Ben-Porath Notes

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Econ 350
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Notes on Ben-Porath Human Capital Model

- Perfect Capital Markets
- No Nonmarket Benefits of Human Capital
- Fixed Labor Supply
- H is human capital
- $I \in [0, 1]$ is investment time
- D is goods input
- F is a strictly concave function in two normal inputs

Human Capital Production Function

- $\dot{H}(t) = F(I(t), H(t), D(t)) - \sigma H(t)$
- $F(I(t), H(t), D(t)) = F(I(t)H(t), D(t))$ (neutrality)
- R is rental rate of human capital.
- Potential earnings: $Y(t) = RH(t)$.
- Observed earnings:

$$E(t) = RH(t) - \underbrace{RI(t)H(t)}_{\substack{\text{earnings} \\ \text{foregone}}} - \underbrace{P_D D(t)}_{\substack{\text{direct goods} \\ \text{costs}}}$$

- Consumer problem (max with respect to $I(t), D(t)$):

$$\int_0^T e^{-rt}E(t)dt \qquad \text{given } H(0) = H_0$$

- Formal solution (Hamiltonian): Flow of value from the optimal lifetime program

$$\underbrace{e^{-rt}[RH(t) - RI(t)H(t) - P_D D(t)]}_{\text{current flow}} + \underbrace{\mu(t)}_{\substack{\text{shadow price}\\ \text{of human capital}}}[\dot{H}]$$

- FOC Conditions (for interior solution):

$$I(t): \quad Re^{-rt}H(t) = \mu(t)F_1H(t)$$

$$D(t): \quad e^{-rt}P_D = \mu(t)F_2$$

$$\dot{\mu}(t) = -e^{-rt}[R - RI(t)] - \mu(t)F_1I(t) + \mu(t)\sigma$$

- Use FOC for investment to obtain:

$$\dot{\mu}(t) = -e^{-rt}R + \mu(t)\sigma.$$

Define $g(t) = \mu(t)e^{+rt}$

$\dot{g}(t) = \dot{\mu}e^{+rt} + r\mu(t)e^{+rt}$

$\dot{g}(t) = (\sigma + r)g(t) - R.$

- Transversality: $\lim_{t \to T} \mu(t)H(t) = 0$

$$\therefore \mu(T) = 0 \Longrightarrow g(T) = 0$$

$$g(t) = \frac{R\left(1 - e^{(\sigma+r)(t-T)}\right)}{\sigma + r}.$$

- Note that g(t) is a discount factor that adjusts for exponential depreciation of gross investment.
- $\dot{H}(t) + \sigma H(t) = F(IH(t), D(t))$.

- $0 < I(t) < 1$, we can set up the problem in a "myopic" way.
- Gross "output" is $F(I(t)H(t), D(t))$.
- Returns on gross output: $g(t)$.
- Costs: $P_D D(t) + RI(t)H(t)$.
- Note: these are costs and returns as of period t.

- The agent's problem is:

$$\max_{I(t),D(t)} [g(t)F(I(t)H(t), D(t)) - P_D D(t) - RI(t)H(t) = 0]$$

FOC:

- $g(t)F_1(I(t)H(t), D(t))H(t) = RH(t)$
- $g(t)F_2(I(t)H(t), D(t)) - P_D = 0.$

Demand functions are inverted first order conditions:

- $I(t)H(t) = I(t)H\left(\frac{R}{g(t)}, \frac{P_D}{g(t)}\right)$
- $D(t) = D\left(\frac{R}{g(t)}, \frac{P_D}{g(t)}\right)$

From normality of inputs, since $\dot{g}(t) < 0$, we have:

- $I\dot{H}(t) < 0,\ \dot{D}(t) < 0.$

- Then, if $\sigma = 0$,
  $\dot{E} = RF(I(t)H(t), D(t)) - R\,I\dot{H}(t) - P_D\dot{D}(t) > 0.$
- Otherwise earnings can rise and then fall over the life cycle. $(\sigma \neq 0)$.
- What about $\ddot{E}(t)$? Ben Porath chose a Cobb-Douglas form for $F(I(t)H(t), D(t))$ and proves $\ddot{E}(t) < 0$.
- $\therefore$ Earnings increase at a decreasing rate over the life cycle.
- To simplify derivations, let $F_2 \equiv 0$ (i.e. ignore $D(t)$).

- First order condition for investment is:

$$g(t)F'(IH) = R.$$

$$\dot{g} = (\sigma + r)g(t) - R$$

- Differentiate the first order condition for investment.
- Set $R = 1$ (for convenience)

(Note that $\dfrac{\dot{g}}{g} = \sigma + r - \dfrac{1}{g}$)

$$\dot{g}(t)F'(I(t)H(t)) + g(t)F''(I(t)H(t))I(t)\dot{H}(t) = 0.$$

Thus $$\dot{IH}(t) = -\left(\frac{\dot{g}(t)}{g(t)}\right)\left[\frac{F'}{F''}\right].$$

- To simplify notation, drop "t" argument for $I(t), H(t), g(t)$ unless it clarifies matters to keep it explicit
- Then $I\ddot{H} = -\left[\frac{\ddot{g}}{g} - \left(\frac{\dot{g}}{g}\right)^2\right]\frac{F'}{F''} - \frac{\dot{g}}{g}\left[I\dot{H} - \frac{F'F'''}{(F'')^2}I\dot{H}\right]$.
- Note that $\ddot{g} = (\sigma + r)\dot{g}$.
- $\therefore \frac{\ddot{g}}{\dot{g}} = (\sigma + r)$ and $\frac{\ddot{g}}{g} = (\sigma + r)\frac{\dot{g}}{g}$ $(\dot{g} \neq 0)$.
- Thus, substituting for $I\dot{H}$ we have

$$I\ddot{H} = -\left[\frac{\ddot{\mathrm{g}}}{\mathrm{g}} - \left(\frac{\dot{g}}{g}\right)^2\right]\frac{F'}{F''} + \left(\frac{\dot{g}}{g}\right)^2\left[1 - \frac{F'F'''}{(F'')^2}\right]\left[\frac{F'}{F''}\right].$$

- Earnings growth is given by (recall $R = 1$)
- $\dot{E} = F(IH) - \dot{IH} - \sigma H$
- $\ddot{E} = F'(IH)\dot{IH} - \ddot{IH} - \sigma \dot{H}$
- Since $F' = \frac{1}{\mathrm{g}}$ we have that
  $$\ddot{E} = \frac{1}{g}\dot{IH} - \ddot{IH} - \sigma \dot{H}$$
- Set $\sigma = 0$ for the moment and use the expression for $\dot{IH}$ given above (including $\dot{IH}$).

Thus

- $\ddot{E} = I\dot{H}\left[\frac{1}{g} + \frac{\dot{g}}{g}\left(1 - \frac{F'F'''}{(F'')^2}\right)\right] + \left(\frac{\ddot{g}}{g} - \left(\frac{\dot{g}}{g}\right)^2\right)\frac{F'}{F''}.$

