Advanced Algorithms

Problem set #3
Due date: 22 Azar 98 (via lms.iut.ac.ir)

Solutions to the problems on this set might be found on the Internet. It is not acceptable to copy such solutions. Write with your own notation and formulation.

Problem 1. Let $\mathcal{A}$ be an algorithm for a minimization **NP**-optimization problem Π such that the expected cost of the solution produced by $\mathcal{A}$ is $\leq \alpha\text{OPT}$, for a constant $\alpha > 1$. What is the best approximation guarantee you can establish for Π using algorithm $\mathcal{A}$?

Problem 2. Given an undirected graph $G = (V, E)$ with nonnegative edge costs, and an integer k, find a partition of V into sets $S_1, \ldots, S_k$ so that the total cost of edges running between these sets is maximized. Give a greedy algorithm for this problem that achieves a factor of $(1 - 1/k)$. Is the analysis of your algorithm tight?

Problem 3. Given an undirected graph $G = (V, E)$, color its vertices with the minimum number of colors so that the two endpoints of each edge receive distinct colors.

(a) Give a greedy algorithm for coloring G with $\Delta + 1$ colors, where Δ is the maximum degree of a vertex in G.

(b) Give an algorithm for coloring a 3-colorable graph with $O(\sqrt{n})$ colors.

Problem 4. In the metric k-Supplier problem we get $m + n$ points on input, where m of them are (in advance) marked as *suppliers* and the rest are *consumers*. Between all those points is a metric (with a triangle inequality as usual). Our task in this case is to select k suppliers so that we minimize the longest distance between a customer and its closest supplier.

(a) Prove that there is no $3 - \epsilon$ approximation algorithm for this problem unless $\mathbf{P} = \mathbf{NP}$.

(b) Find a factor 3-approximation algorithm for this problem.

Problem 5. Consider variants on the metric TSP problem in which the object is to find a simple path containing all the vertices of the graph. Three different problems arise, depending on the number (0, 1, or 2) of endpoints of the path that are specified. Obtain the following approximation algorithms.

(a) If zero or one endpoint is specified, obtain a $3/2$ factor algorithm.

(b) If both endpoints are specified, obtain a $5/3$ factor algorithm.

(c) Let G be a complete undirected graph in which all edge lengths are either 1 or 2 (clearly, G satisfies the triangle inequality). Give a $4/3$ factor algorithm for TSP in this special class of graphs.

Problem 6. An undirected graph is k-*edge-colorable* if each edge can be assigned exactly one of k colors in such a way that no two edges with the same color share an endpoint. We call the assignment of colors to edges a k-*edge-coloring*.

(a) Prove that there is no α-approximation algorithm for the edge coloring problem for $\alpha < 4/3$ unless $\mathbf{P} = \mathbf{NP}$.

(b) Let $G = (V, E)$ be a bipartite graph; that is, V can be partitioned into two sets A and B, such that each edge in E has one endpoint in A and the other in B. Let Δ be the maximum degree of a node in G. Give a polynomial-time algorithm for finding a Δ-edge-coloring of G.

Problem 7(extra credit). In the k-Cut problem, we are given an edge-weighted graph G and an integer k and have to remove a set of edges with minimum total weight so that G has at least k connected components. First, show that getting a $(2 - \epsilon)$-approximation for general k is **NP**-hard (by assuming applicable hypothesis i.e. ETH, PCP, SAT, ...). Second, argue with this question that can we propose a better approximation for k-Cut in **FPT**-time[1] w.r.t k as the parameter? that is, you may show that for some $\epsilon > 0$ there is a $(2 - \epsilon)$-approximation algorithm that runs in **FPT**-time with parameter k.

Good Luck.

[1]FPT contains the *fixed parameter tractable* problems, which are those that can be solved in time $f(k) \cdot n^{O(1)}$ for some computable function f.