$$f : \mathbb{R}^n \longrightarrow \mathbb{R}^m$$

$$f \equiv (f_1, f_2, \ldots, f_n)$$

$$f'(x_0) = (a_{ij})_{m \times n}$$

$$f : [a,b] \xrightarrow{\text{cont}} \mathbb{R} \text{; diff in } (a,b)$$

MVT: $$\frac{f(b) - f(a)}{b-a} = f'(\xi)$$

for some $\xi \in (a,b)$.

MVT for a scalar field: Let $f : \mathbb{R}^n \to \mathbb{R}$ be diff and $\vec{a}, \vec{b} \in \mathbb{R}^n$ then $\exists \vec{z} \in [\vec{a}, \vec{b}]$ s.t. $f(\vec{b}) - f(\vec{a}) = D_{\vec{v}} f(\vec{z})$

$$[\vec{a}, \vec{b}] = \{ t\vec{a} + (1-t)\vec{b} : t \in [0,1] \}$$

$$\vec{v} = \frac{\vec{b} - \vec{a}}{\|\vec{b} - \vec{a}\|} \in \mathbb{R}^n.$$

<u>Hint:</u> Define $g(t) = f(\vec{a} + t(\vec{b} - \vec{a}))$

<u>Then</u> $g : [0,1] \longrightarrow \mathbb{R}$ and apply MVT for g.

<u>Remark:</u> MVT is not applicable for a vector field.

<u>Counter example:</u> $g : [0, 2\pi] \longrightarrow \mathbb{R}^2$

$$g(t) = (\cos t, \sin t)$$

Then $g(2\pi) - g(0) \neq 2\pi g'(c)$ for any $c \in (0, 2\pi)$

Remark: A diff vector field satisfies generalized M.V.T.

Ref: T. M. Apostol - Calculus II.

Chain Rule: Let $f : \mathbb{R}^n \longrightarrow \mathbb{R}^m$ and $g : E \,(\subset \mathbb{R}^m) \longrightarrow \mathbb{R}^p$ where $E \supseteq f(\mathbb{R}^n)$. Let f be diff at $\vec{x_0}$ and g be diff at $f(\vec{x_0})$. Then $g \circ f : \mathbb{R}^n \longrightarrow \mathbb{R}^p$ is diff at $\vec{x_0}$ and $(g \circ f)'(\vec{x_0}) =$

$$\underbrace{g'(f(x_0))}_{p \times m} \circ \underbrace{f'(\vec{x_0})}_{m \times n}.$$

$\underline{\text{Ex:}}$ $f : \mathbb{R}^2 \longrightarrow \mathbb{R}$

$$f(x,y) = \begin{cases} \frac{xy}{x^2+y^2} & (x,y) \neq (0,0) \\ 0 & \text{ow} \end{cases}$$

f is not cont. at $(0,0)$.

$$\frac{\partial f}{\partial x}(0,0) = \lim_{t \to 0} \frac{f((0,0)+t\hat{e}_1) - f(0,0)}{t}$$

$$= 0$$

$$\frac{\partial f}{\partial y}(0,0) = 0$$

Let $f : \mathbb{R}^2 \longrightarrow \mathbb{R}$. which is cont

and $\frac{\partial f}{\partial x} : \mathbb{R}^2 \longrightarrow \mathbb{R}$

$\frac{\partial f}{\partial y} : \mathbb{R}^2 \longrightarrow \mathbb{R}$

Suppose $\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}$ are cont. at (x_0, y_0) then f is diff at (x_0, y_0).

$\Longrightarrow$ A sufficient condition for differentiability.

Directional derivative along a curve:

Let $\gamma : [a,b] \longrightarrow \mathbb{R}^n$ which is cont and $\gamma|_{(a,b)}$ is diff then γ is said to be a smooth curve in $\mathbb{R}^n$.

Let