Topics in Combinatorics

Exam II (out of 10 marks)
(Date: 09 Nov 2020. Timing: 12:00 to 13:05 hours)

Notation. For a graph G, we use $V(G)$ to denote its vertex set and $E(G)$ to denote its edge set. For each vertex $v \in V(G)$, let $N(v) = \{u \in V(G) \ : \ \{u, v\} \in E(G)\}$ denote the *open neighborhood* of v. The *degree* of v, denoted by $deg(v)$, is the number of vertices in the open neighborhood of v. That is, $deg(v) = |N(v)|$. The *maximum degree* of a graph G is $\max\{deg(v) \ : \ v \in V(G)\}$.

If A_1, A_2, A_3 are three pairwise disjoint sets whose union gives a set B, then we denote it by $B = A_1 \uplus A_2 \uplus A_3$.

Theorem (Chernoff Bound)**.** Let $X_1, X_2, \ldots, X_n$ be a sequence of n independent 0-1 random variables. Let $X = \sum_{i=1}^{n} X_i$ and $\mu = E[X]$. Then, for every $R \geq 6\mu$,

$$Pr[X \geq R] \leq \frac{1}{2^R}.$$

Problem

Let G be a simple, undirected, finite graph. Let Δ denote the maximum degree of G. Show that there is a way to partition the vertices of G as $V(G) = V_1 \uplus V_2 \uplus \cdots \uplus V_r$ such that

1. $\forall v \in V(G),\ \forall i \in \{1, 2, \ldots, r\}$, $|N(v) \cap V_i| = O(\log \Delta)$, and
2. $r = O(\Delta / \log \Delta)$.

10 marks