Topics from Sequences and Series of Functions

Introduction to the Approximation Theory

Pradipto Banerjee
pradipto@iith.ac.in

IIT, Hyderabad

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Course Overview and Objectives

Ulterior Goal

- Brief introduction to the basic theory of approximation

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- Brief introduction to the basic theory of approximation
- Theory of **power series**, **Fourier series** and their applications

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Tools

- Knowledge of numerical sequences/series
- Theory of sequence and series of functions:

$$\{f_n(x)\}_{n=1}^{\infty}; \quad \sum_{n=1}^{\infty} f_n(x); \quad x \in [a,b].$$

Prerequisites – Numerical Series

Notations

- The symbol $\mathbb{R}$ will always denote the set of *real* numbers

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- We will use shorthands

$$\{a_n\} \searrow, \quad \{a_n\} \nearrow, \quad \sum_n a_n \searrow \quad \text{and} \quad \sum_n a_n \nearrow$$

to indicate convergence and divergence, respectively.

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- Symbols, S and S_N, are reserved for

$$S = \sum_{n=0}^{\infty} a_n \quad \text{and} \quad S_N = \sum_{n=0}^{N} a_n$$

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- Thus, $S = \sum_n a_n \quad \Longleftrightarrow \quad \lim_{N\to\infty} S_N = S$
- The expression $S < \infty$ indicates that the sum is a finite number

Prerequisites – Numerical Series

- Observe that

$$S < \infty \quad \Longrightarrow \quad \lim_{n\to\infty} a_n = 0$$

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- A series $\sum_n a_n$ is said to converge **absolutely** if

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- Absolute convergence $\Longrightarrow$ convergence, i.e.,

$$\sum_n |\, a_n \,| < \infty \quad \Longrightarrow \quad \sum_n a_n < \infty$$

Prerequisites – Numerical Series

Important Convergence Tests

- The **Comparison** test –

$$\text{If } |a_n| \leq c_n, \quad \text{then} \quad \sum_n c_n \searrow \Longrightarrow \sum_n a_n \searrow$$

Prerequisites – Numerical Series

Important Convergence Tests

- The **Comparison** test –

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- The **Root** test –

$$\text{If } \lim_{n\to\infty} |a_n|^{1/n} < 1 (> 1), \quad \text{then} \quad \sum_n a_n \searrow (\nearrow)$$

Prerequisites – Numerical Series

Important Convergence Tests

- The **Ratio** test – If

$$\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| < 1 (> 1), \quad \text{then} \quad \sum_n a_n \searrow (\nearrow)$$

Prerequisites – Numerical Series

Important Convergence Tests

- The **Ratio** test – If

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| < 1(> 1), \quad \text{then} \quad \sum_n a_n \searrow (\nearrow)$$

- For any $\alpha > 0$ and any β in $\mathbb{R}$, we have

$$\sum_n \frac{(\log n)^\beta}{n^{1+\alpha}} \searrow$$

Weierstrass – Taylor Theorem

Our aim is to study "Bad" functions (somewhat opaque, e.g., solutions to higher order differential equations) by approximating with "Good" ones (well studied, e.g. the polynomials).

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Meaning, if $g(x)$ is a so called "bad" function with domain $[a,b]$, then find a "good" function $f(x)$ having the same domain and s.t.

$$|f(x) - g(x)| = \text{really small} \quad \forall x \in [a,b].$$

Weierstrass – Taylor Theorem

Theorem (Not really a theorem)

Every reasonable function $g(x)$ can be approximated by a continuous function $f(x)$ in a closed interval $[a,b]$.

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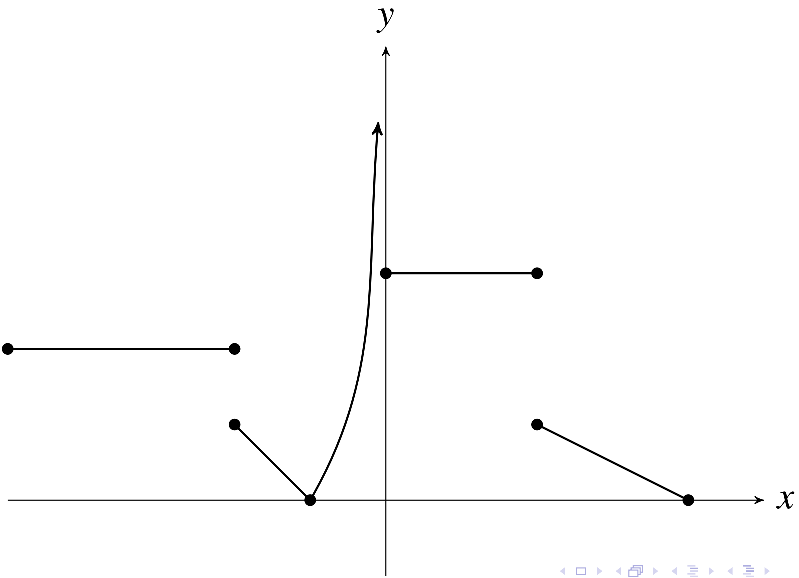
Theorem (Weierstrass Approximation Theorem)

