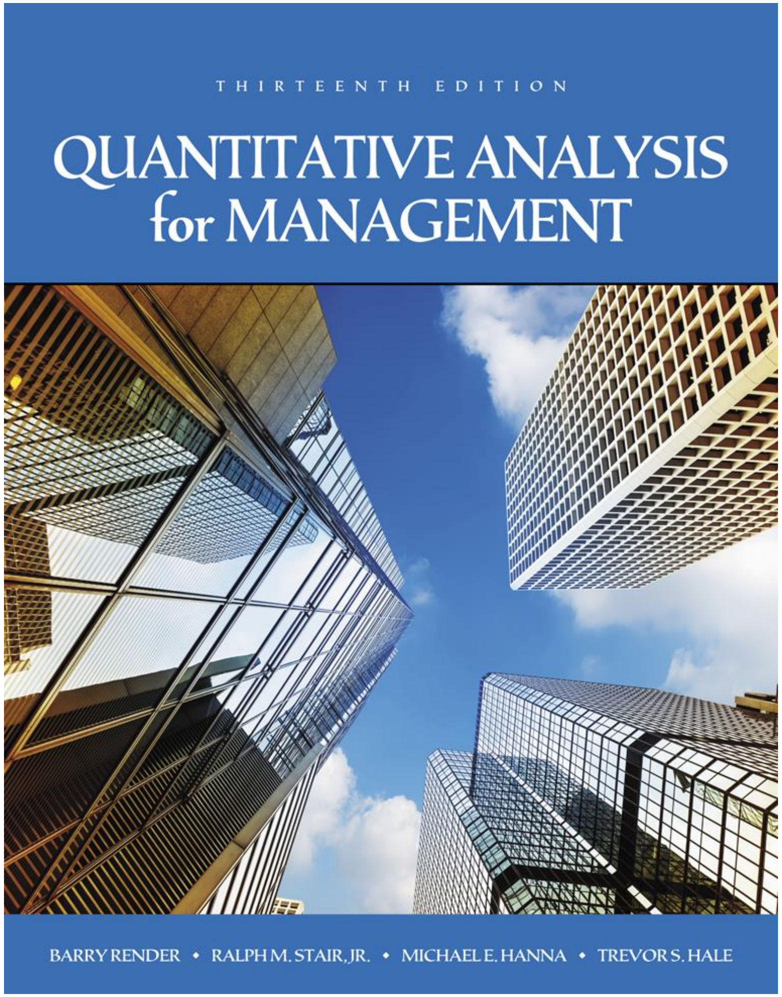
Quantitative Analysis for Management

Thirteenth Edition



Markov Analysis

Pearson

Learning Objectives

After completing this chapter, students will be able to:

14.1 Recognize states of systems and their associated probabilities.

14.2 Compute long-term or steady-state conditions by using only the matrix of transition probabilities.

14.3 Understand the use of absorbing state analysis in predicting future states or conditions.

14.4 Put Markov analysis into practice for the operation of machines.

14.5 Recognize equilibrium conditions and steady state probabilities.

14.6 Understand the use of absorbing states and the fundamental matrix.

Chapter Outline

14.1 States and State Probabilities
14.2 Matrix of Transition Probabilities
14.3 Predicting Future Market Shares
14.4 Markov Analysis of Machine Operations
14.5 Equilibrium Conditions
14.6 Absorbing States and the Fundamental Matrix: Accounts Receivable Application

Introduction (1 of 2)

- Markov analysis deals with the probabilities of future occurrences by analyzing presently known probabilities
- Numerous applications in business
- Assumes the system starts in an initial state or condition
- The probabilities of changing from one state to another are called a **matrix of transition probabilities**
- Requires basic matrix manipulation

Introduction (2 of 2)

- Discussion limited to Markov problems that follow four assumptions:
  1. There are a limited or finite number of possible states.
  2. The probability of changing states remains the same over time.
  3. We can predict any future state from the previous state and the matrix of transition probabilities.
  4. The size and makeup of the system do not change during the analysis.

