Introduction to Sequences and Series of Functions

Pradipto Banerjee

pradipto@iith.ac.in

IIT, Hyderabad/Bhilai

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Pointwise and Uniform Convergence

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Pointwise and Uniform Convergence

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- What does it mean to say that

$$|f(x) - g(x)| < \varepsilon \quad \text{for all} \quad x \in [a,b]$$

- "The graph of g on $[a,b]$ lies completely within the ε-tube around the graph of f on $[a,b]$"

Pointwise and Uniform Convergence

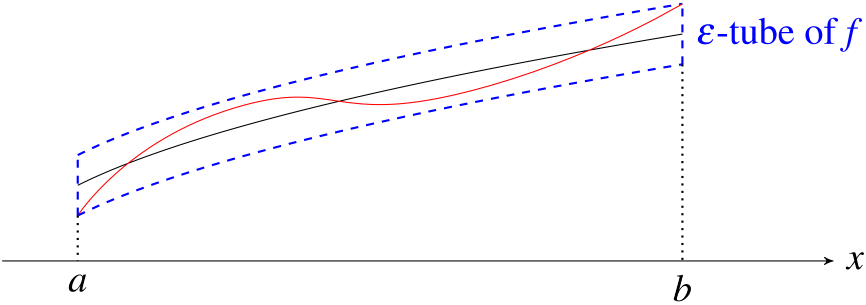

Figure: Example of $|f(x) - g(x)| < \varepsilon$ for all $x \in [a, b]$

Pointwise and Uniform Convergence

A Basic Example

- Consider the sequence of functions

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- Observe that for a fixed $x_0 \in [0,1]$

$$\lim_{n\to\infty} f_n(x_0) = \begin{cases} 0 & \text{if} \quad x_0 \neq 1 \\ 1 & \text{if} \quad x_0 = 1 \end{cases}$$

Pointwise and Uniform Convergence

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- Thus, one can define a function f on $[0,1]$ as

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- The function f, thus defined, is called the **limit function** for the sequence $\{f_n\}_n$

Pointwise and Uniform Convergence

A Basic Example

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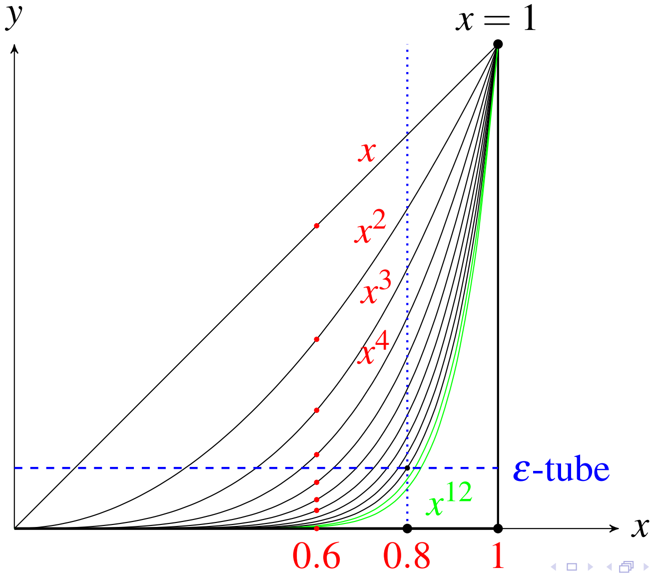
- The sequence $\{f_n\}_n \to f$ at each $x_0 \in [0, 1]$
- Thus $f_n(x_0) \to f(x_0)$ as a numerical sequence for every $x_0 \in [0, 1]$
- In such a case, we say that the sequence of functions $f_n \to f$ **pointwise** on $[0, 1]$

Pointwise and Uniform Convergence



Pointwise and Uniform Convergence

A Basic Example

- In terms of graphs, at every x_0 in $[0, 1]$, after a certain stage (depending on ε), the values of $f_n(x_0)$ fall within the ε-tube about $f(x)$

