Lecture Notes

for

MA 1140

Elementary Linear Algebra

by

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Introduction

Linear algebra[1] is the branch of mathematics concerning linear equations such as

$$a_1x_1 + \cdots + a_nx_n = b$$

linear functions such as

$$(x_1, \ldots, x_n) \mapsto a_1x_1 + \ldots + a_nx_n$$

and their representations through matrices and vector spaces.

Linear algebra is central to almost all areas of mathematics. For instance, linear algebra is fundamental in modern presentations of geometry, including for describing basic objects such as lines, planes and rotations. Also, functional analysis may be basically viewed as the application of linear algebra to spaces of functions. Linear algebra is also used in most sciences and engineering areas, because it allows modeling many natural phenomena, and efficiently computing with such models. For nonlinear systems, which cannot be modeled with linear algebra, linear algebra is often used as a first-order approximation.

Until the 19th century, linear algebra was introduced through systems of linear equations and matrices. In modern mathematics, the presentation through vector spaces is generally preferred, since it is more synthetic, more general (not limited to the finite-dimensional case), and conceptually simpler, although more abstract.

[1] The introduction is noted from Wikipedia [2].

Chapter 1

Vector Spaces

MA 1140 is the study of 'vector spaces' and the 'maps' between them. For now, you may consider $\mathbb{R}^n$ as an example of a vector space, where $\mathbb{R}$ is the set of real numbers. Essentially, it means that you can add two vectors, and what you get is a vector; you can multiply a vector by a scalar, and what you get is a vector.

Definition 1.1. A set V of objects (called vectors) along with vector addition '+' and scalar multiplication '$\cdot$' is said to be a **vector space** over a field $\mathbb{F}$ (say, $\mathbb{F} = \mathbb{R}$, the set of real numbers) if the following hold:

(1) V is closed under '+', i.e. $x + y \in V$ for all $x, y \in V$.

(2) Addition is commutative, i.e. $x + y = y + x$ for all $x, y \in V$.

(3) Addition is associative, i.e. $(x + y) + z = x + (y + z)$ for all $x, y, z \in V$.

(4) Additive identity, i.e. there is $0 \in V$ such that $x + 0 = x$ for all $x \in V$.

(5) Additive inverse, i.e. for every $x \in V$, there is $-x \in V$ such that $x + (-x) = 0$.

(6) V is closed under '$\cdot$', i.e. $c \cdot x \in V$ for all $c \in \mathbb{F}$ and $x \in V$.

(7) $1 \cdot x = x$ for all $x \in V$.

(8) $(ab) \cdot x = a \cdot (b \cdot x)$ for all $a, b \in \mathbb{F}$ and $x \in V$.

(9) $a \cdot (x + y) = a \cdot x + a \cdot y$ for all $a \in \mathbb{F}$ and $x, y \in V$.

(10) $(a + b) \cdot x = a \cdot x + b \cdot x$ for all $a, b \in \mathbb{F}$ and $x \in V$.

The elements of $\mathbb{F}$ are called **scalars**, and the elements of V are called **vectors**.

Remark 1.2. The first five properties are nothing but the properties of abelian group, i.e. $(V, +)$ is an abelian group.

We simply write cx instead of $c \cdot x$ for $c \in \mathbb{F}$ and $x \in V$ when there is no confusion.

From now, we work over the field $\mathbb{R}$.

Example 1.3. The following are examples of vector spaces.

(1) The n-tuple space, $V = \mathbb{R}^n$, where vector addition and scalar multiplication are defined as follows:

$$\begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} + \begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{pmatrix} = \begin{pmatrix} x_1 + y_1 \\ x_2 + y_2 \\ \vdots \\ x_n + y_n \end{pmatrix} \quad \text{and} \quad c \cdot \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} cx_1 \\ cx_2 \\ \vdots \\ cx_n \end{pmatrix}.$$

(2) The space $\mathbb{R}^{m\times n}$ of $m \times n$ matrices

$$\begin{pmatrix} x_{11} & x_{12} & \cdots & x_{1n} \\ x_{21} & x_{22} & \cdots & x_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ x_{m1} & x_{m2} & \cdots & x_{mn} \end{pmatrix} \quad \text{where } x_{ij} \in \mathbb{R}.$$

The vector addition and scalar multiplication are defined by component wise addition and multiplication as in (1).

(3) Let S be any non-empty set. Let V be the set of all functions from S into $\mathbb{R}$. The sum $f + g$ of two vectors f and g in V is defined to be

$$(f + g)(s) := f(s) + g(s) \text{ for all } s \in S.$$

The scalar multiplication $c \cdot f$ is defined by $(c \cdot f)(s) := cf(s)$. Clearly, V is a vector space. Note that the preceding examples are special cases of this one.

(4) The set $\mathbb{R}[x]$ of all polynomials $a_0 + a_1x + \cdots + a_mx^m$, where $a_i \in \mathbb{R}$, x is an indeterminate and m varies over non-negative integers. The vector addition and scalar multiplication are defined in obvious way. Then $\mathbb{R}[x]$ is a vector space over $\mathbb{R}$.

