|| Queueing Theory ||

variations

(i) multiple server multiple queue

(ii) Network of queue

(iii) sequential queue.

Some terminology

arrival rate λ

service " μ

L = avg # of customers in the system

L_Q = " " " " queue

W = avg time customer spends in the system

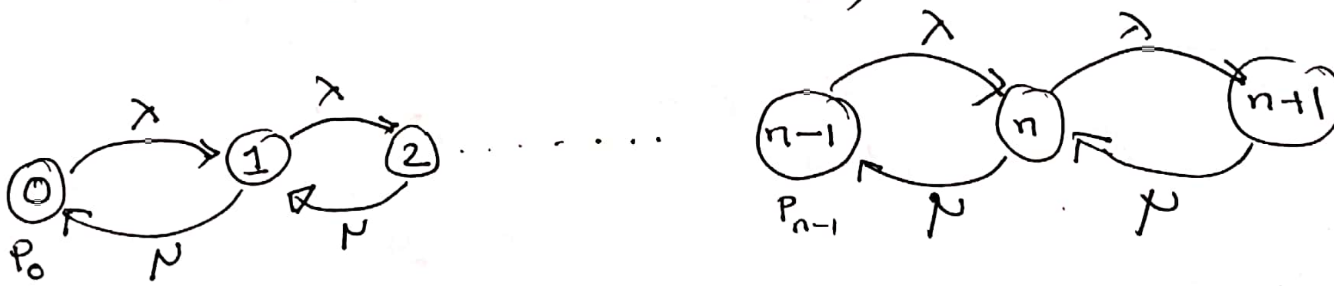
W_Q = " " " " " queue

A. Single-server Exponential Queueing System

customer arrival poisson process with rate λ — M/M/1

service times are independent exp dist with mean $\frac{1}{\mu}$

P_n = probability of n customers in the system. (n-th state)



balance eqn $\rightarrow$ why?

rate at which process leaves = rate at which process enters.

state	eqn
0	$\lambda P_0 = \mu P_1$
n	$\lambda P_{n-1} + \mu P_{n+1} = (\lambda+\mu) P_n$

Solving recurrence relation

$$P_1 = \left(\frac{\lambda}{\mu}\right) P_0$$

$$P_{n+1} = \frac{\lambda}{\mu} P_n + P_n - \frac{\lambda}{\mu} P_{n-1}$$

$$\Rightarrow \quad P_2 = \frac{\lambda}{\mu} P_1 + P_1 - \frac{\lambda}{\mu} P_0 = \frac{\lambda}{\mu} P_1 = \left(\frac{\lambda}{\mu}\right)^2 P_0$$

$$P_3 = \left(\frac{\lambda}{\mu}\right)^3 P_0$$

$$P_n = \left(\frac{\lambda}{\mu}\right)^n P_0$$

$$\sum_{n=0}^{\infty} P_n = 1$$

$$P_0 \left[1 + \left(\frac{\lambda}{\mu}\right) + \left(\frac{\lambda}{\mu}\right)^2 + \dots\dots \right] = 1$$

$$P_0 \cdot \frac{1}{1 - \lambda/\mu} = 1$$

$$P_0 = 1 - \frac{\lambda}{\mu}$$

$$P_n = \left(\frac{\lambda}{\mu}\right)^n \left(1 - \frac{\lambda}{\mu}\right)$$

$$L = \sum_{n=0}^{\infty} n P_n$$

$$= \sum_{n=0}^{\infty} n \left(\frac{\lambda}{\mu}\right)^n \left(1 - \frac{\lambda}{\mu}\right)$$

$$= \left(1 - \frac{\lambda}{\mu}\right) \sum_{n=0}^{\infty} n \left(\frac{\lambda}{\mu}\right)^n$$

$$\sum_{n=0}^{\infty} n x^n = \frac{x}{(1-x)^2}$$

$$= \left(1 - \frac{\lambda}{\mu}\right) \frac{\lambda/\mu}{\left(1 - \frac{\lambda}{\mu}\right)^2}$$