- Use $I\dot{H} = -\frac{\dot{g}}{g}\frac{F'}{F''}$ and $\frac{\ddot{g}}{g} \ = (\sigma \ + r)\frac{\dot{g}}{g}$ to conclude that

$$
\begin{aligned}
\ddot{\mathsf{E}} &= -\frac{\dot{\mathsf{g}}}{\mathsf{g}}\left[\frac{\mathsf{F}'}{\mathsf{F}''}\right]\left\{\frac{1}{\mathsf{g}}+\frac{\dot{\mathsf{g}}}{\mathsf{g}}\left(1-\frac{\mathsf{F}'\mathsf{F}''}{(\mathsf{F}'')^2}\right)\right\} \\
&\quad +\left((\sigma+\mathsf{r})\frac{\dot{\mathsf{g}}}{\mathsf{g}} - \left(\frac{\dot{\mathsf{g}}}{\mathsf{g}}\right)^2\right)\frac{\mathsf{F}'}{\mathsf{F}''} \\
&= -\frac{\dot{\mathsf{g}}}{\mathsf{g}}\left(\frac{\mathsf{F}'}{\mathsf{F}''}\right)\left\{\begin{array}{c} \frac{1}{\mathsf{g}}+\frac{\dot{\mathsf{g}}}{\mathsf{g}}\left(1-\frac{\mathsf{F}'\mathsf{F}'''}{(\mathsf{F}'')^2}\right) \\ -\dfrac{\mathsf{g}(\sigma+\mathsf{r})-\dot{\mathsf{g}}}{\mathsf{g}} \end{array}\right\}
\end{aligned}
$$

but $\dot{g} = (\sigma + r)g - 1 \qquad (\sigma + r)g - \dot{g} = 1.$

Thus

$$\begin{aligned}\ddot{E} &= \left(-\frac{\dot{g}}{g}\frac{F'}{F''}\right)\left(\frac{\dot{g}}{g}\right)\left(1-\frac{F'\,F'''}{(F'')^2}\right) \\ &= \underbrace{-\left(\frac{\dot{g}}{g}\right)^2\frac{F'}{F''}}_{\substack{+\\ \text{(by concavity)}}}\cdot\underbrace{\left(1-\frac{\mathrm{F}'\mathrm{F}'''}{(F'')^2}\right)}_{\substack{\text{Term depends on the}\\ \text{sign of } F'''}}.\end{aligned}$$

- Define $\eta = 1 - \dfrac{F' F'''}{(F'')^2}$.
- Necessary condition for concavity of earnings profiles with age is $F''' > 0$;
- Stronger condition is $-\eta > 0$.

- Note: if $F(x) = \dfrac{Ax^{\alpha}}{\alpha}$, $-\infty < \alpha < 1$, $A > 0$, $F'(x) = Ax^{\alpha-1}$
- $F''(x) = (\alpha - 1)Ax^{\alpha-2}$
- $F'''(x) = (\alpha - 1)(\alpha - 2)Ax^{\alpha-3}$
- $\eta = \frac{\alpha-2}{\alpha-1} < 0$. Thus $\ddot{E}$ is negative (concavity).
- If $F(x) = a - be^{-cx}$, for $b, c > 0$, $\eta = 0$ and $\ddot{E}$ negative.
- Obviously fails with quadratic technologies.

Period of Specialization

- Period of specialization is associated with full time investment.
- Assume $F_2 \equiv 0$ (ignore D).
- Suppose that at time t

$$F'(H_0)g(t) > R.$$

- Then it pays to specialize.
- How to solve? Initially assume $\sigma = 0$.
- Note that marginal returns to investment decline with capital stock growth ($F' \downarrow$) and with time $\dot{g} < 0$.

- Then there is at *most* one period of specialization: $[0,\ t^*]$.
- This is "schooling" in the Ben-Porath model.
- t^* is characterized by

$$F'(H(t^*))g(t^*) = R$$

$$I(t^*) = 1 \text{ (at the endpoint of the interval)}$$

$$H(t^*) = \int_0^{t^*} F\left(H(\tau)\right) d\tau + H_0.$$

- Note that anything that lowers $g(t)$ (and not R) lowers t^*.
- Thus the higher r, the lower t^*.
- Note, also, that the higher H_0, the lower t^*, since it takes less time to acquire $H(t^*)$.

- Now to get $H(\tau)$, notice that $\dot{H} = F(H)$ in the period of specialization.
- Solve jointly to get t^*.
- Now, if $\sigma > 0$, we get the same condition for specialization but could get cycling in the model. (Initially, high σ knocks off capital makes specialization in investment productive again.)
- Let $\sigma = 0$, thus no cycling possible in the model.

Cobb-Douglas example:

$$\dot{H} = A(IH)^{\alpha} - \sigma H, \quad 0 < \alpha < 1, \quad A > 0$$

A period of specialization arises if

$$g(t_0)\alpha A(H_0)^{\alpha-1} \quad > \quad R.$$

Then if

$$\begin{aligned} (H_0)^{\alpha-1} \quad &> \quad \left[\frac{R}{g(t_0)\alpha A}\right] \\ \text{or} \quad H_0 \quad &< \quad \left[\frac{R}{g(t_0)\alpha A}\right]^{\frac{1}{\alpha-1}}, \end{aligned}$$

the agent will specialize.
If $T \to \infty$, the condition simplifies to

$$\begin{aligned} H_0 \quad &< \quad \left(\frac{r}{\alpha A}\right)^{\frac{1}{\alpha-1}} = \left(\frac{\alpha A}{r}\right)^{\frac{1}{1-\alpha}} \\ \text{since } g(t) \quad &= \quad \frac{R}{r} \end{aligned}$$

If the condition required for specialization is satisfied, we obtain:

$$\begin{aligned} \dot{H} &= A(IH)^{\alpha} \\ \therefore \qquad \frac{\dot{H}}{H^{\alpha}} &= A \end{aligned}$$

$$\begin{aligned} H(t)^{1-\alpha} &= (1-\alpha)At + (1-\alpha)K_0 \\ H(t) &= [(1-\alpha)At + (1-\alpha)K_0]^{\frac{1}{1-\alpha}} \\ [K_0(1-\alpha)]^{\frac{1}{1-\alpha}} &= H_0 \\ K_0(1-\alpha) &= H_0^{1-\alpha} \\ K_0 &= \frac{H_0^{1-\alpha}}{(1-\alpha)} \end{aligned}$$

Therefore, we have that during the period of specialization (schooling) human capital is accumulating via the following growth process:

$$\begin{aligned} H(t) &= \left[A(1-\alpha)t + K_0(1-\alpha)\right]^{\frac{1}{1-\alpha}} \\ &= \left[A(1-\alpha)t + H_0^{1-\alpha}\right]^{\frac{1}{1-\alpha}}. \end{aligned}$$

At the end of the period of specialization we have that

$$\alpha g(t^*) A(H(t^*))^{\alpha-1} = R.$$

Let $T \to \infty$, then $g(t^*) = R/r$ and t^* is defined by solving

$$\alpha \frac{R}{r} A \left(A(1-\alpha)t^* + H_0^{1-\alpha} \right)^{-1} = R.$$

Thus,

$$
\begin{aligned}
\left(\frac{\alpha A}{r}\right) &= A\left(1-\alpha\right)t^{*} + H_0^{1-\alpha} \\
\text{Schooling:} \qquad t^{*} &= -\frac{H_0^{1-\alpha}}{A\left(1-\alpha\right)} + \left(\frac{\alpha}{1-\alpha}\right)\frac{1}{r}
\end{aligned}
$$

Higher A, higher t^{*} "ability to learn."
Higher H_0, lower t^{*} "ability to earn."

Define post school work experience as $\tau = t - t^*$. Then

$$E(\tau) = R\int_0^{\tau} \dot{H}(\ell + t^*)\, d\ell + RH(t^*) - RIH(\tau + t^*).$$

At school leaving age and beyond we have

$$\alpha g(t)\, A(IH(t))^{\alpha-1} = R.$$

Thus, we have

$$\begin{aligned}
[IH(t)]^{\alpha-1} &= \frac{R}{\alpha g(t) A} \\
IH(t) &= \left[\frac{\alpha g(t) A}{R}\right]^{\dfrac{1}{1-\alpha}}.
\end{aligned}$$

Thus,

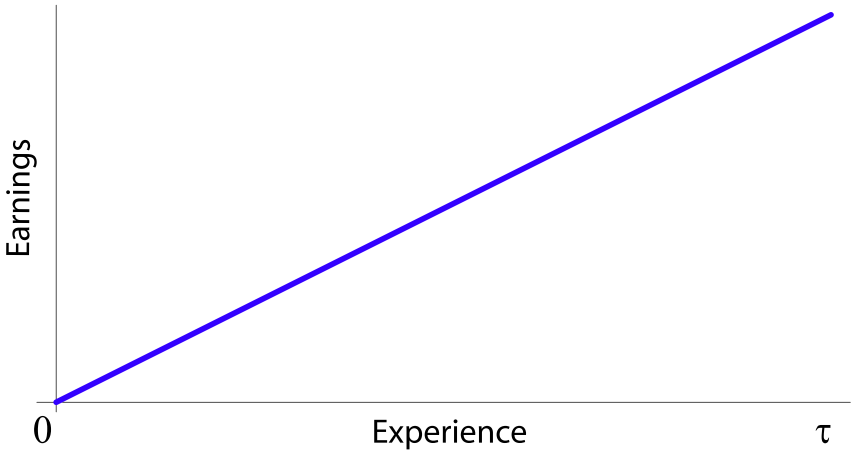
$$\dot{H} = A\left[\frac{\alpha g(t)A}{R}\right]^{\dfrac{\alpha}{1-\alpha}}.$$

