Operations on images

Math operations on images

Arithmetic operations:

$$s(x, y) = f(x, y) + g(x, y)$$

$$d(x, y) = f(x, y) - g(x, y)$$

$$p(x, y) = f(x, y) \times g(x, y)$$

$$v(x, y) = f(x, y) \div g(x, y)$$

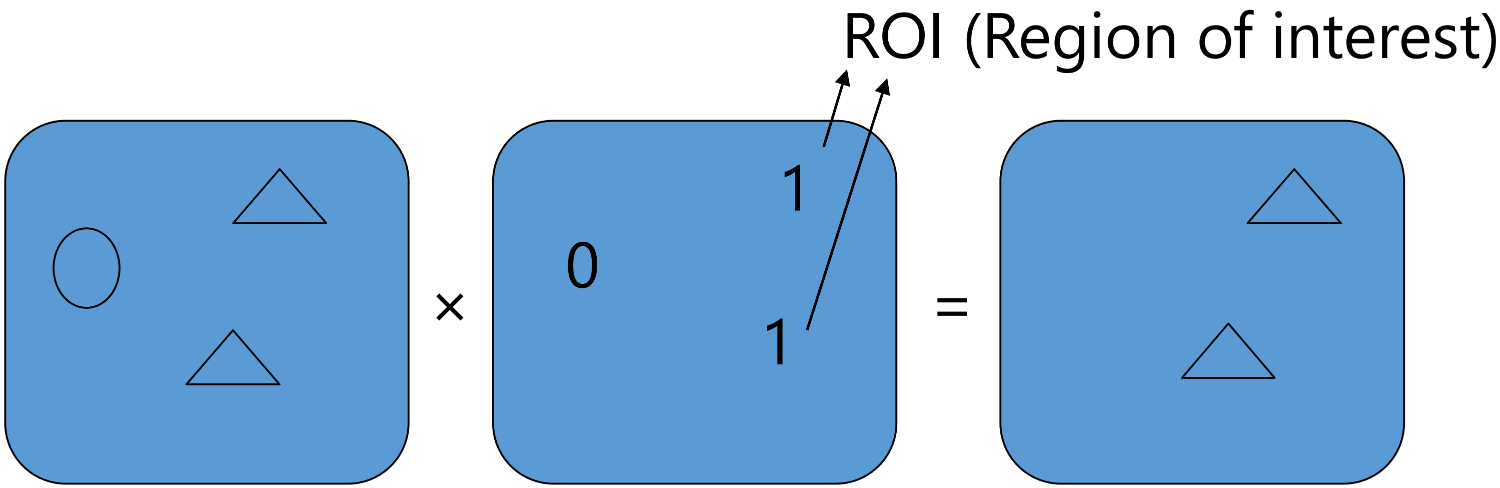
Math operations on images

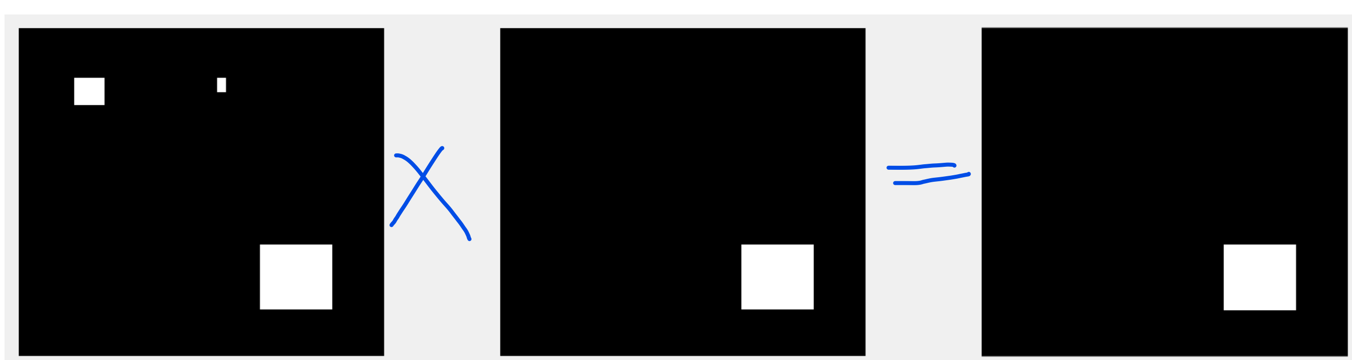
vs

Array operations versus matrix operations

pixel-by-pixel product versus matrix product

$$\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix} = \begin{bmatrix} a_{11}b_{11} & a_{12}b_{12} \\ a_{21}b_{21} & a_{22}b_{22} \end{bmatrix}$$





