Human Capital Accumulation and Earnings Dynamics over the Life Cycle: the Ben-Porath Model and beyond

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Abstract

We analyze the Ben-Porath model of human capital accumulation and several of its variations in order to inform researchers on the consequences of their modeling decisions. We begin with the baseline model which builds on the neutrality assumption, i.e. the assumption that investment and human capital enter in a multiplicative way to the production function of human capital. We explore the properties of the earnings function, the specialization period, the features of the particular case where the production function takes a Cobb-Douglas functional form, and the properties of the returns to schooling. Then we study the Haley-Rosen specification, a finite horizon version with a particular Cobb-Douglas parameterization. We also work on a generalization of the Cobb-Douglas case where neutrality is no longer true, but depreciation is set to zero. Finally, we work on the Shenshinski specification, a general Cobb-Douglas formulation with positive depreciation leading to bang-bang equlibria solutions.

1 Introduction

Human capital investment and accumulation are essential components for many studies on a wide range of topics in Economics, including the economic growth and development of nations, the gender and black-white wage gaps, and the rate of return to schooling. Becker (1962) initiates the formal analysis of human capital studies and offers the first unified and comprehensive framework to study human capital investment with the standard Economic tools.[1]

Following Becker's work, Ben-Porath (1967) proposes a dynamic model that relates human capital accumulation to life-cycle earnings. Currently, this is the workhorse model when it comes to analyzing the relationship between human capital accumulation decisions and life-cycle outcomes. The vast majority of papers that model human capital use some variation of the Ben-Porath model. These variations lead to different time profiles of investment in human capital, human capital accumulation, and earnings.

In this paper, we analyze the Ben-Porath model and several of its variations in order to inform researchers on the consequences of their modeling decisions. We proceed as follows: Section 2 presents the baseline specification. Many implications can be derived in closed form and the intuition can be easily explained with straightforward graphical analysis based on the assumptions of no depreciation and a finite horizon. Section 3 analyzes the Haley-Rosen specification, which

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[1] Becker (2009) collates the work of the author on the subject which covers schooling, learning-by-doing, and on-the-job training.

allows a finite horizon but still keeps tractability. Section 4 studies the model in a more general formulation by relaxing the neutrality assumption. Section 5 studies a specification that generates a Bang-Bang equilibrium. Section 6 allows for depreciation and develops conditions under which investment in human capital happens in different episodes over the life-cycle. Section 7 offers some final comments.

2 Basic Ben-Porath Model

The baseline Ben-Porath model studies how a single representative agent makes optimal life-cycle decisions on human capital investment to maximize her total lifetime disposable earnings. At each point in time, the agent's current stock of human capital, $H(t)$, and the rental rate of human capital, R, determine the amount of her potential earnings: $Y(t) = RH(t)$. The agent chooses two type of inputs in order to produce human capital: (i) a fraction of her current stock of human capital, $I(t)H(t)$, with $I(t) \in [0,1]$; (ii) market goods, $D(t)$. Therefore, the cost of human capital investments includes both foregone earnings, $RI(t)H(t)$, and cost of the purchased market goods, $P_D D(t)$, where P_D is the price of the market goods.

Then, the agent's disposable earnings in period t, $E(t)$, are equivalent to her potential earnings in period t, $Y(t)$, less the total costs:

$$E(t) = RH(t) - RI(t)H(t) - P_D D(t) \tag{1}$$

The observed earnings the agent makes from her work in the labor market is $R(1 - I(t))H(t)$, which are higher than her disposable earnings and lower than her potential earnings. Subtracting this by the cost of purchased market goods, $P_D D(t)$, gives her disposable earnings, $E(t)$.

The agent produces human capital through a production function that takes two inputs.

Assumption 2.1 *(Strict Concavity of the Production Function)* $\forall\ t \in [0,T]$ $F(\cdot,\cdot)$ *is strictly concave in both of its arguments.*

The change in human capital stock at time t, which is summarized by the law of motion for $H(t)$, is defined as:

Definition 2.1 *(Law of Motion for Human Capital Stock in the Basic Ben-Porath Specification)*

$$\dot{H}(t) = F\left(I(t)H(t), D(t)\right) - \sigma H(t). \tag{2}$$

The law of motion for human capital stock embeds a neutrality assumption. Namely, the current stock of human capital at time t, $H(t)$, and the investment time at time t, $I(t)$, appear as a single argument in a multiplicative form in the flow production of human capital stock. This assumption simplifies our calculations by neutralizing the effect of $H(t)$ on the optimal decision of time investment. In particular, a higher level of $H(t)$ increases the marginal return of $I(t)$ in producing human capital and the marginal cost of $I(t)$ in foregone earnings, both in a multiplicative pattern. As a result, $H(t)$ cancels out in the first order condition for $I(t)$ as we show later.

The agent's life-cycle problem is to choose $I(t)$ and $D(t)$ over time to maximize her total disposable earnings subject to the law of motion for human capital. Given an initial condition of human capital, $H(0) = H_0$, the agent's problem is as follows.

Problem 2.1 *(Life-cycle Individual's Problem in the Basic Ben-Porath Model)*

$$\max_{I_t, D_t} \int_0^T \exp^{-rt} RH(t)(1 - I(t))dt$$

s.t.

$$\begin{aligned} H(0) &= H_0 \\ \dot{H}(t) &= F\left(I(t)H(t), D(t)\right) - \sigma H(t) \end{aligned}$$

Therefore, the current value Hamiltonian associated to the agent's maximization problem is

$$\mathcal{H}(\cdot) = \exp^{-rt}\left[RH(t) - RI(t)H(t) - P_D D(t)\right] + \mu(t)\dot{H}(t) \tag{3}$$

where $\mu(t)$ defines the shadow price of the human capital stock. Thus, the following conditions must be satisfied for the interior solution.

Condition 2.1 *(Optimality Conditions for the Life-cycle Individual's Problem in the Basic Ben-Porath Model)*

$$\frac{\partial \mathcal{H}(\cdot)}{\partial I(t)} = 0 \quad \Leftrightarrow \quad \exp^{-rt} R = \mu(t) F_1\left(I(t)H(t), D(t)\right) \tag{4}$$

$$\frac{\partial \mathcal{H}(\cdot)}{\partial D(t)} = 0 \quad \Leftrightarrow \quad \exp^{-rt} P_D = \mu(t) F_2\left(I(t)H(t), D(t)\right) \tag{5}$$

$$\frac{\partial \mathcal{H}(\cdot)}{\partial H(t)} = -\dot{\mu}(t) \quad \Leftrightarrow \quad \exp^{-rt} R\left(1 - I(t)\right) + \mu(t)\left(F_1\left(I(t)H(t), D(t)\right) I(t) - \sigma\right) = -\dot{\mu}(t) \tag{6}$$

$$\frac{\partial \mathcal{H}(\cdot)}{\partial \mu(t)} = \dot{H}(t) \quad \Leftrightarrow \quad \dot{H}(t) = F\left(I(t)H(t), D(t)\right) - \sigma H(t) \tag{7}$$

$$\textit{Transversality} \quad : \quad \lim_{t \to T} \mu(t)H(t) = 0 \tag{8}$$

where F_j is the first order derivative of the production function F with respect to argument j.

To simplify notation, combine the two terms with intertemporal meaning in the life-cycle decision problem into one term through $g(t) \equiv \exp^{rt} \mu(t)$. Then, combine (4) and (6) to get

$$\dot{\mu}(t) = -\exp^{-rt} R + \mu(t)\sigma \tag{9}$$

and note that $\dot{g}(t) = \dot{\mu}(t)\exp^{rt} + r\mu(t)e^{rt}$. Use (9) to obtain

$$\dot{g}(t) = (\sigma + r)g(t) - R. \tag{10}$$

Equation (8) implies that $\mu(T) = 0$, and therefore $g(T) = 0$ provided that $H(T) = 0$ has no economic sense.It is possible, thus, to solve (10) and obtain

$$g(t) = \frac{R}{\sigma + r}\left[1 - \exp^{(\sigma+r)(t-T)}\right], \tag{11}$$

which leads to $\dot{g}(t) < 0$. To wrap up the discussion, note that the optimality conditions for the interior solution are:

$$\begin{aligned} g(t)F_1(I(t)H(t), D(t)) &= R \\ g(t)F_2(I(t)H(t), D(t)) &= P_D. \end{aligned} \tag{12}$$

The system in (12) consists of two equations and two unknowns that solve for the Marshallian demand for $I(t)H(t)$ and $D(t)$. Note that if the optimal solutions are interior and the cross-partial derivative of the production function with respect to its two arguments is assumed to be zero, i.e. $\frac{\partial^2 F}{\partial IH \partial D} = 0$, strict concavity of the production function together with $\dot{g}(t) < 0$ imply that both Marshallian demands are decreasing overtime. This is intuitive, because the agent faces a finite horizon problem and the amount of time left to capture the returns of human capital investment decreases over time.

2.1 Earnings Dynamics

One important question this basic model enables us to ask is how earnings evolve over the life-cycle. Consider both the slope and curvature of the earnings dynamics in the case with no $D(t)$, i.e. $F_{D(t)} = 0$ so that the production function takes the single argument $I(t)H(t)$. Without loss of generality, also assume that $R \equiv 1$.

2.1.1 The Slope of Earnings Dynamics

Claim 2.1 *(Earnings over Time with no Depreciation) Let $\sigma = 0$. Then, when the optimal solution for $I(t)$ is interior, $\dot{E}(t) > 0$.*

Proof: Differentiate (1) and use (2) to write

$$\begin{aligned} \dot{E}(t) &= \dot{H}(t) - I(t)\dot{H}(t) \\ &= F\left(I(t)H(t)\right) - I(t)\dot{H}(t) \\ &> 0 \end{aligned} \tag{13}$$

where the inequality follows because the Marshallian demands for $I(t)H(t)$ is decreasing over time. ⊠

Claim 2.2 *(Earnings over Time with Depreciation) Let $\sigma > 0$. Then, $\dot{E}(t) \lesseqgtr 0$.*

Proof: Follow the same steps as in the proof of Claim 2.1 and note that the term $\sigma H(t)$ appears in the expression for $\dot{E}(t)$. This term could be $\lesseqgtr F\left(I(t)H(t)\right) - I(t)\dot{H}(t)$. ⊠

Claim 2.1 follows because with a positive amount of investment (interior solution) and no depreciation, human capital stock accumulates over time. Moreover, investment in human capital declines over time, and thus disposable earnings increase over time. Claim 2.2 follows because the accumulated stock of human capital may be driven down by a relatively high rate of depreciation.

2.1.2 The Curvature of Earnings Dynamics

We now analyze the curvature of the earnings function for the case in which there is no depreciation.[2].

