# Matrices, Linear equations and solvability

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February 13, 2019

# Equivalent systems of linear equations

- Consider two **equivalent systems:**

$$
\begin{aligned}
A_{11}x_1 + A_{12}x_2 + \cdots + A_{1n}x_n &= b_1 \\
A_{21}x_1 + A_{22}x_2 + \cdots + A_{2n}x_n &= b_2 \\
&\vdots \\
A_{m1}x_1 + A_{m2}x_2 + \cdots + A_{mn}x_n &= b_m
\end{aligned}
$$

and

$$
\begin{aligned}
B_{11}x_1 + B_{12}x_2 + \cdots + B_{1n}x_n &= b'_1 \\
B_{21}x_1 + B_{22}x_2 + \cdots + B_{2n}x_n &= b'_2 \\
&\vdots \\
B_{m1}x_1 + B_{m2}x_2 + \cdots + B_{mn}x_n &= b'_m
\end{aligned}
$$

- That is, every equation in the 2nd system is a linear combination of the equations in the 1st system, vice versa.
- Then they have the same set of solutions.

# Writing a system of linear equations by matrices

- Consider a system:

$$\begin{aligned} A_{11}x_1 + A_{12}x_2 + \cdots + A_{1n}x_n &= b_1 \\ A_{21}x_1 + A_{22}x_2 + \cdots + A_{2n}x_n &= b_2 \\ &\vdots \\ A_{m1}x_1 + A_{m2}x_2 + \cdots + A_{mn}x_n &= b_m \end{aligned}$$

- We write the system by matrices as follows: $Ax = b$, where

$$A = \begin{pmatrix} A_{11} & A_{12} & \cdots & A_{1n} \\ A_{21} & A_{22} & \cdots & A_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ A_{m1} & A_{m2} & \cdots & A_{mn} \end{pmatrix}, \; x = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} \text{ and } b = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{pmatrix}.$$

- For a non-homogeneous system, we apply **elementary row operations** (???) on the augmented matrix $(A \mid b)$.

# A homogeneous system of linear equations

- For a homogeneous system:

$$
\begin{aligned}
A_{11}x_1 + A_{12}x_2 + \cdots + A_{1n}x_n &= 0 \\
A_{21}x_1 + A_{22}x_2 + \cdots + A_{2n}x_n &= 0 \\
&\vdots \\
A_{m1}x_1 + A_{m2}x_2 + \cdots + A_{mn}x_n &= 0
\end{aligned}
$$

- i.e., when the system is $Ax = 0$, then
- It is enough to consider the coefficient matrix $A$.
- So we apply elementary row operations on $A$.

# The elementary row operations (total three)

1. Interchange of two rows of $A$, say $r$th and $s$th rows.
2. Multiplication of one row of $A$ by a non-zero scalar $c \in \mathbb{R}$.
3. Replacement of the $r$th row of $A$ by

$$(r\text{th row} + c \cdot s\text{th row}),$$

where $c \in \mathbb{R}$ and $r \neq s$.

*All the above three* **operations are invertible**, and each has inverse operation of the **same type**.

4. The 1st one is it's own inverse.
5. For the 2nd one, inverse operation is 'multiplication of that row of $A$ by $1/c \in \mathbb{R}$'.
6. For the 3rd one, inverse operation is 'replacement of the $r$th row of $A$ by $(r\text{th row} - c \cdot s\text{th row})$'.

# Row equivalence of matrices

- Let $A$ and $B$ be two $m \times n$ matrices over $\mathbb{R}$.
- We say that $B$ **is row equivalent to** $A$ if $B$ can be obtained from $A$ by a finite sequence of elementary row operations, i.e.,

$$B = e_r \cdots e_2 e_1(A),$$

where $e_i$ are some elementary row operations.
- 'Row equivalence' is an '**equivalence relation**':
- 'Row equivalence' is reflexive, i.e., $A$ is row equivalent to $A$.
- 'Row equivalence' is symmetric, i.e.,

$$B = e_r \cdots e_2 e_1(A) \quad \Longrightarrow \quad A = (e_1)^{-1}(e_2)^{-1} \cdots (e_r)^{-1}(B).$$

In this case, we say that $A$ and $B$ are row equivalent.
- 'Row equivalence' is transitive, i.e., if $B$ is row equivalent to $A$ and $C$ is row equivalent to $B$, then $C$ is row equivalent to $A$.

