

PDF 13 to 16 Integral Formulas

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Chapter 1

PDF 13

1.1 Line Integral

$$\int_C f \, ds = \int_a^b f(\vec{r}(t)) |r'(t)| \, dt \quad (1.1)$$

1.2 Physical Aspect

$$m = \int_C \delta(x, y) \, ds \quad (1.2)$$

$$(\bar{x}, \bar{y}) = \begin{cases} \bar{x} = \frac{1}{m} \int_C x \delta(x, y) \, ds \\ \bar{y} = \frac{1}{m} \int_C y \delta(x, y) \, ds \end{cases} \quad (1.3)$$

1.3 Line Integral of a Vector Field

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \int_a^b \mathbf{F}(\mathbf{r}(t)) \cdot \mathbf{r}'(t) \, dt \quad (1.4)$$

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \int_C p \, dx + q \, dy + r \, dz \quad (1.5)$$

Chapter 2

PDF 14

2.1 Gradient Theorem

$$\mathbf{F} = \nabla f \Rightarrow \begin{cases} P = \frac{\partial f}{\partial x} \\ Q = \frac{\partial f}{\partial y} \\ R = \frac{\partial f}{\partial z} \end{cases} \quad (2.1)$$

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \int_a^b \nabla f \cdot d\mathbf{r} = f(\mathbf{r}(b)) - f(\mathbf{r}(a)) \quad (2.2)$$

$$\frac{\partial p}{\partial y} = \frac{\partial q}{\partial x} \quad \frac{\partial p}{\partial z} = \frac{\partial r}{\partial x} \quad \frac{\partial q}{\partial z} = \frac{\partial r}{\partial y} \quad (2.3)$$

2.2 Green's Theorem

$$\oint_C P dx + Q dy = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dA \quad (2.4)$$

$$\begin{aligned} \int_C P dx + Q dy &= \int_{C_1} P dx + Q dy + \int_{C_2} P dx + Q dy \\ &= \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dA \end{aligned} \quad (2.5)$$

2.3 Area Calculations

$$A = \oint_C x \, dy = - \oint_C y \, dx = \frac{1}{2} \oint_C x \, dy - y \, dx \quad (2.6)$$

Chapter 3

PDF 15

3.1 Piecewise Smooth Parametrized Curve

$$\vec{r}(u, v) = (x(u, v), y(u, v), z(u, v)), \quad (u, v) \in R \quad (3.1)$$

$$\vec{r}_u = \left(\frac{\partial x}{\partial u}, \frac{\partial y}{\partial u}, \frac{\partial z}{\partial u} \right) \quad \vec{r}_v = \left(\frac{\partial x}{\partial v}, \frac{\partial y}{\partial v}, \frac{\partial z}{\partial v} \right) \quad (3.2)$$

3.1.1 Sample

If $z = f(x, y)$, the smooth parametrized curve will be:

$$\vec{r}(u, v) = (u, v, f(u, v)) \quad (3.3)$$

Partial Derivatives are defined as:

$$\begin{aligned} \vec{r}_u &= \left(\frac{\partial u}{\partial u}, \frac{\partial v}{\partial u}, \frac{\partial f(u, v)}{\partial u} \right) \\ &= (1, 0, \frac{\partial f}{\partial u}) = \vec{i} + \frac{\partial f}{\partial u} \vec{k} \end{aligned} \quad (3.4)$$

$$\begin{aligned} \vec{r}_v &= \left(\frac{\partial u}{\partial v}, \frac{\partial v}{\partial v}, \frac{\partial f(u, v)}{\partial v} \right) \\ &= (0, 1, \frac{\partial f}{\partial v}) = \vec{j} + \frac{\partial f}{\partial v} \vec{k} \end{aligned} \quad (3.5)$$

Cross Product of the partial derivatives:

$$\begin{aligned}\vec{r}_u \times \vec{r}_v &= \left(\frac{-\partial f}{\partial u}, \frac{-\partial f}{\partial v}, 1 \right) \\ &= \frac{-\partial f}{\partial u} \vec{i} + \left(\frac{-\partial f}{\partial v} \vec{j} \right) + \vec{k} \neq 0\end{aligned}\tag{3.6}$$