$$= \frac{\lambda}{\mu - \lambda}$$

$$L = \lambda W$$

$$W = \frac{L}{\lambda} = \frac{1}{\mu - \lambda}$$

$$W = W_Q + E[S]$$

$$W_Q = \frac{1}{\mu - \lambda} - \frac{1}{\mu}$$

$$= \frac{\lambda}{\mu(\mu - \lambda)}$$

$$L_Q = \lambda W_Q = \frac{\lambda^2}{\mu(\mu - \lambda)}$$

Single server exp queuing system with finite queue

system capacity N

state	eq^n
0	$\mu P_1 = \lambda P_0$
n ; $0<n<N$	$\lambda P_{n-1} + \mu P_{n+1} = (\lambda+\mu) P_n$
N	$\lambda P_{N-1} = \mu P_N$

$$P_1 = \left(\frac{\lambda}{\mu}\right) P_0$$

$$P_2 = \left(\frac{\lambda}{\mu}\right) P_1 + P_1 - \frac{\lambda}{\mu} P_0 = \left(\frac{\lambda}{\mu}\right)^2 P_0$$

$$P_N = \frac{\lambda}{\mu} P_{N-1} = \left(\frac{\lambda}{\mu}\right)^N P_0$$

$$\sum_{n=0}^{N} P_n = 1$$

$$\sum_{n=0}^{N} \left(\frac{\lambda}{\mu}\right)^n P_0 = 1 \Rightarrow P_0 \cdot \frac{1-(\lambda/\mu)^{N+1}}{1-\lambda/\mu} = 1$$

$$P_n = \left(\frac{\lambda}{\mu}\right)^n \cdot \frac{1-\lambda/\mu}{1-(\lambda/\mu)^{N+1}}$$

$$L = \sum_{n=0}^{N} n P_n$$

$$= \frac{(1-\lambda/\mu)}{1-(\lambda/\mu)^{N+1}} \sum_{n=0}^{N} n\,(\lambda/\mu)^n$$

$$=$$

$$L = \frac{\lambda\left[1 + N(\lambda/\mu)^{N+1} - (N+1)(\lambda/\mu)^N\right]}{(\mu-\lambda)\left(1-(\lambda/\mu)^{N+1}\right)}$$

$$\sum_{n=0}^{N} n x^n = x \sum_{n=0}^{N} \frac{d}{dx}(x^n)$$

$$= x \cdot \frac{d}{dx}\left(\sum_{n=0}^{N} x^n\right)$$

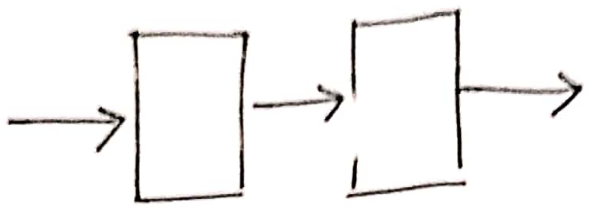
$$=$$

$$= x \cdot \frac{d}{dx}\left[\frac{1-x}{1-x^{N+1}}\right]$$

$$\lambda_a = \lambda(1-P_N)$$

$$W = \frac{L}{\lambda_a}$$

Shoe - Shine Shop Model:



No queue

states:

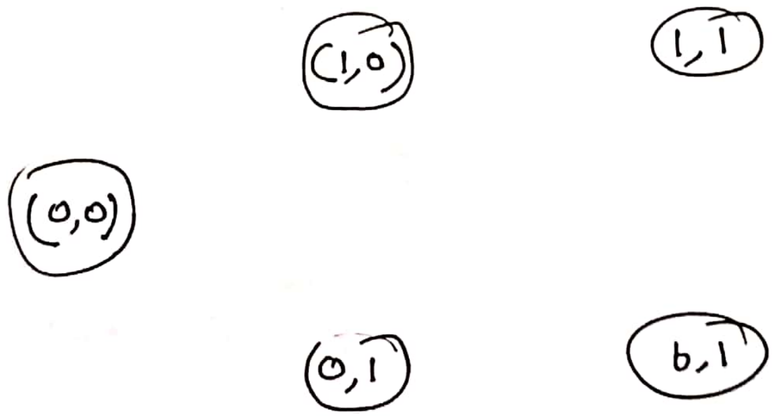
$(0,0)$ = No customer in the system

$(0,1)$ = One " " chair 2

$(1,0)$ = " " " " 1

$(1,1)$ = both chair used

$(b,1)$ = customer in chair 1 waiting



state	eq^n
$(0,0)$	$\lambda P_{00} = \mu_2 P_{01}$
$(1,0)$	$\mu_1 P_{01} = \lambda P_{00} + \mu_2 P_{11}$
$(0,1)$	$(\lambda+\mu_2) P_{01} = \mu_1 P_{10} + \mu_2 P_{b1}$
$(1,1)$	$(\mu_1+\mu_2) P_{11} = \lambda P_{01}$
$(b,1)$	$\mu_2 P_{b1} = \mu_1 P_{11}$

$P_{00} + P_{01} + P_{10} + P_{11} + P_{b1} = 1$

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proportion of customers entering the system

$$= P_{00} + P_{01}$$

Avg # customers in the system

$$L = 1 \times (P_{01} + P_{10}) + 2 \times (P_{11} + P_{b1})$$

Avg time customer spend in the system

$$W = \frac{L}{\lambda_a}$$

$$= \frac{P_{01} + P_{10} + 2(P_{11} + P_{b1})}{\lambda (P_{00} + P_{01})}$$

$\lambda_a = \lambda (P_{00} + P_{01})$

Proportion of entering customer that are blockers

$$\pi_b = \frac{P_{01}}{P_{00} + P_{01}} \cdot \frac{\mu_1}{\mu_1 + \mu_2}$$

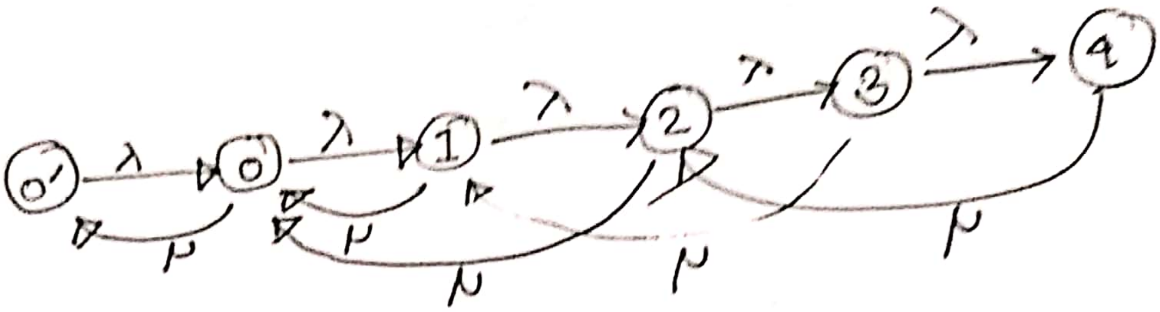
another way, let λ_b the rate at which customer becomes blocker

$$\lambda_b = \mu_1 P_{11}$$

$$\pi_b = \frac{\lambda_b}{\lambda_a} = \frac{\mu_1 P_{11}}{\lambda (P_{00} + P_{01})}$$

Queing with bulk service:

server serves two customers at a time.



state	meaning	eq^n
$0'$	no-one in the system	$\lambda P_0' = \mu P_0$
0	server busy - no-one waiting	$(\lambda+\mu) P_0 = \lambda P_0' + \mu P_1 + \mu P_2$
$n,\ n>0$	n customers waiting	$(\lambda+\mu) P_n = \lambda P_{n-1} + \mu P_{n+2}$

it has the sol^n $P_n = \alpha^n P_0$ (back-word :))