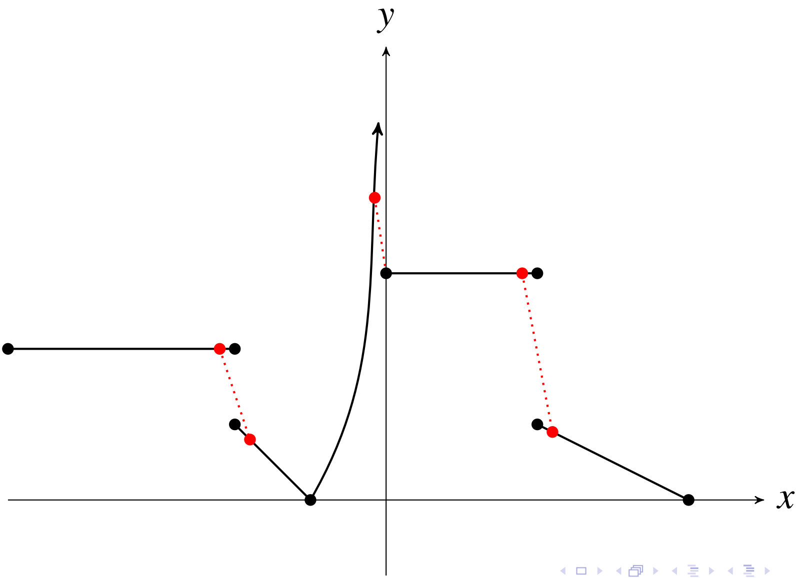
Let $f(x)$ be a continuous function on $[a,b]$, and let $\varepsilon > 0$ (Think $\varepsilon = 10^{-6}$) be given. Then there is a polynomial $p(x)$ with real coefficients such that

$$|f(x) - p(x)| < \varepsilon \quad \text{for any} \quad x \in [a,b].$$

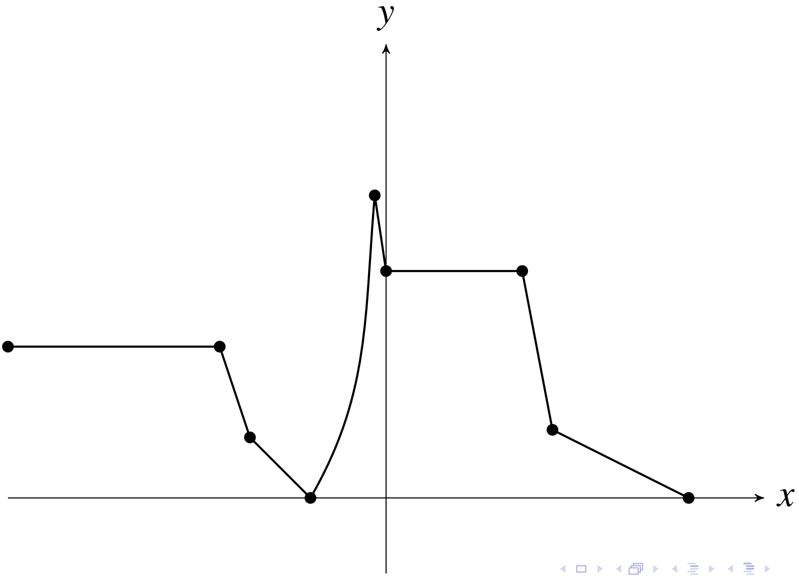
Step 0. Initial Bad Function



Step 1. Slight Perturbations at Bad points



Step 1′. The Continuous Neighbour



Weierstrass – Taylor Theorem

Theorem (Explicit Weierstrass)

Let $f(x)$ be a continuous function on the interval $[0,1]$. *For a positive integer n, define the* ***Bernstein*** *polynomial*

$$B_{f,n}(x) = \sum_{k=0}^{n} \binom{n}{k} f\left(\frac{k}{n}\right) x^k (1-x)^{n-k}.$$

Then one has that

$$\lim_{n\to\infty} B_{f,n}(x) = f(x) \quad \textit{uniformly}.$$

Weierstrass – Taylor Theorem

Upshot

Every bad (but reasonably good) function can be uniformly approximated by a polynomial function on a closed interval.

In this course, we will handle "bad" functions that are slightly *better* than continuous functions.

Weierstrass – Taylor Theorem

The "bad" functions we will deal with here are:

Definition

A function $f(x)$ is called n-**smooth** at a point $x = a$ in its domain if its first n derivatives – $f^{(1)}(x), f^{(2)}(x), \cdots, f^{(n)}(x)$, all exist in a small neighbourhood of a, and $f^{(n)}(x)$ is *continuous* at $x = a$.

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The "good" functions we make use of here are

Definition

A function $f(x)$ is called **smooth** at a point $x = a$ in its domain if derivatives of every order of $f(x)$ exists in a small neighbourhood of a.

Taylor's Approximation Theorem

Theorem (Taylor's Theorem)

Let $f(x)$ be a real valued function on $[a,b]$ such that $f(x)$ is $(n-1)$-smooth on $[a,b]$, and $f^{(n)}(x)$ exists on $[a,b]$.

Taylor's Approximation Theorem

Theorem (Taylor's Theorem)

Let $f(x)$ be a real valued function on $[a,b]$ such that $f(x)$ is $(n-1)$-smooth on $[a,b]$, and $f^{(n)}(x)$ exists on $[a,b]$. Let α and β be two points in $[a,b]$. Define the polynomial $P(x)$ as follows:

$$P(x) = f(\alpha) + \frac{f^{(1)}(\alpha)}{1!}(x-\alpha) + \frac{f^{(2)}(\alpha)}{2!}(x-\alpha)^2 + \cdots + \frac{f^{(n-1)}(\alpha)}{(n-1)!}(x-\alpha)^{n-1}.$$

Taylor's Theorem

Theorem (Taylor's Theorem continued..)

