EP 1027: Maxwell Equations and Electromagnetic waves

Supplementary material for lecture 3

March 30, 2019

1 Checking that $\mathbf{a}\cdot\mathbf{b}$ is a scalar

Under rotation of coordinate axes,

$$a'_i = O_{il}\, a_l, \quad b'_i = O_{im}\, b_m, \tag{1}$$

If we label $\mathbf{a}\cdot\mathbf{b}$ as ϕ. In a rotated frame the product is,

$$\begin{aligned}
\phi' &= \mathbf{a}'\cdot\mathbf{b}' \\
&= a'_i\, b'_i \\
&= \underbrace{O_{il}\, O_{im}}_{=\delta_{lm}}\, a_l\, b_m \\
&= \delta_{lm} a_l b_m \\
&= a_m\, b_m \\
&= \mathbf{a}\cdot\mathbf{b} \\
&= \phi.
\end{aligned}$$

Since this object remains unchanged under rotation of coordinate axes, i.e. $\phi' = \phi$, it is a scalar.

2 Checking that $\mathbf{a}\times\mathbf{b}$ is a vector

Under rotation of coordinate axes,

$$\mathbf{a}\times\mathbf{b} \to \mathbf{a}'\times\mathbf{b}'$$

In terms of index notation, the cross product in the rotated frame is,

$$\begin{aligned}
(\mathbf{a}'\times\mathbf{b}')_k &= \epsilon_{ijk}\, a'_i\, b'_j \\
&= \epsilon_{ijk}\, (O_{il}\, a_l)\, (O_{jm}\, b_m) \\
&= (\epsilon_{ijk}\, O_{il}\, O_{jm})\, a_l\, b_m.
\end{aligned} \tag{2}$$

Now one can write,

$$\epsilon_{ijk} = \epsilon_{ijn}\delta_{nk} = \epsilon_{ijn} O_{pn} O_{pk} \tag{3}$$

where we have used orthogonality of rotation matrix, O,

$$O_{np} O_{kp} = \delta_{nk}.$$

Substituting (3) back in (2), one then has,

$$\begin{aligned}
(\mathbf{a}' \times \mathbf{b}')_k &= (\epsilon_{ijk}\, O_{il}\, O_{jm})\, a_l\, b_m \\
&= (\epsilon_{ijn}\, O_{np}\, O_{kp}\, O_{il}\, O_{jm})\, a_l\, b_m \\
&= O_{kp} \left(\underbrace{\epsilon_{ijn}\, O_{il}\, O_{jm}\, O_{np}}_{=|O|\epsilon_{lmp}} \right) a_l\, b_m \\
&= O_{kp}\ (\epsilon_{lmp} a_l\, b_m) \\
&= O_{kp}\ (\boldsymbol{a} \times \boldsymbol{b})_p
\end{aligned}$$

This indeed looks like the transformation law of a vector,

$$V'_k = O_{kp}\, V_p.$$

3 Checking that $\boldsymbol{\nabla}$ is a vector under rotation of coordinate axes

We defined the del operator as,

$$\boldsymbol{\nabla} \equiv \hat{\mathbf{e}}_k\, \frac{\partial}{\partial x_k}$$

i.e. the k-th component of $\boldsymbol{\nabla}$is,

$$\boldsymbol{\nabla}_k = \frac{\partial}{\partial x_k}.$$

Now under rotation of coordinate axes the coordinates of the point x changes to x', and accordingly in the rotated (primed) coordinate system the del operator components are"

$$\boldsymbol{\nabla}'_k = \frac{\partial}{\partial x'_k}.$$

Using chain rule,

$$\boldsymbol{\nabla}'_k = \frac{\partial}{\partial x'_k} = \frac{\partial x_l}{\partial x'_k}\, \frac{\partial}{\partial x_l}. \tag{4}$$

To further process this expression we first recall that,

$$x'_k = O_{km} x_m$$

or in matrix notation,

$$x' = O.x,$$

inverting both sides of which we get,

$$x = O^{-1}.x.$$

In terms of components,

$$x_l = \left(O^{-1}\right)_{lm} x'_m = O_{ml}\, x'_m,$$

since O is orthogonal. Final taking derivative wrt x'_k of both sides,

$$\frac{\partial x_l}{\partial x'_k} = O_{ml}\ \underbrace{\frac{\partial x'_m}{\partial x'_k}}_{\delta_{mk}} = O_{ml}\delta_{mk} = O_{kl}.$$