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A Basic Example

- In terms of graphs, at every x_0 in $[0,1]$, after a certain stage (depending on ε), the values of $f_n(x_0)$ fall within the ε-tube about $f(x)$
- Rephrasing it - given a $\varepsilon > 0$ and x_0 in $[0,1]$, there is a N_0, depending on both, ε and x_0, such that

$$|f_n(x_0) - f(x_0)| < \varepsilon \quad \text{for all} \quad n \geq N_0$$

Pointwise and Uniform Convergence

Pointwise Convergence

Suppose that a sequence $\{f_n\}_n$ of functions and a function f are defined on $[a,b]$. Then $\{f_n\}$ is said to converge **pointwise** to f on $[a,b]$ if for every $x_0 \in [a,b]$, and any $\varepsilon > 0$, there is a positive integer $N_0 = N_0(x_0, \varepsilon)$ such that

$$|f_n(x_0) - f(x_0)| < \varepsilon \quad \text{for all} \quad n \geq N_0.$$

Pointwise and Uniform Convergence

Roughly speaking, a sequence of functions $f_n \to f$ on $[a,b]$ *uniformly* if, after a certain stage (depending only on ε), the whole graph of $f_n(x)$ falls inside the ε-tube about f on $[a,b]$

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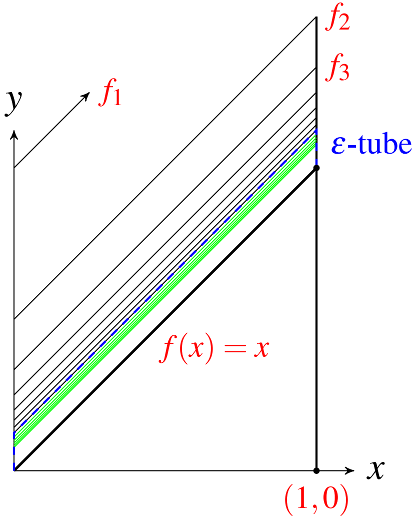
Uniform Convergence

Suppose that $\{f_n\}_n$ and f are defined on $[a,b]$. Then $\{f_n\}$ is said to converge **uniformly** to f on $[a,b]$ if, for a given $\varepsilon > 0$, one can find a $N_0 = N_0(\varepsilon)$ such that

$$|f_n(x_0) - f(x_0)| < \varepsilon$$

for any $n \geq N_0$, and for every $x_0 \in [a,b]$

Uniform Convergence $\frac{1}{n} + x \to x$



Uniform Convergence - An Example

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- The sequence in question is $f_n(x) = \frac{1}{n} + x$ on $[0, 1]$

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$$|f_n(x_0) - f(x_0)| = \frac{1}{n} < \varepsilon \quad \text{for all} \quad n \geq N_0$$

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$$|f_n(x_0) - f(x_0)| = \frac{1}{n} < \varepsilon \quad \text{for all} \quad n \geq N_0$$

- Note that N_0 works for any x_0 in $[0,1]$, and depends only on ε

Notations and Further Remarks

- We shall use the following notations

$$f_n(x) \xrightarrow{\text{pointwise}} f(x); \quad f_n(x) \searrow f(x)$$

$$f_n(x) \xrightarrow{\text{uniformly}} f(x); \quad f_n(x) \searrow\searrow f(x)$$

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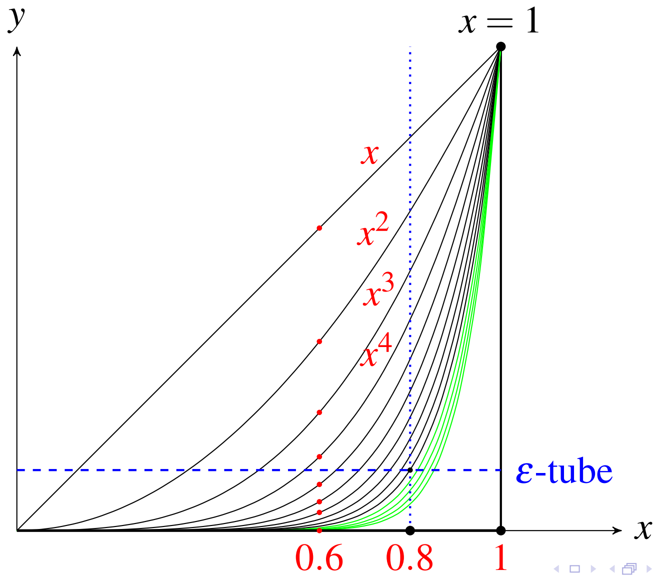
- Clearly

