Understanding turbo codes: A signal processing study

A B S T R A C T

This exposition provides a novel outlook on the remarkable capabilities of turbo codes by analyzing them through the lens of digital signal processing over the complex field. We elucidate the two pivotal elements facilitating the turbo code phenomenon – iterative decoding and interleaving across parallel concatenated codes. A mathematical framework is developed to quantify the decrease in decoded noise power with each iteration, contingent upon the interleaver adequately decorrelating the noise. Exact analytical expressions characterizing the rate of noise power reduction and its limiting value are derived. Remarkably, for a rate ½ turbo code, this limiting decoded noise power is proven to be unattainable by any non-iterative (non-turbo) code of the same rate, underscoring the fundamental advantages of the ingenious turbo architecture. The criticalrole of the interleaver is also examined, providing valuable insights into its design considerations.

1. Introduction

Turbo codes have attracted tremendous research interest (see [1-9]) owing to their ability to operate near the ultimate Shannon limit. Seminal work [2] demonstrated that turbo codes can deliver bit error rates as low as 10^{-5} at the modest SNR of 0.0 dB - a feat once deemed impossible. This astounding "turbo magic" has sparked an intense pursuit to demystify the principles underpinning their excellent performance. As eloquently stated by Forney in [10], "turbo codes work amazingly well, but we don't yet fully understand why." The pivotal question remains: What catalyzes the remarkable power of turbo codes?

A central enabler is the symbiotic relationship between iterative decoding and intelligent interleaving. By iterating the decoding process, with each new pass benefiting from the noise scrambling effects of interleaving, increasingly more errors can be progressively identified and eliminated. This mechanism allows turbo codes to inch ever closer towards the coveted Shannon limit, outperforming conventional codes.

Another key strength lies in the parallel concatenated coding structure itself. By ingeniously combining two or more relatively simple component codes in parallel, with judicious interleaving between them, the resultant overall code can deliver tremendously superior error correction capabilities compared to the individual constituent codes operating alone. It is this architectural innovation that provides the celebrated "turbo" boosting effect.

The core design principles embodied in turbo codes include:

1) Parallel concatenation of relatively simple convolutional codes instead of a single large code, enabling efficient encoding/decoding.
2) Pseudo-random interleaving to disperse burst errors and optimize weight distributions for iterative decoding gains.
3) Iterative decoding where component decoders exchange soft information to synergistically enhance error correction on every pass.
4) Synergistically compounding the interleaving gains from parallel concatenation with the iterative processing gains to approach capacity.

A simple rate 1/2 turbo encoder and decoder structure are illustrated in Fig. 1(a) and 1(b) respectively, where ↓2 denotes downsampling by 2, and z^(-1) is the delay operator D used in coding performance can be outlined as follows:

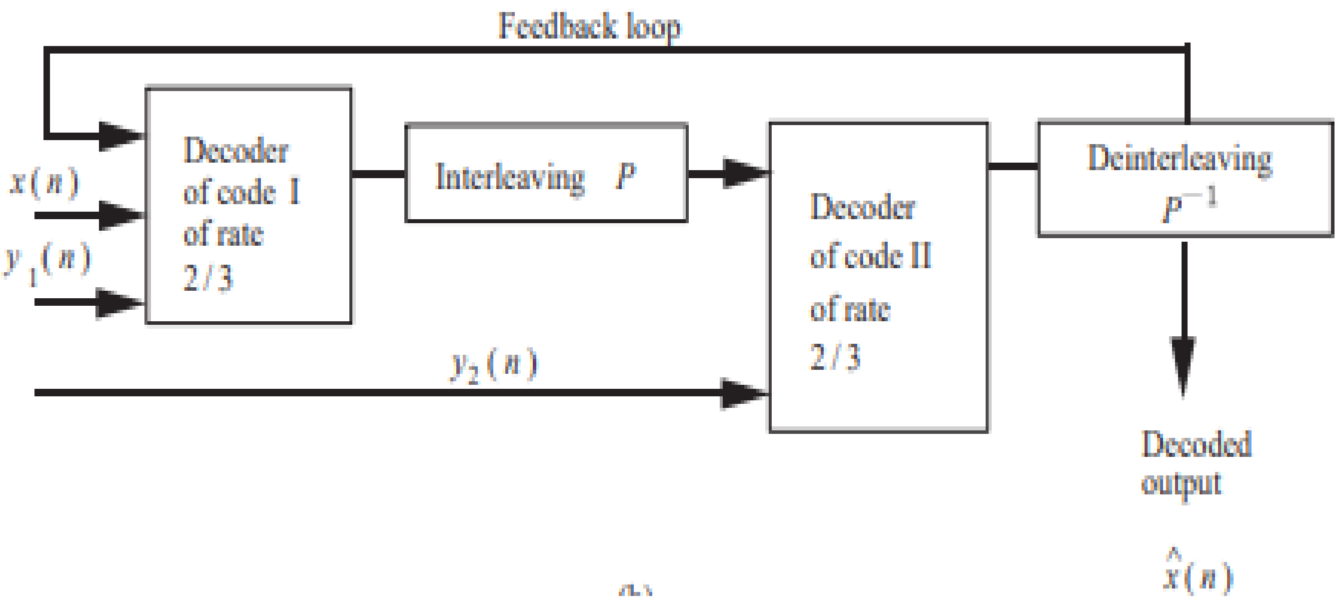
Practical limitations restrict the error correction capability of any finite-length code operating at low SNRs due to their inherently limited minimum distance properties. Consequently, a single decoding pass can only partially reduce errors. The natural recourse of iteratively re-decoding the output is fundamentally limited, as patterned errors like bursts tend to accumulate with each new decoding pass, eventually negating any further gains. The paradigm-shifting breakthrough is the ingenious use of an interleaver between the decoders to randomize the error patterns and maximize the complementary error correction capabilities on every new pass. This strategic interleaver placement then necessitates the deployment of the intricate interleaver architecture between the parallel concatenated encoders as well.

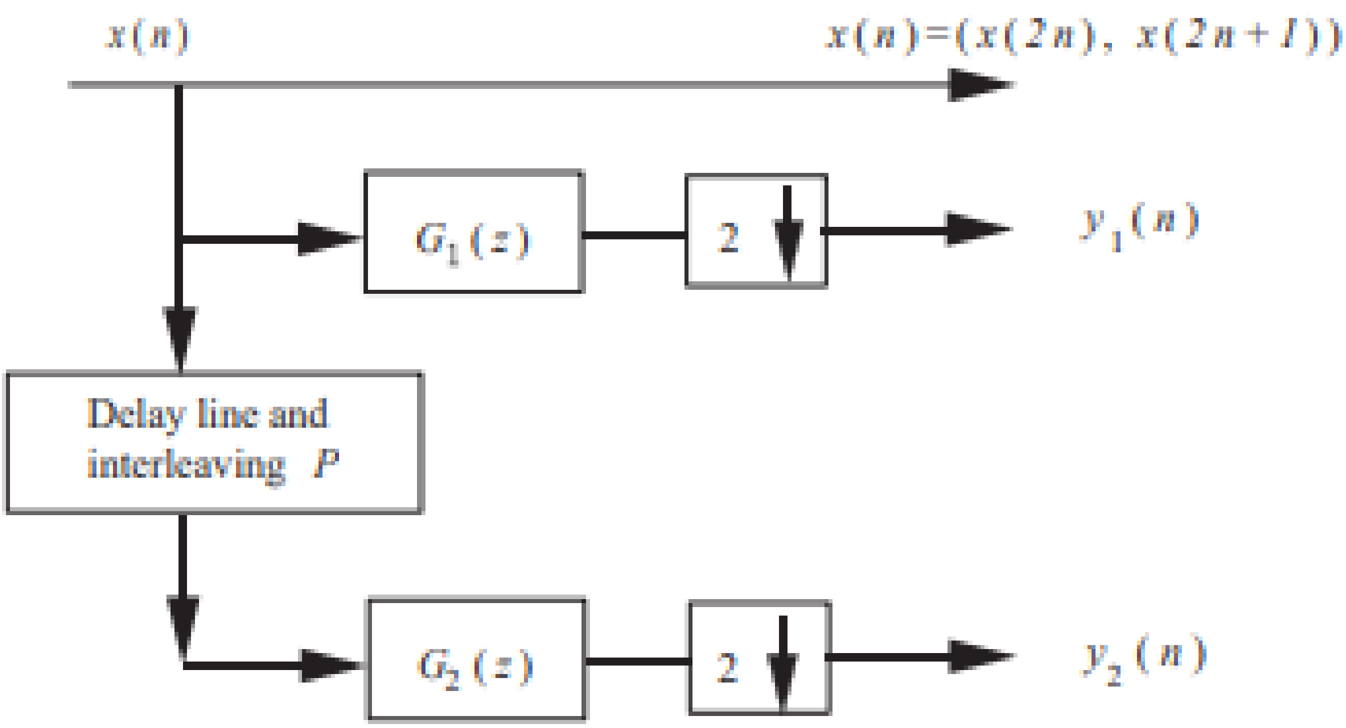
The iterative exchange can be conceptualized as the component decoders passing increasingly refined soft information back-and-forth to progressively improve their estimates of the transmitted message bits. On every new iteration, each decoder benefits from the systematic signal information as well as the parity inputs from the other decoder(s) to synergistically amplify their collective error correction prowess. This iterative process continues until the incremental gains become insignificant or a predetermined iteration limit is reached.