3.2 Area Calculation

$$A(S) = \iint_R |\vec{r}_u \times \vec{r}_v| \, dA\tag{3.7}$$

3.2.1 Theorem

If $z = f(x, y)$, then:

$$A(S) = \iint_R \sqrt{\left(\frac{\partial f}{\partial x}\right)^2 + \left(\frac{\partial f}{\partial y}\right)^2 + 1} \, dA\tag{3.8}$$

3.3 Vector Integral

$$\iint_S \mathbf{G}(x, y, z) \, d\sigma = \iint_R \mathbf{G}(\vec{r}(u, v)) |\vec{r}_u \times \vec{r}_v| \, dA \quad \vec{r}(u, v) \in R\tag{3.9}$$

3.3.1 special cases

1. If $S : \vec{r}(u, v) = (f(u, v), g(u, v), h(u, v))$, then:

$$\iint_S \mathbf{G}(x, y, z) \, d\sigma = \iint_S \mathbf{G}(f(u, v), g(u, v), h(u, v)) |\vec{r}_u \times \vec{r}_v| \, dA\tag{3.10}$$

2. If $z = f(x, y)$, then:

$$\iint_S \mathbf{G}(x, y, z) \, d\sigma = \iint_S \mathbf{G}(x, y, f(x, y)) \sqrt{\left(\frac{\partial f}{\partial x}\right)^2 + \left(\frac{\partial f}{\partial y}\right)^2 + 1} \, dx dy\tag{3.11}$$

Solved Exercise (use it as a template to solve questions)

If $z = \sqrt{x^2 + y^2}$; $(0 \leq z \leq 1)$, then $\iint_S x^2 d\sigma = ?$

Solution:

$$\begin{cases} x = r \cos \theta \\ y = r \sin \theta \\ z = z \end{cases} \Rightarrow \begin{cases} x = r \cos \theta \\ y = r \sin \theta \\ z = r \end{cases} \quad (0 \leq r \leq 1); (0 \leq \theta \leq 2\pi)$$

$$\vec{r}(r, \theta) = (r \cos \theta, r \sin \theta, r)$$

$$\Rightarrow \vec{r}_r = (\cos \theta, \sin \theta, r)$$

$$\vec{r}_\theta = (-r \sin \theta, r \cos \theta, 0)$$

$$\Rightarrow \vec{r}_r \times \vec{r}_\theta = \begin{vmatrix} i & j & k \\ \cos \theta & \sin \theta & r \\ -r \sin \theta & r \cos \theta & 0 \end{vmatrix} = (-r \cos \theta, -r \sin \theta, r)$$

$$\Rightarrow |\vec{r}_r \times \vec{r}_\theta| = \sqrt{\underbrace{(r^2 \cos^2 \theta) + (r^2 \sin^2 \theta)}_{=r^2} + r^2} = \sqrt{2r^2} = r\sqrt{2}$$

$$\Rightarrow \iint_{[0,1] \times [0,2\pi]} r^2 \cos^2 \theta \, r\sqrt{2} \, dr d\theta = \sqrt{2} \int_0^{2\pi} \int_0^1 r^3 \cos^2 \theta \, dr d\theta$$

$$\begin{aligned} &= \frac{\sqrt{2}}{4} \int_0^{2\pi} \cos^2 \theta \, d\theta \\ &= \frac{\sqrt{2}}{2 \times 4} \int_0^{2\pi} 1 + \cos 2\theta \, d\theta \\ &= \frac{\sqrt{2}}{8} \left(\theta + \frac{1}{2} \sin 2\theta \right) \Big|_0^{2\pi} \\ &= \frac{2\pi \times \sqrt{2}}{8} = \frac{\pi\sqrt{2}}{4} \end{aligned}$$

Chapter 4

PDF 16

4.1 Transforms

Del (nabla)

Symbol:

$$\nabla \quad (4.1)$$

Definition:

$$\nabla f := \frac{\partial f}{\partial x} \vec{i} + \frac{\partial f}{\partial y} \vec{j} + \frac{\partial f}{\partial z} \vec{k} \quad (4.2)$$

$$\nabla := \frac{\partial}{\partial x} \vec{i} + \frac{\partial}{\partial y} \vec{j} + \frac{\partial}{\partial z} \vec{k} \quad (4.3)$$