$$f_n(x) \searrow\searrow f(x) \quad \Rightarrow \quad f_n(x) \searrow f(x)$$

- In the example $f_n(x) = x^n$, the convergence is not uniform (although pointwise) on $[0, 1]$, as a part of f_n always lies outside the ε-tube around $f(x)$

Further Remarks



Further Remarks

- In the example $f_n(x) = x^n$, we find that on $[0, 0.8]$, the whole graph of f_n falls within the ε-tube of the limit function (which is the *zero* function) for all $n \geq 11$ (the green curves)

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- Thus

$$x^n \searrow\searrow 0 \quad \text{on} \quad [0, 0.8]$$

- More generally, for any $0 < \beta < 1$,

$$x^n \searrow\searrow 0 \quad \text{on} \quad [0, \beta]$$

Further Remarks

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- Is continuity preserved under uniform convergence?

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- In the example $f_n(x) = x^n$, each f_n is <u>continuous</u> on $[0, 1]$, while the limit function f is clearly <u>not continuous</u> on $[0, 1]$
- Therefore "Continuity is generally <u>not</u> preserved under pointwise convergence"
- Is continuity preserved under uniform convergence?
- The example, $x^n \searrow\searrow 0$ on $[0, 0.8]$, suggests that perhaps the answer is in the affirmative

More Examples

- Consider

$$f_n(x) = \frac{nx}{1+n^2x^2}; \quad x \in [0,1]$$

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$$f_n'(x) = \frac{n(1-n^2x^2)}{(1+n^2x^2)^2}$$
- Thus $f_n(x)$ is increasing $\uparrow$ for $x < 1/n$, flat at $x = 1/n$, and decreasing $\downarrow$ for $x > 1/n$
- Therefore, $x = 1/n$ is a <u>local maxima</u> (also global, check!) for $f_n(x)$ and
$$f_n(1/n) = 1/2$$

Example Contd. ...

- On the other hand, for each x_0 in $[0, 1]$, we have

$$\lim_{n\to\infty} f_n(x_0) = \lim_{n\to\infty} \frac{nx_0}{1+n^2x_0^2} = 0$$

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- But for a ε-tube about $f(x)$ where $\varepsilon < 1/2$, the whole graph of $f_n(x)$ <u>fails</u> to lie inside the tube, precisely the point $(1/n, f_n(1/n)) = (1/n, 1/2)$ lies outside the tube

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- Deduce that the convergence is *not* uniform

“If” condition for Non-Uniform Convergence

Generalizing this phenomena (Hard Exercise!)

"If" condition for Non-Uniform Convergence

Generalizing this phenomena (Hard Exercise!)

Proposition

If each $\{f_n\}$ is continuous and $f_n(x) \searrow f(x)$ on $[a,b]$, and $\{x_n\} \subset [a,b]$ with $x_n \searrow \alpha \in [a,b]$. Then

$$\lim_{n\to\infty} f_n(x_n) \neq f(\alpha) \quad \Longrightarrow \quad f_n(x) \not\searrow\not\searrow f(x).$$

That is, the convergence is not uniform if

$$\lim_{n\to\infty} f_n(x_n) \neq f\left(\lim_{n\to\infty} x_n\right)$$

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The last proposition follows from the following.

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Suppose that each f_n is continuous and $\{f_n(x)\} \searrow\searrow f(x)$ on $[a,b]$, and that $\{x_n\}$ is a sequence in $[a,b]$ with $x_n \searrow \alpha \in [a,b]$. Then

$$\lim_{n\to\infty} f_n(x_n) = f(\alpha)$$

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- The same argument will work for any $g_n(x) \searrow g(x)$ on $[a,b]$, and satisfying

$$|g_n(x) - g(x)| \leq \frac{1}{n} \quad x \in [a,b]$$

Weierstrass test for Sequences

More generally (Not so hard Exercise.)

