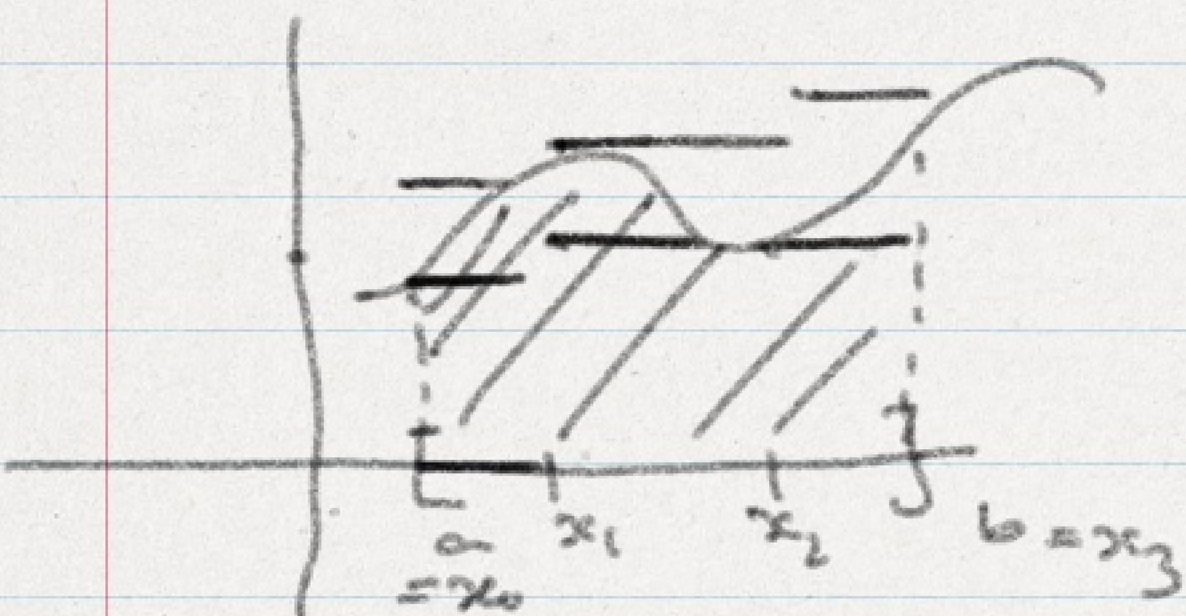
Let $f:[a,b] \longrightarrow \mathbb{R}$

