Limit of a scalar field:

Let f be a real function defined in some deleted neighbourhood of a point $c \in \mathbb{R}^n$. A real number L is said to be a limit of a function f at c if given any $\epsilon > 0$ there exists a $\delta > 0$ such that

$$0 < \|x - c\| < \delta \Rightarrow |f(x) - L| < \epsilon$$

i.e., $x \in S(c, \delta) - \{c\} \Rightarrow f(x) \in (L - \epsilon, L + \epsilon)$

* The value of δ usually depends on ϵ. So we sometimes write $\delta(\epsilon)$ instead of δ to emphasize this dependence
* If L is a Limit of f at c, then we say that 'f converges to L at c'. if the Limit of f at c doesn't exist finitely, we say that 'f diverges at c'.

Uniqueness of limit:

Let f be a real valued function defined in some deleted neighbourhood of a point $c \in \mathbb{R}^n$, and let

$$\underset{x \to c}{Lt} f(x) = L_1 \quad \text{and} \quad \underset{x \to c}{Lt} f(x) = L_2$$

Then, we must have $L_1 = L_2$

proof :-

Let us assume $L_1 \neq L_2$ i.e $|L_1 - L_2| > 0$

we fix $\epsilon = \frac{1}{2}|L_1 - L_2| > 0$

As $\underset{x \to c}{Lt} f(x) = L_1$ $\quad 0 < \|x - c\| < \delta_1 \Rightarrow |f(x) - L_1| < \epsilon$

As $\underset{x \to c}{Lt} f(x) = L_2$ $\quad 0 < \|x - c\| < \delta_2 \Rightarrow |f(x) - L_2| < \epsilon$

we select $\delta = \min\{\delta_1, \delta_2\}$. So both the inequialities satisfy

$$0 < \|x - c\| < \delta$$

Now consider $|L_1 - L_2| = |f(x) - L_2 - f(x) + L_1|$

$$\leq |f(x) - L_2| + |f(x) - L_1|$$

$$< \epsilon + \epsilon$$

$$< 2\epsilon$$

$$< |L_1 - L_2| \quad (\because \epsilon = \tfrac{1}{2}|L_1 - L_2|)$$

Which is a contradiction

Hence, we must have $L_1 = L_2$

Eg1: consider the function $f: \mathbb{R}^2 \to \mathbb{R}$ defined by

$$f(x,y) := \begin{cases} (x^2+y^2)\cos\left(\frac{1}{x^2+y^2}\right) & \text{if } (x,y) \neq (0,0) \\ 0 & \text{if } (x,y) = (0,0) \end{cases}$$

$\forall (x,y) \in \mathbb{R}^2$. We will show that

$$\underset{(x,y)\to(0,0)}{Lt} f(x,y) = 0$$

A: so $L = 0$, $c = (0,0)$

$$0 < \|(x,y)-(0,0)\| = \|(x,y)\| = \sqrt{x^2+y^2} < \delta \Rightarrow |f(x,y) - 0| < \epsilon$$

We consider any $\epsilon > 0$ and take $\delta = \sqrt{\epsilon} > 0$.

$$0 < \sqrt{x^2+y^2} < \delta = \sqrt{\epsilon}$$

$$x^2+y^2 < \epsilon \quad \text{and } (x,y) \neq (0,0)$$

So $$|f(x,y)| = \left|(x^2+y^2)\cos\left(\frac{1}{x^2+y^2}\right)\right| \leq |x^2+y^2|$$
$$< \epsilon$$

Hence it follows $\underset{(x,y)\to(0,0)}{Lt} f(x,y) = 0$

Eg2: consider the function $f: \mathbb{R}^2 \to \mathbb{R}$ defined as

$$f(x,y) = \begin{cases} \frac{xy}{x^2+y^2} & \text{if } (x,y) \neq (0,0) \\ 0 & \text{if } (x,y) = (0,0) \end{cases}$$

$\forall (x,y) \in \mathbb{R}^2$. Check whether limit of f at origin exists or not.

A: Let us approach $(0,0)$ along the path $y = mx$ for some $m \in \mathbb{R}$

the $$\underset{(x,y)\to(0,0)}{Lt} f(x,y) = \underset{x\to 0}{Lt} f(x,mx)$$
$$= \underset{x\to 0}{Lt} \frac{mx^2}{x^2+m^2x^2}$$
$$= \underset{x\to 0}{Lt} \frac{m}{1+m^2}$$
$$= \frac{m}{1+m^2}$$

We can se that if $m=0$ the limit value is 0

if $m = 1$ the limit value is 1

This contradicts, the fact that limit of a function, if exists is unique.

$\therefore \underset{(x,y)\to(0,0)}{Lt} f(x,y)$ doesn't exist

continuity of a Scalar field

Let $f: D(\subset \mathbb{R}^n) \to \mathbb{R}$ be a scalar field and $c \in \text{int} D$. Then, f is said to be continous at c if for every $\epsilon > 0$, there exists a $\delta > 0$ such that

$$0 \le \|x - c\| < \delta \Rightarrow |f(x) - f(c)| < \epsilon$$

$$\text{i.e } x \in S(c, \delta) \Rightarrow f(x) \in (f(c) - \epsilon, f(c) + \epsilon)$$

The following statements are equivalent

1. f is continous on D.
2. if $\{x_n\}$ is a sequence in D which converges to a point $c \in D$, then $\{f(x_n)\}$ is a sequence in R which converges to $f(c)$.

