EP 1027: Maxwell's Equations and Electromagnetic Waves

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Lecture 12

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Agenda

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- Resonant Cavity and Guided Waves

References/Readings

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- Griffiths, D.J., **Introduction to Electrodynamics, Ch. 9**

“Resonant” EM Waves

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- EM Excitations can be stimulated by an antenna (emitter) inside : Can achieve resonant (large amplitude) standing EM waves
- Perfect conducting walls ($\sigma \to \infty$), waves last forever; realistic (Ohmic) conductors, Ohmic heat loss causes waves to dissipate and disappear after a while

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- Maxwell equations

$$\begin{aligned}
\boldsymbol{\nabla} \cdot \mathbf{E} &= 0, \\
\boldsymbol{\nabla} \cdot \mathbf{B} &= 0, \\
\boldsymbol{\nabla} \times \mathbf{E} &= -\frac{\partial \mathbf{B}}{\partial t}, \\
\boldsymbol{\nabla} \times \mathbf{B} &= \frac{1}{c^2}\frac{\partial \mathbf{E}}{\partial t}
\end{aligned}$$

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$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t},$$

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- Wave equations!

$$\nabla^2 \mathbf{E} - \frac{1}{c^2}\frac{\partial^2 \mathbf{E}}{\partial t^2} = 0, \qquad \nabla^2 \mathbf{B} - \frac{1}{c^2}\frac{\partial^2 \mathbf{B}}{\partial t^2} = 0$$

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- Wave equation then becomes an eigenvalue problem:

$$\boldsymbol{\nabla}^2\mathbf{E}(\mathbf{x}) = -\frac{\omega^2}{c^2}\mathbf{E}(\mathbf{x}), \quad \boldsymbol{\nabla}^2\mathbf{B}(\mathbf{x}) = -\frac{\omega^2}{c^2}\mathbf{B}(\mathbf{x})$$

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- However: Wave equation is not everything, not all 6 components of **E** and **B** are independent
- To simplify life: assume cylindrical cavities and wave guides from now on

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- $\sigma, \mathbf{k}$ in conducting walls will adjust themselves to satisfy above, no new b.c.!

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- Axial/Cylindrical Ansatz:

$$\mathbf{E}(t, \mathbf{x}) = \mathbf{E}(x, y)\, e^{-i\omega t \pm ikz}, \quad \mathbf{B}(t, \mathbf{x}) = \mathbf{B}(x, y)\, e^{-i\omega t \pm ikz}$$

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- Maxwell equations (coupled linear diff. eqn.)

$$\begin{aligned}
\frac{\partial B_z}{\partial y} \mp ikB_y &= -\frac{i\omega}{c^2} E_x \\
\pm ikB_x - \frac{\partial B_z}{\partial x} &= -i\frac{\omega}{c^2} E_y \\
\frac{\partial B_y}{\partial x} - \frac{\partial B_x}{\partial y} &= -i\frac{\omega}{c^2} E_z \\
&\vdots
\end{aligned}$$

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- Solve E_x, E_y, B_x, B_y in terms of E_z, B_z.

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- For instance,

$$
\begin{aligned}
E_x &= \frac{1}{\omega^2/c^2 - k^2}\left(\pm ik\frac{\partial E_z}{\partial x} + i\omega\frac{\partial B_z}{\partial y}\right) \\
&= \frac{1}{\omega^2/c^2 - k^2}\left(\frac{\partial^2 E_z}{\partial x \partial z} + i\omega\frac{\partial B_z}{\partial y}\right)
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- Similar for E_y, B_x, B_y

$$E_y = \frac{1}{\omega^2/c^2 - k^2}\left(\frac{\partial^2 E_z}{\partial y \partial z} - i\omega\frac{\partial B_z}{\partial x}\right)$$
$$B_x = \frac{1}{\omega^2/c^2 - k^2}\left(\frac{\partial^2 B_z}{\partial x \partial z} - i\frac{\omega}{c^2}\frac{\partial E_z}{\partial y}\right)$$
$$B_y = \frac{1}{\omega^2/c^2 - k^2}\left(\frac{\partial^2 B_z}{\partial y \partial z} + i\frac{\omega}{c^2}\frac{\partial E_z}{\partial y}\right)$$

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- E_z, B_z determine everything! Themselves determined by EV equations

$$\nabla^2 E_z = -\frac{\omega^2}{c^2}E_z; \quad \nabla^2 B_z = -\frac{\omega^2}{c^2}B_z$$

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- However can have $E_z = 0$ $(B_z \neq 0)$: **TE** waves or can have $B_z = 0$ $(E_z \neq 0)$: **TM** waves
- Rectangular cavities: Cross-section rectangular (a, b)

$$E_z(x, y, z) = E_0 \sin\left(\frac{m\pi x}{a}\right) \sin\left(\frac{n\pi y}{b}\right) \sin kz$$

- w/ dispersion

$$\frac{\omega_{kmn}^2}{c^2} = \left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2 + k^2$$

- For cavity of depth d:

$$k = \frac{l\pi}{d} \quad , \omega_{lmn} = c\sqrt{\left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2 + \left(\frac{l\pi}{d}\right)^2}$$

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- Group velocity

$$v_g = \frac{d\omega}{dk} = \frac{c}{\sqrt{\left(\frac{m\pi}{ak}\right)^2 + \left(\frac{n\pi}{bk}\right)^2 + 1}} < c!$$

- Coaxial Cables: **TEM** modes exist! Cylindrican symmetry reduces the problem to electrostatics and magnetostatics in **2** space dimensions instead of **3.** The solution:

$$\mathbf{E} = \frac{A\cos(kz - \omega t)}{\rho}\hat{\boldsymbol{\rho}}$$

$$\mathbf{B} = \frac{A\cos(kz - \omega t)}{c\rho}\hat{\boldsymbol{\phi}}$$

Full analysis in Sec. 9.5.2 of Griffiths' text