# Row equivalence of two homogeneous systems

- Among three elementary row operations, considering row operation of each type, we observe that
  $A$ and $B$ are row equivalent

  if and only if

  the corresponding homogeneous systems $Ax = 0$ and $Bx = 0$ are equivalent.
- In this case, both the systems have exactly the same solutions.

# Row reduced matrix

## Definition

An $m \times n$ matrix $A$ over $\mathbb{R}$ is called **row reduced** if
(1) the 1st non-zero entry in each non-zero row of $A$ is equal to 1;
(2) each column of $A$ which contains the leading non-zero entry of some row has all its other entries 0.

## Example

(i) $\begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{pmatrix}$ $\surd$ (ii) $\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}$ $\times$

(iii) $\begin{pmatrix} 0 & 2 & 1 \\ 1 & 0 & -3 \\ 0 & 0 & 0 \end{pmatrix}$ $\times$ (iv) $\begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & -3 \\ 0 & 0 & 0 \end{pmatrix}$ $\surd$

Here (ii) and (iii) are not row reduced matrices.

# Row reduced echelon matrix

## Definition

An $m \times n$ matrix $A$ over $\mathbb{R}$ is called **row reduced echelon** matrix if
(1) $A$ is row reduced;
(2) every zero row (?) of $A$ occurs below every non-zero row (?);
(3) if rows $1, \ldots, r$ are the non-zero rows, and if the leading non-zero entry of row $i$ occurs in column $k_i$ for $1 \leq i \leq r$, then $k_1 < k_2 < \cdots < k_r$.

In this case, $(i, k_i)$ are called the pivot positions, and $x_{k_i}$ are called the pivot variables.

## Example

(i) $\begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{pmatrix}$ $\checkmark$ (ii) $\begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & -3 \\ 0 & 0 & 0 \end{pmatrix}$ $\times$

The matrix in (ii) is row reduced, but NOT row reduced echelon.

# Every matrix is row equivalent to a row reduced echelon matrix

## Theorem

*Every $m \times n$ matrix over $\mathbb{R}$ is* **row equivalent** *to a row reduced echelon matrix.*

## Example

$$\begin{bmatrix} 0 & 1 & 1 & 5 \\ 4 & -6 & 0 & -2 \\ -2 & 7 & 2 & 9 \end{bmatrix} \xRightarrow{\textit{Row operations}} \begin{bmatrix} 1 & 0 & 0 & -2 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 6 \end{bmatrix}.$$

# Example: A matrix $\to$ Row reduced echelon matrix

$$\begin{bmatrix} 0 & 1 & 1 & 5 \\ 4 & -6 & 0 & -2 \\ -2 & 7 & 2 & 9 \end{bmatrix} \xrightarrow{R1 \leftrightarrow R2} \begin{bmatrix} 4 & -6 & 0 & -2 \\ 0 & 1 & 1 & 5 \\ -2 & 7 & 2 & 9 \end{bmatrix} \xrightarrow{R1 \to (1/4)R1}$$

$$\begin{bmatrix} 1 & -3/2 & 0 & -1/2 \\ 0 & 1 & 1 & 5 \\ -2 & 7 & 2 & 9 \end{bmatrix} \xrightarrow{R3 \to R3 + 2 \cdot R1} \begin{bmatrix} 1 & -3/2 & 0 & -1/2 \\ 0 & 1 & 1 & 5 \\ 0 & 4 & 2 & 8 \end{bmatrix}$$

$$\xrightarrow{R3 \to R3 - 4 \cdot R2} \begin{bmatrix} 1 & -3/2 & 0 & -1/2 \\ 0 & 1 & 1 & 5 \\ 0 & 0 & -2 & -12 \end{bmatrix} \xrightarrow{R3 \to (-1/2)R3}$$

$$\begin{bmatrix} 1 & -3/2 & 0 & -1/2 \\ 0 & 1 & 1 & 5 \\ 0 & 0 & 1 & 6 \end{bmatrix} \text{(Triangular system with pivot entries 1)}$$

$$\xrightarrow{R2 \to R2 - R3} \begin{bmatrix} 1 & -3/2 & 0 & -1/2 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 6 \end{bmatrix} \xrightarrow{R1 \to R1 + (3/2)R2} \begin{bmatrix} 1 & 0 & 0 & -2 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 6 \end{bmatrix}$$

So it is just combination of forward and backward eliminations.

