Some Diophantine questions

Varieties $X/\mathbb{Q}$.

<u>Qn</u>: $X(\mathbb{Q})$?

<u>Thm</u> (Faltings) X nice curve $g(X)>1$ then $\# X(\mathbb{Q})<\infty$.

[Mordell conj].

$K/\mathbb{Q}$

$$X(K) \ni b \longrightarrow \mathrm{Pic}^0(X)(K) = \frac{\mathrm{Div}^0(X)(K)}{\mathrm{Im}[K(X)^\times]}$$

$$z \longmapsto [z]-[b]$$

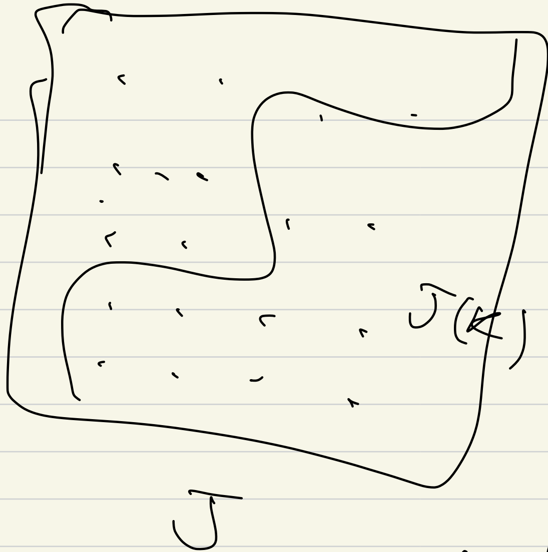
$\mathrm{Pic}^0(X) =: J$ Jacobian of X.

<u>Mordell-Weil thm</u>

$J(K)$ fin gen abelian gp.

unlikely that
$X(\mathbb{C}) \cap J(\mathbb{Q})$ infinite.

$X \longrightarrow$



genus(X)

<u>Thm</u> (Coleman) $\operatorname{rk} J(\mathbb{Q}) < g$!!

$\Rightarrow \forall p > 2g$ prime of good red for X,

$$\#X(\mathbb{Q}) < \#X(\mathbb{F}_p) + 2g - 2.$$

<u>Open Question</u>: Is there $c_g > 0$ s.t.

$$\#X(\mathbb{Q}) < c_g$$

$\forall X$ of genus g.

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Thm (Dimitrov & Gao – Habegger, Kühne, Yuan) 2021?

$\exists\, c(g)$ s.t. $\forall\, X/\mathbb{Q}$ genus g $\mathrm{rk}\, J(\mathbb{Q}) = r$

$$\# X(\mathbb{Q}) < c(g)^{1+r}.$$

Open Qn: Is there an $r(g) > 0$ s.t. $\forall$ AVs $A/\mathbb{Q}$ of dim g, $\mathrm{rk}\, A(\mathbb{Q}) < r(g)$.

p-adic abelian integrals.

$\mathbb{C}$-abelian integrals.

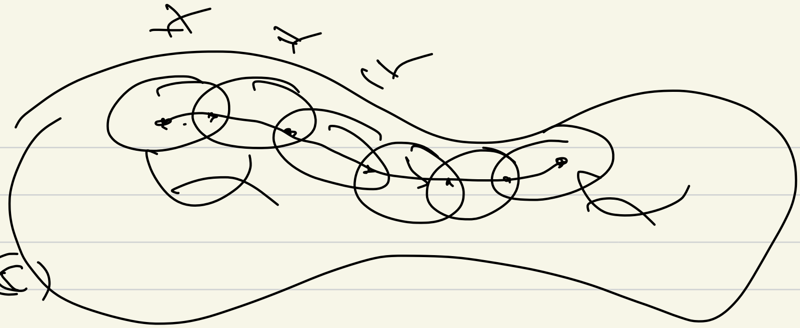
$X/\mathbb{C}$. $\omega \in H^0(X, \Omega_{X|\mathbb{C}})$.

$b \in X(\mathbb{C})$

$z \in X(\mathbb{C})$ choose $\gamma : [0,1] \to X(\mathbb{C})$ path from b to z

$$\rightsquigarrow \int_\gamma \omega \in \mathbb{C}$$

$f(z)\,dz$



$b \in X(\mathbb{C})$

$$X(\mathbb{C}) \longrightarrow H^0(X, \Omega_{X(\mathbb{C})})^*$$

$$\psi \qquad z \longmapsto \Big[\omega \longmapsto \int_{\substack{\gamma \\ b \to z}} \omega \Big]$$

How does this depend on γ.

if γ, γ' 2-different paths from b to z,

then

$$\int_\gamma \omega - \int_{\gamma'} \omega = \int_{(\gamma')^{-1} \circ \gamma} \omega$$

$$\begin{tikzcd}
(\text{Loops based at } b) \arrow[r] \arrow[d] & H^0(X, \Omega_{X(\mathbb{C})})^* \\
\pi_1(X_{\mathbb{C}}, b) \arrow[ur, "\text{hom}"] \arrow[r] & H_1(X_{\mathbb{C}}, b) \arrow[u, "(*)"']
\end{tikzcd}$$

