4.

No.

$$\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \neq \begin{bmatrix} 1 \\ s \\ t \end{bmatrix}, s, t \in \mathbb{R}$$

5.

Yes (I think in fact $U = \left\{ \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \right\}$?)

$$\because x, y \in \mathbb{R} \therefore x^2, y^2 \geq 0$$

$$\because x^2 + y^2 = 0$$

$$\therefore x^2, y^2 = 0$$

$$\therefore x = 0 = y$$

6

$$U \coloneqq \left\{ \begin{bmatrix} x \\ y \end{bmatrix} \in \mathbb{R}^2 \mid x = 0 \vee y = 0 \right\}$$

Pick $\begin{bmatrix} x \\ y \end{bmatrix} \in U$

WLOG let $y = 0, x \neq 0$

Then $r \begin{bmatrix} x \\ 0 \end{bmatrix} = \begin{bmatrix} rx \\ 0 \end{bmatrix} \in U \because 0 \in \begin{bmatrix} rx \\ 0 \end{bmatrix}$

So closed under scalar multiplication.

Pick $\begin{bmatrix} x \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ y \end{bmatrix} \in U, x \neq 0 \neq y$

Then $\begin{bmatrix} x \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ y \end{bmatrix} = \begin{bmatrix} x \\ y \end{bmatrix}$

$$\because 0 \notin \begin{bmatrix} x \\ y \end{bmatrix} \therefore \begin{bmatrix} x \\ y \end{bmatrix} \notin U$$

7.

Part 1 Trivial.

Part 2 trivial by def of subspace? $0 \in V \therefore \{0\} \subseteq V$

8.

(proof of 0 in subspace ignored cause trivial)

Assume x, y are elements of subspace, r is real number/scalar.

Part 1:

$$A(x + ry) = Ax + rAy = 0 + r0 = 0$$

$$\therefore x + ry \in Ker(A)$$

Part 2:

$$\because x, y \in im(A) \therefore \exists x', y' s.t. x = Ax', y = Ay'$$

$$\therefore x + ry = Ax' + rAy' = A(x' + ry')$$

$$\because x' + ry' \in Dom(A)$$

$$\therefore x + ry = A(x' + ry') \in im(A)$$

Part 3:

Note that $E_\lambda = \{x \mid Ax = \lambda x\} \equiv \{x \mid (A - \lambda I)x = 0\} \equiv Ker(A - \lambda I)$

So by part 1 E_λ a subspace.

9.

Consider:

$$T: \mathbb{R}^3 \to \mathbb{R}^3$$

$$T[a, b, c]: [a, \quad a + 2b, \quad b]$$

Consider matrix A induced by T:

$$T(e_1) = [1,1,0]$$

$$T(e_2) = [0,2,1]$$

$$T(e_3) = [0,0,0]$$

So A is:

$$\begin{bmatrix} 1 & 0 & 0 \\ 1 & 2 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

And we done.

11.

Let $V' \coloneqq sp\{v, x_i\}_{i=1}^{n}$

Let $v = \sum_{i=1}^{n} a_i x_i$

Show $U \subseteq V', V' \subseteq U$

$$U \subseteq V'$$

Trivial because $sp\{x_i\}_{i=1}^{n} \subseteq V'$???

$$V' \subseteq U$$

Pick $z \in V'$

Then $z = b_0 v + \sum_{i=1}^{n} b_i x_i = \sum_{i=1}^{n} a_i x_i + \sum_{i=1}^{n} b_i x_i \in sp\{x_i\}_{i=1}^{n} = U$

?

QED.

12.

Then note that:

For $x = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}$

$$Ax = \sum_{i=1}^{n} a_i x_i$$

Then note as x_i is any real number/scalar, we have that $im(A) = sp\{a_i\}_{i=1}^{n}$

QED.

13.

-> Trivial, just in particular 0 has unique lin com

<-

Let $x \in sp\{x_i\}_{i=1}^{n},\ x = \sum_{i=1}^{n} a_i x_i = \sum_{i=1}^{n} b_i x_i$

$$\therefore 0 = \sum_{i=1}^{n} b_i x_i - \sum_{i=1}^{n} a_i x_i$$

$$\therefore 0 = \sum_{i=1}^{n} (b_i - a_i) x_i$$

$$\because assumption \therefore b_i - a_i = 0 \therefore b_i = a_i$$

QED.

15.

WLOG let $x_1 = 0$

Then for any $k \in \mathbb{R}$:

$$kx_1 + \sum_{i=2}^{n} a_i x_i = 0$$

Where $a_i = 0$

Then we have a lin. com. = 0 but not all scalars = 0 (in particular when k non-zero)