Human Capital Accumulation and Earnings Dynamics over the Life Cycle: Lessons from the Ben-Porath Model

Prepared by: Jorge L. García and Yike Wang
The University of Chicago

This Draft: January 31, 2014

Human capital investment and accumulation are essential components of any model that aims to study some of the most relevant topics in Economics. Three representative examples are the economic growth and development of nations, the gender and black-white wage gaps, or the rate of return to schooling. Becker (1962) is the main predecessor in the formal analysis of human capital. His work offers the first unified and comprehensive framework to study human capital investment with the usual tools of Economics.[1]

Following this work, Ben-Porath (1967) proposes a dynamic model that relates human capital accumulation to life-cycle earnings. Currently, this is the workhorse model when it comes to analyze the relation between human capital accumulation decisions and life-cycle outcomes. The vast majority of papers that model human capital use some variation of the so-called Ben-Porath model (**see Figure 1**). These variations lead to very different time profiles of investment in human capital, human capital accumulation, and earnings.

In this paper, we analyze the Ben-Porath model and several of its variations in order to inform researchers on the consequences of their modeling decisions. We proceed as follows: in Section 1 presents the baseline specification. Section 2 specializes the model to a case with no depreciation and infinite horizon, in which many implications are easy to obtain in closed form and provide intuition and the possibility of straightforward graphical analysis. Section 3 analyzes what we call the Haley-Rosen specification, which enables for finite horizon but keeps tractability. Section 4 studies the model in its general formulation. Section 5 allows for depreciation and develops conditions under which investment in human capital happens in different episodes over the life-cycle. Section 6 offers some final comments.

1 Basic Ben-Porath Model

Assumption 1.1 *(Basic Ben-Porath Model) The assumptions of the basic Ben-Porath model are: (i) homogeneity (single representative agent); (ii) perfect capital markets; (iii) no non-market benefits from human capital; (iv) fixed labor supply; (v) constant depreciation of human capital, σ; (vi) finite horizon.*

Let Assumption 1.1 hold for the rest of this section. For each $t \in [0, T]$, H denotes human capital, $I \in [0, 1]$ allocation to investment in human capital, D market goods, and $F(I, D)$ the production function of human capital stock. Thus, the human capital stock is produced through two inputs, D and I.

Assumption 1.2 *(Strict Concavity of the Production Function) $\forall\ t\ in [0, T]$ $F(\cdot, \cdot)$ is strictly concave in both of its arguments.*

Definition 1.3 *(Law of Motion for Human Capital Stock in the Basic Ben-Porath Model) ... is defined as*

$$\dot{H(t)} = F\left(I(t)H(t), D(t)\right) - \sigma H(t). \tag{1}$$

[1] Becker (2009) collates the work of this author on the subject and summarizes his work on schooling, learning-by-doing, and on-the-job training.

Remark 1.4 *(Neutrality) In the basic Ben-Porath model the law of motion for human capital stock embeds a neutrality assumption. Namely, the current stock of human capital at time t, $H(t)$, and the investment time at time t, $I(t)$, appear as a single argument in a multiplicative fashion in the flow production of human capital stock.*

At each point of time, the current stock and the rental rate of human capital, R, define potential earnings as $Y(t) = RH(t)$. In general, earnings and potential earnings differ by two terms: (i) foregone earnings; (ii) direct market goods costs.

Definition 1.5 *(Earnings) Let P_D be the price of markets. Earnings are*

$$E(t) = RH(t) - RI(t)H(t) - P_D D(t) \tag{2}$$

where $RI(t)H(t)$ are foregone earnings and $P_D D(t)$ are direct goods costs.

Definition 1.5 clarifies that $I(t)$ is the allocation to investment in each period of time. In particular, the individual occupies a fraction $I(t)$ of her human capital stock to produce human capital. The individual chooses $D(t)$ and $I(t)$ to maximize her lifetime earnings stream given an initial level of human capital, $H(0) = H_0$ and subject to the law of motion for human capital, (1). Explicitly,

Problem 1.6 *(Life-cycle Individual's Problem in the Basic Ben-Porath Model)*

$$\max_{I_t, D_t} \int_0^T \exp^{-rt} RH(t)(1 - I(t))dt$$

s.t.

$$H(0) = H_0$$
$$(1) \textit{ holds.}$$

The **current value Hamiltonian** associated to Problem 1.6 is

$$\mathcal{H}(\cdot) = \exp^{-rt}\left[RH(t) - RI(t)H(t) - P_D D(t)\right] + \mu(t)\dot{H(t)} \tag{3}$$

where $\mu(t)$ defines the shadow price of the human capital stock. Assumption 1.1 guarantees that the necessary and sufficient conditions for optimality are the following.

Condition 1.7 *(Optimality Conditions for the Life-cycle Individual's Problem in the Basic Ben-Porath Model)*

$$\frac{\partial \mathcal{H}(\cdot)}{\partial I(t)} = 0 \quad \Leftrightarrow \quad \exp^{-rt} R = \mu(t) F_{I(t)H(t)} \tag{4}$$

$$\frac{\partial \mathcal{H}(\cdot)}{\partial D(t)} = 0 \quad \Leftrightarrow \quad \exp^{-rt} RI(t) = \mu(t) F_{D(t)} \tag{5}$$

$$\frac{\partial \mathcal{H}(\cdot)}{\partial H(t)} = -\dot{\mu(t)} \quad \Leftrightarrow \quad \exp^{-rt} R\left(1 - I(t)\right) + \mu(t)\left(F_{I(t)H(t)} - \sigma\right) = -\dot{\mu(t)} \tag{6}$$

$$\frac{\partial \mathcal{H}(\cdot)}{\partial \mu(t)} = \dot{H(t)} \quad \Leftrightarrow \quad \dot{H(t)} = F\left(I(t)H(t), D(t)\right) - \sigma H(t) \tag{7}$$

$$\textit{Transversality} \quad : \quad \lim_{t \to T} \mu(t)H(t) = 0 \tag{8}$$

where $F_j \equiv \frac{\partial F(I(t), D(t))}{\partial j}$ for $j = D(t), I(t)H(t)$.

In order to analyze some aspects of this model, it is useful to solve the period-by-period counterpart of Problem 1.6. We can think of this as a problem in which, each period, the agent maximizes gross investment in human capital less input costs. Provided that we have the adequate discount factor we can write the period-by-period counterpart. $\forall\ t \in [0, T]$ define $g(t)$ as a discount factor and write the period-by-period problem.

Problem 1.8 *(Period-by-Period Individual's Problem in the Baisc Ben-Porath Model)*

$$\max_{I(t),D(t)} \left[g(t)F\left(I(t), H(t)\right) - P_D D(t) - RI(t)H(t)\right].$$

Actually, we can interpret Problem 1.8 as a production problem: the individual is a firm that receives $g(t)$ (return to gross investment in human capital) for the production of human capital investment through the technology $F\left(I(t), H(t)\right)$. It pays the prices P_D, R for the inputs $D(t), I(t)H(t)$.

Claim 1.9 *(Life-cycle and Period-by-period Solution Equivalence) Let $g(t) \equiv \exp^{rt}\mu(t)$. Then, the solution to Problem 1.6 and Problem 1.6 are equivalent.*

Proof: The equivalence follows after comparing the first order conditions (note that $g(t)$ makes equivalent the first order conditions of the two problems). ♣

Economically, $g(t)$ is a discount factor that adjusts for the correct exponential depreciation of gross investment so that the period-by-period and the life-cycle solutions coincide. In order to analyze the solution combine (4) and (6) to get

$$\dot{\mu(t)} = -\exp^{-rt} R + \mu(t)\sigma \tag{9}$$

and note that $\dot{g(t)} = \dot{\mu(t)}\exp^{rt} + r\mu(t)e^{rt}$. Use (9) to obtain

$$\dot{g(t)} = (\sigma + r)g(t) - R. \tag{10}$$

Equation (8) implies that $\mu(T) = 0$ and, therefore, $g(T) = 0$ provided that $H(t) = 0$ because conditions under which $H(T) = 0$ have no economic sense. It is possible, thus, to solve (10) and obtain

$$g(t) = \frac{R}{\sigma + r}\left[1 - \exp^{(\sigma+r)(t-T)}\right]. \tag{11}$$

which leads to $\dot{g(t)} < 0$. To wrap up the discussion note that the optimality conditions for Problem 1.8 are the following.

Condition 1.10 *(Optimality Conditions for the Period-by-period Individual's Problem in the Basic Ben-Porath Model)*

$$\begin{aligned} g(t)F_{I(t)H(t)}H(t) &= RH(t) \\ g(t)F_{D(t)}H(t) &= P_D. \end{aligned} \tag{12}$$

The system in (12) consists of two equations and two unknowns that solve for the Marshallian demands for $I(t)H(t)$ and $D(t)$. Assumption 1.2 together with $\dot{g(t)} < 0$ imply that the both Marshallian demands are decreasing overtime, which is intuitive because the agent faces a finite horizon problem.