Earnings are given by

$$\begin{aligned} E(\tau) &= R\int_0^\tau A\left[\frac{\alpha g(\ell+t^*)A}{R}\right]^{\frac{\alpha}{1-\alpha}} d\ell + RH(t^*) \\ &\quad -R\left[\frac{\alpha g(\tau+t^*)A}{r}\right]^{\frac{1}{1-\alpha}}. \end{aligned}$$

Let $T \to \infty$, then $g(t) = \dfrac{R}{r}$

$$
\begin{aligned}
E(\tau) &= RA\left[\frac{\alpha A}{r}\right]^{\frac{\alpha}{1-\alpha}}\tau + R\left[\frac{\alpha A}{r}\right]^{\frac{1}{1-\alpha}} - R\left[\frac{\alpha A}{r}\right]^{\frac{1}{1-\alpha}} \\
&= RA\left[\frac{\alpha A}{r}\right]^{\frac{\alpha}{1-\alpha}}\tau.
\end{aligned}
$$


Earnings
0
Experience
τ

Human Capital Dynamics

$$t_0 < t < T, \qquad T \to \infty, \qquad t^* = \left(\frac{\alpha}{1-\alpha}\right)\frac{1}{r} - \frac{H_0^{1-\alpha}}{A(1-\alpha)}$$

$$\begin{aligned}
t = t_0 \ &\Rightarrow \ H(t) = H_0 \\
t_0 < t < t^* \ &\Rightarrow \ H(t) = (A(1-\alpha)t + H_0^{1-\alpha})^{\frac{1}{1-\alpha}} \\
t = t^* \ &\Rightarrow \ H(t) = \left(\frac{\alpha A}{r}\right)^{\frac{1}{1-\alpha}} \\
t^* < t \ &\Rightarrow \ H(t) = \left(\frac{\alpha A}{r}\right)^{\frac{\alpha}{1-\alpha}} (t - t^*) + H(t^*)
\end{aligned}$$

Investment Dynamics

$$t_0 < t < T, \qquad T \to \infty, \qquad t^* = \left(\frac{\alpha}{1-\alpha}\right)\frac{1}{r} - \frac{H_0^{1-\alpha}}{A(1-\alpha)}$$

$$\begin{aligned} t = t_0 \quad &\Rightarrow \quad I(t) = 1 \quad \text{if} \quad F'(H_0)g(t) > R \\ t_0 < t \leq t^* \quad &\Rightarrow \quad I(t) = 1 \\ t^* < t \quad &\Rightarrow \quad I(t) = \frac{\left(\frac{\alpha A}{r}\right)^{\frac{1}{1-\alpha}}}{\left(\frac{\alpha A}{r}\right)^{\frac{\alpha}{1-\alpha}}(t-t^*) + H(t^*)} \\ I(t) \quad &= \quad \left(\left(\frac{\alpha A}{r}\right)^{-1}(t-t^*)+1\right)^{-1} \end{aligned}$$

Earnings Dynamics

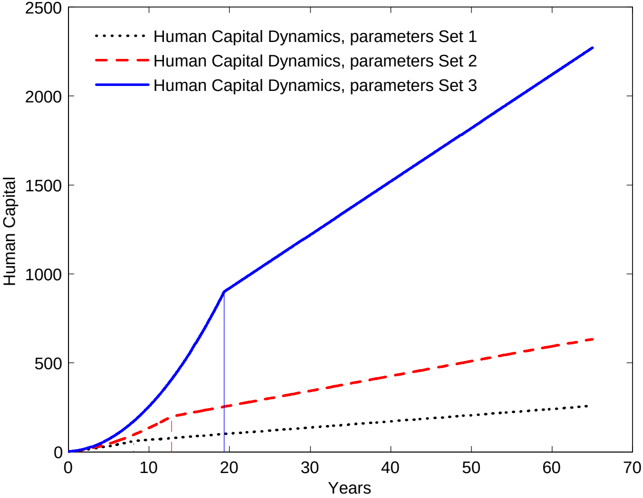
$$t_0 < t < T, \qquad T \to \infty, \qquad t^* = \left(\frac{\alpha}{1-\alpha}\right)\frac{1}{r} - \frac{H_0^{1-\alpha}}{A(1-\alpha)}$$

$E(t) = RH(t) \cdot (1 - I(t))$, so

$$\begin{aligned}
t_0 < t \le t^* \Rightarrow \quad I(t) &= 1 \Rightarrow E(t) = 0 \\
t^* < t \Rightarrow \quad E(t) &= RH(t) - RH(t)I(t) \\
&= RH(t) - \left(\frac{\alpha A}{r}\right)^{\frac{1}{1-\alpha}} \\
&= R(A(1-\alpha)t + H_0^{1-\alpha})^{\frac{1}{1-\alpha}} - \left(\frac{\alpha A}{r}\right)^{\frac{1}{1-\alpha}}
\end{aligned}$$

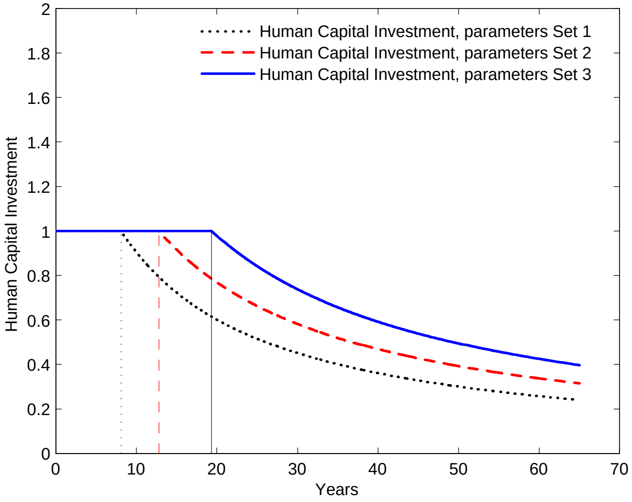
Human capital dynamics, varying α ($A = 3$, $r = 0.05$, $H_0 = 1$)

$\alpha = 0.3$ (dotted line), $\alpha = 0.4$ (dashed line), $\alpha = 0.5$ (solid line)



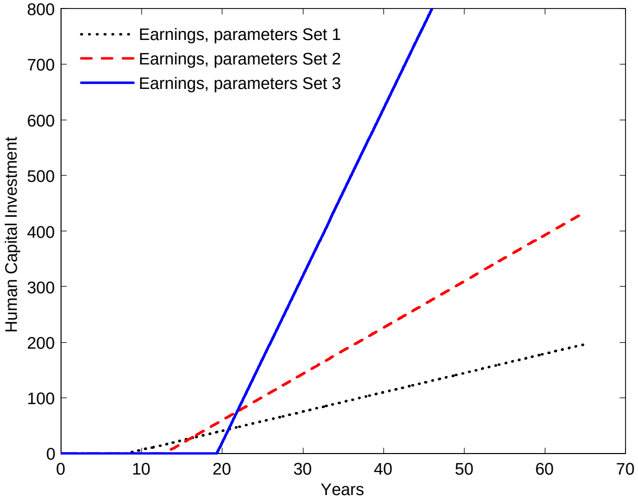
Human investment dynamics, varying α ($A = 3$, $r = 0.05$, $H_0 = 1$, $R = 1$)

$\alpha = 0.3$ (dotted line), $\alpha = 0.4$ (dashed line), $\alpha = 0.5$ (solid line)



