The Accumulation of Human Capital over the Life Cycle: Lessons from the Yoram Ben-Porath Model

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Perhaps the most important conclusion to be drawn from research into the influence of income distribution on consumption is that the effects of inequality depend upon its causes.

Mincer (1958).

Abstract

For several years, economists have worried about theories of income and its prediction of observable outcomes or constructs of social interest. Particular interest they have paid to theories relating the distribution of income and the distribution of abilities (Staehle (1943) offers the first formal treatment on this topic in Economics). The most relevant ingredient of this analysis is human capital and its evolution. Mincer (1958) and Becker (1962) are the two main predecessors of the analysis of human capital investment. The former firstly asked why the distributions of abilities and income differed and argued for market compensation for different worker traits as principal causes. The latter established the first self-contained theoretical analysis of human capital investment. Ben-Porath (1967) generalized Becker (1962) to a dynamic context that allows for a very rich analysis of the production of human capital. Specifically, it enables to analyze variations in (i) specialization periods; (ii) production functions of human capital; (iii)

time horizons; (iv) rates of return. Based on the model in Ben-Porath (1967), we provide a framework to analyze the evolution of earnings and human capital under a wide variety of scenarios. We hope to guide researchers on the consequences of their human capital production and accumulation modeling choices.

1 Baseline Ben-Porath Model

The first and very basic model builds on the following five assumptions: (i) a single representative agent; (ii) perfect capital markets; (iii) no non-market benefits of human capital; (iv) fixed labor supply; (v) constant depreciation of human capital stock, σ. Let T define a time horizon. This model is known as the Ben-Porath model.

For each $t \in T$, H denotes human capital, $I \in [0, 1]$ investment time, D market goods that serve as inputs to the production function, and F a strictly concave production function in two normal inputs. The law of motion for human capital is

$$\dot{H}(t) = F\left(I(t)H(t), D(t)\right) - \sigma H(t) \tag{1}$$

and embeds a neutrality assumption. Namely, the current stock of human capital at time t, $H(t)$, and the investment time at time t, $I(t)$, appear as a single argument in a multiplicative fashion in the flow production of human capital. At each point of time, the current stock and the rental rate of human capital, R, define potential earnings

$$Y(t) = RH(t). \tag{2}$$

In general, observed earnings and potential earnings differ by two terms, foregone earnings and direct market goods costs. Let P_D be the price of markets and define observed earnings, $E(t)$ as

$$E(t) = RH(t) - RI(t)H(t) - P_D D(t) \tag{3}$$

where $RI(t)H(t)$ are foregone earnings and $P_D D(t)$ are direct goods costs. This expression clarifies that $I(t)$ is the fraction of time devoted to investment in each period of time. In particular, the individual occupies a fraction $I(t)$ of her human capital stock to produce a human capital flow.

The individual chooses $D(t)$ and $I(t)$ to maximize her lifetime earnings stream given an initial level of human capital, $H(0) = H_0$ and subject to the law of motion of human capital, (1).

The *current value Hamiltonian* associated to the individual's problem is

$$\mathcal{H}(\cdot) = \exp^{-rt}\left[RH(t) - RI(t)H(t) - P_D D(t)\right] + \mu_t \dot{H(t)} \tag{4}$$

where $\mu(t)$ defines the shadow price of human capital. For a strictly concave function $F(\cdot)$ the necessary and sufficient conditions for optimality are

$$\frac{\partial \mathcal{H}(\cdot)}{\partial I(t)} = 0 \quad \Leftrightarrow \quad \exp^{-rt} R = \mu(t) F_{I(t)H(t)} \tag{5}$$

$$\frac{\partial \mathcal{H}(\cdot)}{\partial D(t)} = 0 \quad \Leftrightarrow \quad \exp^{-rt} RI(t) = \mu(t) F_{D(t)} \tag{6}$$

$$\frac{\partial \mathcal{H}(\cdot)}{\partial H(t)} = -\dot{\mu(t)} \quad \Leftrightarrow \quad \exp^{-rt} R\left(1 - I(t)\right) + \mu(t)\left(F_{I(t)H(t)} - \sigma\right) = -\dot{\mu(t)} \tag{7}$$

$$\frac{\partial \mathcal{H}(\cdot)}{\partial \mu(t)} = \dot{H(t)} \quad \Leftrightarrow \quad \dot{H(t)} = F\left(I(t)H(t), D(t)\right) - \sigma H(t) \tag{8}$$

$$\text{Transversality} \quad : \quad \lim_{t \to T} \mu(t)H(t) = 0 \tag{9}$$

where $F_j \equiv \frac{\partial F(I(t),D(t))}{\partial j}$ for $j = D(t), I(t)H(t)$. Combine (5) and (7) to get

$$\dot{\mu(t)} = -\exp^{-rt} R + \mu(t)\sigma. \tag{10}$$

Let $g(t) \equiv \mu(t)\exp^{rt}$ and note that $\dot{g(t)} = \dot{\mu(t)}\exp^{rt} + r\mu(t)e^{rt}$. Use (10) to obtain

$$\dot{g(t)} = (\sigma + r)g(t) - R. \tag{11}$$

$g(t)$ has fundamental importance to this problem because it is a discount factor that adjusts for exponential depreciation of gross investment and enables to write the individual's problem in a more intuitive way. In particular, note that (9) implies that $\mu(T) = 0$ and, therefore, $g(T) = 0$.[1] Thus, it is possible to solve (11) and obtain

$$g(t) = \frac{R}{\sigma + r}\left[1 - \exp^{(\sigma+r)(t-T)}\right]. \tag{12}$$

Importantly, for an interior solution, this enables to analyze the problem in t and obtain all the features of the investment dynamics and human capital accumulation. In particular, the problem of the agent is to maximize her discounted gross flow of human capital less her costs (foregone earnings plus market goods costs)

$$\max_{I(t),D(t)} \left[g(t)F(I(t)H(t), D(t)) - P_D D(t) - RI(t)H(t)\right] \tag{13}$$

for which the first order conditions are

$$\begin{aligned} g(t)F_{I(t)H(t)}H(t) &= RH(t) \\ g(t)F_{D(t)}H(t) &= P_D. \end{aligned} \tag{14}$$

The system in (14) results on a system of two equations and two unknowns that solves for the Marshallian demands of $I(t)H(t)$ and $D(t)$. Input normality together with the fact that $\dot{g(t)} < 0$ imply that the both Marshallian demands are decreasing overtime, which is intuitive because the agent faces a finite horizon problem.[2] This provides the necessary arguments to analyze earnings dynamics in this first simple model.

[1]Note that $H(T) > 0$ because otherwise the individual losses the possibility to earn in the last period.
[2]Need to think why $g(t)$ is decreasing over time.

1.1 Earnings Dynamics

1.1.1 No Human Capital Depreciation

Suppose that there is no human capital depreciation. Take the definition of earnings, (3), and note that

$$\begin{aligned} \dot{E(t)} &= R\dot{H(t)} - RI(t)\dot{H}(t) - P_D\dot{D(t)} \\ &= RF\left(I(t)H(t), D(t)\right) - RI(t)\dot{H}(t) - P_D\dot{D(t)} \\ &> 0 \end{aligned} \tag{15}$$

where the second equality follows from the law of motion for human capital when $\sigma = 0$.[3]

In the case of human capital depreciation earnings are not necessarily monotonic because $\dot{E(t)} = RF\left(I(t)H(t), D(t)\right) - R\sigma H(t) - RI(t)\dot{H}(t) - P_D\dot{D(t)}$ and $|R\sigma H(t)|$ may be big enough to make $\dot{E(t)}$ over some periods.