Math operations on images

Linear versus nonlinear operations

$$H(af + bg) = aH(f) + bH(g)$$

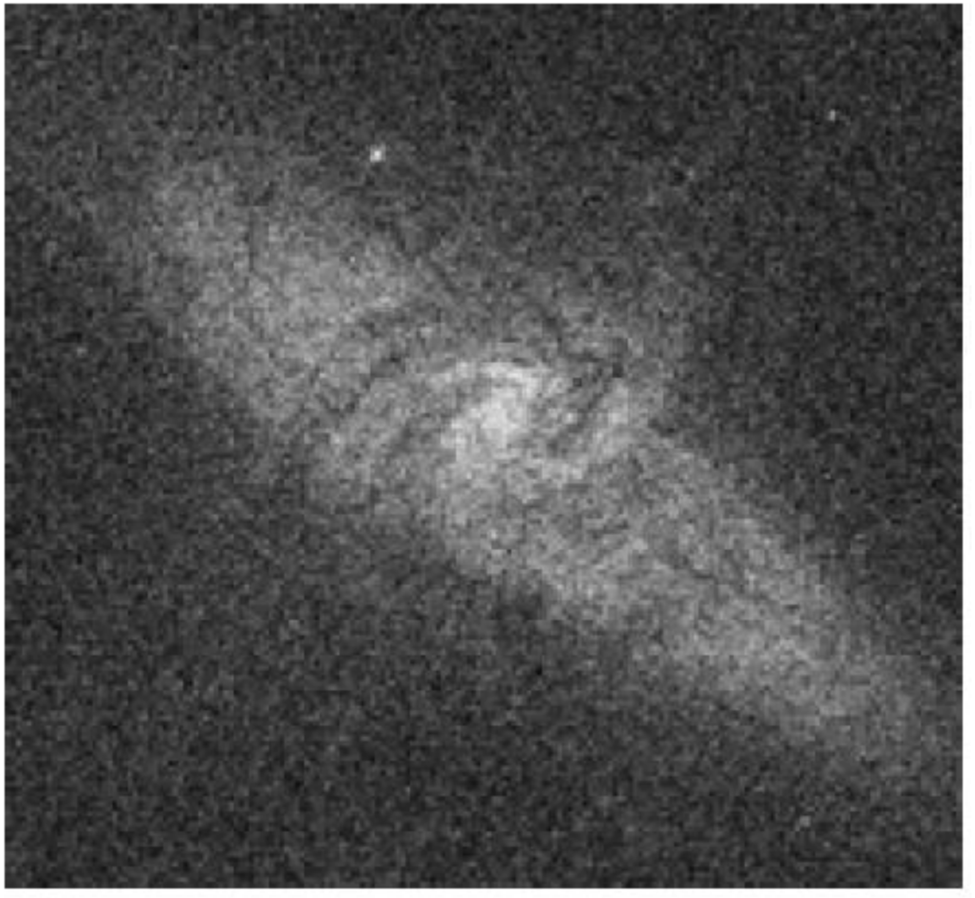
Max: nonlinear

$$max\left\{\begin{bmatrix} 5 & 6 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} 5 & 0 \\ 6 & 1 \end{bmatrix}\right\} = max\left\{\begin{bmatrix} 10 & 6 \\ 6 & 2 \end{bmatrix}\right\} = 10$$

$$max\left\{\begin{bmatrix} 5 & 6 \\ 0 & 1 \end{bmatrix}\right\} + max\left\{\begin{bmatrix} 5 & 0 \\ 6 & 1 \end{bmatrix}\right\} = 6 + 6 = 12$$

$$Then\ 10 \neq 12$$

Math operations on images



Noisy image

Math operations on images

Noisy image

$$g(x, y) = f(x, y) + \eta(x, y)$$

Zero average noise

Averaging over a set of *K* noisy images

$$\bar{g}(x, y) = \frac{1}{K}\sum_{i=1}^{K} g_i(x, y)$$

Math operations on images

$$g(x,y) = f(x,y) + \eta(x,y)$$

$$\bar{g}(x,y) = \frac{1}{k}\sum_{i=1}^{k} g_i(x,y) = \frac{1}{k}\sum_{i=1}^{k}\left[f(x,y) + \eta_i(x,y)\right]$$

