The RHS of the previous equation is clearly nonpositive, so $\alpha'$ must be nonpositive as well. We can also bound the size of $\alpha'$ with

$$
\begin{aligned}
-\varepsilon\alpha' &= \left\lceil \frac{\alpha}{1+\varepsilon} \right\rceil - \alpha \\
1-\varepsilon\alpha' &\geq 1 + \frac{\alpha}{1+\varepsilon} - \alpha \\
-\varepsilon\alpha' &\geq \frac{-\varepsilon\alpha}{1+\varepsilon} \\
(1+\varepsilon)\alpha' &\leq \alpha \\
\alpha' &< \alpha.
\end{aligned}
$$

By (1), if we take the lower bound for the change in $\sum E_k^2$ using $\alpha'$, the value of $\sum E_{k-1}^2$ is equal to or smaller than if we had used $\alpha$ instead. By the same logic that was used to prove Lemma 13, it is always helpful to have a lower $\sum E^2$. $\square$

With Lemma 13 and Lemma 14, we now have a strictly stronger set of actions on $\sum E$ and $\sum E^2$ that we can perform with each move.

**Definition 15.** *A **strong reversed move** that uses $\alpha$ on the pair of values $\sum E_k$ and $\sum E_k^2$ is given by*

$$
\begin{aligned}
\sum E_{k-1} &= \sum E_k - \varepsilon\alpha \\
\sum E_{k-1}^2 &= \sum E_k^2 - 2\sum E_k + (1-\varepsilon)^2\alpha^2.
\end{aligned}
$$

For integers $1 \leq k \leq m$, let $\alpha_k$ denote the value of $\alpha$ used in the strong reversed move to get from $\left(\sum E_k, \sum E_k^2\right)$ to $\left(\sum E_{k-1}, \sum E_{k-1}^2\right)$. Then,

$$
\begin{aligned}
\sum E_m &= \sum E_0 + \varepsilon\alpha_1 + \varepsilon\alpha_2 + \cdots + \varepsilon\alpha_m \\
\sum E_m &= \sum E_0 + \varepsilon\sum_{j=1}^{m}\alpha_j. \qquad (3)
\end{aligned}
$$

The same process is a little more complicated for $\sum E_m^2$.

$$
\sum E_m^2 = \sum E_0^2 + 2\sum_{j=1}^{m}\left(\sum E_j\right) - (1-\varepsilon)^2\sum_{j=1}^{m}\alpha_j^2
$$

We can rewrite the $\sum E_j$ by plugging in (3).

$$
\begin{aligned}
\sum E_m^2 &= \sum E_0^2 + 2\sum_{j=1}^{m}\left(\sum E_j\right) - (1-\varepsilon)^2\sum_{j=1}^{m}\alpha_j^2 \\
\sum E_m^2 &= \sum E_0^2 + 2\sum_{j=1}^{m}\left(\sum E_0 + \varepsilon\sum_{i=1}^{j}\alpha_i\right) - (1-\varepsilon)^2\sum_{j=1}^{m}\alpha_j^2 \\
\sum E_m^2 &= \sum E_0^2 + 2m\sum E_0 + 2\varepsilon\sum_{j=1}^{m}(m-j+1)\alpha_j - (1-\varepsilon)^2\sum_{j=1}^{m}\alpha_j^2 \\
0 &= \sum E_0^2 + 2m\sum E_0 - \sum E_m^2 + 2\varepsilon\sum_{j=1}^{m}(m-j+1)\alpha_j - (1-\varepsilon)^2\sum_{j=1}^{m}\alpha_j^2
\end{aligned}
$$