# Project Module

## Global Solution Methods

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# Aiyagari Model

## Agents

- Agents face idiosyncratic income risk:
  - either employed ($e_{i,t} = 1$) or unemployed ($e_{i,t} = 0$)
  - after-tax wage $w_t \left(1 - \tau_t\right)$
  - unemployment benefits fraction of wage $\mu w_t$
  - government runs balanced budget $\tau_t = \mu \frac{1-L_t}{L_t}$
  - (exogenous) transition probabilities between two states

$$\Pi = \begin{bmatrix} 1 - \pi_{UE} & \pi_{UE} \\ \pi_{EU} & 1 - \pi_{EU} \end{bmatrix}$$

- Agents can accumulate capital $k_{i,t+1}$
  - rental rate $r_t$, depreciation rate $\delta$
  - borrowing constaint $k_{i,t+1} \geq k_{min} K_t$

# Aiyagari Model

## Agents

- Agents maximize utility

$$\max_{\{c_{i,t},k_{i,t+1}\}_{t=0}^{\infty}} \mathbb{E}\sum_{t=0}^{\infty} \beta^t u(c_{i,t})$$
$$c_{i,t} + k_{i,t+1} = (1 + r_t - \delta)\, k_{i,t} + w_t (1 - \tau_t)\, e_{i,t} + \mu w_t (1 - e_{i,t})$$
$$k_{i,t+1} \geq \bar{k}$$

- Euler equation

$$u'(c_{i,t}) \geq \beta \mathbb{E}\left[(1 + r_{t+1} - \delta)\, u'(c_{i,t+1})\right]$$

# Aiyagari Model

## Firms

- Representative firm maximizes profits

$$\max_{K_t, L_t} F_t(K_t, L_t) - r_t K_t - w_t L_t$$

- Factor prices

$$\begin{aligned} \frac{dF_t(K_t, L_t)}{dK} &= r_t \\ \frac{dF_t(K_t, L_t)}{dL} &= w_t \end{aligned}$$

# Aiyagari Model

## General Equilibrium

- Households take (current and future) factor prices as given

$$\frac{dF_t\left(K_t, L_t\right)}{dK} = r_t, \qquad \frac{dF_t\left(K_t, L_t\right)}{dL} = w_t$$

- Policy functions (depending on employment status, wealth, and aggregate state) satisfy first-order conditions

$$c_t\left(e_t, k_t, s_t\right), \qquad k_{t+1}\left(e_t, k_t, s_t\right)$$

  - $s_t$ summarizes aggregate productivity state and cross-sectional distribution

- *Aggregated* capital and labour must be consistent with *aggregate* capital and labour used to get factor prices

$$\sum_i k_{t+1}\left(e_{i,t}, k_{i,t}, s_t\right) = K_{t+1}\left(s_t\right), \qquad \sum_i e_{i,t} = L_t$$

# Solution of Steady State without Aggregate Risk

## Overview

- Aggregate labour can be calculated from transition matrix

$$L = \frac{\pi_{UE}}{\pi_{UE} + \pi_{EU}}$$

- Distribution of capital ($\to$ aggregate capital) must be simulated
  1. Initial guess for $K$
  2. Calculate policy functions

$$c\,(e, k, K)\,, \qquad k'\,(e, k, K)$$

  3. Simulate the economy to find cross-sectional distribution of households
  4. Check if

$$\sum_i k'\,(e_i, k_i, K) = \sum_i k_i \approx K$$

     - ★ If not, update guess for $K$ and repeat from (2)

# Solution of Steady State without Aggregate Risk

## Calculate Policy Functions

- In general, no analytical solution $\rightarrow$ approximate $k'(e, k, K)$
  - $c(e, k, K)$ can then be obtained from budget constraint

- Approaches to function approximation:
  1. Perturbation (local method)
     - Taylor approximation around $k = K$
  2. Projection (global method)
     - Suppose that function is known / can be calculated (approximatively) at points $k_j$ (function values $k'(k_j)$)
     - Fit approximating function (finite number of parameters)
     - Here: piecewise linear function between grid points $\{k_j\}$

$$k'(k) \approx \left(1 - \frac{k - k_j}{k_{j+1} - k_j}\right) k'(k_j) + \frac{k - k_j}{k_{j+1} - k_j} k'(k_{j+1}) \qquad \text{for} \quad k \in [k_j, k_{j+1}]$$