1.1.2 No Depreciation and the Cobb-Douglas Production Function for Human Capital

The simplest version of this model occurs when $\sigma = 0$ and $F(I(t)H(t), D(t))$ is a Cobb-Douglas production function. Ben-Porath (1967) shows that in this case $\dot{E(t)} > 0, \ddot{E(t)} < 0$. This is, earnings increase at a decreasing rate over the life cycle.

1.2 Earnings Growth Dynamics

It is of relevance to understand how earnings behave in cases that are more general relative to Cobb-Douglas. For that sake assume that $F_{D(t)} = 0$, i.e. assume away $D(t)$ and therefore let $F(\cdot)$ take a single argument, $I(t)H(t)$. The first order condition for investment becomes

$$g(t)F'(I(t)H(t)) = R \tag{16}$$

[3]Clear up the intuition of this result.

and we can differentiate with respect to t and get

$$\begin{aligned}
\dot{g(t)}F'(I(t)H(t)) + g(t)F''(I(t)H(t))I(t)\dot{H}(t) &= 0 \\
&\Leftrightarrow \\
I(t)\dot{H}(t) &= -\left(\frac{\dot{g(t)}}{g(t)}\right)\left[\frac{F'}{F''}\right]. \qquad (17)
\end{aligned}$$

Moreover, drop the argument t to shorten notation, and note that

$$I\ddot{H} = -\left[\frac{\ddot{g}}{g} - \left(\frac{\dot{g}}{g}\right)^2\right]\frac{F'}{F''} + \left(\frac{\dot{g}}{g}\right)^2\left[1 - \frac{F'F'''}{F''^2}\right]\left[\frac{F'}{F''}\right] \qquad (18)$$

where we substitute in (17).

Without lost of generality assume that $R = 1$ and note that

$$\begin{aligned}
\dot{E} &= F(IH) - I\dot{H} - \sigma H \\
\ddot{E} &= F'(IH)I\dot{H} - I\ddot{H} - \sigma\dot{H} \\
&= \frac{1}{g}I\dot{H} - I\ddot{H} - \sigma\dot{H}. \qquad (19)
\end{aligned}$$

Now, let $\sigma = 0$ and from (11) obtain $\frac{\ddot{g}}{g} = r\frac{\dot{g}}{g}$. Thus,

$$\begin{aligned}
\ddot{E} &= -\frac{\dot{g}}{g}\frac{F'}{F''}\left[\frac{1}{g} - \frac{\dot{g}}{g}\left(1 - \frac{F'F'''}{F''^2}\right)\right] + \left[r\frac{\dot{g}}{g} - \left(\frac{\dot{g}}{g}\right)^2\right]\frac{F'}{F''} \\
&= -\frac{\dot{g}}{g}\frac{F'}{F''}\left[\frac{1}{g} - \frac{\dot{g}}{g}\left(1 - \frac{F'F'''}{F''^2}\right) - \frac{gr - \dot{g}}{g}\right] \\
&= -\frac{\dot{g}}{g}\frac{F'}{F''}\left[\frac{1}{g} - \frac{\dot{g}}{g}\left(1 - \frac{F'F'''}{F''^2}\right) - \frac{1}{g}\right] \\
&= -\left(\frac{\dot{g}}{g}\right)^2\frac{F'}{F''}\left(1 - \frac{F'F'''}{F''^2}\right) \qquad (20)
\end{aligned}$$

where the third equality uses (11), i.e. $gr - \dot{g} = 1$. F is strictly concave and therefore $-\left(\frac{\dot{g}}{g}\right)^2\frac{F'}{F''} > 0$. The sign of $\eta \equiv \left(1 - \frac{F'F'''}{F''^2}\right)$ depends on F'''. A sufficient condition for $\dot{E}$ to be concave is $\eta < 0$.

Example 1.1 *(Human Capital Production Functions and Earnings Concavity)*

- *Power Production Function 1 : consider the case of* $F(x) = \frac{Ax^{\alpha}}{\alpha}$ *for* $-\infty < \alpha < 1, A > 0$*. Then,* $\eta = \frac{1}{\alpha-1} < 0$*. Under this specification the earnings function is strictly concave with respect to time.*

- *Power Production Function 2 : consider the case of* $F(x) = a - bx^{-\alpha}$ *for* $-1 < \alpha < \infty, a, b, c > 0$*. Then,* $\eta = \frac{-1}{\alpha+1} < 0$*. Under this specification the earnings function is strictly concave with respect to time.*

- *Quadratic Production Function: any quadratic production function has* $F''' = 0$ *and does not induce concavity of earnings with respect to time.*

Importantly, all this examples consider no depreciation of human capital, $\sigma = 0$*.*

1.3 Specialization Period

A period of specialization happens when the agent devotes his complete time to produce a flow of human capital, i.e. when $I(t) = 1$ for $t \in [\underline{t}, \bar{t}]$. In order to illustrate assume away $D(t)$ so that $F_{D(t)} = 0$ and $\sigma = 0$, i.e. no human capital depreciation. First note that the marginal returns to investment decline as the stock of human capital grows and overtime. Concavity of $F(\cdot)$ causes the first implication while $\dot{g} < 0$ causes the second. Then, there is at most one period of specialization at the beginning of the time horizon, if it happens: $[0, t^*]$. This is what Ben-Porath (1967) calls schooling period.

Four conditions hold in the specialization period

$$\begin{aligned} F'(H(t))g(t) &> R \\ F'(H(t^*))g(t^*) &= R \\ I(t^*) &= 1 \\ H(t^*) &= \int_0^{t^*} F(H(\tau))d\tau + H_0 \end{aligned} \tag{21}$$

where $H(t^*)$ is the human capital stock accumulated up to time t^*, the sum over the period $[0, t^*]$ plus the initial stock.

Given that R is fixed at 1, any decrease in $g(t)$ lowers t^* because it lowers the discount to gross investment in human capital. For example, relatively high r implies relatively low t^* because the individual is relatively present oriented. Also, from the (21), note that a high value of t^* implies lower value for t^* because it takes less time to obtain $H(t^*)$.[4] If $\sigma > 0$ the the same conditions characterize specialization. However, the there may be more than one specialization period because, under some scenarios, a high value of σ may knock off capital such that investment cycles are optimal. We stay away from such scenarios and keep $\sigma = 0$ in what follows.

Example 1.2 *(No Depreciation and the Cobb-Douglas Production Function for Human Capital: Initial Human Capital and the Specialization Period) In this case* $\dot{H} = A(IH)^{\alpha}$ *where* $0 < \alpha < 1, A > 0$. *As argued above, specialization happens in the period* $[0, t^*]$. *Thus*

$$\begin{aligned} \alpha A (H(0))^{\alpha-1} g(0) &> R \\ &\Leftrightarrow \\ H(0) &< \left[\frac{R}{g(0)\alpha A}\right]^{\frac{1}{\alpha-1}}. \end{aligned} \tag{22}$$

As the conditions in (21) establish, the time spent in specialization is a decreasing function of $H(0)$. *In this example, actually, the initial human capital needs to be below certain threshold in order for the individual to specialize during one period.*

Example 1.3 *(Infinite Horizon) In the setting of Exercise 1.2 and if the horizon of the problem is infinite:* $H(0) < \left(\frac{\alpha A}{r}\right)^{\frac{1}{1-\alpha}}$ *because* $g(t) = \frac{R}{r}$.