Eg: consider the function $f(x, y) = x^2 + y^2$. Discuss continuity at (x_0, y_0)

A: consider $|f(x, y) - f(x_0, y_0)|$

$$= |x^2 + y^2 - x_0^2 - y_0^2|$$

$$= |x^2 - x_0^2 + y^2 - y_0^2|$$

$$\le |x - x_0||x + x_0| + |y - y_0||y + y_0|$$

$$\le |x - x_0|(|x| + |x_0|) + |y - y_0|(|y| + |y_0|)$$

Now consider $\|(x, y) - (x_0, y_0)\| < \delta$

$$\sqrt{(x - x_0)^2 + (y - y_0)^2} < \delta$$

$$(x - x_0)^2 + (y - y_0)^2 < \delta^2$$

Let us take $\boxed{\delta = \min\{1, \epsilon\}}$ [$\because \delta$ as a function of ϵ]

$\therefore |x - x_0| < 1 \quad |y - y_0| < 1$

Now consider $|f(x, y) - f(x_0, y_0)|$

$$\le |x - x_0|(|x| + |x_0|) + |y - y_0|(|y| + |y_0|)$$

$$\le |x - x_0|((|x - x_0| + |x_0|) + |x_0|) + |y - y_0|((|y - y_0| + |y_0|) + |y_0|)$$

$$\le |x - x_0|(1 + 2|x_0|) + |y - y_0|(1 + 2|y_0|)$$

$$\le M(|x - x_0| + |y - y_0|)$$

where $M > 0$ is a universal constant, essentially, it is the sum of all those bounds one may take

$M = (1 + 2|x_0|) + (1 + 2|y_0|)$

$\boxed{M = 2 + 2|x_0| + 2|y_0|}$

$$\le M\left(\sqrt{|x - x_0|^2 + |y - y_0|^2} + \sqrt{|x - x_0|^2 + |y - y_0|^2}\right)$$

$\left[\because |x - x_0| \le \sqrt{|x - x_0|^2 + |y - y_0|^2}\right]$

$$\le 2M\sqrt{|x - x_0|^2 + |y - y_0|^2}$$

$< 2M\delta \quad \left[\because \sqrt{(x - x_0)^2 + (y - y_0)^2} < \delta\right]$

$< 2M\epsilon \quad [\because \delta = \min(1, \epsilon)]$

$\therefore$ it is continous at (x_0, y_0)

Eg: Consider the function $f(x,y) = \begin{cases} x\sin\frac{1}{y} + y\sin\frac{1}{x} & (x,y) \neq (0,0) \\ 0 & (x,y) = (0,0) \end{cases}$.

Is the function continous at (0,0).

A: Consider $|f(x,y) - f(0,0)|$

$$= \left| x\sin\tfrac{1}{y} + y\sin\tfrac{1}{x} - 0 \right|$$

$$\leq \left|x\sin\tfrac{1}{y}\right| + \left|y\sin\tfrac{1}{x}\right| \quad [\because |x+y| \leq |x| + |y|]$$

$$\leq |x|\left|\sin\tfrac{1}{y}\right| + |y|\left|\sin\tfrac{1}{x}\right| \quad [\because |x||y| = |xy|]$$

$$\leq |x| \cdot 1 + |y| \cdot 1 \quad \left[\because \left|\sin\tfrac{1}{y}\right| \leq 1, \left|\sin\tfrac{1}{x}\right| \leq 1\right]$$

$$\leq \sqrt{|x|^2 + |y|^2} + \sqrt{|x|^2 + |y|^2} \quad \left[\because |x| \leq \sqrt{|x|^2 + |y|^2},\ |y| \leq \sqrt{|x|^2 + |y|^2}\right]$$

$$\leq 2\sqrt{|x|^2 + |y|^2}$$

Now consider $\|(x,y) - (0,0)\| < \delta$

$$\sqrt{x^2 + y^2} < \delta$$

so $|f(x,y) - f(0,0)| \leq 2\sqrt{|x|^2 + |y|^2}$

$$< 2\delta$$

if we consider $\boxed{\delta = \frac{\epsilon}{2}}$ it is obviosly continous

Eg: Consider the function $f(x,y) = \begin{cases} \frac{x^2 - y^2}{x^2 + y^2} & (x,y) \neq (0,0) \\ 0 & (x,y) = (0,0) \end{cases}$. Is the functio

continous at (0,0)

A: Let us approach (0,0) along the line $y = mx$. then

$$f(x,y) = \frac{1-m^2}{1+m^2}$$

as $m = 0$ $Lt\, f(x,y) = 1$

as $m = 1$ $Lt\, f(x,y) = 0$

as Lt shouldn't have multiple values. it is discontinous at (0,0).

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Eg: Consider the function $f(x,y) = xy \cdot \frac{x^2-y^2}{x^2+y^2}$. Is this function continous at $(0,0)$

A: consider $|f(x,y) - f(0,0)|$

$$= \left| xy \cdot \frac{x^2-y^2}{x^2+y^2} - 0 \right|$$

$$= |x||y| \cdot \left| \frac{x^2-y^2}{x^2+y^2} \right| \quad [\because |xyz| = |x||y||z|]$$

$$\leq |x||y| \quad \left[\because \left| \frac{x^2-y^2}{x^2+y^2} \right| \leq 1 \right]$$

$$\leq \sqrt{x^2+y^2} \cdot \sqrt{x^2+y^2} \quad [\because |x| \leq \sqrt{x^2+y^2},\ |y| \leq \sqrt{x^2+y^2}]$$

$$\leq x^2+y^2$$

Now consider $\|(x,y) - (0,0)\| < \delta$

$$\sqrt{x^2+y^2} < \delta$$

$$x^2+y^2 < \delta^2$$

Let $\boxed{\delta^2 = \epsilon} \Rightarrow \delta = \sqrt{\epsilon}$

So $|f(x,y) - f(0,0)| \leq x^2+y^2$

$$< \epsilon$$

Hence $f(x,y)$ is continous at $(0,0)$

Differentiability:

In the single variable Calculus, a real valued function, $f: I \to R$, defined on an open interval I, is said to be differentiable at a point $a \in I$ if the limit $\underset{x \to a}{Lt} \frac{f(x)-f(a)}{x-a}$ exists and we denote it as $f'(a)$ and call it the derivative of f at a.

