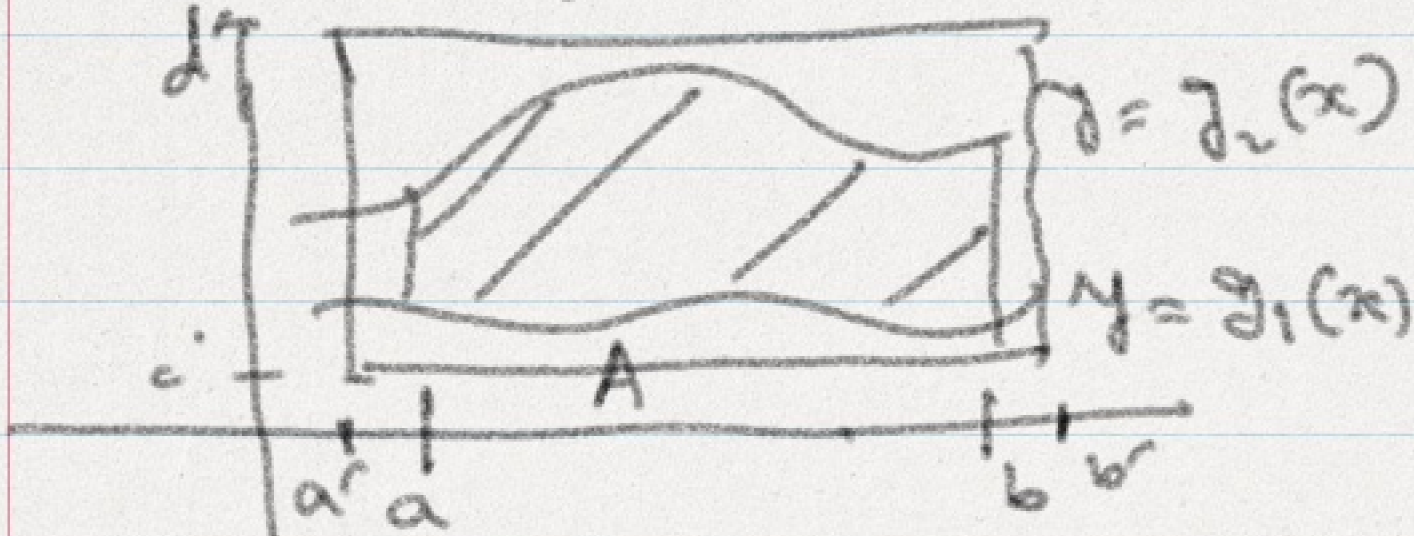
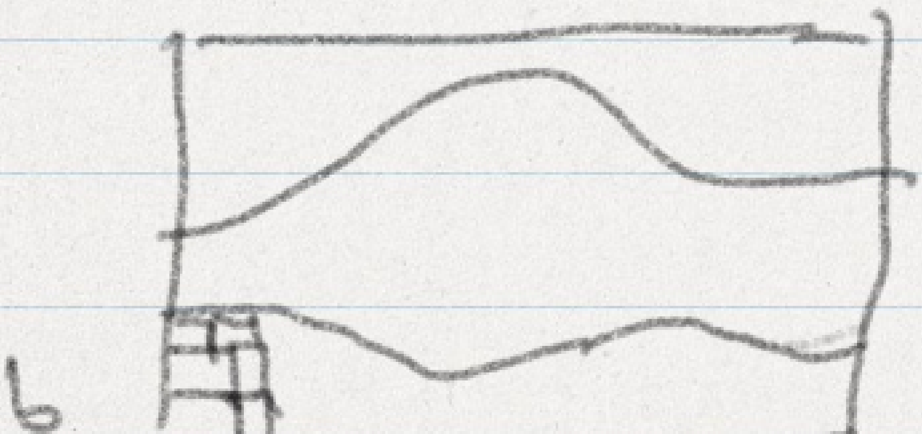
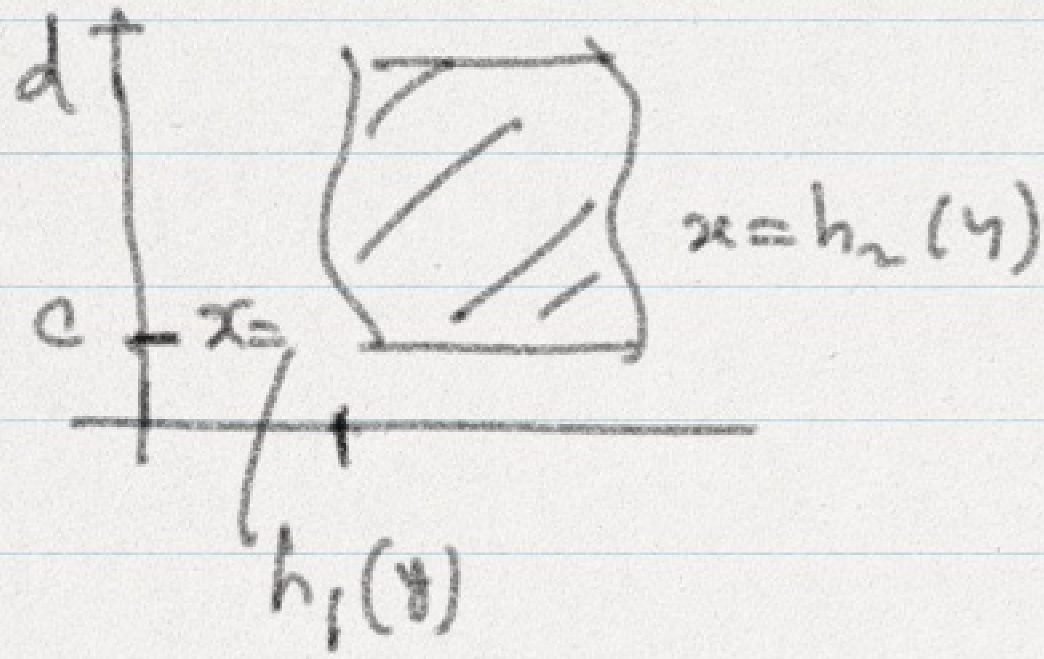
Let $A = \{(x,y) : a \le x \le b,\ g_1(x) \le y \le g_2(x)\}$