1.1 Earnings Dynamics

One of the fundamental questions that this basic model enables to ask is how earnings evolve over the life-cycle. Consider Claim 1.11 and Claim 1.12

Claim 1.11 *(Earnings over Time with no Depreciation) Let $\sigma = 0$. Then, $\dot{E(t)} > 0$.*

Proof: Differentiate (2) and use (1) to write

$$\begin{aligned} \dot{E(t)} &= R\dot{H(t)} - RI(t)\dot{H}(t) - P_D\dot{D(t)} \\ &= RF\left(I(t)H(t), D(t)\right) - RI(t)\dot{H}(t) - P_D\dot{D(t)} \\ &> 0 \end{aligned} \tag{13}$$

where the equality follows because the Marshallian demands for $I(t)H(t)$ and $D(t)$ are decreasing over time. ♣

Claim 1.12 *(Earnings over Time with no Depreciation) Let $\sigma > 0$. Then, $\dot{E(t)} \lesseqgtr 0$.*

Proof: Follow the same steps as in the proof of Claim 1.11 and note that the term $R\sigma H(t)$ appears in the expression for $\dot{E(t)}$. This terms could be $\lesseqgtr RF\left(I(t)H(t), D(t)\right) - RI(t)\dot{H}(t) - P_D\dot{D(t)}$. ♣

Claim 1.11 follows because the solution for human capital investment is interior $\forall\ t \in [0, T]$. Since there is no depreciation, individuals continuously add to their human capital stock and earn R for each accumulated unity. Claim 1.12 follows because the interior solution for human capital investment may be driven down by a relatively high rate of depreciation.

Proof: Differentiate (2) and use (1) to write

$$\begin{aligned} \dot{E(t)} &= R\dot{H(t)} - RI(t)\dot{H}(t) - P_D\dot{D(t)} \\ &= RF\left(I(t)H(t), D(t)\right) - RI(t)\dot{H}(t) - P_D\dot{D(t)} \\ &> 0 \end{aligned} \tag{14}$$

where the equality follows because the Marshallian demands for $I(t)H(t)$ and $D(t)$ are decreasing over time. ♣

1.2 Concavity

We now analyze the curvature of the earnings function for the case in which there is no depreciation.[2]. Without loss of generality we assume away $D(t)$, i.e. $F_{D(t)} = 0$ so that the production function takes the single argument $I(t)H(t)$, and $R \equiv 1$.

Claim 1.13 *(Concavity of the Earnings Function with no Depreciation) Assume $\eta \equiv \left(1 - \frac{F'F'''}{F''^2}\right) < 0$. Then, the earnings function is strictly concave.*

[2] A similar analysis follows when $\sigma > 0$ for the cases in which either $\dot{E(t)} > 0$ or $\dot{E(t)} < 0$

Proof: First note that $\dot{E(t)} > 0$ by Claim 1.11. Since $F_{D(t)} = 0$ we can write the first order for investment becomes

$$g(t)F'(I(t)H(t)) = 1 \tag{15}$$

and we can differentiate it with respect to t to get

$$\begin{aligned} \dot{g(t)}F'(I(t)H(t)) + g(t)F''(I(t)H(t))I(t)\dot{H}(t) &= 0 \\ &\Leftrightarrow \\ I(t)\dot{H}(t) &= -\left(\frac{\dot{g(t)}}{g(t)}\right)\left[\frac{F'}{F''}\right]. \end{aligned} \tag{16}$$

Moreover, drop the argument t to shorten notation, and note that

$$I\ddot{H} = -\left[\frac{\ddot{g}}{g} - \left(\frac{\dot{g}}{g}\right)^2\right]\frac{F'}{F''} + \left(\frac{\dot{g}}{g}\right)^2\left[1 - \frac{F'F'''}{F''^2}\right]\left[\frac{F'}{F''}\right] \tag{17}$$

where we substitute in (16). Further, note that

$$\begin{aligned} \dot{E} &= F(IH) - I\dot{H} - \sigma H \\ \ddot{E} &= F'(IH)I\dot{H} - I\ddot{H} - \sigma\dot{H} \\ &= \frac{1}{g}I\dot{H} - I\ddot{H} - \sigma\dot{H}. \end{aligned} \tag{18}$$

and from (10) obtain $\frac{\ddot{g}}{g} = r\frac{\dot{g}}{g}$. Thus,

$$\begin{aligned} \ddot{E} &= -\frac{\dot{g}}{g}\frac{F'}{F''}\left[\frac{1}{g} - \frac{\dot{g}}{g}\left(1 - \frac{F'F'''}{F''^2}\right)\right] + \left[r\frac{\dot{g}}{g} - \left(\frac{\dot{g}}{g}\right)^2\right]\frac{F'}{F''} \\ &= -\frac{\dot{g}}{g}\frac{F'}{F''}\left[\frac{1}{g} - \frac{\dot{g}}{g}\left(1 - \frac{F'F'''}{F''^2}\right) - \frac{gr - \dot{g}}{g}\right] \\ &= -\frac{\dot{g}}{g}\frac{F'}{F''}\left[\frac{1}{g} - \frac{\dot{g}}{g}\left(1 - \frac{F'F'''}{F''^2}\right) - \frac{1}{g}\right] \\ &= -\left(\frac{\dot{g}}{g}\right)^2\frac{F'}{F''}\left(1 - \frac{F'F'''}{F''^2}\right) \end{aligned} \tag{19}$$

where the third equality uses (10), i.e. $gr - \dot{g} = 1$. F is strictly concave and therefore $-\left(\frac{\dot{g}}{g}\right)^2\frac{F'}{F''} > 0$. Since $\left(1 - \frac{F'F'''}{F''^2}\right) < 0$ the claim follows. ♣

A sufficient condition for $\dot{E(t)}$ to be concave is $\eta < 0$. This crucially depends on how concave F''' is. Intuitively, given that there is no human capital depreciation, the production function for human capital investment needs to be concave enough to make the earnings function concave.

Example 1.14 *(Human Capital Production Functions and Earnings Concavity)*

- *Power Production Function 1 : consider the case of $F(x) = \frac{Ax^\alpha}{\alpha}$ for $-\infty < \alpha < 1, A > 0$. Then, $\eta = \frac{1}{\alpha - 1} < 0$. Under this specification the earnings function is strictly concave with respect to time.*

- *Power Production Function 2 : consider the case of* $F(x) = a - bx^{-\alpha}$ *for* $-1 < \alpha < \infty, a, b, c > 0$*. Then,* $\eta = \frac{-1}{\alpha+1} < 0$*. Under this specification the earnings function is strictly concave with respect to time.*

- *Power Production Function 3: consider the case of* $F(x) = a - b\exp^{-cx}$ *with* $b, c > 0$*. Then,* $\eta = 0$*.*

- *Quadratic Production Function: any quadratic production function has* $F''' = 0$ *and does not induce concavity of earnings with respect to time.*

Importantly, all this examples consider no depreciation of human capital, $\sigma = 0$*.*

1.3 Specialization Period

Specialization happens when the agent devotes his complete human capital to produce human capital stock, i.e. when $I(t) = 1$ for $t \in [\underline{t}, \bar{t}]$. In order to analyze some of the properties of specialization periods we assume away $D(t)$ so that $F_{D(t)} = 0$ and rule out depreciation.

Recall that we can interpret $g(t)$ as the return to investment in human capital investment and that we show above that in this case $\dot{g} < 0$. Then, there is at most one period of specialization at the beginning of the time horizon, if it happens. We denote this by $[0, t^*]$. Ben-Porath (1967) calls this the schooling period and it happens under the conditions that follow.

Condition 1.15 *(Conditions for the Existence of a Period of Specialization in the Basic Ben-Porath Model with no Depreciation)*

$$\begin{aligned} F'(H(t))g(t) &> R \\ F'(H(t^*))g(t^*) &= R \\ I(t^*) &= 1 \\ H(t^*) &= \int_0^{t^*} F(H(\tau))d\tau + H_0 \end{aligned} \tag{20}$$

where $H(t^*)$ *is the human capital stock accumulated up to time* t^**, the sum over the period* $[0, t^*]$ *plus the initial stock.*

Given that R is fixed, any decrease in $g(t)$ lowers t^* because it lowers the return to gross investment in human capital. For example, relatively high r implies relatively low t^* because the individual is relatively present oriented. Also, from (20), note that a high value of t^* implies a lower value for t^* because it takes less time to obtain $H(t^*)$. If $\sigma > 0$ the the same conditions characterize specialization. However, the there may be more than one specialization period because, under some scenarios, a high value of σ may knock off capital such that various investment episodes are optimal. We differ that case for Section 5.2.

Case 1.16 *(No Depreciation and the Cobb-Douglas Production Function for Human Capital: Initial Human Capital and the Specialization Period) In this case $\dot{H} = A\,(IH)^{\alpha}$ where $0 < \alpha < 1, A > 0$. As argued above, specialization happens in the period $[0, t^*]$. Thus*

$$
\begin{aligned}
\alpha A\,(H(0))^{\alpha-1}\,g(0) &> R \\
&\Leftrightarrow \\
H(0) &< \left[\frac{R}{g(0)\alpha A}\right]^{\frac{1}{\alpha-1}}.
\end{aligned} \tag{21}
$$

As the conditions in (20) establish, the time spent in specialization is a decreasing function of $H(0)$. In this example, actually, the initial human capital needs to be below certain threshold in order for the individual to specialize during one period.

