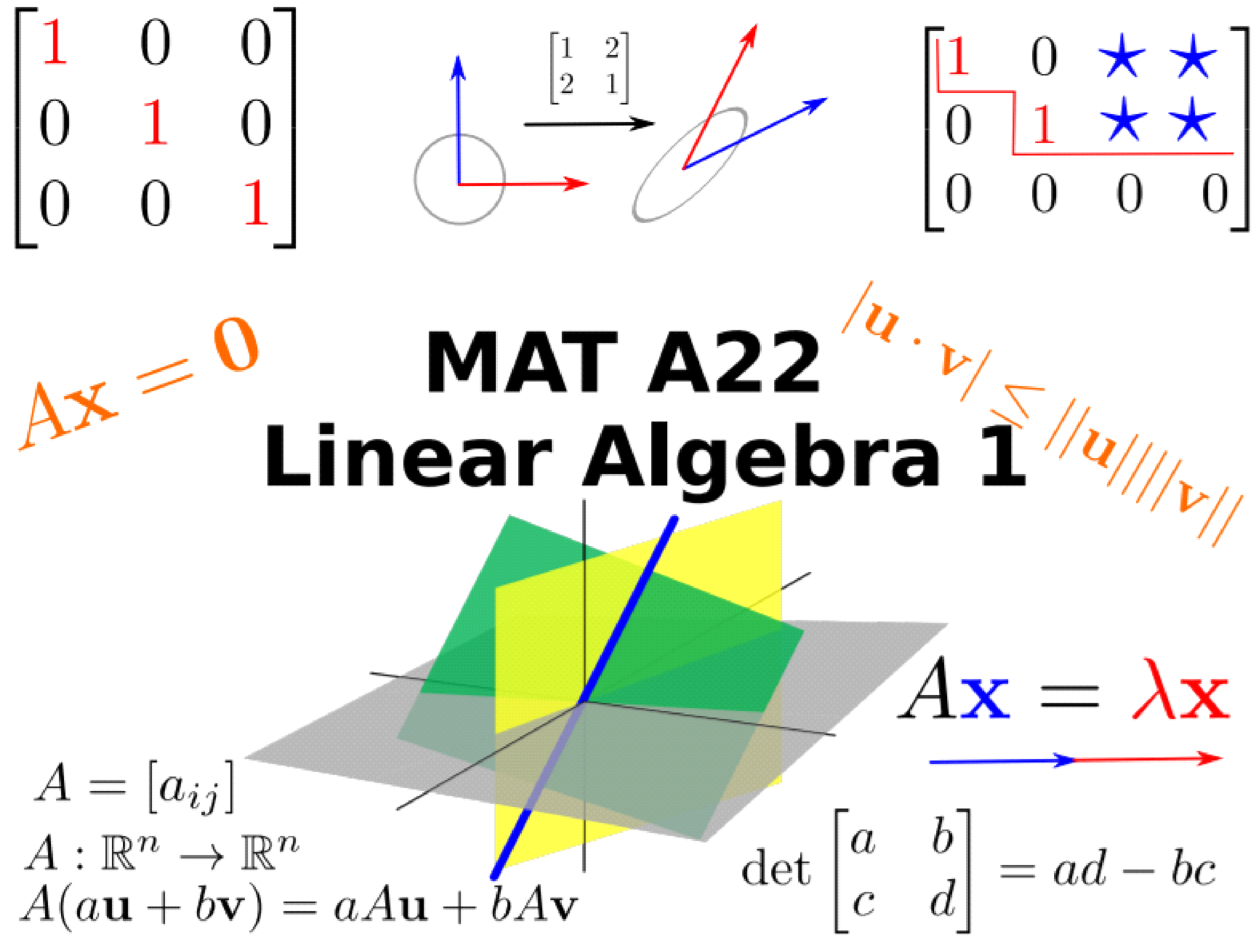
Title Page

February 5, 2022 3:00 PM



Info

February 5, 2022 3:01 PM

MAT A22: Linear Algebra 1 for Mathematical Sciences (Winter 2022)

Welcome to Week 5 of the course.
Questions? Thoughts? Comments?

