MA1130: Vector Calculus

Dr. Bhakti Bhusan Manna

(Lecture Notes)

January 8, 2019

Topics to be covered

- Vectors and Vector algebras.
- Dot and Cross products.
- Multiple integrals: Area, Volume and Change of variables.
- Line and surfce integrals: Divergence, Green's, Stoke's theorem.

Reference: Vector Analysis by Murray R Spiegel (Schaums' outline series).

NOTE: We shall restrict our study till dimension 3. Higher dimensional analysis can be done similarly (for most of the results).

The n-dimensional euclidean space $\mathbb{R}^n$

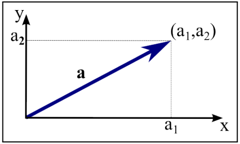
For any positive integer n we defne:

$$\mathbb{R}^n = \{(x_1, \ldots, x_n); x_i \in \mathbb{R}, i = 1, 2, \ldots, n\}$$

- $\mathbb{R}^1 (\equiv \mathbb{R})$ is the set of real numbers.
- $\mathbb{R}^2$ is the 2-dimensional plane.
- $\mathbb{R}^3$ is the 3-dimensional space.

Definition (VECTOR)

Any element of $\mathbb{R}^n$ is called an $n-$ dimensional vector.



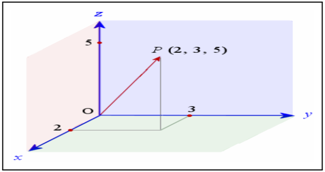
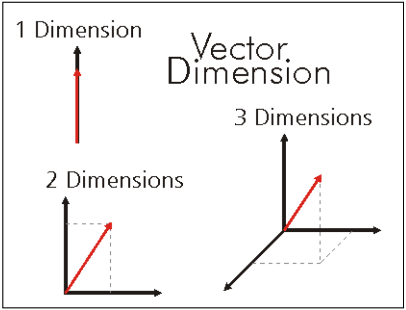
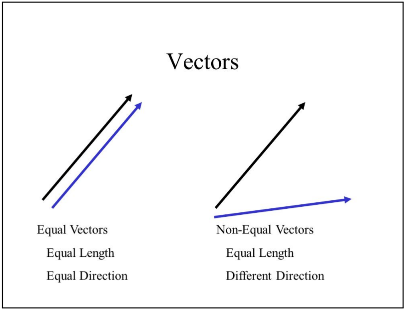


Figure 1: Vectors in 2D and 3D

The vectors in different dimensions



Vectors in different dimensions



vector equality

The n-dimensional euclidean space $\mathbb{R}^n$

Definition (SCALAR)

The element in $\mathbb{R}$ known as the real numbers are a special type of vectors which are called the scalars.

Remark

Classically a vector is a quantity with:

- a length(magnitude),
- a direction (identified by points on sphere of particular dimension of the vector),
- it does not depend upon the starting and ending position i.e. two vectors having same magnitude and direction are identified to be the same vectors even if their starting and ending points are different.
- Hence for any n-dimensional vectors one can assume the starting point to be the origin and the ending point to be a point in $\mathbb{R}^n$. So every point in $\mathbb{R}^n$ represents a n-dimensional vector and every n-dimensional vector can be represented by a point in $\mathbb{R}^n$.

Algebra of vectors

- $\mathbb{R} \subset \mathbb{R}^3$ can be viewed as $\{a \in \mathbb{R} \equiv (a, 0, 0) \in \mathbb{R}^3\}$
- $\mathbb{R}^2 \subset \mathbb{R}^3$ can be viewed as $\{(a, b) \in \mathbb{R}^2 \equiv (a, b, 0) \in \mathbb{R}^3\}$

Definition (Norm or Length of a vector)

For a vector $\mathbf{v} = (v_1, v_2, v_3) \in \mathbb{R}^3$ the magnitude (norm, length) of $\mathbf{v}$ is defined by

$$\|\mathbf{v}\| = \sqrt{v_1^2 + v_2^2 + v_3^2}$$

Definition (Algebra of vectors)

