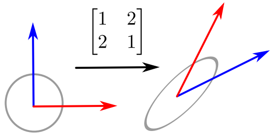
$$\begin{bmatrix} \color{red}{1} & 0 & 0 \\ 0 & \color{red}{1} & 0 \\ 0 & 0 & \color{red}{1} \end{bmatrix}$$

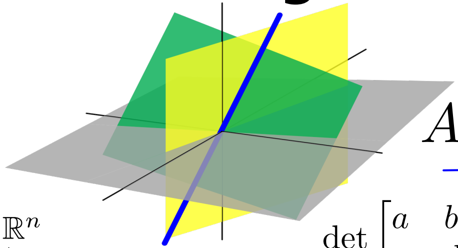


$$\begin{bmatrix} \color{red}{1} & 0 & \star & \star \\ 0 & \color{red}{1} & \star & \star \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$A\mathbf{x} = \mathbf{0}$$

MAT A22
Linear Algebra 1

$$|\mathbf{u} \cdot \mathbf{v}| \leq \|\mathbf{u}\|\|\mathbf{v}\|$$



$$A\mathbf{x} = \lambda\mathbf{x}$$

$$A = [a_{ij}]$$

$$A : \mathbb{R}^n \to \mathbb{R}^n$$

$$A(a\mathbf{u} + b\mathbf{v}) = aA\mathbf{u} + bA\mathbf{v}$$

$$\det \begin{bmatrix} a & b \\ c & d \end{bmatrix} = ad - bc$$

MAT A22: Linear Algebra 1 for Mathematical Sciences (Winter 2022)

Welcome to Week 12 of the course.
Questions? Thoughts? Comments?

Readings:

- LADR: 1.C Subspaces (Direct Sums)
- LADR: 3.E Products and Quotients of Vector Spaces

News and Reminders:

- This is the last tutorial!
- Assignment #6 is due Thursday April 7th at 13:00

Affine Subsets

Definition

Let $U \subseteq V$ be a subspace of V. An <u>affine subset</u> of V is $v + U = \{v + u : u \in U\}$.
"$v + U$ is the subspace U shifted by v"

Example

Let $U = \{(x, y) : y = 2x\} \subset \mathbb{R}^2$. Sketch $S_1 = (0, 1) + U$ and $S_2 = (1, 3) + U$.

TA Notes:

- S_1 is obtained by shifting the line $y = 2x$ up by one unit to obtain $y = 2x + 1$.
- For example, $(2, 4) \in U$ and thus $(2, 5) = (0, 1) + (2, 4) \in (0, 1) + U$.
- Notice $(1, 2) \in U$. This gives:

$$S_2 = (1, 3) + U = (0, 1) + (1, 2) + U = (0, 1) + U = S_1$$

Absorption

Theorem

Suppose that U is a subspace. $v + U = U$ if and only if $v \in U$.

TA Notes:

- Suppose $v + U = U$. We have $v + u \in U$ for all $u \in U$. Thus, $v + 0 = v \in U$.
- Suppose that $v \in U$. We have $v + u$ is a sum of elements in U. Thus, $v + u = u'$ and $v + U \subseteq U$. For any $u \in U$ we can write $u = v + (u - v)$ for some $u - v \in U$. Thus, $U \subseteq v + U$.

Affine Subsets Contain Line Segments

Theorem

A non-empty subset $A \subset V$ is an affine subset if and only if $tv + (1-t)w \in A$ for any $v, w \in A$ and $t \in \mathbb{R}$.

TA Notes:

1. Draw a picture to suggest $tv + (1-t)w$ is a line through v and w.
2. Suppose that $A = v_0 + U$ for some subspace U and vector $v_0 \in V$. We get:

$$tv + (1-t)w = t(v_0 + u_1) + (1-t)(v_0 + u_2) = v_0 + (tu_1 + (1-t)u_2)$$

   U is a subspace and so $(tu_1 + (1-t)u_2) \in U$. Thus, $tv + (1-t)w \in v_0 + U$.
3. Suppose $tv + (1-t)w \in A$ for any $v, w \in A$ and $t \in \mathbb{R}$. Pick $v \in A$ and define $U = A - v = (-v) + A$. This is a subspace (proof on the following slide).

