PDF 13 to 16 Integral Formulas

Mahdi Haghverdi

January 16, 2023

Contents

Chapter 1

PDF 13

1.1 Line Integral

For some scalar field $f\colon U \to \mathbb{R}$ where U R n $U \subseteq \mathbb{R}^n$, the line integral along a *piecewise smooth curve* $C \subset U$ is defined as:

$$\int\limits_C f \,\mathrm{d}s = \int_a^b f\big(\vec{r}(t)\big)\, |r'(t)| \,\mathrm{d}t \tag{1.1}$$

where $\vec{r}\colon [a, b] \to C$ is an arbitrary bijective parametrization of the curve C such that $\vec{r}(a)$ and $\vec{r}(b)$ give the endpoints of C and $a < b$.

1.2 Physical Aspect

$$m = \int\limits_C \delta(x, y) \,\mathrm{d}s \tag{1.2}$$

$$(\bar{x}, \bar{y}) = \begin{cases} \bar{x} = \frac{1}{m} \int\limits_C x\, \delta(x, y)\ \mathrm{d}s \\ \bar{y} = \frac{1}{m} \int\limits_C y\, \delta(x, y)\ \mathrm{d}s \end{cases} \tag{1.3}$$

1.3 Line Integral of a Vector Field

For a vector field $\mathbf{F}\colon U \subseteq \mathbf{R}^n \to \mathbf{R}^n$, the line integral along a piecewise smooth curve $C \subset U$, in the direction of $\vec{r}$, is defined as:

$$\int\limits_C \mathbf{F} \cdot \mathrm{d}\mathbf{r} = \int_a^b \mathbf{F}\big(\mathbf{r}(t)\big) \cdot \mathbf{r}'(t)\,\mathrm{d}t \tag{1.4}$$

where $\cdot$ is the dot product, and $\mathbf{r}\colon [a,b] \to C$ is a bijective parametrization of the curve C such that $\mathbf{r}(a)$ and $\mathbf{r}(b)$ give the endpoints of C.

Line integral of a vector field, also can be shown as below:

$$\int\limits_C \mathbf{F} \cdot \mathrm{d}\mathbf{r} = \int\limits_C \mathtt{p}\,\mathrm{d}x + \mathtt{q}\,\mathrm{d}y + \mathtt{r}\,\mathrm{d}z \tag{1.5}$$

Chapter 2

PDF 14

2.1 Gradient Theorem

The gradient theorem, also known as the fundamental theorem of calculus for line integrals, says that a line integral through a gradient field can be evaluated by evaluating the original scalar field at the endpoints of the curve. The theorem is a generalization of the second fundamental theorem of calculus to any curve in a plane or space (generally n-dimensional) rather than just the real line.

$$\mathbf{F} = \nabla f \Rightarrow \begin{cases} P = \frac{\partial f}{\partial x} \\ Q = \frac{\partial f}{\partial y} \\ R = \frac{\partial f}{\partial z} \end{cases} \tag{2.1}$$

For $\mathbf{F}\colon U \subseteq \mathbf{R}^n \to \mathbf{R}$ as a differentiable function and any continuous curve in U which starts at a point a and ends at a point b, then

$$\int\limits_C \mathbf{F} \cdot \mathrm{d}\mathbf{r} = \int_a^b \nabla f \cdot \mathrm{d}\mathbf{r} = f\big(\mathbf{r}(b)\big) - f\big(\mathbf{r}(a)\big) \tag{2.2}$$

where ∇f denotes the gradient vector field of $\mathbf{F}$.

To check if a vector is a gradient vector:

$$\frac{\partial p}{\partial y} = \frac{\partial q}{\partial x} \qquad \frac{\partial p}{\partial z} = \frac{\partial r}{\partial x} \qquad \frac{\partial q}{\partial z} = \frac{\partial r}{\partial y} \tag{2.3}$$

2.2 Green's Theorem

If C be a positively oriented, piecewise smooth, simple closed curve in a plane, and let D be the region bounded by C. If P and Q are functions of (x, y) defined on an open region containing D and have continuous partial derivatives there, then

$$\oint\limits_C P\,\mathrm{d}x + Q\,\mathrm{d}y = \iint\limits_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) \mathrm{d}A \tag{2.4}$$

where the path of integration along C is anticlockwise.