$$(\lambda+\mu)\alpha^n P_0 = \lambda \alpha^{n-1} P_0 + \mu \alpha^{n+2} P_0$$

$$(\lambda+\mu)\alpha = \lambda + \mu\alpha^3$$

$$\alpha^3\mu - (\lambda+\mu)\alpha + \lambda = 0$$

$$\mu\alpha^2(\alpha-1) + \mu\alpha(\alpha-1) - \lambda(\alpha-1) = 0$$

$$(\alpha-1)(\mu\alpha^2 + \mu\alpha - \lambda) = 0$$

$$\alpha = 1, \quad \alpha = \frac{-\mu \pm \sqrt{\mu^2 + 4\mu\lambda}}{2\mu}$$

$soln$ $\alpha = \frac{-\mu + \sqrt{\mu^2 + 4\mu\lambda}}{2\mu} < 1$

$$\mu^2 + 4\mu\lambda < 9\mu^2$$

$$4\mu\lambda < 8\mu^2$$

$$\lambda < 2\mu$$

$P_n = \alpha^n P_0$, $P_0' = \frac{\mu}{\lambda} P_0$

$$P_0 + P_0' + \sum_{n=1}^{\infty} P_n = 1$$

$$P_0 \left[1 + \frac{\mu}{\lambda} + \sum_{n=1}^{\infty} \alpha^n\right] = 1$$

$$P_0 \left[1 + \frac{\mu}{\lambda} + \frac{\alpha}{1-\alpha}\right] = 1$$

$$P_0 \cdot \frac{\lambda(1-\alpha) + \mu(1-\alpha) + \alpha\lambda}{\lambda(1-\alpha)} = 1$$

$$P_0 = \frac{\lambda(1-\alpha)}{\lambda + \mu(1-\alpha)}$$

P_n , P_0'

proportion of customers that are served alone

rate at which customers are served alone

$$\lambda P_0' + \mu P_1$$

$\therefore$ proportion $= \frac{\lambda P_0' + \mu P_1}{\lambda}$

$$= P_0' + \frac{\mu}{\lambda} P_1$$

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$$L_Q = \sum_{n=0}^{\infty} n P_n$$

$$= \sum_{n=0}^{\infty} n . \alpha^n P_0$$

$$= P_0 . \frac{\alpha}{(1-\alpha)^2}$$

$$= \frac{\lambda(1-\alpha)}{[\lambda+\mu(1-\alpha)]} . \frac{\alpha}{(1-\alpha)^2}$$

