Lecture 10

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Plan

- Last class, we saw Graphs and BFS
- Today, we see proof of correctness of BFS

Graphs

Graph

(directed)

A (directed) graph G is a two tuple (V, E) where:

- V is a set of elements called "vertices".
- $E \subseteq V \times V$ is a binary relation. Elements in E are called "edges".

Note: There are several definitions and variants of graphs.
Graphs are a way to study the relationships among a set of elements.

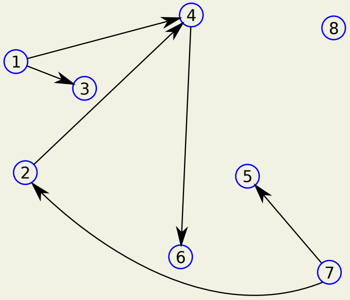
Example - directed graph

Consider:

$$
\begin{aligned}
V =& \{1, 2, 3, 4, 5, 6, 7, 8\} \\
E =& \{\ (1, 3), (1, 4), \\
& (2, 4), (4, 6), \\
& (7, 2), (7, 5)\ \}
\end{aligned}
$$

Example - directed graph

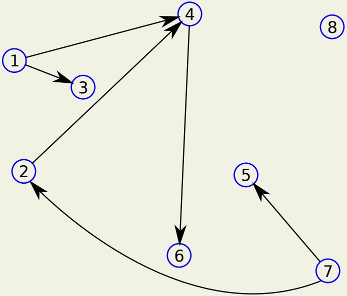
$$V = \{1, 2, 3, 4, 5, 6, 7, 8\}$$
$$E = \{\ (1, 3), (1, 4),$$
$$(2, 4), (4, 6),$$
$$(7, 2), (7, 5)\ \}$$



Example - directed graph

$$V = \{1, 2, 3, 4, 5, 6, 7, 8\}$$

$$E = \{\ (1, 3), (1, 4),$$
$$(2, 4), (4, 6),$$
$$(7, 2), (7, 5)\ \}$$



The vertices can be drawn anywhere! The edges are what matter.

Graph

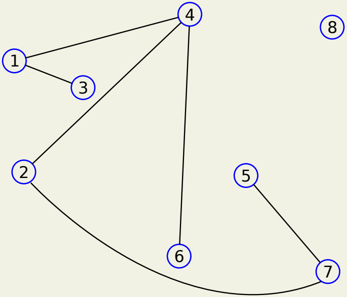
(undirected)

A (undirected) graph G is a two tuple (V, E) where:

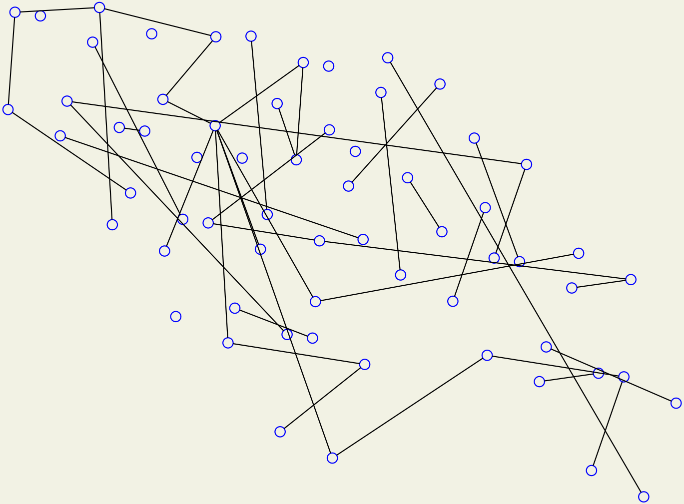
- V is a set of elements called "vertices".
- E is a set of (unordered) pairs of vertices from V.