If curve C is divided into pieces like $C = C_1 \bigcup C_2$, then

$$\begin{aligned} \int\limits_C P\,\mathrm{d}x\ +\ Q\,\mathrm{d}y &= \int\limits_{C_1} P\,\mathrm{d}x\ +\ Q\,\mathrm{d}y + \int\limits_{C_2} P\,\mathrm{d}x\ +\ Q\,\mathrm{d}y \\ &= \iint\limits_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) \mathrm{d}A \end{aligned} \tag{2.5}$$

2.3 Area Calculations

$$A = \oint\limits_C x\,\mathrm{d}y = -\oint\limits_C y\,\mathrm{d}x = \frac{1}{2}\oint\limits_C x\,\mathrm{d}y - y\,\mathrm{d}x \tag{2.6}$$

Chapter 3

PDF 15

3.1 Piecewise Smooth Parametrized Curve

$$\vec{r}(u,v) = \big(x(u,v),\, y(u,v),\, z(u,v)\big), \quad (u,v) \in R \tag{3.1}$$

$$\vec{r}_u = (\frac{\partial x}{\partial u}, \frac{\partial y}{\partial u}, \frac{\partial z}{\partial u}) \qquad \vec{r}_v = (\frac{\partial x}{\partial v}, \frac{\partial y}{\partial v}, \frac{\partial z}{\partial v}) \tag{3.2}$$

3.1.1 Sample

If $z = f(x,y)$, the smooth parametrized curve will be:

$$\vec{r}(u,v) = (u, v, f(u,v)) \tag{3.3}$$

Partial Derivatives are defined as:

$$\begin{aligned} \vec{r}_u &= (\frac{\partial u}{\partial u}, \frac{\partial v}{\partial u}, \frac{\partial f(u,v)}{\partial u}) \\ &= (1, 0, \frac{\partial f}{\partial u}) = \vec{i} + \frac{\partial f}{\partial u}\vec{k} \end{aligned} \tag{3.4}$$

$$\begin{aligned} \vec{r}_v &= (\frac{\partial u}{\partial v}, \frac{\partial v}{\partial v}, \frac{\partial f(u,v)}{\partial v}) \\ &= (0, 1, \frac{\partial f}{\partial v}) = \vec{j} + \frac{\partial f}{\partial v}\vec{k} \end{aligned} \tag{3.5}$$

Cross Product of the partial derivatives:

$$\begin{aligned}\vec{r}_u \times \vec{r}_v &= (\frac{-\partial f}{\partial u}, \frac{-\partial f}{\partial v}, 1) \\ &= \frac{-\partial f}{\partial u}\vec{i} + (\frac{-\partial f}{\partial v}\vec{j}) + \vec{k} \neq 0\end{aligned} \tag{3.6}$$

3.2 Area Calculation

$$A(\mathtt{S}) = \iint\limits_R |\vec{r}_u \times \vec{r}_v| \, \mathrm{d}A \tag{3.7}$$

3.2.1 Theorem

If $z = f(x, y)$, then:

$$A(\mathtt{S}) = \iint\limits_R \sqrt{(\frac{\partial f}{\partial x})^2 + (\frac{\partial f}{\partial y})^2 + 1} \, \mathrm{d}A \tag{3.8}$$

3.3 Vector Integral

$$\iint\limits_S \mathbf{G}(x, y, z) \, \mathrm{d}\sigma = \iint\limits_R \mathbf{G}\big(\vec{r}(u, v)\big) \, |\vec{r}_u \times \vec{r}_v| \, \mathrm{d}A \quad \vec{r}(u, v) \in R \tag{3.9}$$

3.3.1 special cases

1. If $S : \vec{r}(u, v) = \big(f(u, v), g(u, v), h(u, v)\big)$, then:

$$\iint\limits_S \mathbf{G}(x, y, z) \, \mathrm{d}\sigma = \iint\limits_S \mathbf{G}\big(f(u, v), g(u, v), h(u, v)\big) \, |\vec{r}_u \times \vec{r}_v| \, \mathrm{d}A \tag{3.10}$$

2. If $z = f(x, y)$, then:

$$\iint\limits_S \mathbf{G}(x, y, z) \, \mathrm{d}\sigma = \iint\limits_S \mathbf{G}\big(x, y, f(x, y)\big)\sqrt{(\frac{\partial f}{\partial x})^2 + (\frac{\partial f}{\partial y})^2 + 1} \, \mathrm{d}x\mathrm{d}y \tag{3.11}$$

Solved Exercise (use it as a template to solve questions)

If $z=\sqrt{x^2+y^2}$; $(0\le z\le 1)$, then $\iint\limits_S x^2\,\mathrm{d}\sigma = ?$

