CSE-5301 – Data Analysis and Modeling Techniques

Homework 5 (Sample Solutions)

<u>Textbook Exercise Chapter 9</u>

9.7 (a) $\bar{X} \pm z_{0.05}\dfrac{\sigma}{\sqrt{n}} = 37.7 \pm (1.645)\dfrac{9.2}{\sqrt{100}} = \boxed{37.7 \pm 1.5 \text{ or } [36.2, 39.2]}$

(b) Test $H_0 : \mu = 35$ vs $H_A : \mu > 35$. Reject H_0 if the test statistic $Z > z_{0.01} = 2.326$. The observed test statistic is

$$Z = \frac{\bar{X} - \mu_0}{\sigma/\sqrt{n}} = \frac{37.7 - 35}{9.2/\sqrt{100}} = 2.9348,$$

belongs to the rejection region. Therefore, reject H_0 in favor of H_A. Yes, these data provide significant evidence that the mean number of concurrent users is greater than 35.

9.9 (a) The standard deviation is unknown. Therefore, the interval is

$$\bar{X} \pm t_{\alpha/2} s/\sqrt{n},$$

where $\alpha = 1 - 0.90 = 0.10$, $n = 3$, $t_{\alpha/2} = t_{0.05} = 2.920$ (with 2 d.f.), $\bar{X} = (30 + 50 + 70)/3 = 50$, and

$$s = \sqrt{\frac{(30-50)^2 + (50-50)^2 + (70-50)^2}{n-1}} = \sqrt{\frac{800}{2}} = 20,$$

Then, the interval is

$$50 \pm 2.920\frac{20}{\sqrt{3}} = \boxed{50 \pm 33.7 \text{ or } [16.3; 83.7]}$$

(b) Hypothesis $H_0 : \mu = 80$ is *not rejected* against alternative $H_A : \mu \neq 80$ at the 10% level because the 90% confidence interval for μ contains 80.

This is a sufficient explanation, but you may also perform a test,

$$t = \frac{\bar{X} - \mu_0}{s/\sqrt{n}} = \frac{50 - 80}{20/\sqrt{3}} = -2.598$$

It belongs to the acceptance region $[-2.920; 2.920]$, therefore, H_0 is not rejected. The data does *not* provide a significant evidence against H_0.

(c) The 90% confidence interval for σ is

$$\left[\sqrt{\frac{(n-1)s^2}{\chi^2_{\alpha/2}}}, \sqrt{\frac{(n-1)s^2}{\chi^2_{1-\alpha/2}}}\right] = \left[\sqrt{\frac{(2)(400}{5.99}}, \sqrt{\frac{(2)(400}{0.10}}\right]$$

$$= \boxed{[11.6, 89.4] \text{ (thousand dollars)}}$$

9.10 (a) Find $\hat{p} = 24/200 = 0.12$. Then for $\alpha = 1 - 0.96 = 0.04$, find $z_{\alpha/2} = z_{0.02} = 2.054$ (the easiest way is to use Table A5 with ∞ degrees of freedom)

$$\begin{aligned}\hat{p} \pm z_{0.02}\sqrt{\frac{\hat{p}(1-\hat{p})}{n}} &= 0.12 \pm (2.054)\sqrt{\frac{0.12(1-0.12)}{200}} \\ &= \boxed{0.12 \pm 0.047 \text{ or } [0.073, 0.167]}\end{aligned}$$

(b) Test $H_0 : p \le 0.1$ (or $H_0 : p = 0.1$) vs $H_A : p > 0.1$. Disproving the manufacturer's claim means rejecting H_0 in favor of this H_A.

This is a one-sided test, therefore our two-sided confidence interval in (a) cannot be used to conduct this test.

The observed test statistic is

$$Z = \frac{\hat{p} - p_0}{\sqrt{\frac{\hat{p}(1-\hat{p})}{n}}} = \frac{0.12 - 0.1}{\sqrt{\frac{0.12(1-0.12)}{200}}} = 0.8704.$$

In order to consider different significance levels, let us compute the P-value,

$$P = \boldsymbol{P}\{Z > 0.8704\} = 1 - \Phi(0.8704) = 1 - 0.8078 = 0.1922,$$

from Table A4.

