MA 1140: Lecture 5
Basis and Dimension of Vector Spaces, and Linear Transformations

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Linearly independent set has less vectors than spanning set

Theorem

Suppose $V = \mathrm{Span}\{v_1, v_2, \ldots, v_n\}$, *and* $\{u_1, u_2, \ldots, u_m\}$ *is a linearly independent subset of* V. *Then* $m \leq n$.

Proof. If possible, let $n < m$. Note that $u_i \neq 0$. So, by renaming the vectors $v_1, \ldots, v_n$, we have $\{u_1, v_2, v_3, \ldots, v_n\}$ spans V.
In the 2nd step, since $u_2 \in V = \mathrm{Span}\{u_1, v_2, v_3, \ldots, v_n\}$,

$$u_2 = b_1 u_1 + b_2 v_2 + b_3 v_3 + \cdots + b_n v_n \quad \text{for some } b_i \in \mathbb{R}.$$

Then at least one of $\{b_2, \ldots, b_n\}$ is non-zero.
Hence, if necessary, by renaming the vectors $v_2, \ldots, v_n$, we have that $\{u_1, u_2, v_3, \ldots, v_n\}$ spans V.
Continuing in this way, after n steps, we obtain that $\{u_1, u_2, \ldots, u_n\}$ spans V. Hence

$$u_{n+1} \in V = \mathrm{Span}\{u_1, u_2, \ldots, u_n\}.$$

Therefore $\{u_1, u_2, \ldots, u_{n+1}\}$ is linearly dependent, which is a contradiction.

Any two bases of V have the same number of elements

Corollary

If V is a finite dimensional vector space, then any two bases of V have the same number of elements.

Proof. Since V is finite dimensional, it has a finite basis $\{v_1, \ldots, v_n\}$.
If $\{u_1, \ldots, u_m\}$ is another basis of V, then by the last theorem, $m \leq n$.
By the same argument, $n \leq m$. Thus $m = n$. □

Definition

The corollary allows us to define the **dimension** of a finite dimensional vector space as the number of elements in a basis.

We denote the dimension of V by $\dim(V)$.

Consequences of the fact that a linearly independent set has less or equal number of vectors than a spanning set

Corollary

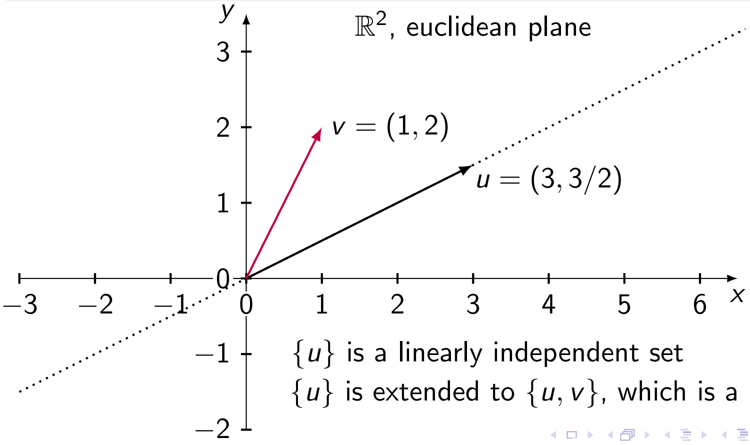
Let V be a finite dimensional vector space, $d = \dim(V)$. Then

1. *any subset of V containing more than d vectors is linearly dependent.*
2. *A subset of V containing fewer than d vectors cannot span V.*

A linearly independent set can be extended to a basis

Theorem

Let W be a subspace of a finite dimensional vector space V, and S be a linearly independent subset of W. Then S is finite, and it is part of a (finite) basis of W.



$\{u\}$ is a linearly independent set

$\{u\}$ is extended to $\{u, v\}$, which is a basis of $\mathbb{R}^2$

A linearly independent set can be extended to a basis

Theorem

Let W be a subspace of a finite dimensional vector space V, and S be a linearly independent subset of W. Then S is finite, and it is part of a (finite) basis of W.

Proof. Since S is also a linearly independent subset of V, S contains at most $\dim(V)$ elements. So S is finite.
If S spans W, then S is a basis of W, and we are done. If $\mathrm{Span}(S) \neq W$, then $\exists\ v_1 \in W \setminus \mathrm{Span}(S)$. Hence, $S_1 := S \cup \{v_1\}$ is linearly independent. If $\mathrm{Span}(S_1) = W$, then we are done. Otherwise, if $\mathrm{Span}(S_1) \neq W$, there is $v_2 \in W \setminus \mathrm{Span}(S_1)$, and hence $S_2 := S \cup \{v_1, v_2\}$ is linearly independent.
This process stops after some finite steps because at most $\dim(V)$ linearly independent vectors can be there in W.
So finally we obtain a set $S \cup \{v_1, v_2, \ldots, v_m\} \subset W$ which is linearly independent and spans W, i.e., it forms a basis of W.