$\exists\, m, M \in \mathbb{R}$ s.t.
$m \leq f(x) \leq M \;\forall x \in [a,b]$



$$\int_a^b f(x)\,dx$$

$P = \{x_0 < x_1 < x_2 < x_3\}$

$[x_{i-1}, x_i]_3$

$$L(P,f) = \sum_{i=1} m_i \cdot \Delta x_i$$

where $m_i = \inf\limits_{x \in [x_{i-1}, x_i]} f(x)$

$\Delta x_i = x_i - x_{i-1}$

$$U(P, f)$$

$$m(b-a) \leq L(P, f) \leq U(P, f) \leq M(b-a)$$

Let P_1 be a finer partition of P of $[a, b]$

$\underline{P_1 \supset P}$

$$L(P, f) \leq L(P_1, f)$$

$$I = [c, d]$$

$$\left[c, \frac{c+d}{2}\right] = I_1 \subset I$$

$$U(P, f) \geq U(P_1, f)$$

$$m(b-a) \leq L(P, f) \leq L(P_1, f) \leq U(P_1, f) \leq U(P, f) \leq M(b-a)$$

$\because L(P, f) \uparrow$ and all $L(P, f) \leq M(b-a)$

Hence $\sup_{P \in \mathcal{P}[a,b]} L(P, f) \leq M(b-a)$

$$\inf_{P \in \mathcal{P}[a,b]} U(P,f) \geq m(b-a)$$

Define $\overline{\int_a^b} f(x)\,dx =$

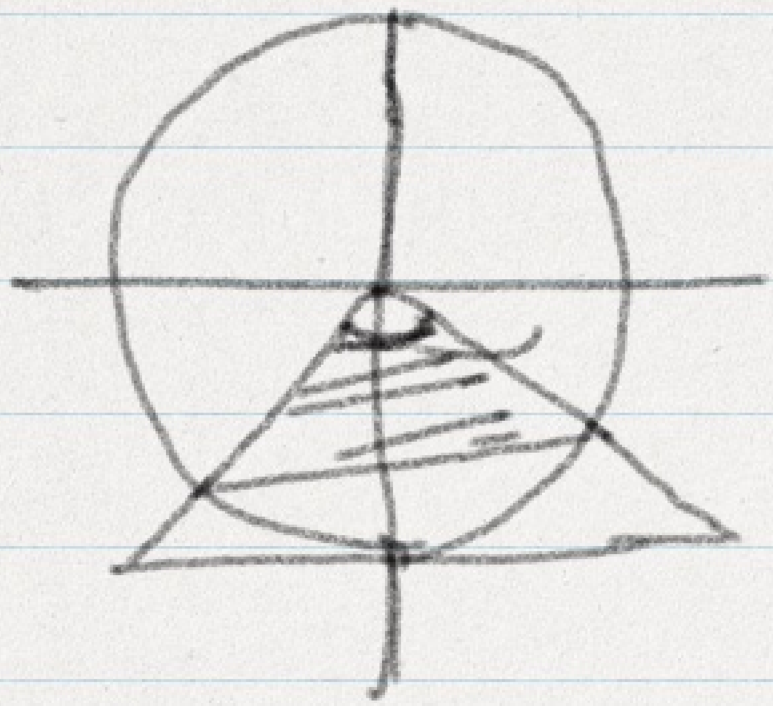
$$\inf U(P,f)$$

& $\underline{\int_a^b} f(x)\,dx = \sup L(P,f)$.

If $\underline{\int_a^b} f(x)\,dx = \overline{\int_a^b} f(x)\,dx$

f is said to be Riemann Int. over $[a,b]$.

<u>Remark</u>: $\underline{\int_a^b} f(x)\,dx \leq \overline{\int_a^b} f(x)\,dx$.



Let A be the area bounded by the circle.

Let $n \in \mathbb{N}$.

$$n\, 2 \cdot \frac{1}{2} \sin \frac{\pi}{n} \cos \frac{\pi}{n} \leq A \leq n\, 2 \frac{1}{2} \tan \frac{\pi}{n}$$

As $n \longrightarrow \infty$, $LHS = RHS = \pi$.

Hence by Sandwitch thm.

$A = \pi$.

Why $\underline{\int_a^b} f(x)\,dx \leq \overline{\int_a^b} f(x)\,dx.$

Let P_1, P_2 be any two partition of $[a,b]$

Let $P = P_1 \cup P_2$ then P a new partition of $[a,b]$

$$L(P,f) \geq L(P_1,f), L(P_2,f)$$
$$U(P,f) \leq U(P_1,f), U(P_2,f)$$

$$L(P_1,f) \leq L(P,f) \leq U(P,f) \leq U(P_2,f)$$

Hence $\underline{\int_a^b} f(x) \leq U(P_2 f)$

$$\leq \overline{\int_a^b} f(x)\,dx$$

<u>Ex:</u> function which is not Riemann Integrable.

$f : [0,1] \longrightarrow \mathbb{R}$.

$$f(x) = \begin{cases} 0 & \text{if } x \in \mathbb{Q} \cap [0,1] \\ 1 & \text{if } x \notin \mathbb{Q} \cap [0,1] \end{cases}$$

Then f is not R-int. over $[a,b]$

$\because$ $U(P,f) = 1$
$L(P,f) = 0$

<u>Thm:</u> Let $f : [a,b] \longrightarrow \mathbb{R}$ be a bounded function. Then f is R-int. over $[a,b]$ iff for given $\varepsilon > 0$ $\exists$ a partition P of $[a,b]$

s.t. $U(P,f) - L(P.f) < \varepsilon$.

