# The local lemma (symmetric case)

Let $A_1, A_2, \dots, A_n$ be events in an arbitrary probability space. Suppose that each $A_i$ is mutually independent of all but at most $d$ other events and $\Pr[A_i] \leq p$, $\forall i \in [n]$. If $ep(d+1) \leq 1$, then $\Pr\left[\bigwedge_{i=1}^{n} \overline{A_i}\right] > 0$.

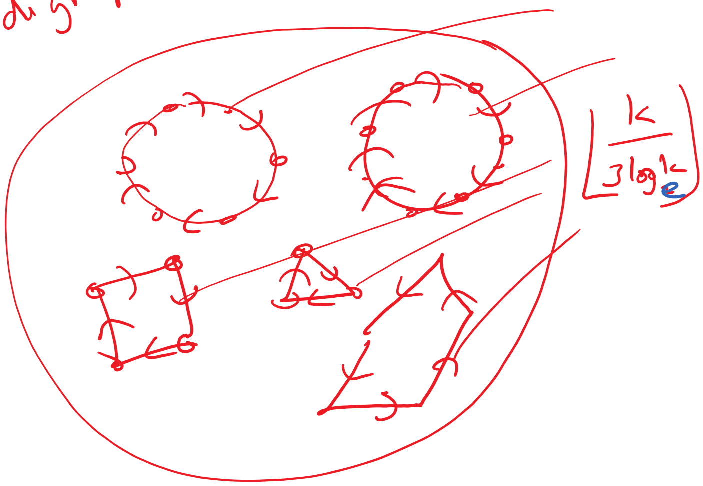
# Application of Local Lemma

**Theorem:** [From Jukna, Chapter 19] Every k-regular digraph has a collection of $\left\lfloor \frac{k}{3\log_e k} \right\rfloor$ vertex disjoint directed cycles.

**Proof:**

in-degree $= k$
out-degree $= k$

directed graph with no multi edges.


Let $r = \left\lfloor \frac{k}{3\log_e k} \right\rfloor$.

Independently and uniformly at random, color each vertex with one of the $r$ colors, namely $c_1, c_2, \ldots, c_r$.

For each vertex $v$,

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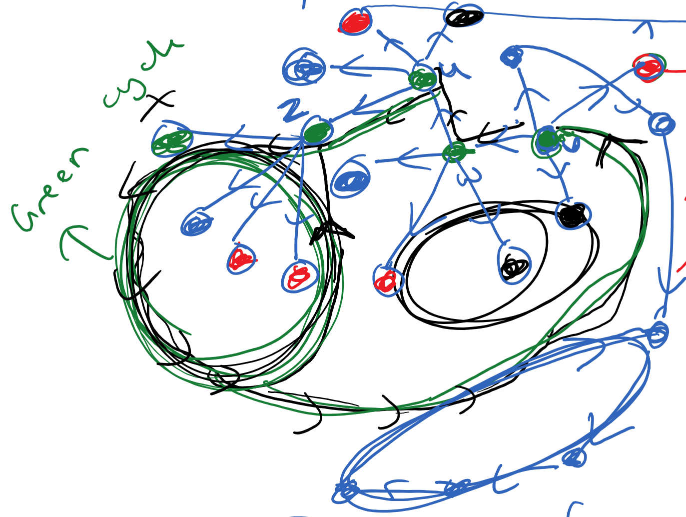
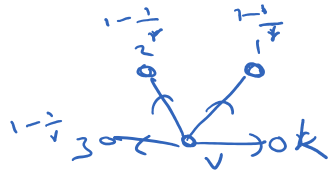
$A_v$ : "BAD" Event that not every color is present in the out-neighborhood of $v$.




$$\Pr[A_v] = \Pr\left[\begin{pmatrix} c_1 \text{ not present} \\ \text{in } N(v) \end{pmatrix} \text{ or } \begin{pmatrix} c_2 \text{ not} \\ \text{present in} \\ N(v) \end{pmatrix} \text{ or} \ldots \begin{pmatrix} c_r \text{ not present} \\ \text{in } N(v) \end{pmatrix}\right]$$

$$\leq \sum_{i=1}^{r} \Pr\left[\begin{pmatrix} c_i \text{ is not present} \\ \text{in } N(v) \end{pmatrix}\right]$$

$$= r\left(1 - \frac{1}{r}\right)^k$$

$$\leq \frac{r}{e^{\frac{k}{r}}} \qquad \left[\text{use } 1+x \leq e^x\right]$$

$$= \frac{r}{e^{\frac{\cancel{k}\, 3\ln k}{\cancel{k}}}} \qquad \left(\text{substitute } r = \left\lfloor \frac{k}{3\log_2 k} \right\rfloor\right)$$

