Data Modeling & Analysis Techniques

Probability Distributions

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Experiment and Sample Space

- A (random) experiment is a procedure that has a number of possible outcomes and it is not certain which one will occur
- The sample space is the set of all possible outcomes of an experiment (often denoted by S).
  - Examples:
    - Coin : S={H, T}
    - Two coins: S={HH, HT, TH, TT}
    - Lifetime of a system: S={0..∞}

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Probability Distributions

- Probability distributions represent the likelihood of certain events
  - Probability “mass” (or density for continuous variables) represents the amount of likelihood attributed to a particular point
  - Cumulative distribution represents the accumulated probability “mass” at a particular point
    - Distributions in probability are usually given and their results are computed
    - Distributions (or their parameters) are usually the items to be estimated in statistics

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PMF, PDF and CDF

	Discrete Data	**Continuous Data**
Distributions	Probability Mass Function Cumulative Distribution Function	Probability Density Function Cumulative Distribution Function

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PMF, PDF and CDF Discussion

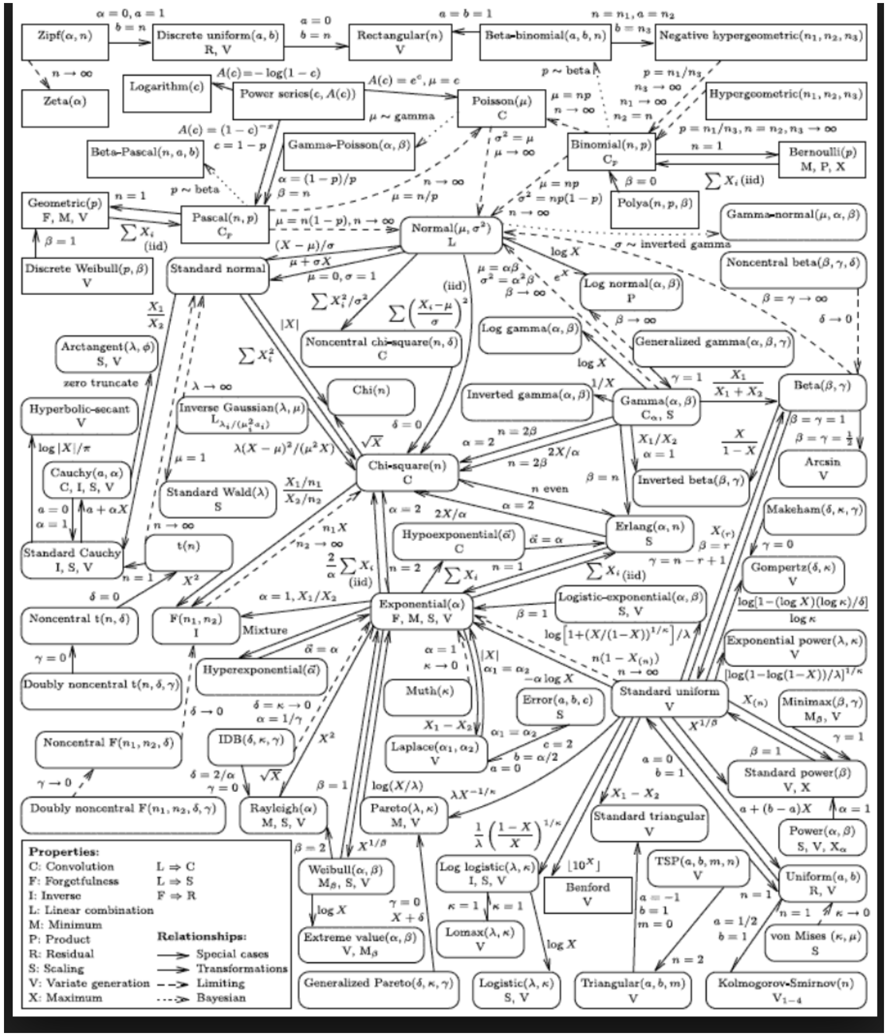
Probability Distributions

- Distributions can be characterized by their moments
  - r^{th} moment: $E\left[(x-a)^r\right]$
  - Important moments:
    - Mean: $E\left[(x-0)^1\right]$
    - Variance: $E\left[(x-\mu)^2\right]$
    - Skewness: $E\left[(x-\mu)^3\right]$