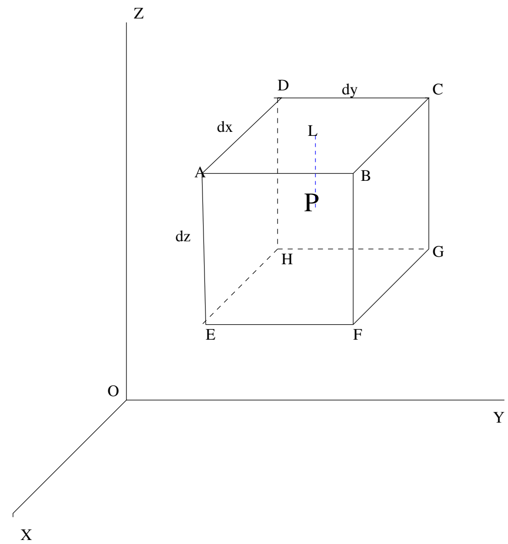

Figure 1: An elementary cuboid centered at P

Plugging this result back in (4), one has,

$$\begin{aligned}\boldsymbol{\nabla}'_k &= \frac{\partial x_l}{\partial x'_k}\,\frac{\partial}{\partial x_l}\\ &= O_{kl}\,\frac{\partial}{\partial x_l}\\ &= O_{kl}\,\boldsymbol{\nabla}_l.\end{aligned}$$

This looks exactly like the transformation law of a vector under rotation of coordinate axes, namely, $V'_k = O_{kl}\,V_l$.

4 Geometric meaning of gradient, divergence and curl

4.1 Gradient of a scalar field

Consider a point P in three dimensional euclidean space ($\mathbb{R}^3$) labeled by the coordinates $\mathbf{x} = (x, y, z)$ wrt to some set of Cartesian axes. Imagine an infinitesimal cuboid centered at P, with side lengths dx, dy, dz the vertices of which are denoted by A, B, C, D, E, F, G, H (refer to figure 1).

Now let's estimate the surface integral of a scalar field ϕ over the surface made up of the union of faces of the cuboid,

$$\begin{aligned}\oiint_S dS\,\hat{\mathbf{n}}\phi &= \sum_{\text{cuboid faces}} \iint dS\,\hat{\mathbf{n}}\phi\\ &= \iint_{ABCD} dS\,\hat{\mathbf{n}}\phi + \iint_{EFGH} dS\,\hat{\mathbf{n}}\phi + \iint_{CBFG} dS\,\hat{\mathbf{n}}\phi + \iint_{AEHD} dS\,\hat{\mathbf{n}}\phi + \iint_{ABFE} dS\,\hat{\mathbf{n}}\phi + \iint_{CGHD} dS\,\hat{\mathbf{n}}\phi\end{aligned}$$

Since the cuboid is infinitesimal, we can assume ϕ does not vary much over a single face and we can use the value of the scalar at the center of a face as an approximate constant value of the scalar over the entire face, e.g.

$$\phi(x)|_{x\in ABCD} \approx \phi(L),$$

where L is the center of the face $ABCD$. Then the surface integral of ϕ over the face $ABCD$ is,

$$\iint_{ABCD} dS\,\hat{\mathbf{n}}\phi \approx \phi(L)\iint_{ABCD} dS\,\hat{\mathbf{n}} \approx \phi(L)\,dx\,dy\,\hat{\mathbf{e}}_3.$$

In terms of coordinates, $\mathbf{x}_L = \left(x, y, z + \frac{dz}{2}\right)$,so,

$$\phi(L) = \phi\left(x, y, z + \frac{dz}{2}\right) \approx \phi(x, y, z) + \frac{\partial\phi}{\partial z}\frac{dz}{2}$$

where at the last step we have Taylor expanded to first order in z. Thus the surface integral of the scalar over the face $ABCD$ (with outward unit normal vector to the face $ABCD$, $\hat{\boldsymbol{n}}|_{ABCD} = \hat{\mathbf{z}}$) is,

$$\iint_{ABCD} dS\,\hat{\mathbf{n}}\phi \approx \left(\phi(x, y, z) + \frac{\partial\phi}{\partial z}\frac{dz}{2}\right)\,dx\,dy\,\hat{\mathbf{z}}.$$