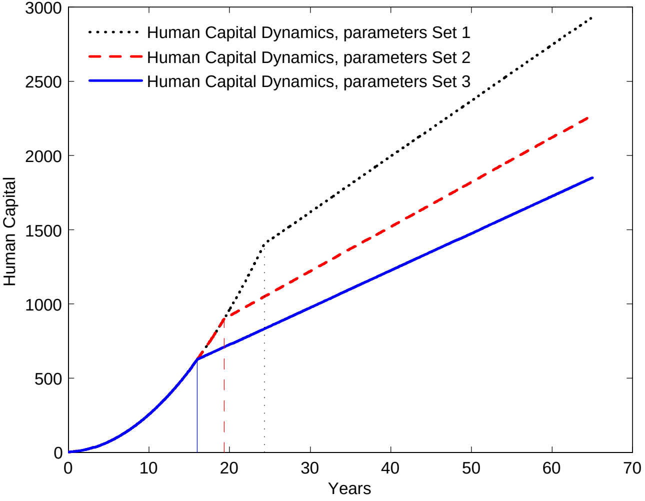
Earnings dynamics, varying α ($A = 3$, $r = 0.05$, $H_0 = 1$)

$\alpha = 0.3$ (dotted line), $\alpha = 0.4$ (dashed line), $\alpha = 0.5$ (solid line)



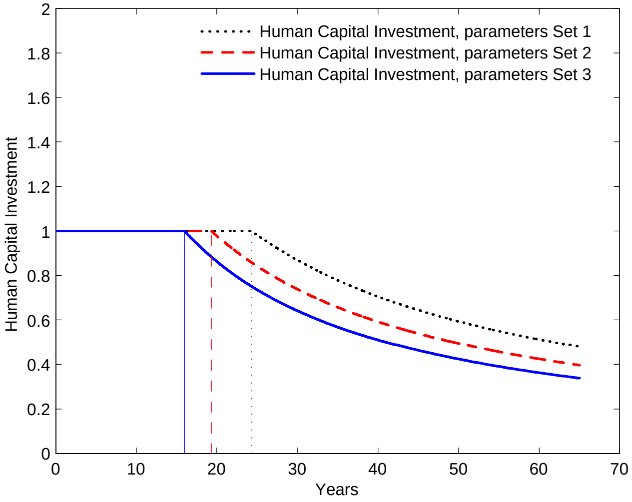
Human capital dynamics, varying r ($A = 3$, $H_0 = 1$, $\alpha = 0.5$)

$r = 0.04$ (dotted line), $r = 0.05$ (dashed line), $r = 0.06$ (solid line)



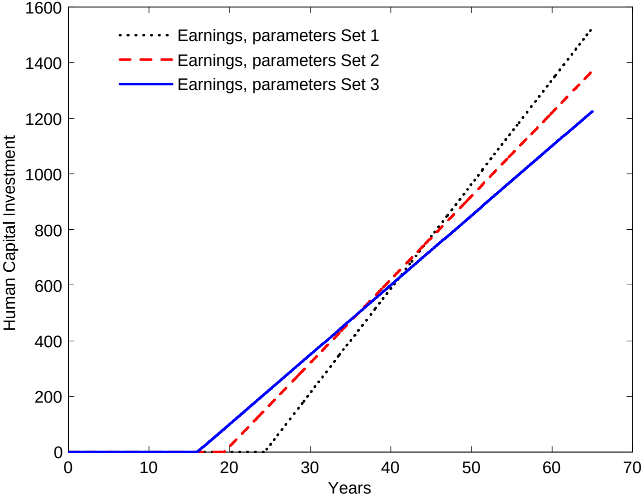
Human investment dynamics, varying r ($A = 3$, $H_0 = 1$, $\alpha = 0.5$)

$r = 0.04$ (dotted line), $r = 0.05$ (dashed line), $r = 0.06$ (solid line)



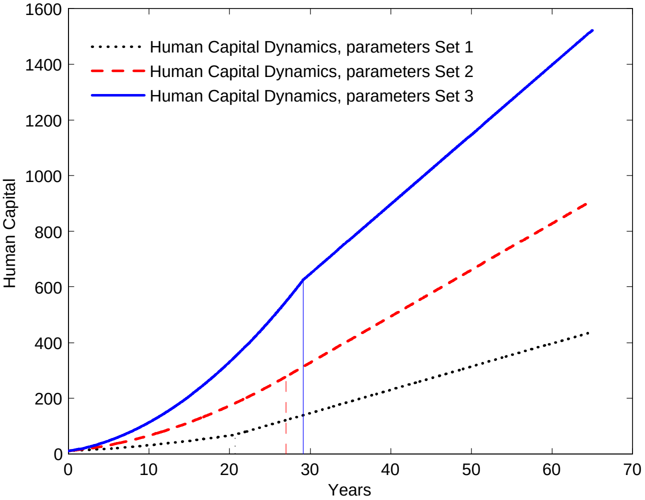
Earnings dynamics, varying r ($A = 3$, $H_0 = 1$, $\alpha = 0.5$, $R = 1$)

$r = 0.04$ (dotted line), $r = 0.05$ (dashed line), $r = 0.06$ (solid line)



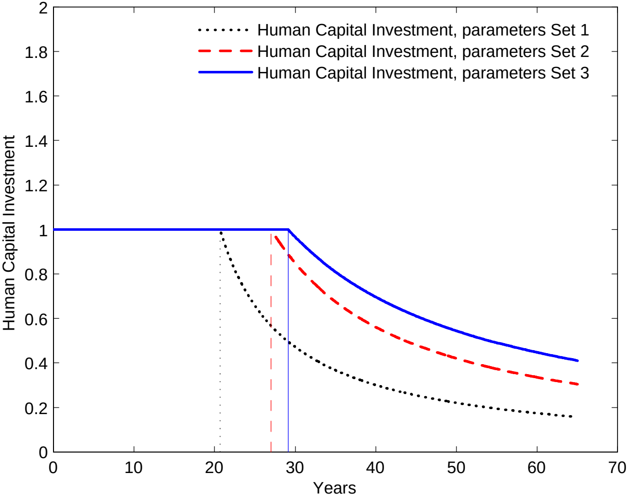
Human capital dynamics, varying A ($r = 0.03$, $H_0 = 10$, $\alpha = 0.5$)

$A = 0.5$ (dotted line), $A = 1.0$ (dashed line), $A = 1.5$ (solid line)



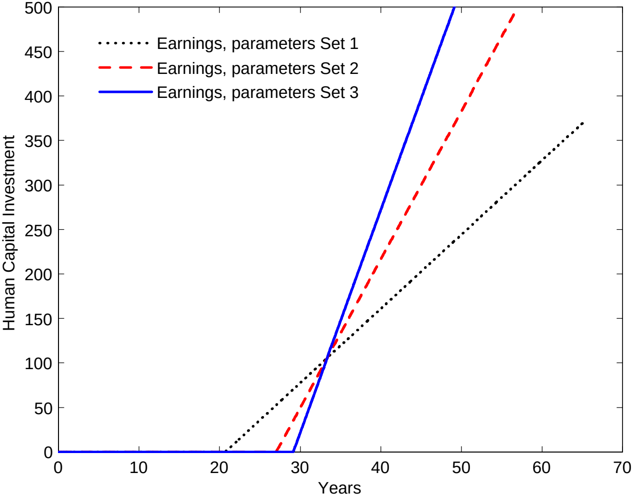
Human investment dynamics, varying A ($r = 0.03$, $H_0 = 10$, $\alpha = 0.5$)

$A = 0.5$ (dotted line), $A = 1.0$ (dashed line), $A = 1.5$ (solid line)



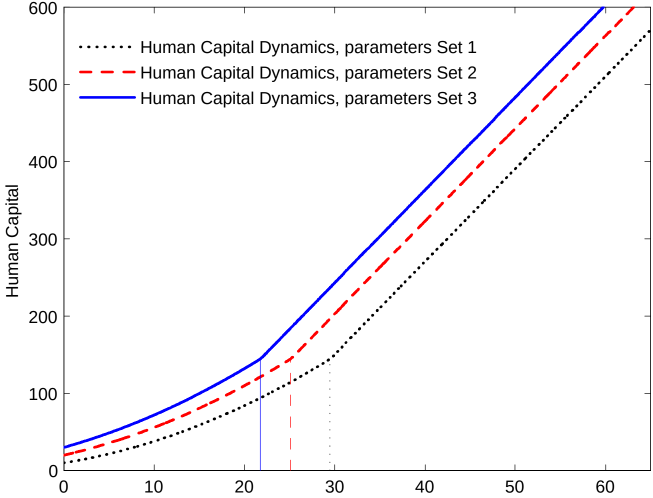
Earnings dynamics, varying A ($r = 0.03$, $H_0 = 10$, $\alpha = 0.5$)

