Coupling of continuous and hybridizable discontinuous Galerkin methods: Application to conjugate heat transfer problem

Mahendra Paipuri[1-2], Sonia Fernández-Méndez[2] and Carlos Tiago[1]

[1]Instituto Superior Técnico (IST)

[2]LaCàN, Universitat Politècnica de Catalunya (UPC-BarcelonaTech)

LABORATORI DE CÀLCUL NUMÈRIC UNIVERSITAT POLITÈCNICA DE CATALUNYA

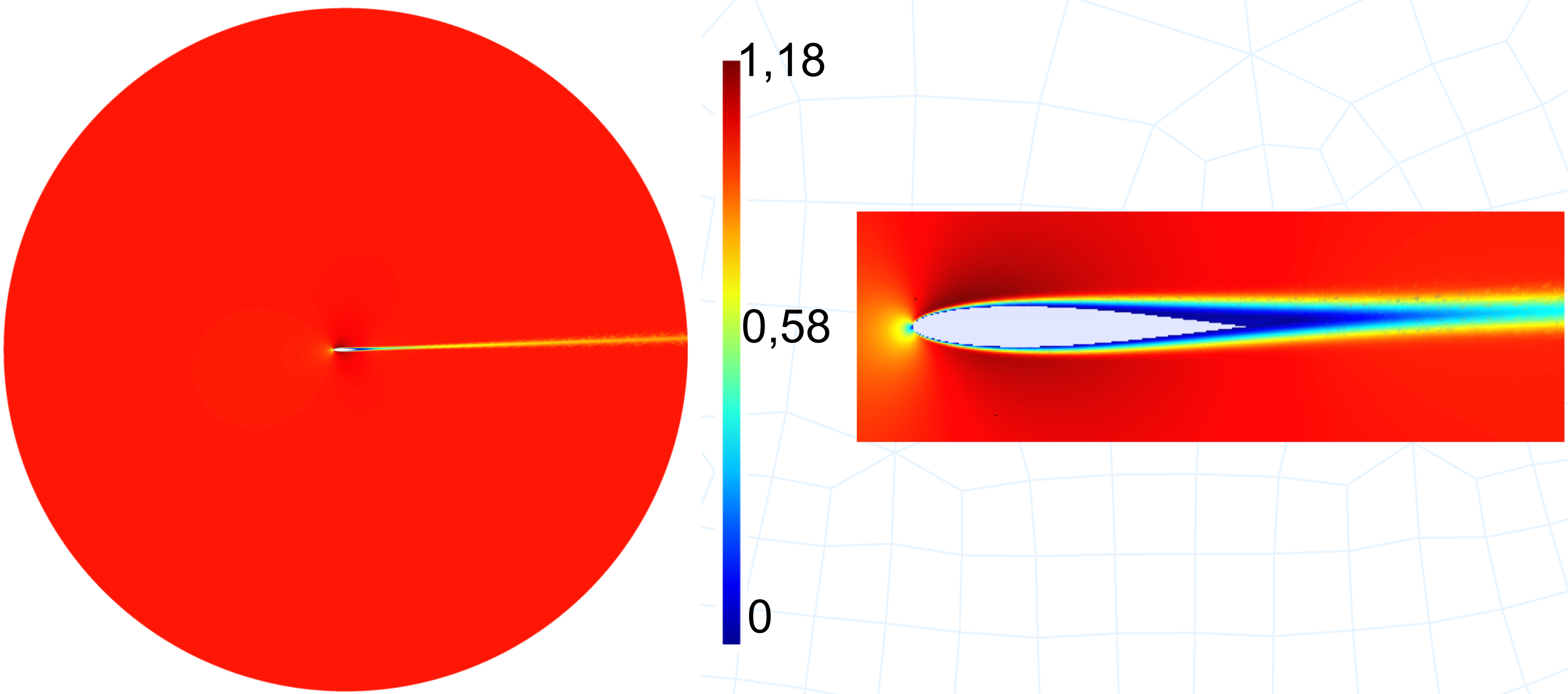
Motivation

Continuous Galerkin (CG)	**DG (DG, IPM, CDG, LDG..)**
✓ Well established	
✓ Computationally efficient	× duplication of nodes
✓ Static condensation	× no static condensation
× Unstable for high convective flows	✓ Stable for convective flows
× Cumbersome p-adaptivity	✓ Suitability for adaptivity, parallel computing,…

Hybridizable Discontinous Galerkin (HDG)

