1-\alpha}(\alpha-\beta)$$

$$\dot{H} = A\left[\frac{R}{A\alpha}\right]^{\frac{\alpha}{\alpha-1}} \mu^{\frac{\alpha}{\alpha-1}} H^{\frac{\alpha(\beta-1)}{1-\alpha}} H^{\beta}$$

$$= A\left[\frac{R}{A\alpha}\right]^{\frac{\alpha}{\alpha-1}} \mu^{\frac{\alpha}{1-\alpha}} H^{(\beta-\alpha)/(1-\alpha)}$$

for (μ, H) jointly. (This is a "split endpoint" problem.)

$$\mu(t) = \int_t^T \dot{\mu}(t)dt + c(3)$$

- Impose condition that $\mu(T) = 0$ for $t > t^* \Rightarrow c(3) = 0$.

$$H(t) = \int_{t^*}^t \dot{H}(\tau)d\tau + H(t^*)$$

- $\mu(t)$ and $H(t)$ must be solved jointly.
- Substitute for (t^*) above and enforce condition on $\mu(0)$. (Thus, $H(t^*)$ depends on $\mu(0)$ and $H(0)$, but $\mu(0)$ set in conjunction with $\mu(T) = 0$.)
- We know $\mu(t), H(t)$ and $I(t)$ continuous.
- $\dot{\mu}(t)$ need not be continuous at t^*.