Distributions

- There are families of important distributions that are useful to model or analyze events
  - Families of distributions are parameterized
  - Different distributions are used to answer different questions about events
    - What is the probability of an individual event
    - How many times would an event happen in a repeated experiment
    - How long will it take until an event happens

Statistical Distributions



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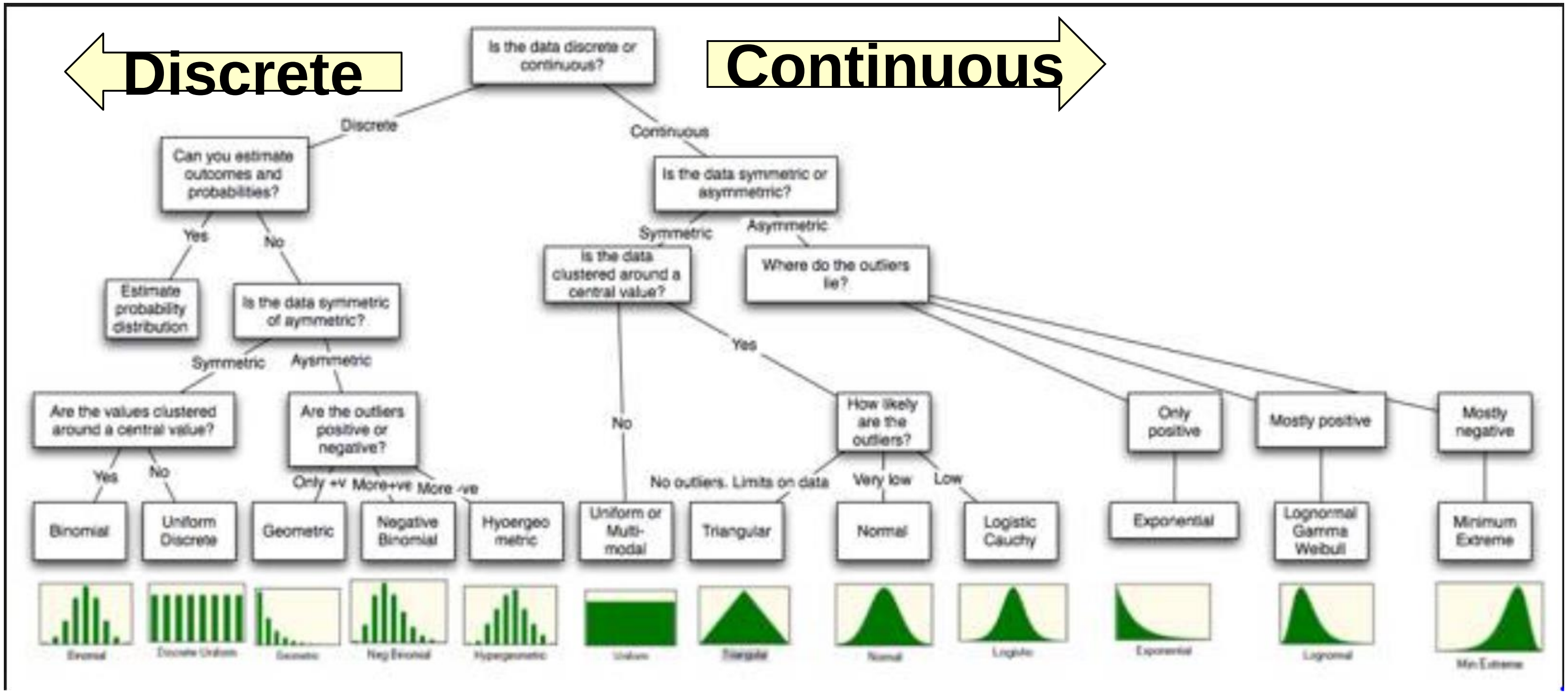
Statistical Distbriutions



