Project Module

Global Solution Methods

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Aiyagari Model

Agents

- Agents face idiosyncratic income risk:
  - either employed ($e_{i,t} = 1$) or unemployed ($e_{i,t} = 0$)
  - after-tax wage $w_t (1 - \tau_t)$
  - unemployment benefits fraction of wage μw_t
  - government runs balanced budget $\tau_t = \mu \frac{1-L_t}{L_t}$
  - (exogenous) transition probabilities between two states

$$\Pi = \begin{bmatrix} 1 - \pi_{UE} & \pi_{UE} \\ \pi_{EU} & 1 - \pi_{EU} \end{bmatrix}$$

- Agents can accumulate capital $k_{i,t+1}$
  - rental rate r_t, depreciation rate δ
  - borrowing constaint $k_{i,t+1} \geq k_{min} K_t$

Aiyagari Model

Agents

- Agents maximize utility

$$\max_{\{c_{i,t},k_{i,t+1}\}_{t=0}^{\infty}} \mathbb{E}\sum_{t=0}^{\infty}\beta^t u\left(c_{i,t}\right)$$
$$c_{i,t}+k_{i,t+1}=\left(1+r_t-\delta\right)k_{i,t}+w_t\left(1-\tau_t\right)e_{i,t}+\mu w_t\left(1-e_{i,t}\right)$$
$$k_{i,t+1}\geq\bar{k}$$

- Euler equation

$$u'\left(c_{i,t}\right)\geq\beta\mathbb{E}\left[\left(1+r_{t+1}-\delta\right)u'\left(c_{i,t+1}\right)\right]$$

Aiyagari Model

Firms

- Representative firm maximizes profits

$$\max_{K_t, L_t} F_t\left(K_t, L_t\right) - r_t K_t - w_t L_t$$

- Factor prices

$$\begin{aligned} \frac{dF_t\left(K_t, L_t\right)}{dK} &= r_t \\ \frac{dF_t\left(K_t, L_t\right)}{dL} &= w_t \end{aligned}$$

Aiyagari Model

General Equilibrium

- Households take (current and future) factor prices as given

$$\frac{dF_t\left(K_t, L_t\right)}{dK} = r_t, \qquad \frac{dF_t\left(K_t, L_t\right)}{dL} = w_t$$

- Policy functions (depending on employment status, wealth, and aggregate state) satisfy first-order conditions

$$c_t\left(e_t, k_t, s_t\right), \qquad k_{t+1}\left(e_t, k_t, s_t\right)$$

    - s_t summarizes aggregate productivity state and cross-sectional distribution

- *Aggregated* capital and labour must be consistent with *aggregate* capital and labour used to get factor prices

$$\sum_i k_{t+1}\left(e_{i,t}, k_{i,t}, s_t\right) = K_{t+1}\left(s_t\right), \qquad \sum_i e_{i,t} = L_t$$

Solution of Steady State without Aggregate Risk

Overview

- Aggregate labour can be calculated from transition matrix

$$L = \frac{\pi_{UE}}{\pi_{UE} + \pi_{EU}}$$

- Distribution of capital ($\rightarrow$ aggregate capital) must be simulated
  1. Initial guess for K
  2. Calculate policy functions

$$c\left(e, k, K\right), \qquad k'\left(e, k, K\right)$$

  3. Simulate the economy to find cross-sectional distribution of households
  4. Check if

$$\sum_i k'\left(e_i, k_i, K\right) = \sum_i k_i \approx K$$

     - ★ If not, update guess for K and repeat from (2)

Solution of Steady State without Aggregate Risk

Calculate Policy Functions

- In general, no analytical solution $\rightarrow$ approximate $k'(e, k, K)$
  - $c(e, k, K)$ can then be obtained from budget constraint

- Approaches to function approximation:
  1. Perturbation (local method)
     - ★ Taylor approximation around $k = K$
  2. Projection (global method)
     - ★ Suppose that function is known / can be calculated (approximatively) at points k_j (function values $k'(k_j)$)
     - ★ Fit approximating function (finite number of parameters)
     - ★ Here: piecewise linear function between grid points $\{k_j\}$

$$k'(k) \approx \left(1 - \frac{k - k_j}{k_{j+1} - k_j}\right) k'(k_j) + \frac{k - k_j}{k_{j+1} - k_j} k'(k_{j+1}) \qquad \text{for} \quad k \in [k_j, k_{j+1}]$$