Case 1.17 *(No Depreciation and the Cobb-Douglas Production Function for Human Capital: Infinite Horizon, Initial Human Capital and the Specialization Period) In the setting of Case 1.16 and if the horizon of the problem is infinite: $H(0) < \left(\frac{\alpha A}{r}\right)^{\frac{1}{1-\alpha}}$ because $g(t) = \frac{R}{r}$.*

Case 1.18 *(No Depreciation and the Cobb-Douglas Production Function for Human Capital: the Specialization Period) In the period of specialization $I(t) = 1$. Then,*

$$\dot{H} = A\,(H)^{\alpha}. \tag{22}$$

The general solution for (22) is

$$H(t) = \left[(1-\alpha)(At + K)\right]^{\frac{1}{1-\alpha}} \tag{23}$$

for some constant K. Given an initial condition $H(0) = H_0$, $K = \frac{H_0^{1-\alpha}}{1-\alpha}$ and

$$H(t) = \left[(1-\alpha)At + H_0^{1-\alpha}\right]^{\frac{1}{1-\alpha}}. \tag{24}$$

At the end of the specialization period, as established in (20):

$$\alpha g(t^*)A\,(H(t^*))^{\alpha-1} = R. \tag{25}$$

If $T \to \infty$, $g(t) = \frac{R}{r}$ and

$$t^* = -\frac{H_0^{1-\alpha}}{A(1-\alpha)} + \frac{\alpha}{1-\alpha}\frac{1}{r}. \tag{26}$$

(26) provides some intuitive results: (i) an individual with relatively high initial human capital specializes during a relatively shorter period: $\frac{\partial t^*}{\partial H_0} < 0$; (ii) a relatively abler individual specializes during relatively longer period: $\frac{\partial t^*}{\partial A} > 0$; a relatively impatient individual specializes for a relatively shorter period: $\frac{\partial t^*}{\partial r} < 0$.

Case 1.19 *(No Depreciation and the Cobb-Douglas Production for Human Capital: Post-experience Earnings) Let $\tau = t - t^*$ define the post-school work experience and write post-school earnings as follows:*

$$E(\tau) = R\int_0^{\tau} \dot{H}(l + t^*)dl + RH(t^*) - RIH(\tau + t^*). \tag{27}$$

Now, from (20) *the following equality holds:*

$$\begin{aligned} \alpha g(t) A \left(IH(t)\right)^{\alpha-1} &= R \\ &\Leftrightarrow \\ IH(t) &= \left[\frac{\alpha g(t) A}{R}\right]^{\frac{1}{1-\alpha}} \end{aligned} \tag{28}$$

Combining (28) *and the law of motion for human capital:*

$$\dot{H} = A\left[\frac{\alpha g(t) A}{R}\right]^{\frac{\alpha}{1-\alpha}}. \tag{29}$$

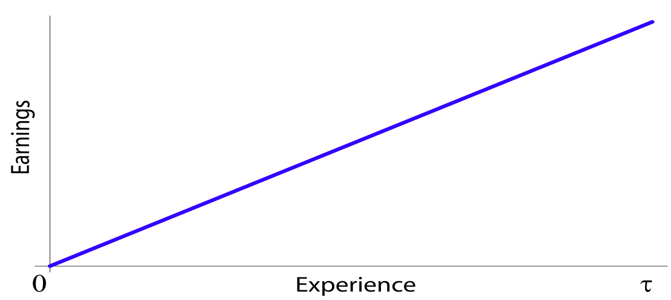
Then,

$$E(\tau) = R\int_0^\tau A\left[\frac{\alpha g(l+t^*)A}{R}\right]^{\frac{\alpha}{1-\alpha}} dl + RH(t^*) - R\left[\frac{\alpha g\left(\tau+t^*\right)A}{r}\right]^{\frac{1}{1-\alpha}} \tag{30}$$

and if $T \to \infty$

$$E(\tau) = RA\left[\frac{\alpha A}{R}\right]^{\frac{\alpha}{1-\alpha}}\tau. \tag{31}$$

Figure 1: Earnings and Experience, Cobb Douglas Technology and No Depreciation



Note:

1.4 The Baseline Model Dynamics under the Cobb-Douglas Specification: a Summary

This section summarizes the dynamics of the main variables in the baseline model when there is no depreciation, market goods are ruled out, and the production function for human capital investment is Cobb-Douglas. We assume that the horizon is infinite to simplify the algebra but it is important to remark that the qualitative properties of the results remain unchanged under finite horizon. To wrap up the section we show simulations that illustrate how the variables of interest behave under various parametrizations (in all of them we set $R = 1$).

1.4.1 Human Capital

- At $t = 0$ an initial condition is given.
- At $0 < t < t^*$ the system (20) provides the conditions that human capital satisfies and its expression is given by (24).
- At $t = t^*$ (24) is still a valid expression for human capital. To obtain the exact quantity it suffices to evaluate the expression for t^*, (26), into (24).
- At $t > t^*$ (20) and the expression for $\dot{H}$, (29), provide the expression for human capital.

Then,

$$H(t) = \begin{cases} H_0 & t = 0 \\ \left[(1-\alpha)At + H_0^{1-\alpha}\right]^{\frac{1}{1-\alpha}}, & 0 < t < t^* \\ \left[\frac{\alpha A}{r}\right]^{\frac{1}{1-\alpha}}, & t = t^* \\ \left[\frac{\alpha A}{r}\right]^{\frac{\alpha}{1-\alpha}}(t-t^*) + \left[\frac{\alpha A}{r}\right]^{\frac{1}{1-\alpha}}, & t > t^*. \end{cases} \tag{32}$$

1.4.2 Investment

We focus on the case in which there is an specialization period, i.e. the case in which (21) holds. The combination of (28) and (32) gives the following

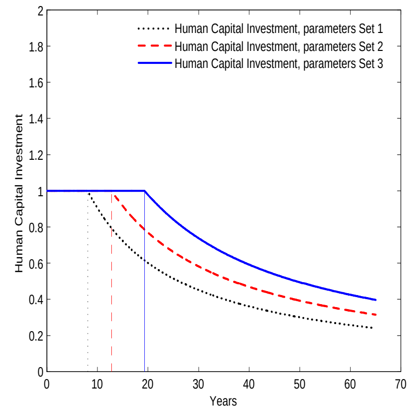
$$I(t) = \begin{cases} 1, & t = 0 \\ 1, & 0 < t < t^* \\ 1, & t = t^* \\ \frac{\left[\frac{\alpha A}{r}\right]^{\frac{1}{1-\alpha}}}{\left[\frac{\alpha A}{r}\right]^{\frac{\alpha}{1-\alpha}}(t-t^*) + \left[\frac{\alpha A}{r}\right]^{\frac{1}{1-\alpha}}}, & t > t^*. \end{cases} \tag{33}$$

1.4.3 Earnings

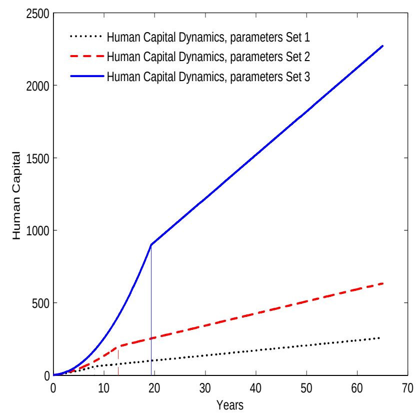
For the case of earnings we also on the case in which there is an specialization period, i.e. the case in which (21) holds. Thus, (2), (32), (33) define earnings as follows

$$E(t) = \begin{cases} 0, & t = 0 \\ 0, & 0 < t < t^* \\ 0, & t = t^* \\ RA\left[\frac{\alpha A}{r}\right]^{\frac{\alpha}{1-\alpha}}(t-t^*), & t > t^*. \end{cases} \tag{34}$$

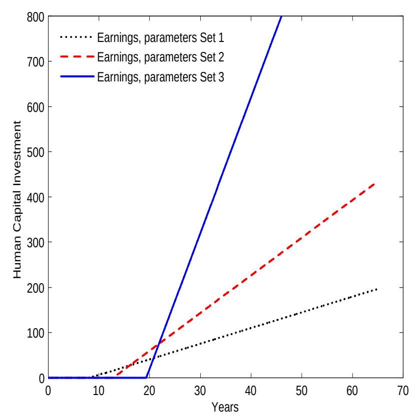
Figure 2: Dynamics for $A = 3, r = .05, H_0 = 1$
$\alpha = .3$ (dotted); $\alpha = .4$ (dashed); $\alpha = .5$ (solid)



