Let X be an alg. variety over $k \subseteq \mathbb{C}$.

$X(\mathbb{C})$ is a top. space. It has the homotopy type of a finite CW-complex

$$H_n^{sing}(X(\mathbb{C}), \mathbb{Q}) \qquad \text{f.d. } \mathbb{Q}\text{-vsp.}$$

Vanishing:

$$H_n^{sing} = 0 \quad \text{if} \quad n > 2 \dim X$$

$$H_n^{sing} = 0 \quad \text{if} \quad n > \dim X$$
and X affine

Let X/k be smooth affine over k of char. 0. $X = \operatorname{spec}(A)$

$$\Omega^0_{A/k} \longrightarrow \Omega^1_{A/k} \longrightarrow \Omega^2_{A/k} \longrightarrow \cdots$$

$$\overset{\shortparallel}{A} \qquad \langle da \mid a \in A \rangle$$

$$dab = adb + bda$$

$$\cdots$$

$\rightsquigarrow$ $H^n_{dR}(X/k)$ alg. deRham coho...

same vanishing

Theorem (Grothendieck - deRham)

$X/k \quad k \subseteq \mathbb{C}$. Then the pairing

$$H^n_{dR}(X/k) \otimes H^{sing}_n(X(\mathbb{C})) \to \mathbb{C}$$

$$\omega \qquad\qquad \sigma \qquad \longmapsto \int_\sigma \omega$$

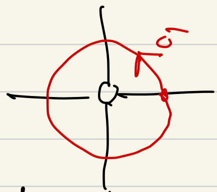
functorial in X! is perfect!

$\rightsquigarrow$ A canonical functorial isomorphism

$$H^n_{dR}(X/k) \otimes_k \mathbb{C} \xrightarrow{\sim} H^n_{sing}(X(\mathbb{C})) \otimes_{\mathbb{Q}} \mathbb{C}$$

Example $\quad X = \operatorname{spec}(\underbrace{k[x, x^{-1}]}_{A}) = \mathbb{G}_m$

$X(\mathbb{C}) = \mathbb{C}^\times$

$H_1(X(\mathbb{C})) = \mathbb{Q}$ gen. by 

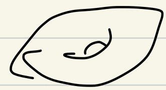
$H^1_{dR}(X/k) = H^1(A \to \Omega^1_{A/k}) = \langle \underbrace{\tfrac{1}{x}dx}_{\omega} \rangle$

$$\int_\sigma \omega = \oint \frac{1}{x} dx = 2\pi i$$

Example $X : y^2 = 4x^3 - ax - b$

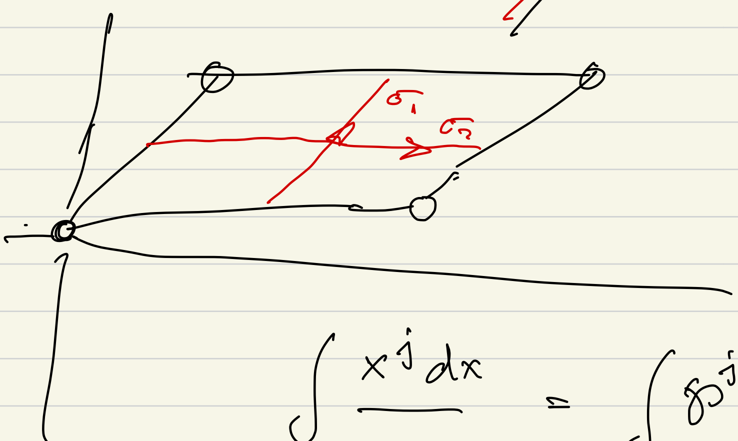
affine elliptic curve

$\langle \sigma_1, \sigma_2 \rangle = H_1^{sing}(X(\mathbb{C}))$



$H^1_{dR} = \left\langle \frac{dx}{y}, \frac{x\,dx}{y} \right\rangle$

$(\wp : \wp')$



$$\int_{\sigma_i} \frac{x^j dx}{y} = \int_{\sigma_i} \wp^j(z)\,dz$$

Y. André

A period structure over $k \subseteq \mathbb{C}$ is $P=(V,W,\alpha)$

$V = \mathbb{Q}$-vsp. f.d.

$W = k$-vsp. f.d.

$\alpha : W \otimes_k \mathbb{C} \xrightarrow{\sim} V \otimes_{\mathbb{Q}} \mathbb{C}$

$\rightsquigarrow$ Abelian, $\mathbb{Q}$-lin. category $\mathbb{P}$

$\otimes$, dual : Tannakian category

$P = (V, W, \alpha)$

fund. group of P is $G_P \subseteq GL_V$

is the largest alg. subgroup respects all subobjects of all $P^{\otimes a} \otimes (P^\vee)^{\otimes b}$