Example 1.4 *(No Depreciation and the Cobb-Douglas Production Function for Human Capital: the*

[4] A more realistic model, perhaps, makes R depend on human capital so that more able individuals find it cheaper to invest in human capital. However, they also have higher foregone earnings and we cannot say without deeper analysis how t^* behaves in such a case.

Specialization Period) In the period of specialization $I(t) = 1$. Then,

$$\dot{H} = A\left(H\right)^{\alpha}. \tag{23}$$

The general solution for (23) is

$$H(t) = \left[(1-\alpha)(At+K)\right]^{\frac{1}{1-\alpha}} \tag{24}$$

for some constant K. Given an initial condition $H(0) = H_0$, $K = \frac{H_0^{1-\alpha}}{1-\alpha}$ and

$$H(t) = \left[(1-\alpha)At + H_0^{1-\alpha}\right]^{\frac{1}{1-\alpha}}. \tag{25}$$

At the end of the specialization period, as established in (21):

$$\alpha g(t^*)A\left(H(t^*)\right)^{\alpha-1} = R. \tag{26}$$

If $T \to \infty$, $g(t) = \frac{R}{r}$ and

$$t^* = -\frac{H_0^{1-\alpha}}{A(1-\alpha)} + \frac{\alpha}{1-\alpha}\frac{1}{r}. \tag{27}$$

(27) provides some intuitive results: (i) an individual with relatively high initial human capital specializes during a relatively shorter period: $\frac{\partial t^*}{\partial H_0} < 0$; (ii) a relatively abler individual specializes during relatively longer period: $\frac{\partial t^*}{\partial A} > 0$; a relatively impatient individual specializes for a relatively shorter period: $\frac{\partial t^*}{\partial r} < 0$.

Example 1.5 *(No Depreciation and the Cobb-Douglas Production for Human Capital: Post-experience Earnings) Let $\tau = t - t^*$ define the post-school work experience and write post-school earnings as follows:*

$$E(\tau) = R\int_0^{\tau} H(l \dot{+} t^*)dl + RH(t^*) - RIH(\tau + t^*). \tag{28}$$

Now, from (21) *the following equality holds:*

$$
\begin{aligned}
\alpha g(t) A \left(IH(t)\right)^{\alpha-1} &= R \\
&\Leftrightarrow \\
IH(t) &= \left[\frac{\alpha g(t) A}{R}\right]^{\frac{1}{1-\alpha}}
\end{aligned}
\tag{29}
$$

Combining (29) *and the law of motion for human capital:*

$$
\dot{H} = A\left[\frac{\alpha g(t) A}{R}\right]^{\frac{\alpha}{1-\alpha}}. \tag{30}
$$

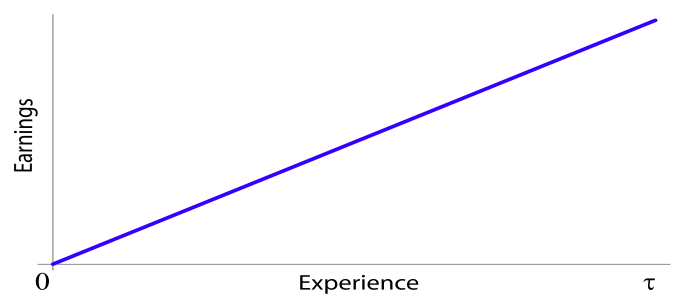
Then,

$$
E(\tau) = R\int_0^\tau A\left[\frac{\alpha g(l+t^*)A}{R}\right]^{\frac{\alpha}{1-\alpha}} dl + RH(t^*) - R\left[\frac{\alpha g\left(\tau+t^*\right)A}{r}\right]^{\frac{1}{1-\alpha}} \tag{31}
$$

and if $T \to \infty$

$$
E(\tau) = RA\left[\frac{\alpha A}{R}\right]^{\frac{\alpha}{1-\alpha}}\tau. \tag{32}
$$

Figure 1: Earnings and Experience, Cobb Douglas Technology and No Depreciation



Note:

1.4 The Baseline Model Dynamics under the Cobb-Douglas Specification: a Summary

This section summarizes the dynamics of the main variables in the baseline model when there is no depreciation and market goods are ruled out as inputs for the production of human capital investment. We assume that the horizon is infinite to simplify the algebra but it is important to remark that the qualitative properties of the results remain unchanged under finite horizon. At the end we show simulations that illustrate how the variables of interest behave under various parametrizations.

1.4.1 Human Capital

- At $t = 0$ an initial condition is given.
- At $0 < t < t^*$ the system (21) provides the conditions that human capital satisfies and its expression is given by (25).
- At $t = t^*$ (25) is still a valid expression for human capital. To obtain the exact quantity it suffices to evaluate the expression for t^*, (27), into (25).
- At $t > t^*$ (21) and the expression for $\dot{H}$, (30), provide the expression for human capital.

Then,[5]

$$H(t) = \begin{cases} H_0 & t = 0 \\ \left[(1-\alpha)At + H_0^{1-\alpha}\right]^{\frac{1}{1-\alpha}}, & 0 < t < t^* \\ \left[\frac{\alpha A}{r}\right]^{\frac{1}{1-\alpha}}, & t = t^* \\ \left[\frac{\alpha A}{r}\right]^{\frac{\alpha}{1-\alpha}}(t - t^*) + \left[\frac{\alpha A}{r}\right]^{\frac{1}{1-\alpha}}, & t > t^*. \end{cases} \tag{33}$$

[5] There is a mistake in the definition of this function in the notes

1.4.2 Investment

We focus on the case in which there is an specialization period, i.e. the case in which (22) holds. The combination of (29) and (33) gives the following

$$I(t) = \begin{cases} 1, & t = 0 \\ 1, & 0 < t < t^* \\ 1, & t = t^* \\ \frac{\left[\frac{\alpha A}{r}\right]^{\frac{1}{1-\alpha}}}{\left[\frac{\alpha A}{r}\right]^{\frac{\alpha}{1-\alpha}}(t-t^*)+\left[\frac{\alpha A}{r}\right]^{\frac{1}{1-\alpha}}}, & t > t^*. \end{cases} \tag{34}$$

1.5 Earnings

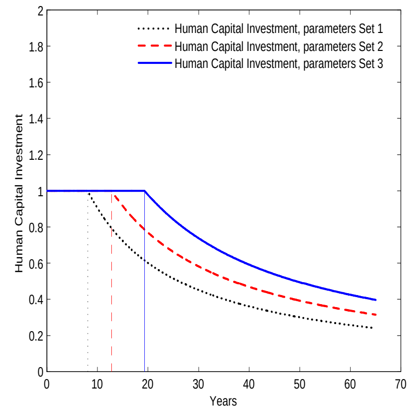
For the case of earnings we also on the case in which there is an specialization period, i.e. the case in which (22) holds. Thus, (3), (33), (34) define earnings as follows

$$E(t) = \begin{cases} 0, & t = 0 \\ 0, & 0 < t < t^* \\ 0, & t = t^* \\ RA\left[\frac{\alpha A}{r}\right]^{\frac{\alpha}{1-\alpha}}(t-t^*), & t > t^*. \end{cases} \tag{35}$$

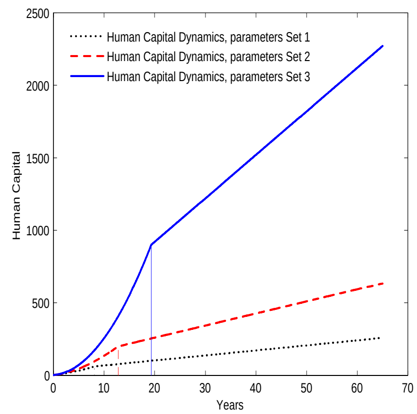
1.5.1 Graphical Analysis

We now produce simulations to illustrate the comparative statics of the model. In all of them we set $R = 1$.

