# EP 1027: Maxwell Equations and Electromagnetic Waves
## Homework Assignment 3

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1. In the last homework set you were asked to compute the field due to an electric dipole. Here you will see/show that a tiny current loop behaves like a magnetic dipole, with a dipole moment given by
$$\mathbf{m} = \frac{I}{2}\oint \mathbf{x}' \times d\mathbf{x}',$$
where $I$ is the current running in the loop and $\mathbf{x}'$ is the position vector of a general point on the loop (see Figure 1).

   To arrive at this conclusion you need to start from the Biot-Savart law for the magnetic field produced by the loop at a far point, $\mathbf{x}$,
$$\mathbf{B}(\mathbf{x}) = \frac{\mu_0 I}{4\pi}\oint d\mathbf{x}' \times \frac{\mathbf{x}-\mathbf{x}'}{|\mathbf{x}-\mathbf{x}'|^3},$$
and show that when the loop is small, i.e. $\mathbf{x}-\mathbf{x}' \approx \mathbf{x}$, the magnetic field looks like the electric field of an electric dipole at the origin, namely,
$$\mathbf{B}(\mathbf{x}) = \frac{3\,(\mathbf{m}\cdot\mathbf{x})\,\mathbf{x}}{|\mathbf{x}|^5} - \frac{\mathbf{m}}{|\mathbf{x}|^3}.$$
(**Hint**: Expand $\frac{1}{|\mathbf{x}-\mathbf{x}'|^3}$ in a Taylor series about $\mathbf{x}'=0$ since it is a very small loop and retain only terms up to first power/linear in $\mathbf{x}'$). (10 points)

**Solution:**
The Taylor expansion of $|\mathbf{x}-\mathbf{x}'|^{-3}$ around $\mathbf{x}'=0$ is,
$$\begin{aligned}\frac{1}{|\mathbf{x}-\mathbf{x}'|^3} &= \frac{1}{|\mathbf{x}|^3} + \mathbf{x}'\cdot\left(\boldsymbol{\nabla}'\frac{1}{|\mathbf{x}-\mathbf{x}'|^3}\right)_{\mathbf{x}'=0} + O(x'^2)\\ &= \frac{1}{|\mathbf{x}|^3} + \frac{3\mathbf{x}'\cdot\mathbf{x}}{|\mathbf{x}|^5} + O(x'^2).\end{aligned}$$

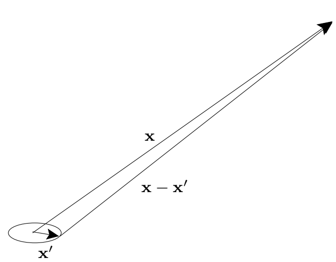


Figure 1: Magnetic field due to a small current loop

Inserting this expression in the Biot-Savart law for a current carrying loop, we get,

$$
\begin{aligned}
\mathbf{B}(\mathbf{x}) &= \frac{\mu_0 I}{4\pi} \oint d\mathbf{x}' \times \frac{\mathbf{x}-\mathbf{x}'}{|\mathbf{x}-\mathbf{x}'|^3} \\
&= \frac{\mu_0 I}{4\pi} \oint d\mathbf{x}' \times (\mathbf{x}-\mathbf{x}') \left[\frac{1}{|\mathbf{x}|^3} + \frac{3\mathbf{x}'\cdot\mathbf{x}}{|\mathbf{x}|^5} + O\left(x'^2\right)\right] \\
&= \frac{\mu_0 I}{4\pi} \oint d\mathbf{x}' \times \left[\frac{\mathbf{x}}{|\mathbf{x}|^3} - \frac{\mathbf{x}'}{|\mathbf{x}|^3} + \frac{3\left(\mathbf{x}'\cdot\mathbf{x}\right)\mathbf{x}}{|\mathbf{x}|^5} + O\left(x'^2\right)\right] \\
&\approx \frac{\mu_0}{4\pi} \left[\left(I \oint d\mathbf{x}'\right) \times \frac{\mathbf{x}}{|\mathbf{x}|^3} - \frac{I \oint d\mathbf{x}' \times \mathbf{x}'}{|\mathbf{x}|^3} + \frac{3\left(I \oint d\mathbf{x}' \times (\mathbf{x}'\cdot\mathbf{x})\,\mathbf{x}\right)}{|\mathbf{x}|^5}\right] \qquad (1)
\end{aligned}
$$