i.e $\underset{x \to a}{Lt} \frac{f(x)-f(a)}{x-a} = f'(a)$

equivalently $\underset{x \to a}{Lt} \left| \frac{f(x)-f(a)}{x-a} - f'(a) \right| = 0$

$$\boxed{\underset{x \to a}{Lt} \frac{|f(x) - (f(a) + f'(a)(x-a))|}{|x-a|} = 0} \quad \text{— (1)}$$

from the calculus I, if f is differentiable at a, the graph of $y = f(x)$ can be approximated by that of the tangent line,

$$L_a(x) = f(a) + f'(a)(x-a) \quad \forall x \in R$$

with an error in the approximation

$$E_a(x-a) = f(x) - L_a(x)$$

So from (1) we can write

$$\underset{x\to a}{Lt} \frac{|E_a(x-a)|}{|x-a|} = 0$$

i.e the error in the linear approximation of f at a goes to 0 more rapidly than $|x-a|$ goes to 0 as x gets closer to a

So we can write

$$f(x+w) - f(x) = m_x w + E_x(w) \quad \text{where } m_x = f'(x), \quad \underset{w\to 0}{Lt} \frac{|E_a(w)|}{|w|} = 0$$

Observe that the map $w \mapsto m_x w$ defines a linear map from R to R

* We then conclude that if f is differentiable at x, there exists a linear map such that the linear map approximates the difference $f(x+w) - f(x)$ in the sense that the error in the approximation goes to 0 as $w \to 0$ at a faster rate than $|w|$ approaches to 0

Def: Let $F: U (\subseteq R^n) \to R^m$ be a ~~scalar field~~ map F is said to be differentiable at $x_0 \in U$ if and only if there exists a linear transformation $T_x : R^n \to R^m$ such that

$$\underset{\|x-x_0\|\to 0}{Lt} \frac{\|F(x) - F(x_0) - T_x(x-x_0)\|}{\|x-x_0\|} = 0$$

similarly $\boxed{F(x+w) = F(x) + T_x(w) + \underset{\text{error term}}{E_x(w)}}$ where $\underset{\|w\|\to 0}{Lt} \frac{\|E_x(w)\|}{\|w\|} = 0$ — (2)

Def: $F: U \to R^m$ be a map. if F is differentiable at $x \in U$ then the unique linear transformation T_x, is called the derivative of F at x and is denoted by $DF(x)$.

So $\quad F(x+w) = F(x) + DF(x)w + E_x(w)$

where $\quad \underset{\|w\|\to 0}{Lt} \frac{\|E_x(w)\|}{\|w\|} = 0$

Theorem: Let $F: U \to \mathbb{R}^m$ be a map if F is differentiable at $x \in U$ then the linear transformation T_x, is unique.

Proof: Suppose there is another linear transformation, $T: \mathbb{R}^n \to \mathbb{R}^m$ in addition to T_x. We then show that T & T_x are the same transformation.

From ② we can write

$$F(x+w) = F(x) + T_x(w) + E_x(w) \quad \text{where} \lim_{\|w\| \to 0} \frac{\|E_x(w)\|}{\|w\|} = 0$$

Similarly $$F(x+w) = F(x) + T(w) + E(w) \quad \text{where} \lim_{\|w\| \to 0} \frac{\|E(w)\|}{\|w\|} = 0$$

this follows $T_x(w) + E_x(w) = T(w) + E(w) \quad \forall\, w \in \mathbb{R}^n$ sufficiently close to $\vec{0}$.

Let $w = t\hat{u}$ for $t \in \mathbb{R}$ sufficiently close to 0

$$T_x(t\hat{u}) + E_x(t\hat{u}) = T(t\hat{u}) + E(t\hat{u})$$

as T is a linear map, by linearity

$$t.T_x(\hat{u}) + E_x(t\hat{u}) = t\,T(\hat{u}) + E(t\hat{u})$$

$$T_x(\hat{u}) + \frac{E_x(t\hat{u})}{t} = T(\hat{u}) + \frac{E(t\hat{u})}{t}$$

Next observe that

$$\lim_{\|w\| \to 0} \frac{\|E_x(w)\|}{\|w\|} = 0 \Rightarrow \lim_{\|t\hat{u}\| \to 0} \frac{\|E_x(t\hat{u})\|}{\|t\hat{u}\|} = 0 \Rightarrow \lim_{|t| \to 0} \frac{\|E_x(t\hat{u})\|}{|t|} = 0$$

Similarly $$\lim_{|t| \to 0} \frac{\|E(t\hat{u})\|}{|t|} = 0$$

So we get $T_x(\hat{u}) = T(\hat{u})$

Hence proved

* We can write

$$f(x+w) = f(x) + \nabla f(x) \cdot w + E_x(w)$$

i.e $$f(x) = f(x_0) + \nabla f(x_0) \cdot (x - x_0) + E_x(x - x_0)$$

Directional Derivatives:

Def: Let $f: U(\subseteq \mathbb{R}^n) \to \mathbb{R}$ denote a scalar field, and $\hat{a}$ be a unit vector in $\mathbb{R}^n$. If the limit

$$\lim_{t \to 0} \frac{f(x+t\hat{a}) - f(x)}{t}$$

exists, we call it the directional derivative of f at x in the direction of the unit vector $\hat{a}$. We denote it by $D_{\hat{a}} f(x)$

i.e. $$\boxed{D_{\hat{a}} f(x) = \lim_{t \to 0} \frac{f(x+t\hat{a}) - f(x)}{t} = \nabla f(x) \cdot \hat{a}}$$

* If we consider $F: U(\subseteq \mathbb{R}^n) \to \mathbb{R}^m$ be a linear map, then the Directional Derivative of F at $x_0 \in U$ in the direction of the unit vector $\hat{v}$ is given by

$$D_{\hat{v}} F(x_0) = \nabla F(x_0) \cdot \hat{v}$$

$$\boxed{D_{\hat{v}} F(x_0) = \underbrace{\begin{pmatrix} \frac{\delta f_1}{\delta x_1}(x_0) & \frac{\delta f_1}{\delta x_2}(x_0) & \cdots & \frac{\delta f_1}{\delta x_n}(x_0) \\ \vdots & \vdots & & \vdots \\ \frac{\delta f_m}{\delta x_1}(x_0) & \frac{\delta f_m}{\delta x_2}(x_0) & \cdots & \frac{\delta f_m}{\delta x_n}(x_0) \end{pmatrix}}_{\text{Jacobian matrix.}} \cdot \begin{pmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{pmatrix}}$$