Divergence

Symbol:

$$\text{div} \quad (4.4)$$

Definition:

If \mathbf{F} be $\mathbf{F}(x, y, z) = P(x, y, z)\vec{i} + Q(x, y, z)\vec{j} + R(x, y, z)\vec{k}$, then:

$$\text{div } \mathbf{F} := \frac{\partial P}{\partial x} \vec{i} + \frac{\partial Q}{\partial y} \vec{j} + \frac{\partial R}{\partial z} \vec{k} \quad (4.5)$$

$$\text{div } \mathbf{F} := \nabla \cdot \mathbf{F} \quad (4.6)$$

curl

Symbol:

$$\text{curl} \tag{4.7}$$

Definition:

If \mathbf{F} be $\mathbf{F}(x, y, z) = P(x, y, z)\vec{i} + Q(x, y, z)\vec{j} + R(x, y, z)\vec{k}$, then:

$$\text{curl } \mathbf{F} := \left(\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z}\right)\vec{i} + \left(\frac{\partial P}{\partial z} - \frac{\partial R}{\partial x}\right)\vec{j} + \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)\vec{k} \tag{4.8}$$

$$\text{curl } \mathbf{F} := \nabla \cdot \mathbf{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ P & Q & R \end{vmatrix} \tag{4.9}$$

4.2 Theorems

1. $\text{curl}(\nabla f) = 0$
2. $\text{div}(\text{curl } \mathbf{F}) = 0$
3. (a) If $\lambda \in \mathbb{R}$, then: $\nabla(f + \lambda g) = \nabla f + \lambda \nabla g$
 (b) $\text{curl}(\mathbf{F} + \lambda \mathbf{G}) = \text{curl } \mathbf{F} + \lambda \text{curl } \mathbf{G}$
 (c) $\text{div}(\mathbf{F} + \lambda \mathbf{G}) = \text{div } \mathbf{F} + \lambda \text{div } \mathbf{G}$
4. $\text{div}(\mathbf{F} \times \mathbf{G}) = \mathbf{G} \cdot \text{curl } \mathbf{F} - \mathbf{F} \cdot \text{curl } \mathbf{G}$
5. $\text{div}(f\mathbf{F}) = f \text{div } \mathbf{F} + \mathbf{F} \cdot \nabla f$
6. $\text{curl}(f\mathbf{F}) = f \text{curl } \mathbf{F} + \nabla f \times \mathbf{F}$

4.3 Surface Integrals of Vector Fields

$$\iint_S \mathbf{F} \cdot \vec{n} \, d\sigma \tag{4.10}$$

$$\begin{aligned}
\iint_S \mathbf{F} \cdot \vec{n} \, d\sigma &= \iint_S \mathbf{F} \cdot \frac{\vec{r}_u \times \vec{r}_v}{|\vec{r}_u \times \vec{r}_v|} \, d\sigma \\
&= \iint_D \left[\mathbf{F} \cdot \frac{\vec{r}_u \times \vec{r}_v}{|\vec{r}_u \times \vec{r}_v|} \right] |\vec{r}_u \times \vec{r}_v| \, dA \\
&= \iint_D \mathbf{F} \cdot (\vec{r}_u \times \vec{r}_v) \, dA \\
\Rightarrow \iint_S \mathbf{F} \cdot \vec{n} \, d\sigma &= \iint_D \mathbf{F} \cdot (\vec{r}_u \times \vec{r}_v) \, dA
\end{aligned} \tag{4.11}$$

If $z = g(x, y)$; $(x, y) \in R$ and $\mathbf{F} = (P, Q, R)$, then:

$$\iint_S \mathbf{F} \cdot \vec{n} \, d\sigma = \iint_R \left(-P \frac{\partial g}{\partial x} - Q \frac{\partial g}{\partial y} + R \right) \, dA \tag{4.12}$$

4.4 Divergence Theorem

$$\iint_S \mathbf{F} \cdot \vec{n} \, d\sigma = \iiint_D \operatorname{div} \mathbf{F} \, dV \tag{4.13}$$

4.5 Stokes' Theorem

$$\int_{\partial S} \mathbf{F} \cdot d\mathbf{r} = \iint_S (\operatorname{curl} \mathbf{F}) \cdot \mathbf{n} \, d\sigma \tag{4.14}$$