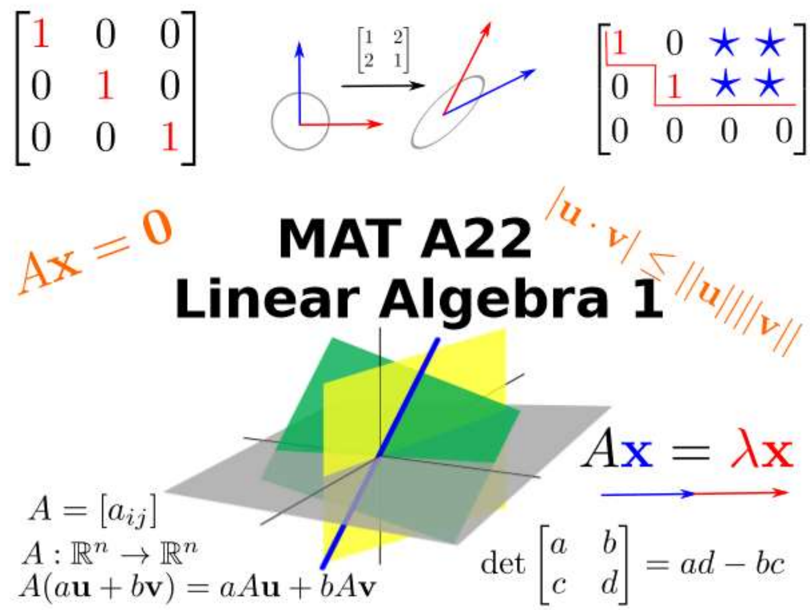
MATA22 W11 SOL



MAT A22: Linear Algebra 1 for Mathematical Sciences (Winter 2022)

Welcome to Week 11 of the course.
Questions? Thoughts? Comments?

Readings:

- 7.3 Isomorphism and Composition
- LADR: 3.A The Vector Space of Linear Maps
- LADR: 3.B Null Spaces and Ranges
- LADR: 3.C Matrices

News and Reminders:

- Assignment #6 is due Thursday April 7th at 13:00

The Space of Linear Maps

Definition (LAWA p.52)

If V and W are vector spaces then $\mathcal{L}(V, W)$ is the set of all linear maps from V to W.

Question

Suppose that W has operations $\boxplus$ and $\boxdot$ and additive identity $\vec{0}_W$.

1. *Identify the operations on $\mathcal{L}(V, W)$ that make it in to a vector space.*
2. *What is the additive identity of $\mathcal{L}(V, W)$?*

TA Notes:

- Suppose $V = \text{span}\{v_1\}$.
- For any linear map $T \in \mathcal{L}(V, V)$ we get: $T(v_1) \in V = \text{span}\{v_1\} \Rightarrow T(v_1) = t_1 v_1$.
- Picking $k = t_1$ gives $T(v) = kv$ for all $v \in V$.

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We can take advantage of the operations in the vector space W and the fact that the images of every element in $\mathcal{L}(V, W)$ are subsets of W. For $T_1, T_2 \in \mathcal{L}(V, W)$, we will define addition as

$$(T_1 + T_2)(\vec{v}) = T_1(\vec{v}) \boxplus T_2(\vec{v})$$

For $T \in \mathcal{L}(V, W)$ and $r \in \mathbb{R}$, And scalar multiplication as

$$(rT)(\vec{v}) = r \boxdot T(\vec{v})$$

Proof of vector space:

A1. Let $T_1, T_2 \in \mathcal{L}(V, W)$. Let $\vec{v}_1, \vec{v}_2 \in V$ and $r \in \mathbb{R}$

$$(\mathrm{T}_1 + T_2)(\vec{v}_1) = T_1(\vec{v}_1) \boxplus T_2(\vec{v}_1) \in W \qquad \text{(this proves } T_1 + T_2 : V \to W\text{)}$$

$$\begin{aligned}(T_1 + T_2)(\vec{v}_1 + r\vec{v}_2) &= T_1(\vec{v}_1 + r\vec{v}_2) \boxplus T_2(\vec{v}_1 + r\vec{v}_2) \\ &= T_1(\vec{v}_1) \boxplus r \boxdot T_1(\vec{v}_2) \boxplus T_2(\vec{v}_1) \boxplus r \boxdot T_2(\vec{v}_2) && \text{(linearity of } T_1, T_2\text{)} \\ &= \big(T_1(\vec{v}_1) \boxplus T_2(\vec{v})\big) \boxplus r \boxdot \big(T_1(\vec{v}_2) \boxplus T_2(\vec{v}_2)\big) && \text{(prop. of } \boxplus, \boxdot\text{)} \\ &= (T_1 + T_2)(\vec{v}_1) \boxplus r \boxdot (T_1 + T_2)(\vec{v}_2)\end{aligned}$$

(this proves linearity of $T_1 + T_2$)

So $T_1 + T_2 \in \mathcal{L}(V, W)$ and so $\mathcal{L}(V, W)$ is closed under $+$

A2. Let $T_1, T_2 \in \mathcal{L}(V, W)$. Then, for any $\vec{v} \in V$,

$$\begin{aligned}(T_1 + T_2)(\vec{v}) &= T_1(\vec{v}) \boxplus T_2(\vec{v}) \\ &= T_2(\vec{v}) \boxplus T_1(\vec{v}) && \text{(commutativity of } \boxplus\text{)} \\ &= (T_2 + T_1)(\vec{v})\end{aligned}$$

So $T_1 + T_2 = T_2 + T_1$

A3. Let $T_1, T_2, T_3 \in \mathcal{L}(V, W)$. Then, for any $\vec{v} \in V$,

$$\begin{aligned}\big(T_1 + (T_2 + T_3)\big)(\vec{v}) &= T_1(\vec{v}) \boxplus (T_2 + T_3)(\vec{v}) \\ &= T_1(\vec{v}) \boxplus \big(T_2(\vec{v}) \boxplus T_3(\vec{v})\big) \\ &= \big(T_1(\vec{v}) \boxplus T_2(\vec{v})\big) \boxplus T_3(\vec{v}) && \text{(associativity of } \boxplus\text{)} \\ &= (T_1 + T_2)(\vec{v}) \boxplus T_3(\vec{v}) \\ &= \big((T_1 + T_2) + T_3\big)(\vec{v})\end{aligned}$$

A4. We define $0_{\mathcal{L}(V,W)}$ as the map:

$0_{\mathcal{L}(V,W)}(\vec{v}) = \vec{0}_W \quad$ for all $\vec{v} \in V$

Let $T \in \mathcal{L}(V,W)$ and $\vec{v} \in V$. Observe that

$$\begin{aligned}
\left(T + 0_{\mathcal{L}(V,W)}\right)(\vec{v}) &= T(\vec{v}) \boxplus 0_{\mathcal{L}(V,W)}(\vec{v}) \\
&= T(\vec{v}) \boxplus \vec{0}_W && \text{(additive identity of } W\text{)} \\
&= T(\vec{v})
\end{aligned}$$

Thus $T + 0_{\mathcal{L}(V,W)} = T$.

A5. Let $T \in \mathcal{L}(V,W)$. We define $-T \in \mathcal{L}(V,W)$ as

$(-T)(\vec{v}) = -\big(T(\vec{v})\big) \quad$ for each $\vec{v} \in V \quad$ (we skip showing $-T \in \mathcal{L}(V,W)$)

Let $T \in \mathcal{L}(V,W)$ and $\vec{v} \in V$. Observe that

$$\begin{aligned}
\big(T + (-T)\big)(\vec{v}) &= T(\vec{v}) \boxplus (-T)(\vec{v}) \\
&= T(\vec{v}) \boxplus -\big(T(\vec{v})\big) \\
&= \vec{0}_W \\
&= 0_{\mathcal{L}(V,W)}(\vec{v})
\end{aligned}$$

Thus $T + (-T) = 0_{\mathcal{L}(V,W)}$

S1. Let $r \in \mathbb{R}, \vec{v} \in V$ and $T \in \mathcal{L}(V,W)$. Then

$(rT)(\vec{v}) = r \boxdot T(\vec{v}) \in W$

So $rT: V \to W$. Now linearity!