The P-value exceeds both 0.04 and 0.15. Therefore, we *do not* have a significance evidence, at the mentioned levels, to disprove the manufacturer's claim.

9.11 Test $H_0 : p_1 = p_2$ vs $H_A : p_1 > p_2$. Higher quality means lower proportion of defective items.

Given $\hat{p}_1 = 0.12$ from a sample of size $n = 200$ and $\hat{p}_2 = 13/150 = 0.0867$ from a sample of size $m = 150$, we compute the pooled proportion

$$\hat{p}(\text{pooled}) = \frac{n\hat{p}_1 + m\hat{p}_2}{n+m} = \frac{24+13}{200+150} = 0.1057.$$

Then, the test statistic is

$$Z = \frac{0.12 - 0.0867}{\sqrt{(0.1057)(1-0.1057)\left(\frac{1}{200} + \frac{1}{150}\right)}} = 1.0027$$

Finally, we compute the P-value

$$P = \boldsymbol{P}\{Z > 1.0027\} = 1 - 0.8413 = 0.1587$$

(Table A4), it is rather large, and we conclude that there is *no significance evidence* that the quality of items produced by the new supplier is higher than the quality of items in Exercise 9.10.

9.14 (a) To see if there is *any* significant difference between servers A and B, we test $H_0 : \mu_A = \mu_B$ (or $\mu_A - \mu_B = 0$) vs $H_A : \mu_A \neq \mu_B$. The 95% confidence interval in Example 9.21 on p. 265 is $[-1.4, -0.2]$. It does not contain the value 0 that we are testing, hence the difference between the two servers *is significant* at the 5% level.

(b) The test statistic is already computed in Example 9.30 on p. 278, and it equals -2.7603. The P-value for this two-sided test is

$$P = 2\boldsymbol{P}\{t > |-2.7603|\}\ \boxed{\text{is between 0.01 and 0.02}}$$

(Table A5 with 25 degrees of freedom, already computed by Satterthwaite approximation in Example 9.21).

We conclude that there is a significant difference between servers A and B at a level of 2% or higher, and the difference is not significant at a level of 1% or lower.

(c) A faster server should have a shorter execution time. Thus we test $H_0 : \mu_A = \mu_B$ vs $H_A : \mu_A < \mu_B$. For this one-sided test, the P-value equals

$$P = \boldsymbol{P}\{t < -2.7603\}\ \boxed{\text{is between 0.005 and 0.01}}$$

This is rather significant. At a 1% level of significance and any level above that, we have a significant evidence that server A is faster than server B.

9.17 For $\hat{p}_1 = 45\%$ support of candidate A, the margin of error is

$$z_{0.025}\sqrt{\frac{\hat{p}_1(1-\hat{p}_1)}{n}} = 1.96\sqrt{\frac{(0.45)(0.55)}{900}} = \boxed{0.0325 \text{ or } 3.25\%}$$

For $\hat{p}_2 = 35\%$ support of candidate B, the margin of error is

$$z_{0.025}\sqrt{\frac{\hat{p}_2(1-\hat{p}_2)}{m}} = 1.96\sqrt{\frac{(0.35)(0.65)}{900}} = \boxed{0.0312 \text{ or } 3.12\%}$$

For $\hat{p}_1 - \hat{p}_2 = 10\%$ lead of candidate A, the margin of error is

$$z_{0.025}\sqrt{\frac{\hat{p}_1(1-\hat{p}_1)}{n} + \frac{\hat{p}_2(1-\hat{p}_2)}{m}} = 1.96\sqrt{\frac{(0.45)(0.55)}{900} + \frac{(0.35)(0.65)}{900}} = \boxed{0.0450 \text{ or } 4.50\%}$$

9.22 We have $n = 20$, $m = 30$, $s_X = 0.6$, and $s_Y = 1.2$.

(a) Test $H_0 : \sigma_X^2 = \sigma_Y^2$ vs $H_A : \sigma_X^2 \neq \sigma_Y^2$.
The F-statistic is

$$F_{\text{obs}} = \frac{s_X^2}{s_Y^2} = 0.25,$$

and the P-value is

$$P = 2\min\left(\boldsymbol{P}\{F > 0.25\}, \boldsymbol{P}\{F < 0.25\}\right) = \boxed{0.0026},$$

using F-distribution with $n-1=19$ and $m-1=29$ d.f. and Matlab command `fcdf(0.25,19,29)` (Table A7 gives $P \approx 0.001$; also see Example 9.48 on p. 298 for details).

