PDF 13 to 16 Integral Formulas

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Chapter 1

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1.1 Line Integral

For some scalar field $f\colon U \to \mathbb{R}$ where U $\quad$ R n $U \subseteq \mathbb{R}^n$, the line integral along a *piecewise smooth curve* $C \subset U$ is defined as:

$$\int_C f \, \mathrm{d}s = \int_a^b f\big(\vec{r}(t)\big) \, |r'(t)| \, \mathrm{d}t \tag{1.1}$$

where $\vec{r}\colon [a,b] \to C$ is an arbitrary bijective parametrization of the curve C such that $\vec{r}(a)$ and $\vec{r}(b)$ give the endpoints of C and $a < b$.

1.2 Physical Aspect

$$m = \int_C \delta(x,y) \, \mathrm{d}s \tag{1.2}$$

$$(\bar{x}, \bar{y}) = \begin{cases} \bar{x} = \frac{1}{m} \int_C x \, \delta(x,y) \, \mathrm{d}s \\ \bar{y} = \frac{1}{m} \int_C y \, \delta(x,y) \, \mathrm{d}s \end{cases} \tag{1.3}$$

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1.3   Line Integral of a Vector Field

For a vector field $\mathbf{F}\colon U \subseteq \mathbf{R}^n \to \mathbf{R}^n$, the line integral along a piecewise smooth curve $C \subset U$, in the direction of $\vec{r}$, is defined as:

$$\int_C \mathbf{F} \cdot \mathrm{d}\mathbf{r} = \int_a^b \mathbf{F}\big(\mathbf{r}(t)\big) \cdot \mathbf{r}'(t)\,\mathrm{d}t \tag{1.4}$$

where $\cdot$ is the dot product, and $\mathbf{r}\colon [a,b] \to C$ is a bijective parametrization of the curve C such that $\mathbf{r}(a)$ and $\mathbf{r}(b)$ give the endpoints of C.

Line integral of a vector field, also can be shown as below:

$$\int_C \mathbf{F} \cdot \mathrm{d}\mathbf{r} = \int_C \mathsf{p}\,\mathrm{d}x + \mathsf{q}\,\mathrm{d}y + \mathbf{r}\,\mathrm{d}z \tag{1.5}$$

Chapter 2

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2.1 Gradient Theorem

The gradient theorem, also known as the fundamental theorem of calculus for line integrals, says that a line integral through a gradient field can be evaluated by evaluating the original scalar field at the endpoints of the curve. The theorem is a generalization of the second fundamental theorem of calculus to any curve in a plane or space (generally n-dimensional) rather than just the real line.

$$\mathbf{F} = \nabla f \Rightarrow \begin{cases} P = \frac{\partial f}{\partial x} \\ Q = \frac{\partial f}{\partial y} \\ R = \frac{\partial f}{\partial z} \end{cases} \tag{2.1}$$

For $\mathbf{F} \colon U \subseteq \mathbf{R}^n \to \mathbf{R}$ as a differentiable function and any continuous curve in U which starts at a point a and ends at a point b, then

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \int_a^b \nabla f \cdot d\mathbf{r} = f\big(\mathbf{r}(b)\big) - f\big(\mathbf{r}(a)\big) \tag{2.2}$$

where ∇f denotes the gradient vector field of $\mathbf{F}$.

To check if a vector is a gradient vector:

$$\frac{\partial p}{\partial y} = \frac{\partial q}{\partial x} \qquad \frac{\partial p}{\partial z} = \frac{\partial r}{\partial x} \qquad \frac{\partial q}{\partial z} = \frac{\partial r}{\partial y} \tag{2.3}$$

2.2   Green's Theorem

If C be a positively oriented, piecewise smooth, simple closed curve in a plane, and let D be the region bounded by C. If P and Q are functions of (x, y) defined on an open region containing D and have continuous partial derivatives there, then

$$\oint_C P\,\mathrm{d}x + Q\,\mathrm{d}y = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) \mathrm{d}A \tag{2.4}$$

where the path of integration along C is anticlockwise.