Likewise one can show that the surface integral of the scalar over the face $EFGH$ to first order in dx, dy, dz is,

$$\iint_{EFGH} dS\,\hat{\mathbf{n}}\phi \approx \left(\phi(x, y, z) - \frac{\partial\phi}{\partial z}\frac{dz}{2}\right)\,dx\,dy\,\left(-\hat{\mathbf{z}}\right)$$

since $\hat{\boldsymbol{n}}|_{EFGH} = -\hat{\mathbf{z}}$. Thus one has the sum of the surface integrals over the faces $ABCD$ and $EFGH$ to be,

$$\iint_{ABCD} dS\,\hat{\mathbf{n}}\phi + \iint_{EFGH} dS\,\hat{\mathbf{n}}\phi = \frac{\partial\phi}{\partial z}\,dx\,dy\,dz\,\hat{\mathbf{z}}.$$

Similarly one can show that,

$$\iint_{CBFG} dS\,\hat{\mathbf{n}}\phi + \iint_{AEHD} dS\,\hat{\mathbf{n}}\phi = \frac{\partial\phi}{\partial y}\,dx\,dy\,dz\,\hat{\mathbf{y}},$$

and,

$$\iint_{ABFE} dS\,\hat{\mathbf{n}}\phi + \iint_{CGHD} dS\,\hat{\mathbf{n}}\phi = \frac{\partial\phi}{\partial x}\,dx\,dy\,dz\,\hat{\mathbf{x}}.$$

Thus gathering contributions from all the six faces of the infinitesimal cuboid, one has

$$\begin{aligned}\oiint\!\!\oiint_S dS\,\hat{\mathbf{n}}\phi &= \sum_{\text{cuboid faces}}\iint dS\,\hat{\mathbf{n}}\phi \\ &= \left(\hat{\mathbf{x}}\frac{\partial\phi}{\partial x} + \hat{\mathbf{y}}\frac{\partial\phi}{\partial y} + \hat{\mathbf{z}}\frac{\partial\phi}{\partial z}\right)\,dx\,dy\,dz,\end{aligned}$$

and dividing both sides by the volume of the cuboid, $\Delta V = dx\,dy\,dz$, in the limit when the side lengths vanish, one has

$$\begin{aligned}\lim_{\Delta V\to 0}\frac{\oiint_S dS\,\hat{\mathbf{n}}\phi}{\Delta V} &= \hat{\mathbf{x}}\frac{\partial\phi}{\partial x} + \hat{\mathbf{y}}\frac{\partial\phi}{\partial y} + \hat{\mathbf{z}}\frac{\partial\phi}{\partial z} \\ &= \boldsymbol{\nabla}\phi.\end{aligned}$$

Hence proved.

4.2 Divergence of a vector field

Now consider the same infinitesimal cuboid in figure 1 centered around the point P except now instead of a scalar field we are concerned about a vector field $\mathbf{A}(\mathbf{x})$. The ***flux*** of the vector field over some surface is defined as the surface integral of the component ***normal*** to the surface. Thus the flux of the vector field over the surface made up of the union of the faces of the cuboid is,

$$\oiint_S dS\,\hat{\mathbf{n}}\cdot\mathbf{A} = \sum_{\text{cuboid faces}} \iint dS\,\hat{\mathbf{n}}\cdot\mathbf{A}$$
$$= \iint_{ABCD} dS\hat{\mathbf{n}}\cdot\mathbf{A} + \iint_{EFGH} dS\hat{\mathbf{n}}\cdot\mathbf{A} + \iint_{CBFG} dS\hat{\mathbf{n}}\cdot\mathbf{A} + \iint_{AEHD} dS\hat{\mathbf{n}}\cdot\mathbf{A} + \iint_{ABFE} dS\hat{\mathbf{n}}\cdot\mathbf{A} + \iint_{CGHD}$$

As before, since the cuboid is infinitesimal, we can assume $\mathbf{A}$ does not vary much over a single face and we can use the value of the vector at the center of a face as an approximate constant value of the vector over the entire face, e.g.

$$\mathbf{A}(\mathbf{x})|_{\mathbf{x}\in ABCD} \approx \mathbf{A}(L),$$

where L is the center of the face $ABCD$. Then the flux of $\mathbf{A}$ over the face $ABCD$ (with outward unit normal vector to the face $ABCD$, $\hat{\boldsymbol{n}}|_{ABCD} = \hat{\mathbf{z}}$) is,

$$\iint_{ABCD} dS\hat{\mathbf{n}}\cdot\mathbf{A} \approx \phi(L) \iint_{ABCD} dS\hat{\mathbf{z}}\cdot\mathbf{A}(L) \approx A_z(L)\,dx\,dy.$$