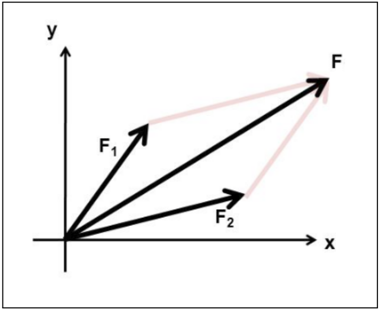
- Vector addition: The sum of two vectors $\mathbf{v}$ and $\mathbf{w}$ is defined by

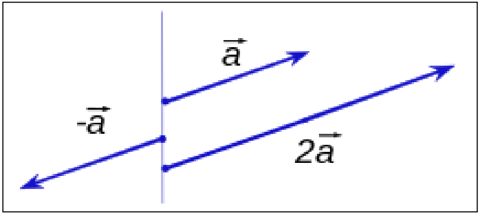
$$\mathbf{v} + \mathbf{w} = (v_1 + w_1, v_2 + w_2, v_3 + w_3)$$

- Scalar multiplication: A scalar multiple of $\mathbf{v}$ by a scalar k is defined by

$$k\mathbf{v} = (kv_1, kv_2, kv_3)$$

Note: sum of two vectors can be defined only when the dimensions of the vectors are same.





Sum of two vectors

Scalar multiplication

The basic laws of vector algebra

Theorem

For any vectors $\mathbf{u}, \mathbf{v}, \mathbf{w}$ *and scalars* k, l *we have*

- $\mathbf{v}+\mathbf{w}=\mathbf{w}+\mathbf{v}$ *Commutative law.*
- $\mathbf{u}+(\mathbf{v}+\mathbf{w})=(\mathbf{u}+\mathbf{v})+\mathbf{w}$ *Associative law.*
- $\mathbf{u}+\mathbf{0}=\mathbf{u}$ *Additive identity.*
- $\mathbf{u}+(-\mathbf{u})=\mathbf{0}$ *Additive inverse.*
- $k(l\mathbf{u})=(kl)\mathbf{u}$ *Associative law.*
- $k(\mathbf{u}+\mathbf{v})=k\mathbf{u}+k\mathbf{v}$ *Disrtibutive law.*
- $(k+l)\mathbf{u}=k\mathbf{u}+l\mathbf{u}$ *Disrtibutive law.*

Proof.

Note

$$\mathbf{v}+\mathbf{w}=(v_1+w_1, v_2+w_2, v_3+w_3)=(w_1+v_1, w_2+v_2, w_3+v_3)=\mathbf{w}+\mathbf{v}$$

Using the property of real numbers. All other proofs are similar and left to the readers. $\square$

Unit and Basis vectors

Definition

An **unit vector** is a vector with magnitude 1.

- For any nonzero vector $\mathbf{v} \neq 0$ one can make it unit vector by dividing with its norm. i.e. the vector $\frac{\mathbf{v}}{\|\mathbf{v}\|}$ is an unit vector.
- The **basis vectors** are special king of unit vectors denoted by

$$\mathbf{i} := (1, 0, 0), \mathbf{j} := (0, 1, 0), \mathbf{k} := (0, 0, 1).$$

- Every vectors $\mathbf{v} = (a, b, c)$ can be written as the unique **linear combination** of unit vectors as

$$\mathbf{v} = (a, b, c) = a\,\mathbf{i} + b\,\mathbf{j} + c\,\mathbf{k} = a(1, 0, 0) + b(0, 1, 0) + c(0, 0, 1)$$

Dot product

Definition (Dot Product)

The **dot product** of two vectors $\mathbf{v} = (v_1, v_2, v_3)$ and $\mathbf{w} = (w_1, w_2, w_3)$ is defined as

$$\mathbf{v} \cdot \mathbf{w} := v_1 w_1 + v_2 w_2 + v_3 w_3$$

- We can see easily $\mathbf{i} \cdot \mathbf{i} = \mathbf{j} \cdot \mathbf{j} = \mathbf{k} \cdot \mathbf{k} = 1$, and $\mathbf{i} \cdot \mathbf{j} = \mathbf{j} \cdot \mathbf{k} = \mathbf{k} \cdot \mathbf{i} = 0$
- Vectors which are in the same direction are known to be collinear vectors. Note there is a unique plane containing two vectors which are non collinear.