A proper subspace has less dimension

Let V be a finite dimensional vector space.

Corollary

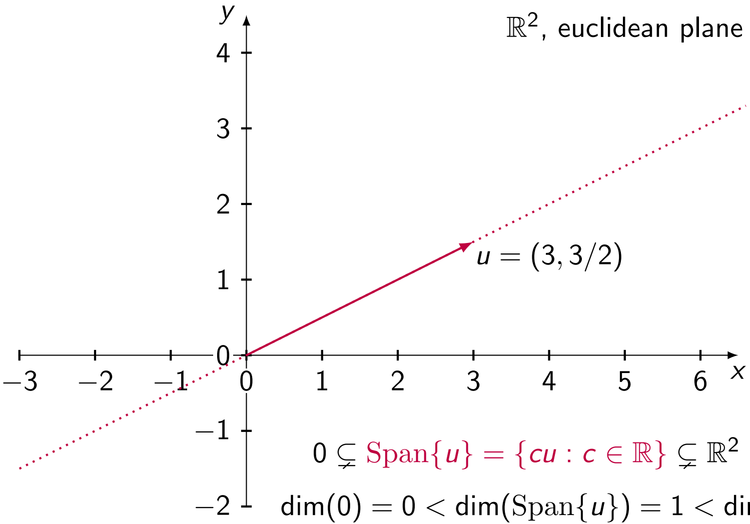
In V, every linearly independent set of vectors is part of a basis.

Corollary

A subspace W of V is PROPER if and only if $\dim(W) < \dim(V)$.

The 'if' part is trivial, because if $W = V$, then $\dim(W) = \dim(V)$.
Proof of 'only if' part. Let $W \subsetneq V$. If $W = 0$, then we are done.
Thus we may assume that $\exists\ u \neq 0$ in W.
Then $\{u\}$ can be extended to a finite basis (say S) of W. So, in particular, W is finite dimensional.
Since $\mathrm{Span}(S) = W \subsetneq V$, there is a vector $v \in V \smallsetminus \mathrm{Span}(S)$.
Then $S \cup \{v\}$ is a linearly independent subset of V.
Hence $S \cup \{v\}$ can be extended to a basis of V.
Therefore $\dim(W) < \dim(V)$.

Example: Proper subspaces of $\mathbb{R}^2$

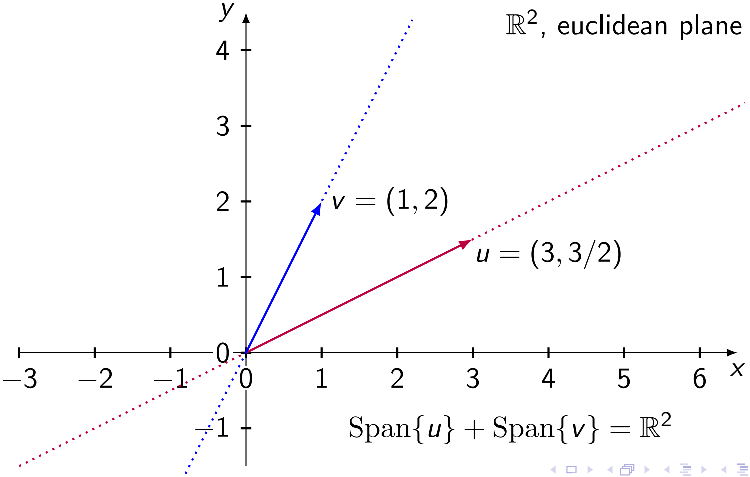


$$0 \subsetneq \operatorname{Span}\{u\} = \{cu : c \in \mathbb{R}\} \subsetneq \mathbb{R}^2$$

$$\dim(0) = 0 < \dim(\operatorname{Span}\{u\}) = 1 < \dim(\mathbb{R}^2) = 2$$

Definition

Let W_1 and W_2 be two subspaces of V. Then
$W_1 + W_2 := \{w_1 + w_2 : w_i \in W_i\}$.



Sum of two subspaces, and its dimension

Theorem

Let W_1 and W_2 be finite dimensional subspaces of V. Then

$$W_1 + W_2 := \{w_1 + w_2 : w_i \in W_i\}$$

is a finite dimensional subspace of V, and

$$\dim(W_1) + \dim(W_2) = \dim(W_1 \cap W_2) + \dim(W_1 + W_2).$$

Proof. Since $W_1 \cap W_2 \subseteq W_1$, it follows that $W_1 \cap W_2$ has a finite basis $\{u_1, \ldots, u_r\}$, which can be extended to a basis

