Complex Analysis

Mathematical Methods for physicists

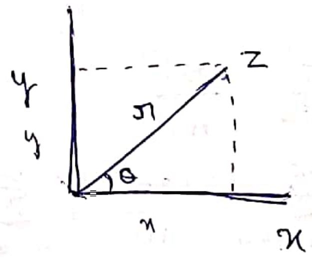
– Arfken & Webber

A complex number is denoted by

$$z = x + iy \ , \quad x, y \in R \text{ and } i = \sqrt{-1}$$

$$x = Re(z) \ , \quad y = Im(z)$$

cartesian coordinates



We can write

$$x = r\cos\theta \ , \quad y = r\sin\theta$$

The terms of polar coordinates

$$z = r(\cos\theta + i\sin\theta)$$

A complex function $f(z)$, depends on z can be resolved into real and imaginary parts

$$f(z) = u(x,y) + iv(x,y)$$

Example:

$$f(z) = z^2$$

$$f(z) = (x+iy)^2 = (x^2 - y^2) + 2ixy$$

$$u(x,y) = x^2 - y^2 \ , \quad v(x,y) = 2xy$$

complex functions can be constructed from functions of real variables.

Taylor series expansion for real function e^x :

$$e^x = 1 + x + \frac{x^2}{2!} + \dots = \sum_{n=0}^{\infty} \frac{x^n}{n!}$$

We can define complex function e^z as

$$e^z = \sum_{n=0}^{\infty} \frac{z^n}{n!}$$

$\sin z$, $\cos z$

For real θ, we have

$$e^{i\theta} = \sum_{n=0}^{\infty} \frac{(i\theta)^n}{n!}$$

$$= \sum_{n=\text{even}}^{\infty} \frac{(i\theta)^n}{n!} + \sum_{n=\text{odd}}^{\infty} \frac{(i\theta)^n}{n!}$$

$$= \sum_{l=0}^{\infty} \frac{(i\theta)^{2l}}{(2l)!} + \sum_{m=0}^{\infty} \frac{(i\theta)^{2m+1}}{(2m+1)!}$$

We have, $(i)^{2l} = (i^2)^l = (-1)^l$

$$(i)^{2m+1} = i(-1)^m$$

$$e^{i\theta} = \sum_{l=0}^{\infty} (-1)^l \frac{\theta^{2l}}{(2l)!} + i \sum_{m=0}^{\infty} (-1)^m \frac{\theta^{2m+1}}{(2m+1)!}$$

$$= \cos\theta + i\sin\theta$$

in polar coordinates

$$z = r(\cos\theta + i\sin\theta)$$

$$= re^{i\theta}$$

For a fixed z, θ can have arbitrary values

$$z = re^{i\theta} = re^{i(\theta+2n\pi)}, \quad n \in \mathbb{Z}$$

$$e^{i2n\pi} = 1$$

'log' of complex number can be written as

$$\ln z = \ln [r e^{i(\theta + 2n\pi)}]$$

$$= \ln r + i(\theta + 2n\pi)$$

$$\left.\begin{array}{l} \sum_{n=0}^{N-1} \cos nx \\ \sum_{n=0}^{N-1} \sin nx \end{array}\right\} \xrightarrow[\text{these}]{\text{put}} \text{Gibbs phenomenon}$$

Cauchy - Riemann conditions for differentiation

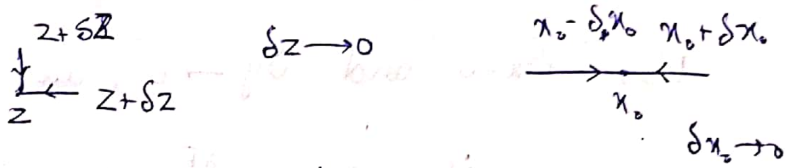
Let $f(z)$ be a complex function

Differentiation of $f(z)$ is defined as

$$\frac{df}{dz} = f'(z) = \lim_{\delta z \to 0} \frac{f(z+\delta z) - f(z)}{z + \delta z - z}$$

$$= \lim_{\delta z \to 0} \frac{\delta f(z)}{\delta z}$$

provided the limit exist irrespective of direction of approach.