$$\iint_A f(x,y)\,dx\,dy$$



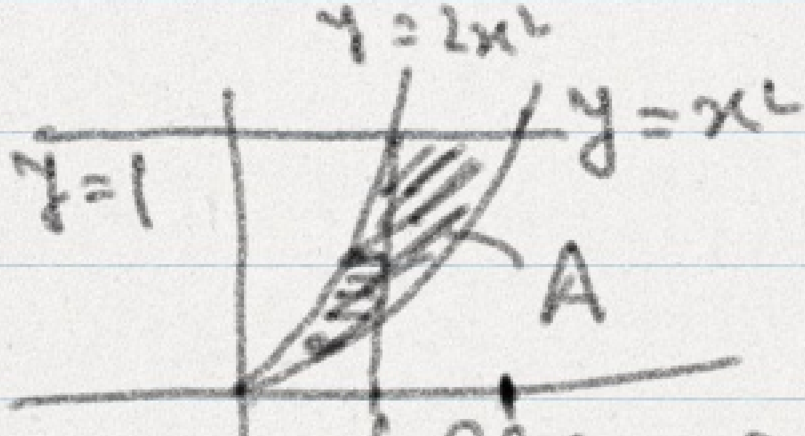
$$\int_{x=a}^{b} \int_{y=g_1(x)}^{g_2(x)} f(x,y)\,dy\,dx$$



$$A = \{(x,y) : c \le y \le d,\ h_1(y) \le x \le h_2(y)\}$$

$$\iint_A f(x,y)\,dx\,dy = \int_{y=c}^{d}\left(\int_{h_1(y)}^{h_2(y)} f(x,y)\,dx\right)dy.$$

Ex:

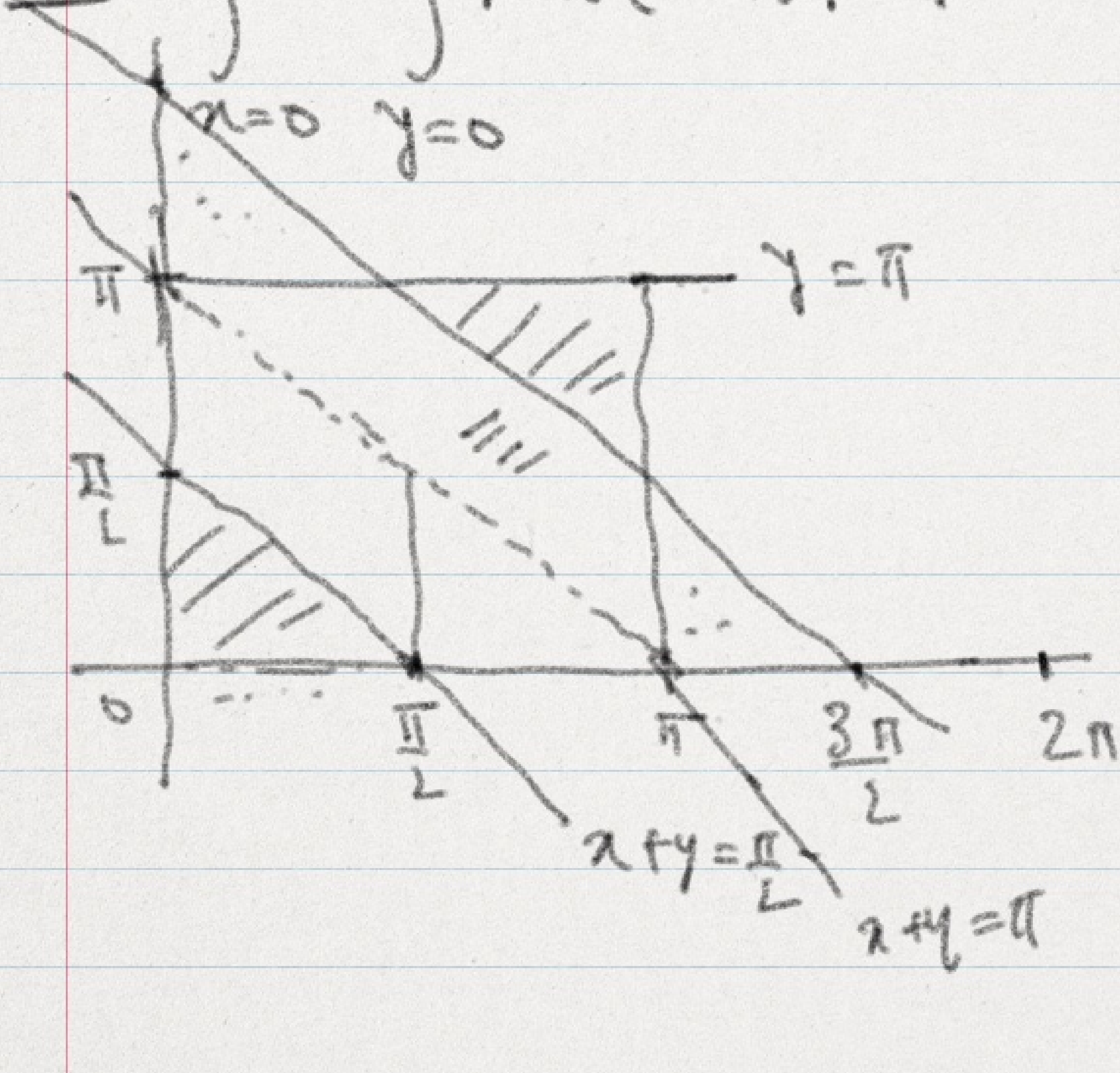


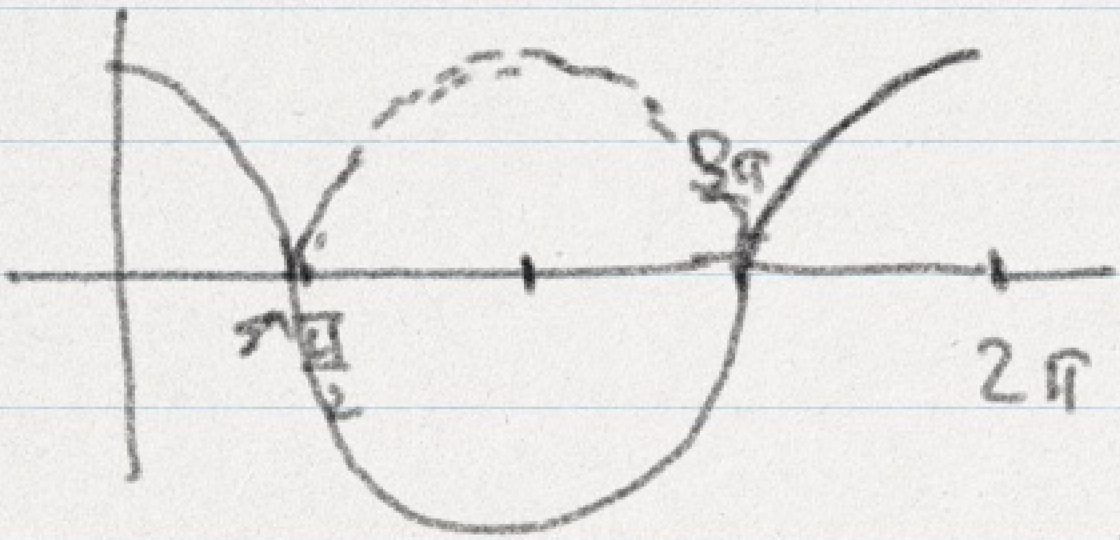
$$\iint_A (x+y)\,dx\,dy.$$

$$I = \int_{x=0}^{\frac{1}{\sqrt{2}}} \int_{y=x^2}^{2x^2} (x+y)\,dy\,dx \;+$$

$$\int_{x=\frac{1}{\sqrt{2}}}^{1} \int_{y=x^2}^{1} (x+y)\,dy\,dx.$$

Ex: $$\int_{x=0}^{\pi} \int_{y=0}^{\pi} |\cos(x+y)|\,dy\,dx.$$





$$I_1 + I_2 + I_3 +$$

$$I_2 = \int_{x=0}^{\pi} \int_{y=\frac{\pi}{2}-x}^{\pi-x} \left(-\cos(x+y)\right) dy\, dx$$