$$\mathrm{E}\{\bar{g}\} = \mathrm{E}\left\{\frac{1}{k}\sum_{i=1}^{k} g_i\right\} = \frac{1}{k}\mathrm{E}\left\{\sum_{i=1}^{k}\left[f + \eta_i\right]\right\} = \frac{1}{k}\left[kf + \mathrm{E}\left\{\sum_{i=1}^{k}\eta_i\right\}\right] = f$$

$$\sigma_{\bar{g}}^2 = \mathrm{E}\left\{\bar{g}^2\right\} - \left[\mathrm{E}\{\bar{g}\}\right]^2 = \mathrm{E}\left\{\left(\frac{1}{k}\sum_{i=1}^{k}\left[f + \eta_i\right]\right)^2\right\} - f^2$$

$$= \frac{1}{k^2}\mathrm{E}\left\{\left(kf + \sum_{i=1}^{k}\eta_i\right)^2\right\} - f^2 = \frac{1}{k^2}\mathrm{E}\left\{(kf)^2 + 2kf\sum_{i=1}^{k}\eta_i + \left(\sum_{i=1}^{k}\eta_i\right)^2\right\} - f^2$$

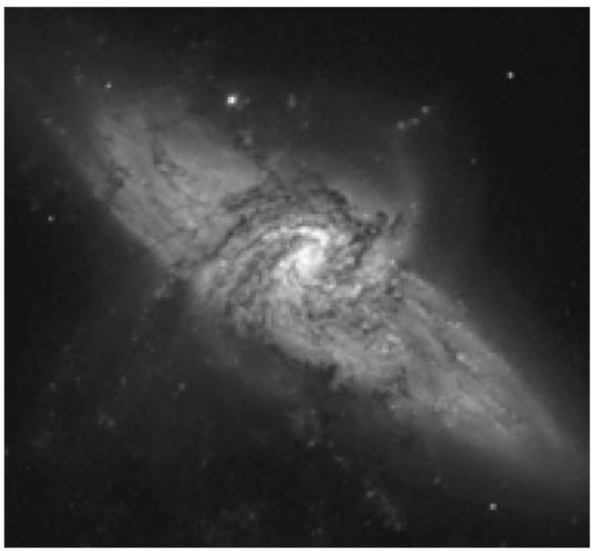
$$= \cancel{f^2} + \frac{2f}{k}\cancel{\mathrm{E}\left\{\sum_{i=1}^{k}\eta_i\right\}} + \frac{1}{k^2}\mathrm{E}\left\{\left(\sum_{i=1}^{k}\eta_i\right)^2\right\} - \cancel{f^2} = \frac{1}{k^2}\mathrm{E}\left\{\left(\sum_{i=1}^{k}\eta_i\right)^2\right\}$$

$$= \frac{1}{k^2}\mathrm{E}\left\{\sum_{i=1}^{k}\eta_i^2\right\} + \frac{1}{k^2}\cancel{\mathrm{E}\left\{2\sum_{j<i}\eta_i\eta_j\right\}} = \frac{1}{k^2}\sum_{i=1}^{k}\mathrm{E}\left\{(\eta_i - 0)^2\right\} = \boxed{\frac{\sigma_\eta^2}{k}}$$