(a) Human Capital Investment

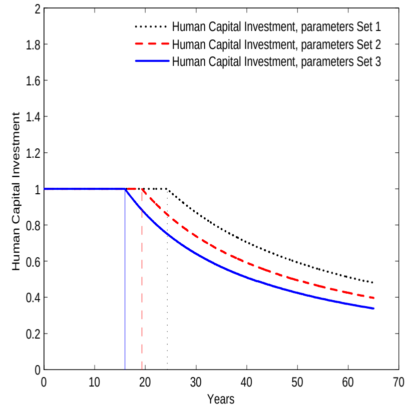


(b) Human Capital Stock

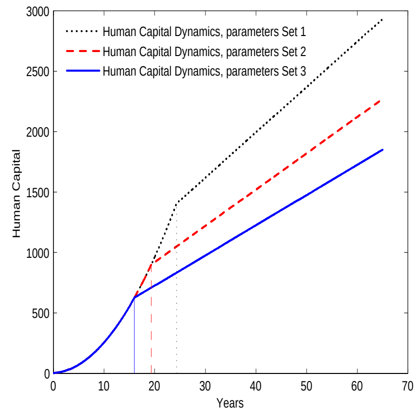


(c) Earnings

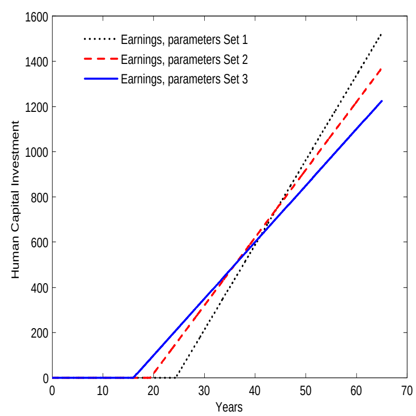
Figure 3: Dynamics for $A = 3, \alpha = .5, H_0 = 1$
$r = .04$ (dotted); $r = .05$ (dashed); $r = .06$ (solid)



(a) Human Capital Investment

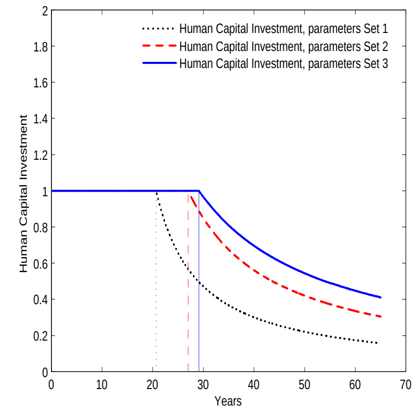


(b) Human Capital Stock

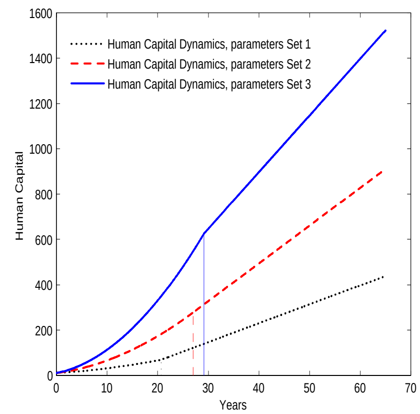


(c) Earnings

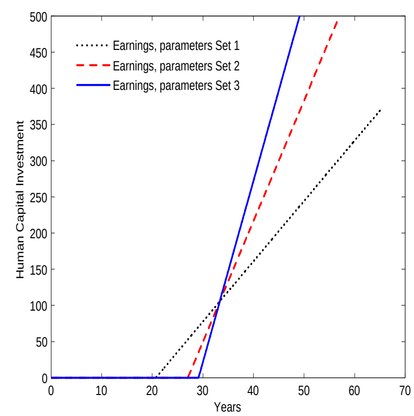
Figure 4: Dynamics for $r = .03, \alpha = .5, H_0 = 10$
$A = .5$ (dotted); $A = 1.0$ (dashed); $A = 1.5$ (solid)



(a) Human Capital Investment

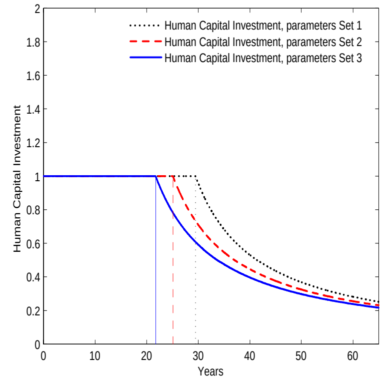


(b) Human Capital Stock

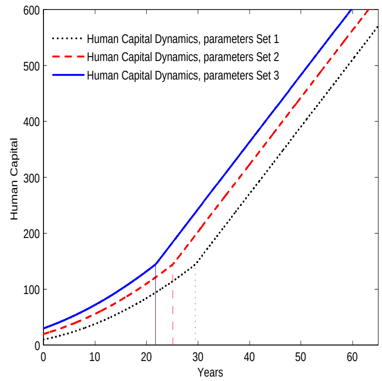


(c) Earnings

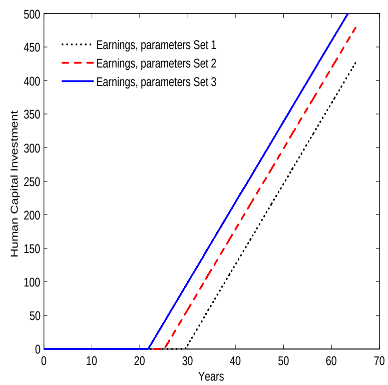
Figure 5: Dynamics for $r = .025, \alpha = .5, A = .6$
$H_0 = 10$ (dotted); $H_0 = 20$ (dashed); $H_0 = 30$ (solid)



(a) Human Capital Investment



(b) Human Capital Stock



(c) Earnings

1.5 Rates of Return under the Cobb-Douglas Specification

We use this model to analyze returns both to schooling and post-schooling. In order to simplify the expressions we let $t \to T$ and we speculate that the analysis is not very sensitive to this assumption.

1.5.1 Schooling

We call schooling the period of specialization in which the individual devotes his complete human capital stock to produce human capital investment. To define the return to schooling consider two scenarios: (i) the individual does not invest either in schooling or in post-schooling. Each τ she earns RH_0; (ii) the individual invests during an specialization period and then does not invest after that. Each τ she earns $R\left[\frac{\alpha A}{r}\right]^{\frac{1}{1-\alpha}}$. Then, we can define the (internal) rate of return of schooling as follows.

Definition 1.20 *("Internal" Rate of Return to Schooling) φ is the (internal) rate of return to schooling and solves the equation*

$$\begin{aligned}
\int_{t^*}^{\infty} \exp^{-\varphi t} R\left[\frac{\alpha A}{r}\right]^{\frac{1}{1-\alpha}} dt &= \int_0^{\infty} \exp^{-\varphi t} RH_0 dt \\
&\Rightarrow \\
\varphi &= \frac{\ln\left[\frac{\alpha A}{r}\right]^{\frac{1}{1-\alpha}} - \ln H_0}{\frac{1}{r} - \frac{1}{2}\frac{H_0^{\frac{1}{2}}}{A}}. \qquad (35)
\end{aligned}$$

1.5.2 Post-schooling

Let $E(\tau)^{NPS}$ and $E(\tau)^{PS}$ denote earnings without and with post-schooling investment, respectively. By (34) we can write

$$\begin{aligned}
E(\tau)^{NPS} &= RH(t^*) \qquad (36) \\
&= R\left(\frac{\alpha A}{r}\right)^{\frac{1}{1-\alpha}} \\
E(\tau)^{PS} &= \left[\frac{\alpha A}{r}\right]^{\frac{\alpha}{1-\alpha}} \tau
\end{aligned}$$

so that the increment in earnings due to post-schooling at τ is

$$\Delta^{E(\tau)} \equiv E(\tau)^{PS} - E(\tau)^{NPS}. \qquad (37)$$

Actually, note that $E(\tau)^{PS} = IH(\tau)$ (see (28)) so that we can interpret $\Delta^{E(\tau)}$ as "returns less costs" from post-schooling. Then, we define the (internal) rate of return to post-schooling as follows.

Definition 1.21 *("Internal" Rate of Return to Post-schooling) ϕ is the (internal) rate of return to schooling and solves the equation*

$$\int_0^{\infty} \exp^{-\phi\tau}\left[\left[\frac{\alpha A}{r}\right]^{\frac{\alpha}{1-\alpha}}\tau - R\left(\frac{\alpha A}{r}\right)^{\frac{1}{1-\alpha}}\right] d\tau = 0 \qquad (38)$$

Using the Laplace transform, (38) implies

$$\begin{aligned}
\frac{1}{\phi^2} RA\left[\frac{\alpha A}{r}\right]^{\frac{\alpha}{1-\alpha}} + \frac{1}{\phi} A\left[\frac{\alpha A}{r}\right]^{\frac{1}{1-\alpha}} &= 0 \\
&\Rightarrow \\
\phi &= \frac{r}{\alpha}. \qquad (39)
\end{aligned}$$

The (internal) rate of return to post-schooling investment is a decreasing function of α. Individuals who are more productive require a smaller return in order to invest in the post-school period. Likewise, relatively patient individuals (relatively low r) require a smaller ϕ to invest in the post-schooling period.

1.6 Earnings Growth and Patience in Finite Horizon

In this section we want to ask, in the same framework, how earnings growth depend on what defines relative patience in this model, the discount rate r. To do that, we investigate $\frac{\partial \dot{E(\tau)}}{\partial r}$.

Claim 1.22 *Assume that* $1 - \frac{F'(\cdot)F'''(\cdot)}{F''^2} < 0$ *(recall from Claim 1.13 that this is a sufficient condition for* $\dot{E(t)}$ *in the current context). Then,* $\frac{\partial \dot{E(\tau)}}{\partial r} < 0$.