Then there is a point γ between α and β such that

$$
\begin{aligned}
f(\beta) = P(\beta) + \frac{f^{(n)}(\gamma)}{n!}(\beta-\alpha)^n \\
= f(\alpha) + \frac{f^{(1)}(\alpha)}{1!}(\beta-\alpha) + \frac{f^{(2)}(\alpha)}{2!}(\beta-\alpha)^2 + \\
\cdots + \frac{f^{(n-1)}(\alpha)}{(n-1)!}(\beta-\alpha)^{n-1} + \frac{f^{(n)}(\gamma)}{n!}(\beta-\alpha)^n.
\end{aligned}
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Taylor's Theorem

The (θ, h) version of Taylor's Theorem

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Taylor's Theorem

The (θ, h) version of Taylor's Theorem

- Set $\beta = \alpha + h$ so that, $|h| > 0$
- Since, γ is in between α and β, there is a $0 < \theta < 1$ s.t. $\gamma = \alpha + \theta h$
- Now, Taylor's theorem can be restated as

$$f(\alpha + h) = f(\alpha) + \frac{f^{(1)}(\alpha)}{1!}h + \frac{f^{(2)}(\alpha)}{2!}h^2 + \cdots + \frac{f^{(n-1)}(\alpha)}{(n-1)!}h^{n-1} + \frac{f^{(n)}(\alpha + \theta h)}{n!}h^n.$$

Remarks on Taylor's Theorem

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- The last term appearing in the expression for $f(\beta)$ is called the **remainder**, or the **error** term and denoted by $E_n(\beta)$. Thus

$$f(\beta) = P(\beta) + E_n(\beta)$$

where $E_n(\beta) = \frac{f^{(n)}(\gamma)}{n!}(\beta - \alpha)^n$

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where $E_n(\beta) = \frac{f^{(n)}(\gamma)}{n!}(\beta - \alpha)^n$

- The function $f(x)$ in question just falls short of being n-smooth, namely that $f^{(n)}(x)$ exists but need not be continuous

Remarks on Taylor's Theorem

- If $n = 1$, then the theorem boils down to that there is a γ between α and β s.t.

$$f(\beta) = f(\alpha) + f^{(1)}(\gamma)(\beta - \alpha),$$

which is simply the *Mean Value Theorem*

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Taylor's theorem in short

Any n-smooth function can be approximated by a polynomial of degree $\leq n-1$.

Remarks on Taylor's Theorem

- If $f(x)$ is smooth on $[a,b]$, then

$$f(\beta) = f(\alpha) + \frac{f^{(1)}(\alpha)}{1!}(\beta - \alpha) + \cdots$$

$$+ \frac{f^{(n)}(\alpha)}{n!}(\beta - \alpha)^n + \cdots$$

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- That is, one gets rid of the error term in this case, but instead the Taylor polynomial becomes a series
- but of course, one can truncate the series at any n to get a Taylor polynomial if one wishes to approximate $f(\beta)$

Taylor Series

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- If f is smooth on $[a,b]$, and if we fix α in $[a,b]$, then the last formula for $f(\beta)$ is valid for all β in $[a,b]$
- i.e., one may substitute β by x, to get the expression

$$f(x) = \sum_{n} \frac{f^{(n)}(\alpha)}{n!}(x-\alpha)^n$$

for $f(x)$ which is valid over $[a,b]$

Taylor Series

Taylor Series

If a real valued function f is smooth on $[a,b]$, and $\alpha \in [a,b]$. Then f can be expressed as a series

$$f(x) = \sum_n \frac{f^{(n)}(\alpha)}{n!}(x-\alpha)^n \quad \text{on} \quad [a,b]$$

The series appearing above is called the **Taylor Series** of (the smooth function) $f(x)$ at $x = \alpha$.

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Remark: f is smooth on $[a,b] \implies f$ has a Taylor series at every $\alpha \in [a,b]$.

Taylor Series

Maclaurin Series

If a real valued function f is defined on $[a, b]$, and smooth on $[a, b]$. If $\underline{0 \in [a, b]}$. Then f can be expressed as a series

$$f(x) = \sum_n \frac{f^{(n)}(0)}{n!} x^n \quad \text{in} \quad [a, b]$$

The series appearing above is called the **Maclaurin Series** of (the smooth function) $f(x)$.

Power Series

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- If the above power series converges for each x in $[a, b]$, then its sum is unique for each x
- Thus, one can define a function f as

$$f(x) = S(x) = \sum_n a_n(x-\alpha)^n, \quad x \in [a, b]$$

Power Series

Uniqueness of power series

If

$$\sum_n a_n(x-\alpha)^n = \sum_n b_n(x-\alpha)^n, \quad x \in [a,b],$$

then $a_n = b_n$ for all $n = 0, 1, 2, \cdots$.

Power Series

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- A convergent power series = the Taylor series of its sum

Power Series

Conclusions Thus Far

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Conclusions Thus Far

- Any n-smooth function can be approximated by a Taylor polynomial of degree $\leq n-1$
- A smooth function $=$ its Taylor series on its domain
- Smooth functions are nothing but sums of convergent power series
- Power series is "better" than polynomial approximations as there are no error terms appearing in the power series

Power Series

Examples

- Consider the examples from high school

$$e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$$

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots$$

Power Series

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$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots$$

- Since the <u>tail</u> of a convergent series is <u>small</u>, any finite truncation of the power series gives a reasonable estimate.

