MA 1140: Lecture 7
Linear transformations and matrices

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February 22, 2019

Definition

A linear map $T : V \to W$ is said to be an **isomorphism** if there is a linear map $S : W \to V$ such that
$S \circ T = 1_V : V \to V$ (identity map) and $T \circ S = 1_W : W \to W$.
If $T : V \to W$ is an isomorphism, we say that V and W are isomorphic, and we write $V \cong W$.

Example

Let A be an $n \times n$ matrix over $\mathbb{R}$. Consider $A : \mathbb{R}^n \to \mathbb{R}^n$ as a linear map. When does it invertible?
Answer: When there is an inverse linear map $B : \mathbb{R}^n \to \mathbb{R}^n$ such that $A \circ B = 1_{\mathbb{R}^n}$ and $B \circ A = 1_{\mathbb{R}^n}$, i.e., when there is an $n \times n$ matrix B over $\mathbb{R}$ such that $AB = I_n$ and $BA = I_n$, i.e., when A is an invertible matrix.

Isomorphism of vector spaces

Theorem

Let $T : V \to W$ be a linear map. The following are equivalent:

1. *T is an isomorphism.*
2. *T is bijective (i.e., as a set map, it is injective and surjective).*

Proof.

(1) $\Rightarrow$ (2): Since T is an isomorphism, there is a linear map $S : W \to V$ such that $S \circ T = 1_V$ and $T \circ S = 1_W$.
Let $T(u) = T(v)$ for some $u, v \in V$.
Apply S on this equality, to get $u = v$. So T is injective.
For surjectivity, note that $T(S(w)) = w$ for every $w \in W$.

(2) $\Rightarrow$ (1): Since T is bijective, there is an inverse SET map $S : W \to V$ such that $S \circ T = 1_V$ and $T \circ S = 1_W$.
All we need to show that $S : W \to V$ is a linear map.
You may try on your own. Otherwise, see the next slide.

Proof of the theorem contd...

$(2) \Rightarrow (1)$:

...

Let $w_1, w_2 \in W$. Want to show $S(w_1 + w_2) = S(w_1) + S(w_2)$.
Set $v_1 := S(w_1)$ and $v_2 := S(w_2)$.
Hence, since T is inverse of S (as a set map), it follows that $T(v_1) = w_1$ and $T(v_2) = w_2$. So $T(v_1 + v_2) = w_1 + w_2$ because T is linear. Therefore $S(w_1 + w_2) = v_1 + v_2 = S(w_1) + S(w_2)$.
Similarly, one can prove that $S(cw) = cS(w)$ for every scalar $c \in \mathbb{R}$ and every vector $w \in W$.

Conditions for a linear transformation to be isomorphism

Theorem

Let $T : V \to V$ be a linear map (or **linear operator***), where* $\dim(V) = n < \infty$*. Then the following statements are equivalent:*

1. *T is an isomorphism (see the definition in the 1st slide).*
2. *T is bijective (as a set map).*
3. *T is injective.*
4. $\mathrm{Ker}(T) = 0$, *i.e.*, $\{T(v) = 0 \;\Rightarrow\; v = 0\}$, *i.e.*, $\mathrm{Null}(T) = 0$.
5. *T is surjective.*

Proof. We already proved (1) $\Leftrightarrow$ (2). The following implications ar trivial: (2) $\Rightarrow$ (3) $\Rightarrow$ (4).
(4) $\Rightarrow$ (5): Since $\mathrm{Null}(T) = 0$, $\mathrm{nullity}(T) = \dim(\mathrm{Null}(T)) = 0$.
Hence, by Rank-Nullity Theorem, $\mathrm{rank}(T) = \dim(V)$.
So $\mathrm{Image}(T) = V$, i.e., T is surjective.
(5) $\Rightarrow$ (2): Since T is surjective, $\mathrm{rank}(T) = \dim(V)$, hence $\mathrm{nullity}(T) = 0$, i.e., $\mathrm{Ker}(T) = 0$. Then, by linearity, T is injective.

Conditions for a square matrix to be invertible

Theorem

Let A be an $n \times n$ matrix over $\mathbb{R}$. The following are equivalent:

1. *A is invertible.*
2. *The homogeneous system $AX = 0$ has only the trivial solution.*
3. *For every $b \in \mathbb{R}^n$, the system $AX = b$ has a solution.*
4. *$\mathbb{R}^n$ is spanned by the column vectors of A.*
5. *The column vectors of A are linearly independent.*
6. *$\mathbb{R}^n$ is spanned by the row vectors of A.*
7. *The row vectors of A are linearly independent.*

Proof. Consider A as a linear map $A : \mathbb{R}^n \to \mathbb{R}^n$. Then (2) is same as $\mathrm{Null}(A) = 0$. Moreover (3) is same as A is surjective. Thus, by the previous theorem, we have (1), (2) and (3) are equivalent. Since AX is nothing but a linear combination of column vectors of A, it follows that (3) and (4) are equivalent.