Solution:

$$\begin{cases} x = r\cos\theta \\ y = r\sin\theta \\ z = z \end{cases} \Rightarrow \begin{cases} x = r\cos\theta \\ y = r\sin\theta \\ z = r \end{cases} \quad (0\le r\le 1)\,;(0\le\theta\le 2\pi)$$

$$\begin{aligned} &\vec{r}\,(r,\theta) = (r\cos\theta, r\sin\theta, r) \\ \Rightarrow \qquad &\vec{r}_r = (\cos\theta,\sin\theta, r) \\ &\vec{r}_\theta = (-r\sin\theta, r\cos\theta, 0) \end{aligned}$$

$$\Rightarrow \vec{r}_r\times\vec{r}_\theta = \begin{vmatrix} i & j & k \\ \cos\theta & \sin\theta & r \\ -r\sin\theta & r\cos\theta & 0 \end{vmatrix} = (-r\cos\theta, -r\sin\theta, r)$$

$$\Rightarrow |\vec{r}_r\times\vec{r}_\theta| = \sqrt{\underbrace{(r^2\cos^2\theta)+(r^2\sin^2\theta)}_{=r^2}+r^2} = \sqrt{2r^2} = r\sqrt{2}$$

$$\begin{aligned} \Rightarrow \iint\limits_{[0,1]\times[0,2\pi]} r^2\cos^2\theta\; r\sqrt{2}\;\mathrm{d}r\mathrm{d}\theta &= \sqrt{2}\int_0^{2\pi}\int_0^1 r^3\cos^2\theta\;\mathrm{d}r\mathrm{d}\theta \\ &= \frac{\sqrt{2}}{4}\int_0^{2\pi}\cos^2\theta\;\mathrm{d}\theta \\ &= \frac{\sqrt{2}}{2\times 4}\int_0^{2\pi} 1+\cos 2\theta\;\mathrm{d}\theta \\ &= \frac{\sqrt{2}}{8}\left(\theta+\frac{1}{2}\sin 2\theta\right)\Big|_0^{2\pi} \\ &= \frac{2\pi\times\sqrt{2}}{8} = \frac{\pi\sqrt{2}}{4} \end{aligned}$$

Chapter 4

PDF 16

4.1 Transforms

Del (nabla)

Symbol:

$$\nabla \tag{4.1}$$

Definition:

$$\nabla f \coloneqq \frac{\partial f}{\partial x}\vec{i} + \frac{\partial f}{\partial y}\vec{j} + \frac{\partial f}{\partial z}\vec{k} \tag{4.2}$$

$$\nabla \coloneqq \frac{\partial}{\partial x}\vec{i} + \frac{\partial}{\partial y}\vec{j} + \frac{\partial}{\partial z}\vec{k} \tag{4.3}$$

Divergence

Symbol:

$$\mathrm{div} \tag{4.4}$$

Definition:
If $\mathbf{F}$ be $\mathbf{F}(x, y, z) = P(x, y, z)\vec{i} + Q(x, y, z)\vec{j} + R(x, y, z)\vec{k}$, then:

$$\mathrm{div}\ \mathbf{F} \coloneqq \frac{\partial P}{\partial x}\vec{i} + \frac{\partial Q}{\partial y}\vec{j} + \frac{\partial R}{\partial z}\vec{k} \tag{4.5}$$

$$\mathrm{div}\ \mathbf{F} \coloneqq \nabla \cdot \mathbf{F} \tag{4.6}$$

curl

Symbol:

$$\text{curl} \tag{4.7}$$

Definition:
If $\mathbf{F}$ be $\mathbf{F}(x,y,z) = P(x,y,z)\vec{i} + Q(x,y,z)\vec{j} + R(x,y,z)\vec{k}$, then:

$$\text{curl } \mathbf{F} := (\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z})\vec{i} + (\frac{\partial P}{\partial z} - \frac{\partial R}{\partial x})\vec{j} + (\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y})\vec{k} \tag{4.8}$$

$$\text{curl } \mathbf{F} := \nabla \cdot \mathbf{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ P & Q & R \end{vmatrix} \tag{4.9}$$

4.2 Theorems

1. $\text{curl}(\nabla f) = 0$
2. $\text{div}(\text{curl}\,\mathbf{F}) = 0$
3. (a) If $\lambda \in \mathbb{R}$, then: $\nabla(f + \lambda g) = \nabla f + \lambda \nabla g$
   (b) $\text{curl}\,(\mathbf{F} + \lambda\mathbf{G}) = \text{curl}\,\mathbf{F} + \lambda\text{curl}\,\mathbf{G}$
   (c) $\text{div}\,(\mathbf{F} + \lambda\mathbf{G}) = \text{div}\,\mathbf{F} + \lambda\text{div}\,\mathbf{G}$
4. $\text{div}\,(\mathbf{F} \times \mathbf{G}) = \mathbf{G} \cdot \text{curl}\,\mathbf{F} - \mathbf{F} \cdot \text{curl}\,\mathbf{G}$
5. $\text{div}\,(f\mathbf{F}) = f\,\text{div}\,\mathbf{F} + \mathbf{F} \cdot \nabla f$
6. $\text{curl}\,(f\mathbf{F}) = f\,\text{curl}\,\mathbf{F} + \nabla f \times \mathbf{F}$