Theorem: Let $f: U(\subset \mathbb{R}^n) \to \mathbb{R}$ denote a scalar field. If f is differentiable at x_0 then it is continous at x_0

proof: Given f is differentiable at x_0 then

$$f(x) = f(x_0) + T(x - x_0) + E(x - x_0)$$

$$f(x+w) = f(x) + T(w) + E_x(w)$$

Now consider $w = t\hat{a}$

$$f(x+t\hat{a}) - f(x) = T(t\hat{a}) + E_x(t\hat{a})$$

$$|f(x) - f(x_0)| = |T(t\hat{a}) + E_x(t\hat{a})|$$

$$\leq |t\,T(\hat{a})| + |E_x(t\hat{a})|$$

$$\leq M\|x - x_0\| \quad [\text{where } M > 1]$$

$$\leq M\delta \qquad \text{So choosing } \delta = \frac{\epsilon}{M} \text{ makes } f \text{ continous}$$

Chain Rule: Let $F: U(\subset R^n) \longrightarrow R^m$ and $G: Q(\subset R^m) \longrightarrow R^k$ be maps suppose that $F(U) \subseteq Q$. If F is differentiable at $x \in U$ and G is differentiable at $y \in Q = F(x)$, then the composition

$$G \circ F: U \longrightarrow R^k$$

is differentiable at x and the derivative map $D(G\circ F)(x): R^n \to R^k$ is given by

$$\boxed{D(G\circ F)(x)\, w = DG(y)\, DF(x) w} \quad \forall\, w \in R^n$$

i.e., $(G\circ F)'(x) = G'(F(x)) \cdot F'(x)$

Mean value theorem: Let $f: U(\subset R^n) \to R$ be a scalar field. Suppose that f is differentiable on U. Then, for any pair of points x and y in B, there exists a point z in the line segment connecting x to y such that

$$\boxed{\begin{aligned} f(y) - f(x) &= D_{\hat{u}} f(z)\, \|y-x\| \\ &= \nabla f(z) \cdot (y-x) \end{aligned}} \quad \text{where } \hat{u} = \frac{y-x}{\|y-x\|}$$

* $\boxed{(\vec{x} - \vec{p}) \cdot \nabla f(\vec{p}) = 0}$ gives the equation of tangent plane to the surface $S = f(x,y,z) = 0$ at point P

Theorem: Let $f: R^2 \to R$ be a continous real-valued function. Then f is continously differentiable if and only if the partial derivative functions $\frac{\delta f}{\delta x}(x,y)$ and $\frac{\delta f}{\delta y}(x,y)$ exist and are continous

Eg: A differentiable function with discontinous partial derivatives

$$f(x,y) = \begin{cases} (x^2+y^2)\sin\left(\frac{1}{\sqrt{x^2+y^2}}\right) & \text{if } (x,y) \neq (0,0) \\ 0 & \text{if } (x,y) = 0 \end{cases}$$

A function having partial derivatives which is not differentiable

$$g(x,y) = \begin{cases} \frac{xy}{\sqrt{x^2+y^2}} & \text{if } (x,y) \neq (0,0) \\ 0 & \text{if } (x,y) = (0,0) \end{cases}$$

Remark: If a function has continous partial derivatives then it is continously differentiable.

Q. The change of variables $x = uv$ & $y = \frac{1}{2}(u^2 - v^2)$ transforms $f(x,y)$ $g(u,v)$. Calculate $\frac{\delta g}{\delta u}$, $\frac{\delta g}{\delta v}$, $\frac{\delta^2 g}{\delta u \delta v}$ in terms of partial derivaties of f.

A. $$\frac{\delta g}{\delta u} = \frac{\delta f}{\delta x}\cdot\frac{\delta x}{\delta u} + \frac{\delta f}{\delta y}\cdot\frac{\delta y}{\delta u} = \frac{\delta f}{\delta x}\cdot v + \frac{\delta f}{\delta y}\cdot u$$

$$\frac{\delta g}{\delta v} = \frac{\delta f}{\delta x}\cdot\frac{\delta x}{\delta v} + \frac{\delta f}{\delta y}\cdot\frac{\delta y}{\delta v} = \frac{\delta f}{\delta x}\cdot u - \frac{\delta f}{\delta y}\, v$$

$$\frac{\delta^2 g}{\delta u \delta v} = \frac{\delta}{\delta v}\left(\frac{\delta g}{\delta u}\right) = \frac{\delta^2 f}{\delta x^2}\cdot\frac{\delta x}{\delta v}\cdot v + \frac{\delta^2 f}{\delta x \delta y}\cdot\frac{\delta y}{\delta v}\cdot v + \frac{\delta f}{\delta x}\cdot\frac{\delta v}{\delta v} +$$
$$\frac{\delta^2 f}{\delta x \delta y}\cdot\frac{\delta x}{\delta v}\cdot u + \frac{\delta^2 f}{\delta y^2}\cdot\frac{\delta y}{\delta v}\cdot u + \frac{\delta f}{\delta y}\cdot\frac{\delta u}{\delta v}$$
$$= \frac{\delta^2 f}{\delta x^2}\cdot u\cdot v + \frac{\delta^2 f}{\delta x \delta y}\cdot(-v)\cdot v + \frac{\delta f}{\delta x} + \frac{\delta^2 f}{\delta x \delta y}\cdot u\cdot u +$$
$$\frac{\delta^2 f}{\delta y^2}\cdot(-v)\cdot u + 0$$

$$\frac{\delta^2 g}{\delta u \delta v} = uv\left(\frac{\delta^2 f}{\delta x^2} - \frac{\delta^2 f}{\delta y^2}\right) + \frac{\delta^2 f}{\delta x \delta y}(u^2 - v^2) + \frac{\delta f}{\delta x}$$

Maxima, minima, saddle points

stable point: The point (x_0, y_0) at which all the partial derivatives of a continous and defferentiable function will be zero (i.e gradient at that point is a zero vector).

stable point
- local maxima
- local minima
- inflection point
- saddle point

saddle points: stable points where the function has a local maximum in one direction, but a local minimum in another direction

Eg: $f(x,y) = x^2 - y^2$ at $(0,0)$

local maximum: A scalar-valued function f has a local maximum at x_0 if there exists some positive number $r > 0$, such that
$$f(x) \le f(x_0) \quad \forall x \text{ such that } \|x - x_0\| < r$$

local minimum: A scalar-valued function f has a local minimum at x_0 if there exists some positive number $r > 0$, such that
$$f(x) \ge f(x_0) \quad \forall x \text{ such that } \|x - x_0\| < r$$

second partial derivative test: it tells us how to verify whether the stable point is a local maximum, local minimum or a saddle point.