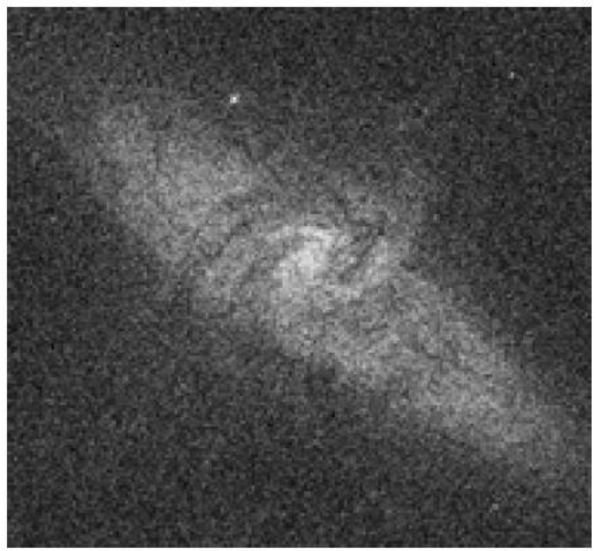
Math operations on images

$$E\{\bar{g}(x,y)\} = f(x,y)$$

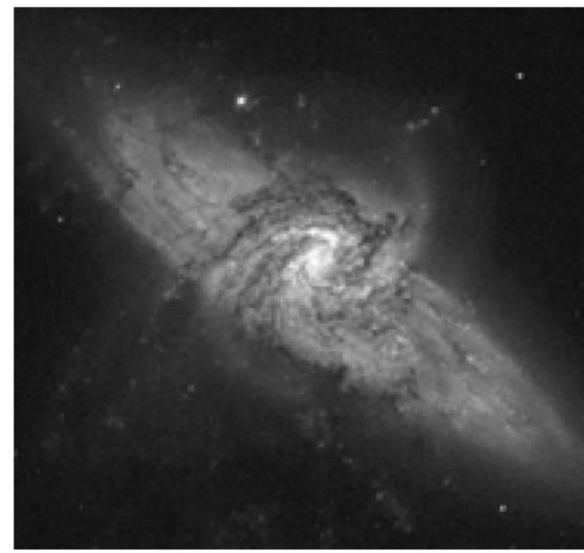
$$\sigma_{\bar{g}(x,y)} = \frac{1}{\sqrt{K}}\,\sigma_{\eta(x,y)}$$



Original



Gaussian
mean=0, s.d.=64



After averaging
128 frames

Image operations

Intensity-based (global) operations

$$s = T(z)$$

Neighborhood (local) operations

$$g(p,q) = T(f(p,q))$$

$$g(p,q) = \frac{1}{(2M+1)(2N+1)} \sum_{j=q-N}^{q+N} \sum_{i=p-M}^{p+M} f(i,j)$$

Geometric (rubber sheet) transformations

$$(p, q) = T\left\{(m, n)\right\}$$

Often used: Affine transformations:

Linear + Translation:

$$\vec{v} = A\vec{u} + \vec{b}$$

Geometric (rubber sheet) transformations

$$(p, q) = T\{(m, n)\}$$

Often used: Affine transformations:

Linear + Translation:

$$\vec{v} = A\vec{u} + \vec{b}$$

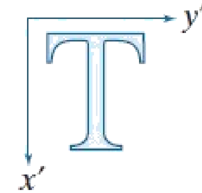
Representation with the affine transformation matrix (example of homogeneous coordinates):

$$\begin{bmatrix} \vec{v} \\ 1 \end{bmatrix} = \begin{bmatrix} A & b \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \vec{u} \\ 1 \end{bmatrix}$$

Affine transformations

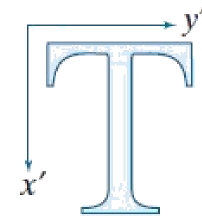
Identity

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$



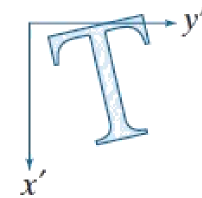
Scaling

$$\begin{bmatrix} c_x & 0 & 0 \\ 0 & c_y & 0 \\ 0 & 0 & 1 \end{bmatrix}$$



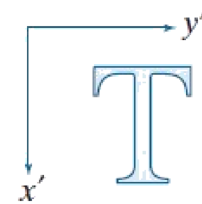
Rotation

$$\begin{bmatrix} cos\theta & -sin\theta & 0 \\ sin\theta & cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$$



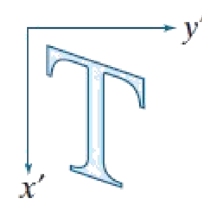
Translation

$$\begin{bmatrix} 1 & 0 & t_x \\ 0 & 1 & t_y \\ 0 & 0 & 1 \end{bmatrix}$$



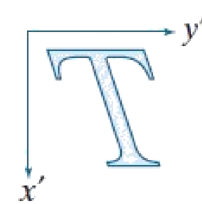
Shear (vertical)

$$\begin{bmatrix} 1 & s_v & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$