[2]A similar analysis follows when $\sigma > 0$ for the cases in which either $\dot{E}(t) > 0$ or $\dot{E}(t) < 0$

Claim 2.3 *(Concavity of the Earnings Function with no Depreciation) Assume that* $\eta \equiv \left(1 - \frac{F'F'''}{F''^2}\right) < 0$. *Then, the earnings function is strictly concave.*

Proof: First note that $\dot{E}(t) > 0$ by Claim 2.1. Since $F_{D(t)} = 0$ we can write the first order condition for investment as

$$g(t)F'(I(t)H(t)) = 1 \tag{14}$$

and differentiate it with respect to t to get

$$\begin{aligned} \dot{g}(t)F'(I(t)H(t)) + g(t)F''(I(t)H(t))I(t)\dot{H}(t) &= 0 \\ &\Leftrightarrow \\ I(t)\dot{H}(t) &= -\left(\frac{\dot{g}(t)}{g(t)}\right)\left[\frac{F'}{F''}\right]. \end{aligned} \tag{15}$$

Moreover, drop the argument t to shorten notation, and note that

$$I\ddot{H} = -\left[\frac{\ddot{g}}{g} - \left(\frac{\dot{g}}{g}\right)^2\right]\frac{F'}{F''} + \left(\frac{\dot{g}}{g}\right)^2\left[1 - \frac{F'F'''}{F''^2}\right]\left[\frac{F'}{F''}\right] \tag{16}$$

where we substitute in (15). Further, note that

$$\begin{aligned} \dot{E} &= F(IH) - I\dot{H} - \sigma H \\ \ddot{E} &= F'(IH)I\dot{H} - I\ddot{H} - \sigma\dot{H} \\ &= \frac{1}{g}I\dot{H} - I\ddot{H} - \sigma\dot{H}. \end{aligned} \tag{17}$$

and from (10) obtain $\frac{\ddot{g}}{g} = r\frac{\dot{g}}{g}$. Thus,

$$\begin{aligned} \ddot{E} &= -\frac{\dot{g}}{g}\frac{F'}{F''}\left[\frac{1}{g} + \frac{\dot{g}}{g}\left(1 - \frac{F'F'''}{F''^2}\right)\right] + \left[r\frac{\dot{g}}{g} - \left(\frac{\dot{g}}{g}\right)^2\right]\frac{F'}{F''} \\ &= -\frac{\dot{g}}{g}\frac{F'}{F''}\left[\frac{1}{g} + \frac{\dot{g}}{g}\left(1 - \frac{F'F'''}{F''^2}\right) - \frac{gr - \dot{g}}{g}\right] \\ &= -\frac{\dot{g}}{g}\frac{F'}{F''}\left[\frac{1}{g} + \frac{\dot{g}}{g}\left(1 - \frac{F'F'''}{F''^2}\right) - \frac{1}{g}\right] \\ &= -\left(\frac{\dot{g}}{g}\right)^2\frac{F'}{F''}\left(1 - \frac{F'F'''}{F''^2}\right) \end{aligned} \tag{18}$$

where the third equality uses (10), i.e. $gr - \dot{g} = 1$. F is strictly concave and therefore $-\left(\frac{\dot{g}}{g}\right)^2\frac{F'}{F''} > 0$. Since $\left(1 - \frac{F'F'''}{F''^2}\right) < 0$ the claim follows. ☒

Therefore, $E(t)$ is concave if and only if $\eta < 0$, which implies a necessary condition for concavity: $F''' > 0$.

Example 2.1 *(Human Capital Production Functions and Earnings Concavity)*

- *Power Production Function 1 : consider the case of* $F(x) = \frac{Ax^\alpha}{\alpha}$ *for* $-\infty < \alpha < 1, A > 0$. *Then,* $\eta = \frac{1}{\alpha - 1} < 0$. *Under this specification the earnings function is strictly concave with respect to time.*

- *Power Production Function 2 : consider the case of* $F(x) = a - bx^{-\alpha}$ *for* $-1 < \alpha < \infty, a, b, c > 0$*. Then,* $\eta = \frac{-1}{\alpha+1} < 0$*. Under this specification the earnings function is strictly concave with respect to time.*

- *Power Production Function 3: consider the case of* $F(x) = a - b\exp^{-cx}$ *with* $b, c > 0$*. Then,* $\eta = 0$*.*

- *Quadratic Production Function: any quadratic production function has* $F''' = 0$ *and does not induce concavity of earnings with respect to time.*

Note, all of these examples consider no depreciation of human capital, $\sigma = 0$*.*

2.2 The Specialization Period

Specialization happens when the agent devotes her entire human capital to produce human capital stock, i.e. when $I(t) = 1$ for $t \in [\underline{t}, \bar{t}]$. In order to analyze some of the properties of specialization periods we assume away $D(t)$ so that $F_{D(t)} = 0$ and rule out depreciation.

Recall that we can interpret $g(t)$ as the return to investment in human capital and that we show above that $\dot{g} < 0$. Then, there is at most one period of specialization at the beginning of the time horizon, if it happens. We denote this by $[0, t^*]$. Ben-Porath (1967) calls this the schooling period and it happens under the conditions that follow.

Condition 2.2 *(Conditions for the Existence of a Period of Specialization in the Basic Ben-Porath Model with no Depreciation)*

$$\begin{aligned} F'(H(t))g(t) &> R \quad \forall\, t \in [0, t^*) \\ F'(H(t^*))g(t^*) &= R \\ I(t) &= 1 \quad \forall\, t \in [0, t^*] \\ H(t^*) &= \int_0^{t^*} F(H(\tau))d\tau + H_0 \end{aligned} \tag{19}$$

where $H(t^*)$ *is the human capital stock accumulated up to time* t^**.*

Given that R is fixed, any decrease in the function $g(t)$ for a given t lowers t^* because it lowers the return to gross investment in human capital. For example, relatively high r implies relatively low t^* because the individual is relatively present oriented. Also, from the system (19), note that a high value of H_0 implies a lower value for t^* because it takes less time to obtain $H(t^*)$. If $\sigma > 0$, similar conditions characterize the specialization period. However, there may be more than one specialization period because, under some scenarios, a high value of σ may knock off capital such that various investment episodes are optimal. We defer that case for Section 6.2.

Case 2.1 *(No Depreciation and the Cobb-Douglas Production Function for Human Capital: Initial Level of Human Capital) In this case* $\dot{H} = A\,(IH)^\alpha$ *where* $0 < \alpha < 1, A > 0$*. As argued above, if it exists, specialization happens in the period* $[0, t^*]$*. Thus*

$$\begin{aligned} \alpha A\,(H(0))^{\alpha-1} g(0) &> R \\ &\Leftrightarrow \\ H(0) &< \left[\frac{R}{g(0)\alpha A}\right]^{\frac{1}{\alpha-1}}. \end{aligned} \tag{20}$$

As the conditions in (19) establish, the time spent in specialization is a decreasing function of $H(0)$. In this example, actually, the initial human capital needs to be below certain threshold in order for the individual to specialize during one period.

Case 2.2 *(No Depreciation and the Cobb-Douglas Production Function for Human Capital with Infinite Horizon: Initial Level of Human Capital) In the setting of Case 2.1 and if the horizon of the problem is infinite: $H(0) < \left(\frac{\alpha A}{r}\right)^{\frac{1}{1-\alpha}}$ because $g(t) = \frac{R}{r}$.*

Case 2.3 *(No Depreciation and the Cobb-Douglas Production Function for Human Capital: the Specialization Period) In the period of specialization $I(t) = 1$. Then,*

$$\dot{H} = A\,(H)^{\alpha}. \tag{21}$$

The general solution for (21) is

$$H(t) = [(1-\alpha)(At + K)]^{\frac{1}{1-\alpha}} \tag{22}$$

for some constant K. Given an initial condition $H(0) = H_0$, $K = \frac{H_0^{1-\alpha}}{1-\alpha}$ and

$$H(t) = \left[(1-\alpha)At + H_0^{1-\alpha}\right]^{\frac{1}{1-\alpha}}. \tag{23}$$

At the end of the specialization period, as established in (19):

$$\alpha g(t^*) A\,(H(t^*))^{\alpha-1} = R. \tag{24}$$

If $T \to \infty$, $g(t) = \frac{R}{r}$ and

$$t^* = -\frac{H_0^{1-\alpha}}{A(1-\alpha)} + \frac{\alpha}{1-\alpha}\frac{1}{r}. \tag{25}$$

(25) provides some intuitive results: (i) an individual with relatively high initial human capital specializes during a relatively shorter period: $\frac{\partial t^*}{\partial H_0} < 0$; (ii) a relatively able individual specializes during relatively long period: $\frac{\partial t^*}{\partial A} > 0$; (iii) a relatively impatient individual specializes for a relatively shorter period: $\frac{\partial t^*}{\partial r} < 0$.

Case 2.4 *(No Depreciation and the Cobb-Douglas Production for Human Capital: Post-school Earnings) Let $\tau = t - t^*$ define the post-school work experience and write post-school earnings as follows:*

$$E(\tau) = R\int_0^{\tau} H(l\dot{+}t^*)dl + RH(t^*) - RIH(\tau + t^*). \tag{26}$$

Now, from (19) the following equality holds:

$$\begin{aligned} \alpha g(t) A\,(IH(t))^{\alpha-1} &= R \\ &\Leftrightarrow \\ IH(t) &= \left[\frac{\alpha g(t) A}{R}\right]^{\frac{1}{1-\alpha}} \end{aligned} \tag{27}$$

Combining (27) *and the law of motion for human capital:*

$$\dot{H} = A\left[\frac{\alpha g(t) A}{R}\right]^{\frac{\alpha}{1-\alpha}}. \tag{28}$$

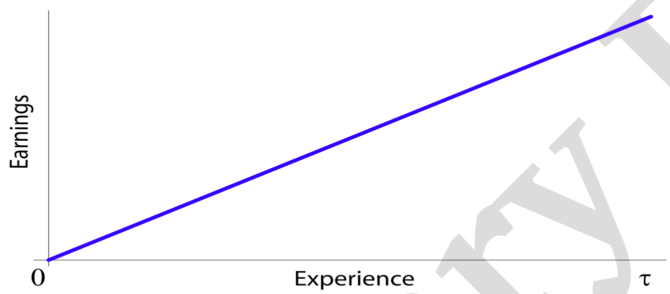
Then,

$$E(\tau) = R\int_0^{\tau} A\left[\frac{\alpha g(l+t^*) A}{R}\right]^{\frac{\alpha}{1-\alpha}} dl + RH(t^*) - R\left[\frac{\alpha g\left(\tau + t^*\right) A}{R}\right]^{\frac{1}{1-\alpha}} \tag{29}$$

and if $T \to \infty$

$$E(\tau) = RA\left[\frac{\alpha A}{r}\right]^{\frac{\alpha}{1-\alpha}} \tau. \tag{30}$$

Figure 1: Earnings and Experience, Cobb Douglas Technology and No Depreciation



When the time horizon is infinite, there is no concern with the reduction in time left for capturing returns to human capital investment and thus $g(t)$ *is fixed over time. When the solution is interior, the optimal choice on* $I(t)H(t)$ *is constant overtime, which implies the increase in* $H(t)$ *overtime is also a constant. This is why* $E(t)$ *increases at a constant rate as well. However, with a finite time horizon, the Cobb-Douglas production function with no depreciation implies a strictly concave earning function* $E(t)$.