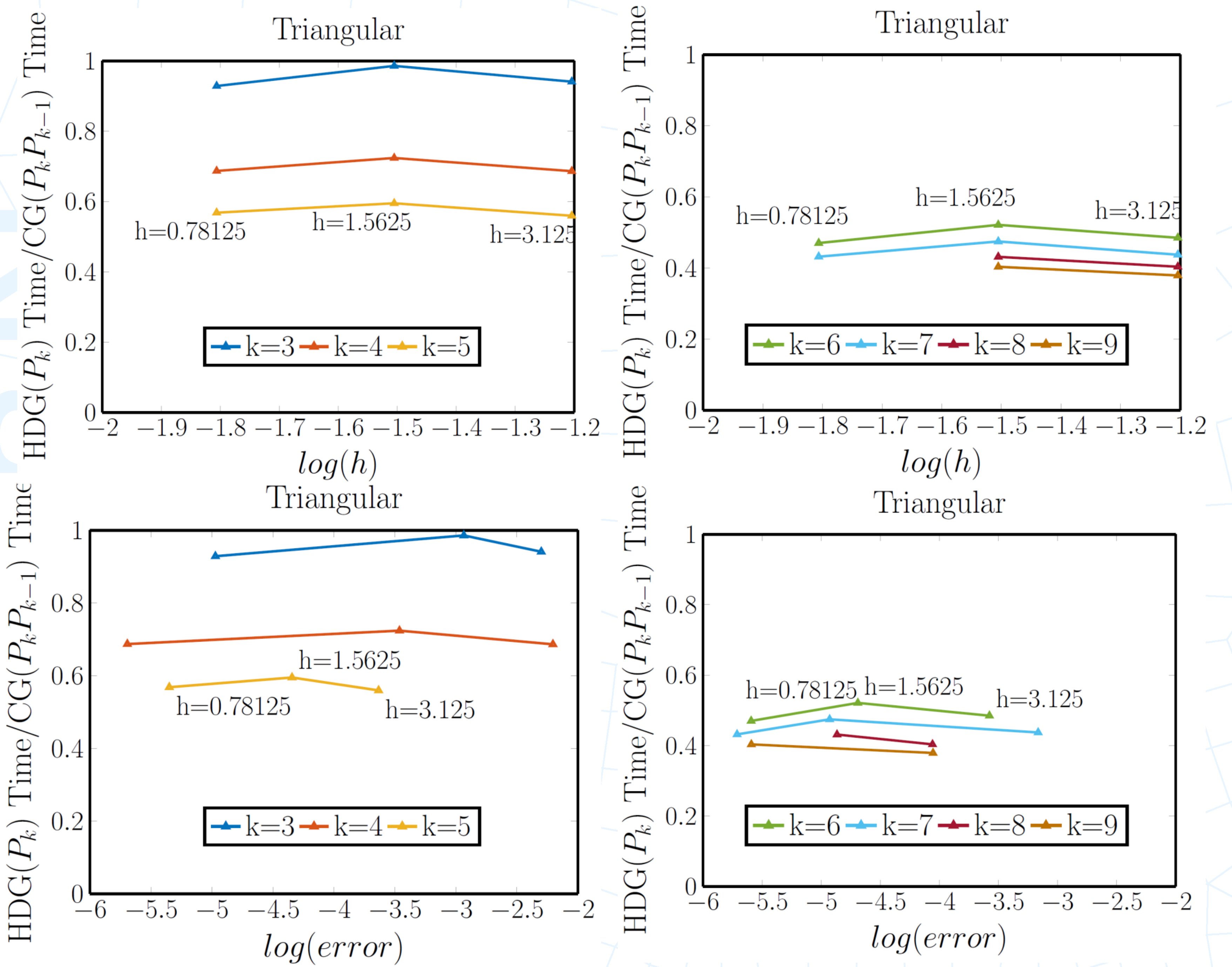
NACA0012 Airfoil

Incompressible Navier-Stokes is solved at Re = 5000 and Angle of attack (AoA) = 2º ; Error in lift coefficient is compared