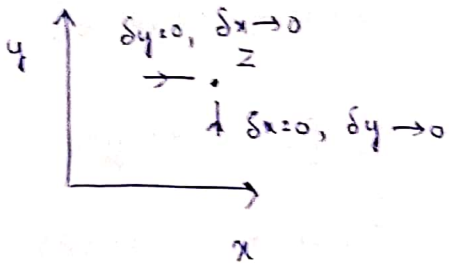
For $f(z) = u(x,y) + iv(x,y)$ and $z = x + iy$,

we can write

$$\delta f = \delta u + i\delta v \text{ and } \delta z = \delta x + i\delta y$$

$$\frac{df}{dz} = \lim_{\delta z \to 0} \frac{\delta u + i\delta v}{\delta x + i\delta y}$$

The limit $\delta z \to 0$ can be ~~wort~~ taken in two different ways



We assume that the ~~particular~~ partial derivatives of $u(x,y)$ & $v(x,y)$ with respect to x, y exist

For $\delta y = 0$ and $\delta x \to 0$.

$$\frac{df}{dz} = \lim_{\delta z \to 0} \frac{\delta f}{\delta z}$$

$$= \lim_{\delta x \to 0} \frac{\delta u + i\,\delta v}{\delta x}$$

$$= \lim_{\delta x \to 0} \frac{\delta u}{\delta x} + i\frac{\delta v}{\delta x}$$

$$= \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} \longrightarrow (1)$$

$\delta y = 0,\ \delta x \to 0$

$$\frac{dF}{dz} = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} \longrightarrow (1)$$

For $\delta x = 0$ and $\delta y \to 0$, we have

$$\frac{dF}{dz} = \lim_{\delta z \to 0} \frac{\partial F}{\partial z}$$

$$= \lim_{\delta y \to 0} \frac{\delta u + i\,\delta v}{i\,\delta y}$$

$$= \lim_{\delta y \to 0} -i\frac{\delta u}{\delta y} + \frac{\delta v}{\delta y}$$

$$\frac{dF}{dz} = -i\frac{\partial u}{\partial y} + \frac{\partial v}{\partial y} \longrightarrow (2)$$

Equating real & imaginary parts of eq ① & ②

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

These are cauchy - Riemann conditions these are necessary if the derivative $\frac{dF}{dz}$ exists.

Conversely, assume that cauchy - Riemann conditions are satisfied and that the partial derivatives of $u(x,y)$ & $v(x,y)$ are continuous. Then, we can prove that derivative $\frac{dF}{dz}$ exists.

We have

$$\delta F = \delta u + i\,\delta v$$

$$= \frac{\partial u}{\partial x}\delta x + \frac{\partial u}{\partial y}\delta y + i\left[\frac{\partial v}{\partial x}\delta x + \frac{\partial v}{\partial y}\delta y\right]$$

$$= \left[\frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}\right]\delta x + \left[\frac{\partial u}{\partial y} + i\frac{\partial v}{\partial y}\right]\delta y$$

Then

$$\frac{\delta F}{\delta z} = \frac{\left(\frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}\right)\delta x + \left(\frac{\partial u}{\partial y} + i\frac{\partial v}{\partial y}\right)\delta y}{\delta x + i\,\delta y}$$

$$= \frac{\frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} + \left(\frac{\partial u}{\partial y} + i\frac{\partial v}{\partial y}\right)\frac{\delta y}{\delta x}}{1 + i\frac{\delta y}{\delta x}} \longrightarrow (1)$$

Using cauchy - Riemann conditions

$$\frac{\partial u}{\partial y} + i\frac{\partial v}{\partial y} = -\frac{\partial v}{\partial x} + i\frac{\partial u}{\partial x}$$

$$= i\left(\frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}\right)$$

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Then

$$\frac{\delta F}{\delta z} = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}$$

Since the above relation doesn't depend on the direction of approach, derivative $\frac{dF}{dz}$ exists

A function $f(z)$ is analytic at $z = z_0$ if $f(z)$ is differentiable at $z = z_0$

Otherwise,

z_0 is a singular point.

Example 1 :: $f(z) = z^2 = u(x,y) + i\,v(x,y)$

$$\Rightarrow u(x,y) = x^2 - y^2, \quad v(x,y) = 2xy$$

$$\frac{\partial u}{\partial x} = 2x = \frac{\partial v}{\partial y}, \quad \frac{\partial u}{\partial y} = -2y = -\frac{\partial v}{\partial x}$$

$f(z) = z^2$ is analytic in the entire complex plane

Example 2 :: $f(z) = z^*$

$$\Rightarrow u(x,y) = x, \quad v(x,y) = -y$$

Now,

$$\frac{\partial u}{\partial x} = 1 \neq \frac{\partial v}{\partial y} = -1$$

$f(z) = z^*$ is not an analytic function.