Definition

The angle between two non collinear vectors is the smallest angle between them in the plane containing the vectors.

Theorem

For two nonzero vectors $\mathbf{v}$ *and* $\mathbf{w}$ *and* θ *being the angle between them one has*

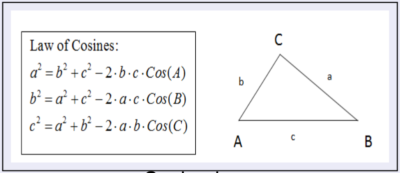
$$\mathbf{v} \cdot \mathbf{w} = \|\mathbf{v}\| \, \|\mathbf{w}\| \, cos\theta$$

Dot product

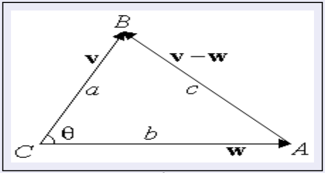
Proof.

By laws of **cosine** we have

$$\|\mathbf{v}-\mathbf{w}\|^2 = \|\mathbf{v}\|^2 + \|\mathbf{w}\|^2 - 2\,\|\mathbf{v}\|\,\|\mathbf{w}\|\,cos\theta$$



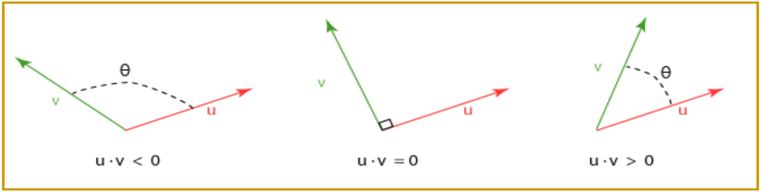
Cosine laws



vectors

Since $\mathbf{v}-\mathbf{w} = (v_1 - w_1, v_2 - w_2, v_3 - w_3)$, expanding $\|\mathbf{v}-\mathbf{w}\|^2$ we get

$$\begin{aligned}
\|\mathbf{v}-\mathbf{w}\|^2 &= (v_1 - w_1)^2 + (v_3 - w_3)^2 + (v_3 - w_3)^2 \\
&= (v_1^2 + v_2^2 + v_3^2) + (w_1^2 + w_2^2 + w_3^2) - 2(v_1 w_1 + v_2 w_2 + v_3 w_3) \\
&= \|\mathbf{v}\|^2 + \|\mathbf{w}\|^2 - 2\mathbf{v}\cdot\mathbf{w} \qquad \textbf{QED}.
\end{aligned}$$

- If the angle between two vectors is $\pi/2$ the their dot product is zero and we say the vectors are **orthogonal** or **perpendicular** to each other and is denoted by $\mathbf{v} \perp \mathbf{w}$.
- 

Theorem (Properties of dot product)

For any vectors $\mathbf{u}, \mathbf{v}, \mathbf{w}$ *and scalar* k *we have*

(a) $\mathbf{v} \cdot \mathbf{w} = \mathbf{w} \cdot \mathbf{v}$

(b) $(k\mathbf{v}) \cdot \mathbf{w} = \mathbf{v} \cdot (k\mathbf{w}) = k(\mathbf{v} \cdot \mathbf{w})$

(c) $\mathbf{v} \cdot \mathbf{0} = 0 = \mathbf{0} \cdot \mathbf{v}$

(d) $\mathbf{u} \cdot (\mathbf{v} + \mathbf{w}) = \mathbf{u} \cdot \mathbf{v} + \mathbf{u} \cdot \mathbf{w}$

(e) $|\mathbf{v} \cdot \mathbf{w}| \leq \|\mathbf{v}\| \, \|\mathbf{w}\|$

Proof.

