Dimension

If $\{v_1, v_2, \ldots, v_n\}$ is a basis for a V.S. V over a field $\mathbb{F}$, then dimension of V, denoted by $\dim(V)$, is n.

Theorem Let $\dim(V) = n$. Then, (V.S. over a field $\mathbb{F}$)

(i) If $S = \{v_1, v_2, \ldots, v_n\}$ is a set of linearly independent vectors of V, then S is a basis for V.

(ii) If $S = \{v_1, v_2, \ldots, v_n\}$ spans V, then S is a basis for V.

Proof Outline

(i)

Basis
- S is L.I. ✓
- $\operatorname{span}(S) = V$. ✓

$S = \{v_1, v_2, \ldots, v_n\}$ Suppose $\operatorname{span}(S) \subsetneq V$.

$\underbrace{w'', w', w, v_1, v_2, \ldots, v_n}_{n+3}$ } L.I.

(ii) S is a Basis
- ~~S is L.I.~~
- ~~$\operatorname{span}(S) = V$~~

If S is not L.I.

$$\underbrace{\{v_1, v_2, \dots, v_k\} \quad \{\cancel{v_{k+1}}, \dots, \cancel{v_n}\}}$$

Theorem: Let $S = \{v_1, v_2, \dots, v_n\}$ be a basis for a V.S. V over a field $\mathbb{F}$. If

$$w = \alpha_1 v_1 + \alpha_2 v_2 + \dots + \alpha_n v_n \quad (1)$$

and $$w = \beta_1 v_1 + \beta_2 v_2 + \dots + \beta_n v_n \quad (2)$$

then $\forall i \in [n],\ \alpha_i = \beta_i$.

Proof: Suppose not.

(1) − (2) is

$$0 = \underbrace{(\alpha_1 - \beta_1)}_{\neq 0} v_1 + \dots + \underbrace{(\alpha_n - \beta_n)} v_n$$

$\square$

Proposition Let K be a field. Then, K^n over K is a V.S.

$$(\_\_, \_\_, \_\_, \_\_, \_\_, \_\_)$$

$\in K$

$(- + - - - -)$

$\in K$

whose dimension is n.

multiplicative identity elmt of K.

$$n \left\{ \begin{array}{l} (1, 0, 0, \ldots, 0) \\ (0, 1, 0, \ldots, 0) \\ (0, 0, 1, \ldots, 0) \\ \vdots \\ (0, 0, 0, \ldots, 0, 1) \end{array} \right.$$

$\square$

Theorem. Let $\{v_1, v_2, \ldots, v_n\}$ be a maximal set of Linearly Independent vectors of a V.S. V over a field $\mathbb{F}$. Then, $\{v_1, v_2, \ldots, v_n\}$ is a basis of V.

Proof:

basis — L-I. ✓

basis — $\text{Span}[\{v_1, v_2, \ldots, v_n\}] = V$.

Take any $v \in V$

$\exists\, \alpha_1, \alpha_2, \ldots, \alpha_n \in \mathbb{F}$, s.t.

$v = \alpha_1 v_1 + \alpha_2 v_2 + \ldots + \alpha_n v_n$

$\underbrace{v, v_1, v_2, \ldots, v_n}$ is L.I.

$\square$

Theorem. Let $\{v_1, v_2, \ldots, v_n\}$ is a minimal set of vectors that spans a vector space V over a field $\mathbb{F}$. Then, $\{v_1, v_2, \ldots, v_n\}$ is a basis for V.

Homogeneous system of Linear Eqns

Recall that $MAT_{m \times n}(K)$ is a v.s. over K.

field

a_{ij}'s belong to a field $\mathbb{F}$

$$
\begin{aligned}
a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= 0 \\
a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= 0 \\
\vdots \qquad\quad \vdots \qquad\qquad\quad \vdots \quad &\ \ \vdots \\
a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n &= 0
\end{aligned}
$$

We know that, such a homogeneous system of L.E. has a non-trivial solution if $m < n$.

$$x_1 \begin{bmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{m1} \end{bmatrix} + x_2 \begin{bmatrix} a_{12} \\ a_{22} \\ \vdots \\ a_{m2} \end{bmatrix} + \cdots + x_n \begin{bmatrix} a_{1n} \\ a_{2n} \\ \vdots \\ a_{mn} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

$$x_1 \overline{A^1} + x_2 \overline{A^2} + \cdots + x_n \overline{A^n} = 0$$

vectors in an m-dimensional V.S. $\in \mathbb{F}^m = \underbrace{\mathbb{F} \times \mathbb{F} \times \cdots \times \mathbb{F}}_{m \text{ times}}$

We know from Proposition ~

$$\dim(\mathbb{F}^m) = m.$$

When $n > m$, $S = \{A^1, A^2, \ldots, A^n\}$ is <u>not Linearly Independent</u> in $\mathbb{F}^m$ over $\mathbb{F}$.

That is, they are Lin Dependent.

That means, there is a non-trivial linear combination of $A^1, A^2, \ldots, A^n$ that yields the 0 vector.

$\square$

<u>Inner Product</u>

Inner Product

$u = (u_1, u_2, \ldots, u_n) \in \mathbb{R}^n$

$v = (v_1, v_2, \ldots, v_n) \in \mathbb{R}^n$

Dot product $u \cdot v = \underbrace{u_1 v_1 + u_2 v_2 + \cdots + u_n v_n}_{\in \mathbb{R}}$

Dot product is a special type of inner product.

Let V be a v.s. over a field K. An inner product on V is a mapping which maps any pair of vectors v, w in V to a scalar. It is denoted by $\langle v, w \rangle$. It satisfies the following properties

$\langle IP1 \rangle$ $\langle v, w \rangle = \langle w, v \rangle$

$\langle IP2 \rangle$ $\langle \alpha v, w \rangle = \alpha \langle v, w \rangle$ and

$\langle v, \alpha w \rangle = \alpha \langle v, w \rangle$

when $\alpha \in K$.

$\langle IP3 \rangle$ $\langle v, w_1 + w_2 \rangle = \langle v, w_1 \rangle +$

$\langle v, w_2 \rangle$

Fisher Inequality

$\hookrightarrow$ 1940s

Ronald Fisher

Theorem: Let k, n be two positive integers with $k \leq n$. Let $\mathcal{F}$ be a family of subsets of $[n]$ such that for every distinct $A, B \in \mathcal{F}$, we have $|A \cap B| = k$.
Then, $|\mathcal{F}| \leq n$.

$\hookrightarrow$ Example.

$$\mathcal{F} = \{ \{1\}, \{1,2\}, \{1,3\}, \{1,4\}, \ldots, \{1,n\} \}$$

$k = 1$.

$$|\{1,2\} \cap \{1,3\}| = 1$$

$|\mathcal{F}| = n$.

History $\hookrightarrow$ 1940s

Fisher $k=1$, $\mathcal{F}$ is r-uniform

Erdős, De Bruijne [1948]

$\hookrightarrow$ $\mathcal{F}$ is

non-uniform

$k = 1$

Bose

Majumder [1950]