2.3 The Baseline Model Dynamics under the Cobb-Douglas Specification: a Summary

This section summarizes the dynamics of the main variables in the baseline model when there is no depreciation, market goods are ruled out, and the production function for human capital investment is Cobb-Douglas. We assume that the horizon is infinite to simplify the algebra, but it is important to note that the qualitative properties of the results remain unchanged under a finite horizon. To wrap up the section, we show simulations that illustrate how the variables of interest behave under various parameterizations (in all of them we set $R = 1$).

2.3.1 Human Capital

- At $t = 0$ an initial condition is given.
- At $0 < t < t^*$ the system (19) provides the conditions that human capital satisfies and its expression is given by (23).

- At $t = t^*$ (23) is still a valid expression for human capital. To obtain the exact quantity it suffices to substitute the expression for t^*, (25), into (23).
- At $t > t^*$ (19) and the expression for $\dot{H}$, (28), provide the expression for human capital.

Then,

$$H(t) = \begin{cases} H_0 & t = 0 \\ \left[(1-\alpha)At + H_0^{1-\alpha}\right]^{\frac{1}{1-\alpha}}, & 0 < t < t^* \\ \left[\frac{\alpha A}{r}\right]^{\frac{1}{1-\alpha}}, & t = t^* \\ A\left[\frac{\alpha A}{r}\right]^{\frac{\alpha}{1-\alpha}}(t - t^*) + \left[\frac{\alpha A}{r}\right]^{\frac{1}{1-\alpha}}, & t > t^*. \end{cases} \tag{31}$$

2.3.2 Investment

We focus on the case in which there is an specialization period, i.e. the case in which (20) holds. The combination of (27) and (31) gives the following

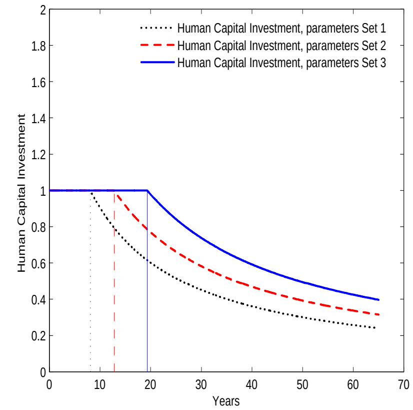
$$I(t) = \begin{cases} 1, & t = 0 \\ 1, & 0 < t < t^* \\ 1, & t = t^* \\ \frac{\left[\frac{\alpha A}{r}\right]^{\frac{1}{1-\alpha}}}{A\left[\frac{\alpha A}{r}\right]^{\frac{\alpha}{1-\alpha}}(t-t^*)+\left[\frac{\alpha A}{r}\right]^{\frac{1}{1-\alpha}}}, & t > t^*. \end{cases} \tag{32}$$

2.3.3 Earnings

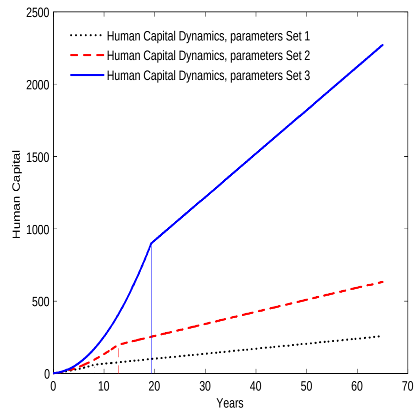
For earnings we also focus on the case with a specialization period, i.e. the case in which (20) holds. Thus, (1), (31), (32) define earnings as follows

$$E(t) = \begin{cases} 0, & t = 0 \\ 0, & 0 < t < t^* \\ 0, & t = t^* \\ RA\left[\frac{\alpha A}{r}\right]^{\frac{\alpha}{1-\alpha}}(t - t^*), & t > t^*. \end{cases} \tag{33}$$

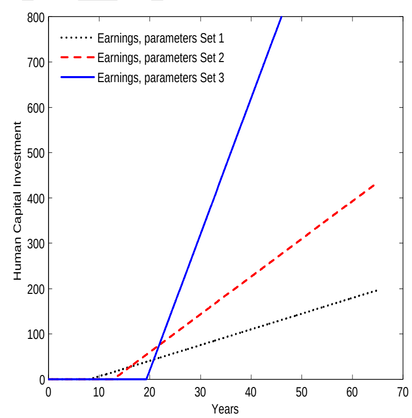
Figure 2: Dynamics with Variations in a Production Technology Parameter
$\alpha = .3$ (dotted); $\alpha = .4$ (dashed); $\alpha = .5$ (solid)
for $A = 3, r = .05, H_0 = 1$



(a) Human Capital Investment

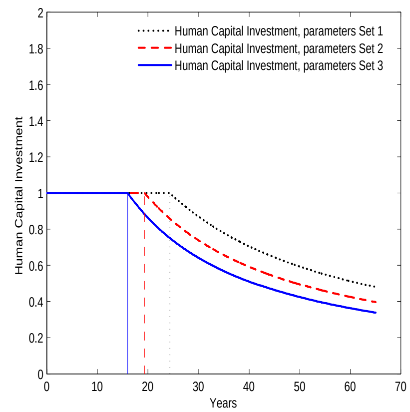


(b) Human Capital Stock

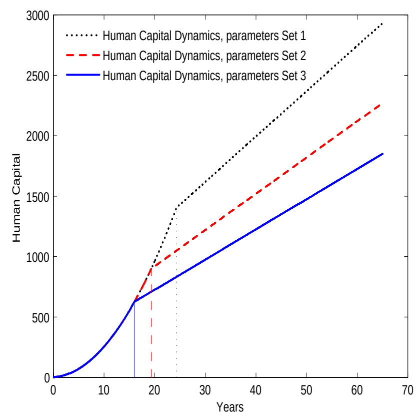


(c) Earnings

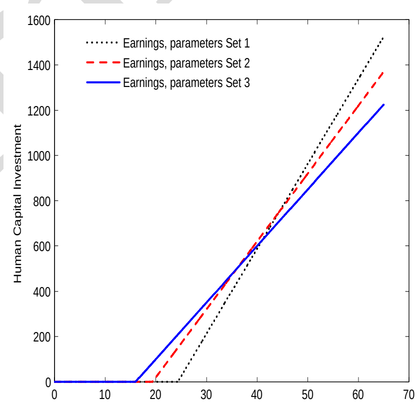
Figure 3: Dynamics with Variations in the Discounting Factor
$r = .04$ (dotted); $r = .05$ (dashed); $r = .06$ (solid)
for $A = 3, \alpha = .5, H_0 = 1$



(a) Human Capital Investment

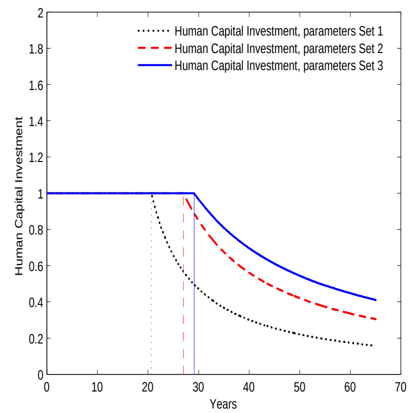


(b) Human Capital Stock

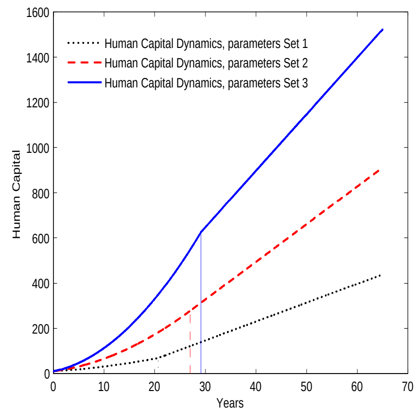


(c) Earnings

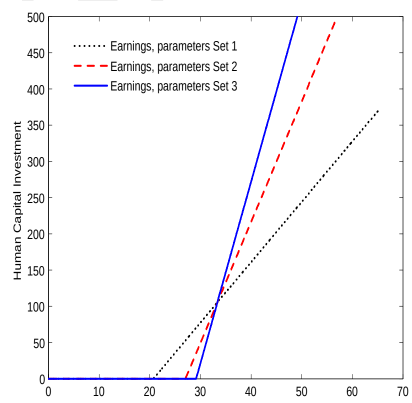
Figure 4: Dynamics with Variations in a Production Technology Parameter
$A = .5$ (dotted); $A = 1.0$ (dashed); $A = 1.5$ (solid)
for $r = .03, \alpha = .5, H_0 = 10$



(a) Human Capital Investment

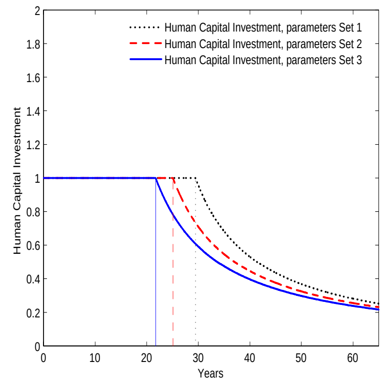


(b) Human Capital Stock

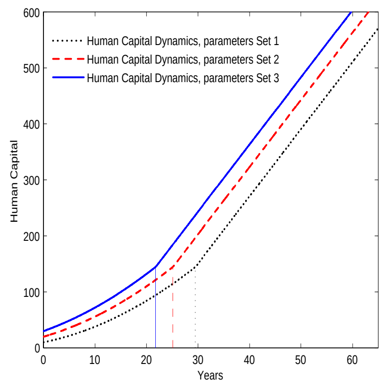


(c) Earnings

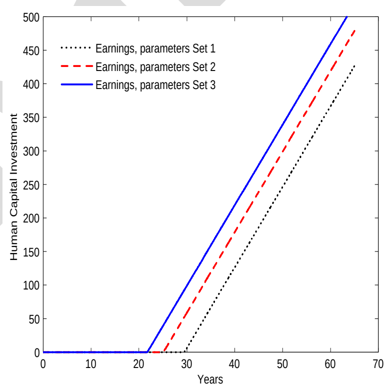
Figure 5: Dynamics with Variations in the Initial Level of Human Capital
$H_0 = 10$ (dotted); $H_0 = 20$ (dashed); $H_0 = 30$ (solid)
for $r = .025, \alpha = .5, A = .6,$



(a) Human Capital Investment



(b) Human Capital Stock



(c) Earnings

2.4 Rates of Return under the Cobb-Douglas Specification

We use this model to analyze returns both to schooling and post-schooling. In order to simplify the expressions we let $t \to \infty$. Similar implications hold for the finite time horizon problem.