Thus, there is a significant evidence that variances are unequal, and we should use the Satterthwaite approximation for the two-sample t-test comparing the two population means.

(b) Find the critical values

$$F_{0.025}(19,29) = 2.23 \text{ and } F_{0.025}(29,19) = 2.40$$

using Matlab commands `finv(0.975,19,29)` and `finv(0.975,29,19)`. The 95% confidence interval for σ_X^2/σ_Y^2 is

$$\left[\frac{s_X^2}{s_Y^2 F_{\alpha/2}(n-1,m-1)},\ \frac{s_X^2 F_{\alpha/2}(m-1,n-1)}{s_Y^2}\right] = \left[\frac{0.6^2}{1.2^2 \cdot 2.23},\ \frac{0.6^2 \cdot 2.40}{1.2^2}\right]$$

$$= \boxed{[0.11,\ 0.60]}$$

Table A7 can also be used to obtain the critical values approximately, using 20, 30 d.f. and 30, 20 d.f. From Table A7, we get

$$F_{0.025}(19,29) \approx 2.2 \text{ and } F_{0.025}(29,19) \approx 2.35,$$

and the approximate 95% confidence interval is

$$\left[\frac{0.6^2}{1.2^2 \cdot 2.2},\ \frac{0.6^2 \cdot 2.35}{1.2^2}\right] = \boxed{[0.11,\ 0.59]}$$

9.23 From the given data, $\bar{X} = 85.00$, $\bar{Y} = 80.00$, $s_X = 12.76$, $s_Y = 3.22$, and $m = n = 6$.

(a) Test $H_0 : \mu_X = \mu_Y$ vs $H_A : \mu_X > \mu_Y$.

To choose a correct method of testing, we compare the variances. The test statistic for $H_0 : \sigma_X = \sigma_Y$ vs $H_A : \sigma_X \neq \sigma_Y$ is

$$F_{\text{obs}} = \frac{s_X^2}{s_Y^2} = 15.65.$$

Comparing with the F-distribution with 5 and 5 d.f. in Table A7, we find the p-value

$$P = 2\min\left(\boldsymbol{P}\left\{F \geq F_{\text{obs}}\right\},\ \boldsymbol{P}\left\{F \leq F_{\text{obs}}\right\}\right) = \text{ between 0.002 and 0.01.}$$

There is a significant evidence that $\sigma_X \neq \sigma_Y$, so we should use the method of Satterthwaite approximation.

The test statistic for testing $H_0 : \mu_X = \mu_Y$ vs $H_A : \mu_X > \mu_Y$ is

$$t_{\text{obs}} = \frac{85.00 - 80.00}{\sqrt{\frac{(12.76)^2}{6} + \frac{(3.22)^2}{6}}} = 0.93.$$

Next, the number of degrees of freedom is estimated by the Satterthwaite approximation,

$$\nu = \frac{\left(\dfrac{s_X^2}{n} + \dfrac{s_Y^2}{m}\right)^2}{\dfrac{s_X^4}{n^2(n-1)} + \dfrac{s_Y^4}{m^2(m-1)}} = \frac{\left(\dfrac{12.76^2}{6} + \dfrac{3.22^2}{6}\right)^2}{\dfrac{12.76^4}{180} + \dfrac{3.22^4}{180}} = 5.64.$$

From Table A5 (we can look at 5 and 6 d.f.) the p-value is

$$P = \boldsymbol{P}\{t > t_{\text{obs}}\} > 0.10.$$

Thus, there is no evidence that Anthony is a stronger student, i.e., that his (population, overall) average grade is higher than Eric's.