In terms of coordinates, $\mathbf{x}_L = \left(x, y, z + \frac{dz}{2}\right)$,so,

$$A_z(L) = A_z\left(x, y, z + \frac{dz}{2}\right) \approx A_z(x,y,z) + \frac{\partial A_z}{\partial z}\frac{dz}{2}$$

where at the last step we have Taylor expanded to first order in dz. Thus the surface integral of the scalar over the face $ABCD$ is,

$$\iint_{ABCD} dS\hat{\mathbf{n}}\cdot\mathbf{A} \approx \left(A_z(x,y,z) + \frac{\partial A_z}{\partial z}\frac{dz}{2}\right)\,dx\,dy.$$

Likewise one can show that the surface integral of the scalar over the face $EFGH$ to first order in dx, dy, dz is,

$$\iint_{EFGH} dS\hat{\mathbf{n}}\cdot\mathbf{A} \approx -\left(A_z(x,y,z) - \frac{\partial A_z}{\partial z}\frac{dz}{2}\right)dx\,dy,$$

since $\hat{\boldsymbol{n}}|_{EFGH} = -\hat{\mathbf{z}}$. Thus the total/net flux of $\mathbf{A}$ over the pair of opposite faces $ABCD$ and $EFGH$ be,

$$\iint_{ABCD} dS\hat{\mathbf{n}}\cdot\mathbf{A} + \iint_{EFGH} dS\hat{\mathbf{n}}\cdot\mathbf{A} = \frac{\partial A_z}{\partial z}\,dx\,dy\,dz.$$

Similarly one can show that the net flux of $\mathbf{A}$ over the pair of opposite faces $CBFG$ and $AEHD$ is

$$\iint_{CBFG} dS\hat{\mathbf{n}}\cdot\mathbf{A} + \iint_{AEHD} dS\hat{\mathbf{n}}\cdot\mathbf{A} = \frac{\partial A_y}{\partial y}\,dx\,dy\,dz,$$

and over the pair of opposite faces $ABFE$ and $CGHD$ is

$$\iint_{ABFE} dS\hat{\mathbf{n}}\cdot\mathbf{A} + \iint_{CGHD} dS\hat{\mathbf{n}}\cdot\mathbf{A} = \frac{\partial A_x}{\partial x}\,dx\,dy\,dz.$$

Finally gathering flux contributions from all the six faces of the infinitesimal cuboid, one has

$$\oiint_S dS\,\hat{\mathbf{n}}\cdot\mathbf{A} = \sum_{\text{cuboid faces}} \iint dS\,\hat{\mathbf{n}}\cdot\mathbf{A}$$
$$= \left(\frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z}\right)\,dx\,dy\,dz.$$

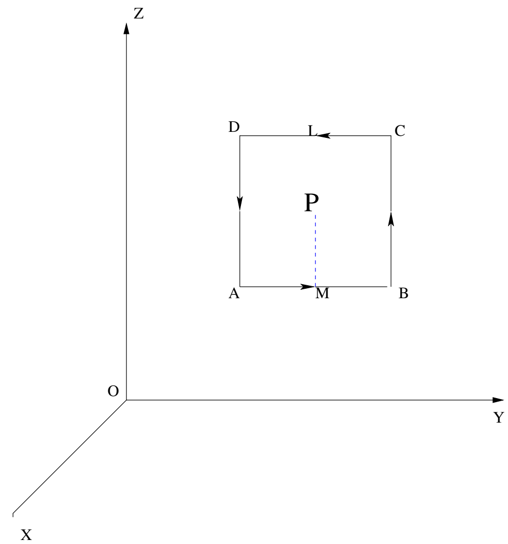

Figure 2: Circulation of a vector field around the loop ABCDA

Dividing both sides by the volume of the cuboid, $\Delta V = dx\, dy\, dz$, in the limit when the side lengths vanish, one has

$$\lim_{\Delta V \to 0} \frac{\oiint_S dS\, \hat{\mathbf{n}} \phi}{\Delta V} = \frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z}$$
$$= \boldsymbol{\nabla} \cdot \mathbf{A}.$$

Thus we have just shown that **Divergence of a vector field at a point P= net flux over an infinitesimal closed surface enclosing P, per unit enclosed volume.**

