

The local lemma (symmetric case)

Let A_1, A_2, \dots, A_n be events in an arbitrary probability space. Suppose that each A_i is mutually independent of all but at most d other events

and $P_i[A_i] \leq p, \forall i \in [n]$. If $\underline{\underline{ep(d+1) \leq 1}}$,
 then $P_i\left[\bigwedge_{i=1}^n \overline{A_i}\right] > 0$.

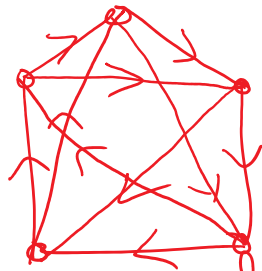
Application of Local Lemma
Theorem → [From Jukna, Chapter 19]
 Every k -regular digraph

has a collection of $\left\lfloor \frac{k}{3 \log_e k} \right\rfloor$ vertex
 disjoint ^{directed} cycles.

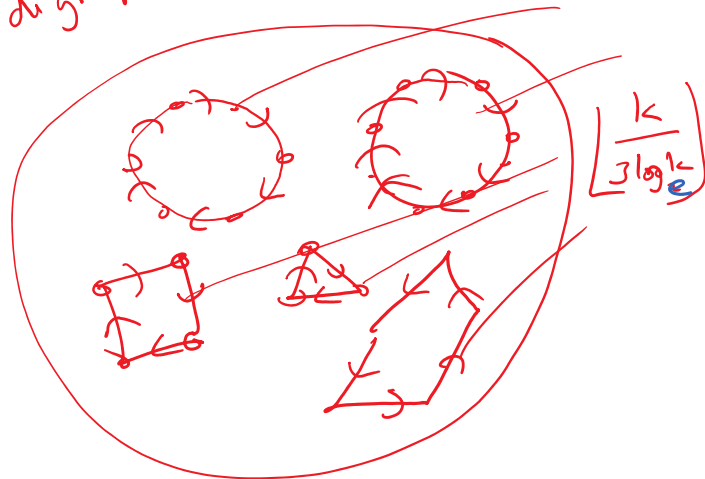
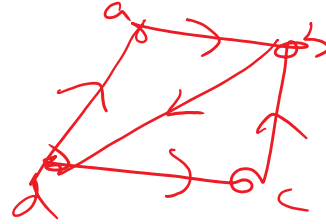
Proof:

in-degree = k
 out-degree = k

directed graph
 with no
 multi edges.



2-regular digraph



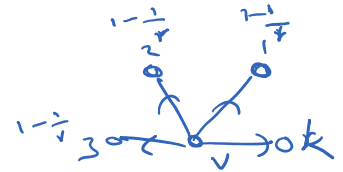
$$\text{Let } r = \left\lfloor \frac{k}{3 \log_e k} \right\rfloor.$$

Independently and uniformly at random,
 color each vertex with one of the
 r colors, namely c_1, c_2, \dots, c_r .

For each vertex v ,

A_v : ^{"BAD"} event that not every color is

A hand-drawn diagram illustrating a network graph. The graph consists of several nodes (circles) and directed edges (arrows). The nodes are colored green, blue, or red. The edges are colored green, blue, or red. The graph is divided into three main regions: a green region on the left, a blue region in the center, and a red region on the right. The green region contains a large, dense, circular structure. The blue region contains a large, dense, circular structure. The red region contains a large, dense, circular structure. The green region is labeled 'Green' in green text. The blue region is labeled 'Blue' in blue text. The red region is labeled 'Red' in red text. The diagram shows a complex network of connections between these three regions.



$$\leq \sum_{i=1}^v P_i \left[(c_i \text{ is not present}) \right. \\ \left. \text{in } \mathcal{N}(v) \right]$$

$$= \gamma \left(1 - \frac{1}{\gamma}\right)^k$$

$$\leq \frac{r}{e^{\frac{k}{r}}} \quad \left[\text{use } 1+n \leq e^n \right]$$

$$= \frac{r}{\cancel{k} \cancel{3 \ln k}} \quad \left(\begin{array}{l} \text{substitute} \\ r = \left[\frac{k}{3 \ln 2 k} \right] \end{array} \right)$$

