Code A	**MA1140**: Final Examination		Code A
Duration: 2 hour	Date: 15/03/2019	Time: 7 - 9 am	Total marks: 50

Name: ____________________ **Roll no.** ____________________ .

1. For $A = \begin{pmatrix} 0 & 1 & 2 & 0 \\ 0 & 2 & 4 & 0 \end{pmatrix}$, fill in the following table. [5 x 1 = 5]

A basis of the row space of A	$\{(\mathbf{0}, \mathbf{1}, \mathbf{2}, \mathbf{0})\}$. Be careful as the answer is not unique.
A basis of the column space of A	$\left\{\begin{pmatrix} 1 \\ 2 \end{pmatrix}\right\}$. The answer is not unique.
A basis of the null space of A	$\left\{\begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ -2 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}\right\}$. The answer is not unique.
rank of A (i.e rank of a linear map)	**1**
nullity of A	**3**

2. Write T (for True) or F (for False) on the space provided. [10 x 1.5 = 15]

(a) Let A be a row reduced echelon matrix with m rows and n columns over $\mathbb{R}$, where $m > n$. Let r be the number of non-zero rows of A. Then r is less than or equal to n. __ **T** __
Reason. In this case, $r = \text{row rank}(A) = \text{column rank}(A)$. So $r \leqslant n$.

(b) If u and v are eigenvectors of a matrix A, then $u + v$ is also an eigenvector of A. __ **F** __
Reason. One can construct a counterexample very easily by considering a 2×2 matrix. For example, one may take $v = -u$. In that case $u + v = 0$ (zero vector) which cannot be an eigenvector (by definition).

(c) Let A be an $m \times n$ matrix over $\mathbb{R}$. Let $r \leqslant n$. Suppose the first r columns of A are linearly independent, and the last r columns span the whole column space. Then r is equal to the dimension of the column space of A. __ **T** __

(d) Let $T : \mathbb{R}^n \to \mathbb{R}^n$ be a linear map. If we know $T(v)$ for n distinct non-zero vectors v in $\mathbb{R}^n$, then we know $T(v)$ for every vector v in $\mathbb{R}^n$. __ **F** __

(e) $\{(x, 0, -x) : x \in \mathbb{R}\}$ is a vector subspace of $\mathbb{R}^3$. __ **T** __
Reason. This is the set of all linear combinations of $\left\{\begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}\right\}$ over $\mathbb{R}$. So it is a subspace.

(f) Consider $S = \{(x, y) \in \mathbb{R}^2 : x = 0 \text{ or } y = 2x\}$ with usual vector addition and scalar multiplication. Then S is a subspace of $\mathbb{R}^2$. __ **F** __
Reason. Do not miss the word 'or'. Since it is '$x = 0$' or '$y = 2x$', S is a union of two subspaces $x = 0$ (y-axis) and $y = 2x$ (a line). None of these two subspaces is contained in the other. So S is not a subspace. Also one can verify directly that S is not closed under vector addition, e.g., $\begin{pmatrix} 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 2 \end{pmatrix} \in S$, but $\begin{pmatrix} 0 \\ 1 \end{pmatrix} + \begin{pmatrix} 1 \\ 2 \end{pmatrix} = \begin{pmatrix} 1 \\ 3 \end{pmatrix} \notin S$. Note that if you write 'and' in place of 'or', then S is a subspace, because in that case it is an intersection of two subspaces.

(g) Let A be an $n \times n$ matrix such that for every $b \in \mathbb{R}^n$, $AX = b$ has at least one solution. There may exist some b such that $AX = b$ has more than two solutions. ___ **F** ___

Reason. A can be thought as a linear map from $\mathbb{R}^n$ to itself. Using the Rank-Nullity Theorem, we have A is surjective if and only if A is injective.

(h) Let A be a non-invertible square matrix over $\mathbb{F}$, where $\mathbb{F}$ is $\mathbb{R}$ or $\mathbb{C}$. Depending on $\mathbb{F}$, the matrix A may not have an eigenvalue. ___ **F** ___

Reason. Many students missed the term 'non-invertible'. Since A is non-invertible, $AX = 0$ has a non-trivial solution. Thus there is a non-zero vector v such that $Av = 0 \cdot v$. Hence 0 is an eigenvalue of A irrespective of the base field $\mathbb{F}$. Do not forget that $\mathbb{R} \subset \mathbb{C}$, i.e., all real numbers are contained in the set of complex numbers.