Discrete Distributions

- Uniform Distribution
- Bernoulli Distribution
- Binomial Distribution
- Poisson Distribution
- Hypergeometric
- Gemoetric Distribution

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Distributions-Uniform Distributions

- Discrete distributions for event probability
  - Uniform distribution
    - Models the likelihood of a set of events assuming they are all equally likely
    - Parameterized by the number of discrete events, *N*
    - Probability function:

$$P(x;N) = P(X = x) = \frac{1}{N}$$

    - If the events are integers in the interval [*a..b*] (with $N=b-a+1$) we can compute a mean and variance
    - Mean: $\mu=(b+a)/2$ Variance: $\sigma^2=(N^2-1)/12$

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Uniform Distribution Example

- Die Example

Distributions-Bernoulli Distribution

- Bernoulli distribution
  - Models the likelihood of one of two possible events happening
    - Ex: good or bad/defective, pass or fail, transmitted or lost signal, benign or malicious attachments, boys or girls, heads or tails, etc..
  - Two outcomes: Successes and Failures
    - Successes do not have to be good or failures do not have to be bad
  - Parameterized by the likelihood, p, of event 1

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Bernoulli Distribution Contd..

- Bernoulli distribution
  - Probability function:

$$P(x;p) = P(X=x) = \begin{cases} p & \text{if } x = 1 \\ 1-p & \text{otherwise} \end{cases}$$

  - Can be easily extended to represent more than two possible events
  - Mean: $\mu=p$ Variance: $\sigma^2=p*(1-p)$
  - *Example*

Ex. Flip a fair coin. Let X = number of heads. Then X is a Bernoulli random variable with p=1/2.
E(X) = 1/2
Var(X) = 1/4

Distributions

- Discrete distributions for event frequency
  - Binomial distribution
    - Models the likelihood that an event will occur a certain number of times in *n* Bernoulli experiments
    - 1. Parameterized by the likelihood, *p*, of event *1* in the Bernoulli experiment and the number of experiments, *n*
    - *2. Two Outcomes (Sucsses/Failure)*
      - *i.e yes/no, Dead/live, treated/untreated, sick/well*
    - *3. P(Success)=p, P(Failure)=q or 1-p*
      - *p+q=1*

Binomial Distribution Contd..

- Probability function: The p(x), the probability that there will be exactly x success in n trilas

$$P(x)=\binom{n}{x}p^{x}q^{n-x}=\frac{n!}{(n-x)!\,x!}p^{x}q^{n-x}$$

where
n = the number of trials (or the number being sampled)
x = the number of successes desired
p = probability of getting a success in one trial
$q = 1 - p$ = the probability of getting a failure in one trial

- Mean: $\mu=np$ Variance: $\sigma^2=np(1-p)$

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Binomial Distribution Contd..Calculating from Tables

n always compute it from the table as

$$P(x) = F(x) - F(x-1).$$

Example 3.16. As part of a business strategy, randomly selected 20% of new internet service subscribers receive a special promotion from the provider. A group of 10 neighbors signs for the service. What is the probability that at least 4 of them get a special promotion?

Solution. We need to find the probability $P\{X \geq 4\}$, where X is the number of people, out of 10, who receive a special promotion. This is the number of successes in 10 Bernoulli trials, therefore, X has Binomial distribution with parameters $n = 10$ and $p = 0.2$. From Table A2,

$$P\{X \geq 4\} = 1 - F(3) = 1 - 0.8791 = \underline{0.1209}.$$

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Distributions

- Poisson distribution
  - Models the likelihood that an event will occur a given number of times in a continuous experiment with constant likelihood that does not depend on the time since the last occurrence
  - Parameterized by the expected number of occurrences, *λ*, of the event within one time period
  - Probability function:

$$P(x;\lambda) = \frac{\lambda^{x} e^{-\lambda}}{x!}$$

  - Mean: $E[x]=\mu=\lambda$ Variance: $\sigma^2=\lambda$

Poisson Distribution Example

Calculating from Tables

Example 3.22 (New accounts). Customers of an internet service provider initiate new accounts at the average rate of 10 accounts per day.