Consider $H = \frac{\delta^2 f}{\delta x^2}(x_0, y_0) \cdot \frac{\delta^2 f}{\delta y^2}(x_0, y_0) - \left(\frac{\delta^2 f}{\delta x \delta y}(x_0, y_0)\right)^2$

$= \text{Det}\begin{pmatrix} \frac{\delta^2 f}{\delta x^2}(x_0,y_0) & \frac{\delta^2 f}{\delta x \delta y}(x_0,y_0) \\ \frac{\delta^2 f}{\delta x \delta y}(x_0,y_0) & \frac{\delta^2 f}{\delta y^2}(x_0,y_0) \end{pmatrix} \rightarrow$ Hessian matrix

* if $H<0$ (x_0,y_0) is a saddle point.
* if $H>0$ & $\frac{\delta^2 f}{\delta x^2}(x_0,y_0) < 0$ (x_0,y_0) is a local maximum
* if $H>0$ & $\frac{\delta^2 f}{\delta x^2}(x_0,y_0) > 0$ (x_0,y_0) is a local minimum
* if $H>0$ & $\frac{\delta^2 f}{\delta x^2}(x_0,y_0) = 0$ (x_0,y_0) is a saddle point
* if $H=0$, we do not have enough information to tell

Lagrange multiplier:

if a function $f(x,y)$ is constrained by a constraint $g(x,y)=b$ and if (x_0,y_0) is a point s.t $f(x_0,y_0)$ is maximized, then for finding (x_0,y_0) we consider

$$\nabla f(x_0,y_0) = \lambda \nabla g(x_0,y_0)$$ where λ is called lagrange multiplier

from this we get (x_0,y_0) in terms of λ which is then substituted in the constraint $g(x,y)=b$ and calculate the value of λ. So we can get (x_0,y_0). Therefore, maximum value of $f(x,y)$ with a constraint $g(x,y)=b$ can be obtained $= f(x_0,y_0)$

Lagrangian:

We define a single function $L(x,y,\lambda)$ which shows all the properties

$$L(x,y,\lambda) = f(x,y) - \lambda(g(x,y)-b)$$

If b is variable or if we want to check in terms of b

$$L(x,y,\lambda,b) = f(x,y,b) - \lambda(g(x,y,b)-b)$$

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1. Find the extreme values of $z = \frac{x}{a} + \frac{y}{b}$ subjected to the condition $x^2+y^2=1$

A. $f(x,y) = \frac{x}{a} + \frac{y}{b}$ $\qquad g(x,y) = x^2+y^2$

Let $f(x,y)$ attains its extremum at (x_0, y_0)

$$\nabla f(x_0, y_0) = \lambda \nabla g(x_0, y_0)$$

$$\begin{bmatrix} \frac{1}{a} \\ \frac{1}{b} \end{bmatrix} = \lambda \begin{bmatrix} 2x_0 \\ 2y_0 \end{bmatrix}$$

$$x_0 = \frac{1}{2a\lambda}, \quad y_0 = \frac{1}{2b\lambda}$$

From the constraint $x_0^2 + y_0^2 = 1$

$$\left(\frac{1}{2a\lambda}\right)^2 + \left(\frac{1}{2b\lambda}\right)^2 = 1$$

$$\boxed{\lambda = \pm \frac{\sqrt{a^2+b^2}}{2ab}}$$

$$(x_0, y_0) = \left(\frac{b}{\sqrt{a^2+b^2}}, \frac{a}{\sqrt{a^2+b^2}}\right) \text{ or } \left(\frac{-b}{\sqrt{a^2+b^2}}, \frac{-a}{\sqrt{a^2+b^2}}\right)$$

$$z_{max} = \frac{b}{\sqrt{a^2+b^2}} \cdot \frac{1}{a} + \frac{a}{\sqrt{a^2+b^2}} \cdot \frac{1}{b}, \quad z_{min} = \frac{-b}{\sqrt{a^2+b^2}} \cdot \frac{1}{a} + \frac{-a}{\sqrt{a^2+b^2}} \cdot \frac{1}{b}$$

$$\boxed{z_{max} = \frac{\sqrt{a^2+b^2}}{ab}} \qquad \boxed{z_{min} = -\frac{\sqrt{a^2+b^2}}{ab}}$$

2. Find the minimum volume bounded by the planes $x=0, y=0, z=0$ and the tangent plane of ellipsoid $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$ at a point in the octant $x>0, y>0, z>0$

A. tangent plane of ellipsoid $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$ at a point (x_0, y_0, z_0)

$$(x-x_0 \quad y-y_0 \quad z-z_0) \cdot \left(\frac{2x_0}{a^2} \quad \frac{2y_0}{b^2} \quad \frac{2z_0}{c^2}\right) = 0$$

$$x\left(\frac{x_0}{a^2}\right) + y\left(\frac{y_0}{b^2}\right) + z\left(\frac{z_0}{c^2}\right) = \frac{x_0^2}{a^2} + \frac{y_0^2}{b^2} + \frac{z_0^2}{c^2} = 1$$

tangent plane is $\boxed{x\left(\frac{x_0}{a^2}\right) + y\left(\frac{y_0}{b^2}\right) + z\left(\frac{z_0}{c^2}\right) = 1}$