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  (To be proved later)

Power Series

- Suppose $\sum_n a_n(x-\alpha)^n \searrow$ for all $x \in [a,b]$
- Then $f(x) = \sum_n a_n(x-\alpha)^n$ is smooth on $[a,b]$
  (To be proved later)
- Thus, the function $g(x) = f(x+\alpha)$, is smooth on $[a-\alpha, b-\alpha]$, and

$$g(x) = f(x+\alpha) = \sum_n a_n x^n; \quad x \in [a-\alpha, b-\alpha]$$

Power Series

- Suppose $\sum_n a_n(x-\alpha)^n \searrow$ for all $x \in [a,b]$
- Then $f(x) = \sum_n a_n(x-\alpha)^n$ is smooth on $[a,b]$ (To be proved later)
- Thus, the function $g(x) = f(x+\alpha)$, is smooth on $[a-\alpha, b-\alpha]$, and

$$g(x) = f(x+\alpha) = \sum_n a_n x^n; \quad x \in [a-\alpha, b-\alpha]$$

- Thus it suffices to study $g(x)$, i.e., w.l.o.g, we may assume $\alpha = 0$ and our typical power series will be a Maclaurin series $\sum_n a_n x^n$

Convergence and Radius of Convergence

- Suppose that

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- How about $\lim_{n\to\infty} a_n$? (**Ex.**)

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Convergence and Radius of Convergence

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- Set

$$R = \begin{cases} 1/\ell & \text{if} \quad 0 < \ell < \infty \\ 0 & \text{if} \quad \ell = \infty \\ \infty & \text{if} \quad \ell = 0 \end{cases}$$

Convergence and Radius of Convergence

Application of Root test to $\sum_n a_n \beta^n$

- Observe that

$$\lim_{n\to\infty} \mid a_n \beta^n \mid^{1/n} = \mid \beta \mid \lim_{n\to\infty} \mid a_n \mid^{1/n} = \mid \beta \mid / R$$

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$$\lim_{n\to\infty} | a_n \beta^n |^{1/n} = | \beta | \lim_{n\to\infty} | a_n |^{1/n} = | \beta | /R$$

- By root test, we deduce that if $|\beta| \neq R$, then

$$\sum_n a_n \beta^n \to \begin{cases} \searrow & \text{if} \quad \beta \in (-R, R) \\ \nearrow & \text{if} \quad \beta \in)-R, R(\end{cases}$$

Convergence and Radius of Convergence

Radius of Convergence

The number R is called the **radius of convergence** of the power series $\sum_n a_n x^n$. Thus, a power series $\sum_n a_n x^n$ represents a smooth function $f(x)$ (its sum function) in $(-R, R)$.

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The end points $x = \pm R$ are the only points where we do not know the behaviour of the series. Even if, say, the series $\searrow$ at $x = R$, it is not entirely clear as to whether we can *continuously* extend the domain of f to $(-R, R]$ by defining $f(R) = \sum_n a_n R^n$!

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- Courtesy a criteria of Abel, we shall find out that this is *always* the case

Remarks on Convergence

- The Radius of convergence can also be computed using the ratio test, i.e., if $\ell' = \lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right|$, set

$$R' = \begin{cases} 1/\ell' & \text{if} \quad 0 < \ell' < \infty \\ 0 & \text{if} \quad \ell' = \infty \\ \infty & \text{if} \quad \ell' = 0 \end{cases}$$

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- Then $R' = R$ (**Ex.**)

Remarks on Convergence

- The series $\sum_n a_n x^n$ converges <u>absolutely</u> on $(-R, R)$ (**Ex.**), that is

$$\sum_n a_n x^n \searrow \quad \Rightarrow \quad \sum_n |a_n x^n| \searrow \text{ on } (-R, R)$$

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$$\sum_n a_n x^n \searrow \quad \Rightarrow \quad \sum_n |a_n x^n| \searrow \text{ on } (-R, R)$$

- If $\boxed{R = 0}$, then the series $\nearrow$ at all but one point, and if $\boxed{R = \infty}$, then it $\searrow$ everywhere, and as such, its sum function is smooth everywhere. Such functions are called **entire** functions

Remarks on Convergence

- For $R \neq 0$, $R \neq \infty$, consider the series

$$g(x) = \sum_n a_n (Rx)^n = f(Rx)$$

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- Note that f and g are the same as far as their analytic properties are concerned (g is a **contraction** of f)

Remarks on Convergence

- For $R \neq 0$, $R \neq \infty$, consider the series

$$g(x) = \sum_n a_n (Rx)^n = f(Rx)$$

- Thus, g has radius of convergence $= 1$
- Note that f and g are the same as far as their analytic properties are concerned (g is a **contraction** of f)
- Therefore, if $R \neq 0$ and $R \neq \infty$, then we may assume w.l.o.g. that $R = 1$

Remarks on Convergence

A Key Aspect of Power Series Convergence

If $\sum_n a_n x^n \searrow$ at $x = \beta$ in $(-R, R)$, then it converges at every point in a *small neighbourhood of* β.

Remarks on Convergence

A Key Aspect of Power Series Convergence

If $\sum_n a_n x^n \searrow$ at $x = \beta$ in $(-R, R)$, then it converges at every point in a *small neighbourhood of* β.

Proof.

For every $\beta \in (-R, R)$, let $\delta = \min\{|\beta \pm R|/2\}$.
Then $\delta \neq 0$. the interval $(\beta - \delta, \beta + \delta) \subset (-R, R)$.
Hence proved. $\square$

Examples

1. The series

$$1 + x + 4x^2 + 27x^3 + 256x^4 + \cdots$$

converges only at $x = 0$ (**Ex.**)

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2. The series

$$1 + \frac{x}{2} + \frac{x^2}{16} + \frac{x^3}{512} + \frac{x^4}{65536} + \cdots$$

converges everywhere on $\mathbb{R}$ (**Ex.**)

Examples

3. The function $f(x) = \dfrac{1}{1-x}$ is smooth on

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$$(-\infty, 1) \bigcup (1, \infty)$$

- In $(-1, 1)$, $f(x)$ has the Maclaurin series

$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + \cdots$$

- How about its power series in $(-\infty, -1)$ and $(1, \infty)$?