Example - undirected graph

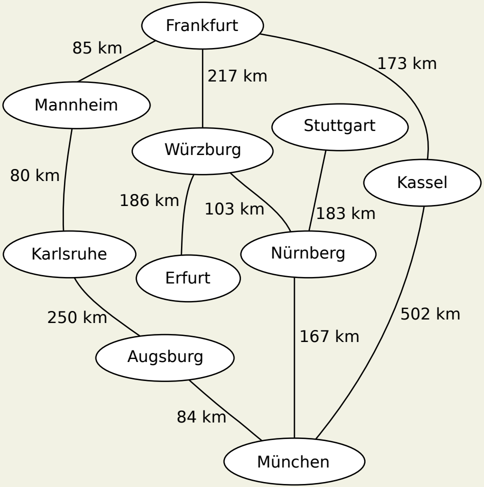
$$
\begin{aligned}
V =& \{1, 2, 3, 4, 5, 6, 7, 8\} \\
E =& \{\ \{1, 3\}, \{1, 4\}, \\
& \{2, 4\}, \{4, 6\}, \\
& \{7, 2\}, \{7, 5\}\ \}
\end{aligned}
$$



Example - undirected graph



Example - undirected graph



(source: wikipedia.org)

Weighted graphs have a *weight* assigned to each edges using a weight function.

Definitions

- Neighbour of v: A vertex u such that $(v, u) \in E$ or $\{v, u\} \in E$
- Neighbourhood of v ($\mathcal{N}(v)$): The set of all neighbours of v
- Degree of v: Cardinality of $\mathcal{N}(v)$

In a directed graph, we can talk about in-neighbours, in-degree, out-neighbours, out-degree etc.

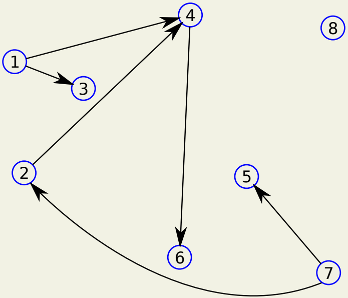
Data structure

Two standard data structures to represent graphs:

- Adjacency matrix
- Adjacency list

Adjacency Matrix

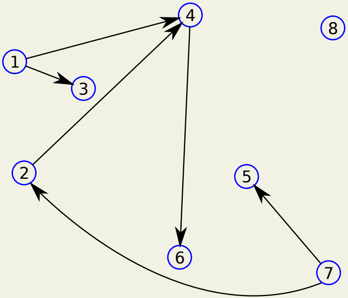
$$
\begin{array}{c|c}
A & \begin{array}{cccccccc} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \end{array} \\
\hline
\begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \\ 7 \\ 8 \end{array} &
\begin{bmatrix}
0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{bmatrix}
\end{array}
$$



$$A[u, v] = 1 \iff (u, v) \in E$$

Adjacency Matrix

$$
\begin{array}{c|cccccccc}
A & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\
\hline
1 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 \\
2 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
4 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
5 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
6 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
7 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 \\
8 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{array}
$$

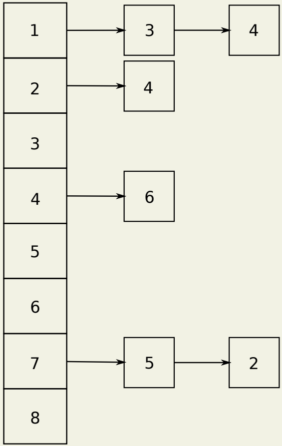


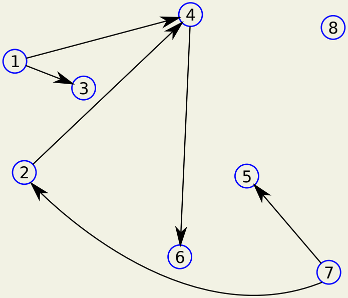
$$A[u, v] = 1 \iff (u, v) \in E$$

For an undirected graph:

- $u, v \in E \iff A[u, v] = A[v, u] = 1$
- The adjacency matrix for an undirected graph is a symmetric matrix

Adjacency Lists





Which representation to choose?

- Is $(u, v) \in E$?
- List all neighbours of v
- List all edges of G
- Add an edge (u, v) or $\{u, v\}$
- Delete an edge (u, v) or $\{u, v\}$

Breadth-first Search

The idea is to explore the graph “radially outward” from the source.