$\therefore$ volume of tetrahedron $= \frac{1}{6}\left(\frac{a^2}{x_0}\right)\left(\frac{b^2}{y_0}\right)\left(\frac{c^2}{z_0}\right) = \frac{a^2b^2c^2}{6x_0y_0z_0}$

$$\therefore f(x,y,z) = \frac{a^2b^2c^2}{6x_0y_0z_0} \qquad g(x,y,z) = \frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2}$$

$\nabla f(x_0, y_0, z_0) = \lambda \nabla g(x_0, y_0, z_0)$

$$\left(\frac{a^2b^2c^2}{6y_0z_0}\left(\frac{-1}{x_0^2}\right), \frac{a^2b^2c^2}{6x_0z_0}\left(\frac{-1}{y_0^2}\right), \frac{a^2b^2c^2}{6x_0y_0}\left(\frac{-1}{z_0^2}\right)\right) = \lambda\left(\frac{2x_0}{a^2}, \frac{2y_0}{b^2}, \frac{2z_0}{c^2}\right)$$

$$\frac{x_0^2}{a^2} \cdot \frac{12x_0y_0z_0}{a^2b^2c^2} = \frac{-1}{\lambda} \qquad \frac{y_0^2}{b^2} \cdot \frac{12x_0y_0z_0}{a^2b^2c^2} = \frac{-1}{\lambda} \qquad \frac{z_0^2}{c^2} \cdot \frac{12x_0y_0z_0}{a^2b^2c^2} = \frac{-1}{\lambda}$$

$$\frac{x_0^2}{a^2} = \frac{y_0^2}{b^2} = \frac{z_0^2}{c^2}$$

$$\frac{x_0}{a} = \frac{y_0}{b} = \frac{z_0}{c} \qquad [\because \text{chosen } x_0>0, y_0>0, z_0>0]$$

So $\frac{x^2}{a^2} + \frac{x^2}{a^2} + \frac{x^2}{a^2} = 1$

$\cancel{\boxed{x = \sqrt{3}a}}$ $\boxed{x = \frac{a}{\sqrt{3}}}$ $\boxed{y = \frac{b}{\sqrt{3}}}$ $\boxed{z = \frac{c}{\sqrt{3}}}$

$$\text{minimum volume} = \frac{1}{6} \cdot \frac{a^{\cancel{2}}b^{\cancel{2}}c^{\cancel{2}}}{\frac{\cancel{a}}{\sqrt{3}} \cdot \frac{\cancel{b}}{\sqrt{3}} \cdot \frac{\cancel{c}}{\sqrt{3}}} = \frac{\sqrt{3}}{2}abc$$

Riemann sum: it is an approximation of the area under a curve by dividing it in to multiple simple shapes (like rectangles or trapezoids) $= \sum_{i=0}^{n-1} f(t_i)(x_{i+1} - x_i)$

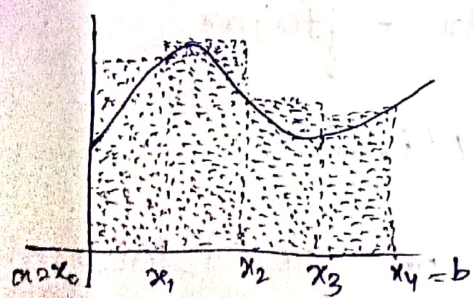
Riemann integration:

Let f be bounded on $[a,b]$ i.e $\exists\, m, M \in \mathbb{R}$ s.t $m \le f(x) \le M \;\forall x \in [a,b]$

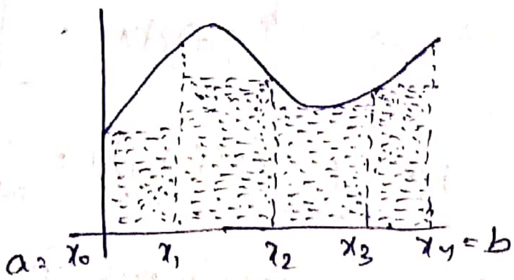
Let $P = \{a = x_0 < x_1 < x_2 < \cdots < x_{n-1} < x_n = b\}$. then P is said to be a partition of $[a,b]$

Define $U(P,f) = \sum_{i=1}^{n} M_i(x_i - x_{i-1})$ where $M_i = \sup f(x)$, $x_{i-1} < x < x_i$

$L(P,f) = \sum_{i=1}^{n} m_i(x_i - x_{i-1})$ where $m_i = \inf f(x)$, $x_{i-1} < x < x_i$



$U(P,f)$



$L(P,f)$

$$m(b-a) \le L(P,f) \le U(P,f) \le M(b-a)$$

Let P_1 be another partition of $[a,b]$ $P_1 \supseteq P$

$$U(P_1,f) \le U(P,f)$$

$$L(P_1,f) \ge L(P,f)$$

$$L(P,f) \le L(P_1,f) \le U(P_1,f) \le U(P,f)$$

$$\sup_{P \in \mathcal{P}[a,b]} L(P,f) = \underline{\int_a^b} f(x)\,dx$$

$$\inf_{P \in \mathcal{P}[a,b]} U(P,f) = \overline{\int_a^b} f(x)\,dx$$

Def: Riemann integral is the limit of the Riemann sums of a function as the partitions get finer. if the limit exists then the function is said to be integrable (or more specifically Riemann-integrable)

i.e if s is the Riemann integral of f, $\forall\, \epsilon > 0$, $\exists\, \delta > 0$ s.t

$$\left| \sum_{i=0}^{n-1} f(t_i)(x_{i+1} - x_i) - s \right| < \epsilon$$

Def: Let f be a bounded function defined on a closed interval $[a,b]$. We say that f is Riemann integrable on $[a,b]$ if the infimum of upper sums through all partitions of $[a,b]$ is equal to the supperimum of all lower sums through all partitions of $[a,b]$

i.e $$\inf_{P \in \mathcal{P}[a,b]} U(P,f) = \sup_{P \in \mathcal{P}[a,b]} L(P,f)$$

$$\Rightarrow \overline{\int_a^b} f(x)\,dx = \underline{\int_a^b} f(x)\,dx = \int_a^b f(x)\,dx$$

$$\text{ie } \lim_{N \to \infty} (U(f,P_N)) = \lim_{N \to \infty} (L(f,P_N))$$

Theorem: Every continous function on a closed interval is Reimann integrable on this interval