While such qualitative insights are valuable, a rigorous quantitative analysis framework is crucial to deepen our understanding of turbo codes. Several pioneering works like [3-9] have made important strides in analytically characterizing their performance, and techniques like EXIT charts [11] have been instrumental in visualizing the dynamics of iterative decoding. However, the development of an elegant yet insightful analytic approach that can lucidly showcase the fundamental ascendancy of turbo codes over conventional codes remains an enticing pursuit.

In this expository work, we adopt an innovative perspective by extending turbo codes and conventional codes (originally defined over finite fields) to the complex field domain. This novel complex-field formulation provides a powerful unified framework enabling comparative analysis of diverse coding schemes on an equal theoretical footing. Since maximum-likelihood decoding over the complex field conveniently reduces to solving a system of linear equations via least-squares techniques, we can precisely quantify the noise power reduction with each iterative decoding step, contingent upon the interleaver providing sufficient decorrelation. For any specified turbo code rate, we derive exact analytic expressions characterizing the rate of noise power reduction with iterations as well as the ultimate limiting noise power. Remarkably, this limiting decoded noise power is proven to be lower than what is fundamentally achievable by any non-iterative (i.e., non-turbo) code of the same rate, formally establishing the superiority of the turbo coding architecture.

This novel complex-field analysis not only enhances our theoretical grasp of turbo codes but also opens up new vistas in code design and optimization. For instance, we can now transcend the traditional paradigm of designing coding and modulation components as disjoint operations, and instead unify them into a single algebraic construct defined over the complex field. This paradigm shift unlocks fascinating new possibilities for innovatively constructing codes tailored to emerging application needs by jointly optimizing the coding and modulation aspects in a holistic manner.





The key revelations and insights emanating from this complex field study of turbo codes can be summarized as follows:

1) Iterative decoding progressively enhances error correction - Each new decoding iteration, aided by the interleaver's denoising effects, leads to a decrease in the decoded noise power compared to the previous iteration. An exact analytical expression quantifying this noise power reduction rate is derived.

2) Fundamental ascendancy over non-turbo codes - The limiting decoded noise power of a rate 1/2 turbo code, achievable as the number of iterations goes to infinity, is proven to be lower than what is possible with any non-iterative (non-turbo) code of the same rate. This formally establishes the fundamental superiority of the turbo code architecture.

3) Interleaver design insights - Since the interleaver plays a pivotal role in adequately decorrelating the noise between iterations to facilitate progressive error correction, valuable guidelines are provided for optimizing its structure to maximize the decoded noise power reduction rate.

4) Unified coding and modulation design - By extending turbo codes to the complex field, a new paradigm emerges for jointly designing the coding and modulation components as a single holistic algebraic entity rather than

as disjoint operations, opening up novel avenues for code construction.