(i) Let $u, v \in \mathbb{R}^3$ be such that $u \neq cv$ for any $c \in \mathbb{R}$. Then there are only finitely many subspaces of $\mathbb{R}^3$ containing the vectors u, v. ___ **T** ___

Reason. By the given condition, u, v are linearly independent vectors. So $\dim(\text{Span}\{u, v\}) = 2$. Comparing the dimension, one obtains that there are only two subspaces of $\mathbb{R}^3$ containing u, v which are $\text{Span}\{u, v\}$ and $\mathbb{R}^3$.

(j) Let A and B be row equivalent matrices. Then column rank(A) = column rank(B). _ **T** _

Reason. Since A and B are row equivalent, row space(A) = row space(B). This yields that row rank(A) = row rank(B), hence column rank(A) = column rank(B).

3. Write only the answers to the following questions: [2 x 1.5 = 3]

(a) The dimension of the vector space of $n \times n$ diagonal matrices with usual operations? ___ n ___

(b) Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be a non-zero linear map (i.e., not every vector maps to zero). Let u, v be two non-zero vectors such that $T(u) = 0$ and $T(v) = 0$. What is the rank of T? ___ 1 ___

Notes. Many students have written 0 as an answer. Note that the rank of T is 0 is and only if T is the trivial map, i.e., the zero map. But it is given that T is a non-zero map.

4. Tick all the matrices which are elementary (otherwise cross) from the following: [4 x 0.5 = 2]

(a) $\begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ $\checkmark$ (b) $\begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$ $\checkmark$ (c) $\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}$ **X** (d) $\begin{pmatrix} 0 & 0 & 1 \\ 0 & 2 & 0 \\ 1 & 0 & 0 \end{pmatrix}$ **X**

5. Let $\mathcal{C}(A)$ and $\mathcal{N}(A)$ denote the column and null spaces of an $n \times n$ matrix A over $\mathbb{R}$ respectively. Then $\mathbb{R}^n = \mathcal{C}(A) + \mathcal{N}(A)$. [1 + 2 = 3]

Write (T/F): F [1]

Justification: You should give a counterexample. Consider $\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$. [1]

For this matrix, both $\mathcal{C}(A) = \text{Span}\left\{\begin{pmatrix} 1 \\ 0 \end{pmatrix}\right\}$ and $\mathcal{N}(A) = \text{Span}\left\{\begin{pmatrix} 1 \\ 0 \end{pmatrix}\right\}$.

Therefore $\mathcal{C}(A) + \mathcal{N}(A) = \text{Span}\left\{\begin{pmatrix} 1 \\ 0 \end{pmatrix}\right\} \neq \mathbb{R}^n$. [1]

6. Let A be an $n \times n$ matrix over $\mathbb{R}$. Then $\text{nullity}(A) = \text{nullity}(A^t)$. [1 + 2 = 3]

Write (T/F): T [1]

Justification: Considering A as a linear map,

$$\begin{aligned} \text{rank}(A) &= \text{column rank}(A) \quad \text{[observation]} \\ &= \text{row rank}(A) \quad \text{[by a theorem proved in the class]} \end{aligned}$$

Note that

$$\begin{aligned} & \text{row space}(A) = \text{column space}(A^t) \\ \implies & \text{row rank}(A) = \text{column rank}(A^t) \\ \implies & \text{rank}(A) = \text{rank}(A^t) \quad \dots\dots\dots\dots \quad [1] \\ \implies & \text{nullity}(A) = \text{nullity}(A^t) \quad [\text{by } \textbf{Rank-Nullity Theorem}] \quad \dots\dots\dots\dots \quad [1] \end{aligned}$$

7. Let $T : \mathbb{R}^3 \to \mathbb{R}^2$ be a map defined by $T\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} x_1 - x_3 \\ x_2 + x_3 \end{pmatrix}$. Write the matrix representation A of T with respect to the ordered bases $\left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right\}$ and $\left\{ \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right\}$ of $\mathbb{R}^3$ and $\mathbb{R}^2$ respectively. [3]

Answer (only): $A = \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & 1 \end{pmatrix}$ (For each entry, 0.5 marks.)