(a) What is the probability that more than 8 new accounts will be initiated today?
(b) What is the probability that more than 16 accounts will be initiated within 2 days?

<u>Solution</u>. (a) New account initiations qualify as rare events because no two customers open accounts simultaneously. Then the number X of today's new accounts has Poisson distribution with parameter $\lambda = 10$. From Table A3,

$$P\{X > 8\} = 1 - F_X(8) = 1 - 0.333 = \underline{0.667}.$$

Poisson Distribution Example

- A radioactive source emits 4 particles on average during a 5 second period
  - A. Calculate the probability that it emits 3 particles during a 5-second period
  - B. Calculate the probability that it emits at least one particle during a 5-second period

Distributions

- Multinomial distribution
  - Models the likelihood that each event, *i*, will occur a certain number of times in *n* independent experiments with l different events
  - Parameterized by the likelihoods, p_i, of the *l* events in the experiment and the number of experiments, *n*
  - Probability function:

$$P(x_1..x_l; n, p_1..p_l) = \frac{n!}{\prod_{i \in [1..l]} x_i!} \prod_{i \in [1..l]} p^{x_i}$$

  - Mean: $\mu_i = np_i$ Variance: $\sigma_i^2 = np_i(1-p_i)$

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Distributions

- Hypergeometric distribution
  - Models the likelihood that an event type will occur a certain number of times in *n* experiments if no specific event can occur twice and they are all equally likely
  - Parameterized by the total number of events, *N*, the number of events of the event type, *M*, and the number of experiments, *n*
  - Probability function:

$$P(x;M,N,n)=\frac{\binom{M}{x}\binom{N-M}{n-x}}{\binom{N}{n}}$$

  - Mean: $\mu=nM/N$ Variance: $\sigma^2=n(M(n-M)(N-n)/(N^2(N-1))$

Distributions

- Discrete distributions for inter-event timing
  - Geometric distribution
    - Models the likelihood that an event will occur for the first time in the x^{th} Bernoulli experiment
    - Parameterized by the probability, *p*, of the event in each Bernoulli experiment
    - Probability function:

$$P(x;p) = (1-p)^{x-1}p$$

    - Mean: $\mu=1/p$ Variance: $\sigma^2=(1-p)/p^2$

Continuous Distributions

- Uniform Distribution
- Normal Distribution
- Exponential Distribution

Continuous Distributions-Uniform

- Continuous distributions for event probability
  - Uniform distribution
    - Models the likelihood that a particular outcome will result from an experiment where every outcome value is equally likely
    - Parameterized by the range of possible outcomes, [a..b]
    - Probability density function:

$$p(x;a,b) = \frac{1}{b - a}$$

    - Mean: $\mu=(a+b)/2$ Variance: $\sigma^2=(b-a)^2/12$

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Distributions

- Normal distribution
  - Models the likelihood of results if the results are either distributed with a "Bell curve" or, alternatively, the result of the summation of a large number of random effects. This is a good approximation for a wide range of natural processes or noise phenomena as we will see a little later
  - Parameterized by a mean, μ, and standard deviation σ
  - Probability density function:

$$p(x;\mu,\sigma) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(x-\mu)^2}{2\sigma^2}}$$

  - Mean: μ Variance: σ^2

Distributions

- Continuous distributions for event frequency
  - Normal distribution
    - Models the number of times an event happens in a very large (infinite) number of experiments
    - Parameterized by a mean, μ, and standard deviation σ
    - Probability density function:

$$p(x;\mu,\sigma) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(x-\mu)^2}{2\sigma^2}}$$