Examples

- Let $\alpha > 1$ be any number, then the Taylor series about α is given by

$$\frac{1}{1-x} = \frac{1}{1-\alpha} \sum_n \frac{(x-\alpha)^n}{(1-\alpha)^n}$$

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- Work out the details and then repeat for $x < -1$ **(Ex.)**

Three Questions on Power Series

Question 1.

- Suppose

$$\sum_n a_n x^n \searrow f(x); \quad \text{in } (-1, 1)$$

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- Define $f(1) = \sum_n a_n$

Three Questions on Power Series

Question 1. Continued..

Then, is it true that

$$\lim_{x\to 1} f(x) =^? f(1) = \sum_n a_n \quad \text{i.e.,}$$

$$\lim_{x\to 1}\left(\sum_n a_n x^n\right) =^? \sum_n \left(\lim_{x\to 1} a_n x^n\right)$$

In other words, does the fact that $\sum_n a_n$ guarantees that the sum function $f(x)$ is *continuous* at the boundary $x = 1$?

Three Questions on Power Series

Question 2.

Is it true that

$$\frac{d}{dx}(f(x)) =^{?} \sum_n n a_n x^{n-1}; \quad x \in (-1,1) \quad \text{i.e.,}$$

$$\frac{d}{dx}\left(\sum_n a_n x^n\right) =^{?} \sum_n \left(\frac{d}{dx}(a_n x^n)\right)$$

Three Questions on Power Series

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Is it true that

$$\frac{d}{dx}(f(x)) \stackrel{?}{=} \sum_n n a_n x^{n-1}; \quad x \in (-1,1) \quad \text{i.e.,}$$

$$\frac{d}{dx}\left(\sum_n a_n x^n\right) \stackrel{?}{=} \sum_n \left(\frac{d}{dx}(a_n x^n)\right)$$

In other words, is <u>term by term</u> differentiation of the series admissible assuming $\sum_n a_n x^n \searrow f(x)$ in order to determine the derivative $f'(x)$ of the sum function?

Three Questions on Power Series

Question 3.

Is it true that

$$\int_0^1 f(x)dx =^? \sum_n \frac{a_n}{n+1}x^{n+1} \quad \text{i.e.,}$$

$$\int_0^1 \left(\sum_n a_n x^n\right) =^? \sum_n \left(\int_0^1 (a_n x^n)dx\right)$$

Three Questions on Power Series

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Is it true that

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In other words, is term by term integration of the series admissible assuming $\sum_n a_n x^n \searrow f(x)$ in order to evaluate the integral of $f(x)$?

Three Questions on Power Series

In order to address these questions, we will study the theory of *sequence and series of functions*.

A tool from Calculus

Theorem (2-nd MVT of Calculus)

Let $H(x)$ and $\phi(x)$ be continuous functions on $[a,b]$ with $\phi(x) \geq 0$ for all x in $[a,b]$. Then for any α and β in $[a,b]$, there exists a γ in (α,β) such that

$$H(\gamma) = \frac{\displaystyle\int_{\alpha}^{\beta} H(t)\phi(t)dt}{\displaystyle\int_{\alpha}^{\beta} \phi(t)dt}.$$

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Remarks on 2nd MVT of Calculus

- A continuous function attains its *weighted average by a nonnegative continuous function* over any closed interval

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- Take $\phi(x) = 1$ and suppose $H(x) = h'(x)$ for some h

A tool from Calculus

Remarks on 2nd MVT of Calculus

- A continuous function attains its *weighted average by a nonnegative continuous function* over any closed interval
- Take $\phi(x) = 1$ and suppose $H(x) = h'(x)$ for some h
- then 2nd MVT $\Longrightarrow$ there is a $\gamma \in (\alpha, \beta)$ s.t.

$$h'(\gamma) = \frac{\int_\alpha^\beta h'(t)dt}{\beta - \alpha} = \frac{h(\beta) - h(\alpha)}{\beta - \alpha}$$

Towards the Proof of Taylor's Theorem

Recall the relaxed version of Taylor's Theorem:

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Theorem (Relaxed version of Taylor's Theorem)

Let $f(x)$ be a real valued function on $[a,b]$ that is n-smooth on $[a,b]$. Then for any α and β in $[a,b]$, there is a γ between α and β such that

$$f(\beta) = f(\alpha) + \frac{f^{(1)}(\alpha)}{1!}(\beta-\alpha) + \frac{f^{(2)}(\alpha)}{2!}(\beta-\alpha)^2 + \cdots + \frac{f^{(n-1)}(\alpha)}{(n-1)!}(\beta-\alpha)^{n-1} + \frac{f^{(n)}(\gamma)}{n!}(\beta-\alpha)^n.$$

Intuitive Proof of Taylor's Theorem

- Extra Assumption – that $f^{(n)}(x)$ is continuous on $[a,b]$ (i.e., integrable), and assume w.l.o.g. that $\alpha < \beta$

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$$f(\beta) = f(\alpha) + \int_{\alpha}^{\beta} f^{(1)}(t)dt$$