Now the first term drops off because,

$$\oint d\mathbf{x}' = 0.$$

The second term can be expressed in terms of the magnetic dipole moment as follows,

$$-\frac{I \oint d\mathbf{x}' \times \mathbf{x}'}{|\mathbf{x}|^3} = \frac{I \oint \mathbf{x}' \times d\mathbf{x}'}{|\mathbf{x}|^3} = \frac{2\mathbf{m}}{|\mathbf{x}|^3}.$$

Before we process the third term we make use of the following identities. First one is,

$$
\begin{aligned}
(\mathbf{m}\times\mathbf{x}) &= \left(\frac{1}{2} I \oint \mathbf{x}' \times d\mathbf{x}'\right) \times \mathbf{x} \\
&= \frac{I}{2} \oint \left[(\mathbf{x}' \times d\mathbf{x}') \times \mathbf{x}\right] \\
&= \frac{I}{2} \oint \left[d\mathbf{x}'\,(\mathbf{x}'\cdot\mathbf{x}) - \mathbf{x}'\,(d\mathbf{x}'\cdot\mathbf{x})\right] \\
&= \frac{I}{2} \oint \left[dx'^k x'^l x^l - x'^k dx'^l x^l\right] \hat{e}_k \\
&= \frac{I}{2} \oint \left[dx'^k x'^l x^l - d(x'^k x'^l)\, x^l + dx'^k x'^l x^l\right] \hat{e}_k \\
&= I \oint dx'^k x'^l x^l \hat{e}_k \\
&= I \oint d\mathbf{x}'\,(\mathbf{x}'\cdot\mathbf{x}),
\end{aligned}
$$

where the term involving line integral of the total derivative $d(x'^k x'^l)$ around a closed loop is zero. Now taking cross product of both sides with $\mathbf{x}$,

$$
\begin{aligned}
(\mathbf{m}\times\mathbf{x})\times\mathbf{x} &= I \oint d\mathbf{x}'\,(\mathbf{x}'\cdot\mathbf{x}) \times \mathbf{x} \\
&= I \oint d\mathbf{x}' \times (\mathbf{x}'\cdot\mathbf{x})\,\mathbf{x},
\end{aligned}
$$

which is the third term in the RHS of equation (1). Thus we can rewrite (1) as,

$$
\begin{aligned}
\mathbf{B}(\mathbf{x}) &= \frac{\mu_0}{4\pi} \left[\frac{2\mathbf{m}}{|\mathbf{x}|^3} + \frac{3\,(\mathbf{m}\times\mathbf{x})\times\mathbf{x}}{|\mathbf{x}|^5}\right] \\
&= \frac{\mu_0}{4\pi} \left[\frac{3\,(\mathbf{m}\cdot\mathbf{x})\,\mathbf{x}}{|\mathbf{x}|^5} - \frac{\mathbf{m}}{|\mathbf{x}|^3}\right],
\end{aligned}
$$

where in the last step we have used BAC minus CAB rule for the vector triple product,

$$(\mathbf{m} \times \mathbf{x}) \times \mathbf{x} = (\mathbf{m} \cdot \mathbf{x})\, \mathbf{x} - \mathbf{m}\left(\mathbf{x}^2\right).$$

2. **d'Alembert's solution to the wave equation in one dimension**: Prove that the general solution to the wave equation in one dimension, namely

$$\left(\frac{\partial^2}{\partial x^2} - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right) f(x,t) = 0 \tag{2}$$

is,

$$f(x,t) = g(x+ct) + h(x-ct),$$

where $g$ and $h$ are some arbitrary functions. (**Hint**: Switch to new variables, $\zeta = x+ct,\ \eta = x-ct$.) (5 points)

**Solution:**
We switch to new variables,

$$\zeta = x + ct, \quad \eta = x - ct,$$

In terms of these the old coordinates,

$$x = \frac{\zeta+\eta}{2}, \quad t = \frac{\zeta-\eta}{2c}.$$

The derivatives transform like,

$$\frac{\partial}{\partial x} = \frac{\partial \zeta}{\partial x}\frac{\partial}{\partial \zeta} + \frac{\partial \eta}{\partial x}\frac{\partial}{\partial \eta} = \frac{\partial}{\partial \zeta} + \frac{\partial}{\partial \eta},$$

$$\frac{1}{c}\frac{\partial}{\partial t} = \frac{1}{c}\left(\frac{\partial \zeta}{\partial t}\frac{\partial}{\partial \zeta} + \frac{\partial \eta}{\partial t}\frac{\partial}{\partial \eta}\right) = \frac{\partial}{\partial \zeta} - \frac{\partial}{\partial \eta}.$$