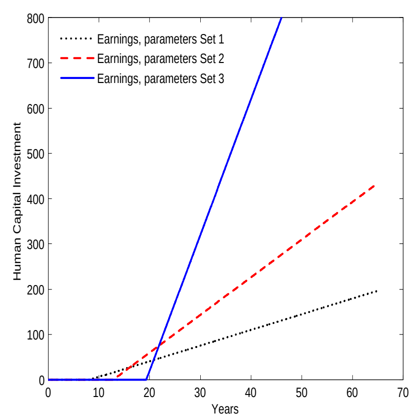
Figure 2: Dynamics for $A = 3, r = .05, H_0 = 1$
$\alpha = .3$ (dotted); $\alpha = .4$ (dashed); $\alpha = .5$ (solid)



(a) Human Capital Investment

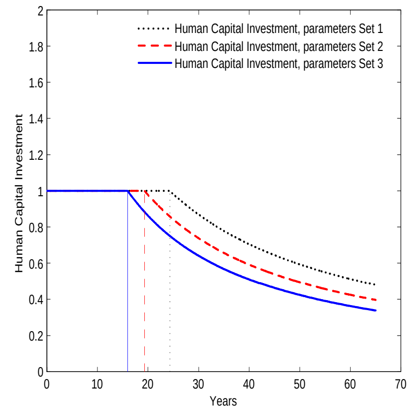


(b) Human Capital Stock

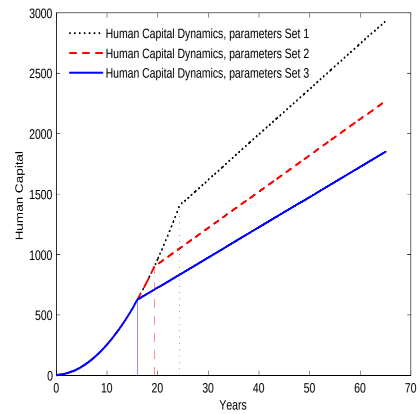


(c) Earnings

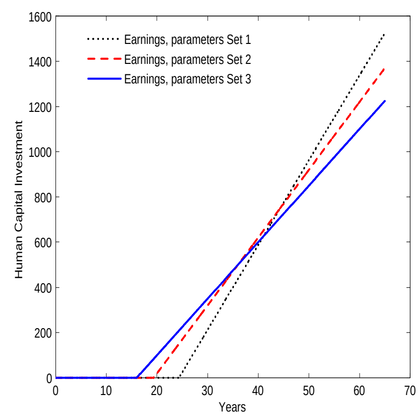
Figure 3: Dynamics for $A = 3, \alpha = .5, H_0 = 1$
$r = .04$ (dotted); $r = .05$ (dashed); $r = .06$ (solid)



(a) Human Capital Investment

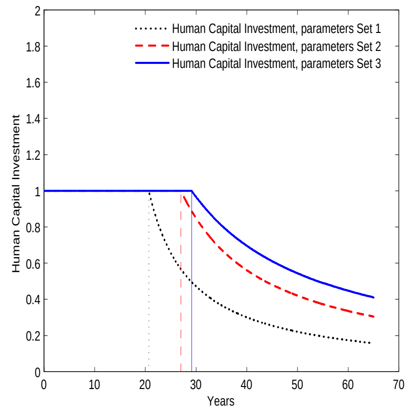


(b) Human Capital Stock

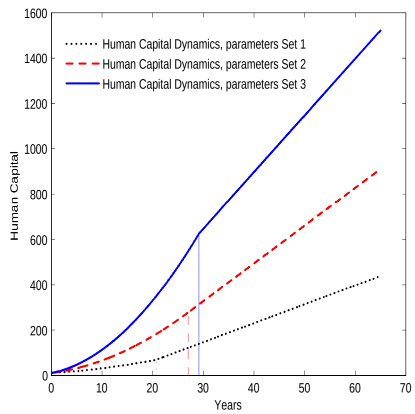


(c) Earnings

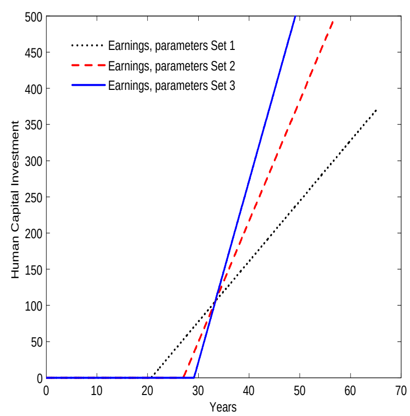
Figure 4: Dynamics for $r = .03, \alpha = .5, H_0 = 10$
$A = .5$ (dotted); $A = 1.0$ (dashed); $A = 1.5$ (solid)



(a) Human Capital Investment

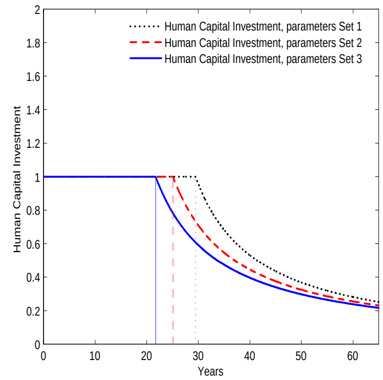


(b) Human Capital Stock

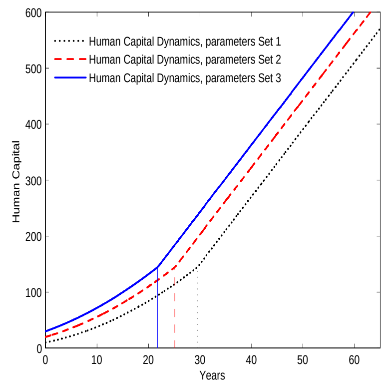


(c) Earnings

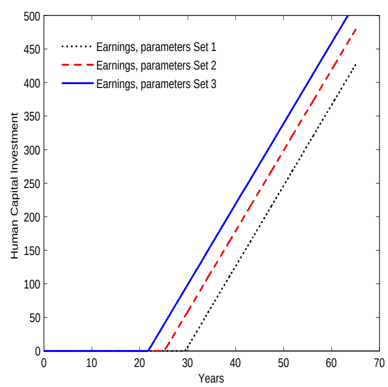
Figure 5: Dynamics for $r = .025, \alpha = .5, A = .6$
$H_0 = 10$ (dotted); $H_0 = 20$ (dashed); $H_0 = 30$ (solid)



(a) Human Capital Investment



(b) Human Capital Stock



(c) Earnings

2 The Haley-Rosen Specification: Finite Horizon and the Autoregression Form

We analyze the finite horizon case under an specification that Haley (1976) and Rosen (1976) use. Specifically, we assume that $\dot{H} = A\left(IH\right)^{\alpha}, \alpha = \frac{1}{2}, \sigma = 0$ and the exact same setting as in Section 1. Actually, in Section 1 we rely on infinite horizon to derive a set of closed form solutions to the individual's problem. In this section we rely on the assumption $\alpha = \frac{1}{1}$ to do exactly the same.