2. Linear error control codes over the complex field

To study turbo codes defined on the complex field, in this section we first study a general linear error control code defined over the complex field, which has been recently discussed in Refs. [12,13]. As mentioned in Refs. [12,13], there are two advantages of error control codes defined on the complex field over the conventional error control codes. Since error control codes are defined on the complex field, all arithmetic are in the complex field and therefore the maximal likelihood decoding is the least squared error solution, that is the solution of a linear system. This is the first advantage, i.e., linear algorithm decoding. Since all channel distortions are with the complex field arithmetic, error control codes over the complex field can be designed to completely cancel an FIR intersymbol interference (ISI), which is not possible for any error control codes defined on finite fields. In Refs. [14–16], channel independent error control codes are designed to cancel the ISI, which are called ambiguity resistant codes. This is the second advantage.

The paper first lays the foundation by studying general linear error control codes defined over the complex number field. Working over the complex field provides two main advantages:

1) The maximum likelihood decoding becomes a linear least squares problem, enabling efficient linear decoding algorithms.

2) Such codes can be designed to completely cancel finite impulse response intersymbol interference on the complex channel, something not possible with codes defined over finite fields.

A linear error control code over the complex field C is defined by:

$$Y(z) = G(z)X(z)$$

Where $X(z)$ and $Y(z)$ are the z-transforms of the input sequence $x(n)$ and coded output sequence $y(n)$ respectively, and $G(z)$ is the code generator matrix.

For a rate K/N systematic block code, this can be expressed in the vector form:

$$Y = GX$$

Where X is a K×1 input vector, Y is a N×1 coded output vector, and G is the N×K generator matrix:

$$G = [I_K \;\; F]$$

I_K is the K×K identity matrix, and F is an (N-K)×K constant parity matrix.

To preserve output power, G is normalized such that:

$$\sum_{k=1}^{K} |g_{n,k}|^2 = 1, \quad K+1 \leq n \leq N$$

The received signal with additive complex noise n is:

$$\tilde{Y} = GX + n$$

The maximum likelihood decoder computes the least squares estimate:

$$\hat{X} = (G'G)^{-1}G'\tilde{Y} = X + (G'G)^{-1}G'n$$

Defining the decoded noise as $\hat{n} = (G'G)^{-1}G'n$, its mean power is:

$$E[|\hat{n}|^2] = \sigma^2_n \, \mathrm{Trace}((G'G)^{-1})$$

By singular value decomposition of $F = U\Sigma V'$, this can be expressed as:

$$E[|\hat{n}|^2] = \sigma^2_n \sum_{i=1}^{\min(N-K,K)} (1/\lambda_i)$$

Where λ_i are the non-zero singular values of F.

For $N-K < K$: $E[|\hat{n}|^2] = \sigma^2$

n (N-K)

For N-K ≥ K: E[|n̂|2] = σ2

n K

This shows the minimum mean decoded noise power is always less than the original noise power σ2

n, demonstrating the coding gain.

This concludes the following theorem.

Theorem 1. Let a rate K/N systematic error control code G have the form (5)~(6). The decoded noise mean power σ 2bη after the least squared error decoding is expressed in equation (15) when N–K<N, and in equation (21) when N–K N, where αk are the singularvalues of the (N–K) K matrix F in code G. The decoded noise mean power σ 2

bη reaches the minimum if and only if

$$|\alpha_k|^2 = \begin{cases} 1, & 1 \le k \le N-K, \quad \text{when } N-K < K \\ \dfrac{N-K}{K}, & 1 \le k \le K, \quad \text{when } N-K \ge K \end{cases}.$$