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Proposition

Suppose, $\{f_n(x)\} \searrow f(x)$ *on* $[a,b]$*. Let* $\{\alpha_n\}$ *is a sequence of positive real numbers satisfying* $\alpha_n \searrow 0$ *such that*

$$|f_n(x) - f(x)| \leq \alpha_n \quad \text{for all } n \geq N_0 \text{ and } x \in [a,b].$$

Then

$$f_n(x) \searrow\searrow f(x) \quad \text{on} \quad [a,b]$$

More Examples

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Problem.

Suppose, that $\alpha_n \geq 0$ satisfies $\alpha_n \searrow 0$, and f be defined on $[a, b]$. Define $f_n(x)$ on $[a, b]$ as $f_n(x) = \alpha_n \sin x + f(x)$. Then show that $f_n(x) \searrow\searrow f(x)$ on $[a, b]$.

More Examples

Problem.

Show that the sequence of functions given by

$$f_n(x) = \frac{\sin nx}{\sqrt{n}}, \quad x \in [a, b],$$

converges uniformly to $f(x) \equiv 0$ on $[a, b]$.

Problem.

Do the same for the sequence

$$f_n(x) = \frac{(\cos x)^n}{\log n}, \quad x \in [a, b],$$

More Examples - Weierstrass Test

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- Is this convergence uniform?

Example Contd. ..

- Observe that from AM $\geq$ GM, we have

$$1 + n^4x^2 \geq 2n^2x$$

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- Since, $f_n(0) = 0 < 1/n$ for any n, deduce that for all x in $[0,1]$,

$$|f_n(x)| \leq \frac{1}{n} \quad \text{for all} \quad n$$

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- By Weierstrass test, $f_n \searrow\searrow 0$ on $[0, 1]$

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$$|f_n(x) - f(x)| = x^n \leq a^n \searrow 0$$

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- Now, consider the sequence $f_n(x) = x^n$ on the interval $[0, a]$, where $a < 1$
- For any $x \in [0, a]$, one has

$$|f_n(x) - f(x)| = x^n \leq a^n \searrow 0$$

- Deduce by Weierstrass test that $f_n \searrow\searrow 0$ on $[0, a]$ for *every* $a < 1$

More Examples

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$$f_n'(x) = n(1-x^2)^{n-1}[1-(2n+1)x^2]$$

- Thus, $f_n(x)$ is increasing for $x < 1/\sqrt{2n+1}$, flat at $x = 1/\sqrt{2n+1}$ and increasing for $x > 1/\sqrt{2n+1}$

Example Contd. ...

- Focus on the numerical sequence

$$f_n\left(\frac{1}{\sqrt{2n+1}}\right) = \frac{n}{\sqrt{2n+1}}\left(1 - \frac{1}{2n+1}\right)^n$$

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- As $n \to \infty$, we have

$$\frac{n}{\sqrt{2n+1}} \nearrow \infty \quad \text{and} \quad \left(1 - \frac{1}{2n+1}\right)^n \searrow \frac{1}{\sqrt{e}}$$

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$$\frac{n}{\sqrt{2n+1}} \nearrow \infty \quad \text{and} \quad \left(1-\frac{1}{2n+1}\right)^n \searrow \frac{1}{\sqrt{e}}$$

- Thus

$$\lim_{n\to\infty} f_n\left(\frac{1}{\sqrt{2n+1}}\right) = \infty$$

Example Contd. ...

- But as $n \to \infty$,

$$\frac{1}{\sqrt{2n+1}} \searrow 0$$

- Therefore, we find that

$$\lim_{n\to\infty} f_n\left(\frac{1}{\sqrt{2n+1}}\right) \neq f(0) = 0$$

- Conclude using an earlier proposition that

$$f_n(x) \not\searrow\!\!\not\searrow f(x) \equiv 0$$

Questions to address

Suppose that $f_n \to f$ (pointwise or uniform) on $[a,b]$. We would like to know whether and when

- Each f_n is *continuous* on $[a,b] \implies f$ is *continuous* on $[a,b]$?