The second derivatives transform like,

$$\begin{aligned}
\frac{\partial^2 f}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right) &= \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial \zeta} + \frac{\partial f}{\partial \eta}\right) \\
&= \left(\frac{\partial}{\partial \zeta} + \frac{\partial}{\partial \eta}\right)\left(\frac{\partial f}{\partial \zeta} + \frac{\partial f}{\partial \eta}\right) \\
&= \frac{\partial^2 f}{\partial \zeta^2} + \frac{\partial^2 f}{\partial \eta\, \partial \zeta} + \frac{\partial^2 f}{\partial \zeta\, \partial \eta} + \frac{\partial^2 f}{\partial \eta^2} \\
&= \frac{\partial^2 f}{\partial \zeta^2} + 2\frac{\partial^2 f}{\partial \eta\, \partial \zeta} + \frac{\partial^2 f}{\partial \eta^2}
\end{aligned}$$

while,

$$\frac{1}{c^2}\frac{\partial^2 f}{\partial t^2} = \left(\frac{\partial}{\partial \zeta} - \frac{\partial}{\partial \eta}\right)^2 f = \frac{\partial^2 f}{\partial \zeta^2} - 2\frac{\partial^2 f}{\partial \eta\, \partial \zeta} + + \frac{\partial^2 f}{\partial \eta^2}.$$

Thus the wave equation in new variables is given by,

$$\frac{\partial^2 f}{\partial x^2} - \frac{1}{c^2}\frac{\partial^2 f}{\partial t^2} = \underline{4\frac{\partial^2 f}{\partial \eta\, \partial \zeta} = 0}.$$

Solving this equation is very easy. From the fact,

$$\frac{\partial^2 f}{\partial \eta\, \partial \zeta} = 0 \implies \frac{\partial}{\partial \eta}\left(\frac{\partial f}{\partial \zeta}\right) = 0$$

Since the $\eta$-derivative of $\frac{\partial f}{\partial \zeta}$ vanishes, it cannot be a function of $\eta$. Thus, Integrating both sides wrt $\eta$, we get,

$$\frac{\partial f}{\partial \zeta} = \tilde{g}(\zeta),$$

where $\tilde{g}(\zeta)$ is an arbitrary function of solely $\zeta$. Further integrating both sides wrt $\zeta$, we get,

$$\begin{aligned} f(\zeta, \eta) &= \underbrace{\int^{\zeta} d\zeta' \, \tilde{g}(\zeta')}_{=g(\zeta)} + h(\eta) \\ &= g(\zeta) + h(\eta) \\ &= g(x - ct) + h(x + ct). \end{aligned}$$

Notice since we did partial integration wrt $\zeta$, the integration "constant" can be a function of $\eta$, namely, $h(\eta)$.

3. **Bernoulli solution to the wave equation in one dimension:** Prove that the general solution (which is finite/regular everywhere in space and time) to the wave equation in one dimension (2) is given by,

$$\begin{aligned} f(x,t) &= \sum_{k \in \mathbb{R}^+} (A_k \sin kx + B_k \cos kx) \, (C_k \cos \omega t + D_k \sin \omega t) \, , \omega = k\, c. \\ &= \sum_{k \in \mathbb{R}^+} A_k \cos (kx - \omega t + \Delta_k) + C_k \cos(kx + \omega t + \tilde{\Delta}_k) \end{aligned}$$

(**Hint:** Take an ansatz of the form, $f(x,t) = X(x)T(t)$, where $X(x)$ and $T(t)$ are purely functions of $x$ and $t$ respectively and show that after plugging this ansatz in the wave equation (2), you get an equation of the form, $X''/X = \ddot{T}/T$ where dot and prime are respectively the time and space derivatives. Now this equation has a function purely of $x$ on one side and purely a function of $t$ on the right side, and hence they can be equal if and only if they are individually equal to a constant. This method is called *Separation of variables method* to solve partial differential equations by converting them into ordinary differential equations.)(5 points)