We focus on the dynamics of post-schooling earnings because one of the less credible consequence of the infinite horizon is the linearity of earnings on experience. From (31) we can write[6]

$$\begin{aligned}
E(\tau) &= RH(t^*) + R\int_0^{\tau} A\left[\frac{1}{2}\frac{g(t^*+l)A}{R}\right]dl - R\left[\frac{1}{2}\frac{g(t^*+l)A}{R}\right]^2 \\
&\Rightarrow \\
\dot{E(\tau)} &= \frac{g(t^*+\tau)A^2}{2R}\left(2R - rg(t^*+\tau)\right) \\
&\Rightarrow \\
\ddot{E(\tau)} &= -\frac{A^2}{R}\dot{g(t^*+\tau)}^2
\end{aligned} \tag{36}$$

where the second and third equalities use (11). Combining (11) and (36) we obtain a second order ODE with constant coefficients:

$$\ddot{E(\tau)} = 2r\dot{E(\tau)} - A^2R \tag{37}$$

where the natural initial and terminal conditions that we impose are $E(0)$ and $\ddot{E(T)} = 0$ and then we guess and verify that $c_2 = \frac{A^2R}{2r}\exp^{2rT}$ in the following general solution to (37)[7]

$$E(\tau) = c_0 + c_1 \exp^{-2r\tau} + c_2\tau \tag{38}$$

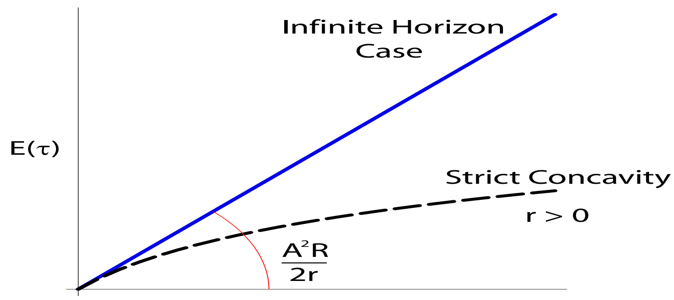
[6] Typo there is Appendix slife 89. Confusin t and τ

[7] There is a typo here in the general solution to the differential equation (slide 50) and in the initial conditions as well. Also, one wrong initial condition is provided.

so that $c_1 + c_0 = 0, 2rc_1 \exp^{2rT} + c_2 = 0$ and, therefore,

$$E(\tau) = \frac{A^2 R}{4r^2} \exp^{-2rT} \left(1 - \exp^{2r\tau}\right) + \frac{A^2 R}{2r}\tau. \tag{39}$$

Figure 6: Post-school Earnings in the Haley-Rosen Specification



Note:

Need to include a discussion here on why this is constant in infinite horizon or $r = 0$.

2.1 Evidence

Brown (1976) estimates (39), which enables to identify r and $A^2 R$. His estimates, however, are imprecise and show that $r \to 0$. Then, he estimates the model for the infinite horizon case. He claims this to be a good approximation because he has a sample of young individuals. However, this disables him to estimate r.

2.2 The Autoregression

From (39) it is possible to write

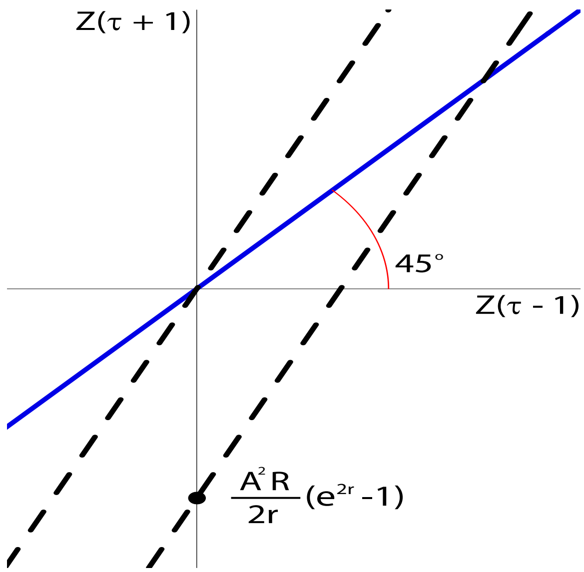
$$E(\tau+1) - E(\tau) = \frac{A^2 R}{2r} + \frac{A^2 R}{4r^2} \exp^{-2rT} \left(\exp^{2r(\tau-1)} \exp^{r\tau}\right) \tag{40}$$

which implies that[8]

$$z(\tau+1) = e^{2r} z(\tau) + \frac{A^2 R}{2r}\left(1 - \exp^{2r}\right) \tag{41}$$

and we can analyze the growth dynamics of earnings. Consider a visual representation of (41)

Figure 7: Earnings Growth in the Haley-Rosen Representation



Note:

Apparently, the dynamics of the earnings growth are explosive. However, note that

$$\begin{aligned} \frac{\partial\left[E(\tau) - E(\tau-1)\right]}{\partial \tau} &= \frac{A^2 R}{2r} \exp 2r(\tau - T)\left[\exp^{-2r} - 1\right] \\ &< 0 \end{aligned} \tag{42}$$

so that even when the growth dynamics of earnings is autoregressive it converges to a constant. More

[8]slide 55 typo when multiplying by $\exp^2 r$

formally, note that $z(0) = E(0) \equiv z_0$ and solve (41) to get

$$z(\tau) = \exp^{2rT} z_0 + \frac{A^2 R}{2r} (1 - \exp 2r) \sum_{j=0}^{T-1} \exp^{2rj} \tag{43}$$

so that the earnings growth converges to the constant $\exp^{2rT} z_0$.