$$= \frac{r}{e^{\ln k^3}}$$

$$= \frac{r}{k^3}$$

$$\le \frac{k}{3k^3 \ln k}$$

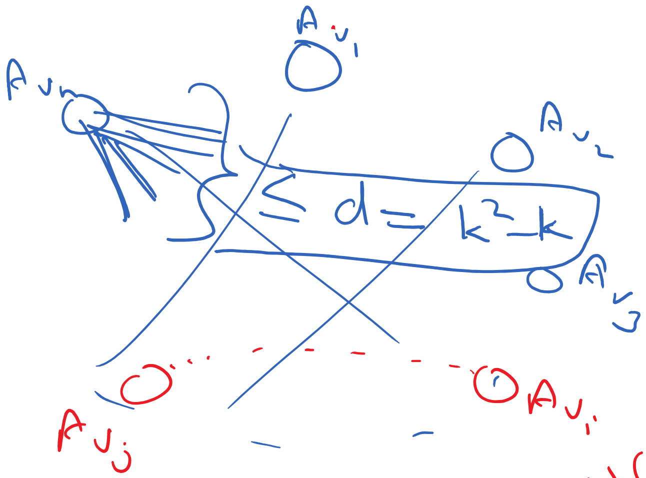
$$= \frac{1}{3k^2 \ln k}$$

Let $p := \frac{1}{3k^2 \ln k}$.

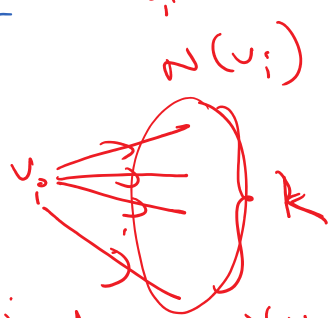
Then, $\forall v$, $\Pr[A_v] \le p$.

Bad events: Suppose $v_1, v_2, \ldots, v_n$ where the vertices of the digraph.

$A_{v_1}, A_{v_2}, \ldots, A_{v_n}$



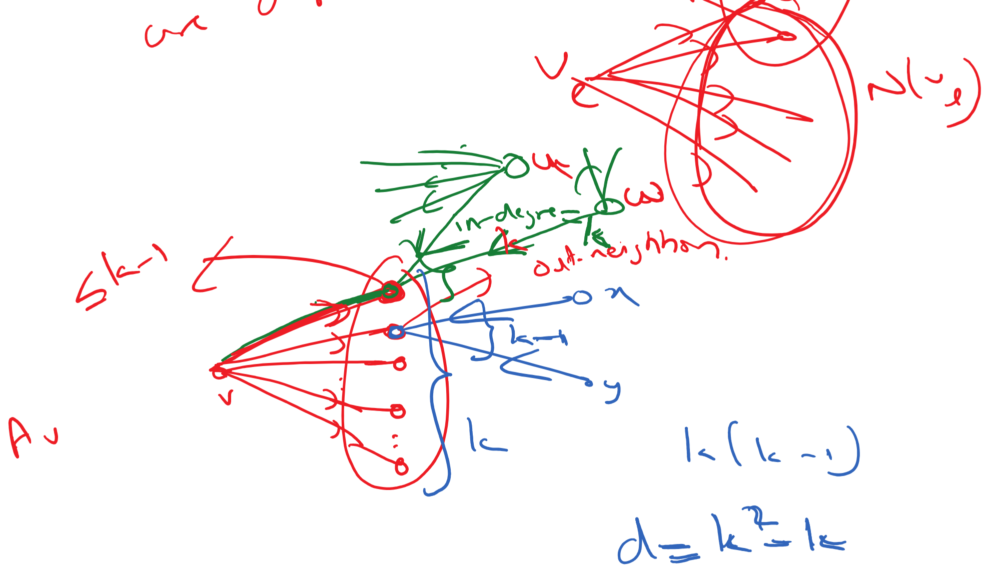

$A_{v_i}$ $A_{v_j}$




If $N(v_i) \cap N(v_j) = \emptyset$, then $A_{v_i}$ and $A_{v_j}$ are mutually independent.


If $N(v_\ell) \cap N(v_k) \ne \emptyset$, then $A_{v_\ell}$ and $A_{v_k}$ ...dant.

then $A_{v_e}$ and ... $v_k$
are dependant.




We have,

$$p = \frac{1}{3k^2 \ln k}$$

$$d = k^2 - k$$

Applying Local lemma:

Since $e\,p(d+1) = e \cdot \frac{1}{3k^2 \ln k}(k^2 - k + 1) \leq 1$ ✓

we have $\Pr\left[\bigwedge_v \bar{A}_v\right] > 0$.

That is, there is a coloring of the vertices of the digraph using $r$ colors such that every vertex sees all the $r$ colors in its out-neighborhood.

$\square$

Later McDiarmid & Molloy showed that $\Omega(k^2)$ vertex disjoint cycles exist.