Velocity

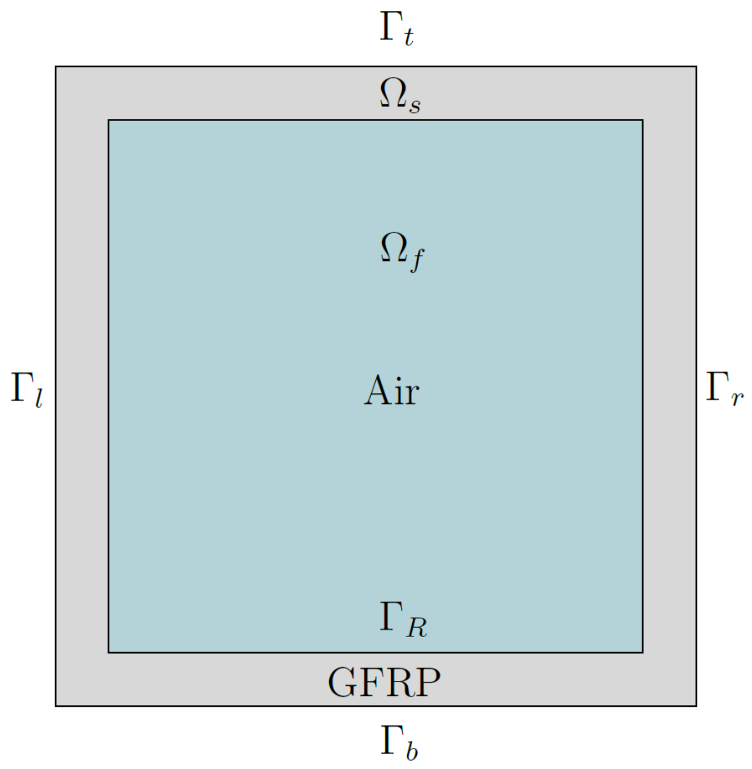
NACA: CPU Time for linear solver



Motivation

GFRP Problem

- Multi-physics transient problem
- Time varying temperature is prescribed on Γ_b.
- Thermophysical properties of GFRP are temperature dependent
- Bousinesseq approximation is used to model the natural buoyancy of air.



Γ_b , Γ_t - Convective and radiative BC

Γ_l, Γ_r - Adiabatic BC

Γ_R - Internal radiation

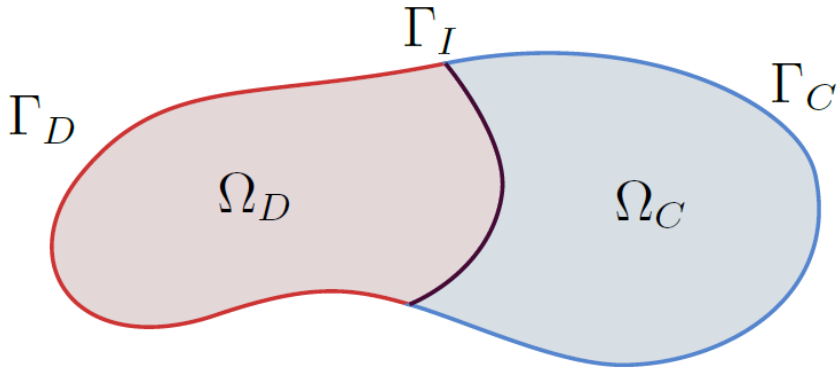
1) Coupling of CG and HDG

2) Coupled Navier-Stokes/heat transfer for HDG

Coupling of CG and HDG

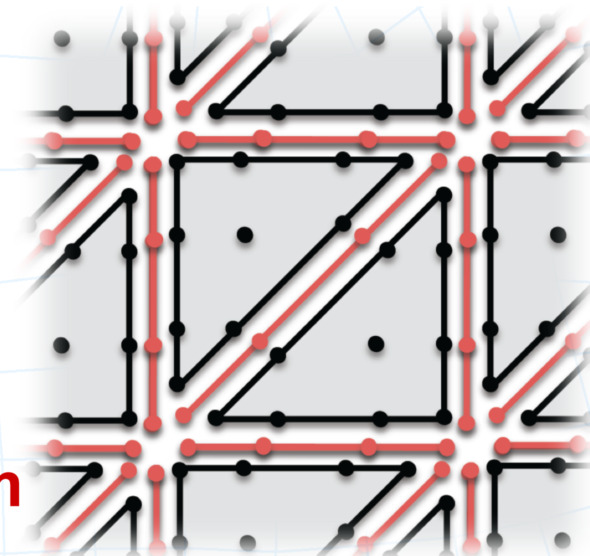
Heat equation

$$
\begin{aligned}
-\boldsymbol{\nabla}\cdot(k\boldsymbol{\nabla} u) &= f && \text{in } \Omega_D\\
u &= u_D && \text{on } \Gamma_D\\
-\boldsymbol{\nabla}\cdot(k\boldsymbol{\nabla} u) &= f && \text{in } \Omega_C\\
u &= u_C && \text{on } \Gamma_C
\end{aligned}
$$



Suffixes *D* and *C* stands for HDG and CG domains. Interface conditions are

$$
\begin{aligned}
u|_D - u|_C &= 0 && \text{on } \Gamma_I\\
-k\boldsymbol{\nabla} u\cdot\boldsymbol{n}|_D - k\boldsymbol{\nabla} u\cdot\boldsymbol{n}|_C &= 0 && \text{on } \Gamma_I
\end{aligned}
$$