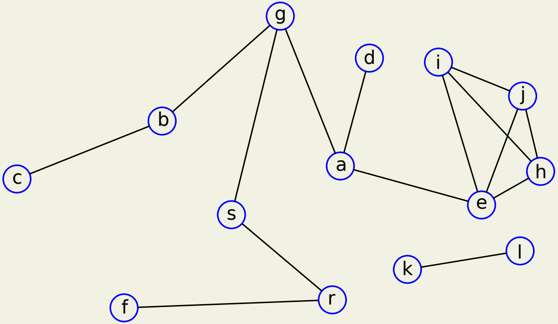
In each step, we expand our exploration by visiting the neighborhood of all explored vertices.

Algorithm 1 Breadth-first Search from vertex s

```
Color all vertices WHITE.
For all u ∈ V, d[u] ← ∞, π[u] ← NIL.
d[s] ← 0, color[s] ← GRAY.
Initialize queue Q ← ∅.
ENQUEUE(Q, s)
while Q ≠ ∅ do
    u ← DEQUEUE(Q)
    for each v ∈ 𝒩(u) do
        if color(v) = WHITE then
            color[v] ← GRAY
            d[v] ← d[u] + 1
            π[v] ← u
            ENQUEUE(Q, v)
        end if
    end for
    color[u] ← BLACK.
end while
```