$A = 0.5$ (dotted line), $A = 1.0$ (dashed line), $A = 1.5$ (solid line)



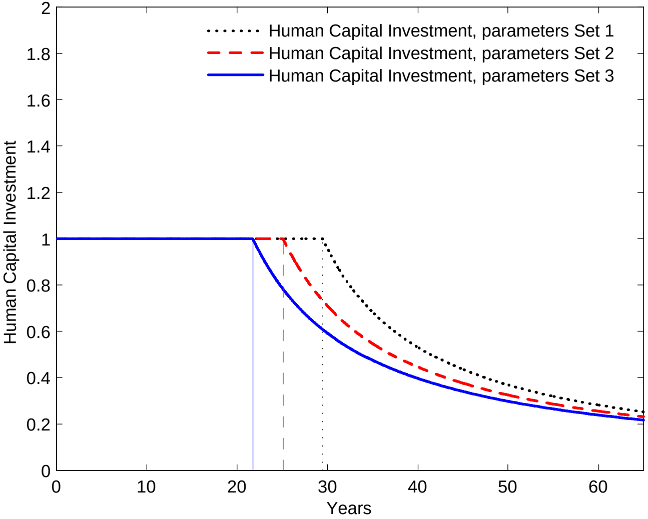
Human capital dynamics, varying H_0 ($A = 0.6$, $r = 0.025$, $\alpha = 0.5$, $R = 1.0$)

$H_0 = 10$ (dotted line), $H_0 = 20$ (dashed line), $H_0 = 30$ (solid line)



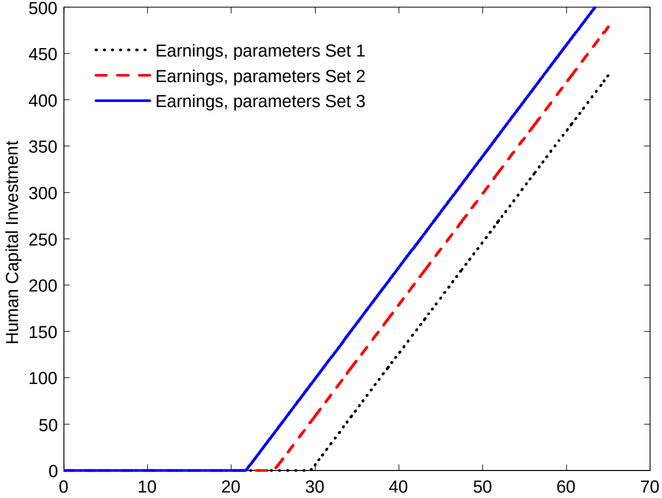
Earnings dynamics, varying H_0 ($A = 0.6$, $r = 0.025$, $\alpha = 0.5$, $R = 1.0$)

$H_0 = 10$ (dotted line), $H_0 = 20$ (dashed line), $H_0 = 30$ (solid line)



Human investment dynamics, varying H_0 ($A = 0.6$, $r = 0.025$, $\alpha = 0.5$, $R = 1.0$)

$H_0 = 10$ (dotted line), $H_0 = 20$ (dashed line), $H_0 = 30$ (solid line)



Finite Horizon Ben Porath Model in Level and Autogressive Form ($\alpha = 1/2$)

- $\dot{H} = A\,(IH)^{\alpha}$
- $\alpha = 1/2$ (Haley, 1976; Rosen, 1976)
- $\sigma = 0$
- $R =$ rental rate

- $$\dot{E}(\tau) = \frac{A^2}{2} g(\tau + t^*) - 2R \left[\frac{A}{2} \frac{g(\tau + t^*)}{R} \right] \left[\frac{A}{2R} \dot{g}(\tau + t^*) \right]$$

$$\dot{g} = rg - R$$

- Thus,

$$\dot{E}(\tau) = \left[\frac{A^2}{2R} \right] g[2R - rg]$$

$$\ddot{E}(\tau) = \frac{-A^2}{R} (\dot{g})^2.$$

- Using $\dot{g} = rg - R$,

$$\dot{E}(\tau) = \frac{A^2}{2R}\left(\frac{\dot{g}+R}{r}\right)\left(2R - r\frac{(\dot{g}+R)}{r}\right)$$

$$= \frac{A^2}{2Rr}(R^2 - (\dot{g})^2).$$

- Thus,

$$\dot{E}(\tau) = \frac{A^2}{2Rr}R^2 - \frac{1}{2r}\frac{A^2}{R}(\dot{g})^2$$

$$= \frac{A^2}{2Rr}R^2 + \frac{1}{2r}\ddot{E}(\tau).$$

- Thus,

$$\ddot{E}(\tau) = 2r\dot{E}(\tau) - A^2 R. \tag{1}$$

- This is a standard ordinary differential equation with constant coefficients. The solution is of the form

$$E(\tau) = c_1 e^{2r\tau} + c_2\tau + c_0.$$

- We can pin this equation down knowing that

$$E(0) = 0 \qquad (\text{so } c_1 + c_0 = 0)$$

$$\dot{E}(T) = 0 \qquad (\text{so } 2rc_1 e^{2rT} + c_2 = 0).$$

- Finally, optimality produces (1) above to get c_0.
- Set

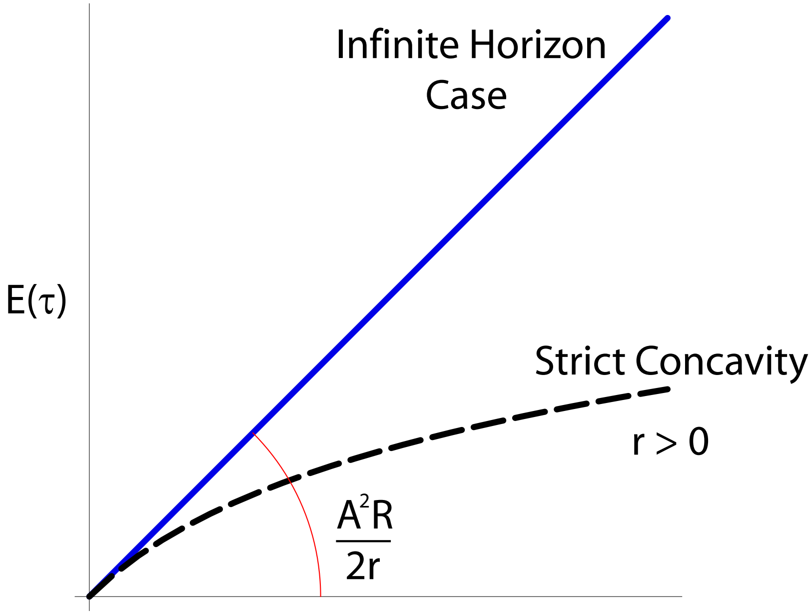
$$c_1 = -c_0$$

$$c_2 = \frac{A^2 R}{2r} e^{2rT},$$

using $E(T) = 0$ and (1).

- Thus

$$E(\tau) = \frac{A^2 R}{(2r)^2} e^{-2rT}(1 - e^{2r\tau}) + \left(\frac{A^2 R}{2r}\right)\tau. \qquad (2)$$


Infinite Horizon Case
E(τ)
Strict Concavity
r > 0
$\frac{A^2R}{2r}$

- This, in its essential form, is the equation that Brown (JPE, 1976) fits; from the τ term, one can identify $\dfrac{A^2 R}{2r}$.
- From the exponential (in τ) one can pick up r and $A^2 R$, but his estimates are poor, $r \to 0$.
- But from Brown, $T \to \infty$ is a good approximation. (His sample is young). Thus

$$E(\tau) \doteq \frac{RA^2}{2}\tau.$$

- Thus "r" is not identified.