States and State Probabilities (1 of 4)

- States are used to identify all possible conditions of a process or system
- Markov analysis assumes states are both collectively exhaustive and mutually exclusive
- Next step is to determine the probability that the system is in this state

States and State Probabilities (2 of 4)

- Information is placed into a **vector of state probabilities**

$\pi(i)$ = vector of state probabilities for period i

$= (\pi_1, \pi_2, \pi_3, \ldots, \pi_n)$

where

n = number of states

$\pi_1, \pi_2, \ldots, \pi_n$ = probability of being in state 1, state 2, …, state n

States and State Probabilities (3 of 4)

- In cases with one item, it is possible to know with complete certainty in which state an item is in
  - Vector of states can then be represented as

$$\pi(1) = (1, 0)$$

where

$\pi(1)$ = vector of states for the machine in period 1
$\pi_1 = 1$ = probability of being in the first state
$\pi_2 = 0$ = probability of being in the second state

States and State Probabilities (4 of 4)

- In cases with one item, it is possible to know with complete certainty in which state an item is in
  - Vector of states can then be represented as

$$\pi(1) = (1, 0)$$

where

$\pi(1)$ = vector of states for the machine in period 1
$\pi_1 = 1$ = probability of being in the first state
$\pi_2 = 0$ = probability of being in the second state

In most cases, we are dealing with more than one item

Grocery Store Example (1 of 5)

- States for people in a small town with three grocery stores
- A total of 100,000 people shop at the three groceries during any given month
  - Forty thousand may be shopping at American Food Store – state 1
  - Thirty thousand may be shopping at Food Mart – state 2
  - Thirty thousand may be shopping at Atlas Foods – state 3

Grocery Store Example (2 of 5)

Probabilities are as follows:

State 1 – American Food Store:	$40{,}000 \div 100{,}000 = 0.40 = 40\%$
State 2 – Food Mart:	$30{,}000 \div 100{,}000 = 0.30 = 30\%$
State 3 – Atlas Foods:	$30{,}000 \div 100{,}000 = 0.30 = 30\%$

- These probabilities can be placed in the following vector of state probabilities

$$\pi(1) = (0.4, 0.3, 0.3)$$

where

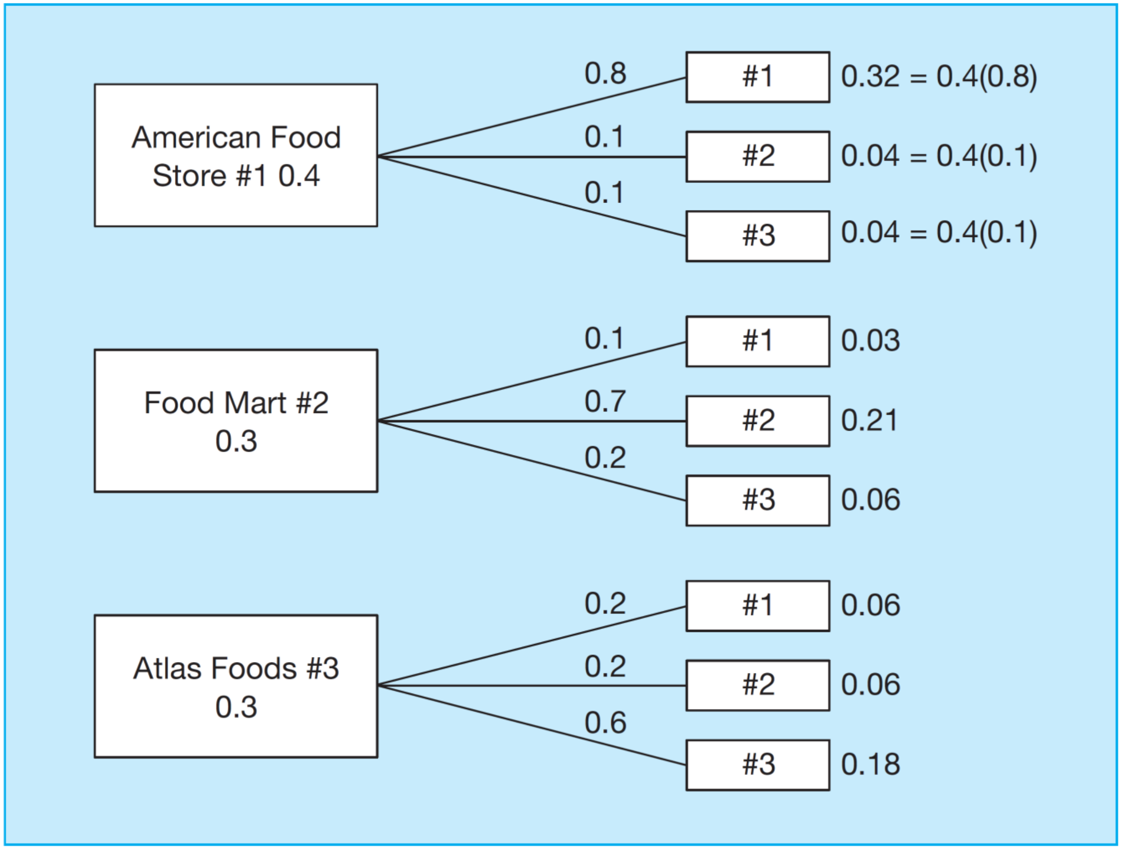
$\pi(1)$ = vector of state probabilities for the three grocery stores for period 1
$\pi_1 = 0.4$ = probability that person will shop at American Food, state 1
$\pi_2 = 0.3$ = probability that person will shop at Food Mart, state 2
$\pi_3 = 0.3$ = probability that person will shop at Atlas Foods, state 3