$$= \frac{r}{\ln k^3}$$

$$\rightarrow \frac{\gamma}{k^3}$$

$$\leq \frac{k}{3k^3 \ln k}$$

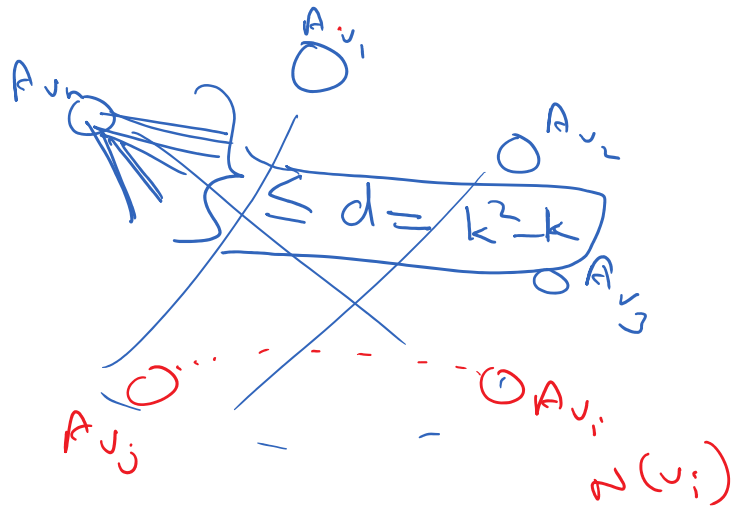
$$= \frac{1}{3k^2 \ln k}$$

Let $p := \frac{1}{3k^2 \ln k}$.

Then, $\forall v, P_i[A_v] \leq p$.

Suppose v_1, v_2, \dots, v_n where the
Bad events:
values of the
digraph.

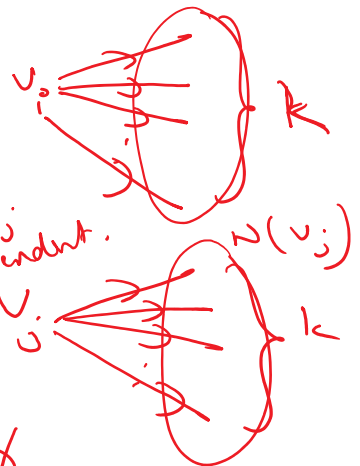
$$A_{v_1}, A_{v_2}, \dots, A_{v_n}$$



$$A_{v_i} \quad A_{v_j}$$

If $N(v_i) \cap N(v_j) = \emptyset$,

then A_{v_i} and A_{v_j}
are mutually independent.

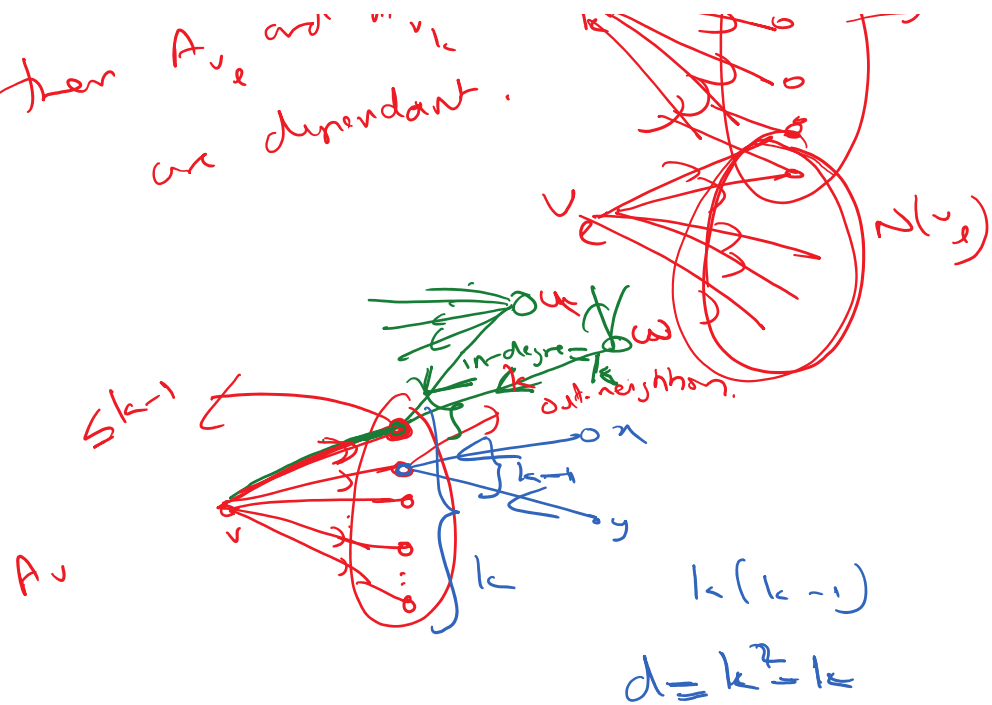


If $N(v_e) \cap N(v_k) \neq \emptyset$,

then A_{v_e} and A_{v_k}
are not independent.



then A_{v_i} and $\dots v_k$
are dependant.



We have,

$$p = \frac{1}{3k^2 \ln k}$$

$$d = k^2 - k$$

Applying Local Lemma.

$$\text{Since } e p(d+1) = e \cdot \frac{1}{3k^2 \ln k} (k^2 - k + 1) \leq 1$$

$$\text{we have } \Pr \left[\bigwedge_v \bar{A}_v \right] > 0.$$

That is, there is a coloring of the vertices of the digraph using r colors such that every vertex sees all the r colors in its out-neighborhood.

□

Later McDiarmid & Molloy showed
that $\Omega(k^2)$ vertex disjoint cycles
exist.