- Write this as an autoregression:

$$E(\tau+1) - E(\tau) = \frac{A^2 R}{(2r)^2} e^{-2rT} \left(e^{2r\tau} - e^{2r(\tau+1)} \right) + \frac{A^2 R}{2r}$$

$$E(\tau) - E(\tau-1) = \frac{A^2 R}{(2r)^2} e^{-2rT} \left(e^{2r(\tau-1)} - e^{2r\tau} \right) + \frac{A^2 R}{2r}.$$

- Multiply second equation by e^{2r}:

$$e^{2r}[E(\tau) - E(\tau-1)] = \frac{A^2R}{2r^2}e^{-2rT}(e^{2r\tau} - e^{2r(\tau+1)}) + e^{2r}\frac{(A^2R)}{2r}$$

$$= E(\tau+1) - E(\tau) - (1 - e^{2r\tau})\frac{A^2R}{2r}.$$

- Thus

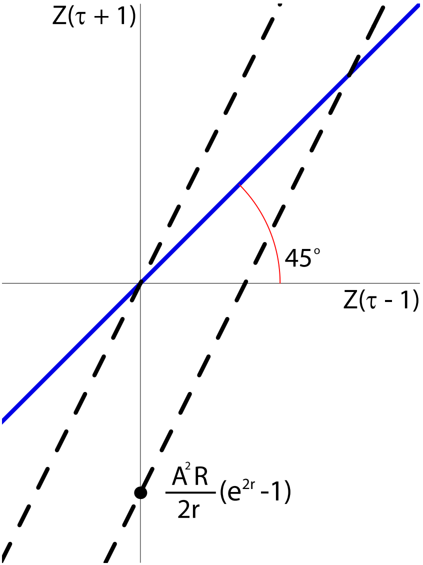
$$E(\tau+1) - E(\tau) = e^{2r}[E(\tau) - E(\tau-1)] - (e^{2r} - 1)\frac{A^2R}{2r}.$$

- Let

$$Z(\tau+1) = E(\tau+1) - E(\tau)$$

$$Z(\tau) = E(\tau) - E(\tau-1)$$

$$Z(\tau+1) = e^{2r}Z(\tau) - (e^{2r}-1)\left(\frac{A^2R}{2r}\right).$$


Z(τ + 1)
45°
Z(τ - 1)
$\frac{A^2R}{2r}(e^{2r}-1)$

- Apparently explosive, it actually converges. Observe:

$$E(\tau) - E(\tau - 1) = \frac{A^2 R}{(2r)^2} e^{-2rT} (e^{2r(\tau-1)} - e^{2r\tau}) + \frac{A^2 R}{2r}$$

$$= \frac{A^2 R}{2r} \left[1 + \frac{e^{-2rT}}{2r} e^{2r\tau} (1 - e^{2r}) \right]$$

- $$\frac{\partial [E(\tau) - E(\tau - 1)]}{\partial \tau} = \frac{A^2 R}{2r} (e^{-2rT} e^{2r\tau} (1 - e^{2r}) < 0$$

- Increments are actually decreasing.
- Let $b = e^{2r}$.
- $$c = -\left(\frac{e^{2r}-1}{2r}\right)\frac{A^2R}{2r} = \left(\frac{1-e^{2r}}{2r}\right)A^2R$$

$$Z(T) = \underbrace{(b)^T\,(Z_0)}_{\text{growing}} + \underbrace{c\sum_{j=0}^{T-1} b^j}_{\text{declining}},$$

but converges to a constant (even though autoregression is "explosive").

Deriving Mincer from Ben Porath

Using (2), we obtain

$$E(\tau) = \left(\frac{A^2 R}{2r}\right)\left[\tau + \frac{e^{-2rT} - e^{2r(\tau - T)}}{2r}\right].$$

In logs,

$$\ln \mathrm{E}(\tau) = \ln\left(\frac{A^2 R}{2r}\right) + \ln\tau + \ln\left[1 + \frac{e^{-2rT} - e^{2r(\tau-T)}}{2r\tau}\right]$$

$$= \ln\left[\frac{A^2 R}{2r}\right] + \ln\tau + \ln\left[1 + \frac{e^{-rT}(1 - e^{2r\tau})}{2r\tau}\right].$$

The Taylor Expansions

$$
\begin{aligned}
\ln(\tau) &\doteq \ln(\tau_0) + \frac{1}{\tau_0}(\tau - \tau_0) - \frac{1}{\tau_0^2}\frac{(\tau - \tau_0)^2}{2!} \\
\ln\left(1 + \frac{e^{-2rT} - e^{2r(\tau - T)}}{2r\tau}\right) &\doteq \xi_0 + \xi_1(\tau - \tau_0) + \xi_2\frac{(\tau - \tau_0)^2}{2!}
\end{aligned}
$$

$$
\begin{aligned}
\xi_0 &\equiv \ln\left(1 + \frac{e^{-2rT} - e^{2r(\tau_0 - T)}}{2r\tau_0}\right) \\
\xi_1 &\equiv -\left(\frac{e^{-2rT} + e^{2r(\tau_0 - T)}(2r\tau_0 - 1)}{\tau_0\left(2r\tau_0 + e^{-2rT} - e^{2r(\tau_0 - T)}\right)}\right) \\
\xi_2 &\equiv \left[\begin{array}{c} \frac{\left(e^{-2rT} + e^{2r(\tau_0 - T)}(2r\tau_0 - 1)\right)}{\left(\tau_0\left(2r\tau_0 + e^{-2rT} - e^{2r(\tau_0 - T)}\right)\right)^2}\left(4r\tau_0 + e^{-2rT} - e^{2r(\tau_0 - T)}(2r\tau_0 + 1)\right) \\ -\left(\frac{(2r)^2\tau_0 e^{2r(\tau_0 - T)}}{\left(\tau_0\left(2r\tau_0 + e^{-2rT} - e^{2r(\tau_0 - T)}\right)\right)}\right) \end{array}\right]
\end{aligned}
$$

Adding the terms together:

$$\ln(\tau) + \ln\left(1 + \frac{e^{-2rT} - e^{2r(\tau - T)}}{2r\tau}\right)$$

$$\doteq \alpha_0 + \alpha_1(\tau - \tau_0) + \alpha_2(\tau - \tau_0)^2$$

$$\begin{aligned}
\alpha_0 &\equiv \ln(\tau_0) + \xi_0 \\
\alpha_1 &\equiv \xi_1 + \frac{1}{\tau_0} \\
\alpha_2 &\equiv \left(-\frac{1}{\tau_0^2} + \xi_2\right) \Big/ 2
\end{aligned}$$

To obtain Mincer Equations:

$$\ln(\tau) + \ln\left(1 + \frac{e^{-2rT} - e^{2r(\tau - T)}}{2r\tau}\right) \doteq k_0 + k_1\tau + k_2\tau^2$$

$$\begin{aligned} k_0 &\equiv \alpha_0 - \tau_0\alpha_1 + \alpha_2\tau_0^2 \\ k_1 &\equiv \alpha_1 - 2\alpha_2\tau_0 \\ k_2 &\equiv \alpha_2 \end{aligned}$$

Mincer Obtained:

- Mincer coefficients

$$\begin{aligned} \hat{k}_1 &= 0.081 \\ \hat{k}_2 &= -0.0012 \end{aligned}$$

- Using $r = 0.0225$, $\tau_0 = 29.54$, $T = 41.43$,

$$\begin{aligned} k_1 &= 0.081 \\ k_2 &= -0.0010 \end{aligned}$$

Parameters			Ben Porath Coefficients	
r	τ_0	T	k_1	k_2
0.0225	29.54	41.43	0.081	-0.0010
0.05	25	60	0.0808	-0.0008
0.05	20	65	0.1002	-0.0013
0.0675	24.70	74.77	0.081	-0.0008
Mincer Coefficients			0.081	-0.0012

$$\text{Model: } \ln(\text{Earnings}) = k_0 + k_1\tau + k_2\tau^2$$

- Suppose
$$rT \doteq 0 \quad \text{and} \quad e^{-rT} = 1\,.$$

- $$\ln \mathrm{E}(\tau) \doteq \ln\left(\frac{A^2R}{2r}\right) + \ln\tau + \ln\left[1 + \frac{1-e^{2r\tau}}{2r\tau}\right]$$