$$I_3 = \int_{x=0}^{\pi} \int_{y=\pi-x}^{\pi} -\left(\cos(x+y)\right) dy\, dx$$

$$I_4 = \int \int \cos(x+y)\, dy\, dx$$

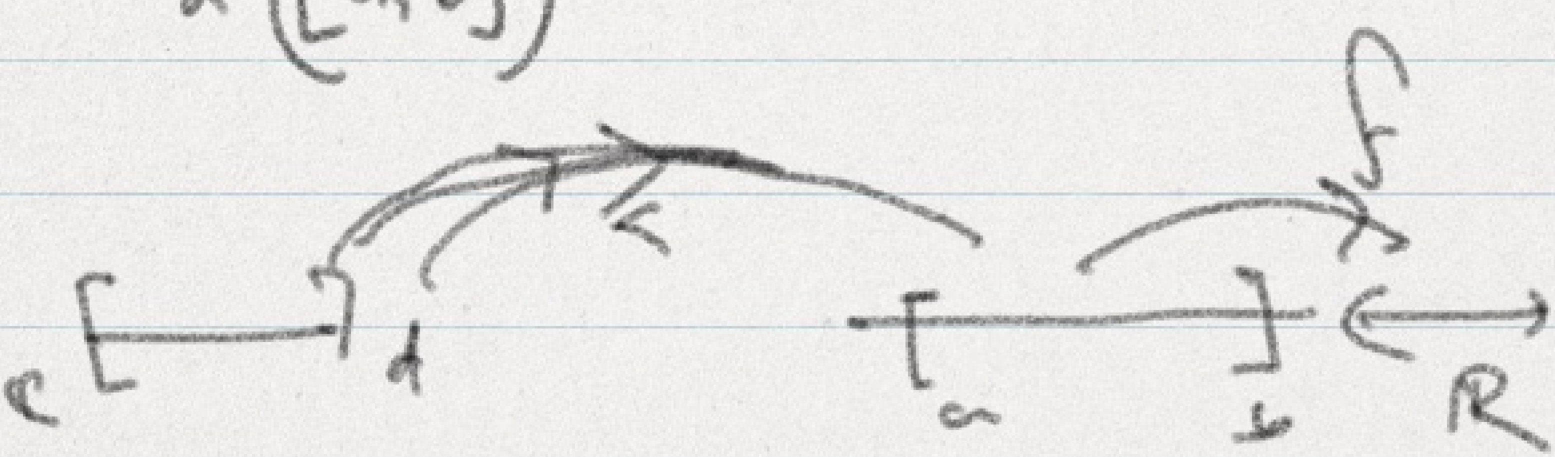
Ex. $$\iint_D (x+y)\,dx\,dy$$

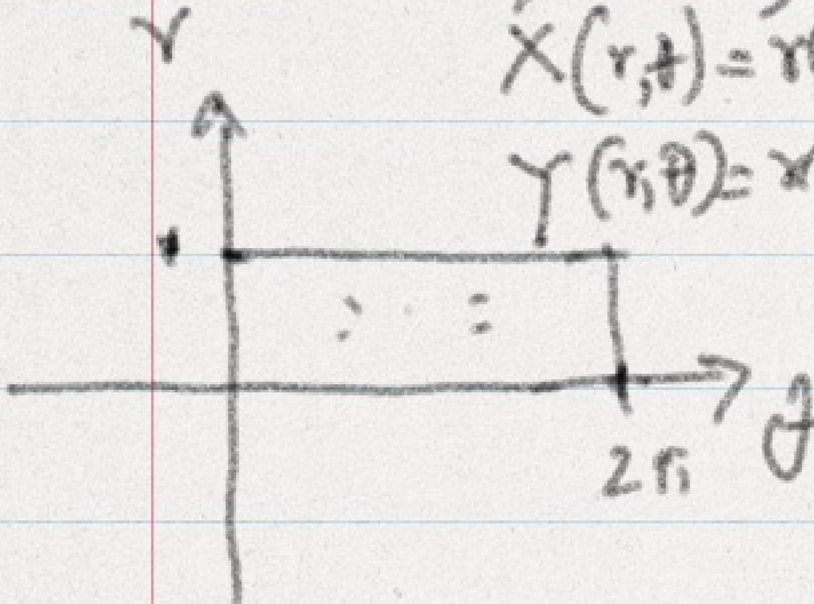
$$D = \{(x,y) : x^2+y^2 \le 1\}$$

$$I = \int_a^b f(x)\,dx$$

Let $x = \alpha(t)$

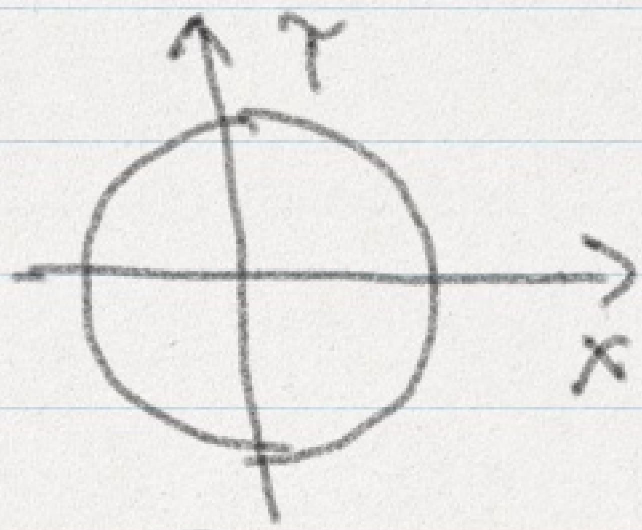
$$I = \int_{\alpha^{-1}([a,b])} f\alpha(t)\,\alpha'(t)\,dt$$





$X(r,\theta) = r\cos\theta$

$Y(r,\theta) = r\sin\theta$



D

$= \{(x,y) : x^2+y^2 \leq 1\}$

$x = X(r,\theta)$

$y = Y(r,\theta)$

$$\iint_D f(x,y)\,dx\,dy$$

$$(X,Y)^{-1}(D) = A$$

$$\iint_A f(X,Y)\,J\,dr\,d\theta$$

$$J = |J(r,\theta)|$$

$$J(r,\theta) = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} \end{vmatrix}$$

Let $f: D \longrightarrow \mathbb{R}$ where $D \subset \mathbb{R}^2$ and $\phi: A \longrightarrow D$ where $A \subset \mathbb{R}^2$ and ϕ is diff and one one. Then

$$\iint_D f(x,y)\,dx\,dy =$$

$$\iint_{\phi^{-1}(D)} f(\phi(u,v)) \left| J_\phi(u,v) \right| du\,dv$$

$$J_\phi(u,v) = \begin{vmatrix} \frac{\partial \phi_1}{\partial u} & \frac{\partial \phi_1}{\partial v} \\ \frac{\partial \phi_2}{\partial u} & \frac{\partial \phi_2}{\partial v} \end{vmatrix}$$

Let A be a bounded region in $\mathbb{R}^n$, and f be integrable over A.

$$\int_A f\,dv = \int_{\varphi^{-1}(A)} f\circ\varphi(t)\,|J_\varphi(t)|\,dt$$

where $\varphi : \mathbb{R}^n \to \mathbb{R}^n$ diff.
$J_\varphi(t)$ is the Jacobian.

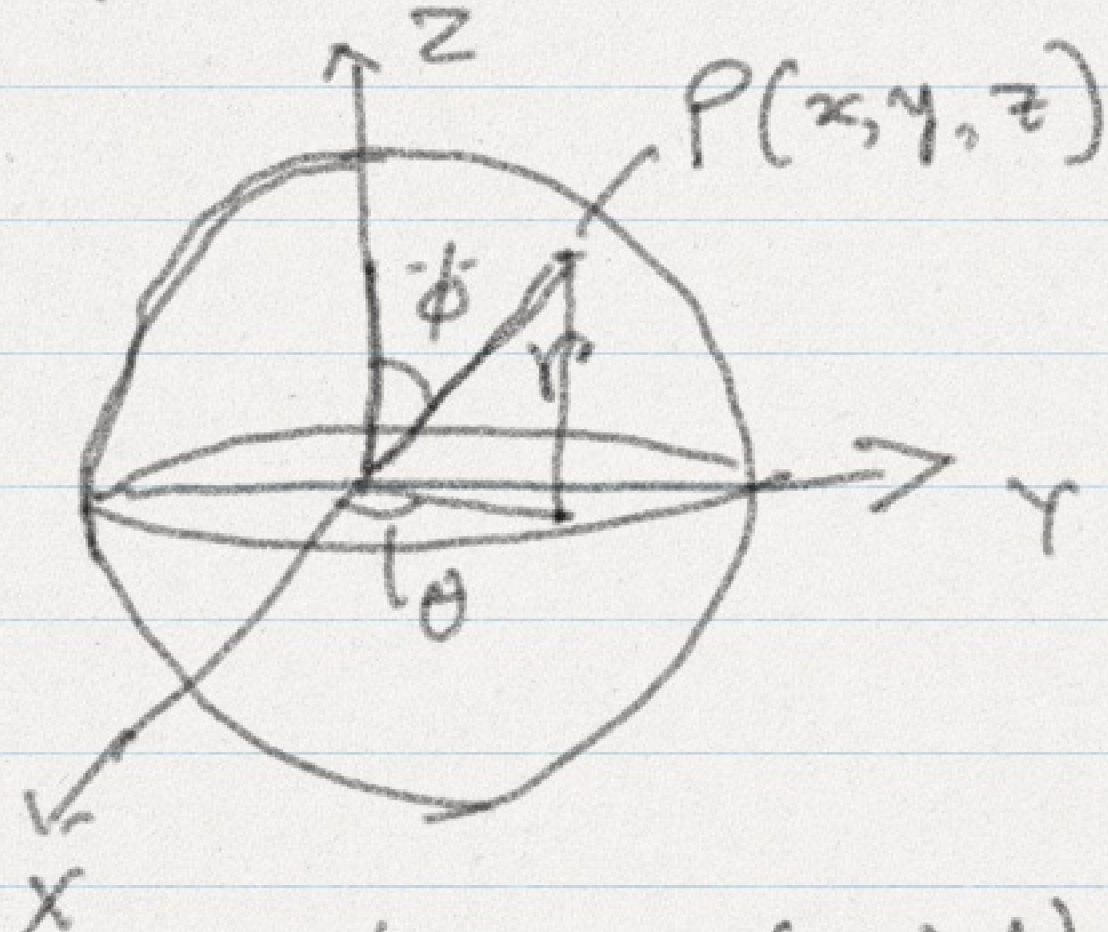
$$\varphi \equiv (\varphi_1, \varphi_2, \cdots, \varphi_n)$$