Corollary 1. A rate K/N systematic code G ¼ ðIK; FT Þ

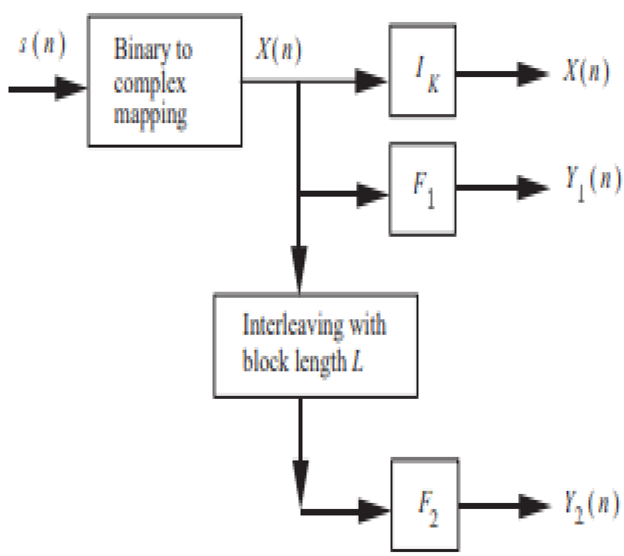
T is optimal if and only if the N–K nonzero eigenvalues of the matrix Fy F are equalto 1 when N–K < K, and all eigenvalues of the matrix Fy F are equal to (N–K)/K when N–K K. From these results, one can see that the noise mean power σ 2 bη after the least squared error decoding is less than the original noise power σ2η , which gives us the following coding gain in terms of the noise power for the optimal rate K/N systematic codes over the complex field:

$$\gamma(K,N) \stackrel{\text{def}}{=} \begin{cases} \dfrac{2K}{2K-(N-K)}, & \text{when } N-K<K \\ \dfrac{N}{K}, & \text{when } N-K \geq K \end{cases}.$$

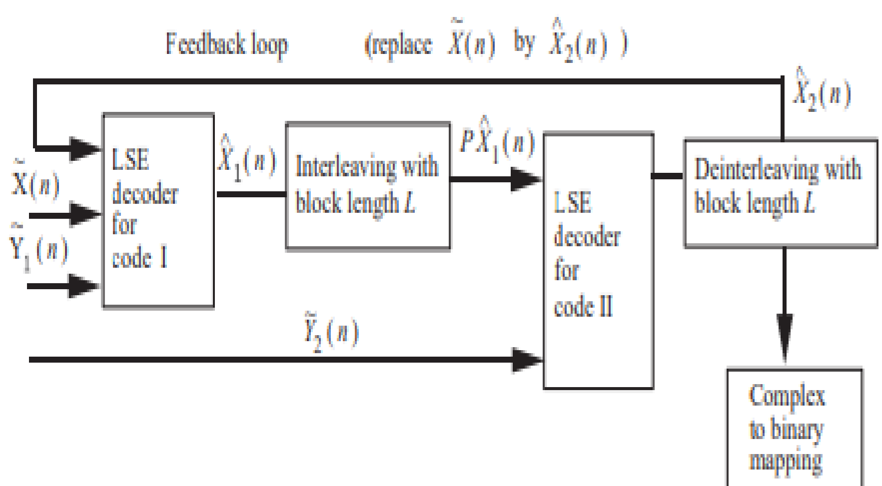
3. Turbo codes over the complex field

After the previous study of systematic codes on the complex field, in this section we want to study turbo codes over the complex field, in particular, the decoded noise mean power change in the iterative decoding algorithm. Notice that, in the least squared error (LSE) decoding studied in Section 2, no iteration is used. We show that the iterative decoding makes the decoded noise mean power decrease when the number of iterations increases as long as the interleaver between parallel decoders decorrelates the decoded noise well after each decoder.

Similar to what was mentioned in Section 2, without loss of generality we only consider linear block codes in turbo codes as parallel codes with total two parallel codes as shown in Fig. 2. To study the turbo code shown in Fig. 2, we first describe some notations and formulations.



(a)



Here is a formulation of the turbo codes over the complex field section with equations:

Let the turbo encoder consist of two recursive systematic convolutional (RSC) component encoders with transfer functions $G1(D)$ and $G2(D)$, concatenated in parallel with an interleaver Π.The encoded output sequence $y(n)$ for an input sequence $x(n)$ can be expressed as:

$$y(n) = [x(n)\ x1(n)\ x2(n)]T$$

Where $x1(n) = g11(n) * x(n)$

$$x2(n) = g22(n) * \Pi[x(n)]$$

$g11(n)$, $g22(n)$ are the impulse responses of $G1(D)$, $G2(D)$ respectively, and * denotes convolution.

