Two Distance set

One distance sets.

$EqDim(\mathbb{R}^n)$ over L_2 norm $= n+1$

$\leq n+1$ (to show)

$\geq n+1$

over L_∞ norm $= 2^n$

over L_1 norm $\leq 2n$ [Kusner conj]

$\geq 2n$

Theorem Let $a_0, a_1, \ldots, a_m$ be m points in $\mathbb{R}^n$. If they are all pairwise equidistant (under L_2-norm) then $m \leq n$. (i.e. # points $\leq n+1$)

Proof:

Proof using independence criterion

$a_0, a_1, \ldots, a_m$ } let d be the distance btwn any two pts.

$\forall i \in \{0,1,2,\ldots,m\}$

$f_i : \mathbb{R}^n \to \mathbb{R}$ defined as

$$f_i(\overset{=(x_1,x_2,\ldots,x_n)}{x}) = \|x - \overset{=(a_{i1},a_{i2},\ldots,a_{in}) \in \mathbb{R}^n}{a_i}\|^2 - d^2$$

$\|x-a_i\|^2 = \langle x-a_i, x-a_i\rangle$

$$f_i(a_i) = -d^2$$

$$j \neq i, \quad f_i(a_j) = 0$$

From Indep Criterion, we know

$f_0, f_1, \ldots, f_m$ are L.I. in the space $\mathbb{R}^{\mathbb{R}^n}$ over $\mathbb{R}$.

Expanding $f_i(x)$

$$f_i(x) = \langle x - a_i, x - a_i\rangle - d^2$$

$$= \langle \overset{(x_1,x_2,\ldots,x_n)}{x}, x\rangle + \langle \overset{(a_{i1},a_{i2},\ldots,a_{in})}{a_i}, a_i\rangle - 2\langle x, a_i\rangle - d^2$$

$$\sum^n x_i^2 \qquad x_i \qquad 1$$

$$\sum_{i=1} x_i^2, \quad x_i, \quad 1$$

$1 \le i \le n$

$n+2$

So,

$$m+1 \le n+2$$

$$\text{or } m \le n+1$$

weaker bound.

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Proof w/o using independence criterion

$a_0, a_1, \dots, a_m$ Equidistant

To show: $m \le n$

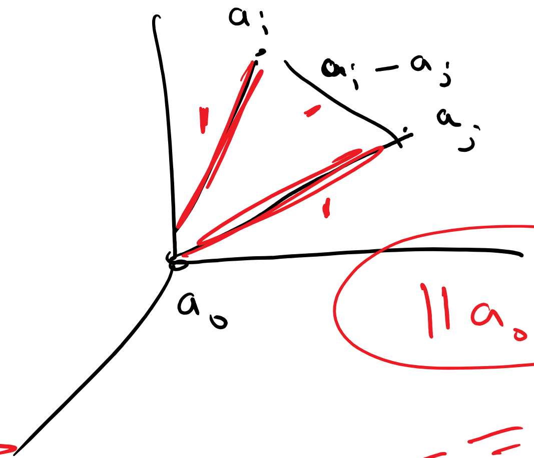
Proof: Let d be the distance btwn any two pts in $a_0, a_1, \dots, a_m$.

Assume $d = 1$.

Assume $a_0 = (\overbrace{0, 0, \dots, 0}^{n})$

We know,

$\|a_i - a_j\| = 1$



$\|a_0\|^2 \neq 1$

When $i > 0$, $j > 0$,

$$\langle a_i - a_j, a_i - a_j \rangle = 1$$

i.e. $\underbrace{\|a_i\|^2}_{=1} + \underbrace{\|a_j\|^2}_{=1} - 2\langle a_i, a_j \rangle = 1$

or $2 - 2\langle a_i, a_j \rangle = 1$

i.e. $\langle a_i, a_j \rangle = \frac{1}{2}$ — (1)

Claim: $a_1, a_2, \ldots, a_m$ are L.I. in $\mathbb{R}^n$. (This would imply that $m \leq n$)

Suppose not. Then,

$$\lambda_1 a_1 + \lambda_2 a_2 + \cdots + \lambda_m a_m = 0 \quad \text{— (2)}$$

has a non-trivial solution for λs.

Take inner product with a_1, on either sides of (2)

$$\langle a_1, \lambda_1 a_1 + \lambda_2 a_2 + \cdots + \lambda_m a_m \rangle = \langle a_1, 0 \rangle$$

$$\lambda_1 \langle a_1, a_1 \rangle + \lambda_2 \langle a_1, a_2 \rangle + \cdots + \lambda_m \langle a_1, a_m \rangle = 0$$

i.e. $\lambda_1 \cdot 1 + \frac{1}{2}\lambda_2 + \frac{1}{2}\lambda_3 + \cdots + \frac{1}{2}\lambda_m = 0$ (from (1))

i.e. $\frac{1}{2} \sum_{j \neq 1} \lambda_j + \lambda_1 = 0$ — (3)

i.e. $\quad 2 \quad j \neq 1$

Similarly taking inner product of either sides of Eqn (2) with $a_2, a_3, \ldots, a_m$ to get,

$$\frac{1}{2} \sum_{j \neq 2} \lambda_j + \lambda_2 = 0 \quad \text{--- (4)}$$

$$\vdots \qquad \vdots$$

$$\frac{1}{2} \sum_{j \neq m} \lambda_j + \lambda_m = 0 \quad \text{--- (m+2)}$$

Adding (3) + (4) + . . . + (m+2)

$$\frac{1}{2}(m-1) \sum_{j=1}^{m} \lambda_j + \sum_{j=1}^{m} \lambda_j = 0$$

i.e. $$\sum_{j=1}^{m} \lambda_j \left(\frac{m+1}{2} \right) = 0$$

i.e. $$\sum_{j=1}^{m} \lambda_j = 0 \quad \text{--- (m+3)}$$

Rewriting (3),

$$\frac{1}{2}\sum_{j\neq 1}\lambda_j + \lambda_1 - \frac{1}{2}\lambda_1 + \frac{1}{2}\lambda_1 = 0$$

i.e. $$\frac{1}{2}\sum_{j=1}^{m}\lambda_j + \frac{1}{2}\lambda_1 = 0$$

from $(m+3)$

i.e. $$0 + \frac{1}{2}\lambda_1 = 0$$

i.e. $\lambda_1 = 0$ //

Similarly, one can show that

$$\lambda_2 = \lambda_3 = \cdots = \lambda_m = 0$$

Thus, our assumption that $a_1, a_2, \ldots, a_m$ are not L.I. in $\mathbb{R}^n$ is false.

This implies that, $m \leq n$.

$\square$

Two Distance Set

Given: d_1, d_2

$\mathbb{R}^n$ Euclidean distance
or L_2-norm

Q. Put as many pts in $\mathbb{R}^n$ s.t. the distance btwn any two of them is either d_1 or d_2.

Theorem: Every two-distance set in $\mathbb{R}^n$ has at most $\binom{n}{2} + 3n + 2$ points.