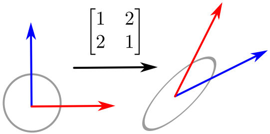
$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

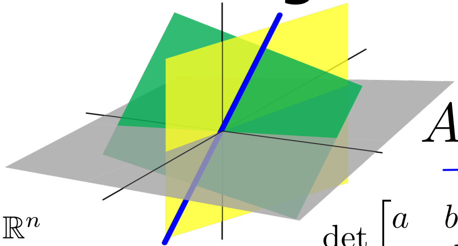


$$\begin{bmatrix} 1 & 0 & \star & \star \\ 0 & 1 & \star & \star \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$A\mathbf{x} = \mathbf{0}$

MAT A22
Linear Algebra 1

$|\mathbf{u} \cdot \mathbf{v}| \leq \|\mathbf{u}\|\|\mathbf{v}\|$



$A\mathbf{x} = \lambda\mathbf{x}$

$A = [a_{ij}]$

$A : \mathbb{R}^n \to \mathbb{R}^n$

$A(a\mathbf{u} + b\mathbf{v}) = aA\mathbf{u} + bA\mathbf{v}$

$$\det \begin{bmatrix} a & b \\ c & d \end{bmatrix} = ad - bc$$

MAT A22: Linear Algebra 1 for Mathematical Sciences (Winter 2022)

Welcome to Week 6 of the course.
Questions? Thoughts? Comments?

Readings:

- Further material on eigenvalues and eigenvectors
- 3.4 An Application to Linear Recurrences

News and Reminders:

- Next week is Reading Week! There will be no classes, tutorials, or office hours.

Cofactor Expansion

Definition

Let $A = [a_{ij}]$ be an $n \times n$ matrix. The sign of the (i,j) position is $(-1)^{i+j}$. Let A_{ij} denote the $(n-1) \times (n-1)$ matrix obtained from A by deleting row i and column j.
The (i,j)-cofactor of A is

$$c_{ij}(A) = (-1)^{i+j} \det(A_{ij}).$$

One formula for the determinant is:

$$\det A = \sum_{k=1}^{n} a_{ik} c_{ik} = a_{11}c_{11}(A) + a_{12}c_{12}(A) + a_{13}c_{13}(A) + \cdots + a_{1n}c_{1n}(A)$$

This is called the cofactor expansion of $\det A$ along row 1.

Note: This is a recursive definition. We define $\det([x]) = x$ as the base case.

Determinants and Invertible Matrices

Question

Find all values of c for which $A = \begin{bmatrix} c & 1 & 0 \\ 0 & 2 & c \\ -1 & c & 5 \end{bmatrix}$ *is invertible.*

Solution

$$\begin{aligned} \det A &= \begin{vmatrix} c & 1 & 0 \\ 0 & 2 & c \\ -1 & c & 5 \end{vmatrix} \\ &= c \begin{vmatrix} 2 & c \\ c & 5 \end{vmatrix} + (-1) \begin{vmatrix} 1 & 0 \\ 2 & c \end{vmatrix} \\ &= c(10 - c^2) - c \\ &= c(9 - c^2) = c(3 - c)(3 + c) \end{aligned}$$

Therefore, A is invertible for all $c \neq 0, 3, -3$.

The Product Formula and Transposes

Theorem (Determinant of Matrix Transpose)

If A is an $n \times n$ matrix, then $\det(A^T) = \det A$.

Question

Suppose A is a 3×3 matrix. Find $\det A$ *and* $\det B$ *if*

$$\det(2A^{-1}) = -4 = \det(A^3(B^{-1})^T).$$

Solution

$$\begin{aligned} \det A &= -2 \\ \det B &= 2 \end{aligned}$$

Proof Practice: Orthogonal Matrices

Definition

A square matrix A is orthogonal if and only if $A^T = A^{-1}$.

Question

What are the possible values of $\det A$ *if* A *is orthogonal?*

Solution

Since $A^T = A^{-1}$,

$$
\begin{aligned}
\det A^T &= \det(A^{-1}) \\
\det A &= \frac{1}{\det A} \\
(\det A)^2 &= 1
\end{aligned}
$$

This implies that $\det A = \pm 1$, *i.e.,* $\det A = 1$ *or* $\det A = -1$.

Cramer's Rule (without proof)

Theorem

Let A be an $n \times n$ invertible matrix, the solution to the system $A\vec{x} = \vec{b}$ of n equations in the variables $x_1, x_2 \cdots x_n$ is given by

$$x_1 = \frac{\det A_1}{\det A}, x_2 = \frac{\det A_2}{\det A}, \cdots, x_n = \frac{\det A_n}{\det A}$$

where, for each k, the matrix A_k is obtained from A by replacing column k with $\vec{b}$.

Cramer's rule is not used in practice but it provides a helpful theoretical corollary:

Theorem

The solutions of a linear system are polynomials in the coefficients of the system.

Cramer's Rule Example

Example

Solve for x_3:

$$\begin{cases} 3x_1 + x_2 - x_3 = -1 \\ 5x_1 + 2x_2 = 2 \\ x_1 + x_2 - x_3 = 1 \end{cases}$$

Solution

By Cramer's rule, $x_3 = \frac{\det A_3}{\det A}$, *where*

$$A = \begin{bmatrix} 3 & 1 & -1 \\ 5 & 2 & 0 \\ 1 & 1 & -1 \end{bmatrix} \text{ and } A_3 = \begin{bmatrix} 3 & 1 & -1 \\ 5 & 2 & 2 \\ 1 & 1 & 1 \end{bmatrix}.$$

Computing the determinants of these two matrices,

$$\det A = -4 \text{ and } \det A_3 = -6.$$

Cramer's Rule Example (Continued)

Example

Use Cramer's rule to solve for x_1 and x_2.

