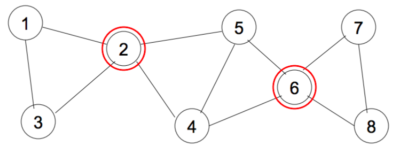
Problem Set 4

1. For a digraph G, $rev(G)$ is the graph obtained by reversing the arcs of G and the strong component graph $scc(G)$ is obtained by contracting the arcs in the strongly connected components of G. Prove that $scc(rev(G)) = rev(scc(G))$ for every directed graph G. Prove that for every directed graph G, the strong component graph $scc(G)$ is acyclic.

2. A vertex v in a connected undirected graph G is called a *cut vertex* or *articulation point* if the subgraph $G - v$ (obtained by removing v from G) is disconnected.

   Articulation points – Example

   (a) Describe a linear time (i.e., $\mathcal{O}(n+m)$ time) algorithm that determines, given an undirected n-vertex m-edge graph G and a vertex v, whether v is a cut vertex in G. What is the running time to find all cut vertices by trying your algorithm for each vertex?

   (b) A vertex u is a cut vertex if and only if in a DFS tree T, either it is the root and has more than one child or one of its proper descendant does not have an edge to a proper ancestor of u. (A vertex v is a proper descendant of u if v belongs to the subtree of T rooted at u and $v \neq u$. A vertex w is a proper ancestor of u, if $w \neq v$ and belongs to the unique path (in T) from u to the root of T).

   (c) Design a linear time algorithm to output all the cut vertices in G.

3. An edge e in a connected undirected graph G is called a bridge (or a cut edge) if the subgraph $G - e$ (obtained by removing e from G) is disconnected. Describe a linear-time algorithm to identify every bridge in G.

4. Let G be a connected graph with distinct edge weights.

   (a) Prove that for any cycle in G, the minimum spanning tree of G excludes the maximum-weight edge in that cycle.

   (b) Prove or disprove: The minimum spanning tree of G includes the minimum-weight edge in every cycle in G.

5. Describe and analyze an algorithm to compute the maximum-weight spanning tree of a given edge-weighted graph.

6. A *feedback edge set* of an undirected graph G is a subset F of the edges such that every cycle in G contains at least one edge in F. In other words, removing every edge in F makes the graph G acyclic. Describe and analyze a fast algorithm to compute the minimum-weight feedback edge set of a given edge-weighted graph.