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Normal Distribution Example

Distributions

- Continuous distributions for inter-event timing
  - Exponential distribution
    - Models the likelihood of an event happening for the first time at time *x* in a Poisson process (i.e. a process where events occur with the same likelihood at any point in time, independent of the time since the last occurrence.
    - Parameterized by event rate, *λ*
    - Probability density function:

$$p(x;\lambda) = \begin{cases} \lambda e^{-\lambda x} & \mathit{if}\ x \geq 0 \\ 0 & \mathit{otherwise} \end{cases}$$

    - Mean: *1/μ*   Variance: *1/λ²*

Exponential Distribution Example

Commonly, car cooling systems are controlled by electrically driven fans. Assuming that the lifetime T in hours of a particular make of fan can be modelled by an exponential distribution with $\lambda = 0.0003$ find the proportion of fans which will give at least 10000 hours service. If the fan is redesigned so that its lifetime may be modelled by an exponential distribution with $\lambda = 0.00035$, would you expect more fans or fewer to give at least 10000 hours service?

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Exponential Distribution Contd..

Answer

We know that $f(t) = 0.0003\mathrm{e}^{-0.0003t}$ so that the probability that a fan will give at least 10000 hours service is given by the expression

$$P(T > 10000) = \int_{10000}^{\infty} f(t)\,dt = \int_{10000}^{\infty} 0.0003\mathrm{e}^{-0.0003t}\,dt = -\left[\mathrm{e}^{-0.0003t}\right]_{10000}^{\infty} = e^{-3} \approx 0.0498$$

Hence about 5% of the fans may be expected to give at least 10000 hours service. After the redesign, the calculation becomes

$$P(T > 10000) = \int_{10000}^{\infty} f(t)\,dt = \int_{10000}^{\infty} 0.00035\mathrm{e}^{-0.00035t}\,dt = -\left[\mathrm{e}^{-0.00035t}\right]_{10000}^{\infty} = e^{-3.5} \approx 0.0302$$

and so only about 3% of the fans may be expected to give at least 10000 hours service.

Hence, after the redesign we expect *fewer* fans to give 10000 hours service.

Exponential Distribution Example

Moments

- Moments represent important aspects of the distribution and can be used to characterize mean, variance, etc.

$$E\left[(x - a)^r\right]$$

- In some cases the standard definition is difficult to compute
  - Moment generating function can sometimes help

Moment Generating Function

- The moment generating function for a random variable *X* is defined as

$$m_X(t) = E\left[e^{xt}\right]$$

  - The r[th] moment of *X* around 0 can then be computed as:

$$\lim_{t\rightarrow 0} \frac{\partial^r}{\partial t^r} m_X(t)$$

    - Note that sometimes this can not be computed since the limit might not be defined

Moment Generating Function

- The moment generating function allows to compute, e.g., the mean and the variance
  - Mean:

$$m = \lim_{t \to 0} \frac{\partial}{\partial t} \int e^{xt} p(x) dx$$

  - Variance:

$$s^2 = \lim_{t \to 0} \frac{\partial^2}{\partial t^2} \int e^{(x-m)t} p(x) dx$$

Example: Poisson Distribution

- Probability mass function

$$P(x;\lambda) = \frac{\lambda^x e^{-\lambda}}{x!}$$

- Moment generating function

$$m_X(t) = E[e^{xt}] = e^{\lambda\left(e^t - 1\right)}$$

- Mean

$$\mu = \lim_{t \to 0} \frac{\partial}{\partial t} e^{\lambda\left(e^t - 1\right)} = \lambda$$

- Variance

$$\sigma^2 = \lim_{t \to 0} \frac{\partial^2}{\partial t^2} e^{\lambda\left(e^{t-\mu} - 1\right)} = \lambda$$

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Multivariate Distributions

- Multivariate distributions sometimes arise when combining the outcomes of multiple random variables
  - Sometimes we are interested of the joint effect of multiple random variables
    - Distribution of the product of two random variables
    - Distribution of the joint additive effect of multiple variables

Multivariate Distributions

- For some operations combining multiple variables we can determine the moments of the distribution relatively easily
  - Usually assumptions made about random variables
    - Independently distributed
    - Moments of the distributions of the individual variables are known
  - If variables are not independent we have to use conditional distributions and the laws of probability