Proof: Without loss of generality, assume that R =1 and note that

$$\frac{\partial \dot{E(\tau)}}{\partial r} = F'(\cdot)\frac{\partial IH}{\partial r} - \frac{\partial}{\partial r}\dot{IH}. \tag{40}$$

From (12) we know that the first order condition of the agent's problem is

$$g(t)F'(\cdot) = 1 \tag{41}$$

which by the implicit function theorem yields

$$\begin{aligned} \frac{\partial IH}{\partial r} &= \frac{\frac{\partial g(t)}{\partial r}F'(\cdot)}{2g(t)F''(\cdot)} \\ &< 0 \end{aligned} \tag{42}$$

where the inequality follows from strict concavity of $F(\cdot)$ and $g(t) > 0, \frac{\partial g(t)}{\partial r} < 0$ (see (45)). Thus, the first term in (40) is negative. If we show that the second term is negative then we can sign (40) and provide a meaning for this results. In order to do that we need $\frac{\partial \dot{IH}}{\partial r} > 0$. From (16) note that

$$\frac{\partial \dot{IH}}{\partial r} = -\frac{\dot{g}}{g}\left[1 - \frac{F'(\cdot)F'''(\cdot)}{F''(\cdot)^2}\right]\frac{\partial IH}{\partial r} + \frac{F'(\cdot)}{F''(\cdot)}\frac{\partial}{\partial r}\left[-\frac{\dot{g}}{g}\right] \tag{43}$$

We know that $1 - \frac{F'(\cdot)F'''(\cdot)}{F''^2} < 0$ and $\dot{g}, \frac{\partial IH}{\partial r} < 0$ the first term in (43) is positive. To sign the second term note that $\dot{g} = rg - 1, -\frac{\dot{g}}{g} = \frac{1}{g} - r$. Then,

$$\frac{\partial}{\partial r}\left[-\frac{\dot{g}}{g}\right] = -\frac{1}{g^2}\frac{\partial g}{\partial r} - 1. \tag{44}$$

To sign (44) note that

$$\begin{aligned} \frac{\partial g}{\partial r} &= \frac{\exp^{r(t-T)}(1 - r(t-T)) - 1}{r^2} \\ &< 0 \end{aligned} \tag{45}$$

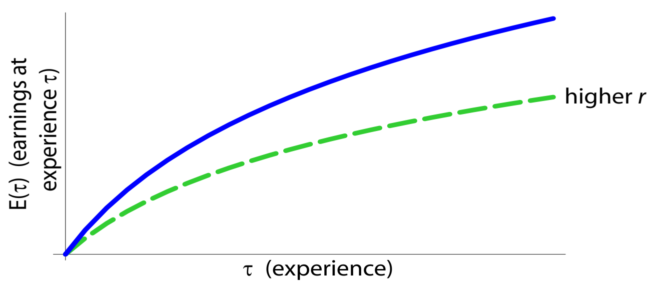
and

$$-\frac{\partial g}{g^2 \partial r} - 1 \;=\; \frac{1}{r^2 g^2} \exp^{r(t-T)} \left(1 + r(t-T) - \exp^{r(t-T)}\right). \\ \;<\; 0 \qquad (46)$$

which implies that $\frac{\partial \dot{E}}{\partial r} < 0$. ♣

The graphical representation of Claim 1.22 is in Figure 6. It implies that the earnings function is relatively "less concave" for relatively impatient individuals (relatively high r). This is a consequence of their investment decisions: they spent less time in the schooling period and accumulate less human capital.

Figure 6: Earnings Profiles in Finite Horizon for Different Values of r



2 The Haley-Rosen Specification: Finite Horizon and the Autoregression Form

Haley (1976) and Rosen (1976) consider a model in which Assumption 1.1 and Assumption 1.2 hold. They further assume that $\dot{H} = A\,(IH)^{\frac{1}{2}}$. This is, they assume that the production function for human capital investment is Cobb-Douglas with an specific value for α and that there is no depreciation.

Definition 2.1 *(Law of Motion for Human Capital Stock in the Haley-Rosen Specification)*

$$\dot{H} = A\,(IH)^{\frac{1}{2}}. \qquad (47)$$

In Section 1, some of the results rely on infinite horizon to derive a set of closed form solutions to the individual's problem. This specification allows for tractability of the finite horizon case. In particular, we focus on the dynamics of post-schooling earnings because one of the implications of the infinite horizon case in Section 1 is the linear relation between earnings and experience an may be a caveat.[3]

[3] Estimates of the so-called Mincer equation usually establish a non-linear relation between earnings an experience (see Heckman et al., 2006)

Claim 2.2 *(Concavity of Earnings in the Haley-Rosen Specification) Consider the Haley-Rosen Specification of the Basic Ben-Porath Model explained in Section 1, i.e. assume that (47) is the law of motion for human capital. Then, the earnings function is strictly concave in experience when the time horizon is finite.*

Proof: From (30) we can write

$$
\begin{aligned}
E(\tau) &= RH(t^*) + R\int_0^\tau A\left[\frac{1}{2}\frac{g(t^*+l)A}{R}\right]dl - R\left[\frac{1}{2}\frac{g(t^*+\tau)A}{R}\right]^2 \\
&\Rightarrow \\
\dot{E(\tau)} &= \frac{g(t^*+\tau)A^2}{2R}\left(2R - rg(t^*+\tau)\right) \\
&\Rightarrow \\
\ddot{E(\tau)} &= -\frac{A^2}{R}\dot{g(t^*+\tau)}^2 \qquad (48)
\end{aligned}
$$

where the second and third equalities use (10). Combining (10) and (48) we obtain a second order ODE with constant coefficients:

$$\ddot{E(\tau)} = 2r\dot{E(\tau)} - A^2R \qquad (49)$$

where the natural initial and terminal conditions that we impose are $E(0)$ and $\ddot{E(T)} = 0$ and then we guess and verify that $c_2 = \frac{A^2R}{2r}\exp^{2rT}$ in the following general solution to (49)

$$E(\tau) = c_0 + c_1\exp^{-2r\tau} + c_2\tau \qquad (50)$$

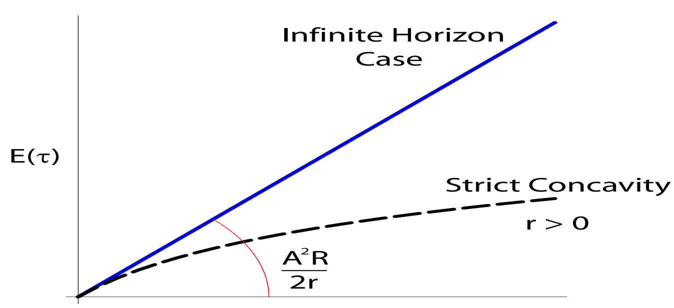
so that $c_1 + c_0 = 0, 2rc_1\exp^{2rT} + c_2 = 0$ and, therefore,

$$E(\tau) = \frac{A^2R}{4r^2}\exp^{-2rT}\left(1 - \exp^{2r\tau}\right) + \frac{A^2R}{2r}\tau \qquad (51)$$

which is strictly concave in τ. ♣

The graphical representation of Claim 2.2 is in Figure 7.

Figure 7: Post-school Earnings in the Haley-Rosen Specification



Note:

As $t \to T$ the earnings function becomes less concave on experience. This follows from the trade-off that the individual faces when investing in the two different time horizons. When the individual faces an infinite horizon it is optimal for him to keep investing in human capital at a constant rate, even after the specialization period, as times goes by. In finite horizon, however, the incentives to invest decrease over time, and therefore, earnings increase at a decreasing rate.

2.1 Evidence

Brown (1976) estimates (51), which enables to identify r and A^2R. His estimates, however, are imprecise and show that $r \to 0$. Then, he estimates the model for the infinite horizon case. He claims this to be a good approximation because he has a sample of young individuals. However, this disables him to estimate r.

2.2 The Autoregression

The Haley-Rosen specification enables straightforward analysis of the earnings dynamics. From (51) it is possible to write

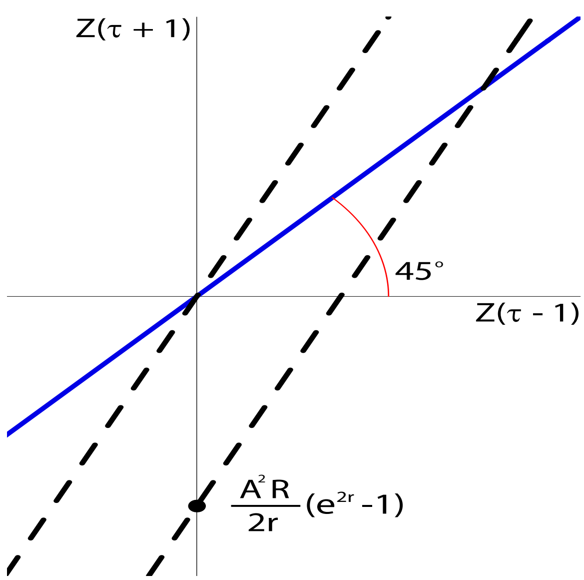
$$E(\tau+1) - E(\tau) = \frac{A^2R}{2r} + \frac{A^2R}{4r^2} \exp^{-2rT} \left(\exp^{2r(\tau-1)} \exp^{r\tau}\right) \tag{52}$$

which implies that

$$z(\tau+1) = \exp^{2r} z(\tau) + \frac{A^2R}{2r} \left(1 - \exp^{2r}\right) \tag{53}$$

where $z(\tau) \equiv E(\tau+1) - E(\tau)$. The graphical representation of (53) in Figure 8 may mislead to the conclusion that earnings growth follow an explosive dynamic. Claim 2.3 clarifies this.

