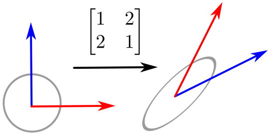
$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

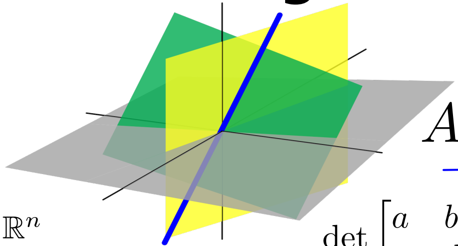


$$\begin{bmatrix} 1 & 0 & \star & \star \\ 0 & 1 & \star & \star \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$A\mathbf{x} = \mathbf{0}$

MAT A22
Linear Algebra 1

$|\mathbf{u} \cdot \mathbf{v}| \leq ||\mathbf{u}||\,||\mathbf{v}||$



$A\mathbf{x} = \lambda\mathbf{x}$

$A = [a_{ij}]$

$A : \mathbb{R}^n \to \mathbb{R}^n$

$A(a\mathbf{u} + b\mathbf{v}) = aA\mathbf{u} + bA\mathbf{v}$

$$\det \begin{bmatrix} a & b \\ c & d \end{bmatrix} = ad - bc$$

MAT A22: Linear Algebra 1 for Mathematical Sciences (Winter 2022)

Welcome to Week 7 of the course.
Questions? Thoughts? Comments?

Readings:

- Subspaces

News and Reminders:

- Welcome back!

Subspaces

Definition

A subset U of $\mathbb{R}^n$ is a subspace of $\mathbb{R}^n$ if:

S1. The zero vector of $\mathbb{R}^n$, $\vec{0}$, is in U;

S2. U is closed under addition: for all $\vec{u}, \vec{w} \in U \Rightarrow \vec{u} + \vec{w} \in U$;

S3. U is closed under scalar multiplication: for all $\vec{u} \in U$ and $k \in \mathbb{R} \Rightarrow k\vec{u} \in U$.

Is It a Subspace? (Part 1)

Question

$$U = \left\{ \begin{bmatrix} 1 \\ s \\ t \end{bmatrix} \;\middle|\; s, t \in \mathbb{R} \right\}$$

Is U a subspace of $\mathbb{R}^3$? Justify your answer.

Is It a Subspace? (Part 2)

Question

$$U = \left\{ \begin{bmatrix} 0 \\ x \\ y \end{bmatrix} \;\middle|\; x^2 + y^2 = 0 \quad and \quad x, y \in \mathbb{R} \right\}$$

Is U a subspace of $\mathbb{R}^3$? Justify your answer.

Build an Almost Subspace

Question

Find a non-empty subset $U \subset \mathbb{R}^2$ such that: U is closed under scalar multiplication but U is NOT closed under addition.

The Zero Subspace

Question

- *Prove that* $\{\vec{0}\}$ *is a subspace of* $\mathbb{R}^n$.
- *Prove that* $\{\vec{0}\} \subseteq V$ *for any subspace* V *of* $\mathbb{R}^n$.

Three Abstract Subspaces

Question

Let A be an $n \times n$ matrix. Prove that the following are all subspaces.

- $\text{null}(A) = \{\vec{x} : A\vec{x} = \vec{0}\}$.
- $\text{image}(A) = \{\vec{y} : A\vec{x} = \vec{y} \textit{ for some } \vec{x}\}$.
- $E_\lambda = \{\vec{x} : A\vec{x} = \lambda\vec{x}\}$.

Express as an Image

Question

Express the following subspace as an image of linear transformation.

$$U = \left\{ \begin{bmatrix} a \\ a+2b \\ b \end{bmatrix} \;\middle|\; a, b \in \mathbb{R} \right\}$$

Formally: Find a matrix A such that $U = \text{image}(A)$.

Spanning Sets

Definition

Let $\vec{x}_1, \vec{x}_2, \ldots, \vec{x}_k \in \mathbb{R}^n$ and $t_1, t_2, \ldots, t_k \in \mathbb{R}$. Then the vector

$$\vec{x} = t_1\vec{x}_1 + t_2\vec{x}_2 + \cdots + t_k\vec{x}_k$$

is called a linear combination of the vectors $\vec{x}_1, \vec{x}_2, \ldots, \vec{x}_k$. The set of **all** linear combinations of $\vec{x}_1, \vec{x}_2, \ldots, \vec{x}_k$ is called the span of $\vec{x}_1, \vec{x}_2, \ldots, \vec{x}_k$, and is written

$$\text{span}\{\vec{x}_1, \vec{x}_2, \ldots, \vec{x}_k\} = \{t_1\vec{x}_1 + t_2\vec{x}_2 + \cdots + t_k\vec{x}_k \mid t_1, t_2, \ldots, t_k \in \mathbb{R}\}.$$

Additional Terminology. If $U = \text{span}\{\vec{x}_1, \vec{x}_2, \ldots, \vec{x}_k\}$, then:

- U is spanned by the vectors $\vec{x}_1, \vec{x}_2, \ldots, \vec{x}_k$.
- the vectors $\vec{x}_1, \vec{x}_2, \ldots, \vec{x}_k$ span U.
- the set of vectors $\{\vec{x}_1, \vec{x}_2, \ldots, \vec{x}_k\}$ is a spanning set for U.

Adding to Linear Combination to a Span Change Nothing

Question

Prove: If $\vec{v}$ *is in* $U = \text{span}\{\vec{x}_1, \ldots, \vec{x}_n\}$ *then* $U = \text{span}\{\vec{v}, \vec{x}_1, \ldots, \vec{x}_n\}$

Images are Spans

Question

Prove: If $A = [\vec{a}_1 \ldots \vec{a}_n]$ is an $n \times n$ matrix written as column vectors then

$$\text{image}(A) = \text{span}\{\vec{a}_1, \ldots, \vec{a}_n\}$$

Independence Warm-Up

Question

Prove: The following are equivalent:

- *Every vector in* $\text{span}\{\vec{x}_1, \dots, \vec{x}_n\}$ *can be written as a linear combination uniquely.*
- *The only way to write* $\vec{0}$ *as a linear combination from* $\text{span}\{\vec{x}_1, \dots, \vec{x}_n\}$ *is:*

$$\vec{0} = 0\vec{x}_1 + 0\vec{x}_2 + \cdots + 0\vec{x}_n$$

Linear Independence

Definition

Let $S = \{\vec{x}_1, \vec{x}_2, \ldots, \vec{x}_k\}$ be a subset of $\mathbb{R}^n$. The set S is linearly independent if:

$$t_1\vec{x}_1 + t_2\vec{x}_2 + \cdots + t_k\vec{x}_k = \vec{0}_n \iff t_1 = t_2 = \cdots = t_k = 0$$

A set that is **not** linearly independent is called dependent.

Zero Guarantees Linear Dependence

Question

Prove: If $\vec{0} \in \{\vec{x}_1, \ldots, \vec{x}_n\}$ *then* $\{\vec{x}_1, \ldots, \vec{x}_n\}$ *is linearly dependent.*