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Distribution of the Product

- The mean and variance of the distribution of the product of two independent random variables can be determined

$$\mu_{XY} = \sum_i \sum_j \left(x_i y_j P(x_i) P(y_j) \right) = \sum_i \left(x_i P(x_i) \sum_j \left(y_j P(y_j) \right) \right)$$

$$= \sum_i \left(x_i P(x_i) \mu_Y \right) = \mu_Y \sum_i \left(x_i P(x_i) \right) = \mu_X \mu_Y$$

Distribution of the Product

$$\sigma_{XY}^2 = \sum_i \sum_j \left((x_i y_j - \mu_X\mu_Y)^2 P(x_i)P(y_j) \right) = \sum_i \left(P(x_i) \sum_j \left(\left(((x_i-\mu_X)+\mu_X)((y_j-\mu_Y)+\mu_Y) - \mu_X\mu_Y \right)^2 P(y_j) \right)\right)$$

$$= \sum_i \left(P(x_i)\sum_j \left(\left(\left((x_i-\mu_X)(y_j-\mu_Y) + (x_i-\mu_X)\mu_Y + (y_j-\mu_Y)\mu_X + \mu_X\mu_Y\right) - \mu_X\mu_Y \right)^2 P(y_j)\right)\right)$$

$$= \sum_i \left(P(x_i)\sum_j \left(\left((x_i-\mu_X)(y_j-\mu_Y) + (x_i-\mu_X)\mu_Y + (y_j-\mu_Y)\mu_X\right)^2 P(y_j)\right)\right)$$

$$= \sum_i \left(P(x_i) \sum_j \left(\left(\begin{aligned} &(x_i-\mu_X)^2(y_j-\mu_Y)^2 + (x_i-\mu_X)^2(y_j-\mu_Y)\mu_Y + (x_i-\mu_X)(y_j-\mu_Y)^2\mu_X + (x_i-\mu_X)(y_j-\mu_Y)\mu_X\mu_Y \\ &+ (x_i-\mu_X)^2\mu_Y^2 + (y_j-\mu_Y)^2\mu_X^2 \end{aligned} \right) P(y_j) \right)\right)$$

$$= \sum_i \left(P(x_i) \left(\begin{aligned} &\sum_j \left((x_i-\mu_X)^2(y_j-\mu_Y)^2P(y_j)\right) + \sum_j \left((x_i-\mu_X)^2(y_j-\mu_Y)\mu_Y P(y_j)\right) + \sum_j \left((x_i-\mu_X)(y_j-\mu_Y)^2\mu_X P(y_j)\right) \\ &+ \sum_j\left((x_i-\mu_X)(y_j-\mu_Y)\mu_X\mu_Y P(y_j)\right) + \sum_j \left((x_i-\mu_X)^2\mu_Y^2 P(y_j)\right) + \sum_j \left((y_j-\mu_Y)^2\mu_X^2 P(y_j)\right) \end{aligned} \right)\right)$$

$$= \sum_i \left(P(x_i) \left(\begin{aligned} &(x_i-\mu_X)^2 \sum_j\left((y_j-\mu_Y)^2P(y_j)\right) + (x_i-\mu_X)^2\mu_Y \sum_j \left((y_j-\mu_Y)P(y_j)\right) + (x_i-\mu_X)\mu_X \sum_j \left((y_j-\mu_Y)^2 P(y_j)\right) \\ &+ (x_i-\mu_X)\mu_X\mu_Y \sum_j\left((y_j-\mu_Y)P(y_j)\right) + (x_i-\mu_X)^2\mu_Y^2\sum_j\left(P(y_j)\right) + \mu_X^2 \sum_j\left((y_j-\mu_Y)^2P(y_j)\right) \end{aligned} \right)\right)$$