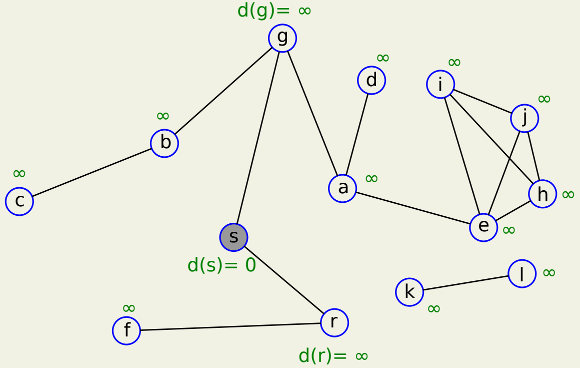
Breadth-first Search

Queue: $\emptyset$



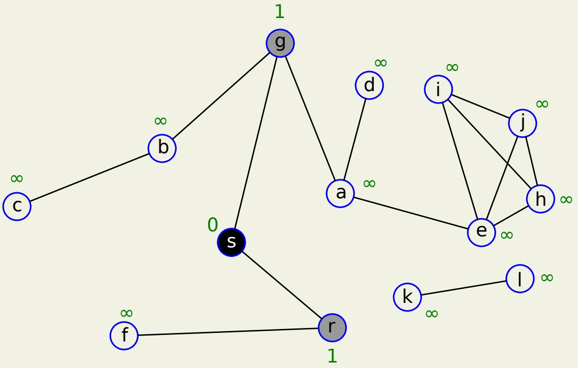
Breadth-first Search

Dequeued vertex: $\boxed{}$ Queue: $\boxed{s}$



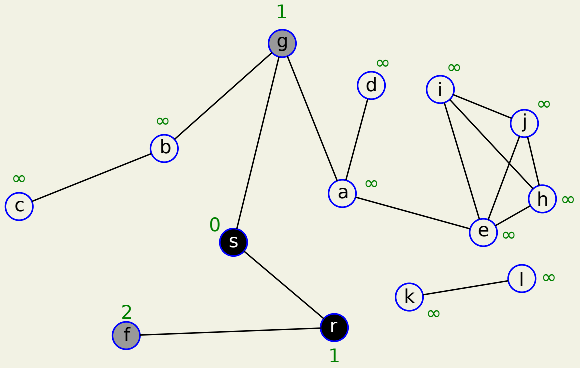
Breadth-first Search

Dequeued vertex: s Queue: r | g



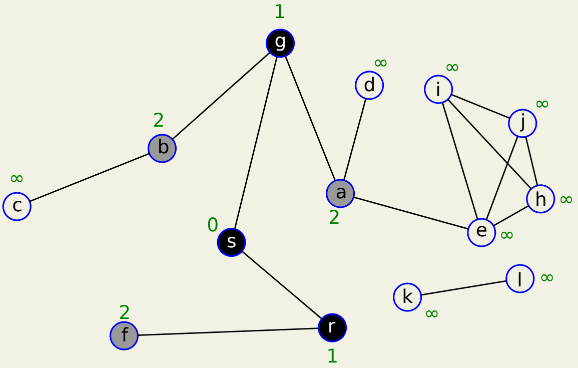
Breadth-first Search

Dequeued vertex: $\boxed{r}$ Queue: $\boxed{g \mid f}$



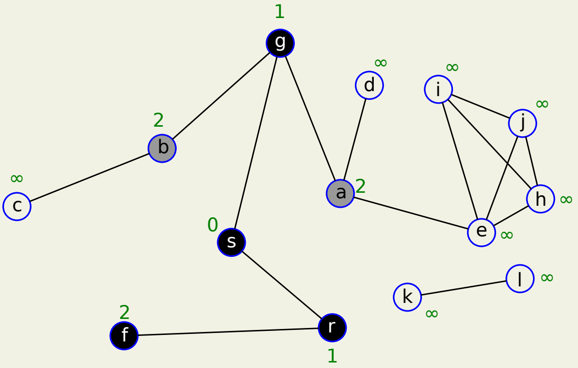
Breadth-first Search

Dequeued vertex: $\boxed{g}$ Queue: $\boxed{f \mid a \mid b}$



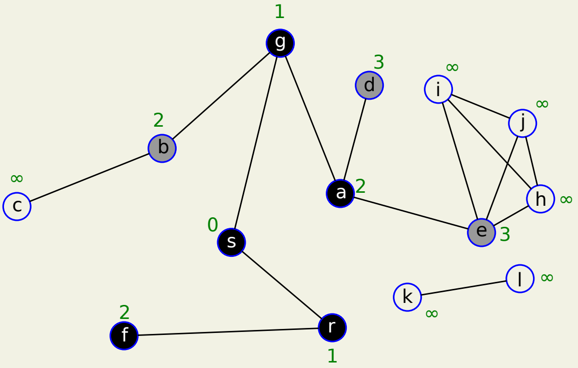
Breadth-first Search

Dequeued vertex: $\boxed{f}$ Queue: $\boxed{a \mid b}$



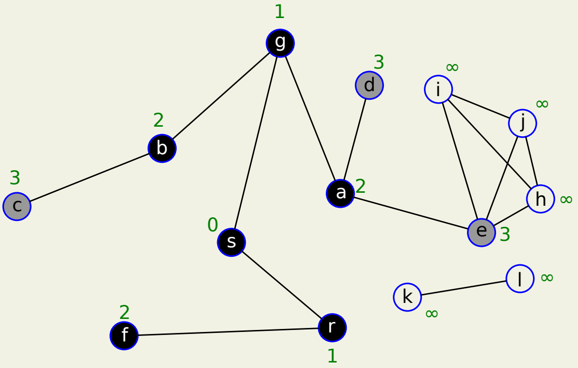
Breadth-first Search

Dequeued vertex: a Queue: b e d



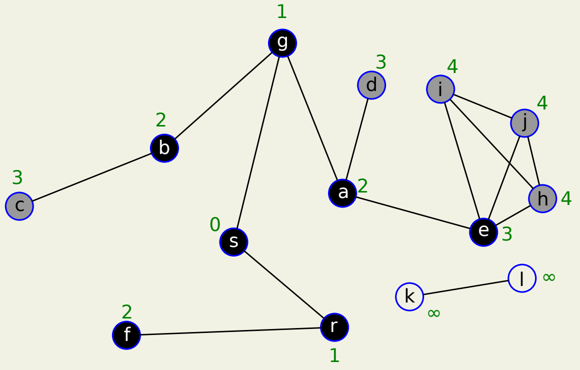
Breadth-first Search

Dequeued vertex: | b | Queue: | e | d | c |



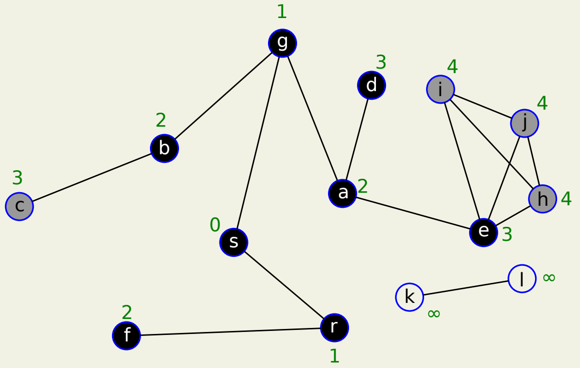
Breadth-first Search

Dequeued vertex: | *e* | Queue: | *d* | *c* | *j* | *h* | *i* |



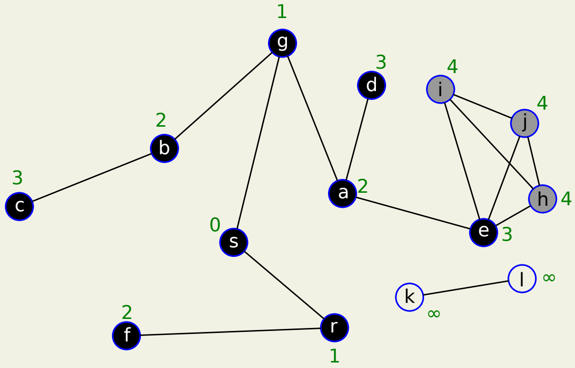
Breadth-first Search

Dequeued vertex: d Queue: c | j | h | i



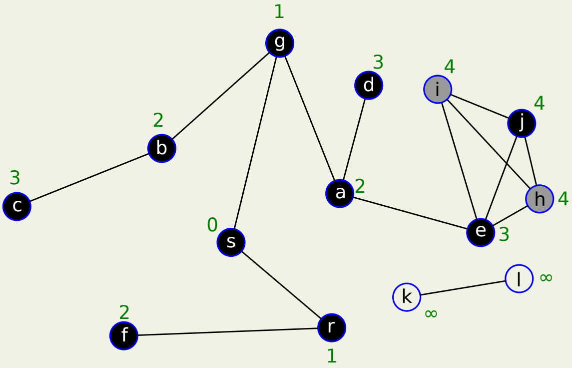
Breadth-first Search

Dequeued vertex: $\boxed{c}$ Queue: $\boxed{j \mid h \mid i}$



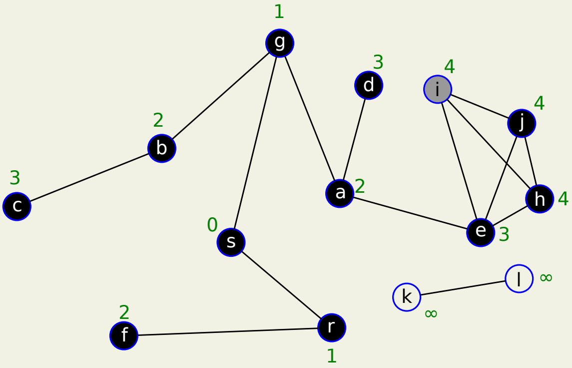
Breadth-first Search

Dequeued vertex: j Queue: h | i



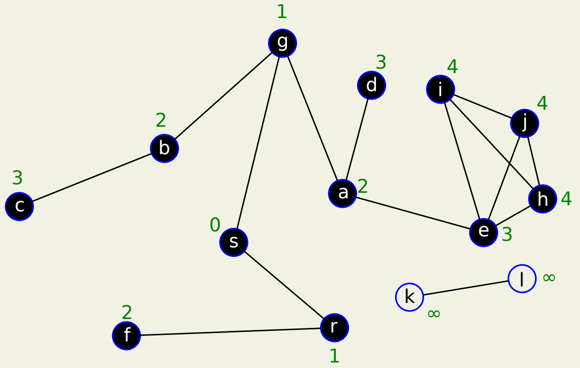
Breadth-first Search

Dequeued vertex: $\boxed{h}$ Queue: $\boxed{i}$



Breadth-first Search

