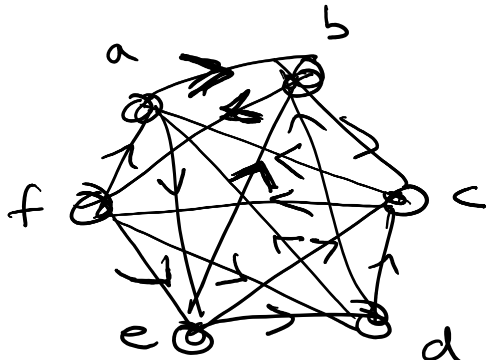
Tournaments

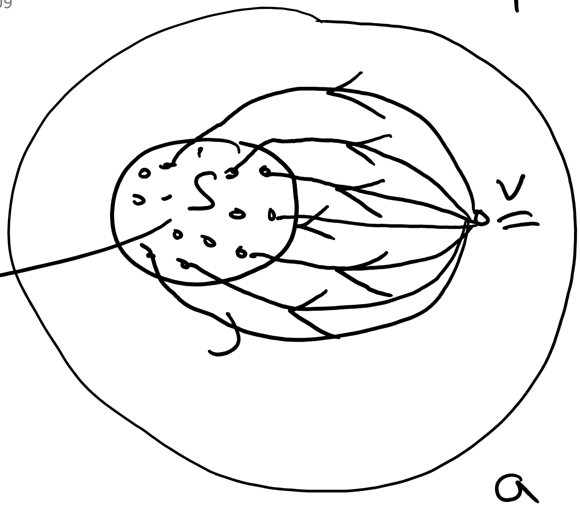
A complete graph
↳ orient every edge.



K_6

Let T be a tournament on n vertices. We say <u>T satisfies the Property P_k</u> if for every set S of k players/vertices in T, there is a player/vertex who has defeated everybody in S.

T on n vertices
$n \geq k$.



$|S| = k$.

Q. For every +ve integer k, does there always exist a tournament on n vertices that satisfies Property P_k.

Yes.

Next Q: Given a k, what is the min n s.t. there is a tournament on n vertices that satisfies P_k.

Theorem: If $\binom{n}{k}\left(1-\frac{1}{2^k}\right)^{n-k} < 1$,

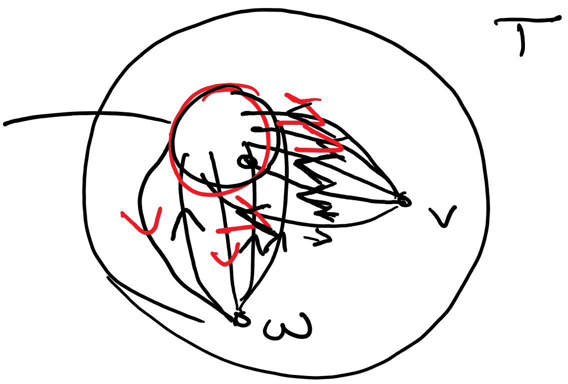
then there is a tournament on n vertices that satisfies Property P_k.

Proof: Construct a random tournament T on n vertices in the following way:

→ Take a complete graph on n vertices

→ For each edge e, independently toss an unbiased coin and orient the edge e based on the outcome of the coin toss.

$|S_1| = k$



$$\Pr[v \text{ defeats all players inside } S_1] = \frac{1}{2^k}$$

$$\Pr[v \text{ does not defeat all players in } S_1] = 1 - \frac{1}{2^k}$$

$$\Pr[w \text{ does not defeat all players in } S_1] = 1 - \frac{1}{2^k}$$

$$\Pr\left[\begin{array}{c}\text{no vertex outside } S_1 \text{ defeats} \\ \text{all players in } S_1\end{array}\right] = \left(1 - \frac{1}{2^k}\right)^{n-k}$$

Bad event when no player defeats everybody in S_1. $\} \to E_1$

$$\Pr[T \text{ does not satisfy } P_k] = \Pr\left[E_1 \cup E_2 \cup \dots \cup E_{\binom{n}{k}}\right]$$

$$\leq \Pr[E_1] + \dots + \Pr\left[E_{\binom{n}{k}}\right]$$

$$= \binom{n}{k}\left(1 - \frac{1}{2^k}\right)^{n-k}$$

$$< 1 \quad (\text{given})$$

$\therefore \Pr[T \text{ satisfies } P_k] = 1 - \Pr[T \text{ does not}]$

So $\Pr[T \text{ satisfies } P_k] = 1 - \Pr[T \text{ does not satisfy } P_k] > 0$ //

$\square$

We showed that there exists a tournament T on n vertices which satisfies P_k provided

$$\binom{n}{k}\left(1-\frac{1}{2^k}\right)^{n-k} < 1.$$

$$\binom{n}{k} \le \left(\frac{en}{k}\right)^k \qquad \left(1-\frac{1}{2^k}\right)^{n-k} \le \frac{1}{e^{\frac{n-k}{2^k}}}$$

$$1+x \le e^x$$

If $\left(\frac{en}{k}\right)^k \frac{1}{e^{\frac{n-k}{2^k}}} < 1$

do the calculations.

whenever $n \ge 2\log_e 2\, k^2 \cdot 2^k$

Let $f(k)$ be the smallest n s.t. there is a tournament on n nodes that satisfies P_k.

