Homework Assignments

MA1130 VECTOR CALCULUS

January 8, 2019

1. If $\mathbf{u} \perp \mathbf{v}$ and $\mathbf{u} \perp \mathbf{w}$ then show that $\mathbf{u} \perp a\mathbf{v} + b\mathbf{w}$ for all $a, b \in \mathbb{R}$

2. Prove the **law of cosine:** For any two nonzero vectors $\mathbf{v}$ and $\mathbf{w}$ one has

$$\|\mathbf{v} - \mathbf{w}\|^2 = \|\mathbf{v}\|^2 + \|\mathbf{w}\|^2 - 2\,\|\mathbf{v}\|\,\|\mathbf{w}\|\,cos\theta,$$

   θ being the angle between $\mathbf{v}$ and $\mathbf{w}$

3. Show that the area of a triangle with adjacent sides $\mathbf{v}$ and $\mathbf{w}$ is $\frac{1}{2}\|\mathbf{v} \times \mathbf{w}\|$

4. For any vectors $\mathbf{u}$, $\mathbf{v}$ and $\mathbf{w}$ show that

$$\mathbf{u} \times (\mathbf{v} \times \mathbf{w}) = (\mathbf{u}\cdot \mathbf{w})\mathbf{v} - (\mathbf{u}\cdot \mathbf{v})\mathbf{w}$$

5. For vectors $\mathbf{u}, \mathbf{v}, \mathbf{w}, \mathbf{z}$ show that

$$(\mathbf{u} \times \mathbf{v})\cdot (\mathbf{w} \times \mathbf{z}) = \begin{vmatrix} \mathbf{u}\cdot \mathbf{w} & \mathbf{u}\cdot \mathbf{z} \\ \mathbf{v}\cdot \mathbf{w} & \mathbf{v}\cdot \mathbf{z} \end{vmatrix}$$

6. Show that for $\mathbf{a} \neq 0$, $\mathbf{x} = \frac{\mathbf{b} \times \mathbf{a}}{\|a\|^2} + k\mathbf{a}$ solves $\mathbf{a} \times \mathbf{x} = \mathbf{b}$ for any scalar k.

7. Prove the **Jacobi Identity:** For any vectors $\mathbf{u}, \mathbf{v}, \mathbf{w}$

$$\mathbf{u} \times (\mathbf{v} \times \mathbf{w}) + \mathbf{v} \times (\mathbf{w} \times \mathbf{u}) + \mathbf{w} \times (\mathbf{u} \times \mathbf{v}) = 0$$

8. Find the point(s) of intersection of the sphere $(x - 3)^2 + (y + 1)^2 + (z - 3)^2 = 9$ and the line $x = -1 + 2t, y = 2 - 3t, z = 3 + t$.

9. Find the intersection of the sphere $x^2 + y^2 + z^2 = 9$ and the cylinder $x^2 + y^2 = 4$.

10. It can be shown that any four non coplanar points (i.e. points that do not lie in the same plane) determine a sphere (Ref: Welchons, A.M. and W.R. Krickenberger, Solid Geometry, pp 160). Assuming this fact find the equation of the sphere that passes through the points $(0, 0, 0), (0, 0, 2), (1, -4, 3)$ and $(0, -1, 3)$.

11. Let S be the sphere with radius 1 centered at $(0, 0, 1)$, and let S be S without the north pole point $(0, 0, 2)$. Let (a, b, c) be an arbitrary point on S. Then the line passing through $(0, 0, 2)$ and (a, b, c) intersects the xy-plane at some point $(x, y, 0)$. Find this point $(x, y, 0)$ in terms of a, b and c.

12. Let P_1 and P_2 be points whose spherical coordinates are $(\rho_1, \theta_1, \phi_1)$ and $(\rho_2, \theta_2, \phi_2)$, respectively. Let $\mathbf{v}_1$ be the vector from the origin to P_1, and let $\mathbf{v}_2$ be the vector from the origin to P_2. For the angle γ between $\mathbf{v}_1$ and $\mathbf{v}_2$, show that

$$\gamma = cos^{-1}(cos\phi_1 cos\phi_2 + sin\phi_1 sin\phi_2 cos(\theta_2 - \theta_1)).$$

13. Show that the distance d between the points P_1 and P_2 with cylindrical coordinates (r_1, θ_1, z_1) and (r_2, θ_2, z_2), respectively, is

$$d = \sqrt{r_1^2 + r_2^2 - 2r_1 r_2 cos(\theta_2 - \theta_1) + (z_2 - z_1)^2}.$$

14. Show that the distance d between the points P_1 and P_2 with spherical coordinates $(\rho_1, \theta_1, \phi_1)$ and $(\rho_2, \theta_2, \phi_2)$ respectively, is

$$d = \sqrt{\rho_1^2 + \rho_2^2 - 2\rho_1\rho_2[cos\phi_1 cos\phi_2 + sin\phi_1 sin\phi_2 cos(\theta_2 - \theta_1)]}.$$