# Solution of a system corresponding to a row reduced echelon matrix

## Example

$$\begin{bmatrix} 0 & 1 & 1 & 5 \\ 4 & -6 & 0 & -2 \\ -2 & 7 & 2 & 9 \end{bmatrix} \xRightarrow{Row\ operations} \begin{bmatrix} 1 & 0 & 0 & -2 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 6 \end{bmatrix}.$$

Considering the corresponding system, we have the solution $u = -2$, $v = -1$ and $w = 6$.

Consider the homogeneous system corr. to the coefficient matrix

$$\begin{bmatrix} 0 & 1 & -3 & 0 & 1/2 \\ 0 & 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \text{ (which is a row reduced echelon matrix).}$$

Here $(1,2)$ and $(2,4)$ are the pivot positions. So $x_2$ and $x_4$ are the pivot variables. The remaining variables are called free variables.

$$\begin{array}{ccccc} x_2 & -3x_3 & & +(1/2)x_5 & = & 0 \\ & & x_4 & +2x_5 & = & 0 \end{array}$$

which yields that

$$\begin{aligned} x_2 &= 3x_3 - (1/2)x_5 \\ x_4 &= -2x_5 \end{aligned}$$

The values of $x_1$, $x_3$ and $x_5$ can be chosen freely.

# Solution of a system corr. to a row reduced echelon matrix

- Consider $Ax = 0$, where $A$ is a row reduced echelon matrix.
- Let rows $1, \ldots, r$ be non-zero, and the leading non-zero entry of row $i$ occurs in column $k_i$.
- The system $Ax = 0$ then consists of $r$ non-trivial equations.
- The variables $x_{k_1}, x_{k_2}, \ldots, x_{k_r}$ are the pivot variables.
- Let $u_1, \ldots, u_{n-r}$ denote the remaining $n - r$ (free) variables.
- Then the $r$ non-trivial equations of $Ax = 0$ can be written as

$$x_{k_1} + \sum_{j=1}^{n-r} C_{1j} u_j = 0$$

$$\ldots$$

$$x_{k_r} + \sum_{j=1}^{n-r} C_{rj} u_j = 0$$

- We may assign any values to $u_1, \ldots, u_{n-r}$. Then $x_{k_1}, \ldots, x_{k_r}$ are determined uniquely by those assigned values.

# Solution to a homogeneous system (when $m < n$)

## Theorem

*Let $A$ be an $m \times n$ matrix over $\mathbb{R}$ with $m < n$. Then the homogeneous system $Ax = 0$ has a non-trivial solution. In fact (over $\mathbb{R}$) it has infinitely many solutions.*

## Proof.

The matrix $A$ is row equivalent to a row reduced echelon matrix $B$. Then $Ax = 0$ and $Bx = 0$ have the same solutions. If $r$ is number of non-zero rows, then $r \leq m < n$. The system $Bx = 0$ can be expressed as

$$x_{k_1} + \sum_{j=1}^{n-r} C_{1j} u_j = 0, \ \ldots, \ x_{k_r} + \sum_{j=1}^{n-r} C_{rj} u_j = 0$$

where $u_1, \ldots, u_{n-r}$ are the free variables. Now assign any values to $u_1, \ldots, u_{n-r}$ to get infinitely many solutions. $\square$

# Solution to a homogeneous system (when $m = n$)

**Theorem**

*Let $A$ be an $n \times n$ matrix over $\mathbb{R}$. Then $A$ is row equivalent to the $n \times n$ identity matrix if and only if the system $Ax = 0$ has only the trivial solution.*

**Proof.**

The matrix $A$ is row equivalent to a row reduced echelon matrix $B$. Then $Ax = 0$ and $Bx = 0$ have the same solutions. If $r$ is number of non-zero rows, then $r \leq n$. The system $Bx = 0$ can be expressed as

$$x_{k_1} + \sum_{j=1}^{n-r} C_{1j} u_j = 0, \ldots, x_{k_r} + \sum_{j=1}^{n-r} C_{rj} u_j = 0,$$

where $u_1, \ldots, u_{n-r}$ are the free variables. Hence it can be observed that $B = I_n$ is the identity matrix if and only if $r = n$ if and only if the system has the trivial solution. $\square$