**Solution:**

We plug in an ansatz of the form,

$$f(x,t) = X(x)T(t)$$

in the wave equation, where $X(x)$ is purely a function of $x$, while $T(t)$ is purely a function of time, $t$. We get,

$$\frac{d^2X}{dx^2}T - X\frac{1}{c^2}\frac{d^2T}{dt^2} = 0 \implies \frac{d^2X}{dx^2}T = X\frac{1}{c^2}\frac{d^2T}{dt^2}$$

Dividing both sides by $X\,T$ we get,

$$\frac{1}{X}\frac{d^2X}{dx^2} = \frac{1}{c^2 T}\frac{d^2T}{dt^2}.$$

Now the two sides of this equation are functions of two different variables, and hence they can only be equal if they are both equal to some constant, say, $\lambda$.

$$\frac{1}{X}\frac{d^2X}{dx^2} = \frac{1}{c^2 T}\frac{d^2T}{dt^2} = \lambda.$$

Now we consider three cases depending on the sign or lambda.
Case I: $\lambda = \kappa^2 > 0$

$$\frac{d^2X}{dx^2} - \kappa^2 X = 0, \quad \frac{d^2T}{dt^2} - \kappa^2 c^2 T = 0$$

Integrating both sides we get the general solutions,

$$X = Ae^{\kappa x} + Be^{-\kappa x}, \quad T = Ce^{\kappa c\, t} + De^{-\kappa c\, t}$$

However we do not want solutions which blow up or decay away, and the above solutions do diverge or decay at $x, t = \pm\infty$, so we abandon the case when $\lambda > 0$.
Case II: $\lambda = 0$

$$\frac{d^2X}{dx^2} = 0, \qquad \frac{d^2T}{dt^2} = 0$$

which can be solved easily to

$$X = Ax + B, \quad T = Ct + D.$$

Again we discard these solutions because they grow unbounded in space and time. (The constants $B$ and $D$ are allowed but are uninteresting).
Case III: $\lambda = -k^2 < 0$
In this case, the equations are that of Simple harmonic motion,

$$\frac{d^2X}{dx^2} + k^2 x = \frac{d^2T}{dt^2} + k^2c^2T = 0$$

and we can write down the solutions,

$$X = A\cos kx + B\sin kx, \quad T = C\cos\omega t + D\sin\omega t, \quad \omega = kc.$$

So the solution is then,

$$f = XT = (A\cos kx + B\sin kx,)\,(C\cos\omega t + D\sin\omega t)\,, \quad \omega = kc.$$

This is finite everywhere including when $x, t \to \pm\infty$. However there is no restriction on the value of $k$ yet, as long as it is real, the solution with hold. So then the most general solution would be a linear superposition (sum) of solutions given by allowed values of $k$,

$$f(x,t) = \sum_{k\in\mathbb{R}} (A_k\cos kx + B_k\sin kx,)\,(C_k\cos\omega t + D_k\sin\omega t)\,, \quad \omega = kc.$$

4. Show that the Fresnel's equations for off-plane polarized EM wave is,

$$\frac{|\tilde{\mathbf{E}}_{0R}|}{|\tilde{\mathbf{E}}_{0I}|} = \frac{1-\alpha\beta}{1+\alpha\beta}, \quad \frac{|\tilde{\mathbf{E}}_{0T}|}{|\tilde{\mathbf{E}}_{0I}|} = \frac{2}{1+\alpha\beta}$$

where $\alpha$ and $\beta$ have been defined in the class. Draw the diagram. (Hint: For concreteness consider the same diagram as in the class, i.e. the interface is the $y$-axis running from up to down and the normal to the interface between the media is the $z$-axis, the left half i.e. the negative $z$-axis is in media 1 while the positive $z$-axis is in the 2nd media. The electric field is perpendicular to the plane of incidence i..e. the $yz$-plane, i.e. entirely along $x$-axis, $\mathbf{E}_{I,R,T} = E^1_{I,R,T}\hat{\mathbf{x}}$). (10 points)