$$J_\varphi(t) = \begin{pmatrix} \frac{\partial \varphi_1}{\partial x_1} & \frac{\partial \varphi_1}{\partial x_2} & \cdots & \frac{\partial \varphi_1}{\partial x_n} \\ \vdots & & & \\ \frac{\partial \varphi_n}{\partial x_1} & & \cdots & \frac{\partial \varphi_n}{\partial x_n} \end{pmatrix}$$

$t \in \varphi^{-1}(A)$

Coordinate System in $\mathbb{R}^3$

1. Cartesian coordinate
2. Spherical coordinate.



$$z = r\cos\phi \quad = Z(r,\theta,\phi)$$
$$x = r\sin\phi\cos\theta = X(r,\theta,\phi)$$
$$y = r\sin\phi\sin\theta = Y(r,\theta,\phi)$$

$$0 \leq r \leq 1$$
$$0 \leq \theta \leq 2\pi$$
$$0 \leq \phi \leq \pi$$

Now consider the transformation

$$\varphi : \mathbb{R}^3 \longrightarrow \mathbb{R}^3$$

$$\varphi(r, \theta, \phi) = \big(X(r,\theta,\phi), Y(r,\theta,\phi), Z(r,\theta,\phi) \big)$$

$$J_\varphi(t) = \begin{vmatrix} \frac{\partial X}{\partial r} & \cdots & \\ & & \\ \frac{\partial Z}{\partial r} & \cdots & \end{vmatrix}$$

$$= r^2 \sin\phi$$

$$I = \int 1 \, dx \, dy \, dz = \frac{4}{3}\pi$$

$$A = \{ (x, y, z) : x^2 + y^2 + z^2 \le 1 \}$$

—

Now

$$\varphi^{-1}(A) = [0,1] \times [0,2\pi] \times [0,\pi]$$

$$\int_{r=0}^{1} \int_{\theta=0}^{2\pi} \int_{\phi=0}^{\pi} 1 . r^2 \sin\phi \, dr \, d\theta \, d\phi$$

$$= \frac{1}{3} . 2\pi \left[\cos\phi \right]_{\pi}^{0} = \frac{4}{3}\pi.$$

$$I = \int_{x=-1}^{1} \int_{y=-\sqrt{1-x^2}}^{\sqrt{1-x^2}} \int_{z=-\sqrt{1-x^2-y^2}}^{\sqrt{1-x^2-y^2}} 1 . dz\,dy\,dx$$

$$= \frac{4}{3}\pi.$$

$$\int_{x=-1}^{1}\int_{y=-\sqrt{1-x^2}}^{\sqrt{1-x^2}}\int_{z=-\sqrt{1-x^2-y^2}}^{\sqrt{1-x^2-y^2}}\int_{w=-\sqrt{}}^{\sqrt{}} 1 \cdot dw\,dz\,dy\,dx$$