Conclusion

- There may be no economic content in Mincer's "rate of return" on post-school investment.
- All of the economic content is in the intercept term.
- Note, however, holding experience constant, there should be no effect of schooling on the earnings function.
- Mincer finds an effect. This would seem to argue against the Ben-Porath model.
- Not necessarily. Look at equation

$$t^* = \frac{1}{r} - \frac{1}{2}\frac{H_0^{1/2}}{A} \quad \text{for } \alpha = 1/2 \text{ and } T \text{ “big.”}$$

- Suppose A is randomly distributed in the population.
- Then, we have that if H_0 is distributed independently of A, the coefficient on t^* (length of schooling) is

$$E\left[\left(-\frac{1}{2}\,\frac{H_0^{1/2}}{A}\right)(2\ln A)\right] > 0.$$

- Thus, the coefficient on schooling is

$$-E\left(H_0^{1/2}\right)E\left(\frac{\ln A}{A}\right).$$

If A is Pareto;

$$F(A) = \left(\frac{\alpha}{A_0}\right)\left(\frac{A_0}{A}\right)^{\alpha+1}, \quad A_0 > 0,\ \alpha > 0.$$

Integrate by parts to reach

$$\begin{aligned} E\left(\frac{\ln A}{A}\right) &= -\frac{(A_0)^{\alpha+1}\alpha}{A_0}(\ln A_0)\, A_0^{-(\alpha+1)} - \frac{1}{\alpha+1} \\ &= -\frac{\alpha \ln A_0}{A_0} - \frac{1}{\alpha+1} \end{aligned}$$

Therefore, the coefficient on schooling is

$$E\left(H_0\right)^{1/2}\left[\frac{1}{\alpha+1}+\frac{\alpha \ln A_0}{A_0}\right]>0.$$

Since units of H_0 are arbitrary, we are done.

Therefore, positive coefficient on schooling solely as a consequence of *not* including ability measures.

Rate of Return to Post-School Investment

Let $T \to \infty$. Without post-school investment, person makes

$$R\left[\frac{\alpha A}{r}\right]^{\frac{1}{1-\alpha}}.$$

Increment in earnings at post-school age τ is simply

$$\underbrace{RA\left[\frac{\alpha A}{r}\right]^{\frac{1}{1-\alpha}}\tau}_{\text{Earnings (above schooling earnings) at } \tau} - \underbrace{R\left[\frac{\alpha A}{r}\right]^{\frac{1}{1-\alpha}}}_{\text{Costs}}.$$

- ϕ is that rate that equates returns and costs. Thus, solve for ϕ.

$$\int_0^\infty e^{-\phi\tau}\left[RA\left[\frac{\alpha A}{r}\right]^{\frac{\alpha}{1-\alpha}}\tau - R\left[\frac{\alpha A}{r}\right]^{\frac{1}{1-\alpha}}\right]d\tau = 0$$

- Use the Laplace transform.

- Then

$$\frac{1}{\phi^2}RA\left[\frac{\alpha A}{r}\right]^{\frac{\alpha}{1-\alpha}} - \frac{1}{\phi}R\left[\frac{\alpha A}{r}\right]^{\frac{1}{1-\alpha}} = 0$$

$$\phi = \frac{r}{\alpha}.$$

- Therefore the rate of return to post-schooling investment is r/α.
- Smaller α, higher ϕ.
- Thus, the lower the productivity (i.e., α), the higher ϕ.

Rate of Return to Schooling (Holding Post-School Investment Fixed)

Person without schooling can earn RH_0. With schooling can earn $RA\left[\frac{\alpha A}{r}\right]^{\frac{\alpha}{1-\alpha}}$. (Assuming no post school investment.)

Recall that (for $T \to \infty$), optimal schooling is given by

$$t^* = \frac{1}{r} - \frac{1}{2}\frac{H_0^{1/2}}{A}.$$

During this period (before t^*), under our assumptions, there are no earnings.

Then the rate of return is given by comparing

$$\int_{t^*}^{\infty} e^{-\phi t}\left[R\left(\frac{\alpha A}{r}\right)^{\frac{1}{1-\alpha}}\right] dt \text{ with } \int_{0}^{\infty} e^{-\phi t} R H_0 \, dt.$$

Solve for ϕ:

$$\begin{aligned}
\left[\frac{\alpha A}{r}\right]^{\frac{1}{1-\alpha}} e^{-\phi t^*} &= H_0 \\
\ln\left[\frac{\alpha A}{r}\right]^{\frac{1}{1-\alpha}} - \phi t^* &= \ln H_0 \\
\phi = \frac{\ln\left[\frac{\alpha A}{r}\right]^{\frac{1}{1-\alpha}} - \ln H_0}{t^*} &= \frac{\ln\left[\frac{\alpha A}{r}\right]^{\frac{1}{1-\alpha}} - \ln H_0}{\frac{1}{r} - \frac{1}{2}\frac{H_0^{1/2}}{A}}
\end{aligned}$$

Has no simple relationship to the rate of return to investment.

Growth of Earnings

- Keep time argument implicit unless being explicit helps.
- E, H, IH all depend on t.
- Growth of earnings:

$$\dot{E} = f(IH) - (\dot{IH})$$

$$\frac{\partial \dot{E}}{\partial r} = \ ?$$

- FOC:

$$g(t)\, f'(IH) = 1$$

$$g(t) = \frac{1 - e^{r(t-T)}}{r}$$

- Totally differentiate FOC with respect to t:

$$\dot{g}f'(IH) + gf''(IH)(\dot{IH}) = 0$$

$$-\left(\frac{\dot{g}}{g}\frac{f'}{f''}\right) = (\dot{IH})$$

- First note that

$$\frac{\partial \dot{E}}{\partial r} = f'\left(\frac{\partial IH}{\partial r}\right) - \frac{\partial}{\partial r}\left[(\dot{IH})\right].$$

- Now observe further that

$$\frac{\partial(IH)}{\partial r} < 0.$$

- Thus the first term is negative.

- Observe that we can show that

$$\frac{\partial(I\dot{H})}{\partial r} > 0$$

if concavity on earnings is satisfied ($\ddot{E} < 0$).

- Intuition: the time rate of decrease in IH is slowed down ($r \uparrow \Rightarrow IH \downarrow$; the function is "less concave").

- If we can establish this, we know that the contribution of the second term is negative and

$$\frac{\partial \dot{E}}{\partial r} < 0.$$

- To show this, observe that

$$\frac{\partial[\dot{IH}]}{\partial r} = \left[-\frac{\dot{g}}{g}\right]\left[1-\frac{f'f'''}{(f'')^2}\right]\frac{\partial(IH)}{\partial r} + \left(\frac{f'}{f''}\right)\frac{\partial}{\partial r}\left[-\frac{\dot{g}}{g}\right].$$

- From the earlier notes, concavity of earnings function in experience ($\ddot{E} < 0$)

$$\left[1-\frac{f'f'''}{(f'')^2}\right] < 0.$$

- The first term is positive, since $\dot{g} < 0$ and

$$\frac{\partial(IH)}{\partial r} < 0.$$

- To investigate the second term, we determine that

$$\dot{g} = rg - 1\,, \qquad \frac{\dot{g}}{g} = r - \frac{1}{g}\,, \qquad -\frac{\dot{g}}{g} = \frac{1}{g} - r\,.$$

- Now,

$$\frac{\partial}{\partial r}\left[-\frac{\dot{g}}{g}\right] = -\frac{1}{g^2}\frac{\partial g}{\partial r} - 1.$$

- This term is negative. Why?