Local Problem

$$
\begin{aligned}
\boldsymbol{\nabla}\cdot\boldsymbol{q} &= f && \text{in } \Omega_{iD}\\
\boldsymbol{q} + k\boldsymbol{\nabla} u &= 0 && \text{in } \Omega_{iD}\\
u &= \widehat{u} && \text{on } \partial\Omega_{iD}
\end{aligned}
$$

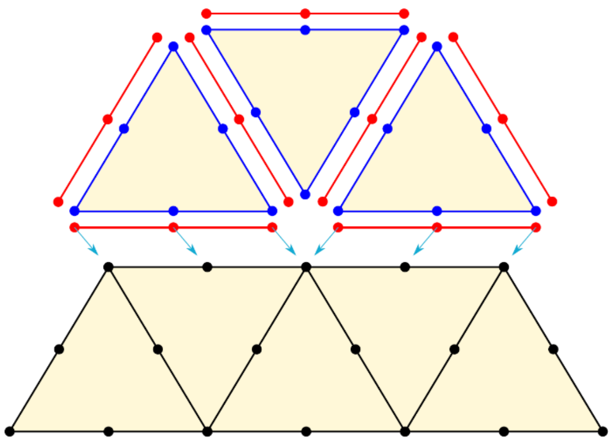
Global Problem

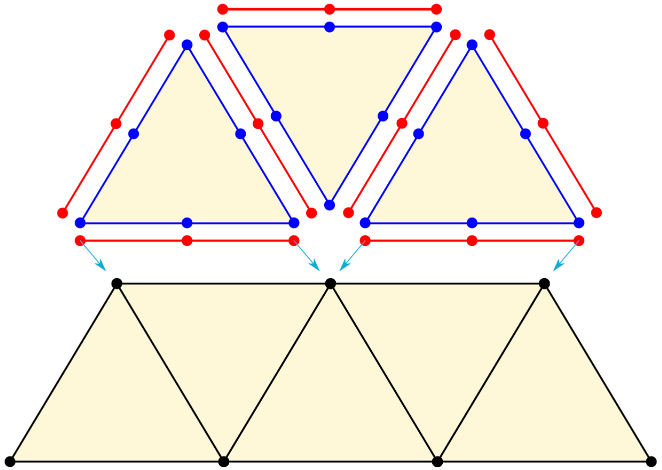
$$
\begin{aligned}
[\![\hat{\boldsymbol{q}}\cdot\boldsymbol{n}]\!] &= 0 && \text{in } \Gamma\\
\hat{u} &= u_D && \text{on } \Gamma_D
\end{aligned}
$$

$$\Gamma = \bigcup_{i=1}^{n_{el}} \partial\Omega_{iD}$$

Coupling of CG and HDG

Continuity of u is strongly imposed between u in Ω_C and $\hat{\mathbf{u}}$ in Ω_D.





If the degree of approximation is different on both domains, the solution is projected from the domain with lower degree to the domain with higher degree of approximation

$$\begin{aligned} \mathbf{u}^D &= \mathbf{P}\,\mathbf{u}^C \\ P_{ij} &= \Psi_j^C(\xi_i^D) \end{aligned}$$

Coupling of CG and HDG

Weak form of CG

$$(\nabla \delta u, \nabla u)_{\Omega_C} - \langle \delta u, k\,(\nabla u \cdot \boldsymbol{n}) \rangle_{\Gamma_I} + (\delta u, f)_{\Omega_C} = 0$$

Weak form of HDG (global equations)