Dequeued vertex: $\boxed{i}$ Queue: $\emptyset$



Algorithm 2 Breadth-first Search from vertex s

Color all vertices WHITE.
For all $u \in V$, $d[u] \leftarrow \infty$, $\pi[u] \leftarrow$ NIL.
$d[s] \leftarrow 0$, color$[s] \leftarrow$ GRAY.
Initialize queue $Q \leftarrow \emptyset$.
ENQUEUE(Q, s)
while $Q \neq \emptyset$ **do**
    $u \leftarrow$ DEQUEUE(Q)
    for each $v \in \mathcal{N}(u)$ **do**
        if color(v) $=$WHITE **then**
            color$[v] \leftarrow$ GRAY
            $d[v] \leftarrow d[u] + 1$
            $\pi[v] \leftarrow u$
            ENQUEUE(Q, v)
        end if
    end for
    color$[u] \leftarrow$ BLACK.
end while

Time Complexity of BFS

- Each enqueue/dequeue takes $O(1)$ time.
- Total queue operations take $O(|V|)$ time.
- Each list in the adj. list is scanned once. This requires total $\Theta(|E|)$. This is assuming the graph is provided using adjacency list.
- Initialization required $\Theta(|V|)$.
- Total running time is $O(|V| + |E|)$.

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- Initialization required $\Theta(|V|)$.
- Total running time is $O(|V| + |E|)$.

- **Note:** The colors can be omitted. Instead, check if $d[v] = \infty$

Correctness of BFS

Notation: Let $\delta(s, v)$ denote the minimum number of edges on a path from s to v.

Theorem

Let $G = (V, E)$ be a graph. When BFS is run on G from vertex $s \in V$:

1. Every vertex that is reachable from s gets discovered.
2. On termination, $d[v] = \delta(s, v)$ for all v.