If curve C is divided into pieces like $C = C_1 \bigcup C_2$, then

$$\begin{aligned}
\int_C P\,\mathrm{d}x + Q\,\mathrm{d}y &= \int_{C_1} P\,\mathrm{d}x + Q\,\mathrm{d}y + \int_{C_2} P\,\mathrm{d}x + Q\,\mathrm{d}y \\
&= \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) \mathrm{d}A
\end{aligned} \tag{2.5}$$

2.3   Area Calculations

$$A = \oint_C x\,\mathrm{d}y = -\oint_C y\,\mathrm{d}x = \frac{1}{2}\oint_C x\,\mathrm{d}y - y\,\mathrm{d}x \tag{2.6}$$

Chapter 3

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3.1 Piecewise Smooth Parametrized Curve

$$\vec{r}(u,v) = \big(x(u,v),\, y(u,v),\, z(u,v)\big), \quad (u,v) \in R \tag{3.1}$$

$$\vec{r}_u = (\frac{\partial x}{\partial u},\, \frac{\partial y}{\partial u},\, \frac{\partial z}{\partial u}) \qquad \vec{r}_v = (\frac{\partial x}{\partial v},\, \frac{\partial y}{\partial v},\, \frac{\partial z}{\partial v}) \tag{3.2}$$

3.1.1 Sample

If $z = f(x,y)$, the smooth parametrized curve will be:

$$\vec{r}(u,v) = (u, v, f(u,v)) \tag{3.3}$$

Partial Derivatives are defined as:

$$\begin{aligned} \vec{r}_u &= (\frac{\partial u}{\partial u},\, \frac{\partial v}{\partial u},\, \frac{\partial f(u,v)}{\partial u}) \\ &= (1, 0, \frac{\partial f}{\partial u}) = \vec{i} + \frac{\partial f}{\partial u}\vec{k} \end{aligned} \tag{3.4}$$

$$\begin{aligned} \vec{r}_v &= (\frac{\partial u}{\partial v},\, \frac{\partial v}{\partial v},\, \frac{\partial f(u,v)}{\partial v}) \\ &= (0, 1, \frac{\partial f}{\partial v}) = \vec{j} + \frac{\partial f}{\partial v}\vec{k} \end{aligned} \tag{3.5}$$

Cross Product of the partial derivatives:

$$\vec{r}_u \times \vec{r}_v = (\frac{-\partial f}{\partial u}, \ \frac{-\partial f}{\partial v}, \ 1)$$
$$= \frac{-\partial f}{\partial u}\vec{i} + (\frac{-\partial f}{\partial v}\vec{j}) + \vec{k} \neq 0 \tag{3.6}$$

3.2   Area Calculation

$$A(\mathtt{S}) = \iint\limits_{R} |\vec{r}_u \times \vec{r}_v| \ \mathrm{d}A \tag{3.7}$$

3.2.1   Theorem

If $z = f(x, y)$, then:

$$A(\mathtt{S}) = \iint\limits_{R} \sqrt{(\frac{\partial f}{\partial x})^2 + (\frac{\partial f}{\partial y})^2 + 1} \ \ \mathrm{d}A \tag{3.8}$$

3.3   Vector Integral

$$\iint\limits_{S} \mathbf{G}(x, y, z)\,\mathrm{d}\sigma = \iint\limits_{R} \mathbf{G}\big(\vec{r}(u, v)\big) |\vec{r}_u \times \vec{r}_v| \ \mathrm{d}A \quad \vec{r}(u, v) \in R \tag{3.9}$$

3.3.1   special cases

1. If $S : \vec{r}(u, v) = \big(f(u, v), g(u, v), h(u, v)\big)$, then:

$$\iint\limits_{S} \mathbf{G}(x, y, z)\,\mathrm{d}\sigma = \iint\limits_{S} \mathbf{G}\big(f(u, v), g(u, v), h(u, v)\big) |\vec{r}_u \times \vec{r}_v| \ \mathrm{d}A \tag{3.10}$$