Let $\vec{v}_1, \vec{v}_2 \in V$ and $k \in \mathbb{R}$. Then

$$\begin{aligned}
rT(\vec{v}_1 + k\vec{v}_2) &= r \boxdot T(\vec{v}_1 + k\vec{v}_2) \\
&= r \boxdot \big(T(\vec{v}_1) \boxplus k \boxdot T(\vec{v}_2)\big) && \text{(linearity of } T_1\text{)} \\
&= r \boxdot T(\vec{v}_1) \boxplus r \boxdot \big(k \boxdot T(\vec{v}_2)\big) && \text{(distributivity of } \boxdot\text{)} \\
&= (rT)(\vec{v}_1) \boxplus k \boxdot \big(r \boxdot T(\vec{v}_2)\big) && \text{(associativity of } \boxdot\text{)} \\
&= (rT)(\vec{v}_1) \boxplus k \boxdot (rT)(\vec{v}_2)
\end{aligned}$$

So $rT \in \mathcal{L}(V,W)$

S2. Let $a \in \mathbb{R}$ and $T_1, T_2 \in \mathcal{L}(V,W)$. Then, for any $\vec{v} \in V$,

$$\begin{aligned}
\big(a(T_1 + T_2)\big)(\vec{v}) &= a \boxdot (T_1 + T_2)(\vec{v}) \\
&= a \boxdot \big(T_1(\vec{v}) \boxplus T_2(\vec{v})\big) \\
&= a \boxdot T_1(\vec{v}) \boxplus a \boxdot T_2(\vec{v}) && \text{(distributivity of } \boxdot\text{)} \\
&= (aT_1)(\vec{v}) \boxplus (aT_2)(\vec{v}) \\
&= (aT_1 + aT_2)(\vec{v})
\end{aligned}$$

So $a(T_1 + T_2) = aT_1 + aT_2$

S3. Let $a, b \in \mathbb{R}$ and $T \in \mathcal{L}(V,W)$. Let $\vec{v} \in V$, then

$$\begin{aligned}
\big((a+b)T\big)(\vec{v}) &= (a+b) \boxdot T(\vec{v}) \\
&= a \boxdot T(\vec{v}) \boxplus b \boxdot T(\vec{v}) && \text{(distributivity of } \boxdot \text{ over scalar } +\text{)} \\
&= (aT)(\vec{v}) \boxplus (bT)(\vec{v}) \\
&= (aT + bT)(\vec{v})
\end{aligned}$$

So $(a+b)T = aT + bT$

S4. Let $a, b \in \mathbb{R}$ and $T \in \mathcal{L}(V,W)$. Let $\vec{v} \in V$, then

$$
\begin{aligned}
\big(a(bT)\big)(\vec{v}) &= a \boxdot (bT)(\vec{v}) \\
&= a \boxdot \big(b \boxdot T(\vec{v})\big) \\
&= (ab) \boxdot T(\vec{v}) && (\boxdot \text{ properties in } W) \\
&= \big((ab)T\big)(\vec{v})
\end{aligned}
$$

So $a(bT) = (ab)T$

S5. Let $T \in \mathcal{L}(V, W)$. Then

$$
\begin{aligned}
(1T)(\vec{v}) &= 1 \boxdot T(\vec{v}) \\
&= T(\vec{v})
\end{aligned}
$$

So $1T = T$

One-Dimensional Vector Spaces

Question

Suppose that V is one dimensional.
Show that every $T \in \mathcal{L}(V, V)$ has the form $T(v) = kv$ for some k.

