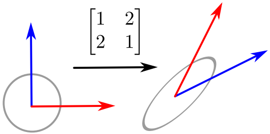
$$\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix}$$

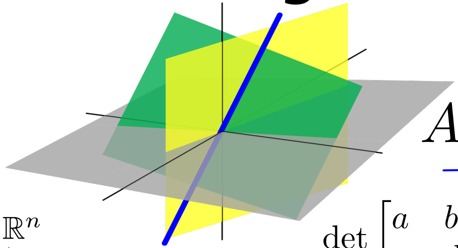


$$\begin{bmatrix}1&0&\star&\star\\0&1&\star&\star\\0&0&0&0\end{bmatrix}$$

$$A\mathbf{x}=\mathbf{0}$$

MAT A22
Linear Algebra 1

$$|\mathbf{u}\cdot\mathbf{v}|\le\|\mathbf{u}\|\|\mathbf{v}\|$$



$$A\mathbf{x}=\lambda\mathbf{x}$$

$$A=[a_{ij}]$$

$$A:\mathbb{R}^n\to\mathbb{R}^n$$

$$A(a\mathbf{u}+b\mathbf{v})=aA\mathbf{u}+bA\mathbf{v}$$

$$\det\begin{bmatrix}a&b\\c&d\end{bmatrix}=ad-bc$$

MAT A22: Linear Algebra 1 for Mathematical Sciences (Winter 2022)

Welcome to Week 10 of the course.
Questions? Thoughts? Comments?

Readings:

- 7.1 Examples and Elementary Properties
- 7.2 Kernel and Image of a Linear Transformation

News and Reminders:

- Three more weeks to go! Woo-hoo!
- Assignment #5 is due Thursday March 24th at 13:00

Definition of Linear Transformations

Definition

Let V and W be vector spaces, and $T : V \to W$ a function. Then T is called a linear transformation if it satisfies the following two properties.

1. T preserves addition.
   For all $\vec{v}_1, \vec{v}_2 \in V$, $T(\vec{v}_1 + \vec{v}_2) = T(\vec{v}_1) + T(\vec{v}_2)$.
2. T preserves scalar multiplication.
   For all $\vec{v} \in V$ and $r \in \mathbb{R}$, $T(r\vec{v}) = rT(\vec{v})$.

Note that the sum $\vec{v}_1 + \vec{v}_2$ is in V, while the sum $T(\vec{v}_1) + T(\vec{v}_2)$ is in W. Similarly, $r\vec{v}$ is scalar multiplication in V, while $rT(\vec{v})$ is scalar multiplication in W.

Important Properties of Linear Transformations

Theorem

Let V and W be vector spaces and $T : V \to W$ a linear transformation.

1. *T preserves the zero vector.*
   $T(\vec{0}) = \vec{0}$.
2. *T preserves additive inverses.*
   For all $\vec{v} \in V$, $T(-\vec{v}) = -T(\vec{v})$.
3. *T preserves linear combinations.*
   For all $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_m \in V$ and all $k_1, k_2, \ldots, k_m \in \mathbb{R}$,

$$T(k_1\vec{v}_1 + k_2\vec{v}_2 + \cdots + k_m\vec{v}_m) = k_1 T(\vec{v}_1) + k_2 T(\vec{v}_2) + \cdots + k_m T(\vec{v}_m).$$

A Linear Map of Polynomials (Solution 1)

Problem

Let $T : \mathcal{P}_2 \to \mathbb{R}$ *be a linear transformation such that*

$$T(x^2 + x) = -1;\ T(x^2 - x) = 1;\ T(x^2 + 1) = 3.$$

Find $T(4x^2 + 5x - 3)$.

Solution 1

Suppose $a(x^2+x)+b(x^2-x)+c(x^2+1)=4x^2+5x-3$. Then

$$(a+b+c)x^2+(a-b)x+c=4x^2+5x-3.$$

Solution 1

Suppose $a(x^2 + x) + b(x^2 - x) + c(x^2 + 1) = 4x^2 + 5x - 3$. Then

$$(a + b + c)x^2 + (a - b)x + c = 4x^2 + 5x - 3.$$

Solving for a, b, and c results in the unique solution $a = 6$, $b = 1$, $c = -3$.

Solution 1

Suppose $a(x^2+x)+b(x^2-x)+c(x^2+1)=4x^2+5x-3$. Then

$$(a+b+c)x^2+(a-b)x+c=4x^2+5x-3.$$

Solving for a, b, and c results in the unique solution $a=6$, $b=1$, $c=-3$. Thus

$$\begin{aligned}
T(4x^2+5x-3) &= T\left(6(x^2+x)+(x^2-x)-3(x^2+1)\right) \\
&= 6T(x^2+x)+T(x^2-x)-3T(x^2+1) \\
&= 6(-1)+1-3(3)=-14.
\end{aligned}$$

A Linear Map of Polynomials (Solution 2)

Solution 2

Notice that $S = \{x^2 + x, x^2 - x, x^2 + 1\}$ is a basis of $\mathcal{P}_2$, and thus x^2, x, and 1 can each be written as a linear combination of elements of S.