$$= \frac{\lambda\alpha}{(1-\alpha)[\lambda+\mu(1-\alpha)]}$$

$$W_Q = \frac{L_Q}{\lambda}$$

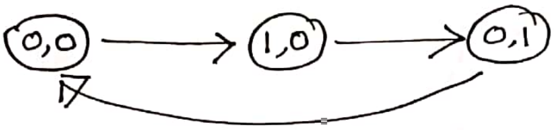
$$W = W_Q + \frac{1}{\mu}$$

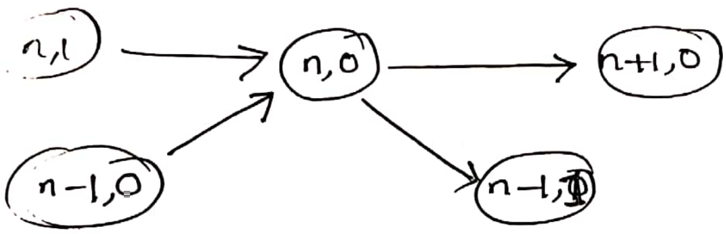
$$L = \lambda W$$

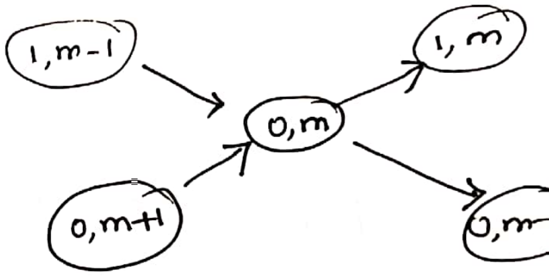
Network of queues

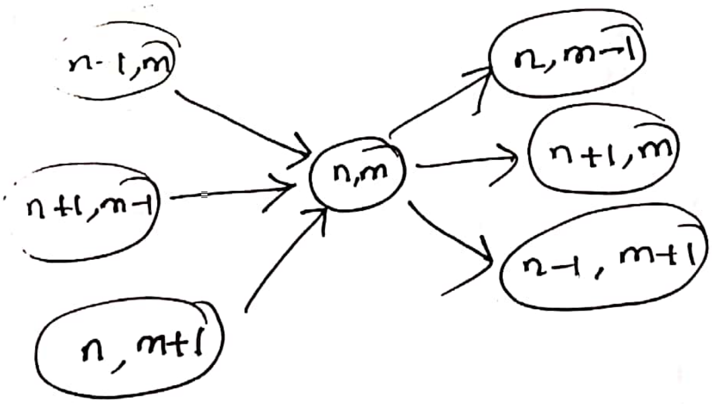
states (n, m) 1st system a customer n

2nd .. .. m









state	eq^n
$(0,0)$	$\lambda P_{00} = \mu_2 P_{01}$
$(n,0),\ n>0$	$(\lambda+\mu_1) P_{n,0} = \mu_2 P_{n,1} + \lambda P_{n-1,0}$
$(0,m),\ m>0$	$(\lambda+\mu_2) P_{0,m} = \mu_1 P_{1,m-1} + \mu_2 P_{0,m+1}$
$(n,m),$	$(\lambda+\mu_1+\mu_2) P_{n,m} = \lambda P_{n-1,m} + \mu_1 P_{n+1,m-1} + \mu_2 P_{n,m+1}$

way 1: solve these eq^n with $\sum_{n,m} P_{n,m} = 1$

way 2: use $P_n \times P_m$ to get $P_{n,m}$ and verify the balance eq^n.

server 1 $\longrightarrow$ M/M/1

departure from S1 is also poisson process with rate λ

so server 2 $\longrightarrow$ M/M/1

for S-1 $P_n = (\lambda/\mu_1)^n \left(1 - \frac{\lambda}{\mu_1}\right)$

S-2 $P_m = (\lambda/\mu_2)^m \left(1 - \frac{\lambda}{\mu_2}\right)$

$$P_{n,m} = P_n \times P_m$$
$$= (\lambda/\mu_1)^n (\lambda/\mu_2)^m \left(1 - \frac{\lambda}{\mu_1}\right)\left(1 - \frac{\lambda}{\mu_2}\right)$$

for $(0,0)$

$$L.H.S. = \lambda \times \left(1 - \frac{\lambda}{\mu_1}\right)\left(1 - \frac{\lambda}{\mu_2}\right)$$

$$R.H.S = \mu_2 \times \left(\frac{\lambda}{\mu_2}\right) \cdot \left(1 - \frac{\lambda}{\mu_1}\right)\left(1 - \frac{\lambda}{\mu_2}\right)$$
$$= L.H.S.$$

$$L = \sum_{n,m} (n+m) P_{n,m}$$
$$= \sum_n n P_n + \sum_m m P_m$$
$$= \frac{\lambda}{\mu_1 - \lambda} + \frac{\lambda}{\mu_2 - \lambda}$$

$$W = \frac{L}{\lambda} = \frac{1}{\mu_1 - \lambda} + \frac{1}{\mu_2 - \lambda}$$

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generalized problem (k. servers)

customer arrival at server i is ind. poisson process with rate r_i

after taking service from server i, probability of joining queue of another server j is $= P_{ij}$

prob of departure from system at serve i $S_i = 1 - \sum_j P_{ij}$

actual arrival rate at server j is

$$\lambda_j = r_j + \sum_{i=1}^{k} P_{ij} \times \lambda_i$$

customers in each server is ind.

$$P\{n \text{ customers in server } j\} = \left(\frac{\lambda_j}{\mu_j}\right)^n \left(1 - \frac{\lambda_j}{\mu_j}\right) \; ; \; n \geq 1$$

$$P\{n_1, n_2, \ldots, n_k\} = \prod_{j=1}^{k} \left(\frac{\lambda_j}{\mu_j}\right)^{n_j} \left(1 - \frac{\lambda_j}{\mu_j}\right)$$

avg # of customers in the system $L = \sum_{j=1}^{k}$ avg # at serve

$$= \sum_{j=1}^{k} \frac{\lambda_j}{\mu_j - \lambda_j}$$

$$W = \frac{L}{\lambda} = \frac{\sum_{j=1}^{k} \frac{\lambda_j}{\mu_j - \lambda_j}}{\sum_{j=1}^{k} r_j}$$

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