2.3 From the Haley-Rosen Specification to the Mince Equation

The earnings function in the Haley-Rosen specification actually leads to the Mincer equation. To see that take take logs of (39) and obtain

$$\ln E(\tau) = \ln\left(\frac{A^2 R}{2r}\right) + \ln\tau + \ln\left[1 + \frac{\exp^{-2rT} - \exp^{2r(\tau - T)}}{2r\tau}\right]. \tag{44}$$

We can approximate around τ_0 the second and third terms in (44) to obtain

$$\begin{aligned} \ln(\tau) &\approx \ln(\tau_0) + \frac{1}{\tau_0}(\tau - \tau_0) - \frac{1}{\tau_0^2}\frac{(\tau - \tau_0)^2}{2!} \\ \ln\left[1 + \frac{\exp^{-2rT} - \exp^{2r(\tau - T)}}{2r\tau}\right] &\approx \xi_0 + \xi_1(\tau - \tau_0) + \xi_2\frac{(\tau - \tau_0)^2}{2!} \end{aligned} \tag{45}$$

for the adequate ξ_0, ξ_1, xi_2. Thus,

$$\ln(\tau) + \ln\left[1 + \frac{\exp^{-2rT} - \exp^{2r(\tau - T)}}{2r\tau}\right] \approx \alpha_0 + \alpha_1(\tau - \tau_0) + \alpha_2(\tau - \tau_0)^2 \tag{46}$$

with $\alpha_0 \equiv= \ln(\tau_0) + \xi_0, \alpha_1 \equiv \frac{1}{\tau_0} + \xi_1, \alpha_2 \equiv \frac{-\frac{1}{\tau_0^2} + \xi_2}{2}$. This leads to the so called Mincer equation (see Mincer, 1974):

$$\ln E(\tau) = k_0 + k_1\tau k_2\tau^2 \tag{47}$$

where $k_0 = \alpha_0 - \tau_0\alpha_1 + \alpha_2\tau_0^2, k_2 = \alpha_2$. This provides a baseline to compare "Ben-Porath" with "Mincer" coefficients. Table 1 provides different combinations of the parameters r, τ_0, T that lead to different values of k_1, k_2 that are close to the estimates that Mincer (1974) obtains.

Table 1: The Ben-Porath and the Mincer Coefficients

Parameters			Ben Porath Coefficients	
r	τ_0	T	k_1	k_2
0.0225	29.54	41.43	0.081	-0.0010
0.05	25	60	0.0808	-0.0008
0.05	20	65	0.1002	-0.0013
0.0675	24.70	74.77	0.081	-0.0008
Mincer Coefficients			0.081	-0.0012

Note: the Mincer model or Mincer equation is $\ln(\mathrm{E}) = k_0 + k_1\tau + k_2\tau^2$, where τ is experience.

Now, if $rT \approx 0$ then $\exp^{-rT} \approx 1$ and (44) becomes

$$\ln E(\tau) \approx \ln\left(\frac{A^2 R}{2r}\right) + \ln\tau + \ln\left[1 + \frac{1-\exp^{2r\tau}}{2r\tau}\right] \tag{48}$$

which leads to various observations. The Haley-Rosen specification of the Ben-Porath model implies no economic content for the Mincerian rate of return on post-school investment. Put differently, an extension of (47) which includes post-school investment does not have a structural counterpart. Actually, this model implies that the entire economic content is in the intercept (see (48). Actually, (48) implies that, *caeteris paribus*, schooling has no effect on earnings. Mincer (1974) finds that the contrary. However, we claim that his finding does not necessarily argues against the Ben-Porath model. It could simply be the case that Mincer (1974) does not include ability measures in his estimations, which appear in (48), and therefore finds a positive coefficient on schooling.

3 Rates of Return

We analyze the rate of return to schooling and post-schooling investments based on the model in Section 1. Recall that this model considers no depreciation, $\sigma = 0$, and $T \to \infty$.[9].

[9]Recall that this implies that $g(t) = \frac{R}{r}$

3.1 Post-school Investment

Let $E(\tau)^{NPS}$ and $E(\tau)^{PS}$ denote earnings without and with post-schooling investment, respectively. By (35) we can write

$$\begin{aligned} E(\tau)^{NPS} &= RH(t*) \\ &= R\left(\frac{\alpha A}{r}\right)^{\frac{1}{1-\alpha}} \\ E(\tau)^{PS} &= \left[\frac{\alpha A}{r}\right]^{\frac{\alpha}{1-\alpha}}\tau \end{aligned} \tag{49}$$

so that the increment in earnings due to post-schooling at τ is

$$\Delta^{E(\tau)} \equiv E(\tau)^{PS} - E(\tau)^{NPS}. \tag{50}$$

Actually, is we note that $E(\tau)^{PS} = IH(\tau)$ (see (29)) then we can interpret $\Delta^{E(\tau)}$ as "returns less costs" from post-schooling and let ϕ be the (internal) rate of return to post-schooling which solves for

$$\int_0^\infty \exp^{-\phi\tau}\left[\left[\frac{\alpha A}{r}\right]^{\frac{\alpha}{1-\alpha}}\tau - R\left(\frac{\alpha A}{r}\right)^{\frac{1}{1-\alpha}}\right]d\tau = 0 \tag{51}$$

Using the Laplace transform, (51) implies

$$\begin{aligned} \frac{1}{\phi^2}RA\left[\frac{\alpha A}{r}\right]^{\frac{\alpha}{1-\alpha}} + \frac{1}{\phi}A\left[\frac{\alpha A}{r}\right]^{\frac{1}{1-\alpha}} &= 0 \\ &\Rightarrow \\ \phi &= \frac{r}{\alpha} \end{aligned} \tag{52}$$

The (internal) rate of return to post-schooling investment is a decreasing function of α. Individuals who are more productive require a smaller return in order to invest in the post-school period. Likewise, relatively patient individuals (relatively low r) require a smaller ϕ to invest in the post-schooling period.

3.2 School Investment

Each τ, an individual with no schooling earns RH_0 while he earns $R\left[\frac{\alpha A}{r}\right]^{\frac{1}{1-\alpha}}$ with schooling and no post-schooling. Let φ be the (internal) rate of return to schooling be defined as the value φ that solves for[10]

$$\begin{aligned}\int_{t^*}^{\infty} \exp^{-\varphi t} R\left[\frac{\alpha A}{r}\right]^{\frac{1}{1-\alpha}} dt &= \int_{0}^{\infty} \exp^{-\varphi t} RH_0 dt \\ &\Rightarrow \\ \varphi &= \frac{\ln\left[\frac{\alpha A}{r}\right]^{\frac{1}{1-\alpha}} - \ln H_0}{\frac{1}{r} - \frac{1}{2}\frac{H_0^{\frac{1}{2}}}{A}} \end{aligned} \tag{53}$$

4 Earnings Growth and Patience in Finite Horizon

In Section 1 that relative patience (i.e., relatively low r) implies a relatively longer schooling period, relatively higher human capital accumulation, and a relatively steeper earnings profile. We now want to ask, in the same framework but with finite horizon, how earnings growth depends on relative patience. In order to answer that we investigate the behavior of $\frac{\partial \dot{E(\tau)}}{\partial r}$. Without loss of generality, assume that R =1 and note that

Note that

$$\frac{\partial \dot{E(\tau)}}{\partial r} = F'(\cdot)\frac{\partial IH}{\partial r} - \frac{\partial}{\partial r}I\dot{H}. \tag{54}$$

From (14) we know that the first order condition of the agent's problem is

$$g(t)F'(\cdot) = 1. \tag{55}$$

[10]Typo in slide 74 in the definition of what do schooling people earn.