Grocery Store Example (3 of 5)

- The probabilities of the vector states represent the **market shares** for the three groceries
- Management will be interested in how their market share changes over time

Grocery Store Example (4 of 5)

FIGURE 14.1 Tree Diagram for Grocery Stores Example



Grocery Store Example (5 of 5)

- A tree diagram for several months would get very large
- Easier to use a matrix of transition probabilities
- A **transition probability** is the probability of moving from one particular state to another particular state

Matrix of Transition Probabilities (1 of 2)

- The matrix of transition probabilities allows us to get from a current state to a future state

  Let P_{ij} = conditional probability of being in state j in the future given the current state of i

  – For example, P_{12} is the probability of being in state 2 in the future given the event was in state 1 in the period before

Matrix of Transition Probabilities (2 of 2)

Let P = the matrix of transition probabilities

$$P = \begin{pmatrix} P_{11} & P_{12} & P_{13} & \cdots & P_{1n} \\ P_{21} & P_{22} & P_{23} & \cdots & P_{2n} \\ \vdots & & & & \vdots \\ P_{m1} & & \cdots & & P_{mn} \end{pmatrix}$$

- Individual P_{ij} values are determined empirically
- The probabilities in each row will sum to 1

Transition Probabilities for the Grocery Store Example (1 of 2)

- Use historical data to develop the matrix

$$P = \begin{pmatrix} 0.8 & 0.1 & 0.1 \\ 0.1 & 0.7 & 0.2 \\ 0.2 & 0.2 & 0.6 \end{pmatrix}$$

Row 1

$0.8 = P_{11}$ = probability of being in state 1 after being in state 1 in the preceding period

$0.1 = P_{12}$ = probability of being in state 2 after being in state 1 in the preceding period

$0.1 = P_{13}$ = probability of being in state 3 after being in state 1 in the preceding period

Transition Probabilities for the Grocery Store Example (2 of 2)

Row 2

0.1 = P_{21} = probability of being in state 1 after being in state 2 in the preceding period

0.7 = P_{22} = probability of being in state 2 after being in state 2 in the preceding period

0.2 = P_{23} = probability of being in state 3 after being in state 2 in the preceding period

Row 3

0.2 = P_{31} = probability of being in state 1 after being in state 3 in the preceding period

0.2 = P_{32} = probability of being in state 2 after being in state 3 in the preceding period

0.6 = P_{33} = probability of being in state 3 after being in state 3 in the preceding

Predicting Future Market Shares (1 of 6)