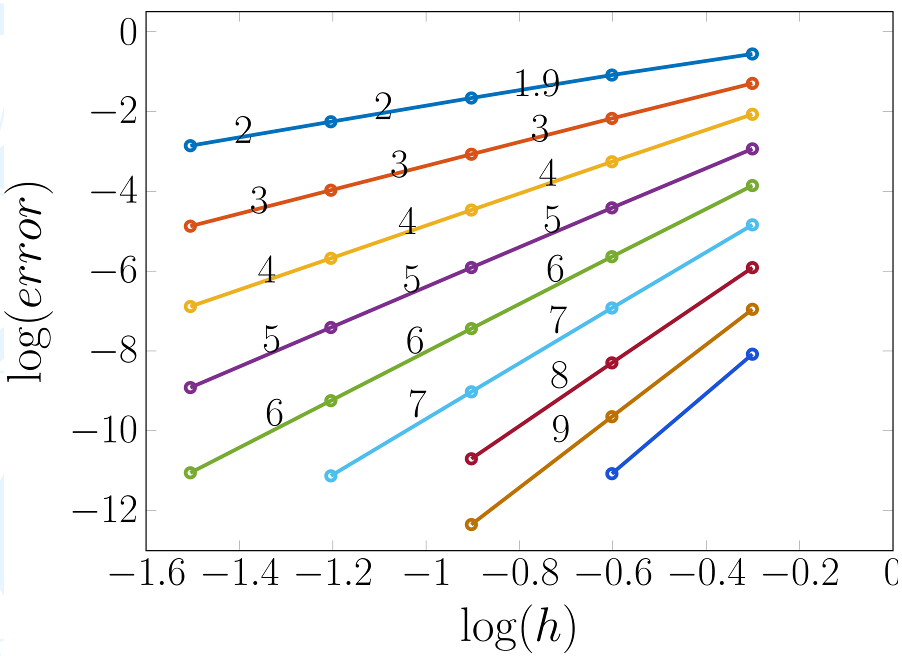
$$\langle \delta \hat{u}, [\![\hat{\boldsymbol{q}} \cdot \boldsymbol{n}]\!] \rangle_{\Gamma} = \langle \delta \hat{u}, [\![\hat{\boldsymbol{q}} \cdot \boldsymbol{n}]\!] \rangle_{\Gamma_i} + \langle \delta \hat{u}, [\![\hat{\boldsymbol{q}} \cdot \boldsymbol{n}]\!] \rangle_{\Gamma_D} + \langle \delta \hat{u}, [\![\hat{\boldsymbol{q}} \cdot \boldsymbol{n}]\!] \rangle_{\Gamma_I}$$

$$\boxed{\langle \delta \hat{u}, [\![\hat{\boldsymbol{q}} \cdot \boldsymbol{n}]\!] \rangle_{\Gamma_I} = \langle \delta \hat{u}, \hat{\boldsymbol{q}} \cdot \boldsymbol{n} \rangle_{\Gamma_I} |_D + \langle \delta \hat{u}, -k \nabla u \cdot \boldsymbol{n} \rangle_{\Gamma_I} |_C}$$

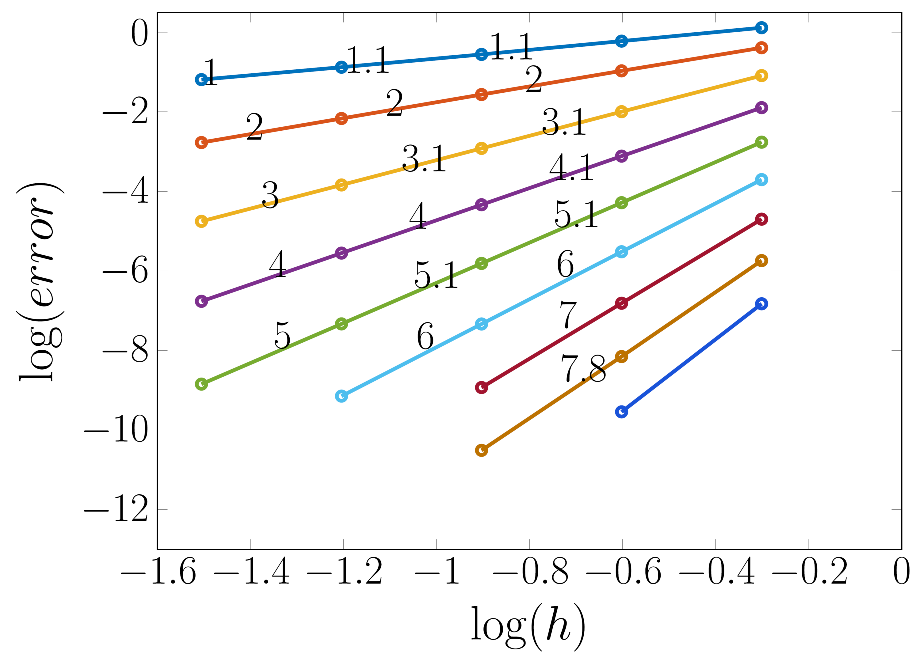
Weak form on interface

$$\boxed{\begin{aligned} &\langle \delta \hat{u}, [\![\hat{\boldsymbol{q}} \cdot \boldsymbol{n}]\!] \rangle_{\Gamma_i} + \langle \delta \hat{u}, [\![\hat{\boldsymbol{q}} \cdot \boldsymbol{n}]\!] \rangle_{\Gamma_D} + \langle \delta \hat{u}, \hat{\boldsymbol{q}} \cdot \boldsymbol{n} \rangle_{\Gamma_I} |_D - \langle \delta \hat{u}, k\,(\nabla u_h \cdot \boldsymbol{n}) \rangle_{\Gamma_I} |_C \\ &\qquad - (\nabla \delta \hat{u}, \nabla v_h)_{\Omega_C} + \langle \delta \hat{u}, k\,(\nabla u_h \cdot \boldsymbol{n}) \rangle_{\Gamma_I} |_C - (\delta \hat{u}, f)_{\Omega_C} = 0 \end{aligned}}$$

