EP 1027: Homework Assignment 1

IITH/Spring19/Shubho Roy

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1. Using the Einstein summation convention for indices, show that the scalar triple product, $\mathbf{A}\cdot(\mathbf{B}\times\mathbf{C})$ is (i) invariant under cyclic permutation of $\mathbf{A},\mathbf{B},\mathbf{C}$, i.e.

$$\mathbf{A}\cdot(\mathbf{B}\times\mathbf{C}) = \mathbf{B}\cdot(\mathbf{C}\times\mathbf{A})$$

(ii)invariant under swap of the "dot" and "cross",

$$\mathbf{A}\cdot(\mathbf{B}\times\mathbf{C}) = (\mathbf{A}\times\mathbf{B})\cdot\mathbf{C}$$

(5+5=10 points)

2. Prove that the determinant of a 3×3 matrix, M can be given in terms of the Levi-Civita tensor

$$|M| = \epsilon^{ijk}M^{i1}M^{j2}M^{k3}.$$

(10 points)

3. Suppose a three component object, $\begin{pmatrix}-2\\1\\4\end{pmatrix}$ becomes $\begin{pmatrix}-2\\1\\0\end{pmatrix}$ when the coordinate axes is rotated around the **x-axis** by 45°. Is this object a vector? (Hint: The matrix for rotation by angle, θ around the **x-axis** is $O=\begin{pmatrix}1&0&0\\0&\cos\theta&\sin\theta\\0&-\sin\theta&\cos\theta\end{pmatrix}$). (5 points)

4. Show that if, $A^{ij}$and $B^{lm}$are two rank 2 tensors, then their product with one pair of indices contracted (i.e. turned dummy), namely,

$$C^{im} = A^{il}B^{lm}$$

is also a tensor of rank 2. (Hint show that C transforms like a rank 2 tensor under orthogonal transformations/rotations). (5 points)

5. Using the rule for partial differentiation, $\frac{\partial x^l}{\delta x^m} = \delta^{lm}$, and the Einstein summation convention, prove the following,
i) $\boldsymbol{\nabla}.\mathbf{x} = 3$,
ii) $\boldsymbol{\nabla}\left(\frac{1}{|\mathbf{x}|}\right) = -\frac{\mathbf{x}}{|\mathbf{x}|^3}$, except when, $\mathbf{x}=(0,0,0)$,
iii) $\boldsymbol{\nabla}^2\left(\frac{1}{|\mathbf{x}|}\right) = 0$, except when, $\mathbf{x}=(0,0,0)$,
iv) $(\mathbf{B}\cdot\boldsymbol{\nabla})\,\mathbf{x} = \mathbf{B}$ (2+3+3+2=10 points)

6. For which integer value of n is,
$$\boldsymbol{\nabla} \cdot (r^n \mathbf{x}) = 0.$$
(5 points)

7. Prove the following identities using the Einstein summation convention (Φ, Ψ are scalar fields, while $\mathbf{A}$ is a vector field)
i) $\boldsymbol{\nabla}(\Phi\Psi) = \Phi\, \boldsymbol{\nabla}\Psi + \Psi\boldsymbol{\nabla}\Phi$,
ii) $\boldsymbol{\nabla}\cdot(\Phi\mathbf{A}) = \Phi(\boldsymbol{\nabla}\cdot\mathbf{A}) + (\mathbf{A}\cdot\boldsymbol{\nabla})\,\Phi$,
iii) $\boldsymbol{\nabla}\times(\Phi\mathbf{A}) = \Phi(\boldsymbol{\nabla}\times\mathbf{A}) + \boldsymbol{\nabla}\Phi\times\mathbf{A}$, (2+3+5 = 10 points)