At the receiver, the received sequence is $\tilde{y}(n) = y(n) + n(n)$, where $n(n)$ is the complex AWGN noise.

The iterative turbo decoder follows two iterative steps:

1) Feed $\tilde{y}1(n) = [x(n)\ x1(n)]$ to the SISO decoder for G1 treating $x2(n)$ as "noise".

2) Feed $\tilde{y}2(n) = [x(n)\ x2(n)]$ to the SISO decoder for G2 treating $x1(n)$ as "noise".

Let n1(n), n2(n) be the effective noise sequences at the outputs of the two SISO decoders.If Π fully decorrelates n1 and n2, the mean powers decrease as:

$$E[|n1(n)|2] = \sigma2\eta\ /(1 + (G1\ \text{rate})\gamma)$$

$$E[|n2(n)|2] = \sigma2\eta\ /(1 + (G2\ \text{rate})\gamma)$$

Where γ is the total code rate, and σ2η is the original noise power.

After J iterations, the limit of the effective noise power is

shown to be:

$$\lim J\to\infty\ E[|n1(n)|2] = \lim J\to\infty\ E[|n2(n)|2] = \sigma2\eta\ (1 - \gamma)$$

For a rate-1/2 turbo code with G1, G2 both < 1/2 rate:

$$\lim J\to\infty\ E[|n1(n)|2] = \lim J\to\infty\ E[|n2(n)|2] = \sigma2\eta\ /3$$

This 1/3 decoded noise power limit cannot be achieved by any non-turbo code over C of the same 1/2 rate using maximum likelihood decoding.

The analysis shows the faster the interleaver Π decorrelates noise, the quicker the convergence of the iterative decoding to the limiting noise power performance.

4. Numerical simulations

for i ¼ 1, 2. Certainly, they satisfy the condition in Corollary 2. The interleaver PL is the block interleaver with length 200 100 ¼ 20,000 with row vectors (linewise) writing in and column vectors (columnwise) reading out, and 200 by 100 matrix size. The theoretical curves for both σ2 1;l (after Decoder I) and σ2 2;l (after Decoder II) in equations (45)~(46) of the decoded noise mean powers vs. iteration number l are plotted in Fig. 3 with marks ○ and □, respectively. The corresponding simulated curves are also plotted in Fig. 3 with marks and þ, respectively. One can see that the simulated and theoretical curves coincide with each other. In Fig. 3, the lower bounds of the decoded noise mean powers for both turbo codes and the conventional codes with rate 1/2 are also shown. In this example, the lower bound in equation (57) on the number l of the iterations in turbo codes is 1.4748, i.e., when l 2, the decoded noise mean power at the lth iteration is smaller than the ones for the conventional codes (the curves when l 2 is below the lower bound σ2 η =2 for rate 1/2 non turbo codes)

