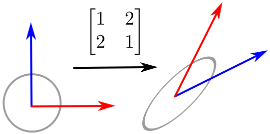
$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

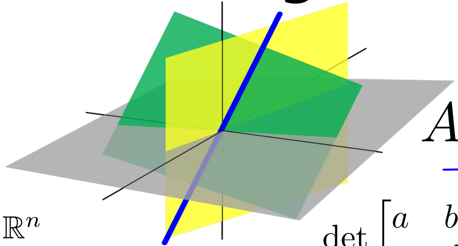



$$\begin{bmatrix} 1 & 0 & \star & \star \\ 0 & 1 & \star & \star \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$A\mathbf{x} = \mathbf{0}$$

# MAT A22
# Linear Algebra 1

$$|\mathbf{u} \cdot \mathbf{v}| \leq \|\mathbf{u}\|\|\mathbf{v}\|$$



$$A\mathbf{x} = \lambda\mathbf{x}$$

$$A = [a_{ij}]$$

$$A : \mathbb{R}^n \to \mathbb{R}^n$$

$$A(a\mathbf{u} + b\mathbf{v}) = aA\mathbf{u} + bA\mathbf{v}$$

$$\det \begin{bmatrix} a & b \\ c & d \end{bmatrix} = ad - bc$$

# MAT A22: Linear Algebra 1 for Mathematical Sciences (Winter 2022)

Welcome to Week 9 of the course.
Questions? Thoughts? Comments?

Readings:

- 6.1 Examples and Basic Properties
- 6.2 Subspaces and Spanning Sets
- 6.3 Linear Independence and Dimension

News and Reminders:

- Midterm #2 is moved to March 26th at 13-15:00.

# What is a vector space?

**Definition**

Let $V$ be a nonempty set of objects (elements) with two operations.

- Vector Addition: for any $\mathrm{v}, \mathrm{w} \in V$, the sum $\mathrm{u} + \mathrm{v} \in V$. ($V$ is closed under vector addition.)
- Scalar Multiplication: for any $\mathrm{v} \in V$ and $k \in \mathbb{R}$, the product $k\mathrm{v} \in V$. ($V$ is closed under scalar multiplication.)

Then $V$ is a vector space if it satisfies the *Axioms of Addition* and the *Axioms of Scalar Multiplication* that follow. In this case, the elements of $V$ are called vectors.

# Axioms of Addition

## Axioms of Addition

A1. Addition is commutative.
$u + v = v + u$ for all $u, v \in V$.

A2. Addition is associative.
$(u + v) + w = u + (v + w)$ for all $u, v, w \in V$.

A3. Existence of an additive identity.
There exists an element $0$ in $V$ so that $u + 0 = u$ for all $u \in V$.

A4. Existence of an additive inverse.
For each $u \in V$ there exists an element $-u \in V$ so that $u + (-u) = 0$.

# Axioms of Scalar Multiplication

## Axioms of Scalar Multiplication

S1. Scalar multiplication distributes over vector addition.
$a(\mathrm{u} + \mathrm{v}) = a\mathrm{u} + a\mathrm{v}$ for all $a \in \mathbb{R}$ and $\mathrm{u}, \mathrm{v} \in V$.

S2. Scalar multiplication distributes over scalar addition.
$(a + b)\mathrm{u} = a\mathrm{u} + b\mathrm{u}$ for all $a, b \in \mathbb{R}$ and $\mathrm{u} \in V$.

S3. Scalar multiplication is associative.
$a(b\mathrm{u}) = (ab)\mathrm{u}$ for all $a, b \in \mathbb{R}$ and $\mathrm{u} \in V$.

S4. Existence of a multiplicative identity for scalar multiplication.
$1\mathrm{u} = \mathrm{u}$ for all $\mathrm{u} \in V$.

## Example

$\mathbb{R}^n$ with matrix addition and scalar multiplication is a vector space.

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**Notes.**

- Notation: the $m \times n$ matrix of all zeros is written $0$ or $0_{mn}$.
- The vector space $\mathrm{M}_{mn}$ "is the same as" the vector space $\mathbb{R}^{mn}$. We will make this notion more precise later on. For now, notice that an $m \times n$ matrix has $mn$ entries arranged in $m$ rows and $n$ columns, while a vector in $\mathbb{R}^{mn}$ has $mn$ entries arranged in $mn$ rows and 1 column.

## Problem

Let $V$ be the set of all $2 \times 2$ matrices of real numbers whose entries sum to zero. We use the usual addition and scalar multiplication of $\mathsf{M}_{22}$. Show that $V$ is a vector space.