Figure 8: Earnings Growth in the Haley-Rosen Representation



Claim 2.3 *Earnings growth, $z(\tau)$, converges to a constant.*

Proof: Informally, note that

$$\begin{aligned} \frac{\partial\left[E(\tau)-E(\tau-1)\right]}{\partial\tau} &= \frac{A^2R}{2r}\exp 2r(\tau-T)\left[\exp^{-2r}-1\right] \\ &< 0. \end{aligned} \tag{54}$$

Formally, note that $z(0)=E(0)\equiv z_0$ and solve (53) to get

$$z(\tau)=\exp^{2rT} z_0+\frac{A^2R}{2r}\left(1-\exp 2r\right)\sum_{j=0}^{T-1}\exp^{2rj} \tag{55}$$

so that the earnings growth converges to the constant $\exp^{2rT} z_0$. ♣

2.3 From the Haley-Rosen Specification to the Mincer Equation

The earnings function in the Haley-Rosen specification actually leads to the Mincer equation. To see that take take logs of (51) and obtain

$$\ln E(\tau)=\ln\left(\frac{A^2R}{2r}\right)+\ln\tau+\ln\left[1+\frac{\exp^{-2rT}-\exp^{2r(\tau-T)}}{2r\tau}\right]. \tag{56}$$

We can approximate around τ_0 the second and third terms in (56) to obtain

$$\begin{aligned} \ln(\tau) &\approx \ln(\tau_0)+\frac{1}{\tau_0}\left(\tau-\tau_0\right)-\frac{1}{\tau_0^2}\frac{\left(\tau-\tau_0\right)^2}{2!} \\ \ln\left[1+\frac{\exp^{-2rT}-\exp^{2r(\tau-T)}}{2r\tau}\right] &\approx \xi_0+\xi_1\left(\tau-\tau_0\right)+\xi_2\frac{\left(\tau-\tau_0\right)^2}{2!} \end{aligned} \tag{57}$$

for the adequate ξ_0, ξ_1, xi_2. Thus,

$$\ln(\tau)+\ln\left[1+\frac{\exp^{-2rT}-\exp^{2r(\tau-T)}}{2r\tau}\right]\approx\alpha_0+\alpha_1\left(\tau-\tau_0\right)+\alpha_2\left(\tau-\tau_0\right)^2 \tag{58}$$

with $\alpha_0\equiv\ln(\tau_0)+\xi_0, \alpha_1\equiv\frac{1}{\tau_0}+\xi_1, \alpha_2\equiv\frac{-\frac{1}{\tau_0^2}+\xi_2}{2}$, which leads to the so-called Mincer equation (see Mincer, 1974):

$$\ln E(\tau)=k_0+k_1\tau k_2\tau^2 \tag{59}$$

where $k_0=\alpha_0-\tau_0\alpha_1+\alpha_2\tau_0^2, k_2=\alpha_2$. This provides a baseline to compare "Ben-Porath" with "Mincer" coefficients. Table 1 provides different combinations of the parameters r, τ_0, T that lead to different values of k_1, k_2 that are close to the estimates that Mincer (1974) obtains.

Table 1: The Ben-Porath and the Mincer Coefficients

Parameters			Ben Porath Coefficients	
r	τ_0	T	k_1	k_2
0.0225	29.54	41.43	0.081	−0.0010
0.05	25	60	0.0808	−0.0008
0.05	20	65	0.1002	−0.0013
0.0675	24.70	74.77	0.081	−0.0008
Mincer Coefficients			0.081	−0.0012

Note: the Mincer model or Mincer equation is $\ln(\mathrm{E}) = k_0 + k_1\tau + k_2\tau^2$, where τ is experience.

Now, if $rT \approx 0$ then $\exp^{-rT} \approx 1$ and (56) becomes

$$\ln E(\tau) \approx \ln\left(\frac{A^2R}{2r}\right) + \ln\tau + \ln\left[1 + \frac{1-\exp^{2r\tau}}{2r\tau}\right] \tag{60}$$

which leads to various observations. The Haley-Rosen specification of the Ben-Porath model implies no economic content for the Mincerian rate of return on post-school investment. Put differently, an extension of (59) which includes post-school investment does not have a structural counterpart. Actually, this model implies that the entire economic content is in the intercept (see (60). Actually, (60) implies that, *caeteris paribus*, schooling has no effect on earnings. Mincer (1974) finds that the contrary. However, we claim that his finding does not necessarily argues against the Ben-Porath model. It could simply be the case that Mincer (1974) does not include ability measures in his estimations, which appear in (60), and therefore finds a positive coefficient on schooling.

3 Generalized Ben-Porath Model

We know generalize the model in Section 1 to a more general production function of human capital investment. We focus our analysis on specialization because the analysis of other conditions is very similar to that of Section 1

Definition 3.1 *(Law of Motion for Human Capital Stock in the Generalized Ben-Porath Model)*

$$\dot{H} = AI^{\alpha}H^{\beta} - \sigma H. \tag{61}$$

The model in Section 1 is a particular case of this general formulation when $\alpha = \beta$. To simplify the analysis of the implications of this model we assume that there is neither discounting nor depreciation, i.e. $r = \sigma = 0$. To ease notation we neglect the argument t when possible. We analyze this model in finite horizon.

The Hamiltonian of the problem is

$$\mathcal{H} = RH(t)\,(1 - I(t)) + \mu\left(AI^{\alpha}H^{\beta}\right) \tag{62}$$

where $\mu(t)$ defines the shadow price of human capital. The following condition describe optimality.

Condition 3.2 *(Optimality Conditions for the Life-Cycle Individual's Problem in the Generalized Ben-Porath Model)*

$$\begin{aligned}
\frac{\partial \mathcal{H}(\cdot)}{\partial I(t)} = 0 \quad &\Leftrightarrow \quad \mu A \alpha I^{\alpha-1} H^{\beta} \geq RG && (63)\\
\frac{\partial \mathcal{H}(\cdot)}{\partial H(t)} = -\dot{\mu(t)} \quad &\Leftrightarrow \quad -R(1-I) - \beta \mu A I^{\alpha} H^{\beta-1} && (64)\\
\frac{\partial \mathcal{H}(\cdot)}{\partial \mu(t)} = \dot{H} \quad &\Leftrightarrow \quad \dot{H(t)} = F\left(I(t)H(t), D(t)\right) - \sigma H(t) && (65)\\
\textit{Transversality} \quad &: \quad \lim_{t \to T} \mu(t) H(t) = 0 && (66)
\end{aligned}$$

Condition 3.2 are equivalent to the Mangasarian sufficient conditions for a global optimum if $\beta \leq 1$ (see Mangasarian, 1966).

3.1 Specialization

If (63) holds with strict inequality the individual specializes, i.e. $I = 1$. Thus, Condition 3.3 guarantees specialization.

Condition 3.3 *(Conditions for Specialization in the Generalized Ben-Porath Model)*

$$\textit{Conditions for Specialization}: \begin{cases} H > \left[\frac{R}{\alpha A \mu}\right]^{\frac{1}{\beta-1}}, & \beta > 1 \\ 1 > \left[\frac{R}{\alpha A \mu}\right]^{\frac{1}{\beta-1}}, & \beta = 1 \\ H < \left[\frac{R}{\alpha A \mu}\right]^{\frac{1}{\beta-1}}, & \beta < 1. \end{cases} \tag{67}$$

During the period(s) of specialization (64), (65) become

$$\begin{aligned}
\dot{\mu} &= -\beta \mu A H^{\beta-1} && (68)\\
\dot{H} &= A H^{\beta} && (69)\\
& && (70)
\end{aligned}$$

and we can solve for the dynamics of human capital stock in this region

$$H(t) = \begin{cases} c_0 \exp^{At}, & \beta = 1 \\ (At + c_1)^{\frac{1}{1-\beta}} (1-\beta)^{\frac{1}{1-\beta}}, & \beta \neq 1. \end{cases} \tag{71}$$

The initial condition for the human capital stock leads to $c_0 = \frac{H_0}{\exp^1}$ and $c_1 = \frac{H_0^{1-\beta}}{1-\beta}$ which implies that

$$H(t) \begin{cases} H_0 \exp^{At-1}, & \beta = 1 \\ \left(At + \frac{H_0^{\frac{1}{1-\beta}}}{1-\beta}\right)^{\frac{1}{1-\beta}} (1-\beta)^{\frac{1}{1-\beta}}, & \beta \neq 1. \end{cases} \tag{72}$$

Also, we can solve (68) and find that

$$\mu(t) = \begin{cases} k_0 \exp^{-At}, & \beta = 1 \\ \frac{k_1}{(At+c_1)^{\frac{\beta}{1-\beta}}}, & \beta \neq 1 \end{cases} \tag{73}$$

for which there is an exact solution given an initial condition $\mu(0) = \mu_0$. This is, we can find k_0, k_1 in (73) provided $\mu_0 > 0$ (it is a price). In particular, note that $k_0 = \mu_0 > 0$ and $k_1 = \mu_0 c_1^{\frac{\beta}{1-\beta}} > 0$ for $0 < \beta < 1$.