Conditions for a square matrix to be invertible contd...

Theorem

Let A be an $n \times n$ matrix over $\mathbb{R}$. The following are equivalent:

1. *A is invertible.*
2. *The homogeneous system $AX = 0$ has only the trivial solution.*
3. *For every $b \in \mathbb{R}^n$, the system $AX = b$ has a solution.*
4. *$\mathbb{R}^n$ is spanned by the column vectors of A.*
5. *The column vectors of A are linearly independent.*
6. *$\mathbb{R}^n$ is spanned by the row vectors of A.*
7. *The row vectors of A are linearly independent.*

Proof. We already proved (1) $\Leftrightarrow$ (2) $\Leftrightarrow$ (3) $\Leftrightarrow$ (4).
(4) $\Leftrightarrow$ (5) and (6) $\Leftrightarrow$ (7): Since $\dim(\mathbb{R}^n) = n$, any collection of n vectors in $\mathbb{R}^n$ is linearly independent if and only if it spans $\mathbb{R}^n$.
(4) $\Leftrightarrow$ (6): It follows from the above equivalences "(4) $\Leftrightarrow$ (5) and (6) $\Leftrightarrow$ (7)" and the fact that $\text{column rank}(A) = \text{row rank}(A)$.

Ordered basis and coordinates

- Let V be a finite dimensional vector space, and $n = \dim(V)$.
- A finite sequence $\mathcal{B} = \{v_1, v_2, \ldots, v_n\}$ of vectors is called an **ordered basis** of V if $\mathcal{B}$ is linearly independent and spans V.
- Let $\mathcal{B} = \{v_1, v_2, \ldots, v_n\}$ be an ordered basis of V, i.e., $\mathcal{B}$ is a basis of V, together with the specified ordering.
- Let $v \in V$. Then there is a unique n-tuple $(x_1, \ldots, x_n) \in \mathbb{R}^n$ such that $v = x_1 v_1 + \cdots + x_n v_n$. Why? Because $\mathcal{B}$ is a basis.
- The n-tuple $(x_1, \ldots, x_n)$ is called the **coordinate** of v with respect to the ordered basis $\mathcal{B}$ if $v = x_1 v_1 + \cdots + x_n v_n$.
- We denote the coordinate of v with respect to $\mathcal{B}$ by $[v]_{\mathcal{B}}$.

A vector space over $\mathbb{R}$ of dimension n is isomorphic to $\mathbb{R}^n$

- Let V be a finite dimensional vector space, and $n = \dim(V)$.
- We will show that $V \cong \mathbb{R}^n$.
- Consider an ordered basis $\mathcal{B} = \{v_1, v_2, \ldots, v_n\}$ of V.
- Define a map $T : V \to \mathbb{R}^n$ as follows:

$$T : V \longrightarrow \mathbb{R}^n$$

$$v \mapsto [v]_{\mathcal{B}} = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$$

- Check that T is well defined. Moreover, T is a linear map.
- Is it an isomorphism?
  Ans: Yes, because T is bijective (prove it).
- Therefore $V \cong \mathbb{R}^n$.

Let A be an $m \times n$ matrix over $\mathbb{R}$.

Definition

- The subspace of $\mathbb{R}^m$ generated by all columns (column vectors) of A is called the **column space** of A.
- The dimension of the column space of A is called **column rank** of A.
- The subspace of $\mathbb{R}^n$ generated by all rows (row vectors) of A is called the **row space** of A.
- The dimension of the row space of A is called **row rank** of A.

Theorem

For an $m \times n$ matrix A over $\mathbb{R}$, $\text{row rank}(A) = \text{column rank}(A)$.

1. Consider A as a linear map $A : \mathbb{R}^n \to \mathbb{R}^m$.
2. An element of $\mathrm{Image}(A)$ looks like AX for some $X \in \mathbb{R}^n$.
3. But $AX = x_1A_1 + \cdots + x_nA_n$, where A_i is the ith column of A.
4. Therefore $\mathrm{Image}(A) = \mathrm{Span}\{A_1, \ldots, A_n\} = \mathrm{Column\ Sp.}(A)$.
5. Hence $\mathrm{rank}(A) := \dim(\mathrm{Image}(A)) = \mathrm{column\ rank}(A)$.
6. **Rank-nullity theorem:** $\mathrm{rank}(A) + \mathrm{nullity}(A) = \dim(\mathbb{R}^n)$.
7. Therefore $\quad \mathrm{column\ rank}(A) = n - \mathrm{nullity}(A)$.

So it is enough to show that

$$\mathrm{row\ rank}(A) = n - \mathrm{nullity}(A).$$

Elementary row operations preserve row space, hence rank

Theorem

Let A and B be row equivalent. Then A and B have the same row space. In particular, $\mathrm{row\ rank}(A) = \mathrm{row\ rank}(B)$.