(b) We now test $H_0 : \sigma_X = \sigma_Y$ vs $H_A : \sigma_X > \sigma_Y$ (notice the one-sided alternative). The test statistic is already computed in (a), $F_{\text{obs}} = 15.65$. From Table A7 with 5 and 5 d.f., we find the p-value is

$$P = \boldsymbol{P}\{F \geq F_{\text{obs}}\} \in (0.001, 0.005).$$

There is a significant evidence that $\sigma_X > \sigma_Y$ supporting Eric's claim that he is more stable.

9.24 Again, from the data, $\bar{X} = 85.00$, $\bar{Y} = 80.00$, $s_X = 12.76$, $s_Y = 3.22$, and $m = n = 6$.

(a) From Table A5 with 5 d.f., $t_{\alpha/2} = t_{0.05} = 2.015$.

Then, the 90% confidence interval for Anthony's mean quiz grade is

$$\bar{X} \pm t_{\alpha/2}\frac{s_X}{\sqrt{n}} = 85.00 \pm (2.015)\frac{12.76}{\sqrt{6}} = \boxed{85.00 \pm 10.50 \text{ or } [74.50,\ 95.50]},$$

and the 90% confidence interval for Eric's mean quiz grade is

$$\bar{Y} \pm t_{\alpha/2}\frac{s_Y}{\sqrt{m}} = 80.00 \pm (2.015)\frac{3.22}{\sqrt{6}} = \boxed{80.00 \pm 2.65 \text{ or } [77.35,\ 82.65]}.$$

(b) From the solution to Exercise 9.23(a), we know that there is a significant evidence of $\sigma_X \neq \sigma_Y$, so we should use the method of Satterthwaite approximation with 5.61 degrees of freedom.

From Table A5, the critical value $t_{0.05}$ with 5.64 d.f. is between 1.943 and 2.015. The exact value $t_{0.05} = 1.9676$ can be obtained by a Matlab command `tinv(0.95,5.61)`.

Then, the 90% confidence interval for $(\mu_X - \mu_Y)$ is

$$\bar{X} - \bar{Y} \pm t_{\alpha/2}\sqrt{\frac{s_X^2}{n} + \frac{s_Y^2}{m}} = 5.00 \pm (1.97)\sqrt{\frac{12.76^2}{6} + \frac{3.22^2}{6}}$$

$$= \boxed{5.00 \pm 10.58 \text{ or } [-5.58, 15.58]}$$

(which shows to us that at the 10% level of significance, there is no significant difference between the two friends' mean grades).

(c) From Table A6, obtain the critical values $\chi^2_{\alpha/2} = \chi^2_{0.05} = 11.1$ and $\chi^2_{1-\alpha/2} = \chi^2_{0.95} = 1.15$, with 5 d.f.

Then, the 90% confidence interval for the variance of Anthony's scores is

$$\left[\frac{(n-1)s_X^2}{\chi^2_{\alpha/2}}, \frac{(n-1)s_X^2}{\chi^2_{1-\alpha/2}}\right] = \left[\frac{5 \cdot 12.76^2}{11.1}, \frac{5 \cdot 12.76^2}{1.15}\right] = \boxed{[73.33,\ 707.8]},$$

and the 90% confidence interval for the variance of Eric's scores is

$$\left[\frac{(n-1)s_Y^2}{\chi^2_{\alpha/2}}, \frac{(n-1)s_Y^2}{\chi^2_{1-\alpha/2}}\right] = \left[\frac{5 \cdot 3.22^2}{11.1}, \frac{5 \cdot 3.22^2}{1.15}\right] = \boxed{[4.68, 45.22]}.$$

(d) From Table A7 with 5 and 5 d.f., $F_{\alpha/2} = F_{0.05} = 5.05$, and by the reciprocal property, $F_{1-\alpha/2} = 1/F_{0.05}$.

Then, the 90% confidence interval for the ratio (σ_X^2/σ_Y^2) is

$$\left[\frac{s_X^2/s_Y^2}{F_{\alpha/2}(n-1, m-1)}, \frac{s_X^2/s_Y^2}{F_{1-\alpha/2}(n-1, m-1)}\right]$$

$$= \left[\frac{12.76^2/3.22^2}{5.05}, (12.76^2/3.22^2)(5.05)\right] = \boxed{[3.10,\ 79.05]}$$

(which shows support, at the 10% level, that $\sigma_X^2 \neq \sigma_Y^2$).