**Solution:**

Since the electric field is "off-plane" is oscillating in a direction perpendicular to the $yz$-plane, we can take,

$$\mathbf{E}_{I,R,T} = \tilde{E}^1_{I,R,T}\hat{\mathbf{x}}$$

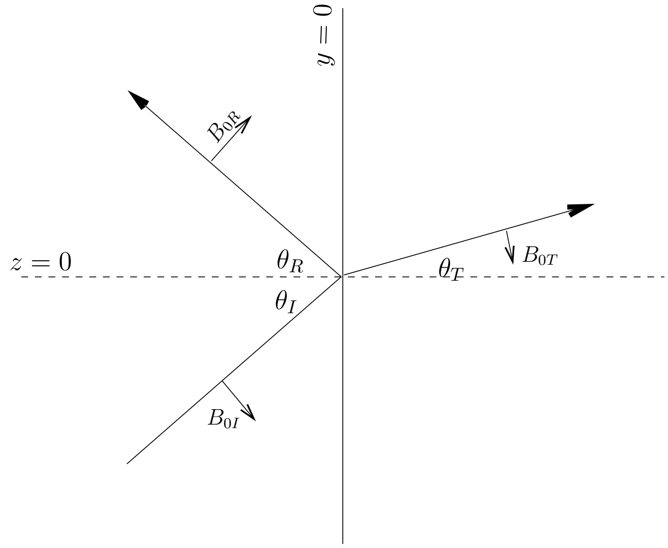


Figure 2: EM wave incident on an interface with off-plane electric polarization

i.e. the electric field is entirely tangential to the interface, $y = 0$. The first boundary condition we have for electric fields is that their tangential components must remain continuous, i.e. in this case the whole field,

$$\tilde{E}^1_{0I} + \tilde{E}^1_{0R} = \tilde{E}^1_{0T}. \tag{3}$$

The normal component of the electric displacement field is continuous as well, however for off-plane electric field polarization, this equation is zero equals to zero and we can forget it. The magnetic fields (incident, reflected, transmitted) however has both tangential and normal components. Continuity of tangential component of the auxiliary field, $H$ gives,

$$\frac{1}{\mu_1}\left(\tilde{B}^2_{0I} + \tilde{B}^2_{0R}\right) = \frac{1}{\mu_2}\tilde{B}^2_{0T}, \tag{4}$$

where the superscript, 2 means it is the $y$-component. From the figure, the $y$-components are (note that the up direction is the positive y-axis)

$$\tilde{B}^2_{0I} = -\left|\tilde{\mathbf{B}}_{0I}\right|\cos\theta_I, \quad \tilde{B}^2_{0R} = \left|\tilde{\mathbf{B}}_{0R}\right|\cos\theta_R, \quad \tilde{B}^2_{0T} = -\left|\tilde{\mathbf{B}}_{0T}\right|\cos\theta_T.$$

Also, recall,

$$\left|\tilde{\mathbf{B}}_{0I,R}\right| = \frac{1}{v_1}\left|\tilde{\mathbf{E}}_{0I,R}\right| = \frac{\tilde{E}^1_{0I,R}}{v_1},$$

while,

$$\left|\tilde{\mathbf{B}}_{0T}\right| = \frac{1}{v_2}\left|\tilde{\mathbf{E}}_{0T}\right| = \frac{\tilde{E}^1_{0T}}{v_2},$$

so Eq. (4) becomes,

$$\frac{1}{\mu_1}\left(-\frac{\tilde{E}^1_{0I}}{v_1}\cos\theta_I + \frac{\tilde{E}^1_{0R}}{v_1}\cos\theta_R\right) = -\frac{1}{\mu_2}\frac{\tilde{E}^1_{0T}}{v_2}\cos\theta_T,$$

or,

$$\begin{aligned}\tilde{E}^1_{0I} - \tilde{E}^1_{0R} &= \left(\frac{\mu_1 v_1}{\mu_2 v_2}\right)\left(\frac{\cos\theta_T}{\cos\theta_I}\right)\tilde{E}^1_{0T} \\ &= \alpha\beta\,\tilde{E}^1_{0T}.\end{aligned} \tag{5}$$

From (3) and (5), we get the Fresnel equations for the off-plane electric polarization,

$$\frac{|\tilde{\mathbf{E}}_{0R}|}{|\tilde{\mathbf{E}}_{0I}|} = \frac{1-\alpha\beta}{1+\alpha\beta}, \quad \frac{|\tilde{\mathbf{E}}_{0T}|}{|\tilde{\mathbf{E}}_{0I}|} = \frac{2}{1+\alpha\beta}.$$