Solution of Steady State without Aggregate Risk

Calculate Policy Functions

- Find $k'\left(e, k_j\right)$ such that Euler equation, budget constraint, and borrowing constraint are satisfied
- Use iteration procedure
  1. guess for $k'\left(e, k_j\right)$
  2. calculate updated guess from FOCs
  3. repeat until convergence

Solution of Steady State without Aggregate Risk

Calculate Policy Functions

- Fixed-point iteration
  - Use old guess to calculate RHS of Euler equation for each (e, k_j)
    1. next period's capital stock $k'(e, k_j)$
    2. next period's choice of capital (depends on next period's employment status) $k'(e', k'(e, k_j))$
    3. calculate next period's consumption from next period's budget constraint
    4. form expecation of marginal utility of consumption using transition matrix
  - New guess for today's consumption from Euler equation
  - New guess for capital from today's budget constraint: $k'^{new}(e, k_j)$
  - Check whether borrowing constraint is satisfied
    - ★ if not: $k'^{new}(e, k_j) = k_{min}$
  - Updated policy function

$$k'(e, k_j) = k'(e, k_j) + \chi\left(k'^{new}(e, k_j) - k'(e, k_j)\right)$$

    - ★ dampening parameter $\chi \in (0, 1)$ to ensure convergence (avoid oscillation)

Solution of Steady State without Aggregate Risk

Calculate Policy Functions

Possible refinement

- "Endogenous" gridpoints method by Christopher Carroll (2006)
  - use grid for k' instead of k
  - calculate implied ("endogenous") value $k\left(k'\right)$ such that Euler equation is satisfied
  - invert this function to get $k'\left(k\right)$
  - don't have to interpolate next period's policy function (inverting $k\left(k'\right)$ typically faster)
  - but not always possible or more complicated (e.g. 2 continuous state variables)

Alternative

- Time iteration
  - only use guess for policy functions on RHS of Euler equation
  - in particular, don't use it to calculate $k'\left(e, k_j\right)$
  - disadvantage: non-linear problem has to be solved ($\rightarrow$ slower)
  - advantage: convergence (without dampening) more generally guaranteed

Solution of Steady State without Aggregate Risk

Simulate the Economy

Ways to calculate ergodic distribution:

- Literally simulate many agents
  - agents have initial capital stock and employment status
  - for each agent calculate $k_{i,t+1} = k'\left(e_{i,t}, k_{i,t}\right)$
  - draw random (according to transition probabilities given $e_{i,t}$) $e_{i,t+1}$
  - simulate many periods until distribution has converged (but for some noise)
    - note that one can also average over time, if sufficiently many initial periods are dropped

Solution of Steady State without Aggregate Risk

Simulate the Economy

- Better alternative that removes sampling error: histogram method (Eric Young (2010))
  - describe distribution by histogram over capital holdings, i.e. mass of agents $m^e_{j,t}$ at each grid point k_j
  - transition between periods:
    - ★ capital holdings: divide mass according to linear interpolation between closest grid points: if $k'(e,k_j) \in [k_n, k_{n+1}]$

      add mass to point n: $\tilde{m}^e_{n,t+1} = \tilde{m}^e_{n,t+1} + \left(1 - \frac{k'(e,k_j)-k_n}{k_{n+1}-k_n}\right) m^e_{j,t}$

      add mass to point $n+1$: $\tilde{m}^e_{n+1,t+1} = \tilde{m}^e_{n+1,t+1} + \frac{k'(e,k_j)-k_n}{k_{n+1}-k_n} m^e_{j,t}$
    - ★ employment status: divide mass according to transition probabilities

$$
\begin{aligned}
m^U_{j,t+1} &= \pi_{EU} \tilde{m}^E_{j,t+1} + (1-\pi_{UE}) \tilde{m}^U_{j,t+1} \\
m^E_{j,t+1} &= (1-\pi_{EU}) \tilde{m}^E_{j,t+1} + \pi_{UE} \tilde{m}^U_{j,t+1}
\end{aligned}
$$

    - ★ either repeat until convergence ($m^e_{j,t+1} \approx m^e_{j,t}$)
    - ★ or calculate transition matrix (jointly for *k and e*) and find Eigenvector corresponding to Eigenvalue=1 → this distribution of masses stays invariant under transition matrix, i.e. is the one after convergence

Solution of Steady State without Aggregate Risk

Simulate the Economy

Building the transition matrix T (size 2N x 2N):