$$= \frac{\pi^2}{2}.$$

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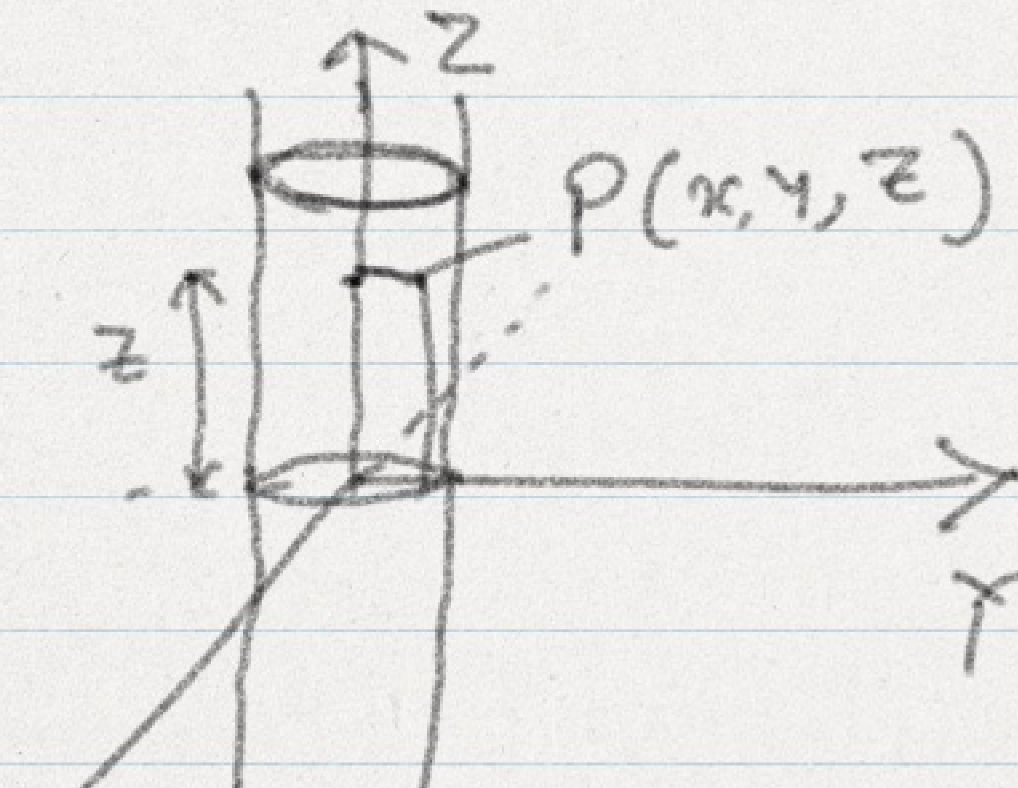
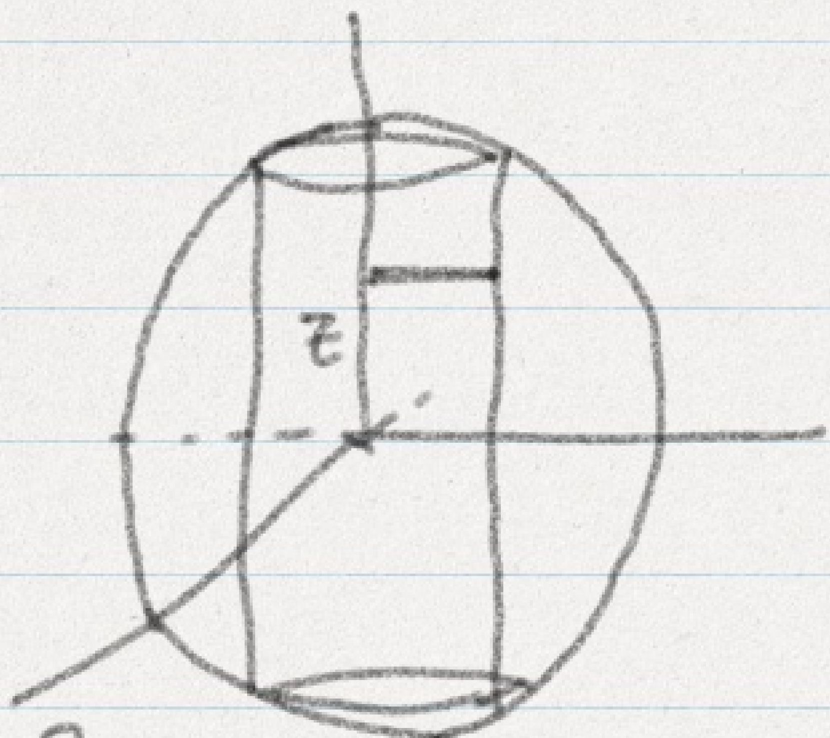
3. Cylindrical coordinate system



$x = r\cos\theta$
$y = r\sin\theta$

$z = z \quad = Z(r, \theta, z)$
$x = r\cos\theta = X(r, \theta, z)$
$y = r\sin\theta = Y(r, \theta, z)$

$$\int_A 1 \, dv =$$



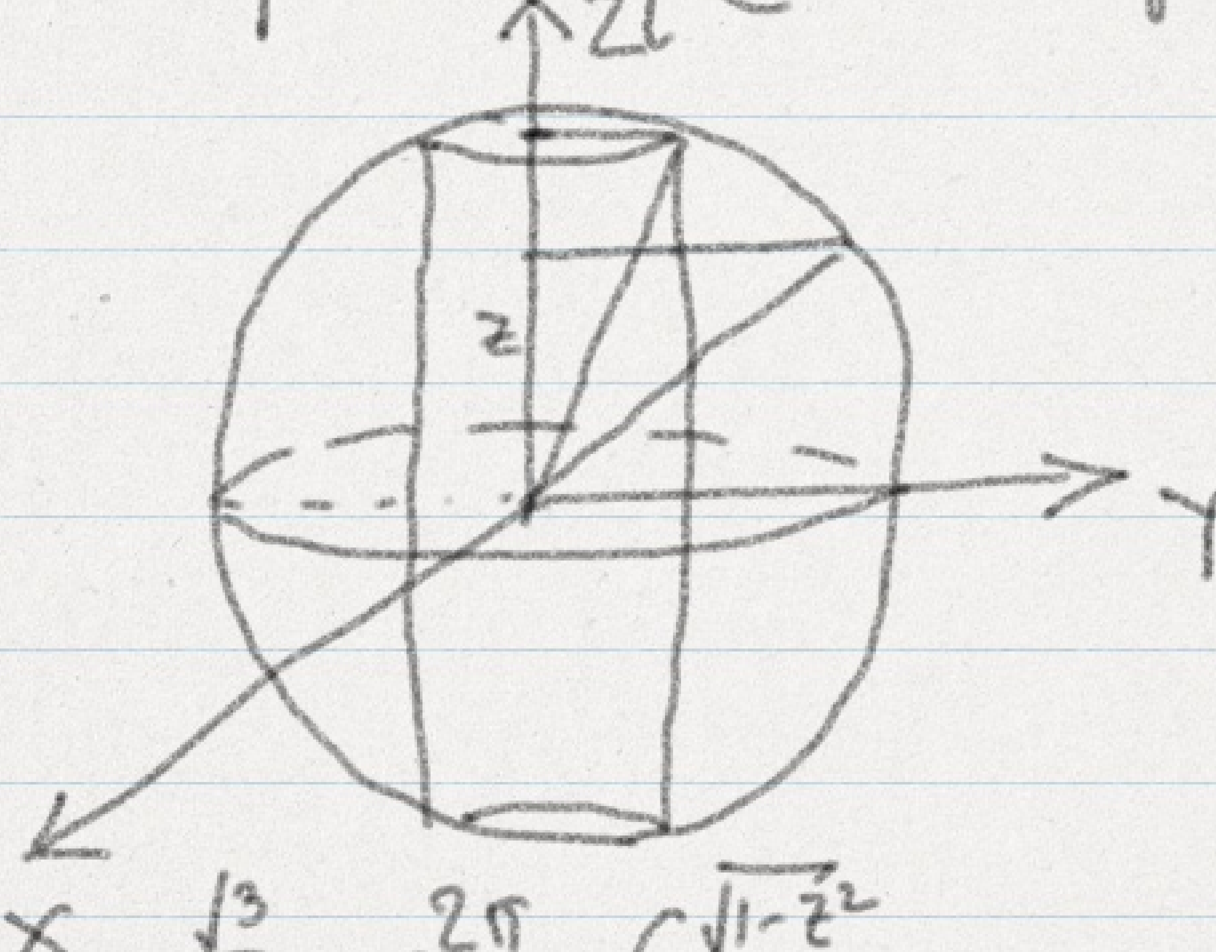
$$\int_{z=-1}^{1} \int_{\theta=0}^{2\pi} \int_{r=0}^{\sqrt{1-z^2}} 1 \; r \, dr \, d\theta \, dz$$