# Solution to a non-homogeneous system $Ax = b$

- Consider the augmented matrix $(A \,|\, b)$ corr. to $Ax = b$.
- Apply elementary row operations on $(A \,|\, b)$ to get row reduced echelon form $(B \,|\, c)$.
- The systems $Ax = b$ and $Bx = c$ are equivalent, and hence they have the same solutions.
- Let $1, \ldots, r$ be the non-zero rows of $B$, and the leading non-zero entry of row $i$ occurs in column $k_i$.
- The variables $x_{k_1}, x_{k_2}, \ldots, x_{k_r}$ are the pivot variables.
- Let $u_1, \ldots, u_{n-r}$ denote the remaining $n - r$ (free) variables.
- Then the system $Bx = c$ can be written as

$$x_{k_1} + \sum_{j=1}^{n-r} C_{1j} u_j = c_1, \ \ldots, \ x_{k_r} + \sum_{j=1}^{n-r} C_{rj} u_j = c_r$$

$$0 = c_{r+1}$$

$$\cdots$$

$$0 = c_m$$

# Solution to a non-homogeneous system $Ax = b$ contd...

- The systems $Ax = b$ and $Bx = c$ are equivalent, and hence they have the same solutions.
- The system $Bx = c$ can be written as

$$x_{k_1} + \sum_{j=1}^{n-r} C_{1j} u_j = c_1, \ \ldots, \ x_{k_r} + \sum_{j=1}^{n-r} C_{rj} u_j = c_r$$

$$0 = c_{r+1}$$

$$\cdots$$

$$0 = c_m$$

- Thus the system $Ax = b$ (equivalenly, $Bx = c$) has a solution if and only if $c_{r+1} = \cdots = c_m = 0$. IN THIS CASE:
- $r = n$ if and only if the system has a unique solution.
- $r < n$ if and only if the system has infinitely many solutions.

# Example: Solution to a non-homogeneous system

- Consider a system $Ax = b$, where $A = \begin{pmatrix} 1 & -2 & 1 \\ 2 & 1 & 1 \\ 0 & 5 & -1 \end{pmatrix}$.
- The corr. augmented matrix is $(A\,|\,b) = \begin{pmatrix} 1 & -2 & 1 & b_1 \\ 2 & 1 & 1 & b_2 \\ 0 & 5 & -1 & b_3 \end{pmatrix}$.
- Applying elementary row operations on $(A\,|\,b)$, we get
$$\begin{pmatrix} 1 & 0 & \frac{3}{5} & \frac{1}{5}(b_1 + 2b_2) \\ 0 & 1 & -\frac{1}{5} & \frac{1}{5}(b_2 - 2b_1) \\ 0 & 0 & 0 & (b_3 - b_2 + 2b_1) \end{pmatrix}.$$
- The system $Ax = b$ has a solution if and only if $b_3 - b_2 + 2b_1 = 0$. In this CASE,
- $x_1 = -\frac{3}{5}x_3 + \frac{1}{5}(b_1 + 2b_2)$ and $x_2 = \frac{1}{5}x_3 + \frac{1}{5}(b_2 - 2b_1)$.
- Assign any value to $x_3$, and compute $x_1, x_2$.

# Elementary matrices

## Definition

An $m \times m$ matrix is called an **elementary matrix** if it can be obtained from the $m \times m$ identity matrix by applying a SINGLE elementary row operation.

## Example

(i) $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$. (obtained by applying 1st type elementary row oper.).

(ii) $\begin{pmatrix} c & 0 \\ 0 & 1 \end{pmatrix}$, $c \neq 0$. (iii) $\begin{pmatrix} 1 & 0 \\ 0 & c \end{pmatrix}$, $c \neq 0$. (2nd type).

(iv) $\begin{pmatrix} 1 & c \\ 0 & 1 \end{pmatrix}$, $c \in \mathbb{R}$. (v) $\begin{pmatrix} 1 & 0 \\ c & 1 \end{pmatrix}$, $c \in \mathbb{R}$. (3rd type).

These are all the $2 \times 2$ elementary matrices.