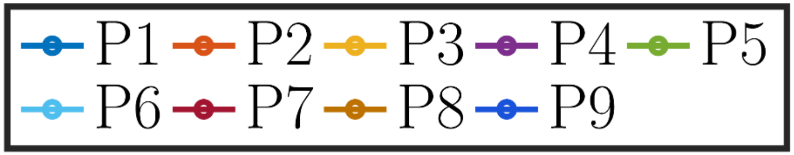
Coupling of CG and HDG



u



∇u



Coupled Navier-Stokes/heat transfer with HDG

Governing equations

Incompressible Navier-Stokes and convection diffusion equations.

$$
\begin{aligned}
\nabla\cdot(\boldsymbol{u}\otimes\boldsymbol{u})-\nabla\cdot(-p\mathbf{I}+\nu\nabla\boldsymbol{u})=\boldsymbol{f}(\theta) &\quad \text{in } \Omega\\
\nabla\cdot\boldsymbol{u}=0 &\quad \text{in } \Omega\\
\boldsymbol{u}\cdot\nabla\theta-\nabla\cdot(k\nabla\theta)=\boldsymbol{b} &\quad \text{in } \Omega\\
\boldsymbol{u}=\boldsymbol{u}_D &\quad \text{on } \partial\Omega\\
\theta=\theta_D &\quad \text{on } \partial\Omega
\end{aligned}
$$

Bousinesseq approximation is used to model the natural buoyancy flows in closed cavities.

$$
\boldsymbol{f}(\theta)=-\beta\boldsymbol{g}(\theta-\theta_0)
$$

The approximation states that change in density of air due to temperature difference acts as body force on fluid itself

Coupled Navier-Stokes/heat transfer with HDG

Governing equations

Kovasznay flow is considered for Navier-Stokes equations in domain [0 2]x[-0.5 1.5].

$$\boldsymbol{u} = \begin{bmatrix} 1 - \exp(\lambda x_1)\cos(2\pi x_2) \\ \frac{\lambda}{2\pi}\exp(\lambda x_1)\sin(2\pi x_2) \end{bmatrix} \qquad \theta = \sin\left(\frac{\pi\, m\, x_1}{2\,a}\right)\sin\left(\frac{\pi\, n\, x_2}{2\,b} + \frac{\pi\, n}{4\,b}\right)$$