$$= \sum_i \left(P(x_i) \left((x_i-\mu_X)^2\sigma_Y^2 + (x_i-\mu_X)^2\mu_Y(\mu_Y-\mu_Y) + (x_i-\mu_X)\mu_X\sigma_Y^2 + (x_i-\mu_X)\mu_X\mu_Y(\mu_Y-\mu_Y) + (x_i-\mu_X)^2\mu_Y^2 + \mu_X^2\sigma_Y^2\right)\right)$$

$$= \sum_i \left(P(x_i)\left((x_i-\mu_X)^2\sigma_Y^2 + (x_i-\mu_X)\mu_X\sigma_Y^2 + (x_i-\mu_X)^2\mu_Y^2 + \mu_X^2\sigma_Y^2\right)\right) = \sum_i \left(P(x_i)\left(\sigma_Y^2\left((x_i-\mu_X)^2 + (x_i-\mu_X)\mu_X + \mu_X^2\right) + (x_i-\mu_X)^2\mu_Y^2\right)\right)$$

$$= \sigma_Y^2 \sum_i (x_i-\mu_X)^2P(x_i) + \sigma_Y^2\mu_X\sum_i(x_i-\mu_X)P(x_i) + \sigma_Y^2\mu_X^2\sum_i P(x_i) + \mu_Y^2\sum_i(x_i-\mu_X)^2P(x_i)$$

$$= \sigma_Y^2\sigma_X^2 + \sigma_Y^2\mu_X(\mu_X-\mu_X) + \sigma_Y^2\mu_X^2 + \mu_Y^2\sigma_X^2 = \sigma_Y^2\sigma_X^2 + \sigma_Y^2\mu_X^2 + \mu_Y^2\sigma_X^2$$

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Distribution of the Sum

- The mean and variance of the distribution of the sum of two independent random variables can be determined

$$
\begin{aligned}
\mu_{X+Y} &= \sum_i \sum_j \left((x_i + y_j)P(x_i)P(y_j)\right) = \sum_i P(x_i) \sum_j \left(x_i P(y_j) + y_j P(y_j)\right) \\
&= \sum_i P(x_i)\left(x_i \sum_j P(y_j) + \sum_j \left(y_j P(y_j)\right)\right) = \sum_i P(x_i)\left(x_i + \mu_Y\right) \\
&= \sum_i P(x_i) x_i + \mu_Y \sum_i P(x_i) = \mu_X + \mu_Y
\end{aligned}
$$

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Distribution of the Sum

$$\sigma^2_{X+Y} = \sum_i \sum_j \left(\left((x_i + y_j) - (\mu_X + \mu_Y) \right)^2 P(x_i) P(y_j) \right) = \sum_i \left(P(x_i) \sum_j \left(\left((x_i + y_j)^2 - 2(x_i + y_j)(\mu_X + \mu_Y) + (\mu_X + \mu_Y)^2 \right) P(y_j) \right) \right)$$

$$= \sum_i \left(P(x_i) \sum_j \left(\left((x_i^2 + 2x_i y_j + y_j^2) - 2(x_i \mu_X + y_j \mu_X + x_i \mu_Y + y_j \mu_Y) + (\mu_X^2 + 2\mu_Y \mu_X + \mu_Y^2) \right) P(y_j) \right) \right)$$

$$= \sum_i \left(P(x_i) \sum_j \left(\left((x_i^2 - 2x_i \mu_X + \mu_X^2) + (y_j^2 - 2y_j \mu_Y + \mu_Y^2) + 2x_i y_j - 2(y_j \mu_X + x_i \mu_Y) + 2\mu_Y \mu_X \right) P(y_j) \right) \right)$$

$$= \sum_i P(x_i) \left((x_i - \mu_X)^2 \sum_j P(y_j) + \sum_j (y_j - \mu_Y)^2 P(y_j) + 2x_i \sum_j y_j P(y_j) - 2\mu_X \sum_j y_j P(y_j) - 2x_i \mu_Y \sum_j P(y_j) + 2\mu_Y \mu_X \sum_j P(y_j) \right)$$

$$= \sum_i P(x_i) \left((x_i - \mu_X)^2 + \sigma_Y^2 + 2x_i \mu_Y - 2\mu_X \mu_Y - 2x_i \mu_Y + 2\mu_Y \mu_X \right)$$

$$= \sum_i P(x_i) \left((x_i - \mu_X)^2 + \sigma_Y^2 \right) = \sum_i (x_i - \mu_X)^2 P(x_i) + \sigma_Y^2 \sum_i P(x_i) = \sigma_X^2 + \sigma_Y^2$$