The proofs of parts (a)-(d) are straightforward applications of the definition of the dot product, and are left to the reader as exercises. We will prove part (e).
If one of **v** or **w** is zero then the result is trivial. Let us suppose both **v** and **w** are nonzero. Then by previous theorem we have

$$\mathbf{v}\cdot\mathbf{w} = \|\mathbf{v}\|\,\|\mathbf{w}\|\,cos\theta$$
$$|\mathbf{v}\cdot\mathbf{w}| = \|\mathbf{v}\|\,\|\mathbf{w}\|\,|cos\theta| \leq \|\mathbf{v}\|\,\|\mathbf{w}\| \text{ as } |cos\theta| \leq 1$$

$\square$

Proposition

For any vectors **v** *and* **w** *we have*

(a) $\|\mathbf{v}\|^2 = \mathbf{v}\cdot\mathbf{v}$

(b) $\|\mathbf{v}+\mathbf{w}\| \leq \|\mathbf{v}\| + \|\mathbf{w}\|$

(c) $\|\mathbf{v}-\mathbf{w}\| \geq \|\mathbf{v}\| - \|\mathbf{w}\|$

Cross product

Definition

Let $\mathbf{v} = (v_1, v_2, v_3)$ and $\mathbf{w} = (w_1, w_2, w_3)$ be any vector in $\mathbb{R}^3$. The cross product of $\mathbf{v}$ and $\mathbf{w}$ denoted by $\mathbf{v} \times \mathbf{w}$ is defined as

$$\mathbf{v} \times \mathbf{w} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{vmatrix} = (v_2 w_3 - v_3 w_2, v_3 w_1 - v_1 w_3, v_1 w_2 - v_2 w_1)$$

Remark

Note that $\mathbf{v}\cdot (\mathbf{v} \times \mathbf{w}) = \begin{vmatrix} v_1 & v_2 & v_3 \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{vmatrix} = 0 = \begin{vmatrix} w_1 & w_2 & w_3 \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{vmatrix} = \mathbf{w}\cdot (\mathbf{v} \times \mathbf{w})$. Hence $\mathbf{v} \times \mathbf{w}$ is perpendicular to both $\mathbf{v}$ and $\mathbf{w}$ and hence it is orthogonal to $a\,\mathbf{v} + b\,\mathbf{w}$ for all $a, b \in \mathbb{R}$

Theorem

If θ is the angle between two nonzero vectors $\mathbf{v}$ *and* $\mathbf{w}$ *then*

$$\|\mathbf{v} \times \mathbf{w}\| = \|\mathbf{v}\| \, \|\mathbf{w}\| \, sin\theta$$

Proof.

First note

$$\|\mathbf{v} \times \mathbf{w}\|^2 = v_1^2(w_2^2 + w_3^2) + v_2^2(w_1^2 + w_3^2) + v_3^2(w_1^2 + w_2^2) - 2(v_1 w_1 v_2 w_2 + v_2 w_2 v_3 w_3 + v_3 w_3 v_1 w_1)$$

Now adding and subtracting $v_1^2 w_1^2, v_2^2 w_2^2, v_3^2 w_3^2$ and using the fact

$$(a+b+c)^2 = a^2 + b^2 + c^2 + 2ab + 2bc + 2ca$$

we get

$$\begin{aligned}\|\mathbf{v} \times \mathbf{w}\|^2 &= \|\mathbf{v}\|^2 \, \|\mathbf{w}\|^2 - (\mathbf{v} \cdot \mathbf{w})^2 = \|\mathbf{v}\|^2 \, \|\mathbf{w}\|^2 \left(1 - \frac{(\mathbf{v} \cdot \mathbf{w})^2}{\|\mathbf{v}\|^2 \, \|\mathbf{w}\|^2}\right) \\ &= \|\mathbf{v}\|^2 \, \|\mathbf{w}\|^2 \, Sin^2\theta\end{aligned}$$

$\square$