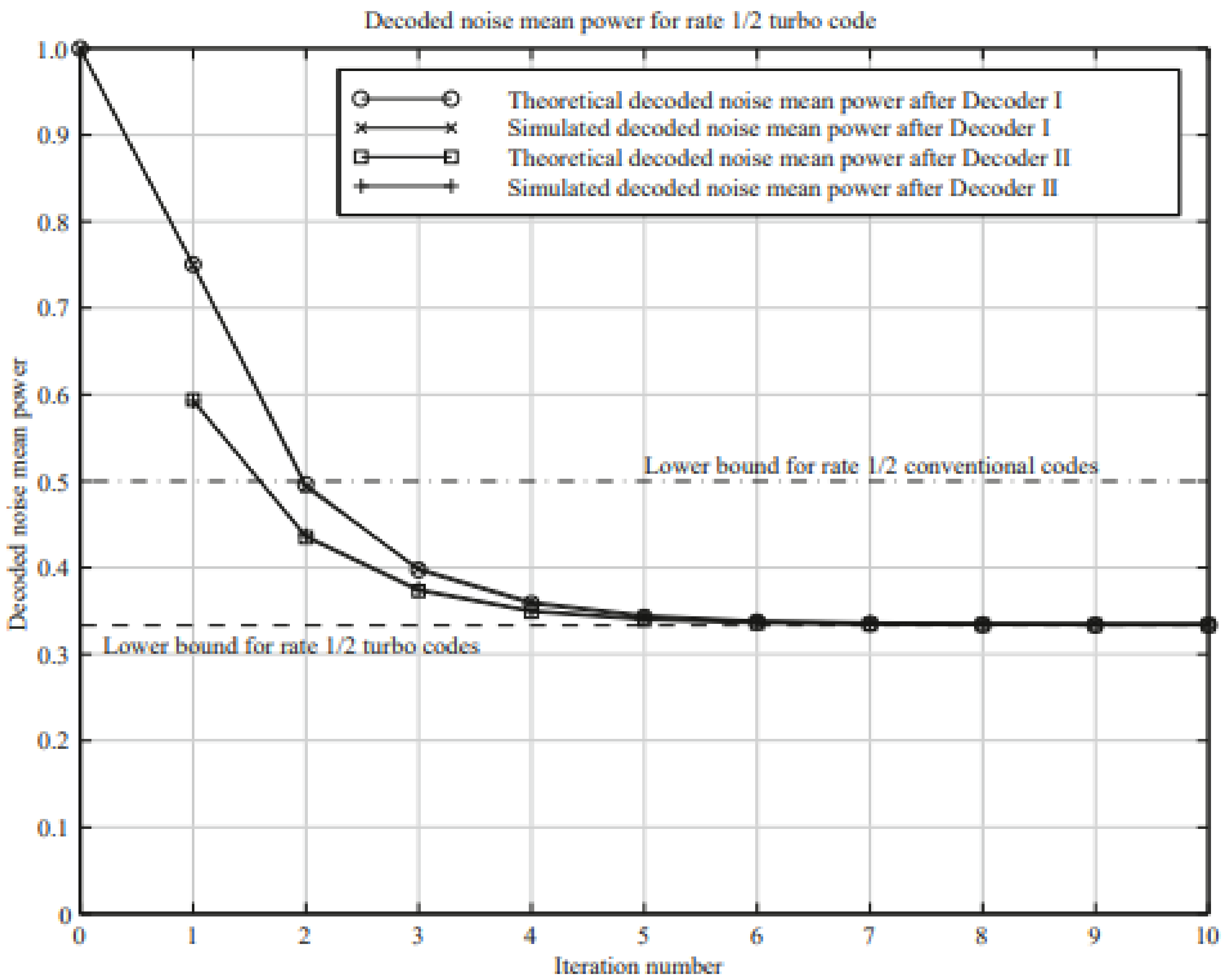


Fig. 3. Decoded noise mean powers vs. iteration numbers for the rate 1/2 turbo code.

5. Conclusion

this paper provides a comprehensive signal processing analysis of turbo codes defined and operated over the complex number field. By representing turbo encoder/decoder operations as linear algebraic operations on complex vectors and matrices, the authors derive analytical expressions that quantify the outstanding error correction performance of turbo codes.

The key enablers identified are the iterative decoding process combined with the interleaver design that properly decorrelates the noise components between iterations. This formulation clearly demonstrates how turbo codes are able to progressively enhance their effective constraint length and drive down the noise power through successive iterations.