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- Each f_n is *continuous* on $[a, b] \implies f$ is *continuous* on $[a, b]$?
- Each f_n is *differentiable* on $[a, b] \implies f$ is *differentiable* on $[a, b]$, and $f_n' \to f'$?
- Suppose, each f_n and f are *integrable* on $[a, b]$, then is it true that for α, β in $[a, b]$

$$\lim_{n\to\infty}\left(\int_\alpha^\beta f_n(x)dx\right) \to \int_\alpha^\beta f(x)dx?$$

Pointwise Convergence and Continuity

- We saw that the sequence of *continuous* functions $\{x^n\}$ on $[0,1]$ converges to a *discontinuous* function $f(x)$ given by

$$f(x) = \begin{cases} 0 & \text{if} \quad x \neq 1 \\ 1 & \text{if} \quad x = 1 \end{cases}$$

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- So the answer is *negative* in general for pointwise convergence

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- Recall the example $f_n(x) = nx(1-x^2)^n$ on $[0,1]$

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- Clearly, $\displaystyle\int_0^1 f_n(x)dx \not\searrow \int_0^1 f(x)dx = 0$

Uniform Convergence and Differentiation

- Also, recall that, on $[-\pi, \pi]$

$$f_n(x) = \frac{\sin nx}{\sqrt{n}} \searrow\searrow 0 = f(x)$$

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- Thus, as $n \to \infty$, at $x = 0$

$$f_n'(0) = \sqrt{n} \nearrow \infty \neq 0 = f'(0)$$

Remarks

- Thus, even under the *uniform convergence* $f_n(x) \searrow\searrow f(x)$, we find that

$$\frac{d}{dx}\left(\lim_{n\to\infty} f_n(x)\right) \neq \lim_{n\to\infty} \frac{d}{dx}\left(f_n(x)\right)$$

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- Need more stringent conditions
- Do we have an affirmative answer to other two questions in the case of uniform convergence?
- Indeed, we have

Uniform Convergence and Continuity

Theorem

Suppose that $\{f_n(x)\}$ is a sequence of continuous functions on $[a,b]$ such that $f_n(x) \searrow\searrow f(x)$ on $[a,b]$. Then $f(x)$ is continuous on $[a,b]$. In other words, for any $\alpha \in [a,b]$,

$$\begin{aligned}\lim_{x\to\alpha} f(x) = \lim_{x\to\alpha}\lim_{n\to\infty} f_n(x) &= \lim_{n\to\infty}\lim_{x\to\alpha} f_n(x)\\ &= \lim_{n\to\infty} f_n(\alpha)\\ &= f(\alpha)\end{aligned}$$

Uniform Convergence and Integration

Theorem

Suppose that $\{f_n(x)\}$ is a sequence of continuous (hence, integrable) functions on $[a,b]$ such that $f_n(x) \searrow\searrow f(x)$ on $[a,b]$. Then $f(x)$ is continuous (hence, integrable) on $[a,b]$ by the previous theorem. Furthermore, for any α and β in $[a,b]$, we have

$$\int_{\alpha}^{\beta} f(x)dx = \int_{\alpha}^{\beta} \left(\lim_{n\to\infty} f_n(x)\right)dx = \lim_{n\to\infty} \int_{\alpha}^{\beta} f_n(x)dx$$

Uniform Convergence and Differentiation

Theorem

Suppose that $\{f_n(x)\}$ is a sequence of continuously differentiable (CD) functions on $[a,b]$ satisfying,

(i) $f_n(x) \searrow f(x)$ *on* $[a,b]$, *and*

(ii) $f_n'(x) \searrow\searrow g(x)$ *on* $[a,b]$,

Then $f(x)$ is CD on $[a,b]$ with $f'(x) = g(x)$ on $[a,b]$. Furthermore, $f_n(x) \searrow\searrow f(x)$ on $[a,b]$. That is, there is a CD function $f(x)$ such that $f_n(x) \searrow\searrow f(x)$ on $[a,b]$ and

$$f'(x) = \frac{d}{dx}\left(\lim_{n\to\infty} f_n(x)\right) = \lim_{n\to\infty} f_n'(x) = g(x)$$