2.4.1 Return to Schooling

We call schooling the period of specialization in which the individual devotes his complete human capital stock to produce new human capital. To define the return to schooling consider two scenarios: (i) the individual does not invest either in schooling or in post-schooling. Then in each period t she earns RH_0; (ii) the individual invests in schooling and does not make any post-schooling investments after that. Then in each after-schooling period τ she earns $R\left[\frac{\alpha A}{r}\right]^{\frac{1}{1-\alpha}}$. We define the (internal) rate of return of schooling as the discount rate at which the present values of the disposable income streams in the two scenarios are equal.

Definition 2.2 *("Internal" Rate of Return to Schooling) φ is the (internal) rate of return to schooling and solves the equation*

$$\begin{aligned}
\int_{t^*}^{\infty} \exp^{-\varphi t} R\left[\frac{\alpha A}{r}\right]^{\frac{1}{1-\alpha}} dt &= \int_0^{\infty} \exp^{-\varphi t} RH_0 dt \\
&\Rightarrow \\
\varphi &= \frac{\ln\left[\frac{\alpha A}{r}\right]^{\frac{1}{1-\alpha}} - \ln H_0}{-\frac{H_0^{1-\alpha}}{A(1-\alpha)} + \frac{\alpha}{1-\alpha}\frac{1}{r}}.
\end{aligned} \tag{34}$$

2.4.2 Return to Post-schooling

Let $E(\tau)^{NPS}$ and $E(\tau)^{PS}$ denote earnings with and without post-schooling investment, respectively. By (33) we can write

$$\begin{aligned}
E(\tau)^{NPS} &= RH(t^*) \\
&= R\left(\frac{\alpha A}{r}\right)^{\frac{1}{1-\alpha}} \\
E(\tau)^{PS} &= RA\left[\frac{\alpha A}{r}\right]^{\frac{\alpha}{1-\alpha}} \tau
\end{aligned} \tag{35}$$

so that the increment in earnings due to post-schooling at τ is

$$\Delta^{E(\tau)} \equiv E(\tau)^{PS} - E(\tau)^{NPS}. \tag{36}$$

We can interpret $\Delta^{E(\tau)}$ as "returns less costs" from post-schooling, with $E(\tau)^{NPS}$ as the costs (i.e. foregone earnings) of post-schooling investments. Then, we define the (internal) rate of return to post-schooling as follows.

Definition 2.3 *("Internal" Rate of Return to Post-schooling) ϕ is the (internal) rate of return to schooling and solves the equation*

$$\int_0^{\infty} \exp^{-\phi\tau}\left[RA\left[\frac{\alpha A}{r}\right]^{\frac{\alpha}{1-\alpha}}\tau - R\left(\frac{\alpha A}{r}\right)^{\frac{1}{1-\alpha}}\right] d\tau = 0 \tag{37}$$

Using the Laplace transform, (37) implies

$$
\begin{aligned}
\frac{1}{\phi^2}RA\left[\frac{\alpha A}{r}\right]^{\frac{\alpha}{1-\alpha}} - \frac{R}{\phi}A\left[\frac{\alpha A}{r}\right]^{\frac{1}{1-\alpha}} &= 0 \\
&\Rightarrow \\
\phi &= \frac{r}{\alpha}.
\end{aligned}
\tag{38}
$$

The (internal) rate of return to post-schooling investment is a decreasing function of α. Recall that the internal rate of return of post-schooling measures the desirability of the investment opportunity, and with a higher α, individuals' levels of human capital are relatively high, even without any post-schooling investment. Therefore, the relative differences (measured by the ratios) between the amount of disposable earnings in cases with and without investment are smaller. Therefore, investment in post-schooling is less desirable for more productive individuals. In addition, relatively patient individuals (who have relatively low r) require a smaller discount rate ϕ to equalize the scenarios with and without post-schooling investment.

2.5 Earnings Growth and Patience in Finite Horizon

In this section we want to ask, in the same framework, how earnings growth depend on what defines relative patience in this model, the discount rate r. To do that, we investigate $\frac{\partial \dot{E(\tau)}}{\partial r}$.

Claim 2.4 *Assume that* $1 - \frac{F'(\cdot)F'''(\cdot)}{F''^2} < 0$ *(recall from Claim 2.3 that this is a sufficient condition for* $\ddot{E(t)} < 0$ *in the current context). Then,* $\frac{\partial \dot{E(\tau)}}{\partial r} < 0$.

Proof: Without loss of generality, assume that R =1 and note that

$$
\frac{\partial \dot{E(\tau)}}{\partial r} = F'(\cdot)\frac{\partial IH}{\partial r} - \frac{\partial}{\partial r}\dot{IH}. \tag{39}
$$

From (12) we know that the first order condition of the agent's problem is

$$
g(t)F'(\cdot) = 1 \tag{40}
$$

which by the implicit function theorem yields

$$
\begin{aligned}
\frac{\partial IH}{\partial r} &= \frac{\frac{\partial g(t)}{\partial r}F'(\cdot)}{2g(t)F''(\cdot)} \\
&< 0
\end{aligned}
\tag{41}
$$

where the inequality follows from strict concavity of $F(\cdot)$ and $g(t) > 0, \frac{\partial g(t)}{\partial r} < 0$ (see (44)). Thus, the first term in (39) is negative. If we show that the second term is negative then we can sign (39) and give meaning to these results. In order to do so we need $\frac{\partial \dot{IH}}{\partial r} > 0$. From (15) note that

$$
\frac{\partial \dot{IH}}{\partial r} = -\frac{\dot{g}}{g}\left[1 - \frac{F'(\cdot)F'''(\cdot)}{F''(\cdot)^2}\right]\frac{\partial IH}{\partial r} + \frac{F'(\cdot)}{F''(\cdot)}\frac{\partial}{\partial r}\left[-\frac{\dot{g}}{g}\right] \tag{42}
$$

We know from $1 - \frac{F'(\cdot)F'''(\cdot)}{F''^2} < 0$ and $\dot{g}$, $\frac{\partial IH}{\partial r} < 0$ that the first term in (42) is positive. To sign the second term note that $\dot{g} = rg - 1, -\frac{\dot{g}}{g} = \frac{1}{g} - r$. Then,

$$\frac{\partial}{\partial r}\left[-\frac{\dot{g}}{g}\right] = -\frac{1}{g^2}\frac{\partial g}{\partial r} - 1. \tag{43}$$

To sign (43) note that

$$\begin{aligned}\frac{\partial g}{\partial r} &= \frac{\exp^{r(t-T)}\left(1 - r(t-T)\right) - 1}{r^2} \\ &< 0 \end{aligned} \tag{44}$$

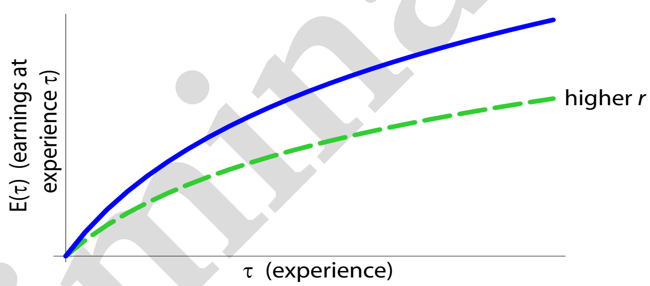
and

$$\begin{aligned}-\frac{\partial g}{g^2 \partial r} - 1 &= \frac{1}{r^2 g^2}\exp^{r(t-T)}\left(1 + r(t-T) - \exp^{r(t-T)}\right). \\ &< 0 \end{aligned} \tag{45}$$

which implies that $\frac{\partial \dot{E}}{\partial r} < 0$. ⊠

The graphical representation of Claim 2.4 is in Figure 6. It implies that the earnings function is relatively "less concave" for relatively impatient individuals (relatively high r). This is a consequence of their investment decisions: they spend less time in the schooling period and accumulate less human capital.

Figure 6: Earnings Profiles in Finite Horizon for Different Values of r



3 The Haley-Rosen Specification: Finite Horizon and the Autoregression Form

We analyze the finite horizon case under the specification that Haley (1976) and Rosen (1976) use. Specifically, we assume that $\dot{H} = A\left(IH\right)^{\alpha}, \alpha = \frac{1}{2}, \sigma = 0$ and the exact same setting as in Section 2. Actually, in Section 2 we rely on an infinite horizon to derive a set of closed form solutions to the individual's problem. In this section we relax the infinite horizon assumption and rely on the assumption $\alpha = \frac{1}{2}$ to gain tractabiliy.

We focus on the dynamics of post-schooling earnings because one of the less credible consequences of the infinite horizon is the linearity of earnings on experience. From (29) we can write

$$
\begin{aligned}
E(\tau) &= RH(t^*) + R\int_0^\tau A\left[\frac{1}{2}\frac{g(t^*+l)A}{R}\right]dl - R\left[\frac{1}{2}\frac{g(t^*+\tau)A}{R}\right]^2 \\
&\Rightarrow \\
E(\dot{\tau}) &= \frac{g(t^*+\tau)A^2}{2R}\left(2R - rg(t^*+\tau)\right) \\
&\Rightarrow \\
E(\ddot{\tau}) &= -\frac{A^2}{R}g(t^*\dot{+}\tau)^2
\end{aligned}
\tag{46}
$$

where the second and third equalities use (10). Combining (10) and (46) we obtain a second order ODE with constant coefficients:

$$E(\ddot{\tau}) = 2rE(\dot{\tau}) - A^2R \tag{47}$$

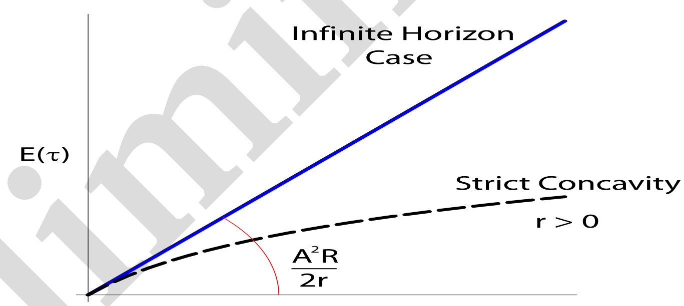
where the natural initial and terminal conditions that we impose are $E(0) = 0$ and $E(\dot{T}) = 0$ and then we guess and verify that the general solution to (47) is the following.

$$E(\tau) = c_0 + c_1 \exp^{-2r\tau} + c_2\tau \tag{48}$$

So $E(0) = 0$ implies $c_1 + c_0 = 0$ and $E(\dot{T}) = 0$ implies $2rc_1 \exp^{2rT} + c_2 = 0$. Together with (47), we can solve for $c_0 = \frac{A^2R}{4r^2\exp^{2rT}}$, $c_1 = -c_0$, and $c_2 = \frac{A^2R}{2r}$. Therefore,

$$E(\tau) = \frac{A^2R}{4r^2}\exp^{-2rT}\left(1 - \exp^{2r\tau}\right) + \frac{A^2R}{2r}\tau. \tag{49}$$

Figure 7: Post-school Earnings in the Haley-Rosen Specification



Note:

From (46), we know that in the finite horizon case, the earnings function is strictly concave unless $t = T$. The intuition behind the linearity of the earnings function in the infinite horizon case is provided in Section 2.2. In contrast with the infinite case, the return to investment is decreasing over time as the agent is approaching the final period of the time horizon, which implies that $I(t)H(t)$ is decreasing overtime. Thus the stock of human capital is increasing at a decreasing rate, which leads to the concavity of earnings dynamics.