- grid for capital k_j
- stack unemployed and employed into one vector

$$\mathbf{m} = \left[m_1^U; \ldots; m_N^U; m_1^E; \ldots; m_N^E\right]$$

- entry (i, j) gives share of mass that moves from i in period t to j in period $t+1$ (sum over each row always equals 1)
- first calculate (deterministic) transition of capital holdings of currently employed and unemployed: T^E, T^U (size N x N)
  - sparse matrices with only 2 (neighbouring) non-zero entries in each row
- then use transition probabilities to build T from 4 NxN blocks:

$$T = \begin{bmatrix} (1-\pi_{UE})\, T^U & \pi_{UE} T^U \\ \pi_{EU} T^E & (1-\pi_{EU})\, T^E \end{bmatrix}$$

Solution of Steady State without Aggregate Risk

Simulate the Economy

Use of transition matrix

- Transition from one period to the next:

$$\mathbf{m_{t+1}} = T'\mathbf{m_t}$$

  - Note that T is transposed! (or $\mathbf{m}'$ has to be multiplied from left)

- Transition over k periods

$$\mathbf{m_{t+k}} = \left(T'\right)^k \mathbf{m_t}$$

- Ways to find the ergodic distribution
  1. Limiting vector starting from *any* initial distribution $\mathbf{m_0}$

$$\mathbf{m} = \lim_{k\to\infty} \left(T'\right)^k \mathbf{m_0}$$

  2. Eigenvector corresponding to Eigenvalue 1 (faster for small n)

$$\mathbf{m} = T'\mathbf{m}$$

$$\Leftrightarrow \left(\mathbb{I}_{2N} - T'\right)\mathbf{m} = \mathbf{0}_{2N}$$

Solution of Steady State without Aggregate Risk

Aggregation

- Distribution **m** (normalized to sum to 1)
- Aggregate employment

$$\tilde{L} = \sum_{j=N+1}^{2N} m_j \stackrel{!}{=} \frac{\pi_{UE}}{\pi_{UE} + \pi_{EU}}$$

- Aggregate capital

$$\tilde{K} = \left[k_1, \ldots k_N, k_1, \ldots k_N\right] * \mathbf{m}$$

Solution of Steady State without Aggregate Risk

Update Guess for Aggregate Capital Stock

Compare $\tilde{K}$ with guess K (that was used to calculate r and w)

- if deviation is too big, update guess and repeat
- either use (dampened) new guess

$$K = K + \chi\left(\tilde{K} - K\right)$$

- or use bisection method (for 1-dimensional problem):
  - if we "know" that $\tilde{K}\left(K\right) - K$ is downward sloping and 0 between $\left[K_{min}, K_{max}\right]$
    1. try $K = \frac{K_{min}+K_{max}}{2}$

    2a. if $\tilde{K}\left(K\right) > K$: set $K_{min} = K$

    2b. if $\tilde{K}\left(K\right) < K$: set $K_{max} = K$

    3. repeat from 1.
  - halves interval in each step
  - important to have accurate $\tilde{K}\left(K\right)$!

Solution of Perfect Foresight Transition to Steady State

- We are interested in solving:
  - some initial distribution of agents in period 1
  - then some periods with deterministic (expected) changes of parameters (productivity, unemployment insurance, transition rates, borrowing constraint,...)
  - afterwards no future shocks expected (some constant set of parameters) $\rightarrow$ (almost) converge to a steady state by period T

- Solution method:
  1. solve for steady state (and corresponding policy functions) at T
  2. initial guess for vector of capital stock in each period: K_t
  3. solve backward in time for policy functions
     1. period T: use policy function $k'_T(e,k)$ from steady state
     2. period t: use policy function $k'_{t+1}(e,k)$ on right hand side of Euler equation and solve for $k'_t(e,k)$ by using policy function iteration
  4. simulate forward in time to get actual capital stock in each period
     - ★ in each period use policy function $k'_t(e_{i,t}, k_{i,t})$ to calculate $k_{i,t+1}$
  5. update guess for K_t and repeat until convergence

Welfare Analysis

- Calculating life-time utility by iteration
  1. initial guess for $U(e,k)$
  2. use guess for continuation value to get updated guess

$$U^{\text{new}}(e,k) = u(c(e,k)) + \beta \mathbb{E} U(e',k')$$

  3. repeat until convergence

Welfare Analysis

Things to consider when comparing welfare in two equilibria $U^1(e,k)$, $U^2(e,k)$