Readings:

- 3.1 The Cofactor Expansion
- 3.2 Determinants and Matrix Inverses
- 3.3 Diagonalization and Eigenvalues
- Extra: 3.6 Proof of the Cofactor Expansion Theorem

News and Reminders:

- Parker is holding office hours Monday 15:15-16:15. See the Zoom Links page for details about how to connect.
- Assignment 3 is due next week Thursday February 17th at 13:00.

A Matrix Multiplication

February 5, 2022 3:01 PM

A Matrix Multiplication

Question

Compute AB where:

$$A = \begin{bmatrix} -1 & 0 & 3 \\ 2 & -1 & 1 \end{bmatrix} \text{ and } B = \begin{bmatrix} -1 & 1 & 2 \\ 0 & -2 & 4 \\ 1 & 0 & 0 \end{bmatrix}$$

$$A\vec{b}_1 = \begin{bmatrix} -1 & 0 & 3 \\ 2 & -1 & 1 \end{bmatrix} \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}, A\vec{b}_2 = \begin{bmatrix} -1 & 0 & 3 \\ 2 & -1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ -2 \\ 0 \end{bmatrix},$$

$$\text{and } A\vec{b}_3 = \begin{bmatrix} -1 & 0 & 3 \\ 2 & -1 & 1 \end{bmatrix} \begin{bmatrix} 2 \\ 4 \\ 0 \end{bmatrix}$$

Thus, $AB = \begin{bmatrix} 4 & -1 & -2 \\ -1 & 4 & 0 \end{bmatrix}$.

Find the (i, j)-entry of a Product

February 5, 2022 3:02 PM

Find the (i,j)-entry of a Product

Question

Find the $(2,3)$*-entry of the product*

$$\begin{bmatrix} -1 & 0 & 3 \\ 2 & -1 & 1 \end{bmatrix} \begin{bmatrix} -1 & 1 & 2 \\ 0 & -2 & 4 \\ 1 & 0 & 0 \end{bmatrix}$$

The (2,3)-entry is computed by the dot product of the second row of the first matrix and the third column of the second matrix:

$$2 \times 2 + (-1) \times 4 + 1 \times 0 = 4 - 4 + 0 = 0.$$

Commutivity of Matrix Multiplication

February 5, 2022 3:02 PM

Commutivity of Matrix Multiplication

Question

Let

$$A = \begin{bmatrix} 1 & 2 \\ -3 & 0 \\ 1 & -4 \end{bmatrix} \text{ and } B = \begin{bmatrix} 1 & -1 & 2 & 0 \\ 3 & -2 & 1 & -3 \end{bmatrix}$$

1. *Does AB exist? If so, compute it.*
2. *Does BA exist? If so, compute it.*

$$AB = \begin{bmatrix} 7 & -5 & 4 & -6 \\ -3 & 3 & -6 & 0 \\ -11 & 7 & -2 & 12 \end{bmatrix}$$

BA does not exist

Proof 1

February 5, 2022 3:02 PM

Proof Practice

Question

Let A, B, and C be $n \times n$ matrices.
Prove that if A and B commute with C, then $A + B$ commutes with C.

Let $A, B, C \in M_{n \times n}$

Given: $AC = CA$ and $BC = CB$

WTS: $(A+B)C = C(A+B)$

LHS $= (A+B)C = AC + BC$ ← (Matrix product distributive property)

RHS $= C(A+B) = CA + CB = AC + BC$

∴ LHS = RHS

Proof 2

February 5, 2022 3:03 PM

Proof Practice

Question

Let A, B and C be $n \times n$ matrices, and suppose that both A and B commute with C, i.e., $AC = CA$ and $BC = CB$. Show that AB commutes with C.

