Fisher Inequality

Theorem [Babai, Frankl, 1992] Let k, n be two positive integers with $k \leq n$. Let $\mathcal{F} = \{A_1, A_2, \ldots, A_m\}$ be a family of subsets of $[n]$ such that $\forall i, j \in [m]$, $i \neq j$, $|A_i \cap A_j| = k$. Then, $|\mathcal{F}| = m \leq n$.

Proof:

(0-1) incidence vectors of the sets

$$A_1 \longrightarrow v_1$$
$$A_2 \longrightarrow v_2$$
$$\vdots \qquad \vdots$$
$$A_m \longrightarrow v_m$$

Observe: dot product

$$\langle v_i, v_j \rangle = |A_i \cap A_j|$$

i.e. when $i = j$,

$$\langle v_i, v_i \rangle = |A_i|$$

Suppose $A_1 = \{1, 3, 7, n\}$

$$\begin{array}{l} \quad\ \ 1\ \ 2\ \ 3\ \ 4\ \ 5\ \ 6\ \ 7\ \ \ldots\ \ n-1\ \ n \\ v_1 = (1, 0, 1, 0, 0, 0, 1, 0, 0, 0, 0, 0, 0, \ldots, 1) \end{array}$$

To show $|\mathcal{F}| = m \leq n$.

In order to show that $m \leq n$, it is enough to show that the it is enough to show that the vectors $\{v_1, v_2, \dots, v_m\}$ are linearly independent in the vector space $\mathbb{R}^n$ over $\mathbb{R}$.

$(\_, \_, \dots, \_)$
0-1 n-bit vector

$\mathbb{F}_2^n$ over $\mathbb{F}_2$

$\mathbb{F}_2$ is the field $(\{0,1\}, +, \cdot)$
modulo 2

Since $\dim(\mathbb{R}^n) = n$, if we show that $\{v_1, v_2, \dots, v_m\}$ are L.I. in $\mathbb{R}^n$, then it would imply that

$|F| = m \leq n$.

To show: $\{v_1, v_2, \dots, v_m\}$ are linearly independent in the vector space $\mathbb{R}^n$ over $\mathbb{R}$.

Suppose they are not L.I. in $\mathbb{R}^n$ over $\mathbb{R}$. Then there in $\mathbb{R}^n$ over $\mathbb{R}$. Then there exist $\lambda_1, \lambda_2, \ldots, \lambda_m$, not all of them being zero, such that

$$\underbrace{\lambda_1 v_1 + \lambda_2 v_2 + \cdots + \lambda_m v_m}_{v} = 0$$

standard dot product.

$$0 = \langle 0, 0 \rangle = \langle v, v \rangle$$

$$= \left\langle \sum_{i=1}^{m} \lambda_i v_i , \sum_{i=1}^{m} \lambda_i v_i \right\rangle$$

$$= \sum_{i=1}^{m} \lambda_i^2 \langle v_i, v_i \rangle + \sum_{1 \le i \ne j \le m} \lambda_i \lambda_j \langle v_i, v_j \rangle$$

$$= \sum_{i=1}^{m} \lambda_i^2 |A_i| + \sum_{1 \le i \ne j \le m} \lambda_i \lambda_j k .$$

since $|A_i \cap A_j| = k$

at least two λ_i's that are greater than zero

$$= \sum_{i=1}^{m} \lambda_i^2 \left(|A_i| - k \right) + k \left(\sum_{i=1}^{m} \lambda_i \right)^2 \geq 0$$

At most one set A_i in $\mathcal{F}$ can be of size equal to k. Every other set has to be of size $> k$.

That means L.H.S $= 0$ and R.H.S > 0. This is a contradiction. Hence, our assumption that $\{v_1, v_2, \ldots, v_m\}$ is a linearly dependent set of vectors in $\mathbb{R}^n$ over $\mathbb{R}$ is FALSE.