Theorem: Every Reimann integrable function may not be continous but it must be piecewise continous

A bounded function f on $[a,b]$ is Riemann Integrable if for $\epsilon>0$ $\exists$ partition P s.t $|U(P,f)-L(P,f)|<\epsilon$

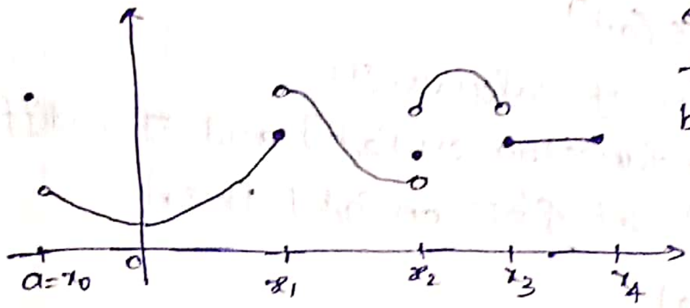
Eg of Riemann integrable function: Any monotonic function is Riemann integrable.

Eg of non Riemann integrable function: Dirichlet function

$$f(x)=\begin{cases} 1 & \text{if } x\in \mathbb{Q}\cap[0,1] \\ 0 & \text{if } x\in \mathbb{Q}^c\cap[0,1] \end{cases}$$

Def: We say that a function f is piecewise continous on interval $[a,b]$ if there is a finite set of points $a=x_0<x_1<\dots<x_N=b$ such that for every segment $[x_{k-1},x_k]$, where $k=1,\dots,N$, the function f has a limit at x_{k-1} from the right, limit at x_k from the left, and is continous on (x_{k-1},x_k).

Eg:



Eg of Riemann integrable funtion, not continous but piecewise continous.

Theorem: if a function is piecewise continous on a closed interval then it is Reimann integrable there.

Proof for any monotonic function is Reimann integrable.

Consider $P_n=\{a<a+\frac{b-a}{n}<a+\frac{2(b-a)}{n}<\dots<b\}$ [consider f is increasing]

then $U(P_n,f)=\frac{1}{n}(M_1+M_2+\dots+M_n)=\frac{1}{n}(f(x_1)+f(x_2)+\dots+f(x_n))$

$L(P_n,f)=\frac{1}{n}(m_1+m_2+\dots+m_n)=\frac{1}{n}(f(x_0)+f(x_1)+\dots+f(x_{n-1}))$

as $n\to\infty$

$$\lim_{n\to\infty}\left(U(P_n,f)-L(P_n,f)\right)=\lim_{n\to\infty}\frac{f(x_n)-f(x_0)}{n}=\lim_{n\to\infty}\frac{f(b)-f(a)}{n}$$
$$=0$$

as $n\to\infty$ $\quad U(P_n,f)-L(P_n,f)=0$

i.e $\lim_{n\to\infty}(U(P_n,f))=\lim_{n\to\infty}(L(P_n,f))$

i.e $\overline{\int_a^b} f(x)\,dx=\underline{\int_a^b} f(x)\,dx$

Hence Riemann integrable.

Riemann sum:

Let $P=\{a=x_0<x_1<\dots<x_n=b\}$ be a partion of $[a,b]$,

$c_i\in(x_{i-1},x_i)$ $1\le i\le n$. Then the Riemann sum

$$\boxed{S(P,f)=\sum_{i=1}^{n} f(c_i)(x_i-x_{i-1})}$$

So $L(P,f)\le S(P,f)\le U(P,f)$

If f is Riemann Integrable, then by sandwich theorem

$S(P,f)\longrightarrow \int_a^b f(x)\,dx$ for finer partition of $[a,b]$

Fundamental theorem of calculus - I:

Let $f:[a,b]\to\mathbb{R}$ be Riemann integrable then $F(x)=\int_a^x f(t)dt$ is cont and if f is continous on $[a,b]$ then F is diff on $[a,b]$

and $F'(x)=f(x)$ $x\in(a,b)$

Fundamental theorem of calculus - II:

Let F be integrable function on $[a,b]$ and $\exists$ a differentiable function f on $[a,b]$. s.t $f'=F$ on $[a,b]$ then

$\int_a^b F(t)dt=f(b)-f(a)$

1. Evaluate $\lim_{n\to\infty}\left[\frac{1}{n}+\frac{1}{n+1}+\dots+\frac{1}{n+n-1}\right]$

A. $\lim_{n\to\infty}\frac{1}{n}\left[1+\frac{n}{n+1}+\dots+\frac{n}{n+n-1}\right]$

$\lim_{n\to\infty}\frac{1}{n}\left[\frac{1}{1+\frac{0}{n}}+\frac{1}{1+\frac{1}{n}}+\dots+\frac{1}{1+\frac{n-1}{n}}\right]$

$=\lim_{n\to\infty}\frac{1}{n}\sum_{r=0}^{n-1}\frac{1}{1+\left(\frac{r}{n}\right)}$

$\Delta x=\frac{b-a}{n}=\frac{1}{n}$ $f\left(a+r\cdot\frac{1}{n}\right)=\frac{1}{1+\frac{r}{n}}$

$b-a=1$ taking $a=0\Rightarrow b=1$

$f\left(r/n\right)=\frac{1}{1+\frac{r}{n}}$

$=\int_0^1\frac{1}{1+x}=\ln 2$

2. Evaluate $\lim_{n\to\infty}\frac{1}{n^3}\left[\sin\frac{\pi}{n}+2^2\sin\frac{2\pi}{n}+\dots+n^2\sin\frac{n\pi}{n}\right]$

A. $\lim_{n\to\infty}\frac{1}{n}\left[\sin\frac{\pi}{n}\cdot\left(\frac{1}{n}\right)^2+\left(\frac{2}{n}\right)^2\sin\frac{2\pi}{n}+\dots+\left(\frac{n}{n}\right)^2\sin\frac{n\pi}{n}\right]$