3.1 The Autoregression

From (49) it is possible to write

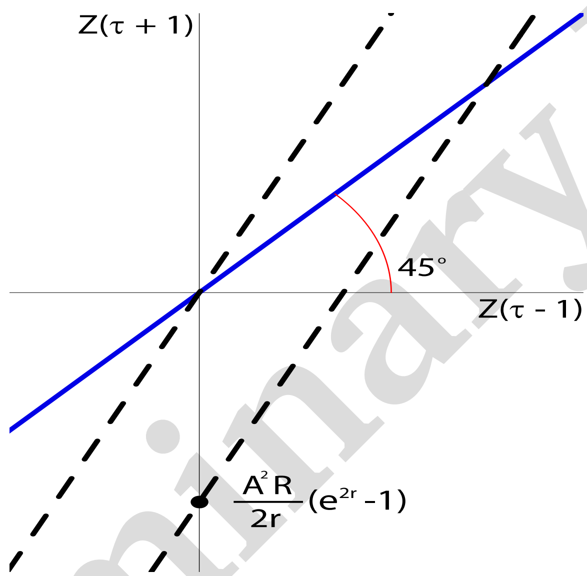
$$E(\tau+1) - E(\tau) = \frac{A^2R}{2r} + \frac{A^2R}{4r^2}\exp^{-2rT}\left(\exp^{2r\tau} - \exp^{2r(\tau+1)}\right) \tag{50}$$

which implies that

$$z(\tau+1) = \exp^{2r} z(\tau) + \frac{A^2R}{2r}\left(1 - \exp^{2r}\right) \tag{51}$$

where $z(\tau) \equiv E(\tau+1) - E(\tau)$ and we can analyze the growth dynamics of earnings. Consider a visual representation of (51)

Figure 8: Earnings Growth in the Haley-Rosen Representation



Note:

Apparently, the dynamics of the earnings growth are explosive. However, note that

$$\begin{aligned} \frac{\partial\left[E(\tau) - E(\tau-1)\right]}{\partial\tau} &= \frac{A^2R}{2r}\exp 2r(\tau - T)\left[\exp^{-2r} - 1\right] \\ &< 0 \end{aligned} \tag{52}$$

so that even when the growth dynamics of earnings is explosive, the earnings dynamics, $E(t)$, can converge over time.

3.2 From the Haley-Rosen Specification to the Mincer Equation

The earnings function in the Haley-Rosen specification actually lead to the Mincer equation. To see that take the log of (49) and obtain

$$\ln E(\tau) = \ln\left(\frac{A^2 R}{2r}\right) + \ln\tau + \ln\left[1 + \frac{\exp^{-2rT} - \exp^{2r(\tau-T)}}{2r\tau}\right]. \tag{53}$$

We can approximate around τ_0 the second and third terms in (53) to obtain

$$\begin{aligned} \ln(\tau) &\approx \ln(\tau_0) + \frac{1}{\tau_0}(\tau-\tau_0) - \frac{1}{\tau_0^2}\frac{(\tau-\tau_0)^2}{2!} \\ \ln\left[1 + \frac{\exp^{-2rT} - \exp^{2r(\tau-T)}}{2r\tau}\right] &\approx \xi_0 + \xi_1(\tau-\tau_0) + \xi_2\frac{(\tau-\tau_0)^2}{2!} \end{aligned} \tag{54}$$

for the adequate ξ_0, ξ_1, ξ_2. Thus,

$$\ln(\tau) + \ln\left[1 + \frac{\exp^{-2rT} - \exp^{2r(\tau-T)}}{2r\tau}\right] \approx \alpha_0 + \alpha_1(\tau-\tau_0) + \alpha_2(\tau-\tau_0)^2 \tag{55}$$

with $\alpha_0 \equiv \ln(\tau_0) + \xi_0, \alpha_1 \equiv \frac{1}{\tau_0} + \xi_1, \alpha_2 \equiv \frac{-\frac{1}{\tau_0^2} + \xi_2}{2}$. This leads to the so called Mincer equation (see Mincer, 1974):

$$\ln E(\tau) = k_0 + k_1\tau + k_2\tau^2 \tag{56}$$

where $k_0 = \alpha_0 - \tau_0\alpha_1 + \alpha_2\tau_0^2, k_2 = \alpha_2$. Also, it provides a baseline to compare "Ben-Porath" with "Mincer" coefficients. Table 1 provides different combinations of the parameters r, τ_0, T that lead to different values of k_1, k_2 that are close to the estimates that Mincer (1974) obtains.

Table 1: The Ben-Porath and the Mincer Coefficients

Parameters			Ben Porath Coefficients	
r	τ_0	T	k_1	k_2
0.0225	29.54	41.43	0.081	−0.0010
0.05	25	60	0.0808	−0.0008
0.05	20	65	0.1002	−0.0013
0.0675	24.70	74.77	0.081	−0.0008
Mincer Coefficients			0.081	−0.0012

Note: the Mincer model or Mincer equation is ln(E) $= k_0 + k_1\tau + k_2\tau^2$, where τ is experience.

Now, if $rT \approx 0$ then $\exp^{-rT} \approx 1$ and (53) becomes

$$\ln E(\tau) \approx \ln\left(\frac{A^2 R}{2r}\right) + \ln\tau + \ln\left[1 + \frac{1 - \exp^{2r\tau}}{2r\tau}\right] \tag{57}$$

which leads to various observations. The Haley-Rosen specification of the Ben-Porath model implies no economic content for the Mincerian rate of return on post-school investment. Put differently, an extension of (56) which includes post-school investment does not have a structural counterpart. Actually, this model implies that the entire economic content is in the intercept (see (57).(57) implies that, *caeteris paribus*, schooling has no effect on earnings. Mincer (1974) finds that the opposite holds. However, we claim that his finding does not necessarily argue against the Ben-Porath model. It could simply be the case that Mincer (1974) does not include ability measures in his estimations, which appear in (57), and therefore finds a positive coefficient on schooling.

4 Generalized Ben-Porath Model

We now generalize the model in Section 2 by relaxing the neutrality assumption so that the production function of human capital is more general. We focus our analysis on specialization because the analysis of other conditions is similar to that of Section 2.

In particular, the law of motion for human capital stock in the generalized Ben-Porath model is

$$\dot{H} = AI^{\alpha}H^{\beta} - \sigma H. \tag{58}$$

So the model in Section 2 is a particular case of this general formulation when $\alpha = \beta$. To simplify the analysis of the implications of this model, we assume that there is neither discounting nor depreciation, i.e. $r = \sigma = 0$. To ease notation, we neglect the argument t when possible. We analyze this model with a finite horizon.

The Hamiltonian of the problem is

$$\mathcal{H} = RH(t)\,(1 - I(t)) + \mu(t)\left(AI(t)^{\alpha}H(t)^{\beta}\right) \tag{59}$$

where $\mu(t)$ defines the shadow price of human capital. The following condition must be satisfied for interior solutions.

Condition 4.1 *(Optimality Conditions for the Life-Cycle Individual's Problem in the Generalized Ben-Porath Model)*

$$\frac{\partial \mathcal{H}(\cdot)}{\partial I(t)} = 0 \quad \Leftrightarrow \quad \mu(t)A\alpha I(t)^{\alpha-1}H(t)^{\beta} = RH(t) \tag{60}$$

$$\frac{\partial \mathcal{H}(\cdot)}{\partial H(t)} = -\dot{\mu(t)} \quad \Leftrightarrow \quad -R(1 - I(t)) - \beta\mu(t)AI(t)^{\alpha}H(t)^{\beta-1} = \dot{\mu(t)} \tag{61}$$

$$\frac{\partial \mathcal{H}(\cdot)}{\partial \mu(t)} = \dot{H} \quad \Leftrightarrow \quad \dot{H(t)} = AI(t)^{\alpha}H(t)^{\beta} \tag{62}$$

$$\textit{Transversality} \quad : \quad \lim_{t \to T} \mu(t)H(t) = 0 \tag{63}$$

Condition 4.1 is equivalent to the Mangasarian sufficient conditions for a global optimum if $\beta \leq 1$ (see Mangasarian, 1966).

4.1 Specialization

If $\frac{\partial \mathcal{H}(\cdot)}{\partial I(t)} > 0$ with $I(t) = 1$, the agent would specialize. Thus the condition that guarantees specialization is as follows.

Condition 4.2 *(Conditions for Specialization in the Generalized Ben-Porath Model)*

$$\textit{Conditions for Specialization}: \begin{cases} H > \left[\frac{R}{\alpha A \mu}\right]^{\frac{1}{\beta-1}}, & \beta > 1 \\ 1 > \left[\frac{R}{\alpha A \mu}\right], & \beta = 1 \\ H < \left[\frac{R}{\alpha A \mu}\right]^{\frac{1}{\beta-1}}, & \beta < 1. \end{cases} \tag{64}$$

During the period(s) of specialization (61), (62) become

$$\dot{\mu} = -\beta \mu A H^{\beta-1} \tag{65}$$
$$\dot{H} = A H^{\beta} \tag{66}$$

and we can solve for the dynamics of human capital stock in this region

$$H(t) = \begin{cases} c_0 \exp^{At}, & \beta = 1 \\ (At + c_1)^{\frac{1}{1-\beta}} (1-\beta)^{\frac{1}{1-\beta}}, & \beta \neq 1. \end{cases} \tag{67}$$

The initial condition for the human capital stock leads to $c_0 = H_0$ and $c_1 = \frac{H_0^{1-\beta}}{1-\beta}$ which implies that

$$H(t) = \begin{cases} H_0 \exp^{At-1}, & \beta = 1 \\ \left(At + \frac{H_0^{\frac{1}{1-\beta}}}{1-\beta}\right)^{\frac{1}{1-\beta}} (1-\beta)^{\frac{1}{1-\beta}}, & \beta \neq 1. \end{cases} \tag{68}$$

Also, we can solve (65) and find that

$$\mu(t) = \begin{cases} k_0 \exp^{-At}, & \beta = 1 \\ \frac{k_1}{(At+c_1)^{\frac{\beta}{1-\beta}}}, & \beta \neq 1 \end{cases} \tag{69}$$

for which there is an exact solution given an initial condition $\mu(0) = \mu_0$. This is, we can find k_0, k_1 in (69) provided $\mu_0 > 0$ (it is a price). In particular, note that $k_0 = \mu_0 > 0$ and $k_1 = \mu_0 c_1^{\frac{\beta}{1-\beta}} > 0$ for $0 < \beta < 1$.