Shear (horizontal)

$$\begin{bmatrix} 1 & 0 & 0 \\ s_h & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$



Affine transformations

Forward mapping:

$$(p, q) = T\left\{(m, n)\right\}$$

Problems:

1) Mapping multiple pixels to the same pixel
2) Having pixels with no intensity assigned

Inverse mapping:

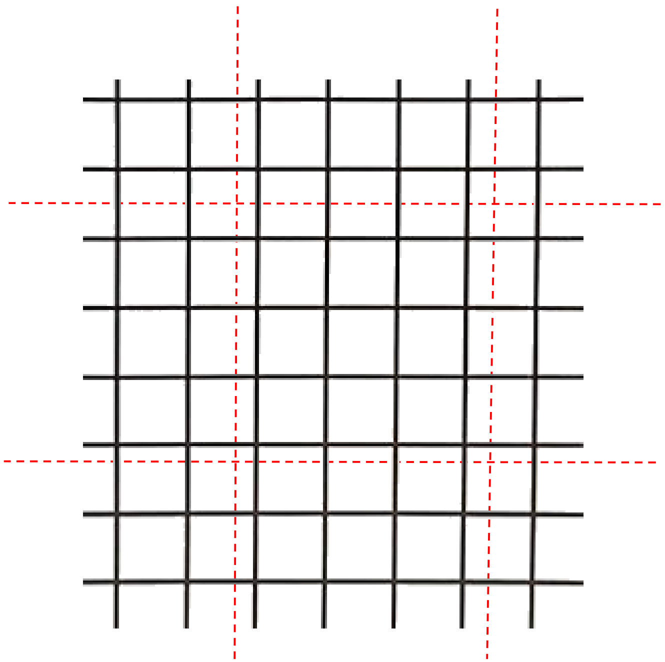
$$(m, n) = T^{-1}\left\{(p, q)\right\}$$

Image interpolation

Applications:
- Shrinking
- Zooming
- Rotating
- Geometric transformations (e.g. polar to cartesian)

Methods:
- Nearest neighbor interpolation
- Bilinear interpolation
- Bicubic interpolation

Image interpolation



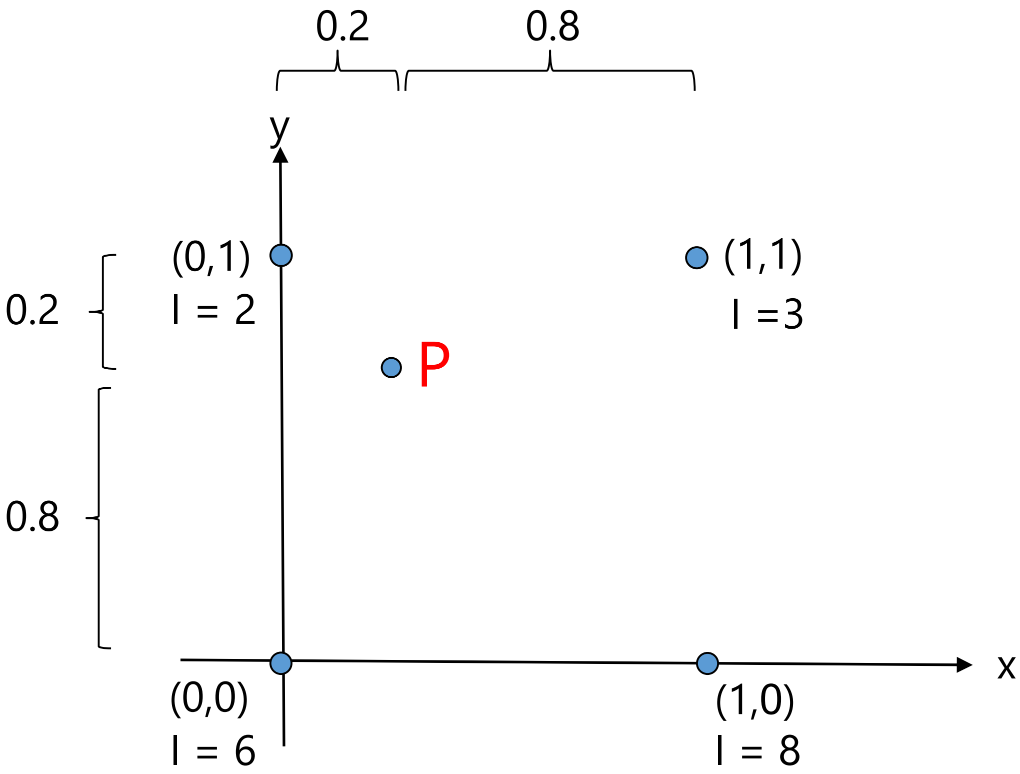


Image interpolation

Bilinear interpolation (n=1)

