CS 435 : Linear Optimization | Fall 2008

Lecture 14: Duality Theorem

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Here is the duality theorem.

1 Duality

Theorem 1 *If the primal is feasible and has a finite optimum then the dual is also feasible, has a finite optimum and their optimums must be equal.*

Proof:
We first show that the dual is feasible. This means we can find a y which satisfies $A^T y = c; y \geq$. What does $A^T y = c$ mean? It means we can write the cost function as a linear combination of the rows of A. Rows of A are also the outward normals to the hyperplanes in $Ax \leq b$. The inequality $y \geq$ means the coefficients should be non-negative.

We showed that at an optimum point of the primal, we can write the cost vector as a non-negative linear combination of the normals to the defining hyperplanes. The coefficients of this non-negative linear combination yield a feasible point in the dual.

In other words, we showed last time that for each point in we can find a feasible point in the dual. The set is non-empty since it contains at least the optimum point x_0. Hence we infer that the dual is feasible.

Next we show that the dual has a finite optimum. Let y be any feasible point in the dual. Let x_0 an optimal point in the primal. Then the cost of y in the dual t is $y^T b \geq y^T Ax_0 = c^T x_0$. Hence the dual cost is bounded from below by $c^T x_0$. To prove the last part, we only need to observe that since x_0 is in there is a feasible y_0 corresponding to it with the same cost. This y_0 has to be the optimum point.

Hence the cost of the two optimums are equal. □

2 The Dual of the Dual is the Primal

Note that the dual is also a linear program. What is its dual? To see this, one way is to put it in the format of the primal and then infer the dual.

$$\min y^T \tag{1}$$
$$A^T y = c \tag{2}$$
$$y \geq 0 \tag{3}$$
$$\tag{4}$$

The dual mentioned above can be rephrased as follows.

$$max -^T y \tag{5}$$
$$A^T y \leq c \tag{6}$$
$$A^T y \geq c \tag{7}$$
$$y \geq 0 \tag{8}$$
$$\tag{9}$$

Exercise: Determine the dual of the above LP and prove that it is the same as the primal.

Here are some tips that let you check your answer. What are the variables in the dual? We started by writing the cost vectors as a non-negative linear combination of the normals. The dual variables correspond to these co-efficients. For instance there is one dual variable for each inequality in the primal.

The dual now has n equalities and m inequalities. What do we do with the equalities? An equality is equivalent to two inequalities. Which means we can use either normal. This means the dual variable corresponding to eqalities in the primal will not be sign constrained. So, there are n variables which are not sign constrained (call them x) and m which are sign constrained (call them z). Also if the cost function is minimization in one, it will be maximisation in the other.

Putting this together we see that the dual will be

$$max^T x \tag{10}$$
$$Ax + z = \tag{11}$$
$$z \geq \tag{12}$$

One can remove z and recover the original primal.

This was just a heuristic construction. We recommend that you do this formally and then use these arguments to check your answer.

Note: For the duality theorem we only say that the cost of points in the Primal and the Dual are bounded for a given cost function. We have not said that they are bounded for *all* cost functions. In fact this will not be true.