Let t^* denote the time when specialization ends. It must be true that, then, (60) holds with strict equality

$$\mu(t^*) A \alpha H(t^*)^{\beta} = R H(t^*) \tag{70}$$

which implies that

$$t^* = \frac{1}{A}\left(\ln\left[\frac{A\alpha}{R} + \ln k_0\right]\right) \tag{71}$$

for $\beta = 1$. For $\beta \neq 1$, t^* solves

$$\frac{k_1}{(At^* + c_0)^{\frac{\beta}{\beta-1}}} \frac{A\alpha}{R} = \left[At^* (1-\beta)^{\frac{1}{1-\beta}} + H_0^{1-\beta} (1-\beta)^{\frac{\beta}{1-\beta}}\right]^{1-\beta}. \tag{72}$$

To wrap up the discussion we ask if the period of specialization is unique for particular cases.

Claim 4.1 *(Uniqueness of the Specialization Period) If the period of specialization exists, it is* *unique when either when $\beta = 1$ or when $\beta \in [0,1]$.*

Proof: In both cases (69) implies that $\dot{\mu(t)} < 0$. Importantly, $\mu(t)$ is the shadow price or value of human capital. Thus, $\dot{I(t)} < 0$ and, if it exists, the period of specialization is unique. ☒

5 The Basic Sheshinski Specification: Bang-Bang Equilibria

The Basic Sheshinski specification is a particular case of the Generalized Ben-Porath model in Section 4 in which $\alpha = \beta = 1$.

Definition 5.1 *(Law of Motion for Human Capital Stock in the Basic Sheshinski Specification)*

$$\dot{H(t)} = AI(t)H(t) - \sigma H(t). \tag{73}$$

Proceeding as in Section 2 and Section 4 we can write down the current value Hamiltonian and obtain the following optimality conditions for the interior solution.

Condition 5.1 *(Optimality Conditions for the Life-cycle Individual's Problem in the Basic Sheshinski Specification)*

$$\begin{aligned}
\frac{\partial \mathcal{H}(\cdot)}{\partial I(t)} = 0 \quad &\Leftrightarrow \quad \mu(t)\exp^{rt} = \frac{R}{A} && (74)\\
\frac{\partial \mathcal{H}(\cdot)}{\partial H(t)} = -\dot{\mu(t)} \quad &\Leftrightarrow \quad -\exp^{-rt} R(1 - I(t)) - \mu(t)\left(AI(t) - \sigma\right) = \dot{\mu(t)} && (75)\\
\frac{\partial \mathcal{H}(\cdot)}{\partial \mu(t)} = \dot{H(t)} \quad &\Leftrightarrow \quad \dot{H(t)} = AI(t)H(t) - \sigma H(t) && (76)\\
\textit{Transversality} \quad &: \quad \lim_{t\to T} \mu(t)H(t) = 0 && (77)
\end{aligned}$$

Claim 5.1 *(Bang-Bang in the Sheshinski Specification) Assume that $\sigma + r < A$ and that there is an initial period of specialization.*[3] *Then, the solution to the problem is Bang-Bang, i.e. either $I = 0$ or $I = 1$.*

Proof: Define $g(t) = \mu(t)\exp^{rt}$ and use (75), (77) to obtain

$$\begin{aligned}
\dot{g} &= -R + (R - Ag)I + (\sigma + r)g && (78)\\
g(T) &= 0. && (79)
\end{aligned}$$

In the specialization period $I(t) = 1$. If $\sigma + r < A$, $\dot{g(t)} < 0$. Actually, by (74), as $g(t)$ decreases to $\frac{R}{A}$, $I(t)$ switches from it's upper bound 1 to its lower bound 0. Then, with $I(t) = 0$ we can use $g(T) = 0$ and write

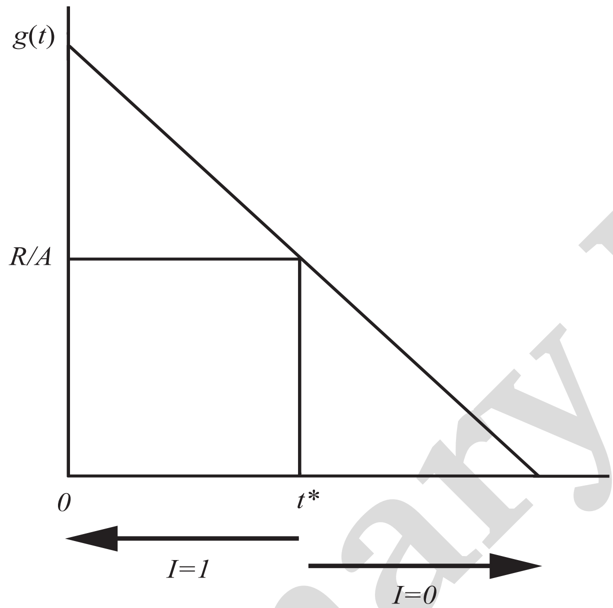
$$\begin{aligned}
\dot{g(t)} &= (\sigma + r)g(t) - R\\
&\Rightarrow\\
g(t) &= \frac{R}{\sigma + r}\left[1 - \exp^{(\sigma+r)(t-T)}\right]. && (80)
\end{aligned}$$

[3]Note that $\sigma + r > A$ implies that $\dot{g(t)} > 0$ and this violates the transversality condition.

for which $\dot{g(t)} < 0$ as well. Therefore, once $I(t)$ reaches zero, it is never positive again. This formulation has a Bang-Bang solution. ⊠

It follows that the schooling period, if it exists, is unique and at the beginning of the investment cycle. If it does not exist the individual does not invest at all in human capital. Figure 9 is a graphical representation of Claim 5.1.

Figure 9: Bang-Bang Equilibrium in the Basic Sheshinski Specification



Note:

We can actually solve for t^*, the length of the schooling period, using the fact that $g(t^*) = \frac{R}{A}$ by (74) and $g(t^*) = \frac{R}{\sigma+r}\left[1 - \exp^{(\sigma+r)(t^*-T)}\right]$ by (80):

$$
\begin{aligned}
\frac{R}{A} &= \frac{R}{\sigma+r}\left[1-\exp^{(\sigma+r)(t^*-T)}\right] \\
&\Leftrightarrow \\
t^* &= \frac{1}{\sigma+r}\ln\frac{A-(\sigma+r)}{A}+T. \qquad (81)
\end{aligned}
$$

Thus, (i) longer life horizons imply more schooling, $\frac{\partial t^*}{\partial T} > 0$; (ii) greater depreciation implies less schooling, $\frac{\partial t^*}{\partial \sigma} < 0$; (iii) higher relative impatience implies less schooling, $\frac{\partial t^*}{\partial r} < 0$; (iv) higher productivity implies more schooling, $\frac{\partial t^*}{\partial A} > 0$; (v) initial human capital does not affect schooling, $\frac{\partial t^*}{\partial H_0} = 0$.

5.1 From the Basic Sheshinski Specification to the Mincer Equation

Assume that there is a period of specialization. From (66) we know that in the period $[0, t^*]$

$$\begin{aligned} \dot{H(t)} &= (A-\sigma)H(t) \\ &\Rightarrow \\ H(t) &= H_0 \exp^{(A-\sigma)t}. \end{aligned} \tag{82}$$

At t^*, actually, $I(t) = 0$ so earnings are $Y(t) = RH(t^*)$ Then,

$$\ln Y(t^*) = \ln(RH_0) + (A-\sigma)t^*. \tag{83}$$

According to this model, the returns to schooling, $A - \sigma$, are given by the productivity of human capital less the human capital depreciation rate.

6 The Modified Sheshinski Specification

The last variation of the Ben-Porath model we consider allows for human capital investment cycles. This is, it allows for investment in human capital stock to happen in different episodes.

Definition 6.1 *(Law of Motion for Human Capital in the Modified Sheshinski Specification)*

$$\dot{H} = AI - \sigma H. \tag{84}$$

Note that the human capital production function does not depend on $H(t)$. The optimality conditions for an interior solution in this model are the following.

Condition 6.1 *(Optimality Condition for the Life-cycle Individual's Problem in the Modified Sheshinski Specification)*

$$\frac{\partial \mathcal{H}(\cdot)}{\partial I(t)} = 0 \quad \Leftrightarrow \quad \mu \exp^{rt} = \frac{RH}{A} \tag{85}$$

$$\frac{\partial \mathcal{H}(\cdot)}{\partial H(t)} = -\dot{\mu(t)} \quad \Leftrightarrow \quad \dot{\mu} = \mu\sigma - \exp^{-rt} R(1-I) \tag{86}$$

$$\frac{\partial \mathcal{H}(\cdot)}{\partial \mu(t)} = \dot{H(t)} \quad \Leftrightarrow \quad \dot{H(t)} = AI - \sigma H \tag{87}$$

$$\textit{Transversality} \quad : \quad \lim_{t \to T} \mu(t)H(t) = 0. \tag{88}$$

6.1 No Depreciation: a Schooling Model

Define $g(t) = \mu(t) \exp^{rt}$ and use (86) to obtain

$$\dot{g} = g(\sigma + r) - R(1-I). \tag{89}$$

Let $\sigma = 0$. Then $\dot{g} = -R(1-I) + rg$. And $\dot{H} = A$ when $I = 1$. So the solution for the human capital trajectory when $I = 1$ is

$$H(t) = At + H_0. \tag{90}$$

At $t = 0$, $I = 1$ if $g(0) > \frac{R}{A} H_0$. Importantly, $I = 1$ implies that $\dot{g(t)} = rg(t) > 0$. As t grows, the return for gross investment grows because the payoff period gets closer. $I = 1$ cannot be a solution forever because the agent receives no earnings if she invests all of the time during the complete life-cycle.