- One of the purposes of Markov analysis is to predict the future
- Use the vector of state probabilities and the matrix of transitional probabilities to determine the state probabilities at a future date
- Allows the computation of the probability that a person will be at one of the grocery stores in the future
- Since this probability is equal to market share, it is possible to determine the future market shares of the grocery stores

Predicting Future Market Shares (2 of 6)

- When the current period is 0, the state probabilities for the next period 1 are determined as follows:

$$\pi(1) = \pi(0)P$$

- For any period n, we can compute the state probabilities for period $n + 1$

$$\pi(n+1) = \pi(n)P$$

Predicting Future Market Shares (3 of 6)

- The computations for the next period's market share are

$$\pi(1) = \pi(0)P$$

$$= (0.4, 0.3, 0.3) \begin{pmatrix} 0.8 & 0.1 & 0.1 \\ 0.1 & 0.7 & 0.2 \\ 0.2 & 0.2 & 0.6 \end{pmatrix}$$

$$= [(0.4)(0.8) + (0.3)(0.1) + (0.3)(0.2), (0.4)(0.1) + (0.3)(0.7) + (0.3)(0.2), (0.4)(0.1) + (0.3)(0.2) + (0.3)(0.6)]$$

$$= (0.41, 0.31, 0.28)$$

Predicting Future Market Shares (4 of 6)

- The market share for American Food and Food Mart have increased and the market share for Atlas Foods has decreased
- We can determine whether this will continue by looking at the state probabilities will be in the future
- For two time periods from now

$$\pi\,(2) = \pi\,(1)P$$

Predicting Future Market Shares (5 of 6)

- Since we know that

$$\pi(1) = \pi(0)P$$

- We have

$$\pi(2) = \pi(1)P = [\pi(0)P]P = \pi(0)PP = \pi(0)P^2$$

- In general

$$\pi(n) = \pi(0)P^n$$

Predicting Future Market Shares (6 of 6)

The question of whether American and Food Mart will continue to gain market share and Atlas will continue to loose is best addressed in terms of equilibrium or steady-state conditions

Markov Analysis of Machine Operations

- The owner of Tolsky Works has recorded the operation of his milling machine for several years
- Over the past 2 years, 80% of the time the milling machine functioned correctly for the current month if it had functioned correctly during the preceding month
- 90% of the time the machine remained incorrectly adjusted if it had been incorrectly adjusted in the preceding month
- 10% of the time the machine operated correctly in a given month when it had been operating incorrectly the previous month

Tolsky Works (1 of 3)

- The matrix of transition probabilities for this machine

$$P = \begin{pmatrix} 0.8 & 0.2 \\ 0.1 & 0.9 \end{pmatrix}$$

where

$P_{11} = 0.8$ = probability that the machine will be *correctly* functioning this month given it was *correctly* functioning last month

$P_{12} = 0.2$ = probability that the machine will *not* be correctly functioning this month given it was *correctly* functioning last month

$P_{21} = 0.1$ = probability that the machine will be *correctly* functioning this month given it was *not* correctly functioning last month

$P_{22} = 0.9$ = probability that the machine will *not* be correctly functioning this month given it was *not* correctly functioning last month

Tolsky Works (2 of 3)

- What is the probability that the machine will be functioning correctly 1 and 2 months from now?

$$\pi(1) = \pi(0)P$$

$$= (1, 0) \begin{pmatrix} 0.8 & 0.2 \\ 0.1 & 0.9 \end{pmatrix}$$

$$= [(1)(0.8) + (0)(0.1), (1)(0.2) + (0)(0.9)]$$

$$= (0.8, 0.2)$$

Tolsky Works (3 of 3)

- What is the probability that the machine will be functioning correctly one and two months from now?

$$\pi(2) = \pi(1)P$$

$$= (0.8, 0.2)\begin{pmatrix} 0.8 & 0.2 \\ 0.1 & 0.9 \end{pmatrix}$$

$$= [(0.8)(0.8) + (0.2)(0.1), (0.8)(0.2) + (0.2)(0.9)]$$

$$= (0.66, 0.34)$$

Equilibrium Conditions (1 of 5)