Solution

$$A_1 = \begin{bmatrix} -1 & 1 & -1 \\ 2 & 2 & 0 \\ 1 & 1 & -1 \end{bmatrix} \textit{ and } A_2 = \begin{bmatrix} 3 & -1 & -1 \\ 5 & 2 & 0 \\ 1 & 1 & -1 \end{bmatrix}$$

We obtain: $x_1 = -1$, $x_2 = \frac{7}{2}$.

Characteristic Polynomials

Definition

The characteristic polynomial of an $n \times n$ matrix A is

$$c_A(\lambda) = \det(\lambda I - A)$$

Question

Find the characteristic polynomial of $A = \left[\begin{array}{rr} 4 & -2 \\ -1 & 3 \end{array}\right]$

Solution

$$\begin{aligned} c_A(\lambda) &= \det\left(\left[\begin{array}{cc} \lambda & 0 \\ 0 & \lambda \end{array}\right] - \left[\begin{array}{rr} 4 & -2 \\ -1 & 3 \end{array}\right]\right) = \det\left[\begin{array}{cc} \lambda - 4 & 2 \\ 1 & \lambda - 3 \end{array}\right] \\ &= (\lambda - 4)(\lambda - 3) - 2 = \lambda^2 - 7\lambda + 10 = (\lambda - 5)(\lambda - 2) \end{aligned}$$

Eigenvalues and Eigenvectors

Definition

Let A be an $n \times n$ matrix.

1. The eigenvalues of A are the roots of $c_A(\lambda)$.
2. The λ-eigenvectors $\vec{x}$ are the nontrivial solutions to $(\lambda I - A)\vec{x} = \vec{0}$.

Example

Find the 2-eigenvectors of $A = \left[\begin{array}{rr} 4 & -2 \\ -1 & 3 \end{array}\right]$.

Solution

To find the 2-eigenvectors of A, solve $(2I - A)\vec{x} = \vec{0}$:

$$\left[\begin{array}{rr|r} -2 & 2 & 0 \\ 1 & -1 & 0 \end{array}\right] \rightarrow \left[\begin{array}{rr|r} 1 & -1 & 0 \\ -2 & 2 & 0 \end{array}\right] \rightarrow \left[\begin{array}{rr|r} 1 & -1 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

Example (Continued)

Example

Find the 5-eigenvectors of $A = \begin{bmatrix} 4 & -2 \\ -1 & 3 \end{bmatrix}$.

Solution

To find the 5-eigenvectors of A, solve $(5I - A)\vec{x} = \vec{0}$:

$$\left[\begin{array}{cc|c} 1 & 2 & 0 \\ 1 & 2 & 0 \end{array}\right] \rightarrow \left[\begin{array}{cc|c} 1 & 2 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

Solution

Summary of the eigenvectors for $A = \begin{bmatrix} 4 & -2 \\ -1 & 3 \end{bmatrix}$.

- *The 2-eigenvectors are solutions of $x - y = 0$.*
- *The 5-eigenvectors are solutions of $x + 2y = 0$.*

Example (Continued)

Specific examples of 2-eigenvectors and 5-eigenvectors for A:

$$\begin{bmatrix} 4 & -2 \\ -1 & 3 \end{bmatrix} \begin{bmatrix} 10 \\ 10 \end{bmatrix} = \begin{bmatrix} 20 \\ 20 \end{bmatrix} = 2 \begin{bmatrix} 10 \\ 10 \end{bmatrix}$$

$$\begin{bmatrix} 4 & -2 \\ -1 & 3 \end{bmatrix} \begin{bmatrix} -2 \\ 1 \end{bmatrix} = \begin{bmatrix} -10 \\ 5 \end{bmatrix} = 5 \begin{bmatrix} -2 \\ 1 \end{bmatrix}$$

Basic Eigenvectors and Repeated Roots

Definition

A basic eigenvector of an $n \times n$ matrix A is any nonzero multiple of a basic solution to $(\lambda I - A)\vec{x} = \vec{0}$, where λ is an eigenvalue of A.

Example

$$\left[\begin{array}{ccc} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{array}\right]$$

Find $c_A(\lambda)$, the eigenvalues of A, and find corresponding basic eigenvectors.

Solution

We have:

$$c_A(\lambda) = \det(\lambda I - A) = \det \left[\begin{array}{ccc} 1-\lambda & 0 & 0 \\ 0 & 1-\lambda & 0 \\ 0 & 0 & 2-\lambda \end{array}\right] = (\lambda-1)^2(\lambda-2) = 0 \Rightarrow \lambda = 1, 2$$

Example (Continued)

Example

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

Find basic eigenvectors for $\lambda = 1$.

Solution

We solve the homogeneous system: $(1I - A)\vec{x} = \vec{0}$.

$$\left[\begin{array}{ccc|c} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \end{array}\right] \longrightarrow \left[\begin{array}{ccc|c} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Thus, $z = 0$ *and we get the basic solutions:*

$$\begin{bmatrix} x \\ y \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} x + \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} y$$

Thus the basic 1*-eigenvectors are:* $\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \right\}$.

Example (Continued)

Example

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

Find basic eigenvectors for $\lambda = 2$.

Solution

We solve the homogeneous system: $(2I - A)\vec{x} = \vec{0}$.

$$\left[\begin{array}{ccc|c} -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \longrightarrow \left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Thus, $x = y = 0$ *and we get the basic solutions:*

$$\begin{bmatrix} 0 \\ 0 \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} z$$

Thus the basic 2*-eigenvector is:* $\left\{ \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\}$.