Let $A, B, C \in M_{n \times n}$

Given: $AC = CA$ and $BC = CB$

WTP: $(AB)C = C(AB)$

$$\begin{aligned} LHS = (AB)C &= \underset{1}{A(BC)} \\ &= A(CB) \quad \text{, from given} \\ &= (AC)B \quad \text{, } 1 \\ &= (CA)B \quad \text{, from given} \\ &= C(AB) \quad \text{, } 1 \\ &= RHS \end{aligned}$$

Aside:
Matrix Product Associativy
$(AB)C = A(BC)$ 1

Find an Inverse

February 5, 2022 3:03 PM

Find an Inverse

Question

Find, if possible, the inverse of $\begin{bmatrix} 1 & 0 & -1 \\ -2 & 1 & 3 \\ -1 & 1 & 2 \end{bmatrix}$.

Using the matrix inversion algorithm

$R_2 = -R_2$

$R_2 = R_2 - 2R_1$

$R_3 = R_3 - R_1$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & -1 & 1 & 0 & 0 \\ -2 & 1 & 3 & 0 & 1 & 0 \\ -1 & 1 & 2 & 0 & 0 & 1 \end{array}\right] \to \left[\begin{array}{ccc|ccc} 1 & 0 & -1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 2 & 1 & 0 \\ 0 & 1 & 1 & 1 & 0 & 1 \end{array}\right] \to$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & -1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 2 & 1 & 0 \\ 0 & 0 & 0 & -1 & -1 & 1 \end{array}\right]$$

From this, we see that A has no inverse.

True or False: Is it Invertible?

February 5, 2022 3:04 PM

True or False: Is it Invertible?

Question

True or false? Justify your answer.

$$\text{If } A^3 = 4I, \text{ then } A \text{ is invertible.}$$

If $A^3 = 4I$, then

$$\frac{1}{4}A^3 = I$$

so

$$\left(\frac{1}{4}A^2\right)A = I \text{ and } A\left(\frac{1}{4}A^2\right) = I$$

Therefore A is invertible, and $A^{-1} = \frac{1}{4}A^2$.

Linear Transformations

February 5, 2022 3:04 PM

Linear Transformations

Definition

Let V and W be vector spaces, and $T : V \to W$ a function.
T is called a linear transformation if it satisfies the following two properties.

1. T preserves addition.
   For all $\vec{v}_1, \vec{v}_2 \in V$, we have: $T(\vec{v}_1 + \vec{v}_2) = T(\vec{v}_1) + T(\vec{v}_2)$.
2. T preserves scalar multiplication.
   For all $\vec{v} \in V$ and $r \in \mathbb{R}$, we have: $T(r\vec{v}) = rT(\vec{v})$.

Note that the sum $\vec{v}_1 + \vec{v}_2$ is in V, while the sum $T(\vec{v}_1) + T(\vec{v}_2)$ is in W.
Similarly, $r\vec{v}$ is scalar multiplication in V, while $rT(\vec{v})$ is scalar multiplication in W.

Computing with Linearity

February 5, 2022 3:04 PM

Computing with Linearity

Question

Let V be a vector space, T a linear operator on V, and $\vec{v}, \vec{w} \in V$. Suppose that

$$T(\vec{v}+\vec{w}) = \vec{v} - 2\vec{w} \text{ and } T(2\vec{v}-\vec{w}) = 2\vec{v}.$$

Find $T(\vec{v})$ and $T(\vec{w})$.

We get: $T(\vec{v}) = \vec{v} - \frac{2}{3}\vec{w}$ and $T(\vec{w}) = -\frac{4}{3}\vec{w}$.

By def of linearity:

$T(\vec{v}+\vec{w}) = T(\vec{v}) + T(\vec{w}) = \vec{v} - 2\vec{w}$

$T(2\vec{v}-\vec{w}) = 2T(\vec{v}) - T(\vec{w}) = 2\vec{v}$

Wow 2 variables and 2 equations!