5. In class I talked about EM waves which are not of infinite extent but are confined in a cavity or a pipe made out of a conducting walls (a waveguide). I mentioned that the boundary conditions for EM waves in the interior of such waveguides are,

$$\mathbf{E}^{\parallel} = 0, \qquad B^{\perp} = 0$$

at the inner boundary of the waveguide. Show that there are two other boundary conditions which are,

$$E^{\perp} = \frac{\sigma}{\epsilon_0}, \mathbf{B}^{\parallel} = \mu_0 \hat{\mathbf{n}} \times \mathbf{K},$$

where $\sigma$ and $\mathbf{K}$ are induced free surface charge density at the conducting walls of the wave-guide.(10 points)

**Solution:**
The conducting walls of the cavity with develop induced charges and let their surface density be, $\sigma$. The normal component of the field inside the cavity very close to the wall, $E^{\perp}$(taken to be directed inwards) can be computed by using the discontinuity of electric field across a surface charge. Inside a conductor, electric field is zero, so the discontinuity in the normal component of electric field (pointing inwards) inside the cavity near the wall is,

$$E^{\perp} = \frac{\sigma}{\epsilon_0}$$

Similarly, the discontinuity of the tangential component of the auxiliary field inside the cavity near the wall gives,

$$H^{\parallel}_{near\, wall} - H^{\parallel}_{inside\, wall} = K$$

where $K$ is the surface density of current due to flow of induced charges. Now in the interior of the walls, the magnetic field vanishes just as the electric field does, so

$$H^{\parallel}_{inside\, wall} = 0.$$

Thus we get,

$$H^{\parallel}_{near\, wall} = K$$

or,

$$\mathbf{B}^{\parallel} = \mu_0 \hat{\mathbf{n}} \times \mathbf{K}.$$

6. **Derivation of Biot-Savart law for steady current configurations:** In this problem you will derive the very well known Biot-Savart law for the magnetic field produced by a steady current distribution using (**??**). (Hint: For a steady current distribution, there is always a fixed/time-independent current density at a given location in the charge/current distribution) (5 points)

**Solution:**

From the result of the last problem, the expression for the magnetic field due to an infinitesimal moving charge,

$$\begin{aligned} d\mathbf{B}(\mathbf{x},t) &= \frac{1}{4\pi\epsilon_0 c^2}\, dq\, \mathbf{v}\times\frac{\mathbf{x}-\mathbf{x}_q(t)}{|\mathbf{x}-\mathbf{x}_q(t)|^3},\ \mathbf{x_q}(t)=\mathbf{v}t \\ &= \frac{1}{4\pi\epsilon_0 c^2}\rho(\mathbf{x}_q(t))\, d^3\mathbf{x}_q(t)\, \mathbf{v}\times\frac{\mathbf{x}-\mathbf{x}_q(t)}{|\mathbf{x}-\mathbf{x}_q(t)|^3}\ \mathbf{x_q}(t)=\mathbf{v}t \end{aligned}$$

But in the case of steady current at a given location, say $\mathbf{x}'$ there is always the same/constant charge density/current density, hence instead of adding up vectorially magnetic fields due to each charge element which is moving around, we can instead add the magnetic field due to the current element located at a fixed location,

$$\begin{aligned} \mathbf{B}(\mathbf{x},t) &= \sum_q \frac{1}{4\pi\epsilon_0 c^2}\rho(\mathbf{x}_q(t))\, d^3\mathbf{x}_q(t)\, \mathbf{v}\times\frac{\mathbf{x}-\mathbf{x}_q(t)}{|\mathbf{x}-\mathbf{x}_q(t)|^3}\ \mathbf{x_q}(t)=\mathbf{v} \\ &= \frac{1}{4\pi\epsilon_0 c^2}\int \rho(\mathbf{x}')\, d^3\mathbf{x}'\, \mathbf{v}(\mathbf{x}')\times\frac{\mathbf{x}-\mathbf{x}'}{|\mathbf{x}-\mathbf{x}'|^3} \\ &= \frac{1}{4\pi\epsilon_0 c^2}\int d^3\mathbf{x}'\mathbf{j}(\mathbf{x}')\times\frac{\mathbf{x}-\mathbf{x}'}{|\mathbf{x}-\mathbf{x}'|^3}, \end{aligned}$$

which is the Biot-Savart law (after we substitute, $\mu_0=\frac{1}{\epsilon_0 c^2}$).