Claim 6.1 *(Uniqueness of the Period of Specialization in the Modified Sheshinski Specification with no Depreciation) If the specialization period exists, then it is unique.*

Proof: Based on (85), if a specialization period exists and if $g(t) - \frac{RH(t)}{A}$ is strictly decreasing overtime, then the specialization period must occur at the beginning of the life cycle and is unique. In the following we show that $\dot{g(t)} - \frac{R}{A}\dot{H(t)} < 0$.

Given that $\dot{g(t)} = rg(t) - R(1 - I(t))$ and $g(T) = 0$, we have:

$$g(t) = R \int_t^T (1 - I(s)) \exp^{r(t-s)} ds \tag{91}$$

Then taking the derivative with respect to time gives:

$$\dot{g(t)} = R \left[-1 + I(t) + r \int_t^T (1 - I(s)) \exp^{r(t-s)} ds \right] \tag{92}$$

Together with (84), we have:

$$\begin{aligned}
\dot{g(t)} - \frac{R}{A}\dot{H(t)} &= -R + Rr \int_t^T (1 - I(s)) \exp^{r(t-s)} ds && (93)\\
&\leq -R + Rr \int_t^T \exp^{r(t-s)} ds && (94)\\
&= -R \exp^{r(t-T)} && (95)\\
&< 0, && (96)
\end{aligned}$$

where the first inequality follows from setting $I(\tau) = 0$. ⊠

Finally, to compute the optimal schooling length, t^*, note that (85) holds with strict equality at t^* and (90) is valid so that

$$g(t^*) = \frac{R}{A}(At^* + H_0). \tag{97}$$

(91) is also valid for t^*. Then,

$$\left(1 - \exp^{r(t^*-T)}\right) = \frac{r}{A}(At^* + H_0) \tag{98}$$

and thus $\frac{\partial t^*}{\partial H_0} < 0$, $\frac{\partial t^*}{\partial A} > 0$ and $\frac{\partial t^*}{\partial r} < 0$ as in the model of Section 2.

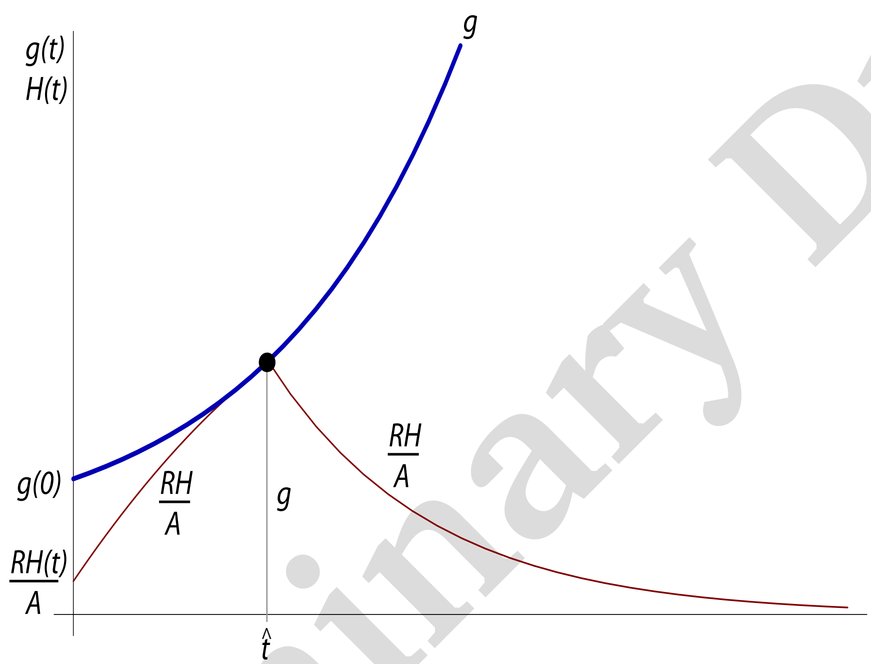
6.2 Depreciation

Let us give some conditions under which human capital investment would have different episodes over the life cycle. First assume that $g(0) > \frac{H_0 R}{A}$ so that there is a specialization period to begin with. We can solve (87) and (89) to obtain

$$
\begin{aligned}
H(t) &= \left[H_0 - \frac{A}{\sigma}\right] \exp^{-\sigma t} + \frac{A}{\sigma} \\
g(t) &= g_0 \exp^{(r+\sigma)t}
\end{aligned}
$$

with $g_0 > 0$. Once the solution becomes interior, $g(t) = \frac{R}{A} H(t)$ by (85). Assume that $\sigma < \frac{A}{H_0}$ so that $\dot{H}(0) > 0$. Then, graphically,

Figure 10: Return to Gross Investment in Human Capital in the Modified Sheshinski Specification



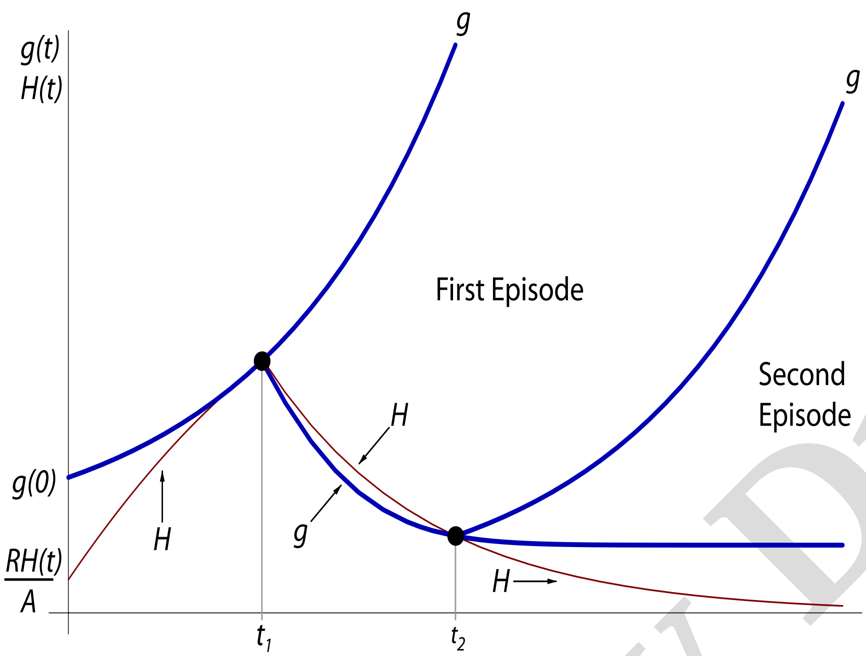
Note:

Let t_1 denote the time in which the first period of specialization ends. If the solution "bangs-out" to $I = 0$ we can use (89) and (84) to get

$$
\begin{aligned}
\dot{g} &= (\sigma + r)g - R \\
H(t) &= H(t_1) \exp^{-\sigma(t-t_1)}
\end{aligned} \tag{99}
$$

for $t_1 < t < t_2$. Likewise, we can define a period t_2 in which the solution "bangs-in" again and so on. Graphically,

Figure 11: Human Capital Investment Episodes in the Modified Sheshinski Specification



Note:

In $t < t_1$, $I = 1$ implies $\dot{g} > 0$. g needs to decrease for the problem to respect the transversality condition. Thus, in the neighborhood of t_1 it has to be that $g(\dot{t}_1) < \frac{RH(\dot{t}_1)}{A}$ (see Figure (11)). If we take the expression from the right of $g(t_1)$ this requires

$$\begin{aligned} -R(\sigma + r)g(t_1) &< \frac{RH(\dot{t}_1)}{A} \\ &= \frac{-\sigma RH(t_1)}{A} \\ &= -\sigma g(t_1) \\ &\Leftrightarrow \\ g(t_1) &< \frac{R}{r}. \end{aligned} \tag{100}$$

We can follow analogous reasoning to construct conditions under which human capital investment happens in different episodes during the life-cycle.

To wrap up this section, note that we have an initial period of specialization if $g_0 > \frac{RH_0}{A}$. At $t = 0$, however, it should be the case that the slope of $\frac{RH_0}{A}$ exceeds $\dot{g}$. Otherwise the expressions for g in the specialization period and the "interior" case do not intersect and the solution violates the transversality condition. This implies that $R\left[1 - \frac{\sigma H_0}{A}\right] > g_0(\sigma + r)$. High initial levels of human capital, low productivity, high discount, high depreciation, and low returns to human capital rule out an initial specialization period. Suppose the conditions described above hold so that specialization happens. We cannot show that $g(t_3) < g(t_1)$ so that it is better to accumulate "all the human capital required for life" in the first period of specialization. This is what we call cycling in the investment on human capital and it is a consequence of depreciation.

7 Literature Extending the Ben-Porath Framework

In this sectio we summarize a variety of research that applies and extends the Ben-Porath model. The common feature of papers in this section is that they all address novel questions which have not been studied before and are frequently cited by other research. The purpose of this section is not to provide a comprehensive literature review. Instead, we aim to shed some light on how the Ben-Porath model can be employed and extended to study a wide range of research questions. Although the original Ben-Porath framework was developed to explain individuals' life cycle earnings profiles, future research adapts the framework to answer questions including life cycle investments in health, occupational choices, rising wage inequality, wage and promotion dynamics inside firms, child labor, childhood investments, wage growth of immigrants, gender gap in wage earnings, and sources of lifetime inequality.

Grossman (1972) is the seminal theory paper studying people's life cycle investment decisions on health. Although health is considered an important dimension of human capital, Grossman (1972) argues that the existing human capital investment framework, including Ben-Porath, is not sufficient to capture people's demand for health. In Ben-Porath, an individual's incentive to invest in human capital is to increase her labor market productivity and thus her wage rate. However, the gain of having additional health is different. According to Grossman (1972), individuals have two incentives to invest in health: (1) Health affects individuals' utilities directly; (2) Health determines the amount of time available for market and non-market activities in each period as well as the length of lifespan. Thus, Grossman (1972) adapts the Ben-Porath model to incorporate this feature of health investment.

By using the adapted framework, Grossman (1972) concludes that the demand for health capital and medical care are different. The model predicts that the demand for health capital would decline over the life cycle, whereas the demand for medical expenditure would rise with age. Both the demand for health and medical care are positively correlated with wage rate. The former is positively correlated with education, while the latter is negatively correlated with education.

In addition to choosing schooling and on-the-job training as in the Ben-Porath model, Keane and Wolpin (1997) also models individuals' choices on occupation. Unlike the Ben-Porath model, which focuses on a representative agent, individual's heterogeneity is explicitly modeled in Keane and Wolpin (1997). Individuals optimally choose general human capital investment (schooling) and occupation-specific human capital investment (occupation-specific work experience) based on their endowment heterogeneity, credit constraint conditions, and occupation-specific pricing of skills. The comparative advantage and sorting argument is applied in the selection process.

To better fit the data on individual's schooling and occupational choices, Keane and Wolpin (1997) extend the model even further. They include skill depreciation for non-work periods, which is also taken in account in Ben-Porath. They introduce market frictions, including job-finding costs and school reentry costs. Non-pecuniary components of occupational payoffs are proved to be empirically relevant.