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- Extra Assumption – that $f^{(n)}(x)$ is continuous on $[a,b]$ (i.e., integrable), and assume w.l.o.g. that $\alpha < \beta$
- **Step 1.** We write

$$f(\beta) = f(\alpha) + \int_{\alpha}^{\beta} f^{(1)}(t)dt$$

- **Step 2.** Now, integrate the integral above by parts, taking

$$u(t) = f^{(1)}(t) \quad \text{and} \quad v(t) = \beta - t$$

Intuitive Proof of Taylor's Theorem

Thus, we have

$$
\begin{aligned}
\int_{\alpha}^{\beta} f^{(1)}(t)dt &= -\int_{\alpha}^{\beta} u(t)d(v(t)) \\
&= -\left(u(t)v(t)\Big|_{\alpha}^{\beta} - \int_{\alpha}^{\beta} v(t)d(u(t)) \right) \\
&= f^{(1)}(\alpha)(\beta-\alpha) + \int_{\alpha}^{\beta} (\beta - t)f^{(2)}(t)dt
\end{aligned}
$$

Intuitive Proof of Taylor's Theorem

- From last slide

$$f(\beta)=f(\alpha)+f^{(1)}(\alpha)(\beta-\alpha)+\int_{\alpha}^{\beta}(\beta-t)f^{(2)}(t)dt$$

Intuitive Proof of Taylor's Theorem

- From last slide

$$f(\beta) = f(\alpha) + f^{(1)}(\alpha)(\beta - \alpha) + \int_{\alpha}^{\beta} (\beta - t) f^{(2)}(t)\,dt$$

- **Step 3.** We evaluate the integral above again by parts, taking

$$u(t) = f^{(2)}(t) \quad \text{and} \quad v(t) = (\beta - t)^2/2!$$

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- **Step 3.** We evaluate the integral above again by parts, taking

$$u(t)=f^{(2)}(t) \quad \text{and} \quad v(t)=(\beta-t)^2/2!$$

- to get that it is equal to

$$f^{(2)}(\alpha)\frac{(\beta-\alpha)^2}{2!}+\int_{\alpha}^{\beta}\frac{(\beta-t)^2}{2!}f^{(3)}(t)dt$$

Intuitive Proof of Taylor's Theorem

Thus,

$$f(\beta) = f(\alpha) + f^{(1)}(\alpha)(\beta - \alpha) + f^{(2)}(\alpha)\frac{(\beta - \alpha)^2}{2!} + \int_{\alpha}^{\beta} \frac{(\beta - t)^2}{2!} f^{(3)}(t)\,dt$$

$$= f(\alpha) + f^{(1)}(\alpha)(\beta - \alpha) + f^{(2)}(\alpha)\frac{(\beta - \alpha)^2}{2!} + f^{(3)}(\alpha)\frac{(\beta - \alpha)^3}{3!} + \int_{\alpha}^{\beta} \frac{(\beta - t)^3}{3!} f^{(4)}(t)\,dt$$

Intuitive Proof of Taylor's Theorem

After $(n-1)$ steps, we get

$$f(\beta) = f(\alpha) + \frac{f^{(1)}(\alpha)}{1!}(\beta-\alpha) + \frac{f^{(2)}(\alpha)}{2!}(\beta-\alpha)^2 + \cdots$$

$$\cdots + \frac{f^{(n-1)}(\alpha)}{(n-1)!}(\beta-\alpha)^{n-1}$$

$$+ \int_\alpha^\beta \frac{(\beta-t)^{n-1}}{(n-1)!} f^{(n)}(t)\,dt.$$

Lagrange's Remainder Formula

- To finish the proof, need to convince ourselves that

$$\frac{f^{(n)}(\gamma)}{n!}(\beta-\alpha)^n = \int_\alpha^\beta \frac{(\beta-t)^{n-1}}{(n-1)!} f^{(n)}(t)\,dt$$

for some γ in (α,β)

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  for some γ in (α, β)

- The above expression for the error is known as the **Lagrange's formula** for the remainder term in Taylor expansion of $f(x)$

Lagrange's Remainder Formula

- Recall the 2nd MVT of integral calculus - for some γ in (α, β)

$$H(\gamma) = \frac{\int_{\alpha}^{\beta} H(t)\phi(t)dt}{\int_{\alpha}^{\beta} \phi(t)dt}; \quad H, \phi - \text{cont.}; \quad \phi \geq 0$$

Lagrange's Remainder Formula

- Recall the 2nd MVT of integral calculus - for some γ in (α, β)

$$H(\gamma) = \frac{\int_{\alpha}^{\beta} H(t)\phi(t)dt}{\int_{\alpha}^{\beta} \phi(t)dt}; \quad H, \phi - \text{cont.}; \quad \phi \geq 0$$

- Set

$$H(x) = f^{(n)}(x) \quad \text{and} \quad \phi(x) = \frac{(\beta - x)^{n-1}}{(n-1)!}$$

Lagrange's Remainder Formula

Noting that $(\beta - x)^{n-1}/(n-1)! \geq 0$ for any $x \in [\alpha, \beta]$, deduce that there is a γ in (α, β) such that

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$$f^{(n)}(\gamma) = \frac{\displaystyle\int_\alpha^\beta \frac{(\beta - t)^{n-1}}{(n-1)!} f^{(n)}(t)\,dt}{\displaystyle\int_\alpha^\beta \frac{(\beta - t)^{n-1}}{(n-1)!}\,dt}$$

$$= \frac{\displaystyle\int_\alpha^\beta \frac{(\beta - t)^{n-1}}{(n-1)!} f^{(n)}(t)\,dt}{\left(\dfrac{(\beta - \alpha)^n}{n!}\right)}$$

Further Remarks

The case $\beta < \alpha$ can be handled similarly; one has to make minor changes while choosing $\phi(x)$. But the conclusion remains the same. This is left as an **Ex.**