- It is easy to imagine that all market shares will eventually be 0 or 1
- *Equilibrium share* of the market values or probabilities generally exist
- Called **steady-state** or **equilibrium probabilities**
- An **equilibrium condition** exists if state probabilities do not change after a large number of periods
- At equilibrium, state probabilities for the next period equal the state probabilities for current period
- Equilibrium state probabilities can be computed by repeating Markov analysis for a large number of periods

State Probabilities

TABLE 14.1 State Probabilities for the Machine Example for 15 Periods

PERIOD	STATE 1	STATE 2
1	1.000000	0.000000
2	0.800000	0.200000
3	0.660000	0.340000
4	0.562000	0.438000
5	0.493400	0.506600
6	0.445380	0.554620
7	0.411766	0.588234
8	0.388236	0.611763
9	0.371765	0.628234
10	0.360235	0.639754
11	0.352165	0.647834
12	0.346515	0.653484
13	0.342560	0.657439
14	0.339792	0.660207
15	0.337854	0.662145

Equilibrium Conditions (2 of 5)

- It is always true that

$$\pi \text{ (next period)} = \pi \text{ (this period)} P$$

- or

$$\pi (n + 1) = \pi (n) P$$

- at equilibrium

$$\pi (n + 1) = \pi (n)$$

- so at equilibrium

$$\pi (n + 1) = \pi (n) P = \pi (n)$$

- or

$$\pi = \pi P$$

Equilibrium Conditions (3 of 5)

- For Tolsky's machine

$$\pi = \pi P$$

$$(\pi_1, \pi_2) = (\pi_1, \pi_2)\begin{pmatrix} 0.8 & 0.2 \\ 0.1 & 0.9 \end{pmatrix}$$

- Using matrix multiplication

$$(\pi_1, \pi_2) = [(\pi_1)(0.8) + (\pi_2)(0.1), (\pi_1)(0.2) + (\pi_2)(0.9)]$$

Equilibrium Conditions (4 of 5)

- The first and second terms on the left side, π_1 and π_2, are equal to the first terms on the right side

$$\pi_1 = 0.8\pi_1 + 0.1\pi_2$$
$$\pi_2 = 0.2\pi_1 + 0.9\pi_2$$

- The state probabilities sum to 1

$$\pi_1 + \pi_2 + \ldots + \pi_n = 1$$

- For Tolsky's machine

$$\pi_1 + \pi_2 = 1$$

Equilibrium Conditions (5 of 5)

- We arbitrarily decide to solve the following two equations

$$\pi_2 = 0.2\pi_1 + 0.9\pi_2$$

$$\pi_1 + \pi_2 = 1$$

- Through rearrangement and substitution

$$0.1\pi_2 = 0.2\pi_1$$

$$\pi_2 = 2\pi_1$$

$$\pi_1 + \pi_2 = 1$$

$$\pi_1 + 2\pi_1 = 1$$

$$3\pi_1 = 1$$

$$\pi_1 = {}^{1}/_{3} = 0.33333333$$

$$\pi_2 = {}^{2}/_{3} = 0.66666667$$

Absorbing States and the Fundamental Matrix (1 of 8)

- Accounts Receivable example
  - Examples so far assume it is possible to go from one state to another
  - If you must remain in a state it is called an **absorbing state**
  - Four typical states

    State 1 (π_1): paid, all bills

    State 2 (π_2): bad debt, overdue more than 3 months

    State 3 (π_3): overdue less than 1 month

    State 4 (π_4): overdue between 1 and 3 months

Absorbing States and the Fundamental Matrix (2 of 8)

- Matrix of transition probabilities

	NEXT MONTH			
THIS MONTH	PAID	BAD DEBT	< 1 MONTH	1 TO 3 MONTHS
Paid	1	0	0	0
Bad debt	0	1	0	0
Less than 1 month	0.6	0	0.2	0.2
1 to 3 months	0.4	0.1	0.3	0.2

Thus,

$$P = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0.6 & 0 & 0.2 & 0.2 \\ 0.4 & 0.1 & 0.3 & 0.2 \end{pmatrix}$$

