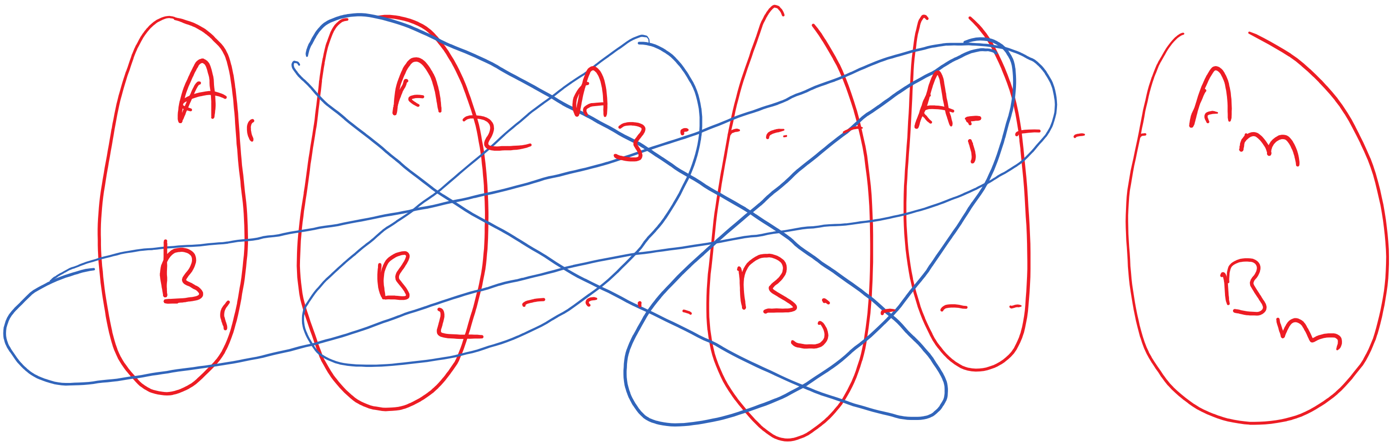
Bollobas Theorem

Theorem: Let $(A_1, A_2, \dots, A_m)$ and $(B_1, B_2, \dots, B_m)$ be two sequences of sets such that $\forall i, j \in [m]$, $A_i \cap \overset{B_j}{\cancel{A_j}} = \emptyset$ if and only if $i = j$. Then,

$$\sum_{i=1}^{m} \frac{1}{\binom{a_i + b_i}{a_i}} \leq 1,$$

where $a_i = |A_i|$, $b_i = |B_i|$.



$$A_i \cap B_i = \emptyset$$

$$A_i \cap A_j \neq \emptyset, \quad i \neq j.$$

How Bollobas Thm $\Rightarrow$ LYM Ineq.

$\mathcal{F} = \{A_1, \ldots, A_m\}$ an antichain made of subsets of $[n]$.

$$\sum_{i=1}^{m} \frac{1}{\binom{n}{|A_i|}} \le 1$$

subsets of $[n]$

$\mathcal{F}$

$A_1 \quad A_2 \quad \ldots \quad A_m$

$[n] \setminus A_1$

$\overline{A_1} \quad \overline{A_2} \quad \ldots \quad \overline{A_m}$

By Bollobas Thm, $\displaystyle\sum_{i=1}^{m} \frac{1}{\binom{a_i+b_i}{a_i}} \le 1$

n

$$\sum_{i=1}^{m} \frac{1}{\binom{n}{|A_i|}} \le 1$$

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Proof of Bollobas Theorem

$A_1 \quad A_2 \quad \cdots \quad A_m$

$B_1 \quad B_2 \quad \cdots \quad B_m$

$A_i \cap B_j = \phi$, iff $i = j$.

Then,

$$\sum_{i=1}^{m} \frac{1}{\binom{a_i + b_i}{a_i}} \leq 1$$

Let $\bigcup_{i=1}^{m} (A_i \cup B_i) = X = \{x_1, x_2, \dots, x_n\}$

Let $|X| = n$.

Let σ be a permutation of $x_1, x_2, \dots, x_n$

$\sigma: \quad x_3 \quad \overline{x_2} \quad x_7 \quad \overline{x_1} \quad x_5 \quad \cdots \quad x_n \quad x_{10} \cdots x_{12}$

$A_i = \{x_2, x_1\} \qquad B_i = \{x_5, x_{10}, x_{11}\}$

The pair (A_i, B_i) is "<u>present</u>" in σ.

<u>Defn</u>: A pair (A_i, B_i) is "<u>present</u>" in a permutation σ of X, if every element of A_i appears before every element of B_i in σ.

