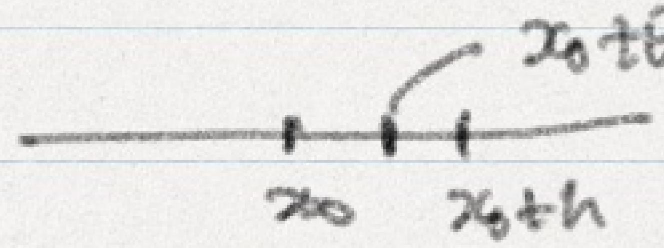
Taylors Thm

Let f has derivatives of order n at x_0. Then

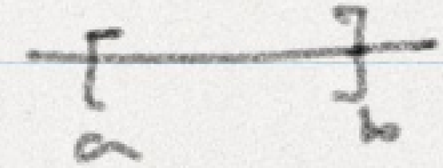
$$f(x_0+h) = f(x_0) + hf'(x_0) + \frac{h^2}{2!}f''(x_0) + \cdots + \frac{h^n}{n!}f^{(n)}(x_0+\theta h)$$

for some $0<\theta<1$



MVT: $f(b)-f(a) = (b-a)f'(\xi)$

$$f(a+h) = f(a) + hf'(a+\theta h).$$



$$f(x_0+h) = f(x_0) + h\underset{\overset{\|}{0}}{f'(x_0)} + \frac{h^2}{2!}f''(x_0+\theta h)$$

Def: Let $f: E (\subset \mathbb{R}^2) \to \mathbb{R}$. Then a point $\vec{x_0} \in E$ is said to be local maxima (minima) if $\exists$ a $\delta > 0$ s.t. $\forall$ $\vec{x} \in B(\vec{x_0}, \delta)$,

$$f(\vec{x}) < f(\vec{x_0}) \quad \left(f(\vec{x}) > f(\vec{x_0}) \right).$$

Defn - - - - - Let $\vec{x_0} \in E$. Then $\vec{x_0}$ is said to be a critical point of f if $\frac{\partial f}{\partial x}(x_0) = 0 = \frac{\partial f}{\partial y}(\vec{x_0})$

Defn - - - - Then a point $\vec{x_0} \in E$ is said to be a saddle point if $\vec{x_0}$ is a critical point but it is neither maxima nor minima.

Cor: A local extremum is a critical point.

<u>Ex</u>: Find 3 positive number a, b, c s.t $a+b+c = 120$ and the sum $ab+bc+ca$ is maximum.

Let $f : E \, (\subseteq \mathbb{R}^2) \longrightarrow \mathbb{R}$.

Let $(a,b) \in E$ and $(h,k) \in \mathbb{R}^2$ s.t $(a+h, b+k) \in E$.

~~Let $z = f(a+th, b+tk)$~~

~~where $t \in [0,1]$~~

$$f(a+h, b+k) = f(a,b) + \left(h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y}\right) f(a,b) + \frac{1}{2!}\left(h^2\frac{\partial^2}{\partial x^2} + 2hk\frac{\partial^2}{\partial x \partial y} + k^2\frac{\partial^2}{\partial y^2}\right) f(a,b) + \cdots$$

$$+ \frac{1}{n!}\left(h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y}\right)^n f(a+\theta h, b+\theta k)$$

for some $\theta \in (0,1)$.

pf: $z = f(a+th, b+tk)$ where $t \in [0,1]$.

$$\frac{dz}{dt} = \frac{\partial f}{\partial x}\cdot h + \frac{\partial f}{\partial y} k$$

$$\frac{d}{dt}\left(\frac{dz}{dt}\right) = \frac{d}{dt}\left(h\frac{\partial f}{\partial x} + k\frac{\partial f}{\partial y}\right)$$

$$= h^2\frac{\partial^2 f}{\partial x^2} + 2hk\frac{\partial^2 f}{\partial x \partial y} + k^2\frac{\partial^2 f}{\partial y^2}.$$

$$\frac{d}{dt}\left(\frac{\partial f}{\partial x}\right) = h\cdot\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right) + k\frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right)$$

$$h\frac{\partial^2 f}{\partial x^2} \quad k\frac{\partial^2 f}{\partial x \partial y} \checkmark$$

$$\frac{d^2 z}{dt^2} = \left(h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y}\right)^2 f$$

$$\frac{d}{dt}\left(\frac{d^{n-1} z}{dt^{n-1}}\right) = \left(h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y}\right)^n f$$

$$\frac{d^3 z}{dt^3} = \frac{d}{dt}\left(\frac{d^2 z}{dt^2}\right)$$

$$= \frac{d}{dt}\left(h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y}\right)^2 f$$