$\Rightarrow$ well-defined map

$$X(\mathbb{C}) \longrightarrow \frac{H^0(X, \Omega_{X(\mathbb{C})})^*}{H_1(X_{\mathbb{C}}, b)}$$

$$z \longmapsto \left[\omega \longmapsto \int_\gamma \omega\right]$$

γ some path from b to z.

eg X an elliptic curve,

$H^0(X, \Omega_{X(\mathbb{C})}) \simeq \mathbb{C}$.

$H_1(X_{\mathbb{C}}, \mathbb{Z}) \simeq \Lambda$.

$\rightsquigarrow X(\mathbb{C}) \simeq \mathbb{C}/\Lambda$.

In general, $H_1(X_{\mathbb{C}}, \mathbb{Z}) \simeq \mathbb{Z}^{2g}$

$$\cap$$

$$H^0(X, \Omega)^* \simeq \mathbb{C}^g$$

As a top. space, this quotient is $\simeq (\mathbb{R}/\mathbb{Z})^{2g}$

In fact $J(\mathbb{C}) \simeq \dfrac{H^0(X,\Omega)^*}{H_1(X,\mathbb{Z})}$

<u>p-adic analogue:</u>

If z_1 & $z_2 \in X(\mathbb{Q}_p)$ are "close together", we can define $\int_{z_1}^{z_2} \omega \in \mathbb{Q}_p$.

eg $X: y^2 = x^5 + 1$.

$\omega = \dfrac{dx}{y} \in H^0(X, \Omega)$

$(0,1) = b$. x is a param at b. $y = \sqrt{1+x^5} = 1 + \dfrac{x^5}{2} + \ldots$

$$\frac{dx}{y} = dx\left(1 - \frac{x^5}{2} + \frac{3}{8}x^{10} - \frac{5}{16}x^{15} + \ldots\right)$$

If $\cancel{z} = (x(z), y(z))$

is s.t. $|x(z)|$ small,

We can evaluate

$$\int_b \frac{dx}{y} = x - \frac{x^6}{3} + \frac{3}{88}x^{11} + \cdots$$

at z. $\quad \uparrow$

$$\mathbb{Q}_p[[x]].$$

$p > 2 \Rightarrow$ this converges whenever $|x(z)| < 1$.

$b \in X(\mathbb{Q}_p)$.

$$\begin{tikzcd}
\left\{\begin{matrix}\text{points close} \\ \text{to } b\end{matrix}\right\} \arrow[r] \arrow[dr] \arrow[d, phantom, "\cap"] & H^0(X_{\mathbb{Q}_p}, \Omega)^* \\
& J^0(\mathbb{Q}_p) \arrow[u, "f_0"] \arrow[d, phantom, "\cap"] & e \arrow[dd] \\
X(\mathbb{Q}_p) \arrow[r] & J(\mathbb{Q}_p) \arrow[dr] \\
& & J(\mathbb{F}_p)
\end{tikzcd}$$

$\#J(\mathbb{F}_p)P \longmapsto 0$

$$0 \to J^0(\mathbb{Q}_p) \to J(\mathbb{Q}_p) \to J(\mathbb{F}_p) \to 0$$

$$\tilde{f} : J(\mathbb{Q}_p) \longrightarrow H^0(X_{\mathbb{Q}_p}, \Omega)^*$$

$$\begin{array}{c} \rotatebox{90}{$\in$} \\ P \mapsto \dfrac{1}{\# J(\mathbb{F}_p)} f(\# J(\mathbb{F}_p) \cdot P) \end{array}$$

extends f.

$$\begin{tikzcd} J(\mathbb{Q}_p) \arrow[r] & H^0(X_{\mathbb{Q}_p}, \Omega)^* \\ X(\mathbb{Q}_p) \arrow[ur] & \end{tikzcd}$$

$$\begin{tikzcd} X(\mathbb{Q})^{b} \arrow[r] \arrow[d, hook] & J(\mathbb{Q}) \simeq \tilde{\mathbb{Z}} \oplus T \arrow[d, hook] \\ X(\mathbb{Q}_p) \arrow[r] & H^0(X_{\mathbb{Q}_p}, \Omega)^* \end{tikzcd}$$

$r < g \Rightarrow \exists\, \underset{\neq 0}{\omega} \in H^0(X_{\mathbb{Q}_p}, \Omega)$

s.t. $\forall z \in X(\mathbb{Q})$,

$$\int_b^{z} \omega = 0.$$

$$\Rightarrow \#X(\mathbb{Q}) \leq \#X(\mathbb{F}_p) + 2g - 2.$$