Further Remarks

- Since $f^{(n)}(x)$ is assumed to be continuous on $[a,b]$, it is also bounded there, say

$$\left|f^{(n)}(x)\right| \leq M \quad \text{for all} \quad x \in [a,b]$$

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- The error $\to 0$ as $n \to \infty$

An Example

- Find an approximate value of $e^{1/3}$ using Taylor expansion. Also estimate the error of approximation. (Use $e < 3$)

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- Find an approximate value of $e^{1/3}$ using Taylor expansion. Also estimate the error of approximation. (Use $e < 3$)
- Since $f(x) = e^x$ is smooth on $\mathbb{R}$, use the Maclaurin expansion, truncating at $n = 4$. Here, we take $\alpha = 0$ and $\beta = 1/3$ so that,

$$e^{1/3} = 1 + \frac{1}{1!}\left(\frac{1}{3}\right) + \frac{1}{2!}\left(\frac{1}{3}\right)^2 + \frac{1}{3!}\left(\frac{1}{3}\right)^3 + \int_0^{1/3} \frac{(\frac{1}{3} - t)^3}{3!} e^t dt$$

Example Contd. ...

- Thus

$$e^{1/3} = 1 + \frac{1}{3} + \frac{1}{18} + \frac{1}{162} + E \approx 1.39506 + E$$

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- Estimating E: Since $e < 3$, we have

$$E < \frac{3}{3!}\int_0^{1/3}\left(\frac{1}{3} - t\right)^3 dt = \frac{1}{2}\left(\frac{-(\frac{1}{3} - t)^4}{4}\Bigg|_0^{1/3}\right)$$
$$= \frac{1}{648} < 0.00155$$

Some Taylor Expansions

The exponential function $f(x) = e^x$

- Note that $f^{(n)}(x) = e^x$ for all $n = 1, 2, 3, \cdots$

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$$e^x = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n = \sum_{n=0}^{\infty} \frac{x^n}{n!}$$

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$$e^x = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n = \sum_{n=0}^{\infty} \frac{x^n}{n!}$$

- Since $\lim_{n\to\infty} \left(\frac{1}{n!}\right)^{1/n} = 0$ (**Ex.**), we deduce that $\boxed{R = \infty}$ (formula valid everywhere)

Some Taylor Expansions

The trigonometric function $f(x) = \sin x$

- In this case, we have

$$f^{(n)}(x) = \sin(x + n\pi/2); \quad \text{for} \quad n = 0, 1, 2, \cdots$$

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- In this case, we have

$$f^{(n)}(x) = \sin(x + n\pi/2); \quad \text{for} \quad n = 0, 1, 2, \cdots$$

- Thus the Maclaurin expansion for $\sin x$ is given by

$$\sin x = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n = \sum_{n=0}^{\infty} \frac{\sin\left(\frac{n\pi}{2}\right)}{n!} x^n$$

Some Taylor Expansions

The trigonometric function $f(x) = \sin x$ contd. ...

- Note that for an integer k,

$$\sin\left(\frac{n\pi}{2}\right) = \begin{cases} 0 & \text{if} \quad n = 2k \\ (-1)^k & \text{if} \quad n = 2k+1 \end{cases}$$

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$$\sin x = \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)!} x^{2k+1}$$

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- Thus

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- The radius of convergence, $\boxed{R = \infty}$ **(Ex.)**

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The natural logarithm function $f(x) = \log x$

- In this case,

$$f^{(n)}(x) = \frac{(-1)^{n-1}(n-1)!}{x^n} \quad \text{for} \quad n = 1, 2, 3, \cdots$$

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- Since, $f(0), f^{(1)}(0), f^{(2)}(0), \cdots$, are not defined, the Maclaurin approach won't work

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$$f^{(n)}(x) = \frac{(-1)^{n-1}(n-1)!}{x^n} \quad \text{for} \quad n = 1, 2, 3, \cdots$$

- Since, $f(0), f^{(1)}(0), f^{(2)}(0), \cdots$, are not defined, the Maclaurin approach won't work
- Instead, find Taylor expansion about some $x \in \mathbb{R}$, in a neighborhood of which $f^{(n)}(x)$ is defined for all $n = 0, 1, 2, \cdots$, say $x = 1$

Some Taylor Expansions

The natural logarithm function $f(x) = \log x$ contd. ...

Expanding $\log x$ about $x = 1$ yields

$$
\begin{aligned}
\log x &= \sum_{n=1}^{\infty} \frac{f^{(n)}(1)}{n!}(x-1)^n \\
&= \sum_{n=1}^{\infty} \frac{(-1)^{n-1}(n-1)!}{n!}(x-1)^n \\
&= \sum_{n=1}^{\infty} (-1)^{n-1} \frac{(x-1)^n}{n}
\end{aligned}
$$

Some Taylor Expansions

The natural logarithm function $f(x) = \log x$ contd. ...

- How about the radius of convergence?

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- How about the radius of convergence?

$$\frac{1}{(1/n)^{1/n}} = n^{1/n} \to 1 \quad \text{as} \quad n \to \infty$$

- For what values of x, does the series converge?

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The natural logarithm function $f(x) = \log x$ contd. ...

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- For what values of x, does the series converge?

$$|x-1| < 1 \quad \text{i.e.,} \quad \boxed{0 < x < 2}$$

Some Taylor Expansions

The natural logarithm function $f(x) = \log x$ contd. ...