$$\{u_1, \ldots, u_r, v_1, \ldots, v_m\} \text{ of } W_1$$

and a basis

$$\{u_1, \ldots, u_r, w_1, \ldots, w_n\} \text{ of } W_2.$$

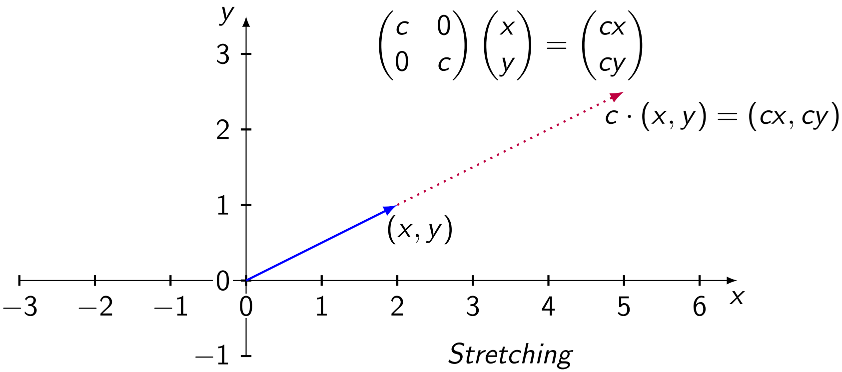
Show that $\{u_1, \ldots, u_r, v_1, \ldots, v_m, w_1, \ldots, w_n\}$ is a basis of $W_1 + W_2$.

A 'Linear Transformation' is nothing but a map between vector spaces. Let us start with some well known maps:

$$T : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$$

$$\begin{pmatrix} x \\ y \end{pmatrix} \mapsto c \begin{pmatrix} x \\ y \end{pmatrix} \quad \text{where } c \in \mathbb{R}$$

$$\begin{pmatrix} c & 0 \\ 0 & c \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} cx \\ cy \end{pmatrix}$$



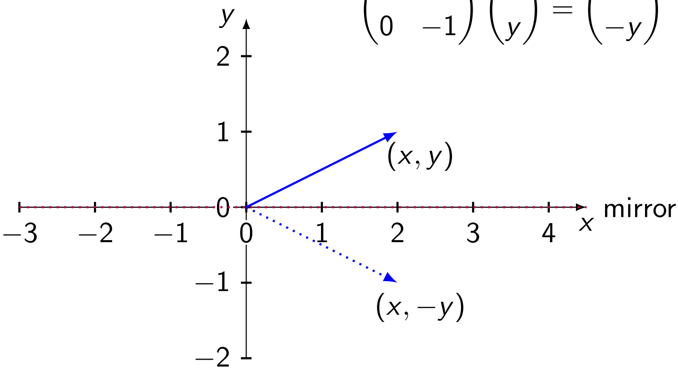
Stretching

Reflection with x-axis as mirror

$$T : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$$

$$\begin{pmatrix} x \\ y \end{pmatrix} \mapsto \begin{pmatrix} x \\ -y \end{pmatrix}$$

$$\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x \\ -y \end{pmatrix}$$



Projection on the x-axis

$$S : \mathbb{R}^2 \longrightarrow \mathbb{R}$$

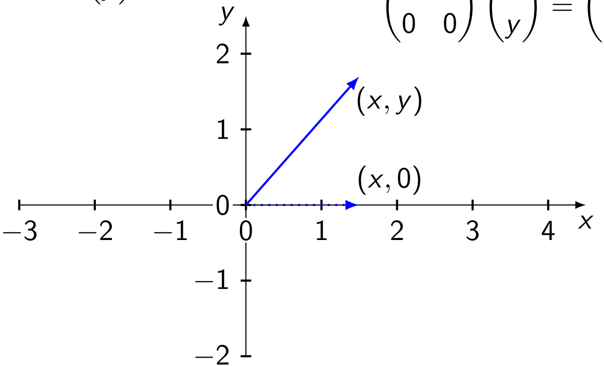
$$\begin{pmatrix} x \\ y \end{pmatrix} \mapsto x$$

$$\begin{bmatrix} 1 & 0 \end{bmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = x$$

$$T : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$$

$$\begin{pmatrix} x \\ y \end{pmatrix} \mapsto \begin{pmatrix} x \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x \\ 0 \end{pmatrix}$$



projection on x-axis

Linear transformation, or linear map

Definition

A transformation (or map) between vector spaces which satisfies the rule of linearity is called linear transformation (or linear map).

More precisely, let V and W be vector spaces over $\mathbb{R}$. A linear transformation $T : V \to W$ is a function such that

$$T(c_1 v_1 + c_2 v_2) = c_1 T(v_1) + c_2 T(v_2)$$

for all $c_1, c_2 \in \mathbb{R}$ and $v_1, v_2 \in V$.

Example

Let A be an $m \times n$ matrix over $\mathbb{R}$. Then the map $T : \mathbb{R}^n \to \mathbb{R}^m$ defined by $T(X) := AX$ for all $X \in \mathbb{R}^n$ is a linear transformation.

Proof. $T(X+Y) = A(X+Y) = AX + AY = T(X) + T(Y)$ and $T(cX) = A(cX) = c(AX) = cT(X)$.

Thank You!