TA Notes:

- Suppose $V = \text{span}\{v_1\}$.
- For any linear map $T \in \mathcal{L}(V, V)$ we get: $T(v_1) \in V = \text{span}\{v_1\} \Rightarrow T(v_1) = t_1 v_1$.
- Picking $k = t_1$ gives $T(v) = kv$ for all $v \in V$.

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Since V has dimension 1, there exists a basis $\{\vec{v}_1\}$ for V. Let $T \in \mathcal{L}(V, V)$ and $\vec{v} \in V$. Since $\vec{v} \in V$ and $T: V \to V$, we must have $T(\vec{v}) \in V$. So, there exists $a, b \in \mathbb{R}$ such that $\vec{v} = a\vec{v}_1$ and $T(\vec{v}) = b\vec{v}_1$. Thus

$$
\begin{aligned}
T(\vec{v}) &= T(a\vec{v}_1) \\
&= aT(\vec{v}_1) \\
&= a(b\vec{v}_1) && (T(\vec{v}) = b\vec{v}_1) \\
&= b(a\vec{v}_1) \\
&= b\vec{v}
\end{aligned}
$$

So take $k = b$ and you will have $T(\vec{v}) = k\vec{v}$ for any $\vec{v} \in V$

Products

Definition (LADR p.55)

If $T \in \mathcal{L}(U, V)$ and $S \in \mathcal{L}(V, W)$ then $ST \in \mathcal{L}(U, W)$ is given:

$$ST(u) = S(T(u))$$

(This will turn out to be the product of the matrices for S and T.)

Question

Verify that $ST \in \mathcal{L}(U, W)$. Check linearity and that the composition makes sense.

1. Let $S, T \in \mathcal{L}(U, W)$ and let $u \in U$. Then

$$\begin{aligned} ST(u) &= S\big(T(u)\big) \\ &= S(v) && (T(u) \in V \text{ by definition of } T) \\ &\in W \end{aligned}$$

So $ST: U \to W$

2. Let $S, T \in \mathcal{L}(U, W)$ and let $u_1, u_2 \in U$ and $r \in \mathbb{R}$

$$\begin{aligned} ST(u_1 + ru_2) &= S\big(T(u_1 + ru_2)\big) \\ &= S\big(T(u_1) + rT(u_2)\big) && \text{(linearity of } T\text{)} \\ &= S\big(T(u_1)\big) + rS\big(T(u_2)\big) && \text{(linearity of } S\text{)} \\ &= ST(u_1) + rST(u_2) \end{aligned}$$

So ST is linear!

Meaning $ST \in \mathcal{L}(U, W)$

Towards Matrices

Question

Suppose that $\dim(V) = n$ *and* $\dim(W) = k$.
Informally show that $\dim(\mathcal{L}(V, W)) = nk$.

- Pick bases on both sides. $V = \text{span}\{v_1, \ldots, v_n\}$ and $W = \text{span}\{w_1, \ldots, w_k\}$.
- $T(v) = T(t_1 v_1 + \cdots + t_n v_n) = t_1 T(v_1) + \ldots t_n T(v_n)$.
  The output of each $T(v_i)$ has at most k non-zero coefficients.
- Any linear map is determind by nk coefficients.

The formal proof is to define the basic maps $T_{ij}(v_i) = w_j$ where v_i is sent to w_j and every other basis element is sent to zero. You then must show these span and are linearly independent.

Suppose $\{v_1, \ldots, v_n\}$ is a basis for V and $\{w_1, \ldots, w_k\}$ is a basis for W. Let $T \in \mathcal{L}(V, W)$ and $v \in V$. Then

$$\begin{aligned} T(v) &= T(a_1 v_1 + \cdots + a_n v_n) \\ &= a_1 T(v_1) + \cdots + a_n T(v_n) \\ &= a_1\left(b_{1,1} w_1 + \cdots + b_{1,k} w_k\right) + \cdots + a_n\left(b_{n,1} w_1 + \cdots + b_{n,k} w_k\right) \\ &= a_1 b_{1,1} w_1 + \cdots + a_1 b_{1,k} w_k + \cdots + a_n b_{n,1} w_1 + \cdots + a_n b_{n,k} w_k \end{aligned}$$

