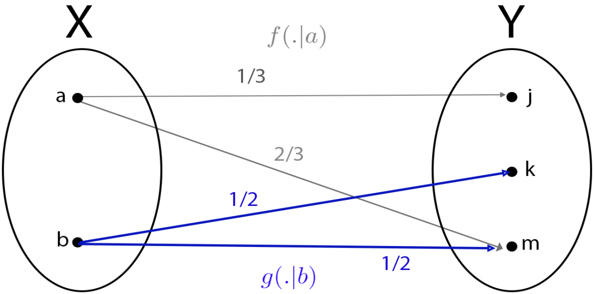
Markov Category : A 10 Minute Introduction

Mahdi Zamani

Nov 2023

Random Morphisms



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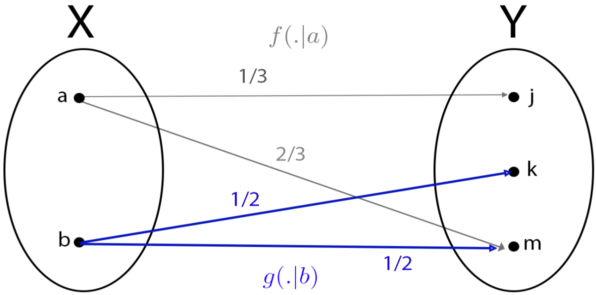
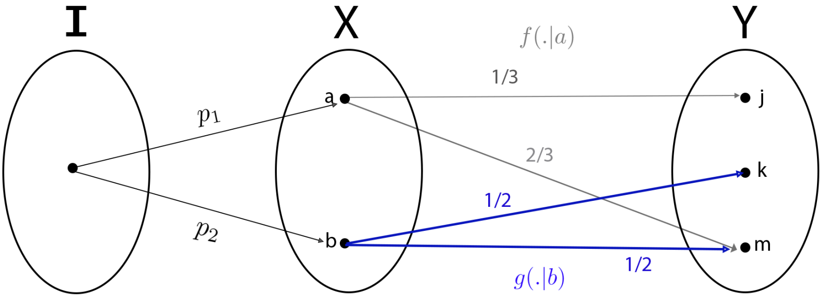


Example

FinStoch Category, with measurable sets as objects, and random morphisms (Markov kernels for Stoch).

$$\begin{array}{cc} \swarrow & \begin{array}{cc} a & b \end{array} \\ \begin{array}{c} j \\ k \\ m \end{array} & \begin{pmatrix} 1/3 & 0 \\ 0 & 1/2 \\ 2/3 & 1/2 \end{pmatrix} \end{array} \qquad f(j|a) = 1/3$$

Probability Distribution



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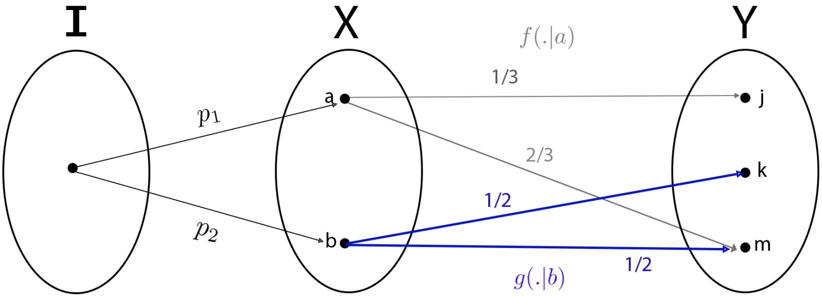
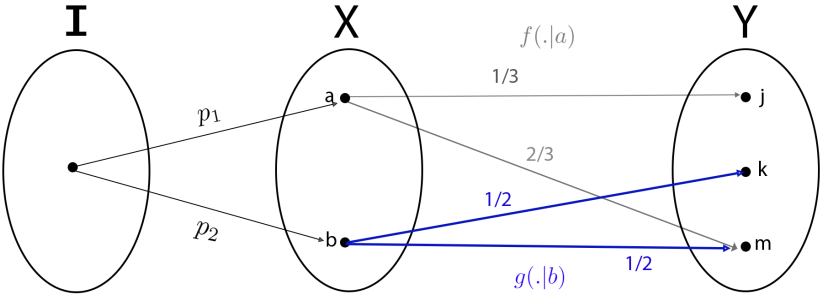