A Linear Map of Polynomials (Solution 2)

Solution 2

Notice that $S = \{x^2+x, x^2-x, x^2+1\}$ is a basis of $\mathcal{P}_2$, and thus x^2, x, and 1 can each be written as a linear combination of elements of S.

$$\begin{array}{rcl} x^2 & = & \frac{1}{2}(x^2+x)+\frac{1}{2}(x^2-x) \\ x & = & \frac{1}{2}(x^2+x)-\frac{1}{2}(x^2-x) \\ 1 & = & (x^2+1)-\frac{1}{2}(x^2+x)-\frac{1}{2}(x^2-x). \end{array}$$

A Linear Map of Polynomials (Solution 2)

Solution 2

Notice that $S = \{x^2 + x, x^2 - x, x^2 + 1\}$ is a basis of $\mathcal{P}_2$, and thus x^2, x, and 1 can each be written as a linear combination of elements of S.

$$\begin{aligned} x^2 &= \tfrac{1}{2}(x^2 + x) + \tfrac{1}{2}(x^2 - x) \\ x &= \tfrac{1}{2}(x^2 + x) - \tfrac{1}{2}(x^2 - x) \\ 1 &= (x^2 + 1) - \tfrac{1}{2}(x^2 + x) - \tfrac{1}{2}(x^2 - x). \end{aligned}$$

Then working with these (alternative basis) elements, we get:

$$\begin{aligned} T(x^2) &= T\left(\tfrac{1}{2}(x^2 + x) + \tfrac{1}{2}(x^2 - x)\right) \\ &= \tfrac{1}{2}(-1) + \tfrac{1}{2}(1) = 0. \end{aligned}$$

A Linear Map of Polynomials (Solution 2)

Solution 2

Notice that $S = \{x^2 + x, x^2 - x, x^2 + 1\}$ is a basis of $\mathcal{P}_2$, and thus x^2, x, and 1 can each be written as a linear combination of elements of S.

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x^2 &= \tfrac{1}{2}(x^2 + x) + \tfrac{1}{2}(x^2 - x) \\
x &= \tfrac{1}{2}(x^2 + x) - \tfrac{1}{2}(x^2 - x) \\
1 &= (x^2 + 1) - \tfrac{1}{2}(x^2 + x) - \tfrac{1}{2}(x^2 - x).
\end{aligned}$$

Then working with these (alternative basis) elements, we get:

$$\begin{aligned}
T(x^2) &= T\left(\tfrac{1}{2}(x^2 + x) + \tfrac{1}{2}(x^2 - x)\right) \\
&= \tfrac{1}{2}(-1) + \tfrac{1}{2}(1) = 0. \\
T(x) &= T\left(\tfrac{1}{2}(x^2 + x) - \tfrac{1}{2}(x^2 - x)\right) \\
&= \tfrac{1}{2}(-1) - \tfrac{1}{2}(1) = -1.
\end{aligned}$$

A Linear Map of Polynomials (Solution 2)

Solution 2

Notice that $S = \{x^2+x, x^2-x, x^2+1\}$ is a basis of $\mathcal{P}_2$, and thus x^2, x, and 1 can each be written as a linear combination of elements of S.

$$\begin{aligned} x^2 &= \tfrac{1}{2}(x^2+x) + \tfrac{1}{2}(x^2-x) \\ x &= \tfrac{1}{2}(x^2+x) - \tfrac{1}{2}(x^2-x) \\ 1 &= (x^2+1) - \tfrac{1}{2}(x^2+x) - \tfrac{1}{2}(x^2-x). \end{aligned}$$

Then working with these (alternative basis) elements, we get:

$$\begin{aligned} T(x^2) &= T\left(\tfrac{1}{2}(x^2+x) + \tfrac{1}{2}(x^2-x)\right) \\ &= \tfrac{1}{2}(-1) + \tfrac{1}{2}(1) = 0. \\ T(x) &= T\left(\tfrac{1}{2}(x^2+x) - \tfrac{1}{2}(x^2-x)\right) \\ &= \tfrac{1}{2}(-1) - \tfrac{1}{2}(1) = -1. \\ T(1) &= T\left((x^2+1) - \tfrac{1}{2}(x^2+x) - \tfrac{1}{2}(x^2-x)\right) \\ &= 3 - \tfrac{1}{2}(-1) - \tfrac{1}{2}(1) = 3. \end{aligned}$$

Action on a Spanning Set

Theorem

Let V and W be vector spaces, where

$$V = \text{span}\{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n\}.$$

Suppose that S and T are linear transformations from V to W. If $S(\vec{v}_i) = T(\vec{v}_i)$ for all i, $1 \leq i \leq n$, then $S = T$.