$$p = -\frac{1}{2}\exp(2\lambda x_1) + C,$$

The body force considered in the analysis is expressed as

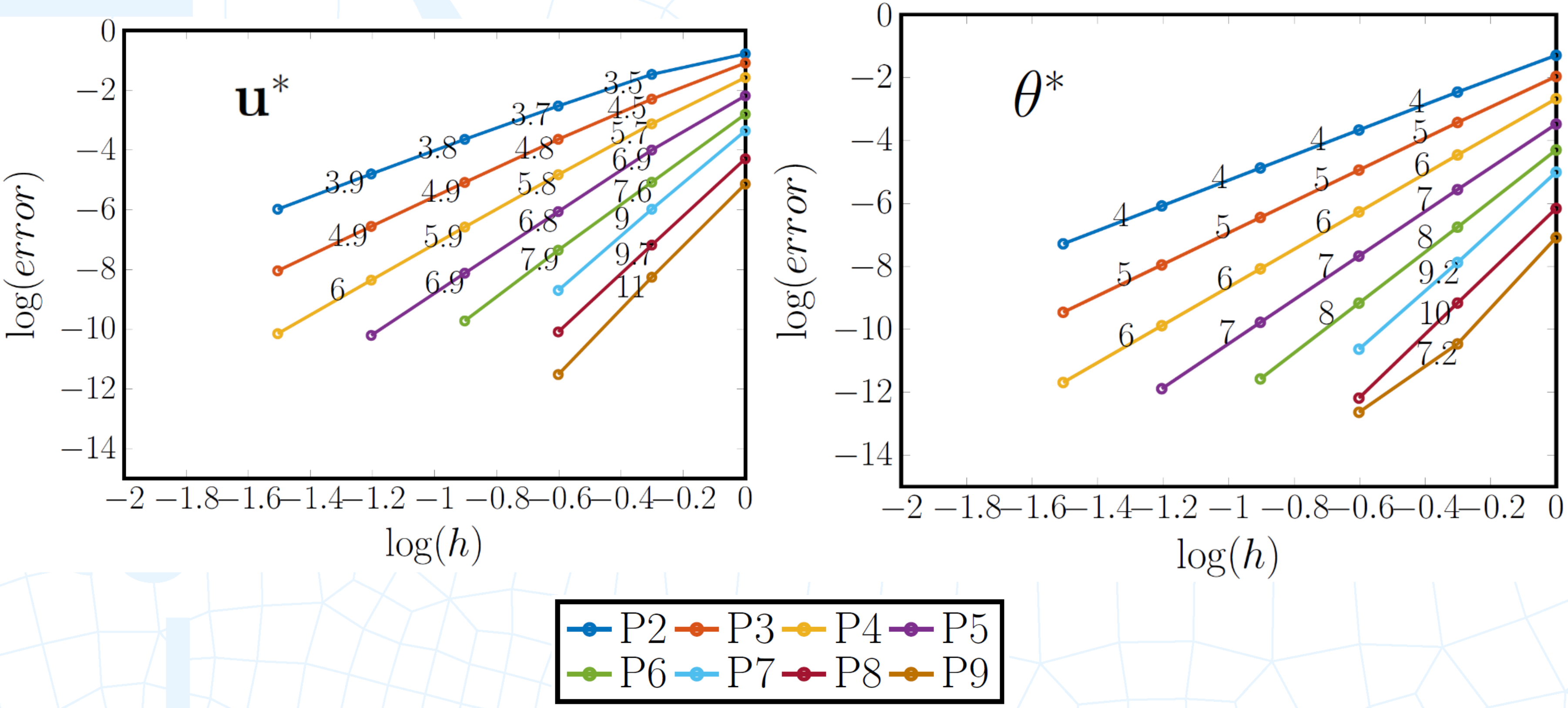
$$\begin{aligned} \boldsymbol{f}(\theta) &= \boldsymbol{f}_n(\theta) - \boldsymbol{f}_a(\theta) \\ \boldsymbol{f}_n(\theta) &= -\beta \boldsymbol{g}\theta \\ \boldsymbol{f}_a(\theta) &= -\beta \boldsymbol{g}\theta_a \end{aligned}$$

m = 3, n = 3; a = b = 2; β = 1; g = (0,-10)

Coupled Navier-Stokes/heat transfer with HDG

Convergence results

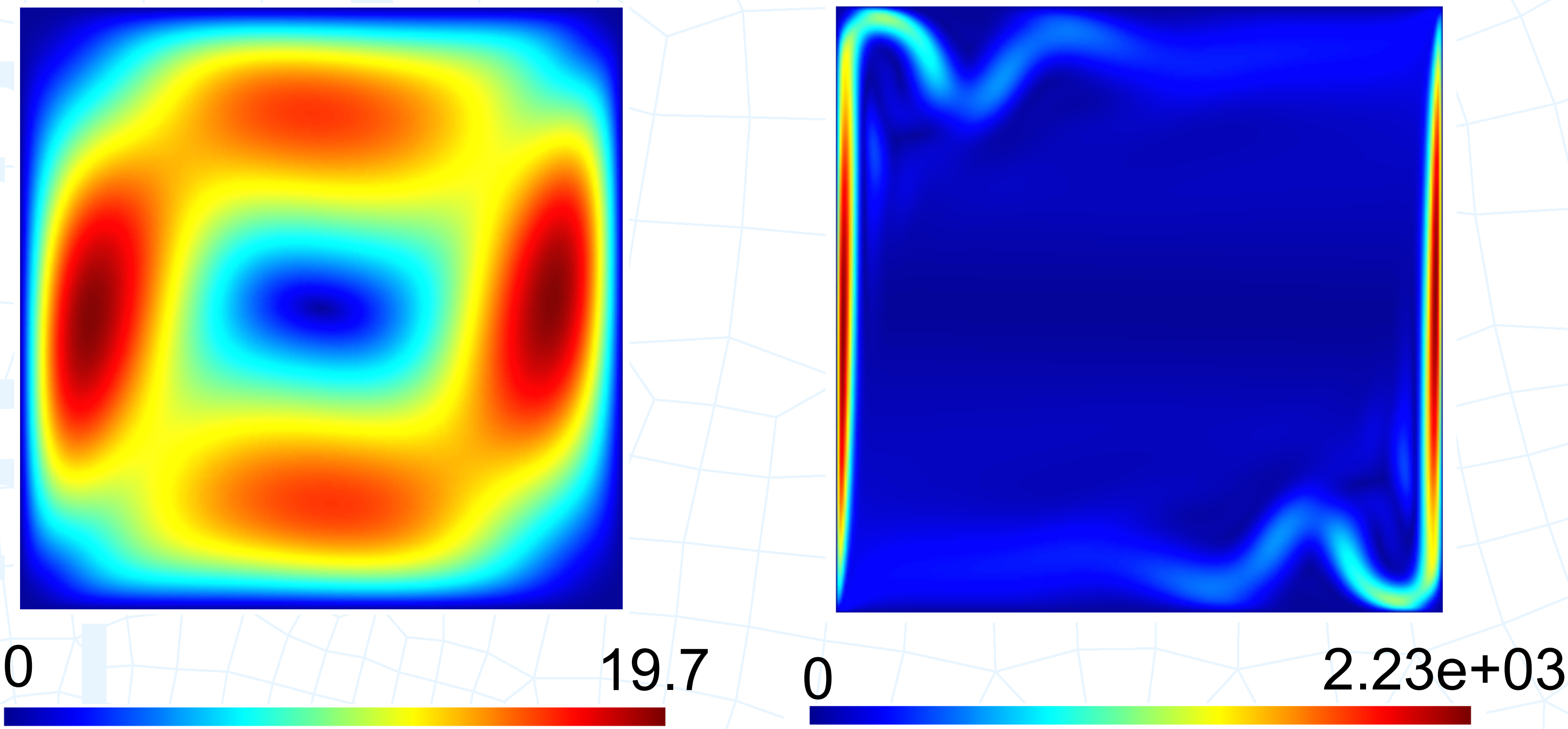
Kovasznay flow is considered for Navier-Stokes equations in domain [0 2]x[-0.5 1.5].