- Now, shift the graph of $\log x$ to the left by 1 unit, i.e., $x \to x+1$, to get

$$\log(1+x) = \sum_{n=1}^{\infty} (-1)^{n-1} \frac{x^n}{n}; \quad -1 < x < 1$$

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- Thus for $\boxed{-1 < x < 1}$, we have

$$\log(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \cdots$$

Some Taylor Expansions

The natural logarithm function $f(x) = \log x$ contd. ...

- Substituting x by $-x$, yields (note that $x \in (-1,1) \Leftrightarrow -x \in (-1,1)$ so that, the formula remains valid)

$$\log(1-x) = \sum_{n=1}^{\infty} (-1)^{n-1} \frac{(-x)^n}{n}$$

$$= \sum_{n=1}^{\infty} (-1)^{2n-1} \frac{x^n}{n}$$

$$= -\left(x + \frac{x^2}{2} + \frac{x^3}{3} + \cdots \right)$$

Abel's Theorem

Theorem (Abel)

Suppose, a power series $\sum_n a_n x^n \searrow$ *to* $f(x)$ *in* $(-1,1)$, *that is*

$$f(x) = \sum_n a_n x^n \quad \textit{for} \quad -1 < x < 1.$$

If $\sum_n a_n < \infty$, *then* $f(x)$ *is* <u>*continuous*</u> *(from left) at* $x = 1$. *That is, if we set* $f(1) = \sum_n a_n$, *then* $f(1) = \lim_{x \to 1} f(x)$. *More precisely,*

$$\lim_{x \to 1} \sum_n a_n x^n = \sum_n \left(\lim_{x \to 1} (a_n x^n) \right)$$

An Application of Abel's Theorem

- Note that, by *Leibniz test*, we have

$$\sum_n \frac{(-1)^{n-1}}{n} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots < \infty$$

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- Thus, upon applying Abel's Theorem to the function $\log(1+x)$, we find that

$$\lim_{x \to 1} \left(\log(1+x) \right) = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots ,$$

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- Note that, by *Leibniz test*, we have

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- Thus, upon applying Abel's Theorem to the function $\log(1+x)$, we find that

$$\lim_{x \to 1} \left(\log(1+x)\right) = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots,$$

- That is

$$\boxed{\log 2 = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots}$$

Binomial Expansion

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- Therefore

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- Thus, $f^{(n)}(0)$ exists for all n, and

$$\begin{aligned} f^{(n)}(0) &= \alpha(\alpha-1)(\alpha-2)\cdots(\alpha-n+1)f(0) \\ &= \alpha(\alpha-1)(\alpha-2)\cdots(\alpha-n+1) \end{aligned}$$

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- How about the radius of Convergence?
- Let us try the ratio Test this time

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- The radius of convergence is given by

$$\frac{1}{R} = \lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n\to\infty} \frac{|\alpha - n|}{n+1} = 1$$

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$$a_n = 0 \quad \text{for any} \quad n > \alpha$$

- In that event, use root rest to deduce that

$$\frac{1}{R} = \lim_{n\to\infty} |a_n|^{1/n} = 0.$$

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- Thus

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- Thus

$$(1+x)^{\alpha} = \sum_{n=1}^{\infty} \frac{\alpha(\alpha-1)(\alpha-2)\cdots(\alpha-n+1)}{n!} x^n$$

- Where the radius of convergence R is given by

$$R = \begin{cases} \infty & \text{if } \alpha \text{ is a nonnegative integer} \\ 1 & \text{otherwise} \end{cases}$$

Differentiation and Integration Rules

Theorem (Friendly Theorem)

For

$$f(x) = \sum_n a_n x^n; \quad -R < x < R,$$

we have

$$\frac{d}{dx}(f(x)) = \sum_n n\, a_n x^{n-1}; \quad x \in (-R, R)$$

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That is to say that

$$\frac{d}{dx}\sum_n a_n x^n = \sum_n \frac{d}{dx}(a_n x^n); \quad x \in (-R, R)$$

Differentiation and Integration Rules

Theorem (Friendly Theorem Contd. ...)

Moreover, for any x in $(-R,R)$

$$\int_0^x f(t)dt = \sum_n \frac{a_n}{n+1}x^n$$

Differentiation and Integration Rules

Theorem (Friendly Theorem Contd. ...)

Moreover, for any x in $(-R, R)$

$$\int_0^x f(t)dt = \sum_n \frac{a_n}{n+1} x^n$$

That is

$$\int_0^x \left(\sum_n a_n t^n \right) dt = \sum_n \int_0^x (a_n t^n) dt.$$

Applications of various Theorems

Problem 1.

Using the geometric series

$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + \cdots; \quad -1 < x < 1,$$

find Maclaurin expansions and the domain of validity for

$$\log(1-x) \quad \text{and} \quad \frac{1}{(1-x)^2}.$$

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Hint: Friendly Theorem allows us to differentiate and integrate the series *term by term* within $(-1, 1)$

Applications of various Theorems

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Use the Maclaurin expansion of $\dfrac{1}{1+x^2}$, to find that of $\arctan x$. Describe its interval of convergence. Now, deduce that

$$\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots .$$

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Hint: Use Leibniz test and Abel's Theorem

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Find an approximate value of the integral

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expressing your answer in up to 5 decimal places. Is it possible to give an error for your application?

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Hint: Routine application of truncated Taylor expansion and Lagrange's remainder formula

Applications of various Theorems

Problem 4.

Let

$$f(x) = \frac{\sin x}{1+x^2}$$

Using Maclaurin expansion, evaluate $f^{(5)}(0)$. You must justify your answer.

Problem 5.

Show that the number e (base of ln) is irrational.

Upcoming Lecture

- Discuss solutions to the 5 problems
- Introduction to the theory of *sequences and series of functions*