Notice that the value of $T(v)$ is determined completely by nk coefficients:

	1	**2**	$\cdots$	$\boldsymbol{n}$
1	$a_1 b_{1,1}$	$a_2 b_{2,1}$	$\cdots$	$a_n b_{n,1}$
2	$a_1 b_{1,2}$	$a_2 b_{2,2}$	$\cdots$	$a_n b_{n,2}$
$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$
$\boldsymbol{k}$	$a_1 b_{1,k}$	$a_2 b_{2,k}$	$\cdots$	$a_n b_{n,k}$

Find a Linear Dependence

Question

Consider the linear maps $T^k \in \mathcal{L}(\mathbb{R}^2, \mathbb{R}^2)$ given by powers of:

$$T = \begin{bmatrix} 4 & 1 \\ 2 & -1 \end{bmatrix}$$

The powers of T are linearly dependent because $\mathcal{L}(\mathbb{R}^2, \mathbb{R}^2)$ is finite dimensional. Find a linear dependence.

TA Note: Computing powers we get:

$$T^0 = I \quad T = \begin{bmatrix} 4 & 1 \\ 2 & -1 \end{bmatrix} \quad T^2 = \begin{bmatrix} 18 & 3 \\ 6 & 3 \end{bmatrix} = 3 \begin{bmatrix} 6 & 1 \\ 2 & 1 \end{bmatrix}$$

This gives:

$$\begin{bmatrix} 4 & 1 \\ 2 & -1 \end{bmatrix}^2 - 3 \begin{bmatrix} 4 & 1 \\ 2 & -1 \end{bmatrix} + 6I = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

Notice that $\{T^0, T^1, T^2, \dots\}$ is an infinite set of vectors in $\mathcal{L}(\mathbb{R}^2, \mathbb{R}^2)$. We argued previously that $\dim(\mathcal{L}(\mathbb{R}^2, \mathbb{R}^2)) = 2 \cdot 2 = 4$. So, it is definitely linearly dependent.

$T^0 = I$
$T = \begin{bmatrix} 4 & 1 \\ 2 & -1 \end{bmatrix}$
$T^2 = \begin{bmatrix} 18 & 3 \\ 6 & 3 \end{bmatrix}$

Set up independence relation:

$c_0 T^0 + c_1 T + c_2 T^2 = \vec{0}$

This gives linear system:

$c_0 + 4c_1 + 18c_2 = 0$
$c_1 + 3c_2 = 0$
$2c_1 + 6c_2 = 0$
$c_0 - c_1 + 3c_2 = 0$

Non-trivial solution: $c_0 = -6, c_1 = -3, c_2 = 1$

Give an Example (1)

Question

Give an example of a linear map T with $\dim(\text{null}(T)) = 3$ *and* $\dim(\text{image}(R)) = 2$.

TA Note: Let $T : \mathbb{R}^5 \to \mathbb{R}^5$.

$$T\left(\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix}\right) = \begin{bmatrix} x_1 \\ x_2 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

Let $T: V \to W$ be a linear map. By Rank-Nullity Theorem (Fundamental Theorem of Linear Transformations),

$$\dim\big(\text{null}(T)\big) + \dim\big(\text{image}(T)\big) = \dim V$$
$$\Rightarrow \dim V = 3 + 2 = 5$$

Let's take V to be the easiest 5-dimensional vector space: $\mathbb{R}^5$
Since $\text{image}(T) \subseteq W$, we need W to be at least 2-dimensional. Let's take $\mathbb{R}^2 = W$

Then define $T: \mathbb{R}^5 \to \mathbb{R}^2$ as

$$T\left(\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix}\right) = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

1. We first show that $\dim\big(\text{null}(T)\big) = 3$.

   We want to solve $T(\vec{x}) = \vec{0} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$. We will show that $\left\{\begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}\right\}$ is a basis for the null space.