4.3 Surface Integrals of Vector Fields

In mathematics, particularly multivariable calculus, a surface integral is a generalization of multiple integrals to integration over surfaces. It can be thought of as the double integral analogue of the line integral. Given a surface, one may integrate a scalar field (that is, a function of position which returns a scalar as a value) over the surface, or a vector field (that is, a function which returns a vector as value). If a region R is not flat, then it is called a surface as shown in the illustration.

Surface integrals have applications in physics, particularly with the theories of classical electromagnetism.

Consider a vector field $\mathbf{F}$ on a surface S, we define the surface integral of vector field as:

$$\iint\limits_S \mathbf{F} \cdot \vec{n} \ \mathrm{d}\sigma \tag{4.10}$$

$$\begin{aligned}
\iint\limits_S \mathbf{F} \cdot \vec{n} \ \mathrm{d}\sigma &= \iint\limits_S \mathbf{F} \cdot \frac{\vec{r}_u \times \vec{r}_v}{|\vec{r}_u \times \vec{r}_v|} \ \mathrm{d}\sigma \\
&= \iint\limits_D \left[\mathbf{F} \cdot \frac{\vec{r}_u \times \vec{r}_v}{|\vec{r}_u \times \vec{r}_v|}\right] |\vec{r}_u \times \vec{r}_v| \ \mathrm{d}A \\
&= \iint\limits_D \mathbf{F} \cdot (\vec{r}_u \times \vec{r}_v) \ \mathrm{d}A
\end{aligned}$$

$$\Rightarrow \iint\limits_S \mathbf{F} \cdot \vec{n} \ \mathrm{d}\sigma = \iint\limits_D \mathbf{F} \cdot (\vec{r}_u \times \vec{r}_v) \ \mathrm{d}A \tag{4.11}$$

If $z = g(x, y)$; $(x, y) \in R$ and $\mathbf{F} = (P, Q, R)$, then:

$$\iint\limits_S \mathbf{F} \cdot \vec{n} \ \mathrm{d}\sigma = \iint\limits_R \left(-P\frac{\partial g}{\partial x} - Q\frac{\partial g}{\partial y} + R\right) \ \mathrm{d}A \tag{4.12}$$

4.4 Divergence Theorem

Suppose D is a subset of $\mathbb{R}^n$ (in the case of $n = 3$, D represents a volume in three-dimensional space) which is compact and has a piecewise smooth boundary S (also indicated with $\partial V = S$). If $\mathbf{F}$ is a continuously differentiable vector field defined on a neighborhood of D, then

$$\iint\limits_S \mathbf{F} \cdot \vec{n} \ \mathrm{d}\sigma = \iiint\limits_D \mathrm{div}\ \mathbf{F} \ \mathrm{d}V \tag{4.13}$$

4.5 Stokes' Theorem

Stokes's theorem, also known as the Kelvin–Stokes theorem after Lord Kelvin and George Stokes, the fundamental theorem for curls or simply the curl theorem, is a theorem in vector calculus on $\mathbf{R}^3$. Given a vector field, the theorem relates the integral of the curl of the vector field over some surface, to the line integral of the vector field around the boundary of the surface. The classical Stokes' theorem can be stated in one sentence: **The line integral of a vector field over a loop is equal to the flux of its curl through the enclosed surface.**

Stokes' theorem is a special case of the generalized Stokes' theorem. In particular, a vector field on $\mathbf{R}^3$ can be considered as a 1-form in which case its curl is its exterior derivative, a 2-form.

Let S be a smooth oriented surface in $\mathbf{R}^3$ with boundary ∂S. If a vector field $\mathbf{F}(x, y, z) = (P(x, y, z), Q(x, y, z), R(x, y, z))$ is defined and has continuous first order partial derivatives in a region containing S, then

$$\int\limits_{\partial S} \mathbf{F} \cdot \mathrm{d}\mathbf{r} = \iint\limits_{S} (\operatorname{curl} \mathbf{F}) \cdot \mathbf{n} \, \mathrm{d}\sigma \tag{4.14}$$