2. If $z = f(x, y)$, then:

$$\iint\limits_{S} \mathbf{G}(x, y, z)\,\mathrm{d}\sigma = \iint\limits_{S} \mathbf{G}\big(x, y, f(x, y)\big)\sqrt{(\frac{\partial f}{\partial x})^2 + (\frac{\partial f}{\partial y})^2 + 1} \ \mathrm{d}x\mathrm{d}y$$

$$\tag{3.11}$$

Solved Exercise (use it as a template to solve questions)

If $z = \sqrt{x^2 + y^2}$; $(0 \leq z \leq 1)$, then $\iint\limits_S x^2 \, \mathrm{d}\sigma =?$

Solution:

$$\begin{cases} x = r\cos\theta \\ y = r\sin\theta \\ z = z \end{cases} \Rightarrow \begin{cases} x = r\cos\theta \\ y = r\sin\theta \\ z = r \end{cases} \quad (0 \leq r \leq 1) ; (0 \leq \theta \leq 2\pi)$$

$$\vec{r}(r, \theta) = (r\cos\theta, r\sin\theta, r)$$
$$\Rightarrow \quad \vec{r}_r = (\cos\theta, \sin\theta, r)$$
$$\vec{r}_\theta = (-r\sin\theta, r\cos\theta, 0)$$

$$\Rightarrow \vec{r}_r \times \vec{r}_\theta = \begin{vmatrix} i & j & k \\ \cos\theta & \sin\theta & r \\ -r\sin\theta & r\cos\theta & 0 \end{vmatrix} = (-r\cos\theta, -r\sin\theta, r)$$

$$\Rightarrow |\vec{r}_r \times \vec{r}_\theta| = \sqrt{\underbrace{(r^2\cos^2\theta) + (r^2\sin^2\theta)}_{=r^2} + r^2} = \sqrt{2r^2} = r\sqrt{2}$$

$$\Rightarrow \iint\limits_{[0,1]\times[0,2\pi]} r^2\cos^2\theta \; r\sqrt{2} \; \mathrm{d}r\mathrm{d}\theta = \sqrt{2} \int_0^{2\pi} \int_0^1 r^3 \cos^2\theta \; \mathrm{d}r\mathrm{d}\theta$$

$$= \frac{\sqrt{2}}{4} \int_0^{2\pi} \cos^2\theta \; \mathrm{d}\theta$$

$$= \frac{\sqrt{2}}{2 \times 4} \int_0^{2\pi} 1 + \cos 2\theta \; \mathrm{d}\theta$$

$$= \frac{\sqrt{2}}{8} \left(\theta + \frac{1}{2}\sin 2\theta\right)\Big|_0^{2\pi}$$

$$= \frac{2\pi \times \sqrt{2}}{8} = \frac{\pi\sqrt{2}}{4}$$

Chapter 4

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4.1 Transforms

Del (nabla)

Symbol:

$$\nabla \tag{4.1}$$

Definition:

$$\nabla f := \frac{\partial f}{\partial x}\vec{i} + \frac{\partial f}{\partial y}\vec{j} + \frac{\partial f}{\partial z}\vec{k} \tag{4.2}$$

$$\nabla := \frac{\partial}{\partial x}\vec{i} + \frac{\partial}{\partial y}\vec{j} + \frac{\partial}{\partial z}\vec{k} \tag{4.3}$$