Notation

$$I \xrightarrow{P} X$$

Probability Distribution



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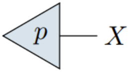

Products

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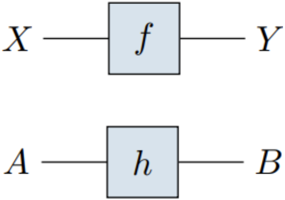

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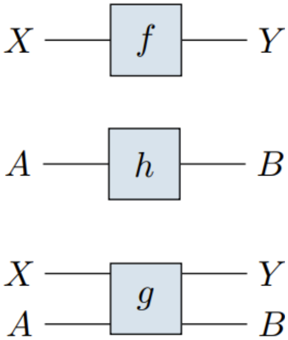

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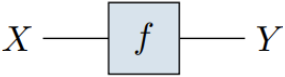

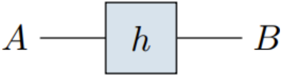

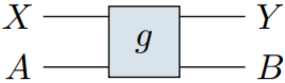

$$g : X \otimes A \to Y \otimes B$$

Markov Category [2]

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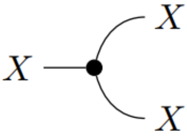

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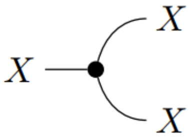
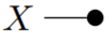


del : $X \to I$



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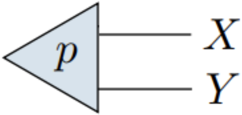

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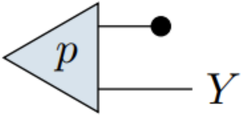
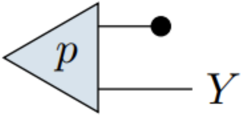


Del

$$\mathbb{P}_Y(y) = \sum_{x \in X} \mathbb{P}_{X,Y}(x, y)$$



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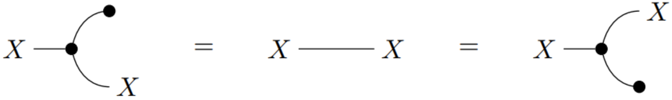

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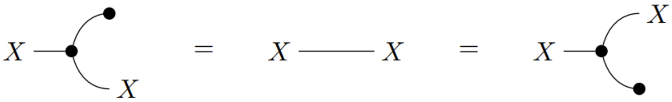

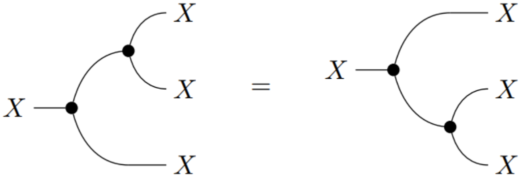

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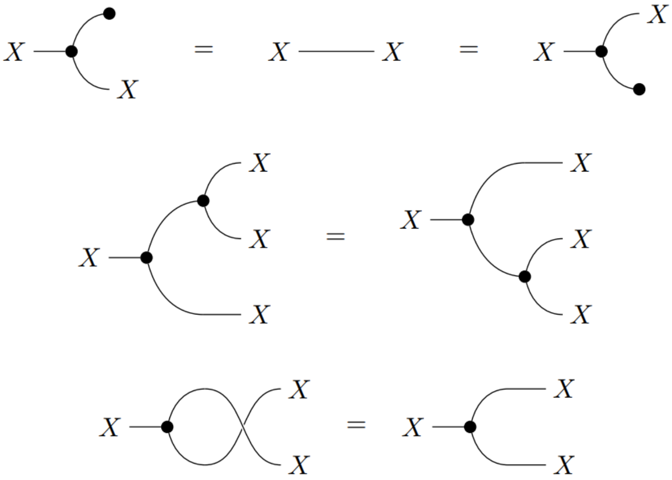

Naturality axiom

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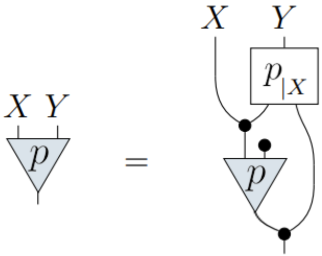