Example 1.4. The set $V = \mathbb{R}^{n\times n}$ of all $n \times n$ matrices with vector addition defined by

$$\begin{pmatrix} x_{11} & x_{12} & \cdots & x_{1n} \\ x_{21} & x_{22} & \cdots & x_{2n} \\ \vdots & \vdots & \cdots & \vdots \\ x_{n1} & x_{n2} & \cdots & x_{nn} \end{pmatrix} \times \begin{pmatrix} y_{11} & y_{12} & \cdots & y_{1n} \\ y_{21} & y_{22} & \cdots & y_{2n} \\ \vdots & \vdots & \cdots & \vdots \\ y_{n1} & y_{n2} & \cdots & y_{nn} \end{pmatrix}$$

$$= \begin{pmatrix} -- & -- & -- \\ -- & (\sum_{k=1}^{n} x_{ik}y_{kj}) & -- \\ -- & -- & -- \end{pmatrix} \quad \text{(matrix multiplication)}$$

and scalar multiplication as before is NOT a vector space. Indeed, the operation '$\times$' is **not commutative** because

$$\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \times \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \neq \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \times \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$$

Moreover, every matrix does not necessarily have multiplicative inverse. For example, there does not exist a matrix A such that

$$\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \times A = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

Definition 1.5. A vector β in V is said to be a **linear combination** of vectors α_1, α_2 and α_r in V if $\beta = c_1\alpha_1 + c_2\alpha_2 + \cdots + c_r\alpha_r$ for some $c_1, c_2, \ldots, c_r \in \mathbb{F}$.

Example 1.6. In $\mathbb{R}^2$, the vector $(1, 2)$ can be written as linear combinations of $\{(1, 0), (0, 2)\}$ and $\{(1, 1), (1, 0)\}$ respectively as

$$\begin{pmatrix} 1 \\ 2 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ 2 \end{pmatrix} = 2\begin{pmatrix} 1 \\ 1 \end{pmatrix} + (-1)\begin{pmatrix} 1 \\ 0 \end{pmatrix}.$$

In view of Definition 1.5, one may ask the following questions. Suppose $\alpha_1, \alpha_2, \ldots, \alpha_r$ and β are given vectors in V. Is it possible to write β as a linear combination of $\alpha_1, \alpha_2, \ldots, \alpha_r$? If yes, then is it a unique way to write that? We find the answers to these questions as we proceed further.

Definition 1.7. Let V be a vector space over a field $\mathbb{F}$. A subspace of V is a subset W of V which is itself a vector space over $\mathbb{F}$ with the same operations of vector addition and scalar multiplication on V.

Theorem 1.8. Let W be a non-empty subset of a vector space V over $\mathbb{F}$. Then W is a subspace of V if and only if for each pair of vectors $\alpha, \beta \in W$ and each scalar $c \in \mathbb{F}$, the vector $c\alpha + \beta$ belongs to W.

Proof. Exercise! Note that many properties of W will be inherited from V. □

Theorem 1.9. Let V be a vector space over a field $\mathbb{F}$. The intersection of any collection of subspaces of V is a subspace of V.

Proof. Exercise! Use Theorem 1.8. □

Example 1.10. (1) The subset W consisting of the zero vector of V is a subspace of V.

(2) In $\mathbb{R}^n$, the set of n-tuples $(x_1, \ldots, x_n)$ with $x_1 = 0$ is a subspace; while the set of n-tuples with $x_1 = 1$ is NOT a subspace.

(3) The set of all 'symmetric matrices' forms a subspace of the space of all $n \times n$ matrices. Recall that an $n \times n$ square matrix A is said to be symmetric if (i, j)th entry of A is same as its (j, i)th entry, i.e. $A_{ij} = A_{ji}$ for each i and j.

Definition 1.11. Let S be a set of vectors in a vector space V. The **subspace spanned** by S is defined to be THE smallest subspace of V containing S.

What is the guarantee for existence of a subspace spanned by a given set? The following theorem is giving us that guarantee.

Theorem 1.12. Let S be a set of vectors in a vector space V. The following subspaces are equal.

(1) The intersection of all subspaces of V containing S.

(2) The set of all linear combinations of vectors in S, i.e. $\{c_1v_1+\cdots+c_rv_r : c_i \in \mathbb{R},\ v_i \in S\}$. (One can check that it is a subspace.)

(3) The subspace spanned by S, i.e. the smallest subspace of V containing S.

Proof. Let W_1, W_2 and W_3 be the subspaces described as in (1), (2) and (3) respectively. Then W_1 is contained in any subspace of V containing S. Therefore, since W_1 is a subspace (by Theorem 1.9), W_1 is the smallest subspace of V containing S, i.e. $W_1 = W_3$.

Using Theorem 1.8, it can be shown that W_2 is a subspace. Therefore, since W_2 contains S, we have $W_1 \subseteq W_2$. Notice that any subspace of V containing S also contains all linear combinations of vectors in S, i.e. any subspace of V containing S also contains W_2. Hence it follows that $W_2 \subseteq W_1$. Therefore $W_1 = W_2$. Thus $W_1 = W_2 = W_3$. $\square$

Bibliography

[1] K. Hoffman and R. Kunze, *Linear Algebra*, 2nd Edition.

[2] https://www.wikipedia.org

[3] G. Strang, *Linear Algebra and Its Applications*, 4th Edition.