Divergence

Symbol:

$$\text{div} \tag{4.4}$$

Definition:
If $\mathbf{F}$ be $\mathbf{F}(x, y, z) = P(x, y, z)\vec{i} + Q(x, y, z)\vec{j} + R(x, y, z)\vec{k}$, then:

$$\text{div } \mathbf{F} := \frac{\partial P}{\partial x}\vec{i} + \frac{\partial Q}{\partial y}\vec{j} + \frac{\partial R}{\partial z}\vec{k} \tag{4.5}$$

$$\text{div } \mathbf{F} := \nabla \cdot \mathbf{F} \tag{4.6}$$

curl

Symbol:

$$\text{curl} \tag{4.7}$$

Definition:
If $\mathbf{F}$ be $\mathbf{F}(x, y, z) = P(x, y, z)\vec{i} + Q(x, y, z)\vec{j} + R(x, y, z)\vec{k}$, then:

$$\text{curl } \mathbf{F} := \left(\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z}\right)\vec{i} + \left(\frac{\partial P}{\partial z} - \frac{\partial R}{\partial x}\right)\vec{j} + \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)\vec{k} \tag{4.8}$$

$$\text{curl } \mathbf{F} := \nabla \cdot \mathbf{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ P & Q & R \end{vmatrix} \tag{4.9}$$

4.2   Theorems

1. $\text{curl}(\nabla f) = 0$

2. $\text{div}(\text{curl } \mathbf{F}) = 0$

3.   (a) If $\lambda \in \mathbb{R}$, then: $\nabla(f + \lambda g) = \nabla f + \lambda \nabla g$
     (b) $\text{curl}(\mathbf{F} + \lambda \mathbf{G}) = \text{curl } \mathbf{F} + \lambda \text{curl } \mathbf{G}$
     (c) $\text{div}(\mathbf{F} + \lambda \mathbf{G}) = \text{div } \mathbf{F} + \lambda \text{div } \mathbf{G}$

4. $\text{div}(\mathbf{F} \times \mathbf{G}) = \mathbf{G} \cdot \text{curl } \mathbf{F} - \mathbf{F} \cdot \text{curl } \mathbf{G}$

5. $\text{div}(f\mathbf{F}) = f \, \text{div } \mathbf{F} + \mathbf{F} \cdot \nabla f$

6. $\text{curl}(f\mathbf{F}) = f \, \text{curl } \mathbf{F} + \nabla f \times \mathbf{F}$

4.3   Surface Integrals of Vector Fields

$$\iint_S \mathbf{F} \cdot \vec{n} \, \mathrm{d}\sigma \tag{4.10}$$

$$\iint_S \mathbf{F} \cdot \vec{n} \ \mathrm{d}\sigma = \iint_S \mathbf{F} \cdot \frac{\vec{r_u} \times \vec{r_v}}{|\vec{r_u} \times \vec{r_v}|} \ \mathrm{d}\sigma$$

$$= \iint_D \left[\mathbf{F} \cdot \frac{\vec{r_u} \times \vec{r_v}}{|\vec{r_u} \times \vec{r_v}|} \right] |\vec{r_u} \times \vec{r_v}| \ \mathrm{d}A$$

$$= \iint_D \mathbf{F} \cdot (\vec{r_u} \times \vec{r_v}) \ \mathrm{d}A$$

$$\Rightarrow \iint_S \mathbf{F} \cdot \vec{n} \ \mathrm{d}\sigma = \iint_D \mathbf{F} \cdot (\vec{r_u} \times \vec{r_v}) \ \mathrm{d}A \tag{4.11}$$

If $z = g(x, y)$; $(x, y) \in R$ and $\mathbf{F} = (P, Q, R)$, then:

$$\iint_S \mathbf{F} \cdot \vec{n} \ \mathrm{d}\sigma = \iint_R \left(-P \frac{\partial g}{\partial x} - Q \frac{\partial g}{\partial y} + R \right) \ \mathrm{d}A \tag{4.12}$$

4.4 Divergence Theorem

$$\iint_S \mathbf{F} \cdot \vec{n} \ \mathrm{d}\sigma = \iiint_R \mathrm{div} \ \mathbf{F} \ \mathrm{d}V \tag{4.13}$$

4.5 Stokes Theorem

$$\int_{\partial S} \mathbf{F} \cdot \mathrm{d}\vec{r} = \iint_S (\mathrm{curl} \ \mathbf{F}) \cdot \vec{n} \ \mathrm{d}\sigma \tag{4.14}$$