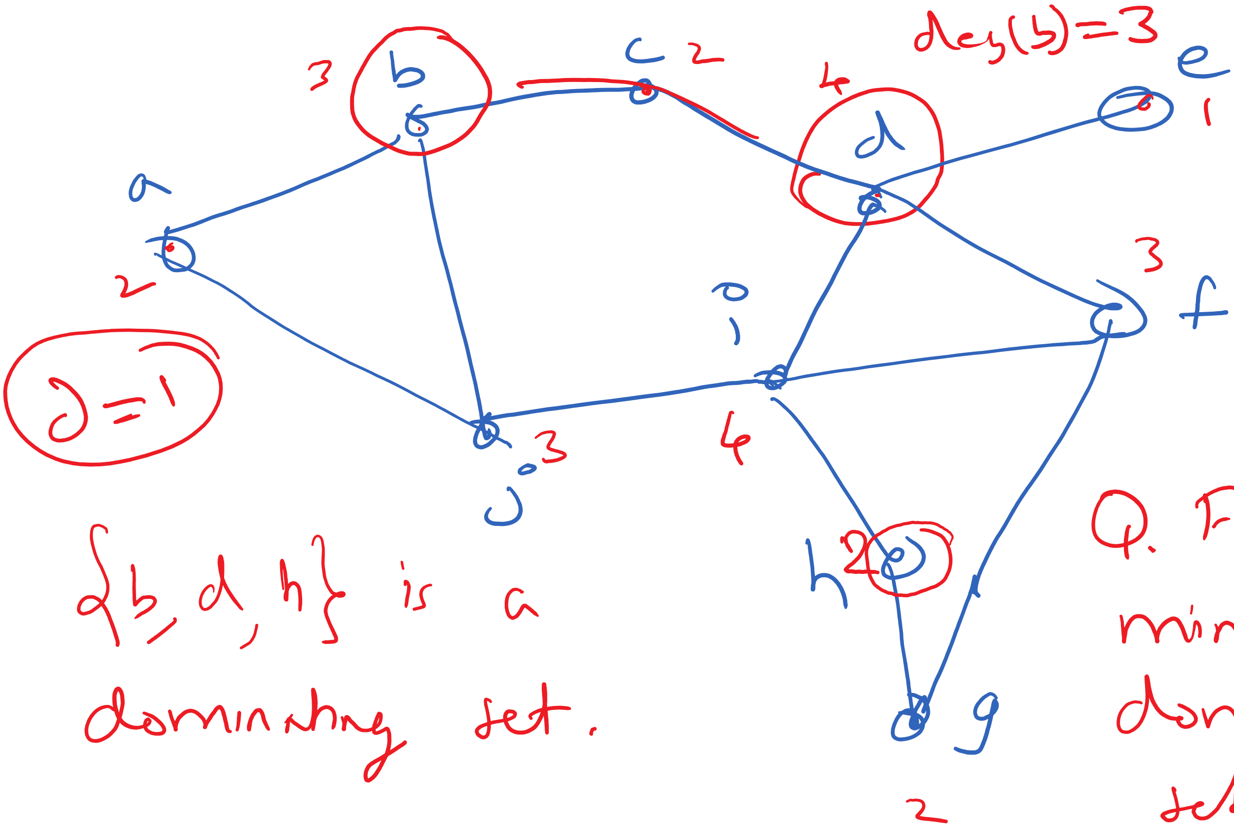
$$c_1 k 2^k \leq f(k) \leq 2 \log_e^2 k^2 \cdot 2^k$$

[Szekeres].

where c_1 is a constant.

Dominating set

$deg(c) = 2$
$deg(b) = 3$



$\{b, d, h\}$ is a dominating set.

Q. Finding min dominating sets.

Definition: Let G be a graph with vertex set $V(G)$ and edge set $E(G)$. A set $S \subseteq V(G)$ is a dominating set for G if every vertex in G is either present in S or is a neighbor of some vertex in S.

OR

$$\left(S \text{ is a dominating set in } G\right) \Rightarrow \forall v \in V(G) \left((v \in S) \text{ or } (\exists u \in S \text{ s.t. } uv \in E(G)) \right)$$

set in G

uv ∈ E(G)

Definition: Min degree = $\min\{\deg(v) : v \in V(G)\}$

δ

Theorem: Let $G=(V,E)$ be a graph on n vertices with minimum degree $\delta \geq 1$. Then G has a **dominating set** of size at most $\frac{n}{\delta+1}\left(1+\log_e(\delta+1)\right)$.

Recalling probability fundamentals

Sample space Ω : set of all outcomes

Event $\rightarrow$ a subset of Ω.

Random Variable

$X : \Omega \rightarrow \mathbb{R}$ reals

Example. Dice throw

$$\Omega = \{1, 2, 3, 4, 5, 6\}$$

$$X = \begin{cases} 10, & \text{if outcome is } 1 \\ 20, & \text{if outcome is } 2 \\ \vdots & \\ 60, & \text{if outcome is } 6 \end{cases}$$

$$E[X] = \sum_{X=x} \Pr[X=x] \cdot x$$

$$E[X] = \frac{1}{6}\cdot 10 + \frac{1}{6}\cdot 20 + \dots + \frac{1}{6}\cdot 60$$

$$\rightarrow E[X] = \frac{1}{6}\cdot 10 + \frac{1}{6}\cdot 20 + \dots + \frac{1}{6}\cdot 60$$

Take two R.V X and Y.

Linearity of expectation.

$$E[X+Y] = E[X] + E[Y]$$

In fact,

$$E[\alpha X + \beta Y] = \alpha E[X] + \beta E[Y]$$

where α and β two reals.