> Supplementary code for the subsection "In terms of first integrals"

> Again, we consider a differential system with the following right-hand side

```
> eqs := [
   -kn1 * X * T1 - kn2 * X * T2 + kf1 * D1 + kf2 * D2, # X
   -kn1 * X * T1 + kf1 * D1 - kn1 * T1 * D2 + kf1 * Y, # T1
   -kn2 * X * T2 + kf2 * D2 - kn2 * T2 * D1 + kf2 * Y, # T2
   kn1 * X * T1 - kf1 * D1 - kn2 * T2 * D1 + kf2 * Y, # D1
   kn2 * X * T2 - kf2 * D2 - kn1 * T1 * D2 + kf1 * Y, # D2
   kn1 * T1 * D2 + kn2 * T2 * D1 - (kf1 + kf2) * Y # Y
 ];
```

$$eqs := [-kn1\, X\, T1 - kn2\, X\, T2 + kf1\, D1 + kf2\, D2, -kn1\, T1\, D2 - kn1\, X\, T1 + kf1\, D1 + kf1\, Y, -kn2\, T2\, D1 - kn2\, X\, T2 + kf2\, D2 + kf2\, Y, -kn2\, T2\, D1 + kn1\, X\, T1 - kf1\, D1 + kf2\, Y, -kn1\, T1\, D2 + kn2\, X\, T2 - kf2\, D2 + kf1\, Y, kn1\, T1\, D2 + kn2\, T2\, D1 - (kf1 + kf2)\, Y] \tag{1}$$

> We observe that the system has the following first integrals:

```
> first_int := [
   c1 - Y - T1 - D1,
   c2 - Y - T2 - D2,
   c3 - D1 - D2 - X - Y
 ];
```

$$first\_int := [c1 - Y - T1 - D1, c2 - Y - T2 - D2, c3 - D1 - D2 - X - Y] \tag{2}$$

> Our goal is to show that, for every value of c1, c2, c3, there is at most one positive steady state, and we will also give expressions for its coordinates.

```
> with(Groebner) :
  gb := Basis([op(eqs), op(first_int)], plex(X, Y, D1, D2, T1, T2, c1, c2, c3, kf1, kf2, kn1, kn2)) :
```

> Expression for T2:

```
> gb[1];
```

$$T2^2\, kn2 - T2\, c2\, kn2 + T2\, c3\, kn2 + T2\, kf2 - c2\, kf2 \tag{3}$$

> Since the constant term is negative, the equation has exactly one positive solution. Now for T1:

```
> gb[2];
```

$$T1^2\, kn1 - T1\, c1\, kn1 + T1\, c3\, kn1 + T1\, kf1 - c1\, kf1 \tag{4}$$

> Again, only one positive solution. For D2:

```
> factor(gb[3]);
```

$$(c1^2\, kf2\, kn1^2\, kn2 + c1\, c2\, kf1\, kn1^2\, kn2 + c1\, c2\, kf2\, kn1\, kn2^2 - c1\, c3\, kf1\, kn1^2\, kn2 - 2\, c1\, c3\, kf2\, kn1^2\, kn2 - c1\, c3\, kf2\, kn1\, kn2^2 + c2^2\, kf1\, kn1\, kn2^2 - c2\, c3\, kf1\, kn1^2\, kn2 \tag{5}$$

$- 2\, c2\, c3\, kf1\, kn1\, kn2^2 - c2\, c3\, kf2\, kn1\, kn2^2 + c3^2\, kf1\, kn1^2\, kn2 + c3^2\, kf1\, kn1\, kn2^2$
$+ c3^2\, kf2\, kn1^2\, kn2 + c3^2\, kf2\, kn1\, kn2^2 + c1\, kf1\, kf2\, kn1^2 + 2\, c1\, kf1\, kf2\, kn1\, kn2$
$+ c1\, kf2^2\, kn1\, kn2 + c2\, kf1^2\, kn1\, kn2 + 2\, c2\, kf1\, kf2\, kn1\, kn2 + c2\, kf1\, kf2\, kn2^2 - c3\, kf1^2\, kn1^2$
$- c3\, kf1^2\, kn1\, kn2 - c3\, kf1\, kf2\, kn1^2 - 2\, c3\, kf1\, kf2\, kn1\, kn2 - c3\, kf1\, kf2\, kn2^2 - c3\, kf2^2\, kn1\, kn2$
$- c3\, kf2^2\, kn2^2 + kf1^2\, kf2\, kn1 + kf1^2\, kf2\, kn2 + kf1\, kf2^2\, kn1 + kf1\, kf2^2\, kn2\big)\, (D2\, c3 + T1\, T2$
$- T1\, c2 - T2\, c1 + T2\, c3 + c1\, c2 - c2\, c3)$

> **The first bracket does not contain the state variables and is nonzero in the generic case. The second bracket yields a linear equation for D2. Now D1:**

> $gb[10];$

$$D1 - D2 + T1 - T2 - c1 + c2 \tag{6}$$

> **Y:**

> $gb[11];$

$$-c2 + Y + T2 + D2 \tag{7}$$

> **X:**

> $gb[12];$

$$X + D2 - T1 + c1 - c3 \tag{8}$$

>