CS6350: Topics in Combinatorics
Assignment 7

Abburi Venkata Sai Mahesh - CS18BTECH11001

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1. Show that there is a finite n_0 such that any directed graph on $n > n_0$ vertices in which each outdegree is at least log(n) [logarithm to the base 2] contains an even simple directed cycle.

A. Let us consider a directed graph G with n vertices and m edges with outdegree of each vertex is atleast $\log_2 n$. Now let us apply a random bi-coloring(red, blue) on the graph.
Now consider a vertex v_i and find a neighbour v_{i+1} of v_i with different colour as of v_i. Now find a neighbour v_{i+2} of v_{i+1} with different colour as of v_{i+1} and so on. When the process continues we end up with an already chosen vertex implying an even cycle.
For this to happen we have to choose the graph colouring such that neighbourhood of each vertex $v \in V$ should contains atleast one vertex of different colour that of v (non-monochromatic neighbourhood).
Lemma: There exist a graph colouring such that, for every vertex $v \in V$ with a non-monochromatic neighbourhood .
Proof: Let us consider an event X(v) for a vertex $v \in V$, such that N(v) contains all vertices of same colour.

$$\begin{aligned} Pr(X(v)|v \in V) &= \left(\frac{1}{2}\right)^{|N(v)|} \\ &\leq \left(\frac{1}{2}\right)^{(\log_2 n)} \quad [\because |N(v)| \geq \log_2 n] \\ &= \frac{1}{n} \end{aligned}$$

Now consider Expectation of X,

$$E[X] = \sum_{i=1}^{n} p(X(v_i)|v_i \in V)$$
$$\leq \sum_{i=1}^{n} \frac{1}{n} = 1$$

This expectation also includes the graph colouring where all the n vertices are coloured same(either all vertices are red or all vertices are blue). This makes that there are n monochromatic neighbourhood colourings.
As the expectation of number of monochromatic neighbourhood colouring over all graph colourings is atmax 1, *there should exists atleast one graph colouring with all the vertices having non-monochromatic neighbourhood coloring* to counterbalance the effect of above said n monochromatic neighbourhood colouring.
So, there exists a simple even cycle $\forall n > 1$
$\Rightarrow \boxed{n_0 = 1}$.
Hence Proved.