$$= \frac{4}{3}\pi$$

Ex. Find the volume of the solid bounded by the sphere

$\{(x,y,z) : x^2+y^2+z^2=1\}$ and

the cylinder $\{(x,y) : x^2+y^2=\frac{1}{4}\}$.



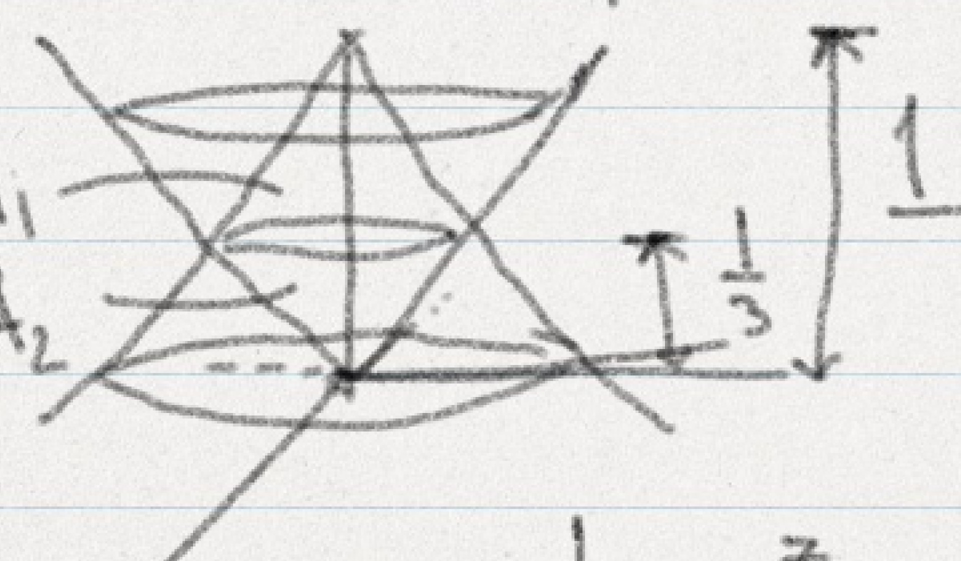
$$\frac{4}{3}\pi - \int_{z=-\frac{\sqrt{3}}{2}}^{\frac{\sqrt{3}}{2}} \int_{\theta=0}^{2\pi} \int_{r=\frac{1}{2}}^{\sqrt{1-z^2}} r\,dr\,d\theta\,dz = A$$

(Sorry)

— By cylindrical coordinate.

$$- \iint_{\left\{(x,y)\,:\,x^2+y^2 \le \frac{1}{4}\right\}} \sqrt{1-x^2-y^2}\,dx\,dy \quad = A$$

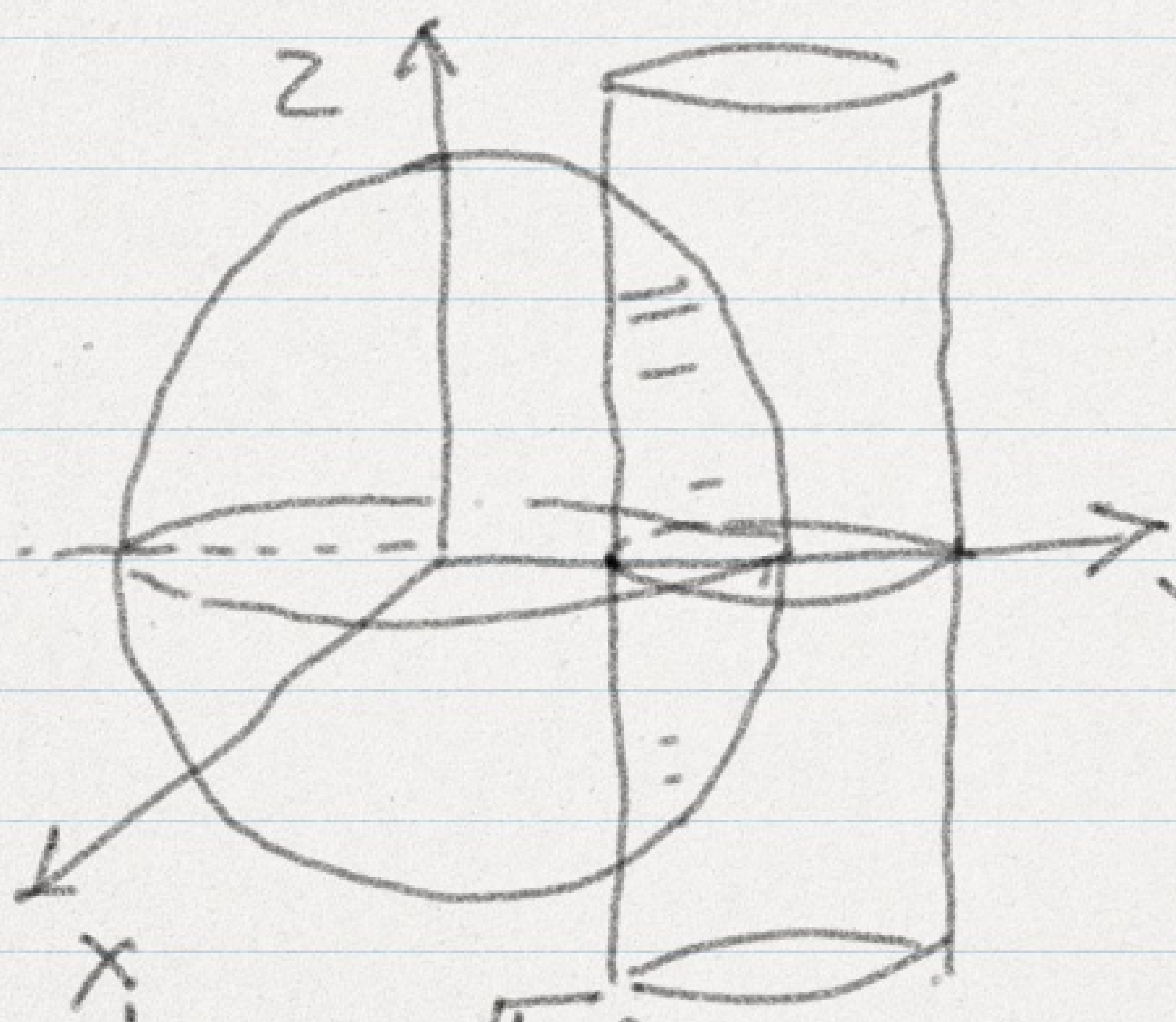
Ex. Find the vol of the solid bounded by the surfaces $z = r$ and $z = 1-2r$ where $r = \sqrt{x^2+y^2}$