α **Ex:** Let $f : [0,1] \to \mathbb{R}$ be defined by $f(x) = x$.
Show that f is R-int, find the value of $\int_a^b f(x)\,dx$.

Thm: Let $(P_n)_{n=1}^{\infty}$ be a sequence of partition of $[a,b]$ such that $\|P_n\| \to 0$ as $n \to \infty$. Suppose
$\lim_n L(P_n, f) = \lim_n U(P_n, f)$.

Then f is R-int over $[a,b]$ and $\int_a^b f(x)\,dx = \lim_n L(P_n, f)$

$\|P_n\|$ = maximum length of the subinterval of P_n.

$\underline{Ex}$: $$L(P_n, f) = \sum_{i=1}^{n} \frac{i-1}{n} \cdot \frac{1}{n}$$

$$= \frac{1}{n^2} \sum_{i=1}^{n} (i-1)$$

$$U(P_n, f) = \frac{1}{n^2} \sum_{i=1}^{n} i$$

$$\lim_n L(P_n, f) = \lim_n U(P_n f) = \frac{1}{2}$$

$\underline{Thm}$: Let $f : [a, b] \longrightarrow \mathbb{R}$ be continuous / monotone then f is Riemann Integrable over $[a, b]$

$$U(P, f) - L(P, f) = \sum_{i=1}^{n} (M_i - m_i) \Delta x_i$$

$$< \varepsilon (b-a)$$

Fundamental theorem of Calculus (I)

Let f be Riemann Int on $[a,b]$. Define

$$F(x) = \int_a^x f(t)\,dt$$

then F is continuous on $[a,b]$.
If f is cont at x_0 then F is differentiable at x_0.

Pf: $x_1, x_2 \in [a,b]$, $x_1 < x_2$

$$|F(x_1) - F(x_2)|$$
$$= \left| \int_{x_1}^{x_2} f(t)\,dt \right|$$
$$\le \int_{x_1}^{x_2} |f(t)|\,dt$$
$$\le \int_{x_1}^{x_2} M\,dt$$

$= M(x_1 - x_2)$

$\leq M|x_1 - x_2|$

$|F(x_1) - F(x_2)| \leq M|x_1 - x_2|$

ie F is u.c. on $[a, b]$

<u>Case 2</u> : When f is cont.

$$\left| \frac{F(x) - F(x_0)}{x - x_0} - f(x_0) \right|$$

$$= \left| \frac{1}{x - x_0} \int_{x_0}^{x} f(t)\, dt - f(x_0) \right|$$

$$= \left| \frac{1}{x - x_0} \int_{x_0}^{x} (f(t) - f(x_0))\, dt \right|$$

$\vdots$

$< \varepsilon$

Finally,

$$\lim_{x \to x_0} \frac{F(x) - F(x_0)}{x - x_0} = f(x_0)$$

FTC (II)