4.3 Curl of a vector field

To extract the geometric meaning of the curl of vector field $\mathbf{A}$ at a point P in three dimensional euclidean space ($\mathbb{R}^3$) labeled by the coordinates $\mathbf{x} = (x, y, z)$ wrt to some set of Cartesian axes, consider an infinitesimal rectangular loop $ABCD$ centered at P parallel to YZ plane with side lengths dy and dz as shown in figure (2). The ***circulation*** of the vector field is defined as the line integral of the ***tangential*** component of the vector field along the loop: $\oint \mathbf{A} \cdot d\mathbf{l}$ in the anti-clockwise direction. The circulation of the vector field $\mathbf{A}$ over the loop $ABCD$ is the sum of the line integrals along the four segments AB, BC, CD, DA,

$$\oint_{ABCD} \mathbf{A} \cdot d\mathbf{l} = \int_A^B \mathbf{A} \cdot d\mathbf{l} + \int_B^C \mathbf{A} \cdot d\mathbf{l} + \int_C^D \mathbf{A} \cdot d\mathbf{l} + \int_D^A \mathbf{A} \cdot d\mathbf{l}.$$

Let's first consider the contribution from the segment AB while going around the loop in anticlockwise direction. For this $d\mathbf{l} = dy\, \hat{\mathbf{y}}$, and $\mathbf{A} \cdot d\mathbf{l} = A_y\, dy$. Since the segment AB is infinitesimal we can assume that the this does not vary much over the segment and we can approximate it by its value at the mid-point M of the segment. Thus,

$$\int_A^B \mathbf{A} \cdot d\mathbf{l} = \int_A^B A_y\, dy \approx A_y(M)\, dy.$$

Recall that the coordinates of M are $\left(x, y, z - \frac{dz}{2}\right)$ and hence,

$$A_y(M) = A_y\left(x, y, z - \frac{dz}{2}\right) \approx A_y(x, y, z) - \frac{\partial A_y}{\partial z}\frac{dz}{2}$$

after Taylor expanding to first order in dz. Thus,

$$\int_A^B \mathbf{A}\cdot d\mathbf{l} \approx A_y(M)\,dy \approx \left[A_y(x,y,z) - \frac{\partial A_y}{\partial z}\,\frac{dz}{2}\right] dy.$$

Likewise one can show that the contribution to the circulation from the opposite edge i.e. CD is,

$$\int_C^D \mathbf{A}\cdot d\mathbf{l} \approx -\left[A_y(x,y,z) + \frac{\partial A_y}{\partial z}\,\frac{dz}{2}\right] dy.$$

The negative sign is because for going along CD in anticlockwise direction, $d\mathbf{l} = -dy\,\hat{\mathbf{y}}$. Adding the contribution from this pair of opposite segments AB and CD, we get,

$$\int_A^B \mathbf{A}\cdot d\mathbf{l} + \int_C^D \mathbf{A}\cdot d\mathbf{l} = -\frac{\partial A_y}{\partial z}\,dy\,dz.$$

Similarly one show that the contribution from the pair of opposite segments BC and DA is,

$$\int_B^C \mathbf{A}\cdot d\mathbf{l} + \int_D^A \mathbf{A}\cdot d\mathbf{l} = \frac{\partial A_z}{\partial y}\,dy\,dz.$$

And finally gathering contributions from all four segments one has,

$$\begin{aligned}\oint_{ABCD} \mathbf{A}\cdot d\mathbf{l} &= \int_A^B \mathbf{A}\cdot d\mathbf{l} + \int_B^C \mathbf{A}\cdot d\mathbf{l} + \int_C^D \mathbf{A}\cdot d\mathbf{l} + \int_D^A \mathbf{A}\cdot d\mathbf{l} \\ &= \left(\frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z}\right) dy\,dz.\end{aligned}$$

Dividing both sides by the area of the loop, $\Delta S_{yz} = dydz$ and taking the limit in which the segments vanish we have the result,

$$\lim_{\Delta S \to 0} \frac{\oint_{ABCD} \mathbf{A}\cdot d\mathbf{l}}{\Delta S_{yz}} = \frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z}.$$

It is obvious that the right hand side is the x-component of the curl of $\mathbf{A}$,

$$\lim_{\Delta S \to 0} \frac{\oint_{ABCD} \mathbf{A}\cdot d\mathbf{l}}{\Delta S_{yz}} = (\boldsymbol{\nabla} \times \mathbf{A})_x\,.$$

Thus we have arrived at the result that the **Component of Curl of a vector field at P along some direction $\hat{\mathbf{n}}$ = Anticlockwise circulation in an infinitesimal loop around P per unit normal area bounded by the loop .**