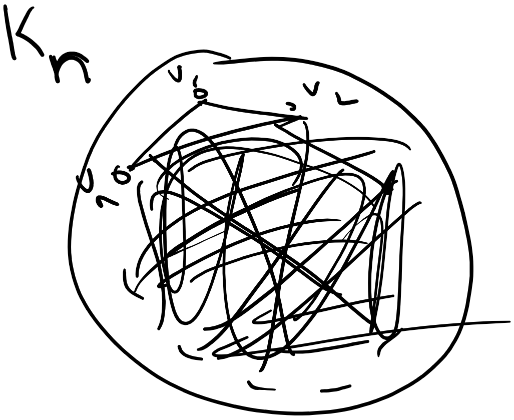
This implies, $m \leq n$.

$\square$

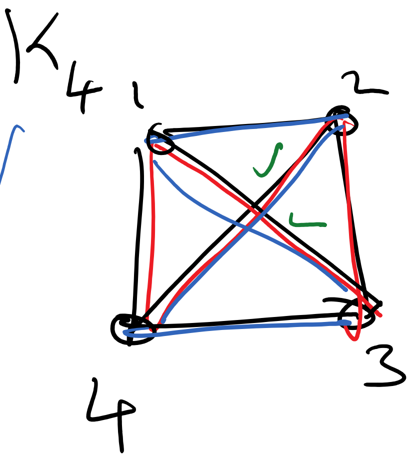
01 November 2020 17:28

Graph Decomposition

K_n — complete graph on n vertices

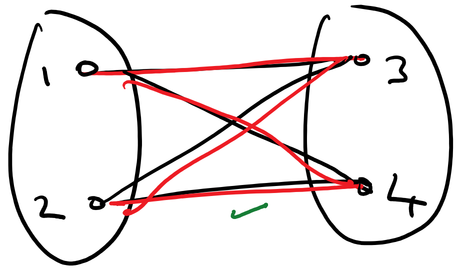


Q. Cover the edges of K_n using only complete bipartite graphs. How many complete bipartite graphs do you need?

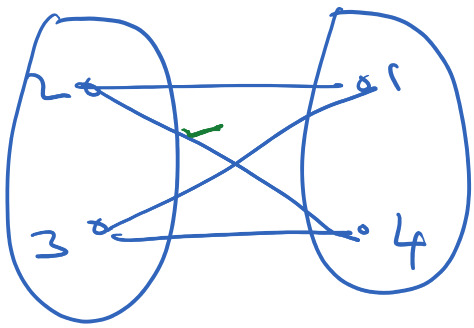


Ans: 2.

(i)