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## Solution

The matrices in $V$ may be described as follows:

$$V = \left\{ \left[ \begin{array}{cc} a & b \\ c & d \end{array} \right] \in \mathrm{M}_{22} \;\middle|\; a + b + c + d = 0 \right\}.$$

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What needs to be shown is closure under addition (for all $\mathrm{v}, \mathrm{w} \in V$, $\mathrm{v} + \mathrm{w} \in V$), and closure under scalar multiplication (for all $\mathrm{v} \in V$ and $k \in \mathbb{R}$, $k\mathrm{v} \in V$), as well as showing the existence of an additive identity and additive inverses in the set $V$.

## Solution (continued)

- Closure under addition

  Suppose
  $$A = \begin{bmatrix} w_1 & x_1 \\ y_1 & z_1 \end{bmatrix} \text{ and } B = \begin{bmatrix} w_2 & x_2 \\ y_2 & x_2 \end{bmatrix}$$
  are in $V$. Then $w_1 + x_1 + y_1 + z_1 = 0, w_2 + x_2 + y_2 + z_2 = 0$, and
  $$A + B = \begin{bmatrix} w_1 & x_1 \\ y_1 & z_1 \end{bmatrix} + \begin{bmatrix} w_2 & x_2 \\ y_2 & z_2 \end{bmatrix} = \begin{bmatrix} w_1 + w_2 & x_1 + x_2 \\ y_1 + y_2 & z_1 + z_2 \end{bmatrix}.$$
  Since
  $$\begin{aligned} &(w_1 + w_2) + (x_1 + x_2) + (y_1 + y_2) + (z_1 + z_2) \\ &= (w_1 + x_1 + y_1 + z_1) + (w_2 + x_2 + y_2 + z_2) \\ &= 0 + 0 = 0, \end{aligned}$$
  $A + B$ is in $V$, so $V$ is closed under addition.

## Solution (continued)

- Closure under scalar multiplication

  Suppose $A = \begin{bmatrix} w & x \\ y & z \end{bmatrix}$ is in $V$ and $k \in \mathbb{R}$.
  Then $w + x + y + z = 0$, and

$$kA = k \begin{bmatrix} w & x \\ y & z \end{bmatrix} = \begin{bmatrix} kw & kx \\ ky & kz \end{bmatrix}.$$

  Since

$$kw + kx + ky + kz = k(w + x + y + z) = k(0) = 0,$$

  $kA$ is in $V$, so $V$ is closed under scalar multiplication.

## Solution (continued)

- Existence of an additive identity

  The additive identity of $M_{22}$ is the $2 \times 2$ matrix of zeros,

$$0 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix};$$

  Since $0 + 0 + 0 + 0 = 0$, $0$ is in $V$, and has the required property (as it does in $M_{22}$).

## Solution (continued)

- Existence of an additive inverse

  Let $A = \begin{bmatrix} w & x \\ y & z \end{bmatrix}$ be in $V$.

  Then $w + x + y + z = 0$, and its additive inverse in $\mathrm{M}_{22}$ is

  $$-A = \begin{bmatrix} -w & -x \\ -y & -z \end{bmatrix}.$$

  Since

  $$(-w) + (-x) + (-y) + (-z) = -(w + x + y + x) = -0 = 0,$$

  $-A$ is in $V$ and has the required property (as it does in $\mathrm{M}_{22}$).

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5. Verify that $a \odot (b \odot (x_1, y_1)) = (ab) \odot (x_1, y_1)$.
6. Verify that $1 \odot (x, y) = (x, y)$.

### Definition

A vector space $V$ is finite dimensional if it is spanned by a finite set of vectors. Otherwise it is called infinite dimensional.

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## Example

$\mathbb{R}^n$ and $\mathrm{M}_{mn}$ are examples of finite dimensional vector spaces.
$F(\mathbb{N}) = \{f : \mathbb{N} \to \mathbb{R}\}$ is an infinite dimensional vector space.

## Question

*Prove that $F(\mathbb{N})$ is infinite dimensional.*
*That is, show that $F(\mathbb{N})$ cannot be spanned by finitely many vectors.*

# Extra Questions

## Question

$$\vec{v}_1 = (2,3,0,0) \quad \vec{v}_2 = (0,0,1,-1) \quad \vec{v}_3 = (1,0,0,4) \quad \vec{v}_4 = (0,0,0,2)$$

*Show that* $\{\vec{v}_1, \vec{v}_2, \vec{v}_3, \vec{v}_4\}$ *form a basis for* $\mathbb{R}^4$*. Find the coordinates of each of the standard basis vectors of* $\mathbb{R}^4$ *in the basis* $\{\vec{v}_1, \vec{v}_2, \vec{v}_3, \vec{v}_4\}$*.*

# Extra Questions

## Question

*Let $\mathcal{P}_n(\mathbb{R})$ be the vector space of polynomials of degree $n$.*
*Let $f^{(k)}$ denote the $k^{th}$ derivative of $f$.*
*Prove that $\{f, f', f'', \ldots, f^{(n)}\}$ is a basis for $\mathcal{P}_n(\mathbb{R})$.*