which by the implicit function theorem yields

$$
\begin{aligned}
\frac{\partial IH}{\partial r} &= \frac{\frac{\partial g(t)}{\partial r}F'(\cdot)}{2g(t)F''(\cdot)} \\
&< 0
\end{aligned}
\tag{56}
$$

where the inequality follows from strict concavity of $F(\cdot)$ and $g(t) > 0, \frac{\partial g(t)}{\partial r} < 0$ (see (59)). Thus, the first term in (54) is negative. If we show that the second term is negative then we can sign (54) and provide a meaning for this results. In order to do that we need $\frac{\partial \dot{IH}}{\partial r} > 0$. From (17) note that

$$
\frac{\partial \dot{IH}}{\partial r} = -\frac{\dot{g}}{g}\left[1 - \frac{F'(\cdot)F'''(\cdot)}{F''(\cdot)^2}\right]\frac{\partial IH}{\partial r} + \frac{F'(\cdot)}{F''(\cdot)}\frac{\partial}{\partial r}\left[-\frac{\dot{g}}{g}\right]
\tag{57}
$$

From Section 1.2 we know that a sufficient condition for earnings strict concavity, i.e. $\ddot{E} < 0$ is $1 - \frac{F'(\cdot)F'''(\cdot)}{F''^2} < 0$. This together with $\dot{g}, \frac{\partial IH}{\partial r} < 0$ implies that the first term in (57) is positive. To sign the second term note that $\dot{g} = rg - 1, -\frac{\dot{g}}{g} = \frac{1}{g} - r$. Then,

$$
\frac{\partial}{\partial r}\left[-\frac{\dot{g}}{g}\right] = -\frac{1}{g^2}\frac{\partial g}{\partial r} - 1.
\tag{58}
$$

To sign (58) note that

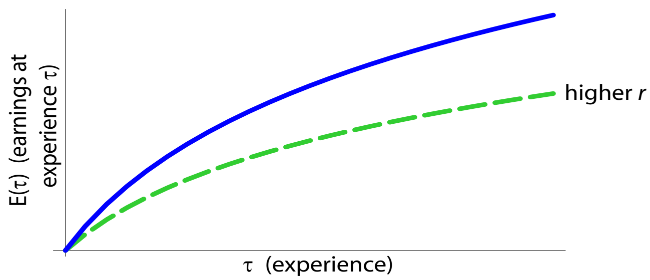
$$
\begin{aligned}
\frac{\partial g}{\partial r} &= \frac{\exp^{r(t-T)}(1 - r(t-T)) - 1}{r^2} \\
&< 0
\end{aligned}
\tag{59}
$$

and

$$
\begin{aligned}
-\frac{\partial g}{g^2 \partial r} - 1 &= \frac{1}{r^2 g^2}\exp^{r(t-T)}\left(1 + r(t-T) - \exp^{r(t-T)}\right). \\
&< 0
\end{aligned}
\tag{60}
$$

Therefore, if the earnings function is concave, $\frac{\partial \dot{OH}}{\partial r} > 0$ which implies that $\frac{\partial \dot{E}}{\partial r} < 0$. This implies that the earnings function is relatively "less concave" for relatively impatient individuals (relatively high r). This is a consequence of their investment decisions: they spent less time in the schooling period and accumulate less human capital.

Figure 8: Earnings Profiles in Finite Horizon for Different Values of r



Note:

5 Generalized Ben-Porath Model

Let us consider a generalization of the model in Section 1 and allow for the law of motion of human capital to be

$$\dot{H} = AI^{\alpha}H^{\beta} - \sigma H. \tag{61}$$

This baseline Ben-Porath model is a particular case of this general formulation when $\alpha = \beta$. To simplify the analysis of the implications of this model we assume that there neither discounting nor depreciation, i.e. $r = \sigma = 0$. To ease notation we neglect the argument t when possible. We analyze this model in finite horizon and follow the same notation as in Section 1. Thus, the agent's problem is to maximize

$$\max_{I(t)} \int_0^T \left[RH(t)\left(1 - I(t)\right)\right] dt \tag{62}$$

subject to an initial condition for the stock of human capital, $H(0) = H_0$, and (61). The Hamiltonian of the problem is

$$\mathcal{H} = RH(t)\,(1 - I(t)) + \mu\left(AI^{\alpha}H^{\beta}\right) \tag{63}$$

where $\mu(t)$ defines the shadow price of human capital. The first order conditions are

$$\frac{\partial \mathcal{H}(\cdot)}{\partial I(t)} = 0 \quad \Leftrightarrow \quad \mu A \alpha I^{\alpha-1} H^{\beta} \geq RG \tag{64}$$

$$\frac{\partial \mathcal{H}(\cdot)}{\partial H(t)} = -\dot{\mu(t)} \quad \Leftrightarrow \quad -R(1-I) - \beta\mu A I^{\alpha} H^{\beta-1} \tag{65}$$

$$\frac{\partial \mathcal{H}(\cdot)}{\partial \mu(t)} = \dot{H} \quad \Leftrightarrow \quad \dot{H(t)} = F\left(I(t)H(t), D(t)\right) - \sigma H(t) \tag{66}$$

$$\text{Transversality} \quad : \quad \lim_{t \to T} \mu(t)H(t) = 0 \tag{67}$$

Importantly, these conditions are equivalent to the Mangasarian sufficient conditions for a global optimum if $\beta \leq 1$ (see Mangasarian, 1966).

5.1 Specialization

The specialization or schooling period when $I = 1$ happens if (64) holds with strict inequality for $I = 1$.

$$\text{Conditions for Specialization} : \begin{cases} H > \left[\frac{R}{\alpha A \mu}\right]^{\frac{1}{\beta-1}}, & \beta > 1 \\ 1 > \left[\frac{R}{\alpha A \mu}\right]^{\frac{1}{\beta-1}}, & \beta = 1 \\ H < \left[\frac{R}{\alpha A \mu}\right]^{\frac{1}{\beta-1}}, & \beta < 1. \end{cases} \tag{68}$$

During this period (65), (66) become

$$\dot{\mu} = -\beta\mu A H^{\beta-1} \tag{69}$$

$$\dot{H} = AH^{\beta}. \tag{70}$$

$$\tag{71}$$

We can solve for (70) and get

$$H(t) = \begin{cases} c_0 \exp^{At}, & \beta = 1 \\ (At + c_1)^{\frac{1}{1-\beta}} (1-\beta)^{\frac{1}{1-\beta}}, & \beta \neq 1. \end{cases} \tag{72}$$

The initial condition for the human capital stock leads to $c_0 = \frac{H_0}{\exp^1}$ and $c_1 = \frac{H_0^{1-\beta}}{1-\beta}$ which implies that

$$H(t) \begin{cases} H_0 \exp^{At-1}, & \beta = 1 \\ \left(At + \frac{H_0^{\frac{1}{1-\beta}}}{1-\beta} \right)^{\frac{1}{1-\beta}} (1-\beta)^{\frac{1}{1-\beta}}, & \beta \neq 1. \end{cases} \tag{73}$$

Also, we can solve (69) and find that

$$\mu(t) = \begin{cases} k_0 \exp^{-At}, & \beta = 1 \\ \frac{k_1}{(At+c_1)^{\frac{\beta}{1-\beta}}}, & \beta \neq 1 \end{cases} \tag{74}$$

for which there is an exact solution given an initial condition $\mu(0) = \mu_0$. This is, we can find k_0, k_1 in (74) provided $\mu_0 > 0$ (it is a price). In particular, note that $k_0 = \mu_0 > 0$ and $k_1 = \mu_0 c_1^{\frac{\beta}{1-\beta}} > 0$ for $0 < \beta < 1$.