Remarks

- *Pointwise* convergence of CD functions $\{f_n\}$ + the *uniform convergence* of $\{f'_n\}$ yields the desired scenario, i.e.,

$$\frac{d}{dx}\left(\lim_{n\to\infty} f_n(x)\right) = \lim_{n\to\infty} \frac{d}{dx}\Big(f_n(x)\Big)$$

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- Note that, $f_n'(x)$-continuous $\Longrightarrow$ $g(x)$-continuous, and as such, g-integrable
- Then the theorem asserts that $f_n \searrow\searrow$ to the antiderivative of $g(x)$ on $[a,b]$

Series of Functions

- Given sequence of functions $\{f_n(x)\}$, defined on $[a,b]$, define the series as

$$\sum_{n=0}^{\infty} f_n(x); \quad x \in [a,b]$$

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- If the *series* converges at a given x_0 in $[a,b]$, we define its sum to be $f(x_0)$, that is

$$f(x_0) = \sum_{n=0}^{\infty} f_n(x_0)$$

Series of Functions

- Further, define the sequence of *partial sum functions* as

$$S_N(x) = \sum_{n=0}^{N} f_n(x); \quad x \in [a, b]$$

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- Note that

$$\sum_n f_n(x_0) \searrow f(x_0) \quad \Longleftrightarrow \quad S_N(x_0) \searrow f(x_0)$$

Series of Functions

- We say the series $\sum_n f_n(x)$ converges *uniformly* to $f(x)$ on $[a,b]$ if $\{S_N(x)\} \searrow\searrow f(x)$ on $[a,b]$, and denote it by

$$\sum_{n=0}^{\infty} f_n(x) \searrow\searrow f(x) \quad \text{on} \quad [a,b]$$

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- A series converges *absolutely* pointwise/uniformly, according as

$$\sum_{n=0}^{\infty} |f_n(x)| \searrow f(x) \quad \text{or} \quad \sum_{n=0}^{\infty} |f_n(x)| \searrow\searrow f(x)$$

Uniform Convergence of Series

Weierstrass Test for Series

- A series $\sum_n f_n(x)$ is said to be *dominated* by a numerical series $\sum_n \alpha_n$ on $[a,b]$ if

$$|f_n(x)| \leq \alpha_n \quad \forall\, x \in [a,b] \quad \text{and} \quad n \in \mathbb{N}$$

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- Moreover, if $\sum_n \alpha_n \searrow$, then there is a function $f(x)$ defined on $[a,b]$ such that

$$\sum_{n=0}^{\infty} f_n(x) \searrow\searrow f(x) \quad \text{on} \quad [a,b]$$

Proof of Weierstrass Test

- Pointwise limit f exists due to dominance of $\sum_n f_n(x_0)$ by $\sum_n \alpha_n$ at all $x_0 \in [a,b]$ (Recall the comparison test)

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- Now,

$$
\begin{aligned}
|S_N(x) - f(x)| = \left| \sum_{n \geq N} f_n(x) \right| &\leq \sum_{n \geq N} |f_n(x)| \\
&\leq \sum_{n \geq N} \alpha_n \searrow 0
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$$|S_N(x) - f(x)| = \left| \sum_{n \geq N} f_n(x) \right| \leq \sum_{n \geq N} |f_n(x)|$$

$$\leq \sum_{n \geq N} \alpha_n \searrow 0$$

- Conclude that $S_N \searrow\searrow f$ on $[a,b]$

Example

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- Since $\sum_n \frac{1}{n^2} \searrow$, deduce that on $[0, 2\pi]$,

$$\sum_{n=0}^{\infty} \frac{\sin x}{n^2} \searrow\searrow \quad \text{some function} \quad f(x)$$