Proof that $U = (-v) + A$ is a subspace

1. We picked $v \in A$, thus $0 = -v + v \in (-v) + A$.
2. Thus, $(-v) + A$ contains the zero vector.
3. Pick $-v + a \in (-v) + A$.
4. $k(-v + a) = -kv + ka = -v - (k-1)v + ka$
5. Set $t = -(k-1)$. This gives $1 - t = 1 - [-(k-1)] = k$.
6. We obtain that $-(k-1)v + ka = tv + (1-t)a$.
7. By the closure of A under lines we get: $k(-v + a) = -v + a'$ for $a' \in A$.
8. Thus, $-v + A$ is closed under scaling.
9. Pick $a_1, a_2 \in A$. We have $v \in A$ by hypothesis.
10. We need $(-v + a_1) + (-v + a_2) = -v + a' \in -v + A$.
11. Closure under lines (from a_1 to a_2) implies $\frac{1}{2}a_1 + \frac{1}{2}a_2 \in A$.
12. Closure under lines (from $\frac{1}{2}a_1 + \frac{1}{2}a_2$ to $-v$) implies:
$$2\left(\frac{1}{2}a_1 + \frac{1}{2}a_2\right) + (1-2)v = a_1 + a_2 - v \in A$$
13. This gives: $(-v + a_1) + (-v + a_2) = -v + (a_1 + a_2 - v) = -v + a' \in -v + A$.
14. Thus, $(-v) + A$ is closed under addition.

Quotient Spaces

Definition

Suppose that $U \subseteq V$ is a subspace. The quotient space $V/U = \{v + U : v \in V\}$ is the set of all affine subspaces. The vector space operations of V/U are defined as:

1. $k(v + U) = kv + U$
2. $(v_1 + U) + (v_2 + U) = (v_1 + v_2) + U$

Question

What's the additive identity for V/U? What's the additive inverse of $v + U$?

TA Notes:

1. $0 + U = U$ is the additive identity
2. $(v + U) + (-v + U) = (v - v) + U = 0 + U = U$ gives that $-(v + U) = (-v) + U$

Playing with Quotients

Example

Let $U = \{(x, y) : x = y\} \subset \mathbb{R}^2$. Sketch the elements of the quotient $\mathbb{R}^2/U$.

Question

Write $((2, 1) + U) + 4\,((1, -1) + U)$ *in the form* $(0, y) + U$*.*

- The affine subsets are $(a, b) + U = (a, b) + \{(x, y) : x = y \in \mathbb{R}\}$.
- These look like lines parallel to $x = y$.
- Each line is determined by its y-intercept.
- The calculation:

$$((2, 1) + U) + 4\,((1, -1) + U) = (2, 1) + (4, -4) + U = (6, -3) + U$$

We put this answer in the desired form:

$$(6, -3) + U = (0, -9) + (6, 6) + U = (0, -9) + U$$

From V/U to V

Question

Suppose that $U \subset V$ is a subspace. Let $\{v_1 + U, \dots v_n + U\}$ be a basis of V/U and $\{u_1, \dots, u_k\}$ be a basis for U. Show that $\{v_1, \dots, v_n, u_1 \dots, u_k\}$ is a basis for V.

TA Notes:

- We show that $\operatorname{span}\{v_1, \dots, v_n, u_1, \dots, u_k\} = V$.
- Pick $v \in V$ and write $v + U$ in the basis $\{v_1 + U, v_2 + U, \dots, v_n + U\}$.

$$v + U = t_1 (v_1 + U) + \cdots + t_n (v_n + U) \Rightarrow v - t_1 v_1 - \cdots - t_n v_n + U = U$$

- It follows that $v - t_1 v_1 - \cdots - t_n v_n \in U$.
- We can write $v - t_1 v_1 - \cdots - t_n v_n$ in the basis for U.

$$v - t_1 v_1 - \cdots - t_n v_n = s_1 u_1 + s_2 u_2 + \dots s_k u_k$$

- We obtain:

$$v = t_1 v_1 + \cdots + t_n v_n + s_1 u_1 + \cdots + s_k u_k$$

Linear Independence

TA Notes:

- As always, we apply the independence test.

$$0 = t_1 v_1 + \cdots + t_n v_n + \underbrace{s_1 u_1 + \cdots + s_k u_k}_{\in U} \quad (\star)$$

- In the quotient space, we get:

$$0 + U = t_1 v_1 + \cdots + t_n v_n + U$$

  Equivalently,

$$0 + U = t_1 (v_1 + U) + t_2 (v_2 + U) + \cdots + t_n (v_n + U)$$

  By the linear independence of $\{v_1 + U, \ldots, v_n + U\}$ we get $t_1 = t_2 = \cdots = t_n$.

- Plugging this in to $(\star)$ we get:

$$0 = 0 + \cdots + 0 + s_1 u_1 + \cdots + s_k u_k$$

  By the linear independence of $\{u_1, \ldots, u_k\}$ we get $s_1 = \cdots = s_k = 0$.