   Let $x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} \in \text{null}(T)$. Then $T(x) = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$. Note then that $x_1 = x_2 = 0$. So $x = \begin{bmatrix} 0 \\ 0 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix}$.
   We now suppose

   $$x = c_1 \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} + c_2 \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} + c_3 \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 0 \\ 0 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ c_1 \\ c_2 \\ c_3 \end{bmatrix}$$

Thus $c_1 = x_3, c_2 = x_4, c_3 = x_5$ is the unique linear combination of the vectors in $\left\{ \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \right\}$, proving the null space has dimension 3.

2. By Rank-Nullity, we must have $\dim(\text{null}(T)) + \dim(\text{image}(T)) = 5$

$$\Rightarrow 3 + \dim(\text{image}(T)) = 5$$
$$\Rightarrow \dim(\text{image}(T)) = 2$$

Give an Example (2)

Question

Give an example of a linear map $T \in \mathcal{L}(\mathbb{R}^4, \mathbb{R}^4)$ with $\dim(\text{null}(T)) = \dim(\text{image}(T))$.

TA Note: Let $T : \mathbb{R}^4 \to \mathbb{R}^4$.

$$T\left(\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} \right) = \begin{bmatrix} x_1 \\ 0 \\ x_3 \\ 0 \end{bmatrix}$$

By Rank-Nullity, notice that

$$\dim(\text{null}(T)) + \dim(\text{image}(T)) = \dim(\mathbb{R}^4)$$
$$\Rightarrow 2\dim(\text{null}(T)) = 4$$
$$\Rightarrow \dim(\text{null}(T)) = 2$$

So we need a transformation with nullity 2 and rank 2. Let's take

$$T\left(\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} \right) = \begin{bmatrix} x_1 \\ 0 \\ x_3 \\ 0 \end{bmatrix}$$

1. We first show that $\dim(\text{null}(2)) = 3$.

We want to solve $T(\vec{x}) = \vec{0} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$. We will show that $\left\{ \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \right\}$ is a basis for the null space.

Let $x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} \in \text{null}(T)$. Then $T(x) = \begin{bmatrix} x_1 \\ 0 \\ x_3 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$. Note then that $x_1 = x_3 = 0$. So $x = \begin{bmatrix} 0 \\ x_2 \\ 0 \\ x_4 \end{bmatrix}$. We now suppose

$$x = c_1 \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} + c_2 \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 0 \\ x_2 \\ 0 \\ x_4 \end{bmatrix} = \begin{bmatrix} 0 \\ c_1 \\ 0 \\ c_2 \end{bmatrix}$$

Thus $c_1 = x_2, c_2 = x_4$ is the unique linear combination of the vectors in $\left\{ \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \right\}$, proving the null space has dimension 2.

2. By Rank-Nullity, we must have $\dim(\text{null}(T)) + \dim(\text{image}(T)) = 4$

$$\Rightarrow 2 + \dim(\text{image}(T)) = 4$$
$$\Rightarrow \dim(\text{image}(T)) = 2$$

Prove There is No Example (2)

Question

Prove there is no example of a linear map $T \in \mathcal{L}(\mathbb{R}^5, \mathbb{R}^5)$ with

$$\dim(\text{null}(T)) = \dim(\text{image}(T))$$

TA Note: By rank-nullity:

$$5 = \dim(\mathbb{R}^5) = \dim(\text{null}(T)) + \dim(\text{image}(T))$$

However, dimension is always a whole number and five is odd.
Therefore, there is no such map.

Suppose, for contradiction, there is such $T \in \mathcal{L}(\mathbb{R}^5, \mathbb{R}^5)$. By rank-nullity,

$$\dim(\text{null}(T)) + \dim(\text{image}(T)) = \dim(\mathbb{R}^5)$$
$$\Rightarrow 2\dim(\text{null}(T)) = 5$$

$$\Rightarrow \dim(\text{null}(T)) = \frac{5}{2} \notin \mathbb{Z}$$

We obtain a contradiction since dimension must be an integer! So such a T cannot exist.