Let t^* denote the time when specialization ends. It must be true that, then, (63) holds with strict equality

$$\mu(t^*) A\alpha H(t^*)^\beta = RH(t^*) \tag{74}$$

which implies that

$$t^* = \frac{1}{A}\left(\ln\left[\frac{A\alpha}{R} + \ln k_0\right]\right) \tag{75}$$

for $\beta = 1$. For $\beta \neq 1$, t^* solves

$$\frac{k_1}{(At^* + c_0)^{\frac{\beta}{\beta-1}}} \frac{A\alpha}{R} = \left[At^* (1-\beta)^{\frac{1}{1-\beta}} + H_0^{1-\beta} (1-\beta)^{\frac{\beta}{1-\beta}}\right]^{1-\beta}. \tag{76}$$

To wrap up the discussion we ask if the period of specialization is unique for some particular cases.

Claim 3.4 *(Uniqueness of the Specialization Period) The period of specialization is unique when either $\beta = 1$ or $\beta \in [0, 1]$.*

Proof: In both cases (73) implies that $\dot{\mu(t)} < 0$. Importantly, $\mu(t)$ is the shadow price or value of human capital. Thus, $\dot{I(t)} < 0$ and, if it exists, the period of specialization is unique. ♣

4 The Basic Shenshinski Specification: Bang-Bang E quilibria

The Basic Shenshinski specification is a particular case of the Generalized Ben Porath model in Section 3 in which $\alpha = \beta = 1$.

Definition 4.1 *(Law of Motion for Human Capital Stock in the Basic Shenshinski Specification)*

$$\dot{H(t)} = AI(t)H(t) - \sigma H(t). \tag{77}$$

Proceeding as in Section 1 and Section 3 we can write down the current value Hamiltonian and obtain the following conditions for optimality.

Condition 4.2 *(Optimality Conditions for the Life-cycle Individual's Problem in the Basic Shenshinski Specification)*

$$\frac{\partial \mathcal{H}(\cdot)}{\partial I(t)} = 0 \quad \Leftrightarrow \quad \mu(t) \exp rt \geq \frac{R}{A} \tag{78}$$

$$\frac{\partial \mathcal{H}(\cdot)}{\partial H(t)} = -\dot{\mu(t)} \quad \Leftrightarrow \quad -\exp^{rt} R(1 - I(t)) - \mu(t)\,(AI(t) - \sigma) = \dot{\mu(t)} \tag{79}$$

$$\frac{\partial \mathcal{H}(\cdot)}{\partial \mu(t)} = \dot{H(t)} \quad \Leftrightarrow \quad \dot{H(t)} = AI(t)H(t) - \sigma H(t) \tag{80}$$

$$\textit{Transversality} \quad : \quad \lim_{t \to T} \mu(t)H(t) = 0 \tag{81}$$

Claim 4.3 *(Bang-Bang in the Shenshinski Specification) Assume that $\sigma + r < A$ and that there is an initial period of specialization.* [4] *Then, the solution to the problem is Bang-Bang, i.e. either $I = 0$ or $I = 1$.*

Proof: Define $g(t) = \mu(t) \exp^{rt}$ and use (79), (81) to obtain

$$\dot{g} = -R + (R - Ag)I + (\sigma + r)g \tag{82}$$

$$g(T) = 0. \tag{83}$$

In the specialization period $I(t) = 1$ If $\sigma + r < A$, $\dot{g(t)} < 0$. Actually, by (78), $g(t)$ decreases to $\frac{R}{A}$ and $I(t) = 0$ onwards. This happens because $g(t)$ represents returns to gross investment in human capital net accounting for time and depreciation and $I(t)$ is bounded below by zero. Thus, if $g(t)$ reaches a constant phase $I(t)$ remains constant at 0. Now, when $I(t) = 0$ we can use $g(T) = 0$ and write

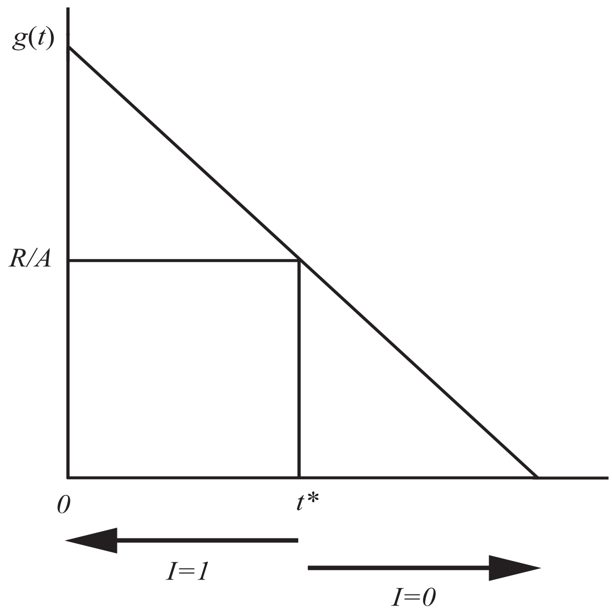
$$\begin{aligned} \dot{g(t)} &= (\sigma + r)g(t) - R \\ &\Rightarrow \\ g(t) &= \frac{R}{\sigma + r}\left[1 - \exp^{(\sigma + r)(t - T)}\right]. \end{aligned} \tag{84}$$

for which $\dot{g(t)} < 0$ as well. Then, once $I(t)$ reaches zero it never goes back again to a positive value. This formulation has a Bang-Bang solution. ♣

It follows that the schooling period, if it exists, is unique and at the beginning of the investment cycle. If it does not exist the individual does not invest at all in human capital. Figure 9 is a graphical representation of Claim 4.3.

[4] Note that $\sigma + r > A$ implies that $\dot{g(t)} > 0$ and this violates the transversality condition.

Figure 9: Bang-Bang Equilibrium in the Basic Shenshinski Specification



Note:

We can actually solve for t^*, the length of the schooling period, using the fact that $g(t^*) = \frac{R}{A}$ by (78) and $g(t^*) = \frac{R}{\sigma+r}\left[1 - \exp^{(\sigma+r)(t^*-T)}\right]$ by (84):

$$\begin{aligned} \frac{R}{A} &= \frac{R}{\sigma+r}\left[1 - \exp^{(\sigma+r)(t^*-T)}\right] \\ &\Leftrightarrow \\ t^* &= \frac{1}{\sigma+r}\ln\frac{A-(\sigma+r)}{A} + T. \end{aligned} \tag{85}$$

Thus, (i) longer investment horizons imply more schooling, $\frac{\partial t^*}{\partial T} > 0$; (ii) greater depreciation implies less schooling, $\frac{\partial t^*}{\partial \sigma} < 0$; (iii) higher relative impatience implies less schooling, $\frac{\partial t^*}{\partial r} > 0$; (iv) higher productivity implies more schooling; (v) initial human capital does not affect schooling, $\frac{\partial t^*}{\partial H_0} = 0$.

4.1 From the Basic Shenshinski Specification to the Mincer Equation

Assume that there is a period of specialization. From (69) we know that in the period $[0, t^*]$

$$\begin{aligned} \dot{H}(t) = (A-\sigma)H(t) \\ \Rightarrow \\ H(t) = H_0 \exp^{A-\sigma}. \end{aligned} \tag{86}$$

At t^*, actually, $I(t) = 0$ so earnings are $Y(t) = RH(t^*)$ Then,

$$\ln Y(t^*) = \ln(RH_0) + (A - \sigma)t* \tag{87}$$

According to this model, the returns to schooling are given by the productivity of the human capital investment production function less the human capital depreciation.

5 The Modified Shenshinski Specification

Consider, now, the law of motion for human capital $\dot{H} = AI - \sigma H$.[5]. The Hamiltonian of the problem is

$$\mathcal{H}(\cdot) = \exp^{-rt} R(1 - I)H + \mu(AI - \sigma). \tag{88}$$

The corresponding optimality conditions are

$$\frac{\partial \mathcal{H}(\cdot)}{\partial I(t)} = 0 \quad \Leftrightarrow \quad \mu \exp^{rt} \geq \frac{RH}{A} \tag{89}$$

$$\frac{\partial \mathcal{H}(\cdot)}{\partial H(t)} = -\dot{\mu(t)} \quad \Leftrightarrow \quad \dot{\mu} = \mu\sigma - \exp^{-rt} R(1 - I) \tag{90}$$

$$\frac{\partial \mathcal{H}(\cdot)}{\partial \mu(t)} = \dot{H(t)} \quad \Leftrightarrow \quad \dot{H(t)} = AI - \sigma H \tag{91}$$

$$\text{Transversality} \quad : \quad \lim_{t \to T} \mu(t)H(t) = 0. \tag{92}$$

Define $g(t) = \mu(t) \exp^{rt}$ and use (90) to obtain

$$\dot{g} = g(\sigma + r) - R(1 - I). \tag{93}$$

5.1 No Depreciation: a Schooling Model

Let $\sigma = 0$. Then $\dot{g} = -R(1 - I) + rg$ ans $\dot{H} = A$ and we obtain the solution for the human capital trajectory when $I = 1$

$$H(t) = At + H_0. \tag{94}$$

At $t = 0$, $I = 1$ if $g(0) > \frac{R}{A}H_0$. Importantly, $I = 1$ implies that $\dot{g(t)} = rg(t) > 0$. As t grows, the return for gross investment grows because the payoff period gets closer. $I = 1$ cannot be a solution forever because the agent receives no earnings if he invest all of the time during the complete life-cycle.