$$= \frac{d}{dt}\left(h^2\frac{\partial^2 f}{\partial x^2} + 2hk\frac{\partial^2 f}{\partial x \partial y} + k^2\frac{\partial^2 f}{\partial y^2}\right)$$

$$\vdots$$

$$= \left(h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y}\right)^3 f$$

$z = f(a+th, b+tk)$

$z : [0,1] \longrightarrow \mathbb{R}$

Apply Taylor's Thm for z. then

$$z(1) = z(0) + z'(0) + \frac{1}{2!} z''(0) + \cdots + \frac{1}{n!} z^{(n)}(0+\theta)$$

for some $\theta \in (0,1)$

$$f(a+h, b+k) = f(a,b) + \left(h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y}\right) f(a,b) + \cdots + \frac{1}{n!}\left(h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y}\right)^n f(\cdot)$$

$$f(a+\theta h, b+\theta k)$$

For $n=2$,

$$f(a+h, b+k) = f(a,b) + \left(h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y}\right) f(a,b) + \frac{1}{2}\left(h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y}\right)^2 f(a+\theta h, b+\theta k)$$

If (a,b) is a local extremum of f then

$$f(a+h, b+k) = f(a,b) + \frac{1}{2!}\left(h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y}\right)^2 f(a+\theta h, b+\theta k)$$

One can derive from here that if $D = \det\begin{pmatrix} f_{xx} & f_{xy} \\ f_{xy} & f_{yy} \end{pmatrix} > 0$

Then (1) f has local maxima if $f_{xx} < 0$

(2) f has local minima if $f_{xx} > 0$

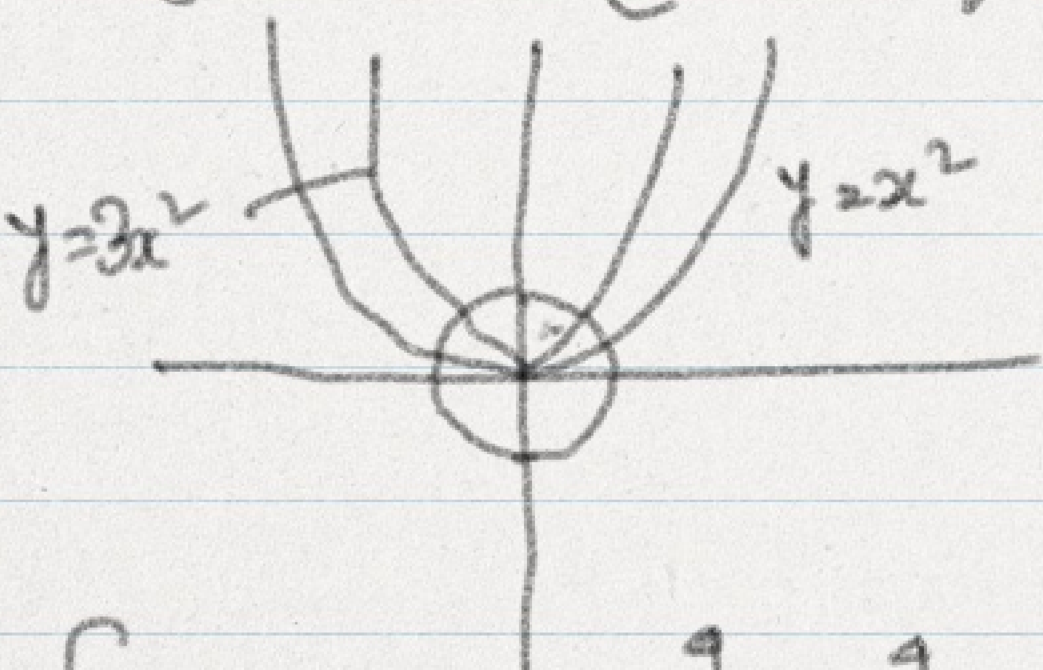
(3) f has saddle point if $f_{xx} < 0$.

Remark: If $D = 0$ then this test is inconclusive

Remark: If $D<0$ then f has a saddle point at (a,b).

Exc: Let $f(x,y) = 3x^4 - 4x^2y + y^2$
Show that $(0,0)$ is neither maxima nor minima for f.