- $$\begin{aligned}\frac{\partial g}{\partial r} &= \frac{-(t-T)e^{r(t-T)}}{r} - \frac{1-e^{r(t-T)}}{r^2} \\ &= \frac{1}{r^2}\left[e^{r(t-T)}\left(1-r(t-T)\right)-1\right]\end{aligned}$$

- Now observe that

$$e^{r(T-t)} > 1 + r(T-t) \qquad \text{for } T \geq t\,.$$

- Thus

$$\frac{\partial g}{\partial r} < 0.$$

- Consider next that

$$\frac{-\partial g}{g^2 \partial r} - 1 = \frac{1}{r^2}\left[\frac{1 - e^{r(t-T)}\left(1 - r(t-T)\right)}{g^2}\right] - 1$$

$$= \frac{1}{g^2 r^2}\left[1 - e^{r(t-T)}\left(1 - r(t-T)\right) - \left(1 - e^{r(t-T)}\right)^2\right]$$

$$= \frac{1}{(rg)^2}\left[1 - e^{r(t-T)}\left(1 - r(t-T)\right) - 1 + 2e^{r(t-T)} - e^{2r(t-T)}\right]$$

$$= \frac{1}{(rg)^2}\left[e^{r(t-T)}\right]\left[1 + r(t-T) - e^{r(t-T)}\right].$$

- This expression is clearly negative.
- Set $x \equiv T - t$:

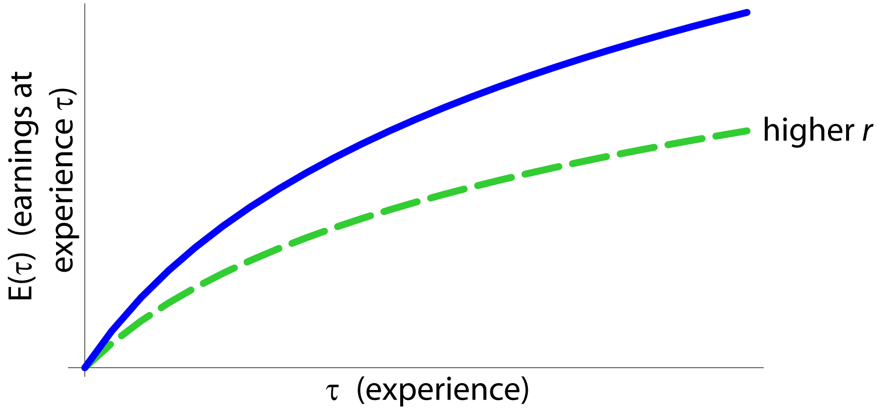
(1)
$$1 - rx - e^{-rx} = 0 \quad \text{when } x = 0.$$

(2)
$$\frac{\partial}{\partial x}\left(1 - rx - e^{-rx}\right) = -r + re^{-rx} < 0.$$

- Thus from concavity ($f'' < 0$),

$$\left(\frac{f'}{f''}\right)\frac{\partial}{\partial r}\left[-\frac{\dot{g}}{g}\right] > 0.$$

- Now the proposition is proved for $\sigma = 0$ with $\ddot{E} < 0$ everywhere. *Q.E.D.*


E(τ) (earnings at experience τ)
higher r
τ (experience)

Appendix: Haley-Rosen: Let $\alpha = 1/2$.

$$E(\tau) = RH(t^*) + R\int_0^{\tau} A\left(\frac{1}{2}\frac{g(t^*+\ell)A}{R}\right)\,d\ell - R\left[\frac{1}{2}\frac{g(\tau+t^*)A}{R}\right]^2.$$

This can be written as a simple autoregression in earnings:

$$\begin{aligned}\dot{E}(\tau) &= R\left[A\left(\frac{1}{2}\frac{g(t^*+\tau)A}{R}\right) - 2R\left[\frac{1}{2}\frac{g(\tau+t^*)A}{R}\right]\frac{A}{2R}\dot{g}(\tau+t^*)\right] \\ &= \frac{1}{2}A^2[g(t^*+\tau)(R-\dot{g}(t^*+\tau)].\end{aligned}$$

$$\dot{g} = rg - R$$

Thus

$$\dot{E}(\tau) = \frac{A^2}{2R}\left[g(t^* + \tau)\left(R - \dot{g}(t^* + \tau)\right)\right]$$

$$\begin{aligned} \dot{g} &= rg - R \\ \ddot{g} &= r\dot{g}. \end{aligned}$$

Haley-Rosen: $\alpha = \beta = 1/2$

$$
\begin{aligned}
E(\tau) &= RH(t^*) + R\int_0^{\tau} A\left(\frac{1}{2}\frac{g(t^*+\ell)A}{R}\right)\, d\ell - R\left[\frac{A}{2}\frac{g(\tau+t^*)}{R}\right]^2 \\
\dot{E}(\tau) &= \frac{A^2}{2}g(\tau+\tau^*) - 2R\left[\frac{A}{2}\frac{g(\tau+t^*)}{R}\right]\left[\frac{A}{2R}\dot{g}\right] \\
&= \frac{A^2}{2}g(\tau+t^*) - \frac{1}{2}\frac{A^2}{R}g\dot{g} \\
&= \frac{1}{2}A^2 g\left[1-\frac{\dot{g}}{R}\right] \qquad \text{use: } \dot{g} = rg - R \\
&= \frac{1}{2}\frac{A^2}{R}g[R-\dot{g}] = \frac{A^2}{2R}g[R-rg+R] \\
&= \frac{A^2}{2R}g[2R-rg]
\end{aligned}
$$

$$
\begin{aligned}
\ddot{E}(\tau) &= \frac{A^2}{2R}[\dot{g}(2R - rg) + g(-r\dot{g})] \\
&= \frac{A^2}{2R}\dot{g}[2R - 2rg] = \frac{A^2}{R}\dot{g}(R - rg) \\
&= -\frac{A^2}{2}(\dot{g})^2.
\end{aligned}
$$

Notice that $\dot{E}(\tau)$ can be written as

$$
\begin{aligned}
\dot{E}(\tau) &= \frac{A^2}{2R}\left(\frac{\dot{g} + R}{r}\right)\left(2R - r\frac{(\dot{g} + R)}{r}\right) \\
&= \frac{A^2}{2R}\left(\frac{\dot{g} + R}{r}\right)(2R - \dot{g} - R) \\
&= \frac{A^2}{2R}\left(\frac{\dot{g} + R}{r}\right)(R - \dot{g}) = \frac{A^2}{2Rr}(R^2 - (\dot{g})^2).
\end{aligned}
$$

Thus we conclude that

$$\begin{aligned}\dot{E}(\tau) &= \frac{A^2}{2Rr}R^2 - \frac{1}{2r}\frac{A^2}{R}(\dot{g})^2 \\ &= \frac{A^2}{2Rr}R^2 + \frac{1}{2r}\ddot{E}\end{aligned}$$

so that

$$\ddot{E}(\tau) - 2r\dot{E}(\tau) + A^2R = 0.$$

Integrate once to reach

$$\dot{E}(\tau) - 2rE(\tau) + A^2R\tau + c_0 = 0$$

where c_0 is a constant of integration.

Then “reduced equation” is

$$\dot{E}(\tau) = 2rE(\tau)$$

so that

$$E(\tau) = c_1 e^{2r\tau},$$

c_1 is constant of integration.

The general solution is thus:

$$E(\tau) = c_0 + c_2\tau + c_1 e^{2r\tau}.$$

For a period of specialization, $E(0) = 0$ so that $c_1 + c_0 = 0$.

$$\dot{E}(\tau) = 2rc_1 e^{2r\tau} + c_2$$

so that at $\tau = 0$,

$$(2rc_1 e^{2r\tau} + c_2) - 2r[c_1 e^{2r\tau} + c_0 + c_2\tau] + A^2 R\tau + c_0 = 0.$$

Thus we conclude that

$$c_2 = \frac{A^2 R}{2r}$$

To this point, the equation looks like

$$E(\tau) = c_0(1 - e^{2r\tau}) + \frac{A^2R}{2r}\tau.$$

Now there is no investment at the end of life.

$$\dot{E}(\tau) = 0.$$

Thus

$$\dot{E}(T) = 0 = -2rc_0e^{2rT} + \frac{A^2R}{2r}$$

so $c_0 = \frac{A^2R}{(2r)^2}e^{-2rT}$. Thus

$$E(\tau) = \frac{A^2R}{(2r)^2}e^{-2rT}(1 - e^{2r\tau}) + \frac{A^2R}{2r}\tau.$$