Advanced Algorithms

Problem set #2
Due date: 30 Aban 98 (via lms.iut.ac.ir)

Solutions to the problems on this set might be found on the Internet. It is not acceptable to copy such solutions. Write with your own notation and formulation.

Problem 1. For each of the following planar edge-coloring problems, either show that the problem is **NP**-hard, or show that there exists a polynomial-time algorithm for the problem (e.g., by reducing to shortest paths, minimum spanning tree, matching, network flow, etc.).

(a) Given a multiset of non-negative integers $A = \{a_1, \cdots, a_{2n}\}$ that sum to tn, find a partition of A into n groups $S_1, \cdots, S_n$ of size 2 such that each group sums to t.

(b) Given a 3-regular planar undirected multigraph[1] (allowing parallel edges and self-loops), color each edge either red or blue such that, at each vertex, the multiset of colors of the incident edges is $\{R, B, B\}$.

Problem 2. Consider the problem that given a finite set S and some of its subset $A_1, \ldots, A_m$. The question is whether there exist a partition (S_1, S_2) of S, such that for all $i = 1, \ldots, m$, A_i intersects both S_1 and S_2 ?

(a) Show that the above problem is **NP**-complete.

(b) Explain why this problem belongs to **P**, provided that $|A_1| = \ldots |A_m| = 2$.

Problem 3. In the *Balanced Partition* we are given a set S of numbers and we want to check if the numbers can be divide into sets A and $A' = S - A$ where

$$\sum_{x \in A} x = \sum_{x \in A'} x.$$

Show that *Balanced Partition* is **NP**-complete.

Problem 4. Consider a set $A = \{a_1, \ldots, a_n\}$ and a collection $B_1, B_2, \ldots, B_m$ of subsets of A (i.e., $B_i \subseteq A$ for each i). We say that a set $H \subseteq A$ is a *hitting set* for the collection $B_1, \ldots, B_m$ if H contains at least one element from each B_i, that is, if $H \cap B_i$ is not empty for each i (so H "hits" all the sets B_i). We now define the *Hitting Set Problem* as follows. We are given a set $A = \{a_1, \ldots, a_n\}$, a collection $B_1, \ldots, B_m$ of subset of A, and a number k. We are asked: Is there a hitting set $H \subseteq A$ for $B_1, \ldots, B_m$ so that the size of H is at most k? Prove that Hitting Set Problem is **NP**-complete.

[1] A *regular graph* with vertices of degree k is called a *k-regular* graph or regular graph of degree k. A graph is said to be *embeddable in the plane*, or *planar*, if it can be drawn in the plane so that its edges intersect only at their ends.

Problem 5. Let A and B be two problems. Then show that:

(a) If A and B are in **NP**, then so are $A \cup B$ and $A \cap B$.

(b) Recall the definition of **NP**-completeness that presented in the class. Now we give a new definition here as follow

" a problem like A is para-**NP**-complete if; (1) $A \in \mathbf{NP}$ and (2) there exist a polynomial like p such that $\forall A' \in \mathbf{NP}$ there exist a function g where

- $g(x)$ is computable in time $p(|x|)$ and
- $x \in A'$ if and only if $g(x) \in A$."

Prove that para-**NP**-complete languages do not exist.

(c) For a language L, we define the *Kleene star* of the language is

$$L^* = \{x_1 \dots x_k | x_i \in L, k \geq 0\}.$$

we say that for a complexity class $\mathcal{C}$ closed under Kleene star if $L \in \mathcal{C} \Rightarrow L^* \in \mathcal{C}$.

Show that **P** and **NP** are closed under Kleene star.

Problem 6. Let 2-CNF-SAT be the set of satisfiable boolean formulas in CNF with exactly 2 literals per clause. Show that 2-CNF-SAT $\in$ **P**. Make your algorithm as efficient as possible.

Problem 7(optional-unsolved problem). Prove that Sudoku (or, more formally, the generalization to $n \times n$ boards) is **NP**-complete. Note: by using Google you can find a solution, but I am looking for a simpler solution that relies on a reduction from a more well-known NP-complete problem. I don't know how difficult it is to find a "simpler" solution, so work on this problem at your own risk!

Good Luck.