Ex: $f(x,y) = (3x^2-y)(x^2-y)$

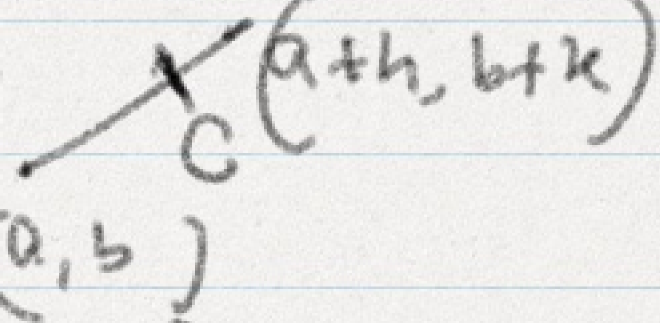


Ex: $f(x,y) = -x^4 - y^4$

$$\begin{pmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{pmatrix}$$

$$f(a+h, b+k) = f(a,b) + \underline{\left(h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y}\right)} f(a,b)$$

$$+ \left(h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y}\right)^2 f(c)$$



$$f(a+h, b+k) - f(a,b) = h^2\frac{\partial^2 f}{\partial x^2}(c) +$$

$$2hk\frac{\partial^2 f}{\partial x \partial y}(c) +$$

$$k^2\frac{\partial^2 f}{\partial y^2}(c)$$

$$= Q(c)$$

$$\text{LHS} \times 2\frac{\partial^2 f}{\partial x^2}(a,b) = Q(c) \times 2\frac{\partial^2 f}{\partial x^2}(a,b)$$

$$=$$

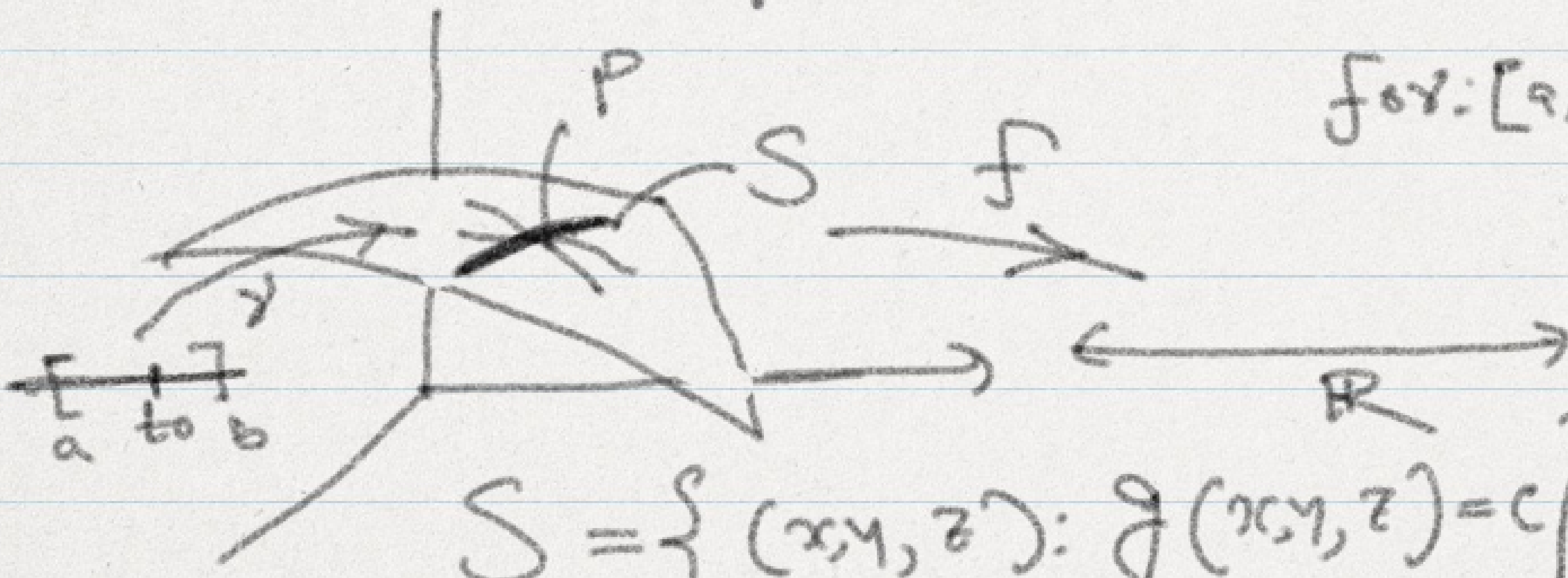
RHS =

$$\left(\left(h f_{xx} + k f_{xy} \right)(c) \right)^2 +$$

$$k^2 \left(f_{xx} f_{yy} - f_{xy}^2 \right)(c)$$

Let $D > 0$

Method of Lagrange Multiplier.



for $\gamma : [a,b] \to \mathbb{R}$

$$S = \{ (x,y,z) : g(x,y,z) = c \}$$

for some $c \in \mathbb{R}$.