# Elementary matrices vs elementary row operation

- Consider a matrix $A = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix}$.
- $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix} = \begin{pmatrix} 4 & 5 & 6 \\ 1 & 2 & 3 \end{pmatrix}$. $(R1 \leftrightarrow R2.)$
- $\begin{pmatrix} c & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix} = \begin{pmatrix} c & 2c & 3c \\ 4 & 5 & 6 \end{pmatrix}$. $(R1 \to c \cdot R1.)$
- $\begin{pmatrix} 1 & 0 \\ c & 1 \end{pmatrix} \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 3 \\ 4 + c & 5 + 2c & 6 + 3c \end{pmatrix}$.
  $$(R2 \to R2 + c \cdot R1.)$$
- So applying an *elementary row operation* on a matrix is same as left multiplying by the corresponding *elementary matrix*.

# Theorem on elementary matrices and elementary row operation

## Theorem

*Let $e$ be an elementary row operation. Let $E$ be the corresponding $m \times m$ elementary matrix, i.e., $E = e(I_m)$, where $I_m$ is the $m \times m$ identity matrix. Then, for every $m \times n$ matrix $A$,*

$$EA = e(A).$$

## Corollary

*Let $A$ and $B$ be two $m \times n$ matrices. Then*
*$A$ and $B$ are equivalent*

*if and only if*

*$B = PA$, where $P$ is a product of some $m \times m$ elementary matrices.*

# Elementary matrices are invertible

## Theorem

*Every elementary matrix is invertible.*

## Proof.

Let $E$ be an elementary matrix corresponding to the elementary row operation $e$, i.e., $E = e(I)$. Note that $e$ has an inverse operation, say $e'$. Set $E' := e'(I)$. Then

$$EE' = e(E') = e(e'(I)) = I \text{ and}$$
$$E'E = e'(E) = e'(e(I)) = I.$$

$\square$

# Invertible matrices

## Theorem

*Let $A$ be an $n \times n$ matrix. Then the following are equivalent:*
*(1) $A$ is invertible.*
*(2) $A$ is row equivalent to the $n \times n$ identity matrix.*
*(3) $A$ is a product of some elementary matrices.*

## Proof.

Let $A$ be row-equivalent to a row-reduced echelon matrix $B$. Then

$$B = E_k \cdots E_2 E_1 A \tag{1}$$

Since elementary matrices are invertible, we have

$$E_1^{-1} E_2^{-1} \cdots E_k^{-1} B = A. \tag{2}$$

Hence $A$ is invertible if and only if $B$ is invertible if and only if $B = I$ if and only if $E_1^{-1} E_2^{-1} \cdots E_k^{-1} = A$. $\square$

# Invertible matrices

**Theorem**

*Let $A$ be an $n \times n$ invertible matrix. If a sequence of elementary row operations reduces $A$ to the identity $I$, then that same sequence of operations when applied to $I$ yields $A^{-1}$.*

**Proof.**

Note that if $E_k \cdots E_2 E_1 A = I$, then $A^{-1} = (E_k \cdots E_2 E_1)$. $\square$

# Example: How to compute inverse of a matrix

Let $A = \begin{pmatrix} 2 & -1 \\ 1 & 3 \end{pmatrix}$. Want to compute $A^{-1}$. Consider

$$\begin{pmatrix} 2 & -1 & 1 & 0 \\ 1 & 3 & 0 & 1 \end{pmatrix} \xRightarrow{\frac{1}{2}R1} \begin{pmatrix} 1 & -\frac{1}{2} & \frac{1}{2} & 0 \\ 1 & 3 & 0 & 1 \end{pmatrix} \xRightarrow{R2 \to R2 - R1}$$

$$\begin{pmatrix} 1 & -\frac{1}{2} & \frac{1}{2} & 0 \\ 0 & \frac{7}{2} & -\frac{1}{2} & 1 \end{pmatrix} \xRightarrow{\frac{2}{7}R2}$$

$$\begin{pmatrix} 1 & -\frac{1}{2} & \frac{1}{2} & 0 \\ 0 & 1 & -\frac{1}{7} & \frac{2}{7} \end{pmatrix} \xRightarrow{R1 \to R1 + \frac{1}{2}R2} \begin{pmatrix} 1 & 0 & \frac{3}{7} & \frac{1}{7} \\ 0 & 1 & -\frac{1}{7} & \frac{2}{7} \end{pmatrix}$$

So $A^{-1} = \begin{pmatrix} \frac{3}{7} & \frac{1}{7} \\ -\frac{1}{7} & \frac{2}{7} \end{pmatrix}$.

Thank You!