Let t^* denote the time when specialization ends. It must be true that, then, (64) holds with strict equality[11]

$$\mu(t^*) A\alpha H(t^*)^{\beta} = RH(t^*) \tag{75}$$

which implies that

$$t^* = \frac{1}{A} \left(\ln \left[\frac{A\alpha}{R} + \ln k_0 \right] \right) \tag{76}$$

[11] typo in slide 18. All the time arguments in the first equation should be t^*.

for $\beta = 1$. For $\beta \neq 1$, $t*$ solves

$$\frac{k_1}{(At^* + c_0)^{\frac{\beta}{\beta-1}}} \frac{A\alpha}{R} = \left[At^* (1-\beta)^{\frac{1}{1-\beta}} + H_0^{1-\beta} (1-\beta)^{\frac{\beta}{1-\beta}}\right]^{1-\beta}. \tag{77}$$

To wrap up the discussion we ask, if there is at least one period of specialization, how many periods of specialization there are. Consider two cases: (i) $\beta = 1$; (ii) $0 < \beta < 1$. In both cases, from (74) we note that $\dot{mu}(t) < 0$. Importantly, $\mu(t)$ is the shadow price or value of human capital. Thus, $\dot{I(t)}$ and, if it exists, the period of specialization is unique.

6 The Basic Shenshinski Specification

The Basic Shenshinski specification is a particular case of the Generalized Ben Porath model in Section 5 in which $\alpha = \beta = 1$. Then, $\dot{H(t)} = AI(t)H(t) - \sigma H(t)$. Proceeding in Section 1 and Section 5 we can write down the current value Hamiltonian and obtain the following conditions for optimality

$$\begin{aligned}
\frac{\partial \mathcal{H}(\cdot)}{\partial I(t)} = 0 \quad &\Leftrightarrow \quad \mu(t) \exp rt \geq \frac{R}{A} && (78)\\
\frac{\partial \mathcal{H}(\cdot)}{\partial H(t)} = -\dot{\mu(t)} \quad &\Leftrightarrow \quad -\exp^{rt} R(1 - I(t)) - \mu(t)\left(AI(t) - \sigma\right) = \dot{\mu(t)} && (79)\\
\frac{\partial \mathcal{H}(\cdot)}{\partial \mu(t)} = \dot{H(t)} \quad &\Leftrightarrow \quad \dot{H(t)} = AI(t)H(t) - \sigma H(t) && (80)\\
\text{Transversality} \quad &: \quad \lim_{t \to T} \mu(t)H(t) = 0 && (81)
\end{aligned}$$

We want to analyze the investment dynamics. To do that, define $g(t) = \mu(t)\exp^{rt}$. Use (79) and (81) to obtain

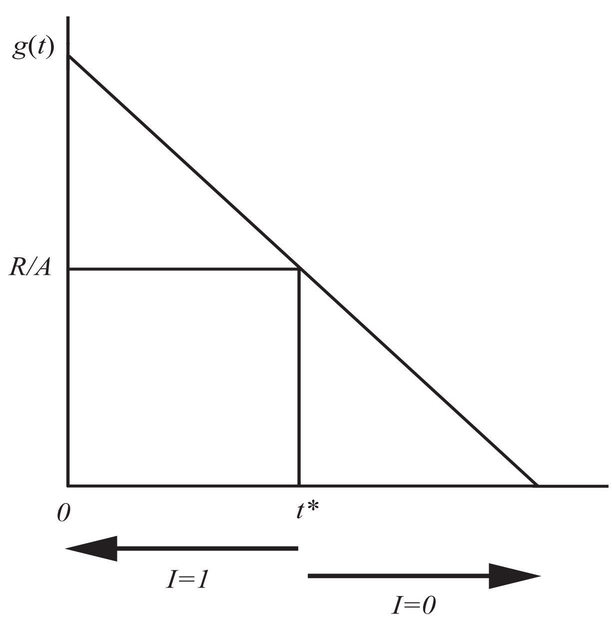
$$\begin{aligned}
\dot{g} &= -R + (R - Ag)I + (\sigma + r)g && (82)\\
g(T) &= 0. && (83)
\end{aligned}$$

In the specialization period $I(t) = 1$ and from (82) and (78) we have $g(t) = g_0 \exp^{(\sigma+r-A)t}$. If $\sigma + r > A$ the solution violates (81): $\dot{g(t)} > 0$ and $I(t) = 1$ $\forall t \in [0,T]$. If $\sigma + r < A$, $\dot{g(t)} < 0$. Actually, by (78), $g(t)$ decreases to $\frac{R}{A}$ and $I(t) = 0$ onwards. This happens because $g(t)$ represents returns to gross investment in human capital net accounting for time and depreciation and $I(t)$ is bounded below by zero. Thus, if $g(t)$ reaches a constant phase $I(t)$ remains constant at 0. Now, when $I(t) = 0$ we can use $g(T) = 0$ and write

$$
\begin{aligned}
(\dot{g(t)}) &= (\sigma + r)g(t) - R \\
&\Rightarrow \\
g(t) &= \frac{R}{\sigma + r}\left[1 - \exp^{(\sigma+r)(t-T)}\right]. \qquad (84)
\end{aligned}
$$

for which $\dot{g(t)} < 0$ as well. Then, once $I(t)$ reaches zero it never goes back again to a positive value. This formulation has a Bang-Bang equilibrium.

Figure 9: Bang-Bang Equilibrium in the Basic Shenshinski Specification



Note:

It follows that the schooling period is unique and at the beginning of the investment cycle.[12] We can actually solve for t^* the length of the schooling period using the fact that $g(t^*) = \frac{R}{A}$ by (78) and $g(t^*) = \frac{R}{\sigma+r}\left[1 - \exp^{(\sigma+r)(t^*-T)}\right]$ by (84). This implies that (i) longer investment horizons imply more schooling, $\frac{\partial t^*}{\partial T} > 0$; (ii) greater depreciation implies less schooling, $\frac{\partial t^*}{\partial \sigma} < 0$; (iii) higher relative impatience implies less schooling, $\frac{\partial t^*}{\partial r} > 0$; (iv) higher productivity implies more schooling; (v) initial human capital does not affect schooling, $\frac{\partial t^*}{\partial H_0} = 0$. This is because

$$\begin{aligned} \frac{R}{A} &= \frac{R}{\sigma+r}\left[1 - \exp^{(\sigma+r)(t^*-T)}\right] \\ &\Leftrightarrow \\ t^* &= \frac{1}{\sigma+r}\ln\frac{A-(\sigma+r)}{A} + T. \end{aligned} \tag{85}$$

6.1 From the Basic Shenshinski Specification to the Mincer Equation

From (70) we know that in the period $[0, t^*]$[13]

$$\begin{aligned} \dot{H}(t) &= (A-\sigma)H(t) \\ &\Rightarrow \\ H(t) &= H_0 \exp^{A-\sigma}. \end{aligned} \tag{86}$$

At t^*, actually, $I(t) = 0$ so earnings are $Y(t) = RH(t^*)$.[14] Then,

$$\ln Y(t^*) = \ln(RH_0) + (A-\sigma)t * . \tag{87}$$

According to this model, the returns to schooling are given by the productivity of the human capital investment production function less the human capital depreciation.

[12]Mistake in slide 7 when solving for t^*.

[13]Mistake in slide 8. Why add φ? Makes no sense.

[14]Miss-defined in slide 8

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