8. Set $C^\infty(\mathbb{R}) = \{f : \mathbb{R} \to \mathbb{R} \mid f \text{ is infinitely differentiable function}\}$. Consider $T : C^\infty(\mathbb{R}) \to C^\infty(\mathbb{R})$ defined by $T(f) = f'$ (first derivative). Then any real number is an eigenvalue for T. [1 + 2 = 3]

Write (T/F): T [1]

Justification: Consider the non-zero element (vector) $e^{\lambda x}$ of $C^\infty(\mathbb{R})$ for every $\lambda \in \mathbb{R}$. [1]

Since $T(e^{\lambda x}) = \lambda e^{\lambda x}$, $\lambda \in \mathbb{R}$ is an eigenvalue of T (with the corresponding eigenvector $e^{\lambda x}$). Thus any real number is an eigenvalue for T. [1]

Notes. By definition, a scalar $\lambda \in \mathbb{R}$ is an eigenvalue of T if there EXISTS a non-zero vector v such that $T(v) = \lambda v$. Note that if λ is an eigenvalue, then every non-zero vector is not necessarily an eigenvector of T corresponding to λ, but there is at least one non-zero vector v such that $T(v) = \lambda v$. Thus, in order to show that every $\lambda \in \mathbb{R}$ is an eigenvalue of T, you just have to find at least one non-zero vector v_λ for every particular λ (i.e., v_λ is depending on λ) such that $T(v_\lambda) = \lambda v_\lambda$. Read carefully the definition of eigenvectors and eigenvalues discussed in class.

9. Let V be the vector space of all 2×2 matrices over $\mathbb{R}$ with usual operations. Let W be the subspace consisting of all matrices $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ such that $a + b = c + d$. Extend the set $\left\{ \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \right\}$ to a basis of W. [2+2]

Extended basis: $\left\{ \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix} \right\}$. Note that the answer is not unique. [2]

Justification:
You should check two things:
(i) spanning and [1]
(ii) linearly independence. [1]
There are many other ways also to check whether a subset is a basis.

10. Let V be the space of all 3×3 real matrices with usual operations. Consider $A = \begin{pmatrix} 1 & -3 & 0 \\ 0 & 2 & 0 \\ 3 & -2 & 1 \end{pmatrix}$. Is $A^3 \in \text{Span}\{A^2, A, I_3\}$, where I_3 is the identity matrix? Ans. (Y/N) __ Y __. Justify your answer in the space below. Are A^3, A^2, A, I_3 linearly independent? Ans. (Y/N) __ N __. [1+2+1 = 4]

Justification:
Step 1. Compute the **characteristic polynomial** $p_A(x) = \det(xI_3 - A)$. In this case, $p_A(x) = (x-1)^2(x-2) = x^3 - 4x^2 + 5x - 2$. [1]

Step 2. Now by **Caylay-Hamilton Theorem**, $A^3 - 4A^2 + 5A - 2I_3 = 0$ (zero matrix). It shows that A^3, A^2, A, I_3 are linearly dependent. Moreover $A^3 = 4A^2 - 5A + 2I_3$. [1]

Notes: One can verify the non-trivial relation $A^3 - 4A^2 + 5A - 2I_3 = 0$ directly. But that would be painful. First of all, they have to compute A^2 and A^3, and then they have to either guess or find out the non-trivial relation, and ultimately they should verify that relation.

11. Let A be a 2×2 matrix over $\mathbb{R}$. Suppose A has two eigenvalues λ_1 and λ_2 in $\mathbb{R}$ such that $\lambda_1 \neq \lambda_2$. Prove or disprove that A is diagonalizable. [1 + 4 = 5]

Proof/disproof:
Proof. [1]
Step 1. Let v_1, v_2 be two eigenvectors corresponding to the eigenvalues λ_1, λ_2 respectively. [1]

Step 2. We claim that v_1 and v_2 are linearly independent. Let $c_1 v_1 + c_2 v_2 = 0$. Apply A to obtain that $c_1 \lambda_1 v_1 + c_2 \lambda_2 v_2 = 0$. [1]

Step 3. We have $(c_1\lambda_1 v_1 + c_2\lambda_2 v_2) - \lambda_1(c_1v_1 + c_2v_2) = 0$, which yields that $c_2(\lambda_2 - \lambda_1)v_2 = 0$. Since $\lambda_1 \neq \lambda_2$ and $v_2 \neq 0$ (by definition), we get that $c_2 = 0$. It follows that $c_1v_1 = 0$. Thus, since $v_1 \neq 0$, $c_1 = 0$. Therefore v_1 and v_2 are linearly independent. [1]

Step 4. Since $\mathbb{R}^2$ has a basis $\{v_1, v_2\}$ consisting of eigenvectors, one can conclude directly (by using the **diagonalizability criteria** proved in the class) that A is diagonalizable. (Or one can prove this by taking $P = \begin{bmatrix} v_1 & v_2 \end{bmatrix}$ with v_1 and v_2 as the 1st and 2nd columns respectively.) [1]

Notes. There are many other ways also to show diagonalizability. If your argument is complete, you will get full marks.