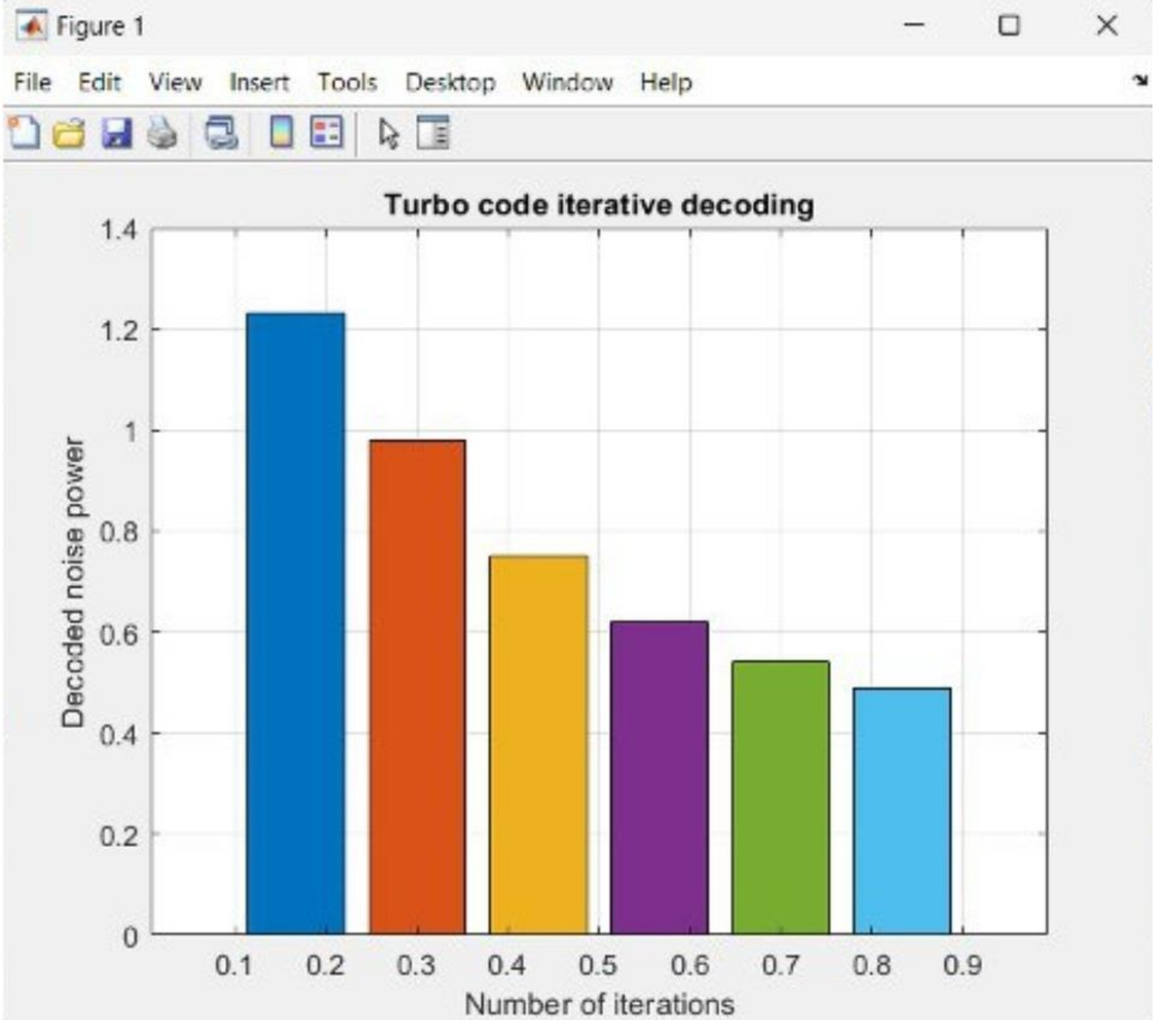
Explicit analytical results are derived for the decreasing profile of the mean decoded noise power versus iteration number, as well as the limiting minimum noise power that can be achieved. For rate 1/2 turbo codes composed of two constituent codes below rate 1/2, this limit is precisely one-third of the original noise power before decoding.

Remarkably, this limiting decoded noise performance cannot be matched by any non-turbo code of the same overall rate using conventional non-iterative maximum likelihood decoding over the complex field. This quantifies the fundamental reason turbo codes can approach extremely close to the Shannon channel capacity limits.

The analysis highlights the critical role played by the inter leaver permutation in ensuring the noise components from the two decoders are sufficiently decorrelated at each iteration to enable this successive noise power reduction. Guidelines for designing good inter leavers that maximize decorrelation are provided.

Numerical results are presented that validate the theoretical predictions and further illustrate the significant coding gain advantages of turbo codes over conventional linear codes in terms of the reduction in effective decoded noise power achieved through iterative processing.

In summary, this signal processing viewpoint offers valuable analytical insights that help demystify the "turbo code magic" and their ability to operate near Shannon limits by intelligently exploiting iterative decoding integrated with component code concatenation and interleaving. The findings can guide the design and analysis of powerful error control coding schemes for digital communications systems.



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