# Solution of Steady State without Aggregate Risk

## Calculate Policy Functions

- Find $k'\left(e, k_j\right)$ such that Euler equation, budget constraint, and borrowing constraint are satisfied
- Use iteration procedure
  1. guess for $k'\left(e, k_j\right)$
  2. calculate updated guess from FOCs
  3. repeat until convergence

# Solution of Steady State without Aggregate Risk

## Calculate Policy Functions

- Fixed-point iteration
  - Use old guess to calculate RHS of Euler equation for each $(e, k_j)$
    1. next period's capital stock $k'(e, k_j)$
    2. next period's choice of capital (depends on next period's employment status) $k'(e', k'(e, k_j))$
    3. calculate next period's consumption from next period's budget constraint
    4. form expecation of marginal utility of consumption using transition matrix
  - New guess for today's consumption from Euler equation
  - New guess for capital from today's budget constraint: $k'^{new}(e, k_j)$
  - Check whether borrowing constraint is satisfied
    - ★ if not: $k'^{new}(e, k_j) = k_{min}$
  - Updated policy function

$$k'(e, k_j) = k'(e, k_j) + \chi\left(k'^{new}(e, k_j) - k'(e, k_j)\right)$$

★ dampening parameter $\chi \in (0, 1)$ to ensure convergence (avoid oscillation)

# Solution of Steady State without Aggregate Risk

## Calculate Policy Functions

Possible refinement

- ”Endogenous” gridpoints method by Christopher Carroll (2006)
  - use grid for $k'$ instead of $k$
  - calculate implied (”endogenous”) value $k\left(k'\right)$ such that Euler equation is satisfied
  - invert this function to get $k'\left(k\right)$
  - don’t have to interpolate next period’s policy function (inverting $k\left(k'\right)$ typically faster)
  - but not always possible or more complicated (e.g. 2 continuous state variables)

Alternative

- Time iteration
  - only use guess for policy functions on RHS of Euler equation
  - in particular, don’t use it to calculate $k'\left(e, k_j\right)$
  - disadvantage: non-linear problem has to be solved ($\rightarrow$ slower)
  - advantage: convergence (without dampening) more generally guaranteed

# Solution of Steady State without Aggregate Risk

## Simulate the Economy

Ways to calculate ergodic distribution:

- Literally simulate many agents
  - agents have initial capital stock and employment status
  - for each agent calculate $k_{i,t+1} = k'(e_{i,t}, k_{i,t})$
  - draw random (according to transition probabilities given $e_{i,t}$) $e_{i,t+1}$
  - simulate many periods until distribution has converged (but for some noise)
    - note that one can also average over time, if sufficiently many initial periods are dropped

# Solution of Steady State without Aggregate Risk

## Simulate the Economy

- Better alternative that removes sampling error: histogram method (Eric Young (2010))
  - describe distribution by histogram over capital holdings, i.e. mass of agents $m^e_{j,t}$ at each grid point $k_j$
  - transition between periods:
    - ★ capital holdings: divide mass according to linear interpolation between closest grid points: if $k'(e, k_j) \in [k_n, k_{n+1}]$

      add mass to point $n$: $\tilde{m}^e_{n,t+1} = \tilde{m}^e_{n,t+1} + \left(1 - \frac{k'(e,k_j)-k_n}{k_{n+1}-k_n}\right) m^e_{j,t}$

      add mass to point $n+1$: $\tilde{m}^e_{n+1,t+1} = \tilde{m}^e_{n+1,t+1} + \frac{k'(e,k_j)-k_n}{k_{n+1}-k_n} m^e_{j,t}$
    - ★ employment status: divide mass according to transition probabilities

$$
\begin{aligned}
m^U_{j,t+1} &= \pi_{EU}\tilde{m}^E_{j,t+1} + (1-\pi_{UE})\,\tilde{m}^U_{j,t+1} \\
m^E_{j,t+1} &= (1-\pi_{EU})\,\tilde{m}^E_{j,t+1} + \pi_{UE}\tilde{m}^U_{j,t+1}
\end{aligned}
$$