Coupled Navier-Stokes/heat transfer with HDG

Rayleigh-Bernard flow: Velocity contour plots

Square cavity of unit length and using Rayleigh number ($g\beta\Delta\theta L^3/\nu\alpha$). Triangular elements with h = 0.03125, k = 5.

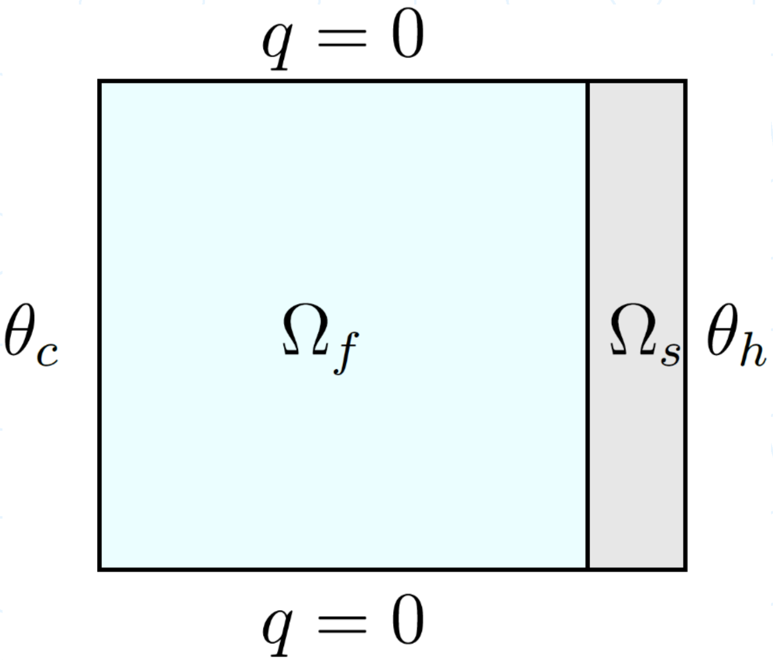


Ra = 10^4 Ra = 10^8

Coupled HDG and CG method for conjugate HT

Governing equations

- Solid part of the conjugate heat transfer problem is discretized using CG.
- Fluid part is solved using HDG.
- Coupling presented for heat equation is extended to the problem.
- A benchmark results
  are presented for domain.
- Different Raleigh numbers
  are considered.



Concluding remarks

- Computational efficiency between HDG and CG for Stokes problem
  - High order-coarser mesh is more efficient than low order-finer mesh in both HDG and CG.
  - HDG is more efficient at order $k \geq 3$ for same level of accuracy, in terms of CPU time of linear solver
- Similar computational efficiency is observed in Navier-Stokes example (NACA).