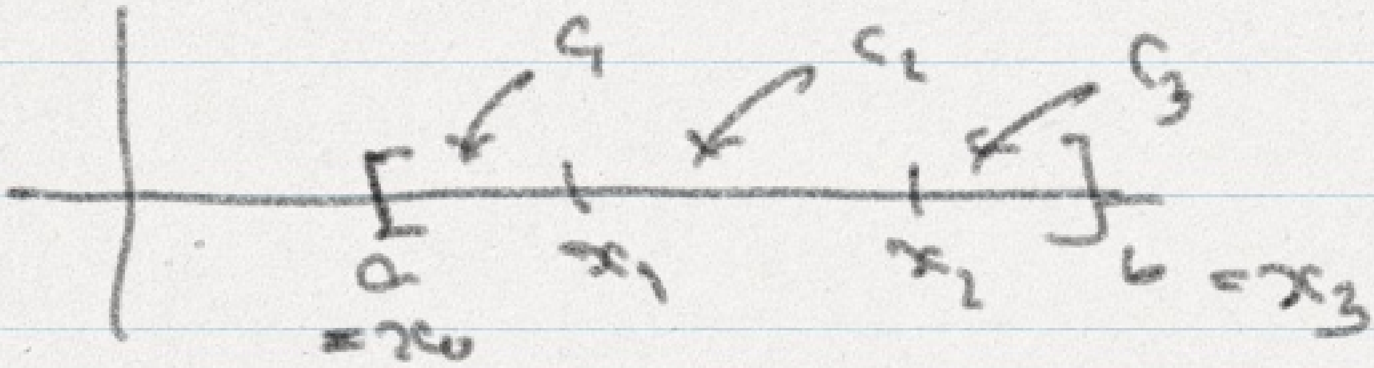
Let F is diff on (a,b) and Cont on $[a,b]$. Let F' be integrable on $[a,b]$, then

$$\int_a^b F'(t)\,dt = F(b) - F(a).$$

Fact: Let $f: [a,b] \to \mathbb{R}$ be R-integrable. Let P be any partion of $[a,b]$. and $c_i \in [x_{i-1}, x_i]$ Then

$$\lim_{\|P\| \to 0} \sum_{i=1}^{n} f(c_i)\Delta x_i = \int_a^b f(t)\,dt$$

where $\|P\|$ = maximum length of the subintervals of P.



$$L(P,f) \le \sum_{1}^{n} f(c_i)\Delta x_i \le U(P,f)$$

$$L(P,f) \le \int_a^b f(t)\,dt \le U(P,f)$$

Application :

Let f be cont on $\mathbb{R}$ and p be a fixed real no. for which $f(x+p) = f(x)$ $\forall x \in \mathbb{R}$

Then $\int_a^{a+p} f(t)\,dt$ is constant $\forall a \in \mathbb{R}$.

Define $F(x) = \int_0^x f(t)\,dt$

~~We have to prove~~ $\cancel{F'(a) = 0}$.

$$\cancel{\frac{d}{da} F(a) = f(a+p) - f(a)}$$

$$\cancel{F'(a+p) = f(a+p)}$$

$$F'(x) = f(x)$$

$$\frac{d}{da}\left(\int_a^{a+p} f(t)\,dt\right)$$

$$= \frac{d}{da}\left(\int_0^{a+p} f(t)\,dt - \int_0^{a} f(t)\,dt\right)$$

$$= f(a+p) - f(a)$$

$$= 0$$

Hence $\int_a^{a+p} f(t)\,dt$ constant $\forall a \in \mathbb{R}$.

2. Find the limit

$$\lim_{x \to 0} \frac{1}{x^3} \int_0^x \frac{t^2}{1+t^4}\,dt$$

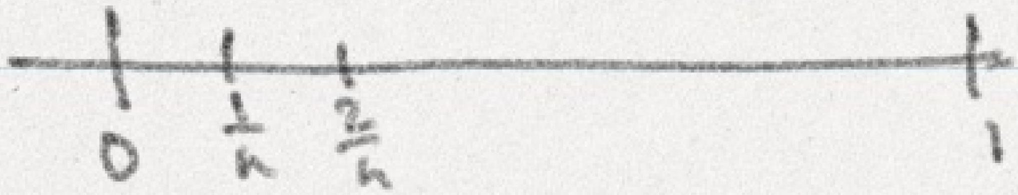
3. Find $\lim_{n\to\infty} x_n$

$$x_n = \frac{1}{n} + \frac{1}{n+1} + \cdots + \frac{1}{2n-1}$$

$n+n-1$

Now $x_n =$

$$\frac{1}{n}\left[\frac{1}{1} + \frac{1}{1+\frac{1}{n}} + \cdots + \frac{1}{1+\frac{n-1}{n}}\right]$$