Why does this matter?

This theorem tells us that a linear transformation is completely determined by its actions on a spanning set.

Determined by a Spanning Set

Proof.

We must show that $S(\vec{v}) = T(\vec{v})$ for each $\vec{v} \in V$. Let $\vec{v} \in V$. Then (since V is spanned by $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n$), there exist $k_1, k_2, \ldots, k_n \in \mathbb{R}$ so that

$$\vec{v} = k_1 \vec{v}_1 + k_2 \vec{v}_2 + \cdots + k_n \vec{v}_n.$$

Determined by a Spanning Set

Proof.

We must show that $S(\vec{v}) = T(\vec{v})$ for each $\vec{v} \in V$. Let $\vec{v} \in V$. Then (since V is spanned by $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n$), there exist $k_1, k_2, \ldots, k_n \in \mathbb{R}$ so that

$$\vec{v} = k_1\vec{v}_1 + k_2\vec{v}_2 + \cdots + k_n\vec{v}_n.$$

It follows that

$$\begin{aligned} S(\vec{v}) &= S(k_1\vec{v}_1 + k_2\vec{v}_2 + \cdots + k_n\vec{v}_n) \\ &= k_1S(\vec{v}_1) + k_2S(\vec{v}_2) + \cdots + k_nS(\vec{v}_n) \\ &= k_1T(\vec{v}_1) + k_2T(\vec{v}_2) + \cdots + k_nT(\vec{v}_n) \\ &= T(k_1\vec{v}_1 + k_2\vec{v}_2 + \cdots + k_n\vec{v}_n) \\ &= T(\vec{v}). \end{aligned}$$

Therefore, $S = T$. □

Isomorphisms

Example

$\mathcal{P}_1 = \{ax + b \mid a, b \in \mathbb{R}\}$ has addition and scalar multiplication defined as follows:

$$\begin{array}{rcl} (a_1x + b_1) + (a_2x + b_2) & = & (a_1 + a_2)x + (b_1 + b_2), \\ k(a_1x + b_1) & = & (ka_1)x + (kb_1), \end{array}$$

for all $(a_1x + b_1), (a_2x + b_2) \in \mathcal{P}_1$ and $k \in \mathbb{R}$. This feels a lot like:

$$\mathbb{R}^2 = \left\{ \left[\begin{array}{c} a \\ b \end{array} \right] \;\middle|\; a, b \in \mathbb{R} \right\}$$

where addition and scalar multiplication are defined as follows:

$$\left[\begin{array}{c} a_1 \\ b_1 \end{array} \right] + \left[\begin{array}{c} a_2 \\ b_2 \end{array} \right] = \left[\begin{array}{c} a_1 + a_2 \\ b_1 + b_2 \end{array} \right] \qquad k \left[\begin{array}{c} a_1 \\ b_1 \end{array} \right] = \left[\begin{array}{c} ka_1 \\ kb_1 \end{array} \right]$$

for all $\left[\begin{array}{c} a_1 \\ b_1 \end{array} \right]$ and $\left[\begin{array}{c} a_2 \\ b_2 \end{array} \right] \in \mathbb{R}^2$ and $k \in \mathbb{R}$.

The Definition of Isomorphism

Definition

Let V and W be vector spaces and $T : V \to W$ a linear transformation. T is an isomorphism if and only if T is both one-to-one and onto (i.e., $\text{null}(T) = \{0\}$ and $\text{image}(T) = W$). If $T : V \to W$ is an isomorphism, then the vector spaces V and W are said to be isomorphic, and we write $V \cong W$.

Example

The identity operator $I : V \to V$ given $I(\vec{v}) = \vec{v}$ on any vector space is an isomorphism.

Onto but Not One-to-One

Problem

Let $T : M_{22} \to \mathbb{R}^2$ *be a linear transformation defined by*

$$T \left[\begin{array}{cc} a & b \\ c & d \end{array} \right] = \left[\begin{array}{c} a+d \\ b+c \end{array} \right] \text{ for all } \left[\begin{array}{cc} a & b \\ c & d \end{array} \right] \in M_{22}.$$

Prove that T *is onto but not one-to-one.*

Solutions

Let $\begin{bmatrix} x \\ y \end{bmatrix} \in \mathbb{R}^2$. Since $T\begin{bmatrix} x & y \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} x \\ y \end{bmatrix}$, T is onto.