Hypergeometric to Binomial

- If the population is large and the number of samples drawn is small, then the Hypergeometric distribution can be approximated by the Binomial distribution.
  - $p=M/N$

Normal to Standard Normal

- We usually denote Normal as: $N(\mu,\sigma^2)$
- The standard normal as: $N(0,1)=Z$
- If random variable X is normally distributed, i.e., $X= N(\mu,\sigma^2)$ then $Z=(X-\mu)/\sigma$

Binomial to Poisson

- Binomial pdf:

$$P(x;n,p) = \binom{n}{x} p^x (1-p)^{(n-x)}$$

- Binomial is hard to calculate for large *n*
- Poisson asks a similar question but in continuous time (no discrete time steps)

$$P(x;\lambda) = \frac{\lambda^x e^{-\lambda}}{x!}$$

- If n is lage and p is small, then the binomial can be approximated by a Poisson distribution with rate $\lambda = np$

Binomial to Normal

- We cannot use Poisson to approximate binomial if p is not very small (as np goes towards infinity).
- However, we can use the Normal distribution: $N(np, np(1-p))$
- Thus we can also approximate the Poisson as $N(\lambda, \llcorner\ )$ for large λ-s

Distributions

- If a function is always positive and converges to 0, $\lim_{x \to \infty} f(x) = 0$, can we make it into a pdf ?
- E.g. $f(x) = a/x$ between 1 and ∞ and 0 otherwise.

- No
  - There are functions of this type that do not represent probability distributions

Heavy Tailed Distributions

- How about "quicker" convergence:
  - $f(x)=a/x^2$ between 1 and ∞ and 0 otherwise.
- Can this be made into a pdf
  - Yes
- What is its mean
  - Infinite – the tail is too heavy
  - i.e. there are distributions that do not have numeric mean
- What is its variance
  - Infinite

Lower Polynomial Powers

- How about even “quicker” convergence:
  - $f(x)=a/x^3$ between 1 and ∞ and 0 otherwise.
- Can this be made into a pdf
  - Yes
- What is its mean
  - 2
- What is its variance
  - Infinite
  - i.e. there are distributions that have a numeric mean but do no numeric variance

Lower Polynomial Powers

- How about even "quicker" convergence:
  - $f(x)=a/x^4$ between 1 and ∞ and 0 otherwise.
- Can this be made into a pdf
  - Yes
- Does it have a finite mean
  - Yes
- Does it have a finite variance
  - Yes

Pareto Distribution

- The Pareto distribution has two parameters, a shape parameter α and a minimum x_m
  - Models many social and physical phenomena
    - Wealth distribution (80-20 rule), heard drive failures, daily maximum rainfalls, size of fires, etc.
  - Probability density

$$p(x; a, x_m) = \begin{cases} \dfrac{a x_m^a}{x^{(a+1)}} & x \geq x_m \\ 0 & otherwise \end{cases}$$

  - Cumulative density function

$$P(y < x) = \begin{cases} 1 - \left(\dfrac{x_m}{x}\right)^a & x \geq x_m \\ 0 & otherwise \end{cases}$$

Pareto Distribution

- The Pareto distribution is heavy tailed for some parameter settings
  - Infinite mean for $\alpha \leq 1$
  - Infinite variance for $\alpha \leq 2$
- For many interesting problems the parameters fall into this region
  - E.g. 80-20 rule has $\alpha \approx 1.161$
- Heavy tailed distributions exist and model existing problems
  - Has implications on sums and products of functions and the central limit theorem