(Problem 2)

If vectors $\vec{v}_1, \vec{v}_2, \dots, \vec{v}_m$ are a basis for V, but not an orthogonal basis. However, V does have an orthonormal basis. Let $\vec{u}_1, \vec{u}_2, \dots \vec{u}_m$ be this basis. Now define $Q = [\vec{u}_1\ \vec{u}_2 \dots \vec{u}_m]$, $\vec{u}_i$ s are orthonormal therefore $Q^T Q = I_m$. Now, let $\vec{v}_j = \sum_i R_{ij} \vec{u}_i$. We can write $A = QR$.

R is square and invertible. Lets look at P_A, now:

$$P = A(A^T A)^{-1} A^T$$

$$= (QR)((QR)^T (QR))^{-1} (QR)^T$$

$$= QR(R^T Q^T QR)^{-1} R^T Q^T \qquad \# \ Q^T Q = I$$

$$= QR(R^T R)^{-1} R^T Q^T \qquad \# \ R \text{ and } R^T \text{ are invertible}$$

$$= QR R^{-1} (R^T)^{-1} R^T Q^T$$

$$= QQ^T$$

Where the columns of Q are an orthonormal basis. Then $Q^T Q$ is the matrix of orthogonal projection onto V. #

(Problem 3) $\hat{x} = (A^T A)^{-1} A^T b$

$= A^{\dagger} b$

1/ if $x \neq \hat{x}$, then $\|Ax - b\|_2 \geq \|A\hat{x} - b\|_2$

Proof: $\|Ax - b\|_2 = \|A(x - \hat{x}) + (A\hat{x} - b)\|_2$

$= \|A(x - \hat{x})\|_2 + \|A\hat{x} - b\|_2$

$> \|A\hat{x} - b\|_2$

2/ $A(x - \hat{x}) \perp A(\hat{x} - b)$

$(A(x - \hat{x}))^T (A\hat{x} - b) = (x - \hat{x})^T (A^T A \hat{x} - A^T b) = 0$

3/ $A(x - \hat{x}) \neq 0$ (if $x \neq \hat{x}$)

$f(x) = \|Ax - b\|_2 = \sum_{i=1}^{m} \left(\sum_{j=1}^{n} A_{ij} x_j - b_i \right)^2$

$\frac{\partial f(x)}{\partial x_k} = 2 (A^T (Ax - b))_k$

$\Rightarrow \nabla f(x) = 2 A^T (Ax - b) \Rightarrow$ minimizer $\hat{x}$ of $f(x)$:

$(A\hat{x} - b = 0)$ #