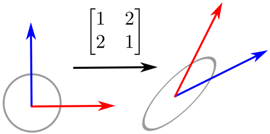
$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

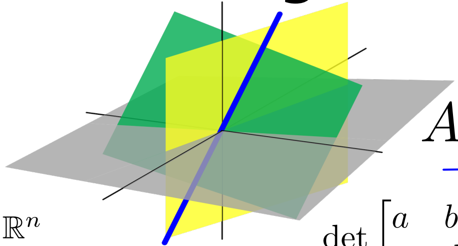


$$\begin{bmatrix} 1 & 0 & \star & \star \\ 0 & 1 & \star & \star \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$A\mathbf{x} = \mathbf{0}$$

MAT A22
Linear Algebra 1

$$|\mathbf{u} \cdot \mathbf{v}| \leq \|\mathbf{u}\|\|\mathbf{v}\|$$



$$A\mathbf{x} = \lambda\mathbf{x}$$

$$A = [a_{ij}]$$

$$A : \mathbb{R}^n \to \mathbb{R}^n$$

$$A(a\mathbf{u} + b\mathbf{v}) = aA\mathbf{u} + bA\mathbf{v}$$

$$\det \begin{bmatrix} a & b \\ c & d \end{bmatrix} = ad - bc$$

MAT A22: Linear Algebra 1 for Mathematical Sciences (Winter 2022)

Welcome to Week 8 of the course.
Questions? Thoughts? Comments?

Readings:

- 5.4 Rank of a Matrix
- 5.5 Similarity and Diagonalization

News and Reminders:

- Homework #4 Thursday March 10th at 13:00.

The Tiniest Example

Question

Let $\vec{u} \in \mathbb{R}^n$ and let $S = \{\vec{u}\}$.
For which vectors $\vec{u}$ is S linearly independent?

If $\vec{u} = \vec{0}_n$, then S is dependent.
If $\vec{u} \neq \vec{0}_n$, then S is independent.
As a consequence, $S = \{\vec{u}\}$ is independent if and only if $\vec{u} \neq \vec{0}_n$.

An Explicit Example

Question

Is $S = \left\{ \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 3 \\ 5 \end{bmatrix} \right\}$ *linearly independent?*

Abstract Example

Question

Let $\{\vec{u}, \vec{v}, \vec{w}\}$ *be an independent subset of* $\mathbb{R}^n$*.*
Is $\{\vec{u}+\vec{v}, 2\vec{u}+\vec{w}, \vec{v}-5\vec{w}\}$ *linearly independent?*

Suppose $a(\vec{u}+\vec{v}) + b(2\vec{u}+\vec{w}) + c(\vec{v}-5\vec{w}) = \vec{0}_n$ for some $a, b, c \in \mathbb{R}$. Then

$$(a+2b)\vec{u} + (a+c)\vec{v} + (b-5c)\vec{w} = \vec{0}_n.$$

Abstract Example

Question

Let $\{\vec{u}, \vec{v}, \vec{w}\}$ be an independent subset of $\mathbb{R}^n$.
Is $\{\vec{u}+\vec{v}, 2\vec{u}+\vec{w}, \vec{v}-5\vec{w}\}$ linearly independent?

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$$(a+2b)\vec{u} + (a+c)\vec{v} + (b-5c)\vec{w} = \vec{0}_n.$$

Since $\{\vec{u}, \vec{v}, \vec{w}\}$ is independent,

$$\begin{aligned} a+2b &= 0 \\ a+c &= 0 \\ b-5c &= 0. \end{aligned}$$

Abstract Example

Question

Let $\{\vec{u}, \vec{v}, \vec{w}\}$ be an independent subset of $\mathbb{R}^n$.
Is $\{\vec{u}+\vec{v}, 2\vec{u}+\vec{w}, \vec{v}-5\vec{w}\}$ linearly independent?

Suppose $a(\vec{u}+\vec{v}) + b(2\vec{u}+\vec{w}) + c(\vec{v}-5\vec{w}) = \vec{0}_n$ for some $a, b, c \in \mathbb{R}$. Then

$$(a+2b)\vec{u} + (a+c)\vec{v} + (b-5c)\vec{w} = \vec{0}_n.$$

Since $\{\vec{u}, \vec{v}, \vec{w}\}$ is independent,

$$\begin{aligned} a+2b &= 0 \\ a+c &= 0 \\ b-5c &= 0. \end{aligned}$$

This system of three equations in three variables has the unique solution $a=b=c=0$. Therefore, $\{\vec{u}+\vec{v}, 2\vec{u}+\vec{w}, \vec{v}-5\vec{w}\}$ is independent.

An Abstract Example

Question

Let $\vec{x}, \vec{y} \in \mathbb{R}^n$, $U_1 = \text{span}\{\vec{x}, \vec{y}\}$, and $U_2 = \text{span}\{2\vec{x} - \vec{y}, 2\vec{y} + \vec{x}\}$. Prove that $U_1 = U_2$.

Since $2\vec{x} - \vec{y}, 2\vec{y} + \vec{x} \in U_1$, it follows that: $\text{span}\{2\vec{x} - \vec{y}, 2\vec{y} + \vec{x}\} \subseteq U_1$.
That is: $U_2 \subseteq U_1$.
We calculate:

$$\begin{aligned}
\vec{x} &= \frac{2}{5}(2\vec{x} - \vec{y}) + \frac{1}{5}(2\vec{y} + \vec{x}), \\
\vec{y} &= -\frac{1}{5}(2\vec{x} - \vec{y}) + \frac{2}{5}(2\vec{y} + \vec{x}),
\end{aligned}$$

Therefore, $\vec{x}, \vec{y} \in U_2$. We get: $\text{span}\{\vec{x}, \vec{y}\} \subseteq U_2$. That is: $U_1 \subseteq U_2$.