    - ★ either repeat until convergence ($m^e_{j,t+1} \approx m^e_{j,t}$)
    - ★ or calculate transition matrix (jointly for *k and e*) and find Eigenvector corresponding to Eigenvalue=1 $\rightarrow$ this distribution of masses stays invariant under transition matrix, i.e. is the one after convergence

# Solution of Steady State without Aggregate Risk

## Simulate the Economy

Building the transition matrix $T$ (size 2N x 2N):

- grid for capital $k_j$
- stack unemployed and employed into one vector

$$\mathbf{m} = \left[ m_1^U; \ldots; m_N^U; m_1^E; \ldots; m_N^E \right]$$

- entry $(i, j)$ gives share of mass that moves from $i$ in period $t$ to $j$ in period $t+1$ (sum over each row always equals 1)
- first calculate (deterministic) transition of capital holdings of currently employed and unemployed: $T^E$, $T^U$ (size N x N)
  - sparse matrices with only 2 (neighbouring) non-zero entries in each row
- then use transition probabilities to build $T$ from 4 NxN blocks:

$$T = \begin{bmatrix} \left(1 - \pi_{UE}\right) T^U & \pi_{UE} T^U \\ \pi_{EU} T^E & \left(1 - \pi_{EU}\right) T^E \end{bmatrix}$$

# Solution of Steady State without Aggregate Risk

## Simulate the Economy

Use of transition matrix

- Transition from one period to the next:

$$\mathbf{m_{t+1}} = T'\mathbf{m_t}$$

  - Note that $T$ is transposed! (or $\mathbf{m}'$ has to be multiplied from left)
- Transition over $k$ periods

$$\mathbf{m_{t+k}} = \left(T'\right)^k \mathbf{m_t}$$

- Ways to find the ergodic distribution
  1. Limiting vector starting from *any* initial distribution $\mathbf{m_0}$

$$\mathbf{m} = \lim_{k\to\infty} \left(T'\right)^k \mathbf{m_0}$$

  2. Eigenvector corresponding to Eigenvalue 1 (faster for small $n$)

$$\mathbf{m} = T'\mathbf{m}$$

$$\Leftrightarrow \left(\mathbb{I}_{2N} - T'\right)\mathbf{m} = \mathbf{0}_{2N}$$

# Solution of Steady State without Aggregate Risk

## Aggregation

- Distribution **m** (normalized to sum to 1)
- Aggregate employment

$$\tilde{L} = \sum_{j=N+1}^{2N} m_j \overset{!}{=} \frac{\pi_{UE}}{\pi_{UE} + \pi_{EU}}$$

- Aggregate capital

$$\tilde{K} = [k_1, \ldots k_N, k_1, \ldots k_N] * \mathbf{m}$$

# Solution of Steady State without Aggregate Risk

## Update Guess for Aggregate Capital Stock

Compare $\tilde{K}$ with guess $K$ (that was used to calculate $r$ and $w$)

- if deviation is too big, update guess and repeat
- either use (dampened) new guess

$$K = K + \chi \left(\tilde{K} - K\right)$$

- or use bisection method (for 1-dimensional problem):
  - if we "know" that $\tilde{K}\left(K\right) - K$ is downward sloping and 0 between $\left[K_{min}, K_{max}\right]$
    1. try $K = \frac{K_{min}+K_{max}}{2}$
    
    2a. if $\tilde{K}\left(K\right) > K$: set $K_{min} = K$
    
    2b. if $\tilde{K}\left(K\right) < K$: set $K_{max} = K$
    
    3. repeat from 1.
  - halves interval in each step
  - important to have accurate $\tilde{K}\left(K\right)$!