Augmented Matrix

$$\left[\begin{array}{cc|c} 1 & 1 & \vec{v}-2\vec{w} \\ 2 & -1 & 2\vec{v} \end{array}\right] \quad R_2 = R_2 - 2R_1$$

$$\left[\begin{array}{cc|c} 1 & 1 & \vec{v}-2\vec{w} \\ 0 & -3 & 2\vec{v}-2(\vec{v}-2\vec{w}) \end{array}\right]$$

$$\left[\begin{array}{cc|c} 1 & 1 & \vec{v}-2\vec{w} \\ 0 & -3 & 4\vec{w} \end{array}\right] \quad R_2 = -\frac{R_2}{3}$$

$$\left[\begin{array}{cc|c} 1 & 1 & \vec{v}-2\vec{w} \\ 0 & 1 & -\frac{4}{3}\vec{w} \end{array}\right] \quad R_1 = R_1 - R_2$$

$$\left[\begin{array}{cc|c} 1 & 0 & \vec{v}-\frac{2}{3}\vec{w} \\ 0 & 1 & -\frac{4}{3}\vec{w} \end{array}\right]$$

$T(\vec{v}) = \vec{v} - \frac{2}{3}\vec{w}$

$T(\vec{w}) = -\frac{4}{3}\vec{w}$

Proof 3

February 5, 2022 3:04 PM

Proof Practice: Examples of Linear Transformations

Question

Prove that the following transformations $T_i : \mathbb{R}^n \to \mathbb{R}^n$ are linear:

1. $T_0(\vec{v}) = \vec{0}$
2. $T_1(\vec{v}) = \vec{v}$
3. $T_\lambda(\vec{v}) = \lambda\vec{v} \leftarrow \lambda$ *is the Greek letter "lambda". It will come up a lot in this course!*

Let $\vec{x}, \vec{y} \in \mathbb{R}^n$ Let $c \in \mathbb{R}$

1 $T_0(\vec{x} + \vec{y}) = \vec{0} = \vec{0} + \vec{0} = T_0(\vec{x}) + T_0(\vec{y})$

$T_0(c\vec{x}) = \vec{0} = c\,\vec{0} = c\,T_0(\vec{x})$

2 $T_1(\vec{x}+\vec{y}) = \vec{x} + \vec{y} = T_1(\vec{x}) + T_1(\vec{y})$

$T_1(c\vec{x}) = c\vec{x} = c\,T_1(\vec{x})$

3 $T_\lambda(\vec{x} + \vec{y}) = \lambda(\vec{x} + \vec{y}) = \lambda\vec{x} + \lambda\vec{y} = T_\lambda(\vec{x}) + T_\lambda(\vec{y})$

$T_\lambda(c\vec{x}) = \lambda c\vec{x} = c\,\lambda\vec{x} = c\,T_\lambda(\vec{x})$

Proof 4

February 5, 2022 3:05 PM

Proof Practice: Examples of Linear Transformations

Question

Let $M_{n\times n}(\mathbb{R})$ denote the vector space of $n \times n$ matrices with real entries. Prove that the following transformations are linear:

1. $T_1(M) = M^T$ *where* $T_1 : M_{n\times n}(\mathbb{R}) \to M_{n\times n}(\mathbb{R})$
2. $T_2([a_{ij}]) = \text{trace}([a_{ij}]) = \sum_{i=1}^{n} a_{ii}$ *where* $T_2 : M_{n\times n}(\mathbb{R}) \to \mathbb{R}$.

Let $A, B \in M_{n\times n}(\mathbb{R})$ Let $c \in \mathbb{R}$

1. $T_1(A+B) = (A+B)^T = A^T + B^T = T_1(A) + T_1(B)$
   $T_1(cA) = (cA)^T = c\,A^T = c\,T_1(A)$

2. $T_2(A+B) = tr(A+B) = tr(A) + tr(B) = \sum_{i=1}^{n} a_{ii} + \sum_{i=1}^{n} b_{ii} = T_2(A) + T_2(B)$
   $T_2(cA) = tr(cA) = c\,tr(A) = c\sum_{i=1}^{n} a_{ii} = c\,T_2(A)$