$$\text{vol. of } A_2 = \int_{z=0}^{\frac{1}{3}} \int_{r=0}^{z} \int_{\theta=0}^{2\pi} r\,dr\,d\theta\,dz$$

$$\text{vol of } A_1 = \int_{z=\frac{1}{3}}^{1} \int_{r=0}^{\frac{1-z}{2}} \int_{\theta=0}^{2\pi} r\,dr\,d\theta\,dz.$$

Ex. Find the volume of the solid bounded by the sphere $\{(x,y,z) : x^2+y^2+z^2 = 1\}$ and the cylinder $\{(x,y) : x^2+(y-1)^2 = \frac{1}{4}\}$

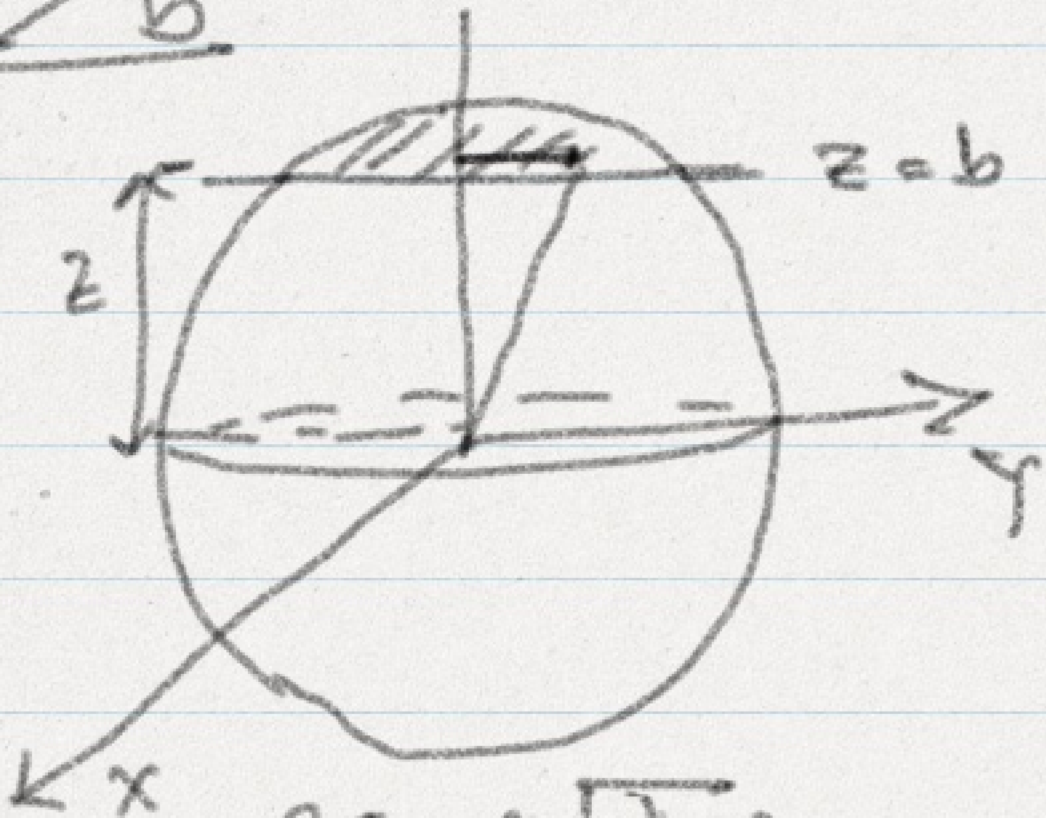


$$4\int_{x=0}^{\frac{1}{2}} \int_{y=0}^{1+\sqrt{\frac{1}{4}-x^2}} \cancel{\sqrt{1-x^2-y^2}\,dy\,dx}$$

X

Ex. Find the vol. of the solid bounded above by the sphere $x^2+y^2+z^2=a^2$ and below by the plane $z=b$.

assume $a > b$



$$\int_{z=b}^{a}\int_{\theta=0}^{2\pi}\int_{r=0}^{\sqrt{a^2-z^2}} r\,dr\,d\theta\,dz$$