nonempty open subinterval J of I such that $A = \{p : i_p = i \text{ and } n_p = n\}$ contains the closure of J.

Again by Lemma 1.2 we have

$$\bigcup_{p \in A \cap J} U'(p, n) = \bigcup_{q \in \mathbb{Q} \cap J} U'(q, n)$$

This now implies that $f[U'(q, n)] \subseteq K_i$, and hence that $i \cdot \varepsilon \leqslant f(q, 0)$, whenever $q \in \mathbb{Q} \cap J$. But $\inf\{f(q, 0) : q \in \mathbb{Q} \cap J\} = 0$ and hence $i = -1$ or $i = 0$.

We find that $f[U(x, n)] \subseteq [0, 2\varepsilon]$, and hence $f(x, 0) \leqslant 2\varepsilon < 2^{-k}$, for all $x \in J$. □

2. A Book

We apply Jones' method from [3] to N to obtain a regular, non-completely regular space, as follows.

We start with the product $N \times \mathbb{N}$, where $\mathbb{N}$ carries its discrete topology.

For every even $n \in \mathbb{N}$ and $q \in \mathbb{Q}$ we identify the points $\langle\langle q, 0\rangle, n\rangle$ and $\langle\langle q, 0\rangle, n+1\rangle$, so that $Q \times \{n\}$ and $Q \times \{n+1\}$ become one copy of Q, which we denote Q_n.

Likewise for every odd $n \in \mathbb{N}$ and $p \in \mathbb{P}$ we identify $\langle\langle p, 0\rangle, n\rangle$ and $\langle\langle p, 0\rangle, n+1\rangle$, thus creating out of $P \times \{n\}$ and $P \times \{n+1\}$ one copy of P that we denote P_n.

The resulting space we call B and we let $\pi : N \times \mathbb{N} \to B$ denote the quotient map.

Remark 2.1. In [5, Examples 90 and 91] Tychonoff's example, mentioned in the introduction, is represented pictorially as a corkscrew.

Our space B looks more like a book made from infinitely many sheets of paper sewn together alternately along the rational and irrational points on the x-axis, hence the title of the present section. We shall call B a *book* from now on.

Lemmas 1.3 and 1.4 above imply that if $f : B \to [0, 1]$ is continuous and equal to 0 on the set Q_0 then, by induction, the following holds for every n, where we let $F : N \times \mathbb{N} \to [0, 1]$ be the composition $f \circ \pi$.

- if n is even then $\inf\{F(x, 0, n) : x \in O \cap \mathbb{Q}\} = 0$ whenever O is an open interval in $\mathbb{R}$, and
- if n is odd then there is a dense G_δ-set G_n, such that $F(x, 0, n+1) = F(x, 0, n) = 0$ whenever $x \in G_n \cap P_n$.

Remark 2.2. Note that by the second item there is in fact a single dense G_δ-set G, to wit $\bigcap\{G_n : n \text{ is odd}\}$, such that $F(x, 0, n+1) = F(x, 0, n) = 0$ whenever $x \in G \cap P_n$ and n is odd.

This implies that if we were to add a 'point at infinity' ∞ to $N \times \mathbb{N}$ with basic neighbourhoods $U_m = \{\infty\} \cup (N \times \{n \in \mathbb{N} : n \geqslant m\})$ and apply the quotient operation above to the new space $(N \times \mathbb{N}) \cup \{\infty\}$, then the resulting quotient space $B \cup \{\infty\}$ is not completely regular at ∞.

Lemma 2.3. *The map $\pi : (N \times \mathbb{N}) \cup \{\infty\} \to B \cup \{\infty\}$ is closed.*

Proof. If F is closed in $(N \times \mathbb{N}) \cup \{\infty\}$ then $\pi^{\leftarrow}\bigl[\pi[F]\bigr]$ is closed as well. If $n \in \mathbb{N}$ then its intersection with the sheet $N \times \{n\}$ consists of three parts (two if $n = 0$):

- $F \cap (N \times \{n\})$,
- the set $\{\langle\langle x, 0\rangle, n\rangle : \langle\langle x, 0\rangle, n+1\rangle \in F\}$, and
- the set $\{\langle\langle x, 0\rangle, n\rangle : \langle\langle x, 0\rangle, n-1\rangle \in F\}$ if also $n > 0$.

The union of these intersections is closed in $N \times \mathbb{N}$.

If $\infty \notin F$ then F intersects only finitely many sheets and so ∞ is not in the closure of $\pi^{\leftarrow}\bigl[\pi[F]\bigr]$.

If $\infty \in F$ then all is well. □