Heckman et al. (1998) develops a dynamic general equilibrium model with heterogeneous agents to explain the rising wage inequality in the U.S. The Ben-Porath model is extended in several ways. First, the original model does not distinguish human capital generated through schooling and human capital accumulated through on the job training. However, in Heckman et al. (1998), schooling human capital is an input in the production of on the job training human capital. In particular, individuals with different levels of schooling could produce qualitatively different skills through on the job training. Second, skills corresponding to different levels of schooling have different prices. As in Keane and Wolpin (1997), this helps with introducing the story of comparative advantage and sorting. Moreover, Heckman et al. (1998) assumes that intial stocks of human capital are heterogeneous across individuals, and individuals have heterogeneous abilities to produce job-specific human capital, whereas the Ben-Porath model studies the behavior of a representative agent without introducing any individual heterogeneity. Finally, a general equilibrium model is constructed to capture the relationship between the capital market and the markets of different skill levels.

Although many extensions are applied to the Ben-Porath model, one important feature is maintained, namely the skill prices and wage earnings are explicitly distinguished and allowed to move in different directions. With the general equilibrium framework, Heckman et al. (1998) is able to explain the main facts of U.S. wage inequality in the past 35 years.

Gibbons and Waldman (1999) aims at explaining a variety of empirical evidence on firms' wage and promotion dynamics. For example, the paper tends to justify the empirical finding that real wage decreases are not uncommon but demotions are very uncommon. Empirical findings, including that wage increases are serially correlated and promotions are associated with large wage increases, are also addressed. Moreover, the paper studies why workers receiving large wage increases early at a given job level are promoted faster.

To explain these empirical findings systematically, the authors use the idea of the Ben-Porath model to incorporate individuals' on the job human capital acquisition behaviors. However, they also claim that using the Ben-Porath model alone is not sufficient. Two other principle elements are taken into account. In particular, they model how firms assign different jobs to different workers by using the idea of comparative advantage. They also include the component of learning, given that firms do not have perfect information on workers' productivity. By using the Ben-Porath model with two additional principal ingredients, the authors conclude that they are able to explain the main findings in the empirical literature on wage and promotion dynamics inside firms.

Baland and Robinson (2000) provides a formal analysis on the issue of child labor. In their context, parents could decide how to allocate their child's time: either allowing her to study at school or requesting her to work as a child labor. Not surprisingly, the trade-off between child's working time and schooling time can be modeled by using a Ben-Porath type of framework.

If the market were perfect, namely parents could borrow and lend as much as they want, leave debts to their child, and sign a formal contract with their child on the future returns of the human capital investments on their child, then parental decisions on child's education and labor force participation are proved efficient. However, if the market is imperfect, altruistic parents who face a binding budget constraint may choose to over spend their child's time on working instead of schooling. Then the inefficient amount of human capital investment in their child generates total social welfare loss.

In developed countries where child labor barely exists, the opportunity cost of a child's time is irrelevant. Instead, the opportunity cost of the parent's time used in producing the child's human capital matters a lot. Accordingly, the Ben-Porath model can still be used to capture this trade-off. Leibowitz (1974) is a pioneering study investigating the effect of family investments on child's future outcomes including ability, schooling and earnings. Following Ben-Porath, Leibowitz (1974) applies a Cobb-Douglas human capital production technology, which assumes that the inputs, including the current level of human capital stock and investments, are complementary. This strong assumption is restrictive and has been relaxed in later research.

In Cunha and Heckman (2007) and Cunha et al. (2010), empirical evidence is provided to show that the human capital formation process is governed by a multistage technology. Technologies change across different stages of the child development, and qualitatively different inputs can be used over time. The single stage Ben-Porath model oversimplifies the dynamics of the skill formation process.

Eckstein and Weiss (2004) studies the mechanism causing the differences in wage growth patterns between natives and immigrants in Israel from 1990-2000. Applying Ben-Porath's idea, the observed workers' wage earnings in the labor market are equal to their potential wage earnings minus the costs of investing in human capital acquisition. Assuming that the value of the imported skills brought by immigrants cannot be applied immediately after they arrive in the host country, then the price of the imported skills is low at the beginning. This gives immigrants an incentive to invest more in local skills when they arrive, since the opportunity cost of investment is low. As the price of the imported skills rises over time since the arrival, the incentive to invest in local skills is reduced accordingly. Therefore, both the rising price of the imported skills and the investment incentives in local skills explain the steeper growth of the observed earnings profiles for the immigrants. The levels of the earnings for the immigrants also depend on the quality of the imported skills. With relatively low quality of the imported skills, as in the case discussed in Eckstein and Weiss (2004), the immigrants' earnings levels cannot catch up the natives' earnings levels.

In Manning and Swaffield (2008), the authors tend to solve the puzzle that whereas the average earnings of the males and females are similar when they enter the labor market, the growth of the earnings for males is much faster than for females in the first ten years after labor market entry. The Ben-Porath model provides an answer to justify the puzzle. In particular, if women expect to work less or even leave the labor market in order to specialize more in the household production after marriage, they have less incentive to take on the job training and thus their earnings profile grows less. However, by using a U.K. data-set, the authors find that the human capital investment explanation can only capture 50% of the gender wage gap.

Moreover, the authors test two other explanations. The first one assumes that women are less concerned with money but more concerned with other non-pecuniary issues when they seek to change jobs, which leads to lower returns for job changes for women. The second explanation uses the psychological differences between men and women to justify the differences in their wage growth. It has been well documented that men and women are very different in terms of attitudes to risk-taking, competition, self-esteem, and selfishness. Nevertheless, it is shown that empirically, both the job shopping and psychological explanations can explain only a small percentage of the gender wage gap. Thus a sizable proportion of the gender wage difference remains unexplained.

Huggett et al. (2011) documents that the mean of individuals' earnings is hump-shaped over the life cycle and the dispersion of individuals' earnings is increasing with age. The authors justify the hump-shaped mean earnings profile simply by using the Ben-Porath model. To justify the increasing pattern of the dispersion, the authors add features to the Ben-Porath framework, including idiosyncratic human capital production shocks, different amounts of human capital stocks at age 23, which is the starting age of the life cycle earnings profiles studied in this paper, and differences in individuals' learning ability and wealth. The show that differences in individuals' learning ability explain a large part of the rising dispersion.

The estimated Ben-Porath framework with a risky human capital production technology shows that the differences in individuals' lifetime earnings are mainly due to the differences in individuals' initial conditions at age 23, rather than the idiosyncratic shocks experienced over the rest of their lives. And among the initial conditions, the variations in the initial human capital stocks are shown to be much more relevant than the variations in individuals' learning abilities and wealth for explaining the dispersion in individuals' lifetime earnings.

In this section, a variety of research which addresses different research questions by using the Ben-Porath framework is summarized. Although it is natural to extend the basic framework to incorporate additional features when tackling different questions, the fundamental idea behind the original framework remains in all these studies. In particular, the essential trade-off faced by decision makers when choosing the optimal amount of human capital investment is to compare the future gains from having an additional stock of human capital and the opportunity costs of making such an investment. Although the form and dynamic of the relevant benefits and costs vary between different contexts, leading to different conclusions on the trajectory of human capital investment, we can see that since the same basic trade-off underlies all the research topics summarized above, the Ben-Porath framework has a rather wide usage and still has potential to be further extended to solve new research questions in the future.

Appendix 0.1 Optimal Control

Optimal control theory is used to solve continuous time optimization problems formulated using three major components: (i) *state variables*, which describe the state of the system at any given point in time, (ii) *control variables*, which are analogous to the choice variables of a discrete problem and (iii) one or more *laws of motion* or *production function.* The typical problem is formulated as follows:

Problem Appendix 0.1 *(Basic Formulation)*

$$\max_{x(t),u(t)} \int_0^T f(x(t), u(t), t)\,\mathrm{d}t \tag{1}$$

s.t.

$$\begin{aligned} \dot{x}(t) &= g(x(t), u(t)) \\ x(0) &= x_0 \\ x(T) &= T \end{aligned}$$

where $0 \leq t \geq T$ and $\dot{x}(t) = \frac{\mathrm{d}x(t)}{\mathrm{d}t}$. In this case, $x(t)$ is a vector of state variables, $u(t)$ is a vector of control variables and $\dot{x}(t)$ is the law of motion. The terminal condition may also be left free with $T \to \infty$; however, we only deal with finite horizon problems in this document and thus, focus on that case. Typically, the objective function and law of motion are assumed to be continuous, twice differentiable, strictly increasing and concave in their arguments and to satisfy the Inada conditions. A function $y(x)$ satisfies the Inada conditions if:

$$\begin{aligned} \lim_{x\to\infty} \frac{\partial y(x)}{\partial x} &= 0 \\ \lim_{x\to 0} \frac{\partial y(x)}{\partial x} &= \infty \end{aligned}$$

Where in discrete time we would use a Lagrangian to solve the problem, in continuous time we use a Hamiltonian function. We can choose to set up the Hamiltonian as a *current value* function or a *present value* function. Both result in the same optimal paths; however, a present value Hamiltonian considers values discounted to the present value, while the current value Hamiltonian doesn't. Because we use the present value formulation throughout this document, we will provide an overview of the present value formulation here. [4]

The present value Hamiltonian is

$$H(f(\bullet), u(\bullet), \Lambda, t) = f(x(t), u(t), t) + \lambda g(x(t), u(t)) \tag{2}$$

Where λ is called the *co-state variable* and is analogous to the Lagrangian multiplier. The necessary conditions are the following:

[4] Switching between the present (H_P) and current (H_C) value formulations is simple, as $H_P = e^{-\rho t} H_C$.

$$
\begin{aligned}
\frac{\partial H(\bullet)}{\partial u(t)} &= 0 \qquad (3) \\
\frac{\partial H(\bullet)}{\partial x(t)} &= -\dot{\lambda}(t) \qquad (4) \\
\lambda(T) \geq 0 \quad , &\quad \lambda(T)x^*(T) = 0 \qquad (5)
\end{aligned}
$$

Where (5) are called the transversality conditions. Here, as in the rest of this document, we provide the transversality conditions for a finite horizon problem.

(Mangasarian, 1966) proves that if $(x*(t), u*(t))$ is an admissible pair for 1 and if $H(\bullet)$ is a concave function over an open convex set of all the admissible values of all x, u, then there is a global maximum of $\int_0^T f(x(t), u(t), t)\,\mathrm{d}t$ at $(x*(t), u*(t))$. If $H(\bullet)$ is strictly concave, then $(x*(t), u*(t))$ yields the unique global maximum of $\int_0^T f(x(t), u(t), t)\,\mathrm{d}t$.

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