CS6350: Topics in Combinatorics
Assignment 6

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1. **Prove that every three-uniform hypergraph with n vertices and m $\geq \frac{n}{3}$ edges contains an independent set (i.e., a set of vertices containing no edges) of size at least $\frac{2n^{3/2}}{3\sqrt{3}\sqrt{m}}$**

A. Let us consider a three-uniform hypergraph H with n vertices and m edges. Now let us construct a set X with elements chosen randomly independently from the vertices of the hypergraph H each with a probability p.

$\Rightarrow \boxed{E[|X|] = n * p}$ $\quad (n\ vertices\ with\ probability\ p\ for\ each\ vertex)$

Now let Y be the edges of the hypergraph H present in the set X. Now let us remove one of the each three vertices for each edge of Y. Then the resultant hypergraph would become independent(hypergraph with no edges).

$\Rightarrow \boxed{E[|Y|] = m * p * p * p}$ $\quad (m\ edges\ with\ probability\ p\ for\ each\ of\ three\ vertices\ per\ edge)$

So, the expected value for independent vertices is

$$\begin{aligned} E[|X| - |Y|] &= E[|X|] - E[|Y|] \\ &= np - mp^3 \end{aligned}$$

Now for the maximum value of $E[|X| - |Y|]$ consider its derivative equal to zero

$$\begin{aligned} \frac{d}{dp}(np - mp^3) &= 0 \\ n - 3mp^2 &= 0 \\ p^2 &= \frac{n}{3m} \\ \boxed{p = \sqrt{\frac{n}{3m}}} \end{aligned}$$

So the maximum value of $E[|X| - |Y|]$ is

$$\begin{aligned} E[|X| - |Y|] = np - mp^3 &= p(n - mp^2) \\ &= \sqrt{\frac{n}{3m}}(n - m(\frac{n}{3m})) \end{aligned}$$

$$\Rightarrow \boxed{E[|X| - |Y|]_{max} = \frac{2n^{3/2}}{3\sqrt{3}\sqrt{m}}}$$

Hence it is proved that a any hypergraph with n vertices and m edges contains an independent set of size atleast $\frac{2n^{3/2}}{3\sqrt{3}\sqrt{m}}$