$$f(x) = \frac{1}{1+x}$$

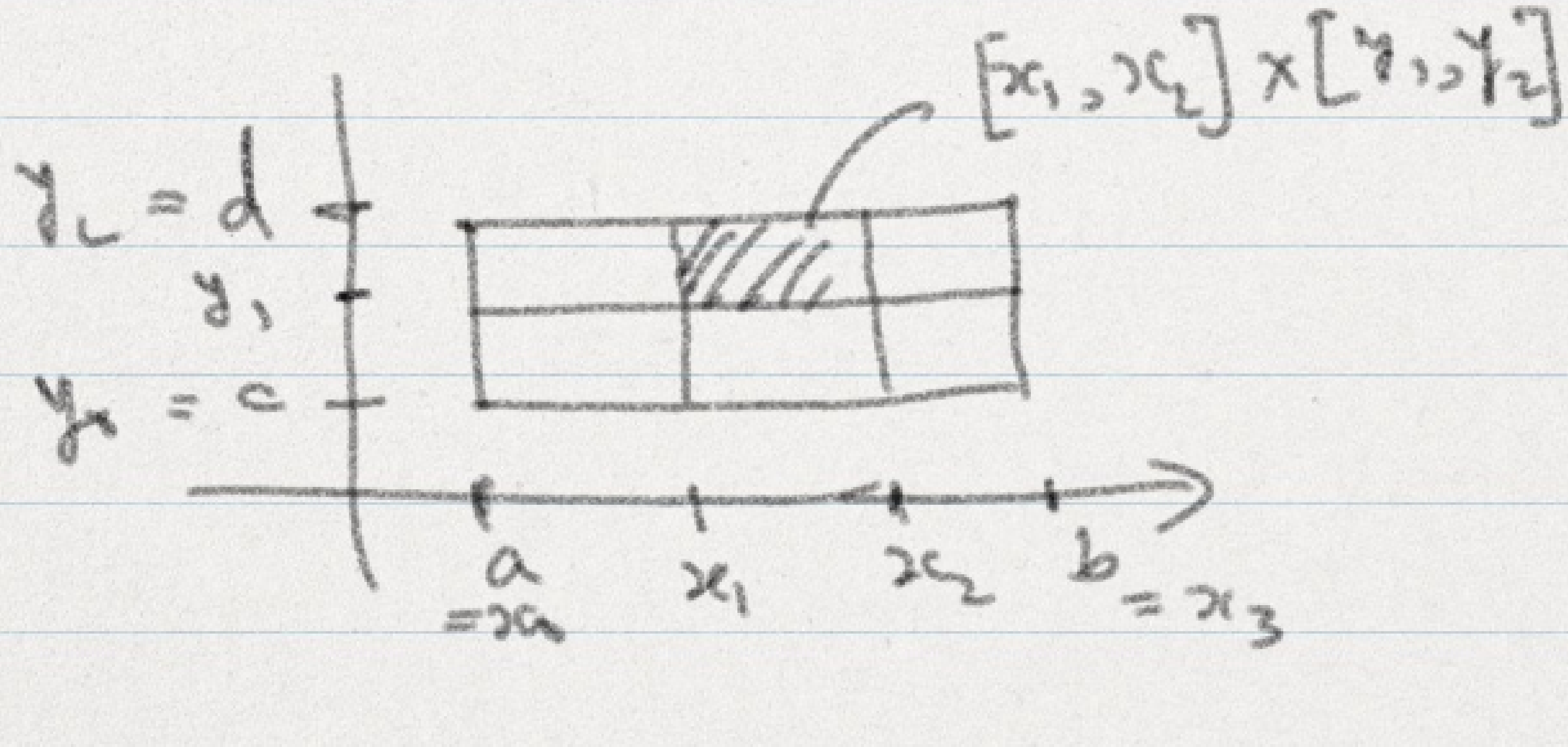
$$\lim_{n\to\infty} x_n = \int_0^1 \frac{1}{1+x}\,dx.$$