Absorbing States and the Fundamental Matrix (3 of 8)

- To obtain the fundamental matrix, it is necessary to partition the matrix of transition probabilities

$$P = \begin{array}{c} \begin{array}{cc} I & 0 \\ \downarrow & \downarrow \end{array} \\ \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0.6 & 0 & 0.2 & 0.2 \\ 0.4 & 0.1 & 0.3 & 0.2 \end{pmatrix} \\ \begin{array}{cc} \uparrow & \uparrow \\ A & B \end{array} \end{array}$$

$$I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \qquad 0 = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

$$A = \begin{pmatrix} 0.6 & 0 \\ 0.4 & 0.1 \end{pmatrix} \qquad B = \begin{pmatrix} 0.2 & 0.2 \\ 0.3 & 0.2 \end{pmatrix}$$

where

I = an identity matrix

0 = a matrix with all 0s

Absorbing States and the Fundamental Matrix (4 of 8)

- The fundamental matrix can be computed as

$$F = (I - B)^{-1}$$

$$F = \left(\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} - \begin{pmatrix} 0.2 & 0.2 \\ 0.3 & 0.2 \end{pmatrix}\right)^{-1}$$

$$F = \begin{pmatrix} 0.8 & -0.2 \\ -0.3 & 0.8 \end{pmatrix}^{-1}$$

The inverse of the matrix $= \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ is $\begin{pmatrix} a & b \\ c & d \end{pmatrix}^{-1} = \begin{pmatrix} \frac{d}{r} & \frac{-b}{r} \\ \frac{-c}{r} & \frac{a}{r} \end{pmatrix}$

where

$$r = ad - bc$$

Absorbing States and the Fundamental Matrix (5 of 8)

- To find the matrix *F*, we compute

$$r = ad - bc = (0.8)(0.8) - (-0.2)(-0.3) = 0.64 - 0.06 = 0.58$$

- With this, we have

$$F = \begin{pmatrix} 0.8 & -0.2 \\ -0.3 & 0.8 \end{pmatrix}^{-1} = \begin{pmatrix} \dfrac{0.8}{0.58} & \dfrac{-(-0.2)}{0.58} \\ \dfrac{-0.3}{0.58} & \dfrac{0.8}{0.58} \end{pmatrix} = \begin{pmatrix} 1.38 & 0.34 \\ 0.52 & 1.38 \end{pmatrix}$$

Pearson

Absorbing States and the Fundamental Matrix (6 of 8)

- We can use the matrix *FA* to answer questions such as how much of the debt in the less than one month category will be paid back and how much will become bad debt

$$M = (M_1, M_2, M_3, \dots , M_n)$$

where

n = number of nonabsorbing states

M_1 = amount in the first state or category

M_2 = amount in the second state or category

M_n = amount in the nth state or category

Absorbing States and the Fundamental Matrix (7 of 8)

- If we assume there is \$2,000 in the less than one month category and \$5,000 in the one to three month category, M would be

$$M = (2{,}000,\ 5{,}000)$$

$$\begin{aligned}\text{Amount paid and amount in bad debts} &= MFA \\ &= \begin{pmatrix}2{,}000, 5{,}000\end{pmatrix}\begin{pmatrix}0.97 & 0.03 \\ 0.86 & 0.14\end{pmatrix} \\ &= \begin{pmatrix}6{,}240, & 760\end{pmatrix}\end{aligned}$$

Absorbing States and the Fundamental Matrix (8 of 8)

- If we assume there is $2,000 in the less than one month category and $5,000 in the one to three month category, *M* would be

$$M = (2{,}000,\ 5{,}000)$$

$$\begin{aligned}\text{Amount paid and amount in bad debts} &= MFA \\ &= \left(2{,}000, 5{,}000\right)\begin{pmatrix}0.97 & 0.03 \\ 0.86 & 0.14\end{pmatrix} \\ &= \left(6{,}240,\ 760\right)\end{aligned}$$

Out of the total of $7,000, $6,240 will eventually be paid and $760 will end up as bad debt

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