$\mathcal{S} = \left\{ \left(\sigma, (A_i, B_i) \right) : \right.$ σ is a permutation of X, A_i, B_i are sets present in the two sequences given in the theorem, (A_i, B_i) is "present" in σ $\}$

σ : $x_3 \quad x_2 \quad x_{10} \quad x_{11} \quad x_1 \quad x_{12} \quad x_9 \quad x_6 \quad x_4 \ldots$

$A_i = \{x_3, x_{10}\}$ $\quad A_j = \{x_{11}, x_{12}\}$

$B_i = \{x_{11}, x_1\}$ $\quad B_j = \{x_{10}, x_6, x_4\}$

(A_i, B_i) is present in σ.

(A_j, B_j) is not present in σ.

$$\sum_{i=1}^{m} \frac{n!}{\binom{a_i+b_i}{a_i}} \leq |\mathcal{S}| \leq n! \cdot 1 \Rightarrow \sum_{i=1}^{m} \frac{1}{\binom{a_i+b_i}{a_i}} \leq 1 \quad \square$$

Q. In how many σs can a given (A_i, B_i) pair be present?

$$|X \setminus \{A_i \cup B_i\}| = n - a_i - b_i := k$$

$\overbrace{\qquad\qquad}^{a_i} \quad \overbrace{\qquad\qquad}^{b_i}$

$a_i! \cdot b_i!$

take any one such arrangement

$\underbrace{\qquad\qquad}_{a_i} \quad \underbrace{\qquad\qquad}_{b_i}$

$a_i + b_i$ elements

Let the remaining k elements be

$x_1, x_2, \ldots, x_k$. $(k = n - a_i - b_i)$

$k!$ ways

take any one such arrangement

$x_2 \quad x_3 \quad x_7 \quad x_1 \ldots x_k \ldots x_{10}$

Choose $(a_i + b_i)$ locations from the above $k+1$ locations. Repetition allowed

choosing r elements from n elements with rep allowed

$(n+r-1)$

$$\binom{k+1+a_i+b_i-1}{a_i+b_i}$$

$$\left(\quad a_i + b_i \quad \right) \qquad \omega \qquad \binom{n+r-1}{r}$$

$$k = n - a_i - b_i$$

$$\binom{n}{a_i + b_i}$$

$$\frac{\underbrace{a_i!\,b_i!}_{1}\,\underbrace{(n - a_i - b_i)!}_{2}}{\binom{n}{a_i+b_i}}$$

$$= \frac{a_i\, b_i\, \cancel{(n - a_i - b_i)!}\; n!}{(a_i + b_i)!\, \cancel{(n - a_i - b_i)!}}$$

$$= \frac{n!}{\binom{a_i + b_i}{a_i}}$$

<u>Corollary</u>: Let $(A_1, A_2, \dots, A_m)$ and $(B_1, \dots, B_m)$ be two sequences of sets such that $\forall i,j \in [m]$, $A_i \cap B_j = \phi$ <u>iff</u> $i = j$. Let $|A_i| \le a$, $|B_i| \le b$, $\forall i \in [m]$. Then,

$$m \le \binom{a+b}{a}$$

<u>Proof</u>:

$$\frac{m}{\binom{a+b}{a}} \le \sum_{i=1}^{m} \frac{1}{\binom{a_i+b_i}{a_i}} \le 1$$

$\square$

a-sized $\rightarrow$ $A_1 \quad A_2 \quad \dots \quad A_j \quad \dots \quad A_m$

b-sized $\rightarrow$ $B_1 \quad B_2 \quad \dots \quad B_j \quad \dots \quad B_m$

Consider the spl case, $|A_i| = a$, $|B_i| = b$, $\forall i \in [m]$

$\rightarrow$ Hitting set for $\{B_1, B_2, \dots, B_m\}$

$\{B_2, B_3, \ldots, B_m\}$

$\rightarrow$ Hitting set for $\{B_1, B_3, B_4, \ldots, B_m\}$

Property of $\{A_1, A_2, \ldots, A_m\}$ is that if we remove any set from it, then the resulting family has a hitting set of size b.

In other words, $\{A_1, A_2, \ldots, A_m\}$ is a minimal family that has no hitting set of size b.

"Skew" version of Bollobas Thm

Let $(A_1, A_2, \ldots, A_m)$ and $(B_1, B_2, \ldots, B_m)$ be two sequences of sets such that $\forall i,j \in [m]$, (i) $A_i \cap B_j = \emptyset$, if $i=j$, and (ii) $A_i \cap B_j \neq \emptyset$, if $i<j$. Then,

$i \neq j$

$$\sum_{i=1}^{m} \frac{1}{\binom{a_i+b_i}{a_i}} \le 1, \quad \text{where}$$

$a_i = |A_i|$, $b_i = |B_i|$.