We will first show (2).

Proof of correctness

Proof

Suppose, for the sake of contradiction, (2) does not hold.
Let v be the vertex with smallest $\delta(s, v)$ such that $d[v] \neq \delta(s, v)$.
Claim 1: $d[v] \geq \delta(s, v)$
Choose a *shortest* path from s to v.
Let u be the vertex immmediately preceding v.
Then $\delta(s, v) = \delta(s, u) + 1 = d[u] + 1$.
So we have:

$$d[v] > \delta(s, v) = \delta(s, u) + 1 = d[u] + 1$$

Proof of correctness

Proof cont...

We have:

$$d[v] > \delta(s, v) = \delta(s, u) + 1 = d[u] + 1$$

Consider the time step when u is dequeued.

- Case 1: v was white.
  The algo sets $d[v] = d[u] + 1$.
  This contradicts the eq above.
- Case 2: v is black.
  Then, v was dequeued before u.
  Claim 2: If v was dequeued before u, then $d[v] \leq d[u]$.

Proof of correctness

Proof cont...

- Case 3: v was gray.
  Vertex v was colored gray after dequeuing some vertex w earlier.
  So $d[v] = d[w] + 1$.
  By Claim 2, $d[w] \leq d[u]$ since w was dequeued before u.
  This gives: $d[v] = d[w] + 1 \leq d[u] + 1$.

Exercise

Show (1) using (2). That is, given that $d[v] = \delta(s, v)$, show that every vertex reachable from s gets discovered.

Proof of correctness

Claim 3

Let $(u, v) \in E$. Then we have:

$$\delta(s, v) \leq \delta(s, u) + 1$$

Proof

If u is reachable from s, then:
Take the shortest path from s to u. Then take the edge (u, v).
This gives a path from s to v.
The shortest path from s to v can only be shorter than the above path.