$\nabla g(x_0, y_0, z_0)$ $\quad P = (x_0, y_0, z_0)$

Let $f: S \longrightarrow \mathbb{R}$, where f is a smooth scalar field

Consider the optimization problem

$$\text{Max } f(\vec{x})$$
$$\text{Sub. to } g(\vec{x}) = c.$$

<u>Hence</u> $\exists\, \lambda \in \mathbb{R}$ s.t

$$\nabla f(x_0, y_0, z_0) = \lambda \nabla g(x_0, y_0, z_0)$$

<u>Ex:</u> Consider the plane $2x+3y+z=1$
Find the distance from the origin to this plane.

Ex: Consider the surface

$$S = \{ (x, y, z) : x^2 + y^2 + z^2 = 1 \}$$

and $f : \mathbb{R}^3 \longrightarrow \mathbb{R}$ s.t

$$f(x, y, z) = ax + by + cz.$$

$$\nabla f = \lambda \nabla g$$

$$(a, b, c) = \lambda (2x, 2y, 2z)$$

$$x = \frac{a}{2\lambda}, \quad y = \frac{b}{2\lambda}, \quad z = \frac{c}{2\lambda}$$

$$a^2 + b^2 + c^2 = 4\lambda^2$$

$$2\lambda = \pm \frac{\sqrt{a^2+b^2+c^2}}{\cancel{2}}$$

$$x = \pm \frac{a}{\sqrt{}}, \quad y = \pm \frac{b}{\sqrt{}}, \quad z = \pm \frac{c}{\sqrt{a^2+b^2+c^2}}$$

$$\left(\frac{a}{\sqrt{}}, \frac{b}{\sqrt{}}, \frac{c}{\sqrt{}}\right) = (x_0, y_0, z_0)$$

Maximum value of f at (x_0, y_0, z_0) is $\frac{f}{\sqrt{a^2+b^2+c^2}}$.

<u>Ex:</u> Let $S = \{(x,y,z) : |x|^3 + |y|^3 + |z|^3 = 1\}$

$f : \mathbb{R}^3 \longrightarrow \mathbb{R}$

$f(x,y,z) = ax + by + cz.$

$\nabla f = \lambda \nabla g$

$(a,b,c) = \lambda\left(3|x|^2 sgn(x), 3|y|^2 sgn(y),\right.$

$\left. 3|z|^2 sgn(z)\right)$

$$|a| = 3|\lambda||x|^2 \Rightarrow |x| = \left(\frac{|a|}{3|\lambda|}\right)^{\frac{1}{2}} \text{ etc}$$

$$|b| = 3|\lambda||y|^2 \Rightarrow |y|$$

$$|c| = 3|\lambda||z|^2 \Rightarrow |z|$$

$$|a|^{\frac{3}{2}} + |b|^{\frac{3}{2}} + |c|^{\frac{3}{2}} = (3|\lambda|)^{\frac{3}{2}}$$

$$3|\lambda| = \left(|\ |^{\frac{3}{2}} + |\ |^{\frac{3}{2}} + |\ |^{\frac{3}{2}}\right)^{\frac{2}{3}}$$

Hence

$$|x| = \frac{|a|^{\frac{1}{2}}}{\left(|a|^{\frac{3}{2}} + |b|^{\frac{3}{2}} + |c|^{\frac{3}{2}}\right)^{\frac{1}{3}}}$$

$$|y| = \frac{|b|^{\frac{1}{2}}}{\quad}$$

$$|z| = \frac{|c|^{\frac{1}{2}}}{\quad}$$

$$f(x,y,z) = ax + by + cz$$

$$x = \frac{\operatorname{sgn}(a)\,|a|^{\frac{1}{2}}}{(\qquad)}$$

$$y = \frac{\operatorname{sgn}(b)\,|b|^{\frac{1}{2}}}{(\qquad)}$$

$$z = \frac{\qquad}{(\qquad)}$$

$$\frac{|a|^{\frac{3}{2}} + |b|^{\frac{3}{2}} + |c|^{\frac{3}{2}}}{\left(|a|^{\frac{3}{2}} + |b|^{\frac{3}{2}} + |c|^{\frac{3}{2}}\right)^{\frac{1}{3}}}$$

$$= \left(|a|^{\frac{3}{2}} + |b|^{\frac{3}{2}} + |c|^{\frac{3}{2}}\right)^{\frac{2}{3}}$$