8. Prove the following identities using the Einstein summation convention ($\mathbf{A}$and $\mathbf{B}$ are vector fields)
i) $\boldsymbol{\nabla}\cdot(\mathbf{A}\times\mathbf{B}) = \mathbf{B}\cdot(\boldsymbol{\nabla}\times\mathbf{A}) - \mathbf{A}\cdot(\boldsymbol{\nabla}\times\mathbf{B})$,
ii) $\boldsymbol{\nabla}\times(\boldsymbol{\nabla}\times\mathbf{A}) = \boldsymbol{\nabla}(\boldsymbol{\nabla}\cdot\mathbf{A}) - \boldsymbol{\nabla}^2\mathbf{A}$, where $\boldsymbol{\nabla}^2 = \boldsymbol{\nabla}\cdot\boldsymbol{\nabla}$ is the Laplacian
iii) $\boldsymbol{\nabla}\times(\mathbf{A}\times\mathbf{B}) = (\mathbf{B}\cdot\boldsymbol{\nabla})\,\mathbf{A} - (\mathbf{A}\cdot\boldsymbol{\nabla})\,\mathbf{B} + \mathbf{A}\,(\boldsymbol{\nabla}\cdot\mathbf{B}) - \mathbf{B}\,(\boldsymbol{\nabla}\cdot\mathbf{A})$. (5+5+10=20 points)

9. Consider a scalar field,
$$\Phi(\mathbf{x}) = \frac{2z^2 - x^2 - y^2}{(x^2+y^2+z^2)^{5/2}},$$
evaluate, $\nabla\Phi$ and $\nabla^2\Phi$. (4+6=10 points)

10. For two scalar fields, $\Phi(\mathbf{x})$ and $\Psi(\mathbf{x})$, show that the following result holds for a closed surface, S enclosing a volume, V
$$\iiint_V d^3\mathbf{x}\,\left(\Psi\boldsymbol{\nabla}^2\Phi - \Phi\boldsymbol{\nabla}^2\Psi\right) = \oiint_S dS\,\hat{\mathbf{n}}\cdot(\Psi\boldsymbol{\nabla}\Phi - \Phi\boldsymbol{\nabla}\Psi)\,.$$
This very useful result is known as **Green's Theorem**. (Hint: Try to apply Gauss divergence theorem to a vector field, $\mathbf{A}$ made out of the two scalar fields, namely, $\mathbf{A} = \Psi\boldsymbol{\nabla}\Phi - \Phi\boldsymbol{\nabla}\Psi$.) (10 points)

11. Stokes' theorem is proposed for an *open* surface i.e. a surface with a boundary, e.g. the surface of a hemisphere, which has a boundary, namely, the equator. What if we try to propose a version of Stokes theorem for a closed surface i.e. without boundaries, e.g. a full sphere, S^2. What does the result look like, i.e. the flux integral over curl of a vector over a closed surface equals to what?
$$\oiint_{S^2} dS\,\hat{\mathbf{n}}.\,(\boldsymbol{\nabla}\times\mathbf{A}) = ?$$
(5 points)

12. Gauss theorem and Stoke's theorem both apply to vector fields. But what about scalar fields? Show that the following results hold which are counterparts of Gauss and Stokes theorem for scalar fields, say $\Phi(\mathbf{x})$,
$$\iiint_V d^3\mathbf{x}\,\boldsymbol{\nabla}\Phi = \oiint_S dS\,\hat{\mathbf{n}}\,\Phi$$
$$\iint_S dS\,\hat{\mathbf{n}}\times\nabla\Phi = \oint_C \Phi\,d\mathbf{l}$$
(Hint: Cook up a vector field from the scalar, Φ by multiplying it with a constant vector, say $\mathbf{C}$ i.e. whose components are same everywhere, and then apply Gauss theorem and Stokes theorem to that cooked up vector field, $\mathbf{A} = \mathbf{C}\Phi$.) (5+5 = 10 points)

13. Prove that,

$$\iiint_B d^3\mathbf{x}\, \boldsymbol{\nabla}^2 \left(\frac{1}{|\mathbf{x}|}\right) = -4\pi.$$

where the volume over which we are integrating is a ball of unit radius, B with the origin of coordinates at the center of the ball. (Hint: Use Gauss' divergence theorem after acting with one of the ∇ on $\frac{1}{|\mathbf{x}|}$, and use spherical polar coordinates) (10 points)

14. Suppose that the electric current density in a region filled with plasma is $\mathbf{j} = C(xr\,\hat{\mathbf{x}} + yr\,\hat{\mathbf{y}})$, where C is a constant and $r = |\mathbf{x}|$. What is the rate of change of the electric charge in the spherical region bounded by $r = R$? (5 points)

15. Show that for an *incompressible* fluid (i.e. the mass density ρ is constant), the fluid current density vector is divergenceless aka solenoidal. Geometrically this means fluid flow lines are closed loops (5 points)