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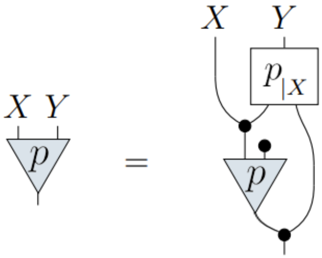

$$\mathbb{P}_{X,Y}(x,y) = \mathbb{P}_X(x)\mathbb{P}_{Y|X}(y|x)$$

Bayes

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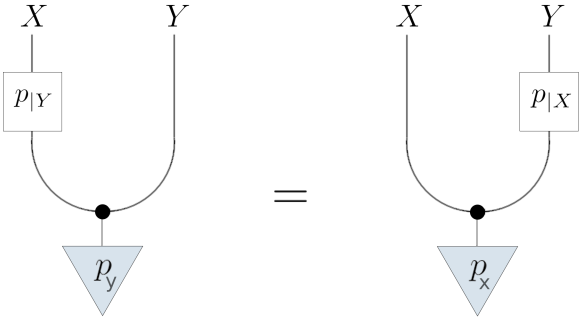

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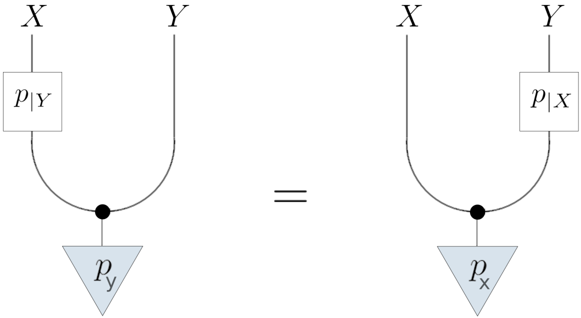

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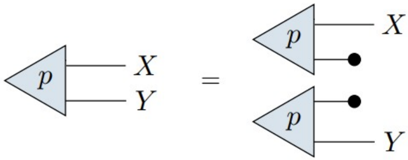

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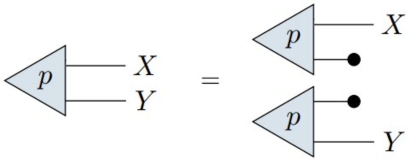

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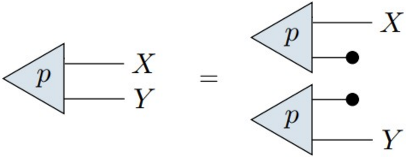

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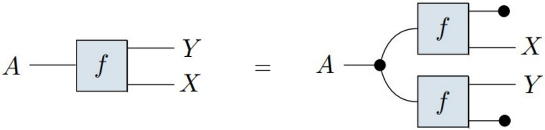

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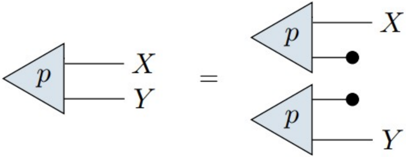

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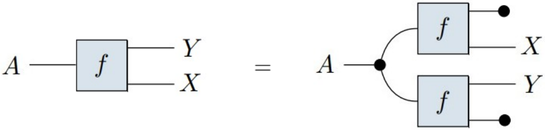

$$\mathbb{P}_{X,Y|\,A}(x, y|a) = \mathbb{P}_X(x|a)\mathbb{P}_Y(y|a)$$

Applications : Sufficient statistics [2]

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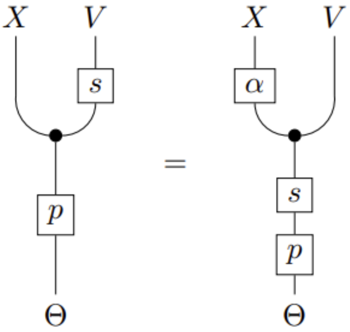
Which means that we have , $\alpha : V \to X$, s.t



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- and much more! [7]

References

[1] Brendan Fong. *Causal Theories: A Categorical Perspective on Bayesian Networks*. 2013. eprint: arXiv:1301.6201.

[2] Tobias Fritz. “A synthetic approach to Markov kernels, conditional independence and theorems on sufficient statistics”. In: (2019). DOI: 10.1016/j.aim.2020.107239. eprint: arXiv:1908.07021.

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Thank You

Questions