<u>Ex 1</u> Let $f: [0,1] \to \mathbb{R}$ be a cont function. S.T.

$$\lim_{n\to\infty} \int_0^1 x^n f(x) = 0$$

<u>Ex 2</u>: $f: [0,1] \to \mathbb{R}$ be cont.
S.T. $\lim_{n\to\infty} \int_0^1 f(x^n)\, dx = f(0)$

Let $f: [a,b] \times [c,d] \to \mathbb{R}$ bounded.



Hence

$$m \leq f(x,y) \leq M$$

Define $U(P,f)$ and $L(P,f)$

$$m(b-a)(d-c) \leq L(P,f) \leq U(P,f) \leq M(b-a)(d-c).$$

If $P_1 \supseteq P$ then

$$L(P_1,f) \geq L(P_1,f)$$

$$U(P_1,f) \leq U(P_1,f)$$

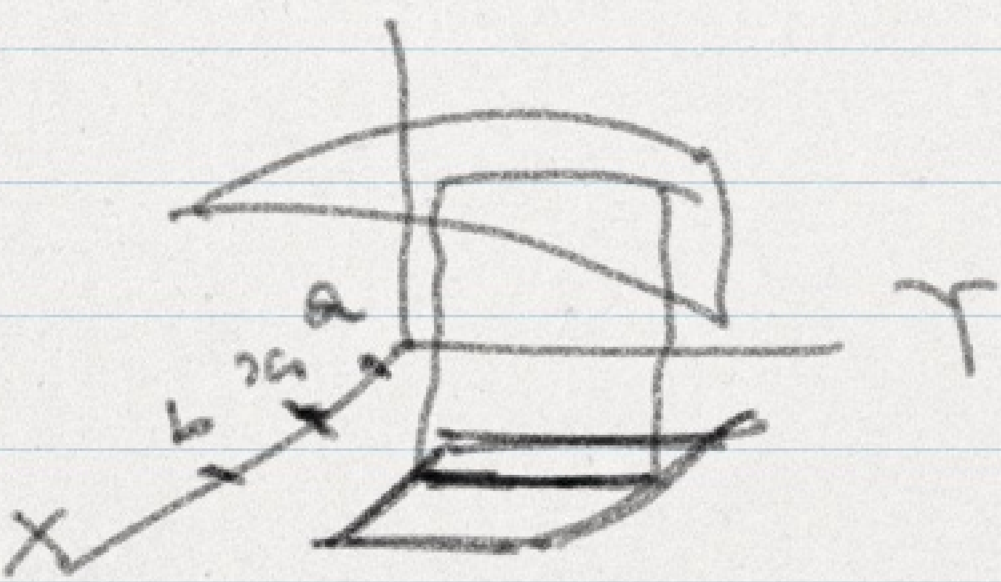
If $\inf U(P,f) = \sup L(P,f)$

then f is said to be R integrable over $[a,b]\times[c,d]$

Fubini's thm: Suppose $f: [a,b] \times [c,d] \to \mathbb{R}$ be cont then $\int_a^b \int_c^d f(x,y)\,dx\,dy$ exists and it is same as

$$\int_{x=a}^{b} \left(\int_{y=c}^{d} f(x,y)\,dy \right) dx =$$

$$\int_{y=c}^{d} \left(\int_{x=a}^{b} f(x,y)\,dx \right) dy$$



Ex: $f(x,y) = x^{19} y^{9} - 9\sin(x)\cos(y)$

$$\int_a^b \left(\int_c^d f(x,y)\,dy \right) dx$$

$$= \int_c^d \left(\int_{x=a}^{b} f(x,y)\,dx \right) dy$$