Null is a Span

Example

Let A be an $m \times n$ matrix, and let $\{\vec{x}_1, \vec{x}_2, \ldots, \vec{x}_k\}$ denote a set of basic solutions to $A\vec{x} = \vec{0}_m$. Then $\vec{x}_i \in \text{null}(A)$ for each i, $1 \leq i \leq k$. It follows that

$$\text{span}\{\vec{x}_1, \vec{x}_2, \ldots, \vec{x}_k\} \subseteq \text{null}(A).$$

Conversely, every solution to $A\vec{x} = \vec{0}_m$ can be expressed as a linear combination of basic solutions, implying that

$$\text{null}(A) \subseteq \text{span}\{\vec{x}_1, \vec{x}_2, \ldots, \vec{x}_k\}.$$

Therefore, $\text{null}(A) = \text{span}\{\vec{x}_1, \vec{x}_2, \ldots, \vec{x}_k\}$.

Image is a Span

Example

Let A be an $m \times n$ matrix with columns $\vec{c}_1, \vec{c}_2, \ldots, \vec{c}_n$.

Suppose $\vec{y} \in \text{image}(A)$. Then (by definition) there is a vector $\vec{x} \in \mathbb{R}^n$ so that $\vec{y} = A\vec{x}$. Write $\vec{x} = \left[\begin{array}{cccc} x_1 & x_2 & \ldots & x_n \end{array} \right]^T$. Then

$$\vec{y} = A\vec{x} = \left[\begin{array}{cccc} \vec{c}_1 & \vec{c}_2 & \ldots & \vec{c}_n \end{array} \right] \left[\begin{array}{c} x_1 \\ x_2 \\ \vdots \\ x_n \end{array} \right] = x_1\vec{c}_1 + x_2\vec{c}_2 + \cdots + x_n\vec{c}_n.$$

Therefore, $\vec{y} \in \text{span}\{\vec{c}_1, \vec{c}_2, \ldots, \vec{c}_n\}$, implying that

$$\text{image}(A) \subseteq \text{span}\{\vec{c}_1, \vec{c}_2, \ldots, \vec{c}_n\}.$$

Find a Basis

Question

Let

$$U = \left\{ \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} \in \mathbb{R}^4 \;\middle|\; a - b = d - c \right\}.$$

Show that U is a subspace of $\mathbb{R}^4$, find a basis of U, and find $\dim(U)$.

1. U is the null space of $T_A : \mathbb{R}^4 \to \mathbb{R}$ given by:

$$T_A\left(\begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix}\right) = \begin{bmatrix} 1 & -1 & 1 & -1 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} = a - b + c - d$$

   Therefore, it is a subspace.

2. U is spanned by the basic solutions of $a - b + c - d = 0$:

$$U = \operatorname{span}\left\{ \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix} \right\}$$

   Thus, $\dim(U) = 3$.

Invertible Matrices and Bases

Question

Suppose that $B = \{\vec{x}_1, \vec{x}_2, \ldots, \vec{x}_n\}$ is a basis of $\mathbb{R}^n$ and that A is an $n \times n$ invertible matrix. Let $D = \{A\vec{x}_1, A\vec{x}_2, \ldots, A\vec{x}_n\}$. Prove that D is a basis of $\mathbb{R}^n$.

Context for Extra Problems

Definition

If U and V are subspaces of $\mathbb{R}^n$ then we can form a sum subspace:

$$U + V = \{u + v : u \in U \text{ and } v \in V\}$$

If every vector in $U + V$ can be written uniquely, then we call the sum a direct sum and write:

$$U + V = U \oplus V$$

Example

$\mathbb{R}^2 = \{[x, 0]^T : x \in \mathbb{R}\} \oplus \{[0, y]^T : y \in \mathbb{R}\}$

Question

What's an example of a sum of subspaces that is not direct?

Extra Problems (1)

Question

Let $V = F(\mathbb{R})$ be the real vector space of functions from $\mathbb{R}$ to $\mathbb{R}$. Define

$$V_e = \{f \in V \mid f(-x) = f(x)\ \forall x \in \mathbb{R}\}$$

$$V_o = \{f \in V \mid f(-x) = -f(x)\ \forall x \in \mathbb{R}\}.$$

- *Prove that $V = V_e \oplus V_o$.*
- *Give the decomposition of the function $f(x) = e^x$ according to the above direct sum.*

Extra Problems (2)

Question

Suppose U and W are both five-dimensional subspaces of $\mathbb{R}^9$. Prove that $U \cap W \neq \{0\}$.

Extra Problems (3)

Question

Check if the following statement is true. If it's true, prove it. If not , show a counter example.

$$\{M_{n\times n}\} = \{Sy_{n\times n}\} \oplus \{Sk_{n\times n}\}$$

where $\{Sy_{n\times n}\}$ *means the set of all* n *by* n *symmetric matrices and* $\{Sk_{n\times n}\}$ *means the set of all* n *by* n *skew symmetric matrices.*