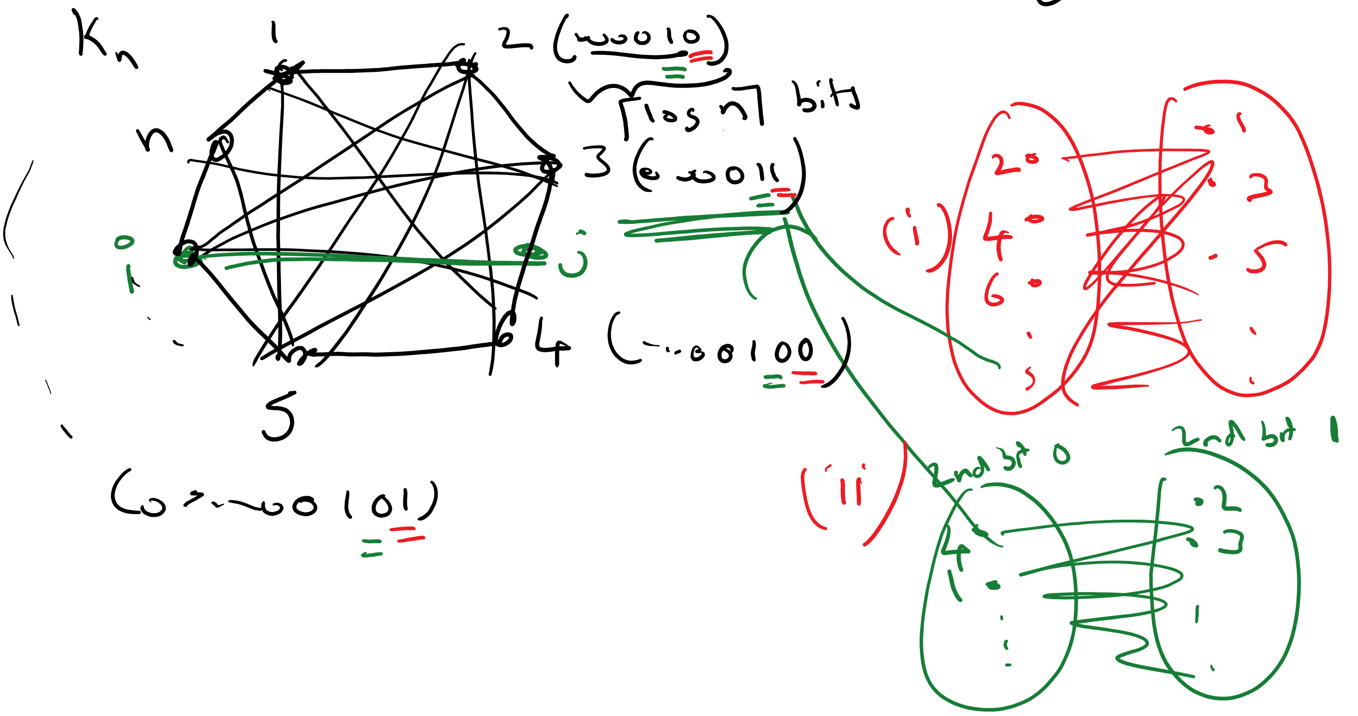
(ii)



$K_n \longrightarrow$ claim:

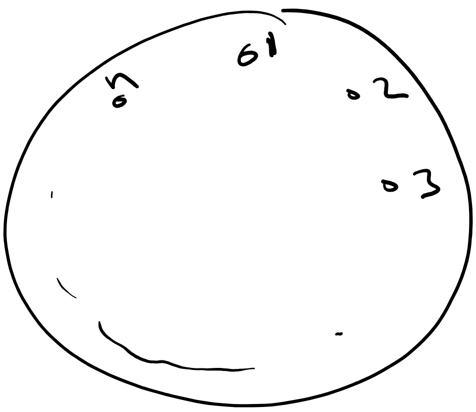
min no. of complete

min no. of complete bipartite graphs needed to cover edges of K_n $\Big\} = \lceil \log_2 n \rceil$



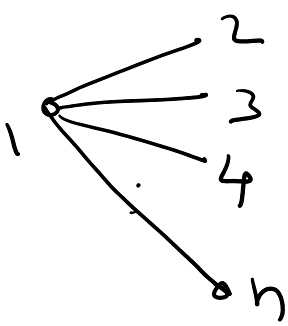
Another question

Partition the edges of a K_n into complete bipartite graphs

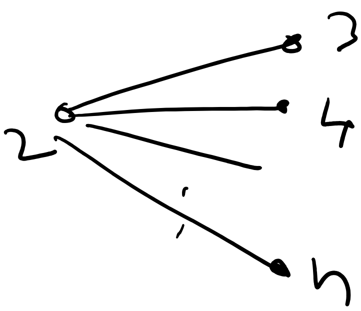


Using $(n-1)$ complete bipartite graphs

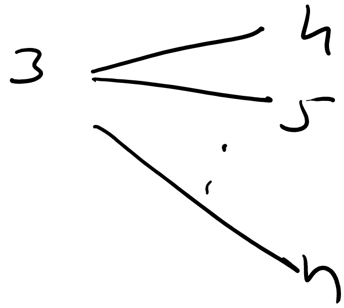
(i)



(ii)



(iii)



$\vdots$

(n-1) $n-1$ ——— n.