Uniform Convergence of Series and Continuity

Theorem

Suppose, $\{f_n(x)\}$ *is a sequence of continuous functions on* $[a,b]$ *such that* $\sum_n f_n(x) \searrow\searrow f(x)$ *on* $[a,b]$. *Then* $f(x)$ *is continuous on* $[a,b]$. *In other words, for any* $\alpha \in [a,b]$,

$$\lim_{x\to\alpha} f(x) = \lim_{x\to\alpha} \sum_{n=1}^{\infty} f_n(x) = \sum_{n=0}^{\infty} \lim_{x\to\alpha} f_n(x)$$

$$= \sum_{n=0}^{\infty} f_n(\alpha) = f(\alpha)$$

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Theorem

Suppose, $\{f_n(x)\}$ is a sequence of continuous functions on $[a,b]$ such that $\sum_n f_n(x) \searrow\searrow f(x)$ on $[a,b]$. Then $f(x)$ is continuous on $[a,b]$, and for any α, β in $[a,b]$, we have

$$\int_\alpha^\beta f(x)dx = \int_\alpha^\beta \left(\sum_{n=0}^{\infty} f_n(x)\right)dx = \sum_{n=0}^{\infty} \int_\alpha^\beta f_n(x)dx$$

Uniform Convergence of Series and Differentiation

Suppose, $\{f_n(x)\}$ is a sequence of *CD* functions on $[a,b]$ such that (i) $\sum_n f_n(x) \searrow f(x)$ pointwise in $[a,b]$, and (ii) $\sum_n f_n'(x) \searrow\searrow g$ on $[a,b]$. Then,we have the following diagram:

$$
\begin{array}{ccccc}
\sum_{n=1}^{\infty} f_n(x) & \xrightarrow{\text{uniformly!}} & \int_a^x g(t)\,dt & = & f(x) \\
\uparrow & & \uparrow & & \uparrow \\
\sum_{n=1}^{\infty} f_n'(x) & \xrightarrow{\text{uniformly}} & g(x) & = & f'(x)
\end{array}
$$

Power Series - Revisited

Theorem

Let $\sum_n a_n x^n$ be a power series with the radius of convergence R, and let f be its sum. Let $\alpha \in (-R, R)$. Then the series converges uniformly *on $[-\alpha, \alpha]$.*

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Theorem

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Proof.

One has $|a_n x^n| \leq |a_n||\alpha|^n$ for all $x \in [-\alpha, \alpha]$. Since a power series also converges *absolutely* in $(-R, R)$, deduce that $\sum_{n=1}^{\infty} |a_n||\alpha|^n < \infty$. Conclude by Weierstrass test that the series $\searrow\searrow f$ on $[-\alpha, \alpha]$. $\square$

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Integration Rule for Power Series

- Fix $x_0 \in (-R, R)$

Power Series - Revisited

Integration Rule for Power Series

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- Note that each $a_n x^n$ and the sum function f are continuous (integrable), and $\sum_n a_n x^n \searrow\searrow f(x)$ on $[-R+\varepsilon, R-\varepsilon]$ for any $\varepsilon > 0$

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- Note that each $a_n x^n$ and the sum function f are continuous (integrable), and $\sum_n a_n x^n \searrow\searrow f(x)$ on $[-R+\varepsilon, R-\varepsilon]$ for any $\varepsilon > 0$
- One has

$$\int_0^{x_0} f(x)dx = \sum_n \frac{a_n}{n+1} x_0^{n+1}$$

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Differentiation Rule for Power Series

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- Note that $|na_n x^n| \leq |na_n||\alpha|^n$ for all $x \in [-\alpha, \alpha]$ and all $n \geq 0$
- Since $\sum_n |na_n||\alpha|^{n-1} < \infty$ (**Ex.**), deduce that the series

$$\sum_{n=1}^{\infty} na_n x^{n-1} \searrow\searrow \text{ some } g(x) \quad \text{on} \quad [-\alpha, \alpha]$$

Power Series - Revisited

Differentiation Rule for Power Series

- By the differentiation rule theorem for series of functions, we deduce that

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- By *uniqueness* of limit, conclude that the anti-derivative of g must be f, i.e., on $(-R,R)$

$$f'(x) = \frac{d}{dx}\left(\sum_n a_n x^n\right) = \sum_n n a_n x^{n-1} = g(x)$$