$\square$

Proof of correctness

Claim 1

$\forall v \in V, d[v] \geq \delta(s, v)$

Proof

Induction on the number of enqueue operations.
Hypothesis: same as claim.
Base case: The time when the first vertex enqueued.
The first vertex enqueued is s. At this time we have:

- $\forall v \in V \setminus \{s\}, d[v] = \infty$
- $d[s] = \delta(s, s) = 0$.

Hence the claim holds for the base case.

Proof of correctness

Proof

Hypothesis: $\forall v \in V, d[v] \geq \delta(s, v)$
Step: A white (undiscovered) vertex v gets discovered while we are visiting a vertex u with $(u, v) \in E$.
From induction, we have: $d[u] \geq \delta(s, u)$.
The algorithm assigns $d[v] \leftarrow d[u] + 1$. So:

$$\begin{aligned} d[v] &= d[u] + 1 \\ &\geq \delta(s, u) + 1 \\ &\geq \delta(s, v) \end{aligned}$$

Last inequality follows from Claim 3.

□

Proof of correctness

Claim 2

If v was dequeued before u, then $d[v] \leq d[u]$.

We will show a stronger claim:

Claim 4

If at some point, the queue contained $v_1, v_2, \ldots, v_r$ where v_1 was the head. Then:

(a) $d[v_1] \leq d[v_2] \leq \cdots \leq d[v_r]$

(b) $d[v_r] \leq d[v_1] + 1$

Proof of Claim 2:

Write down vertices in the order they went through the queue.
By claim 4 (a), the calculated d values for them are non-decreasing.
Vertex v will appear before u in this order.
Hence claim 2 follows. $\square$

Proof of correctness

Claim 4

If queue contains $v_1, v_2, \ldots, v_r$ where v_1 is the head. Then:

(a) $d[v_1] \leq d[v_2] \leq \cdots \leq d[v_r]$

(b) $d[v_r] \leq d[v_1] + 1$

Proof

Induction on number of queue operations.
Hypothesis: Same as claim. We show that the claim holds after every enqueue and dequeue.
Base case: The first queue operation - enqueuing s.
The claim trivially holds.

Proof of correctness

Claim 4

If queue contains $v_1, v_2, \ldots, v_r$ where v_1 is the head. Then:

(a) $d[v_1] \leq d[v_2] \leq \cdots \leq d[v_r]$

(b) $d[v_r] \leq d[v_1] + 1$

Proof

Step:

- **Dequeue:** After v_1 is dequeued, v_2 is the new head.
  Part (a): From induction,
  $d[v_1] \leq d[v_2] \leq d[v_3] \leq \cdots \leq d[v_r]$.
  Hence (a) holds.
  Part (b): From induction, $d[v_r] \leq d[v_1] + 1$. And so:

$$\begin{aligned} d[v_r] &\leq d[v_1] + 1 \\ &\leq d[v_2] + 1 \end{aligned}$$

Proof of correctness

Proof

- **Enqueue:** When a vertex v is enqueued:
  It was enqueued because:
  - it was undiscovered so far.
  - it was present in the adjacency list of a vertex u that was just dequeued.

  Since u was the previous head of the list, from induction we have:
  - $d[u] \leq d[v_1] \leq d[v_2] \leq \cdots \leq d[v_r]$.
  - $d[v_r] \leq d[u] + 1$.

  We assign $d[v] \leftarrow d[u] + 1$ and then enqueue v. Hence, we have:
  - $d[v_r] \leq d[u] + 1 = d[v]$
  - $d[v_1] \leq d[v_2] \leq \cdots \leq d[v_r] \leq d[v]$.

$\square$

Loop Invariant

Claim 4

If queue contains $v_1, v_2, \ldots, v_r$ where v_1 is the head. Then:

(a) $d[v_1] \leq d[v_2] \leq \cdots \leq d[v_r]$

(b) $d[v_r] \leq d[v_1] + 1$

Claim 4 is actually a loop invariant!

Another loop invariant

The queue Q consists of the set of GRAY vertices.