$$V(x,y) = \sum_{j=0}^{n} \sum_{i=0}^{n} a_{ij} x^i y^j \begin{Bmatrix} a_{00} = f(0,0) \\ a_{10} = f(1,0) - f(0,0) \\ a_{01} = f(0,1) - f(0,0) \\ a_{11} = f(1,1) + f(0,0) - f(0,1) - f(1,0) \end{Bmatrix}$$

Bicubic interpolation (n=3)

$$V(x,y) = \sum_{j=0}^{n} \sum_{i=0}^{n} a_{ij} x^i y^j$$

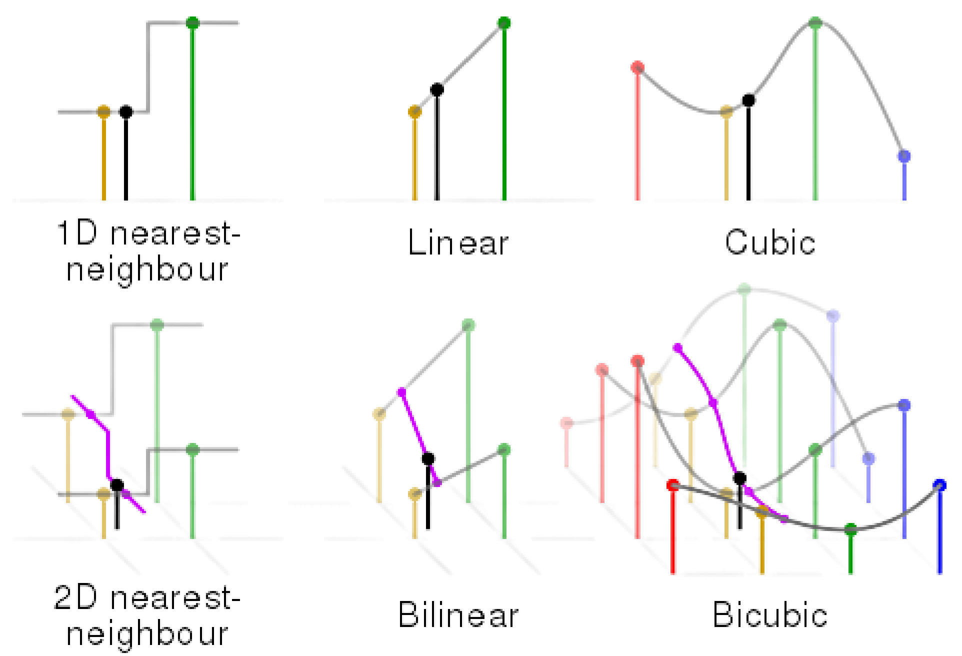


Image interpolation

Bilinear interpolation

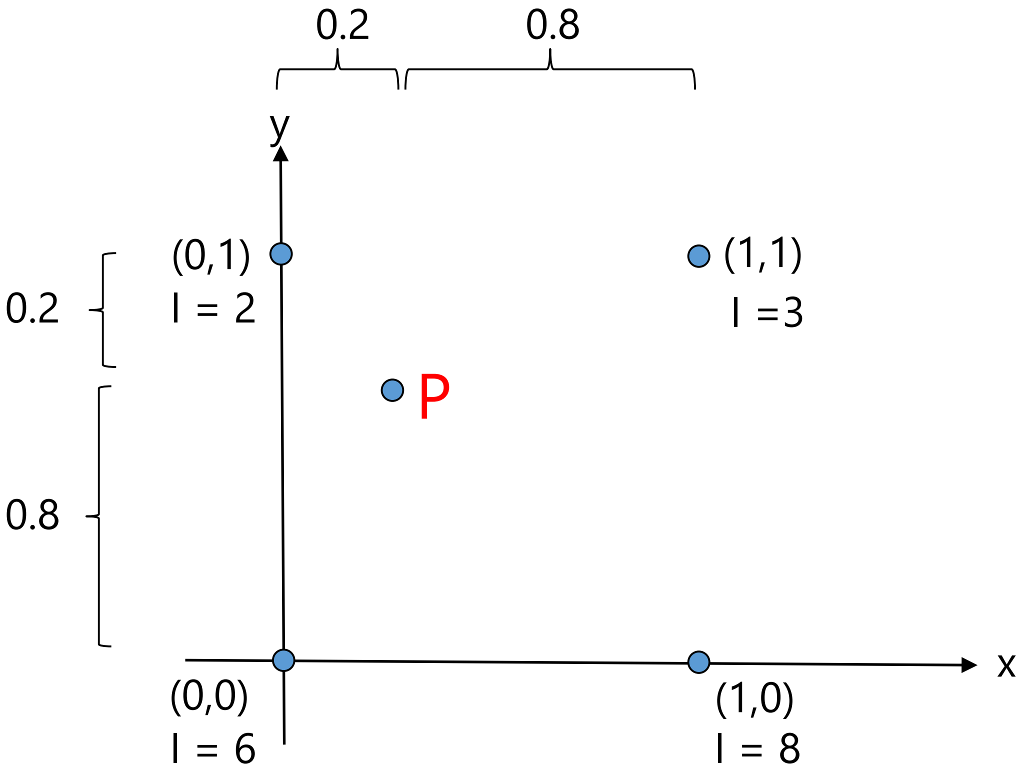
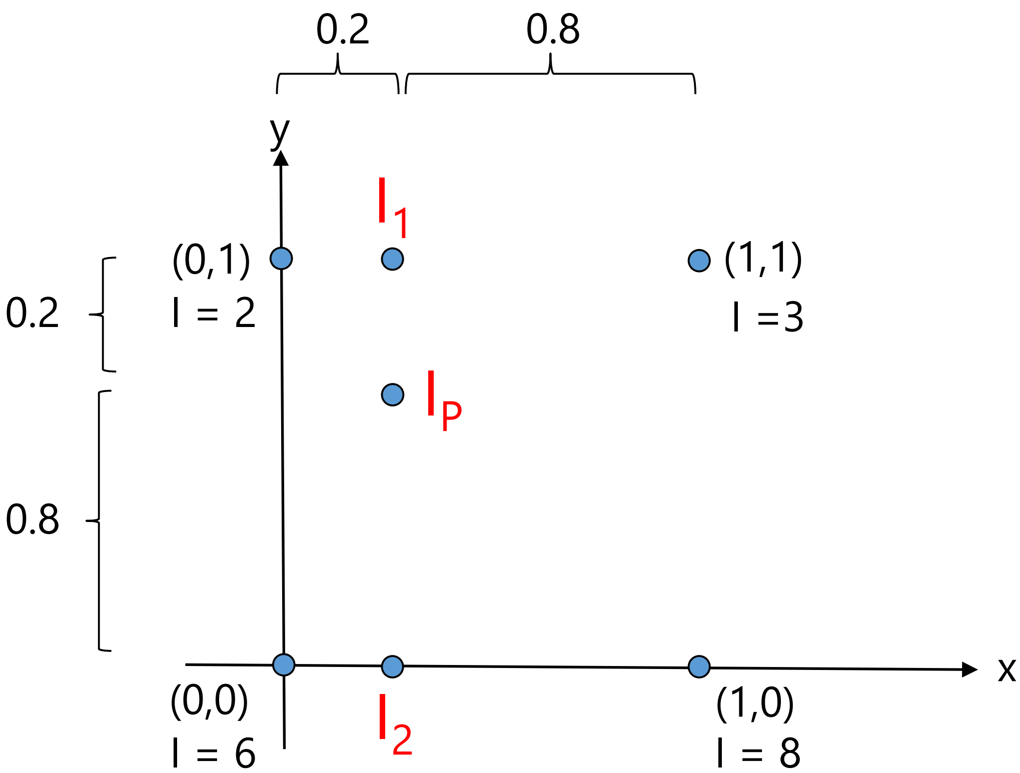