Observe that $\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \in \ker(T)$, so $\ker(T) \neq \vec{0}_{22}$.

Invertible Matrices

Problem

Suppose U is an invertible $m \times m$ matrix and let $T : M_{mn} \to M_{mn}$ be defined by

$$T(A) = UA \text{ for all } A \in M_{mn}.$$

Prove the following:

1. *T is linear*
2. *T is one-to-one*
3. *T is onto*

Polynomials and $\mathbb{R}^{n+1}$

Problem

The map $T : \mathcal{P}_n \to \mathbb{R}^{n+1}$ defined below is an isomorphism.

$$T(a_0 + a_1x + a_2x^2 + \cdots + a_nx^n) = \begin{bmatrix} a_0 \\ a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix}$$

To verify this, prove that:

1. *T is a linear transformation*
2. *T is one-to-one*
3. *T is onto*

2×2 Symmetric Matrices

Problem

Let V denote the set of 2×2 real symmetric matrices. Then V is a vector space with dimension three. Find an isomorphism $T : \mathcal{P}_2 \to V$ with the property that $T(1) = I_2$ (the 2×2 identity matrix).

Solution

$$V = \left\{ \begin{bmatrix} a & b \\ b & c \end{bmatrix} \;\middle|\; a, b, c \in \mathbb{R} \right\} = \operatorname{span} \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right\}.$$

Solution

$$V = \left\{ \left[\begin{array}{cc} a & b \\ b & c \end{array} \right] \;\middle|\; a, b, c \in \mathbb{R} \right\} = \mathrm{span} \left\{ \left[\begin{array}{cc} 1 & 0 \\ 0 & 0 \end{array} \right], \left[\begin{array}{cc} 0 & 1 \\ 1 & 0 \end{array} \right], \left[\begin{array}{cc} 0 & 0 \\ 0 & 1 \end{array} \right] \right\}.$$

Let

$$B = \left\{ \left[\begin{array}{cc} 1 & 0 \\ 0 & 0 \end{array} \right], \left[\begin{array}{cc} 0 & 1 \\ 1 & 0 \end{array} \right], \left[\begin{array}{cc} 0 & 0 \\ 0 & 1 \end{array} \right] \right\}.$$

Then B is independent, and $\mathrm{span}(B) = V$, so B is a basis of V. Also, $\dim(V) = 3 = \dim(\mathcal{P}_2)$.

Solution

$$V = \left\{ \left[\begin{array}{cc} a & b \\ b & c \end{array} \right] \;\middle|\; a, b, c \in \mathbb{R} \right\} = \operatorname{span} \left\{ \left[\begin{array}{cc} 1 & 0 \\ 0 & 0 \end{array} \right], \left[\begin{array}{cc} 0 & 1 \\ 1 & 0 \end{array} \right], \left[\begin{array}{cc} 0 & 0 \\ 0 & 1 \end{array} \right] \right\}.$$

Let

$$B = \left\{ \left[\begin{array}{cc} 1 & 0 \\ 0 & 0 \end{array} \right], \left[\begin{array}{cc} 0 & 1 \\ 1 & 0 \end{array} \right], \left[\begin{array}{cc} 0 & 0 \\ 0 & 1 \end{array} \right] \right\}.$$

Then B is independent, and $\operatorname{span}(B) = V$, so B is a basis of V. Also, $\dim(V) = 3 = \dim(\mathcal{P}_2)$. However, we want a basis of V that contains I_2.

Solution (Continued)

Let

$$B' = \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right\}.$$

Since B' consists of $\dim(V)$ symmetric independent matrices, B' is a basis of V. Note that $I_2 \in B'$.

Solution (Continued)

Let

$$B' = \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right\}.$$

Since B' consists of $\dim(V)$ symmetric independent matrices, B' is a basis of V. Note that $I_2 \in B'$. Define

$$T(1) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, T(x) = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, T(x^2) = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}.$$

Then for all $ax^2 + bx + c \in \mathcal{P}_2$,

$$T(ax^2 + bx + c) = \begin{bmatrix} c & b \\ b & a + c \end{bmatrix},$$

and $T(1) = I_2$.

Extra Problems: An Unexpected Isomorphism

Problem

Let $\mathbb{R}^+ = \{x \in \mathbb{R} : x > 0\} = (0, \infty)$ *denote the positive numbers.*
Consider the following operations "box plus" and "box dot" on $\mathbb{R}^+$*.*

$$\lambda \boxdot x = x^\lambda \qquad x \boxplus y = xy$$

Prove that $\mathbb{R}^+$ *(with* $\boxplus$ *and* $\boxdot$*) is isomorphic* $\mathbb{R}$ *(with* $+$ *and* $\cdot$*) via* $T(2) = 1$*.*