Proof. Note that A and B have the same order (say, $m \times n$). Let $R_1, \ldots, R_m \in \mathbb{R}^n$ be the row vectors of A. We observe that the elementary row operations preserve the row space:

1. Effect of the **1st type** elementary row operation, e.g., $\mathrm{Span}\{R_1, R_2, R_3, \ldots, R_m\} = \mathrm{Span}\{R_2, R_1, R_3, \ldots, R_m\}$.
2. Effect of the **2nd type** elementary row operation, e.g., $\mathrm{Span}\{R_1, R_2, R_3, \ldots, R_m\} = \mathrm{Span}\{R_1, c \cdot R_2, R_3, \ldots, R_m\}$, where $c \neq 0$ (important!).
3. Effect of the **3rd type** elementary row operation, e.g., $\mathrm{Span}\{R_1, R_2, R_3, \ldots, R_m\} =$ $\mathrm{Span}\{R_1, R_2 - c \cdot R_1, R_3, \ldots, R_m\}$, where $c \in \mathbb{R}$.

Elementary row operations preserve the nullity of a matrix

1. Let A and B be row equivalent matrices over $\mathbb{R}$.
2. Then $AX = 0$ and $BX = 0$ are equivalent system.
3. Hence $AX = 0$ and $BX = 0$ have the same solution set, i.e., $\mathrm{Null}(A) = \mathrm{Null}(B)$.
4. Therefore $\mathrm{nullity}(A) = \mathrm{nullity}(B)$.

Proof of "$\mathrm{row\ rank}(A) = n - \mathrm{nullity}(A)$"

1. Let A be an $m \times n$ matrix over $\mathbb{R}$.
2. Then A is row-equivalent to a row-reduced echelon matrix B.
3. Since $\mathrm{row\ rank}(A) = \mathrm{row\ rank}(B)$ and $\mathrm{nullity}(A) = \mathrm{nullity}(B)$, it is enough to prove that
$$\mathrm{row\ rank}(B) = n - \mathrm{nullity}(B).$$
4. Let r be the number of non-zero rows of B.
5. The proof is complete once we show that
$$\mathrm{row\ rank}(B) = r \quad \text{and} \quad \mathrm{nullity}(B) = n - r.$$
6. I leave it as an exercise to verify the last statement.

Example: $\mathrm{row\ rank}(A) = n - \mathrm{nullity}(A)$

$A=\begin{pmatrix} 1 & -2 & 1 & 1 \\ 2 & 1 & 1 & 3 \\ 0 & 5 & -1 & 1 \end{pmatrix}$ is row-equivalent to $B=\begin{pmatrix} 1 & 0 & \frac{3}{5} & \frac{7}{5} \\ 0 & 1 & -\frac{1}{5} & \frac{1}{5} \\ 0 & 0 & 0 & 0 \end{pmatrix}$.

- We shall show that $\mathrm{row\ rank}(B) = 4 - \mathrm{nullity}(B)$.
- It can be observed that $\mathrm{row\,rank}(B)$ is the number of non-zero rows of B, i.e., the number of pivots of B. So $\mathrm{row\,rank}(B) = 2$.
- Consider the system $BX = 0$. The pivot variables are x_1, x_2. The free variables are x_3 and x_4.

$$x_1 = -\frac{3}{5}x_3 - \frac{7}{5}x_4$$
$$x_2 = \frac{1}{5}x_3 - \frac{1}{5}x_4$$

- We see that $\mathrm{nullity}(B)$ is the number of free variables, because ...

How to solve $BX = 0$ when B is row-reduced echelon?

- Consider $B = \begin{pmatrix} 1 & 0 & \frac{3}{5} & \frac{7}{5} \\ 0 & 1 & -\frac{1}{5} & \frac{1}{5} \\ 0 & 0 & 0 & 0 \end{pmatrix}$, a row-reduced echelon matrix.
- The corresponding homogeneous system can be written as

$$\begin{aligned} x_1 &= -\frac{3}{5}x_3 - \frac{7}{5}x_4 \\ x_2 &= \frac{1}{5}x_3 - \frac{1}{5}x_4 \end{aligned}$$

- The solutions of the system are given by

$$\begin{pmatrix} -\frac{3}{5}x_3 - \frac{7}{5}x_4 \\ \frac{1}{5}x_3 - \frac{1}{5}x_4 \\ x_3 \\ x_4 \end{pmatrix} = x_3 \begin{pmatrix} -3/5 \\ 1/5 \\ 1 \\ 0 \end{pmatrix} + x_4 \begin{pmatrix} -7/5 \\ -1/5 \\ 0 \\ 1 \end{pmatrix}$$

- $\mathrm{nullity}(B) = \dim(\mathrm{Null}(B)) =$ the number of free variables.

Thank You!