1. Calculate consumption equivalent or cash equivalent of policy change
   - permanent change of consumption by factor Δ that increases life-time utility by same amount as policy change:

$$
\begin{aligned}
U^2(e,k) &= \mathbb{E}\sum_{t=0}^{\infty}\beta^t \frac{\left(c^1_{i,t}\Delta\right)^{1-\sigma}-1}{1-\sigma} \\
&= \Delta^{1-\sigma}\mathbb{E}\sum_{t=0}^{\infty}\beta^t \frac{\left(c^1_{i,t}\right)^{1-\sigma}-1}{1-\sigma} + \frac{\Delta^{1-\sigma}-1}{(1-\sigma)(1-\beta)} \\
&= \Delta^{1-\sigma}U^1(e,k) + \frac{\Delta^{1-\sigma}-1}{(1-\sigma)(1-\beta)} \\
\Delta &= \left(\frac{U^2(e,k)(1-\sigma)(1-\beta)+1}{U^1(e,k)(1-\sigma)(1-\beta)+1}\right)^{\frac{1}{1-\sigma}}
\end{aligned}
$$

   - transfer of Δ units of wealth in first period

$$U^2(e,k) = U^1(e,k+\Delta)$$

Welfare Analysis

Things to consider when comparing welfare in two equilibria $U^1(e,k)$, $U^2(e,k)$

2. Welfare gains differ across agents
   - calculate average, median, ... of consumption equivalent
   - aggregate cash equivalent $\rightarrow$ redistribution that makes everybody better off possible?

3. Idiosyncratic and aggregate transition
   - can't compare 2 steady states directly
   - approximation that takes care of idiosyncratic (but not aggregate) transition
     - Krusell, Mukoyama, and Sahin (2010)
     - For each agent in first economy, consider that they are placed with current assets in second economy (i.e. given new parameters, prices,...) and compare $U^1(e,k)$ and $U^2(e,k)$
   - exact way: calculate (deterministic) transition path and then calculate value function in first period

Solution with Aggregate Risk

- Aggregate risk: e.g. aggregate productivity shocks z_t
- Challenge: need to know r_t and w_t in any current and future state
  - need to forecast L_t and K_t
  - demand for capital depends on whole cross-sectional distribution
    - because marginal propensity to consume differs
    - distribution is infinite-dimensional $\Rightarrow$ have to approximate it
- Krusell and Smith (1998) algorithm:
  - forecast K_{t+1} using K_t and z_t. Aggregate law of motion

$$\log K_{t+1} \approx \left[1, \log z_t, \log K_t\right] \mathbf{b}^{\text{guess}}$$

  - in addition to idiosyncratic state variables, need grids for z and K
  - solve household problem: policy function $k'\left(e, k, z, K\right)$
  - simulate economy $\rightarrow$ simulated series $\left\{z_t, K_t\right\}$
  - estimate coefficient $\mathbf{b}^{\text{new}}$ in aggregate law of motion
  - repeat until $\mathbf{b}^{\text{new}} \approx \mathbf{b}^{\text{guess}}$ and check that forecast is accurate

References

Christopher D. Carroll, The method of endogenous gridpoints for solving dynamic stochastic optimization problems, Economics Letters, Volume 91, Issue 3, June 2006, Pages 312-320, ISSN 0165-1765, http://dx.doi.org/10.1016/j.econlet.2005.09.013.

Per Krusell, Toshihiko Mukoyama, and Ayşegül Şahin, Labour-Market Matching with Precautionary Savings and Aggregate Fluctuations, Review of Economic Studies, Volume 77, Issue 4, October 2010, Pages 1477-1507, http://dx.doi.org/10.1111/j.1467-937X.2010.00700.x.

Per Krusell and Anthony A. Smith, Jr., Income and Wealth Heterogeneity in the Macroeconomy, Journal of Political Economy, Volume 106, Issue 5, October 1998, Pages 867–896, http://dx.doi.org/10.1086/250034.

Eric R. Young, Solving the incomplete markets model with aggregate uncertainty using the Krusell–Smith algorithm and non-stochastic simulations, Journal of Economic Dynamics and Control, Volume 34, Issue 1, January 2010, Pages 36-41, ISSN 0165-1889, http://dx.doi.org/10.1016/j.jedc.2008.11.010.