# Solution of Perfect Foresight Transition to Steady State

- We are interested in solving:
  - some initial distribution of agents in period 1
  - then some periods with deterministic (expected) changes of parameters (productivity, unemployment insurance, transition rates, borrowing constraint,...)
  - afterwards no future shocks expected (some constant set of parameters) $\rightarrow$ (almost) converge to a steady state by period T

- Solution method:
  1. solve for steady state (and corresponding policy functions) at T
  2. initial guess for vector of capital stock in each period: $K_t$
  3. solve backward in time for policy functions
     1. period T: use policy function $k'_T\left(e,k\right)$ from steady state
     2. period t: use policy function $k'_{t+1}\left(e,k\right)$ on right hand side of Euler equation and solve for $k'_t\left(e,k\right)$ by using policy function iteration
  4. simulate forward in time to get actual capital stock in each period
     - ★ in each period use policy function $k'_t\left(e_{i,t},k_{i,t}\right)$ to calculate $k_{i,t+1}$
  5. update guess for $K_t$ and repeat until convergence

# Welfare Analysis

- Calculating life-time utility by iteration
  1. initial guess for $U(e, k)$
  2. use guess for continuation value to get updated guess

$$U^{\text{new}}(e, k) = u(c(e, k)) + \beta \mathbb{E} U(e', k')$$

  3. repeat until convergence

# Welfare Analysis

Things to consider when comparing welfare in two equilibria $U^1(e,k)$, $U^2(e,k)$

1. Calculate consumption equivalent or cash equivalent of policy change
   - permanent change of consumption by factor $\Delta$ that increases life-time utility by same amount as policy change:

$$\begin{aligned}
U^2(e,k) &= \mathbb{E}\sum_{t=0}^{\infty}\beta^t\frac{\left(c_{i,t}^1\Delta\right)^{1-\sigma}-1}{1-\sigma} \\
&= \Delta^{1-\sigma}\mathbb{E}\sum_{t=0}^{\infty}\beta^t\frac{\left(c_{i,t}^1\right)^{1-\sigma}-1}{1-\sigma}+\frac{\Delta^{1-\sigma}-1}{(1-\sigma)(1-\beta)} \\
&= \Delta^{1-\sigma}U^1(e,k)+\frac{\Delta^{1-\sigma}-1}{(1-\sigma)(1-\beta)} \\
\Delta &= \left(\frac{U^2(e,k)(1-\sigma)(1-\beta)+1}{U^1(e,k)(1-\sigma)(1-\beta)+1}\right)^{\frac{1}{1-\sigma}}
\end{aligned}$$

   - transfer of $\Delta$ units of wealth in first period

$$U^2(e,k)=U^1(e,k+\Delta)$$

# Welfare Analysis

Things to consider when comparing welfare in two equilibria
$U^1(e,k)$, $U^2(e,k)$

2. Welfare gains differ across agents
   - calculate average, median, ... of consumption equivalent
   - aggregate cash equivalent $\rightarrow$ redistribution that makes everybody better off possible?

3. Idiosyncratic and aggregate transition
   - can't compare 2 steady states directly
   - approximation that takes care of idiosyncratic (but not aggregate) transition
     - Krusell, Mukoyama, and Sahin (2010)
     - For each agent in first economy, consider that they are placed with current assets in second economy (i.e. given new parameters, prices,...) and compare $U^1(e,k)$ and $U^2(e,k)$
   - exact way: calculate (deterministic) transition path and then calculate value function in first period

# Solution with Aggregate Risk

- Aggregate risk: e.g. aggregate productivity shocks $z_t$
- Challenge: need to know $r_t$ and $w_t$ in any current and future state
  - need to forecast $L_t$ and $K_t$
  - demand for capital depends on whole cross-sectional distribution
    - ★ because marginal propensity to consume differs
    - ★ distribution is infinite-dimensional $\Rightarrow$ have to approximate it
- Krusell and Smith (1998) algorithm:
  - forecast $K_{t+1}$ using $K_t$ and $z_t$. Aggregate law of motion

$$\log K_{t+1} \approx \left[1, \log z_t, \log K_t\right] \mathbf{b}^{\text{guess}}$$

  - in addition to idiosyncratic state variables, need grids for $z$ and $K$
  - solve household problem: policy function $k'\left(e, k, z, K\right)$
  - simulate economy $\rightarrow$ simulated series $\left\{z_t, K_t\right\}$
  - estimate coefficient $\mathbf{b}^{\text{new}}$ in aggregate law of motion
  - repeat until $\mathbf{b}^{\text{new}} \approx \mathbf{b}^{\text{guess}}$ and check that forecast is accurate

# References

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