Image interpolation

Bilinear interpolation



$$I_1 = 0.8 \times 2 + 0.2 \times 3 = 2.2$$

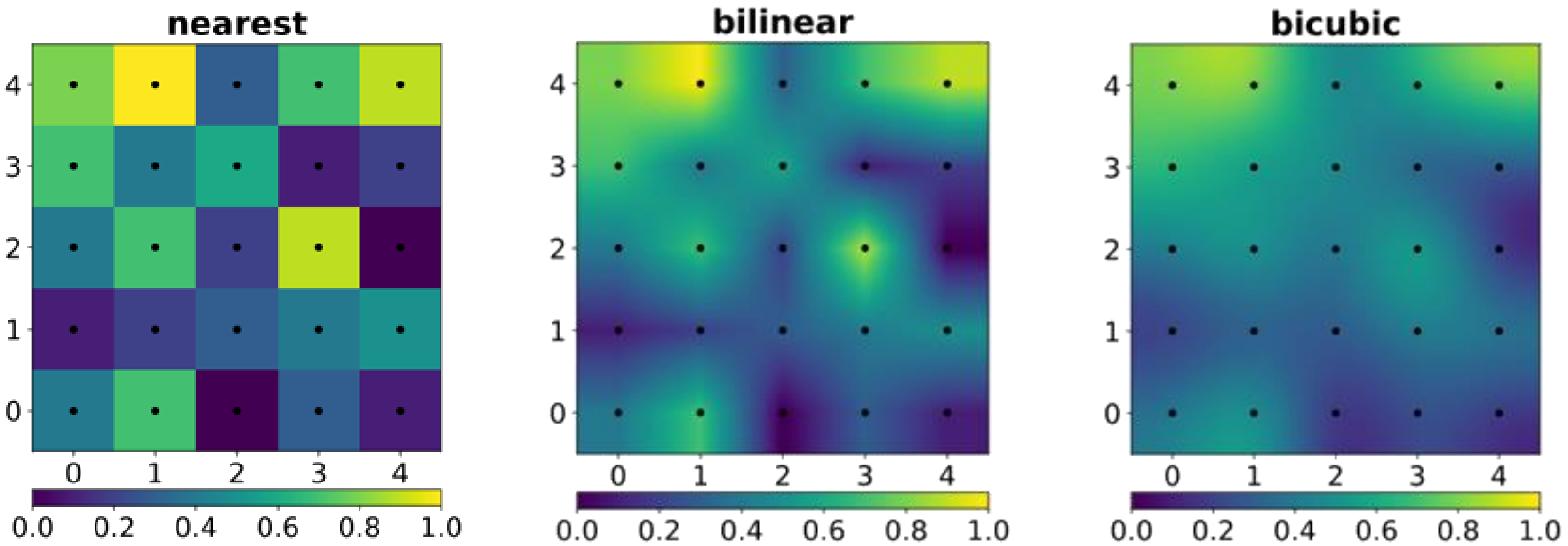
$$I_2 = 0.8 \times 6 + 0.2 \times 8 = 6.4$$

$$I_p = 0.8 \times 2.2 + 0.2 \times 6.4$$

Image interpolation

Example results:

blurr



Transform domain processing

2D linear transform:

$$T(u,v) = \sum_{y=0}^{N-1} \sum_{x=0}^{M-1} f(x,y)r(x,y,u,v)$$

(x,y) → Spatial variables
(u,v) → Transform variables
M,N → Row and column dimensions of f
$r(x,y,u,v)$ → Forward transformation kernel

Inverse transform:

$$f(x,y) = \sum_{v=0}^{N-1} \sum_{u=0}^{M-1} T(u,v)s(x,y,u,v)$$

Transform domain processing

2D linear transform:

Separable:

$$r(x, y, u, v) = r_1(x, u) r_2(y, v)$$

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Symmetry:

$$r_1(s, t) = r_2\ (s, t) \rightarrow r(x, y, u, v) = r_1(x, u) r_1(y, v)$$

Probabilistic methods

Intensity levels:

$$z_i, i = 0,1,\ldots,L-1$$

Number of occurrences of z_k in an MxN image: n_k

Probability of occurrence of z_k :

$$p(z_k) = \frac{n_k}{MN}$$

$$\sum_{k=0}^{L-1} p(z_k) = 1$$

Probabilistic methods

Mean:

$$\mu = \sum_{k=0}^{L-1} z_k p(z_k)$$

Variance:

$$\sigma^2 = \sum_{k=0}^{L-1} (z_k - \mu)^2 p(z_k)$$