The question, then, is if there is more than one period of specialization. We can use (92) and solve (93) for any $I \in [0, 1)$ to obtain

$$g(t) = \frac{R(1 - I)}{r} \left[1 - \exp^{r(t-T)}\right] \tag{95}$$

[5] Typo in slide 21 in the definition of the law of motion and in the definition of the optimality conditions

and note that $\dot{g} < 0$.[6] Thus, there is at most one period of specialization and it happens at the beginning of the life cycle. This is a model of schooling.

Finally, note that (89) holds with strict equality at t^* and (94) is valid so that

$$g(t^*) = \frac{R}{A}\left(At^* + H_0\right). \tag{96}$$

(95) is also valid for t^*. Then,

$$\left(1 - \exp^{r(t^*-T)}\right) = \frac{r}{A}\left(At^* + H_0\right) \tag{97}$$

and $\frac{\partial t^*}{\partial H_0} < 0$, $\frac{\partial t^*}{\partial A} > 0$ as in the model of Section 1.

5.2 Depreciation

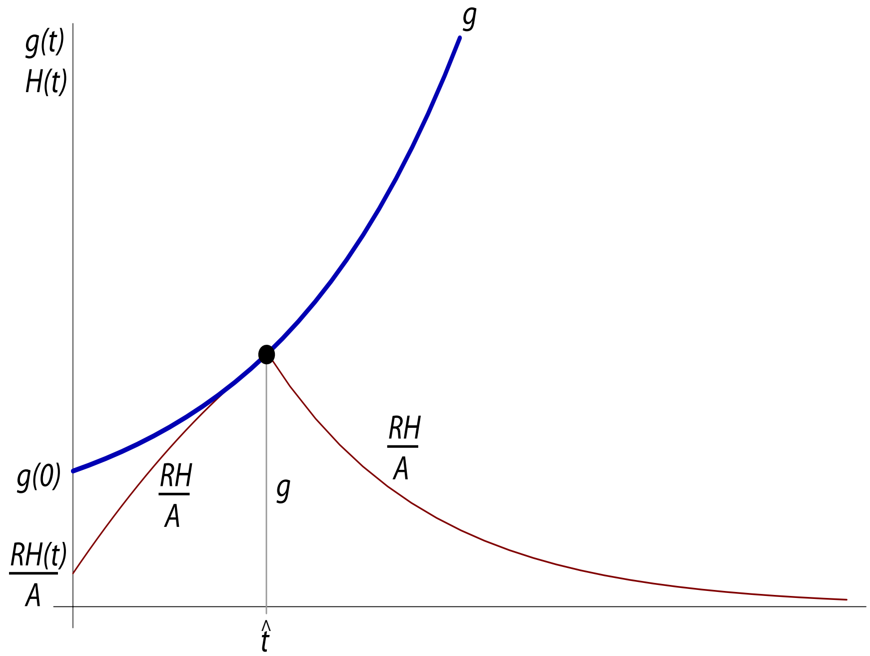
Let us give some conditions under which human capital investment would have different episodes over the life cycle. First assume that $g(0) > \frac{H_0 R}{A}$ so that there is an specialization period to begin with. We can solve (91) and (93) to obtain

$$\begin{aligned} H(t) &= \left[H_0 - \frac{A}{\sigma}\right]\exp^{-\sigma t} + \frac{A}{\sigma} \\ g(t) &= g_0 \exp^{(r+\sigma)t} \end{aligned}$$

with $g_0 > 0$. Once the solution becomes interior, $g(t) = \frac{R}{A}H(t)$ by (89). Assume that $\sigma < \frac{A}{H_0}$ so that $\dot{H}(0) > 0$. Then, graphically,

[6] This needs to be completely rewritten in slide 22 - 24. Nothing makes sense. Simply copy all this. Conditions have mistakes in the HO.

Figure 10: Return to Gross Investment in Human Capital in the Modified Shenshinski Specification



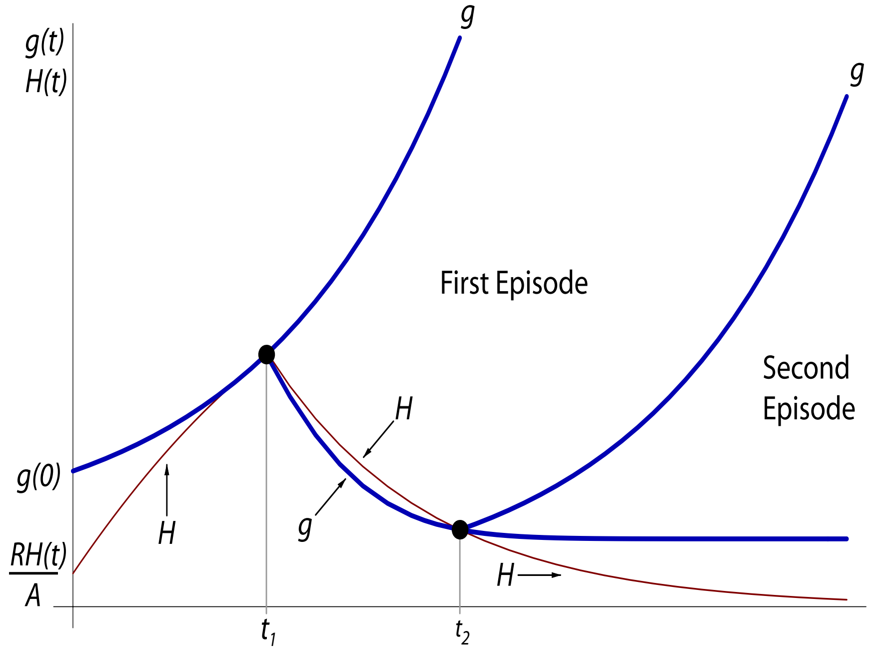
Note:

Let t_1 denote the period in which the solution first period of specialization finishes. If the solutions "bangs-out" to $I = 0$ we can use (93) and the low of motion for human capital to obtain

$$\begin{aligned} \dot{g} &= (\sigma + r)g - R \\ H(t) &= H(t_1)\exp^{-\sigma(t-t_1)} \end{aligned} \tag{98}$$

for $t_1 < t < t_2$. Likewise, we can define a period t_2 in which the solutions "bangs-in" again and so on. Graphically,

Figure 11: Human Capital Investment Episodes in the Modified Shenshinski Specification



Note:

We informally build some conditions for this kind of cycling to happen. In $t < t_1$, $I = 1$ implies $\dot{g} > 0$. g needs to decrease for the problem to respect the transversality condition. Thus, in the neighborhood of t_1 it has to be that $\dot{g}(t_1) < \frac{R\dot{H}(t_1)}{A}$ (see Figure (11)). If we take the expression from the right of $g(t_1)$ this requires

$$
\begin{aligned}
-R(\sigma + r)g(t_1) &< \frac{R\dot{H}(t_1)}{A} \\
&= \frac{-\sigma RH(t_1)}{A} \\
&= -\sigma g(t_1) \\
g(t_1) &< \frac{R}{r}. \qquad (99)
\end{aligned}
$$

To have an initial period of specialization we need $g_0 > \frac{RH_0}{A}$. At $t = 0$, however, it should be the case that the slope of $\frac{RH_0}{A}$ exceeds $\dot{g}$. Otherwise the expressions for g in the specialization period and the "interior" case do not intersect and the solution violates the transversality condition. This implies that $R\left[1 - \frac{\sigma H_0}{A}\right] > g_0(\sigma + r)$. High initial levels of human capital, low productivity, high discount, high depreciation, and low returns to human capital rule out an initial specialization period. Suppose the conditions described above hold so that specialization happens. We cannot show that $g(t_3) < g(t_1)$ so that is better to accumulate "all the human capital required for life" in the first period of specialization. This is what we call cycling in the investment on human capital and is a consequence of depreciation.

References

Becker, G. S. (1962). Investment in Human Capital: A Theoretical Analysis. *The Journal of Political Economy 70*(5), 9–49.

Becker, G. S. (2009). *Human capital: A theoretical and empirical analysis, with special reference to education*. University of Chicago Press.

Ben-Porath, Y. (1967). The Production of Human Capital and the Life Cycle of Earnings. *The Journal of Political Economy 75*(4), 352–365.

Brown, C. (1976). A Model of Optimal Human Capital Accumulation and the Wages of Young High-school Graduates. *The Journal of Political Economy*, 299–316.

Haley, W. J. (1976). Estimation of the Earnings Profile from Optimal Human Capital Accumulation. *Econometrica: Journal of the Econometric Society*, 1223–1238.

Heckman, J. J., L. J. Lochner, and P. E. Todd (2006). Earnings Functions, Rates of Return and Treatment Effects: The Mincer Equation and Beyond. *Handbook of the Economics of Education 1*, 307–458.

Mangasarian, O. L. (1966). Sufficient Conditions for the Optimal Control of Non-linear Systems. *SIAM Journal on Control 4*(1), 139–152.

Mincer, J. A. (1974). Schooling, Experience, and Earnings. In *Schooling, experience, and earnings*, pp. 41–63. Columbia University Press.

Rosen, S. (1976). A Theory of Life Earnings. *The Journal of Political Economy 84*(4), S45–S67.