$=\lim_{n\to\infty}\frac{1}{n}\sum_{r=1}^{n}\left(\frac{r}{n}\right)^2\sin\left(\frac{r}{n}\right)\pi$

$=\int_0^1 x^2\sin\pi x\,dx=\left[-\frac{x^2\cos\pi x}{\pi}+\frac{2x\sin\pi x}{\pi^2}+\frac{2\cos\pi x}{\pi^3}\right]_0^1=\frac{1}{\pi}-\frac{4}{\pi^3}$

3. Let $f:[a,b]\to R$ be Riemann integrable, then show that
$\int_a^b f(t)\,dt = \lim_{x\to b}\int_a^x f(t)\,dt.$

A: consider $F(x) = \int_a^x f(t)\,dt$

$$|F(b)-F(x)| = \left|\int_x^b f(t)\,dt\right|$$
$$\le \int_x^b |f(t)|\,dt$$
$$\le M(b-x) \quad [\because f(x) \text{ is bounded upwards by } M]$$

as $x\to b$ $|F(b)-F(x)|\to 0$

$$\therefore \lim_{x\to b} F(x) = F(b)$$
$$= \int_a^b f(t)\,dt$$

4. Let $f:[0,1]\to R$ be continous. Let $|f(x)| \le \int_0^x f(t)\,dt$. $\forall x\in[0,1]$, then show that $f\equiv 0$ on $[0,1]$.

A: f is bounded. So Let $M>0$, be s.t $|f(t)|\le M$, $\forall x\in[0,1]$

$$|f(x)| \le \int_0^x f(t)\,dt$$
$$\le \int_0^x M\,dt$$
$$\le Mx.$$

$$|f(x)| \le \int_0^x f(t)\,dt$$
$$\le \int_0^x Mt\,dt$$
$$\le \frac{Mx^2}{2}$$

$$|f(x)-0| \le \frac{M}{2} \quad \text{as } x\in[0,1]$$
$$\therefore f(x)\equiv 0$$

5. Let $f,g:(a,b]\to R$ be cont and $\int_a^b f(x)\,dx = \int_a^b g(x)\,dx$. show that $\exists\, c\in(a,b)$. s.t $f(c)=g(c)$

A. consider $F(x) = \int_a^x (f(x)\,\cancel{dx} - g(x))\,dx$

then $F(a) = \int_a^a (f(x)-g(x))\,dx = 0$

$F(b) = \int_a^b (f(x)-g(x))\,dx = 0$ [$\because$ given]

given f & g are continous on $[a,b]$ so F is differentiable

From mean value theorem $\exists\, c\in(a,b)$ s.t $F'(c) = \frac{F(b)-F(a)}{b-a} = 0$

i.e., $\exists\, c \in (a,b)$ $f(x) - g(x) = 0$

i.e., $\exists\, c \in (a,b)$ $f(c) = g(c)$

6. Evaluate $\underset{x \to 0}{Lt} \frac{1}{x^3} \int_0^x \frac{t^2}{1+t^4}\, dt$

A. $\int_0^x \frac{t^2}{1+t^4}\, dt \to 0$ as $x \to 0$

$$\underset{x\to 0}{Lt} \frac{\int_0^x \frac{t^2}{1+t^4}\, dt}{x^3}$$

By L Hospital

$$\underset{x\to 0}{Lt} \frac{\frac{\cancel{x^2}}{1+x^4}}{3\cancel{x^2}} = \underset{x\to 0}{Lt} \frac{1}{3(1+x^4)} = \frac{1}{3}$$

Double Integral:

$f : [a,b] \times [c,d] \to R$

$\int_a^b \int_c^d f(x,y)\, dx\, dy$ is called double integral

$$m(b-a)(d-c) \le L(P,f) \le U(P,f) \le M(b-a)(d-c)$$

where m & M are min & max bounds of $f(x,y)$.

Fubini's theorem:

$$\int_{x=a}^{b} \left(\int_{y=c}^{d} f(x,y)\, dy \right) dx = \int_{y=c}^{d} \left(\int_{x=a}^{b} f(x,y)\, dx \right) dy.$$

1. Find the volume of tetrahedron bounded by the planes $3x+6y+4z=12$ and the coordinate axes.

Sol: $z = f(x,y) = \frac{12-3x-6y}{4}$

if x-y plane $z=0$

$3x + 6y = 12$

$x + 2y = 4$

$\boxed{x = 4-2y}$

as y varies from 0 (coordinate axes) to 2 (meeting point of plane with y-axis) $\Rightarrow$ x varies from $(0, 4)$

$$\therefore V = \int_{x=0}^{4}\int_{y=0}^{2} \frac{12-3x-6y}{4}\, dy\, dx$$

$$= \int_{x=0}^{4} \left[\frac{12y-3xy-3y^2}{4}\right]_0^2 dx$$

$$= \int_{x=0}^{4} \left[\frac{24-6x-12-0}{4}\right] dx$$

$$= \int_0^4 \left[\frac{6-3x}{2}\right] dx$$

$$= \frac{1}{2}\left\{6x-\frac{3x^2}{2}\right\}_0^4$$

$$= \frac{1}{2}\{24-24$$

$$V = \int_0^4 \int_0^{\frac{4-x}{2}} \frac{12-3x-6y}{4}\, dy\, dx$$

$$= \int_0^2 \int_0^{4-2y} \frac{12-3x-6y}{4}\, dx\, dy$$

$$= \int_0^2 \left[\frac{12x-\frac{3x^2}{2}-6xy}{4}\right]_0^{4-2y} dy$$

$$= \int_0^2 \left[\frac{(12-6y)(4-2y)-\frac{3}{2}(4-2y)^2}{4}\right] dy$$

$$= \int_0^2 \left[\frac{(6-3y)(4-2y)-3(2-y)^2}{2}\right] dy$$

$$= \int_0^2 \left[\frac{24-24y+6y^2-12-3y^2+12y}{2}\right] dy$$

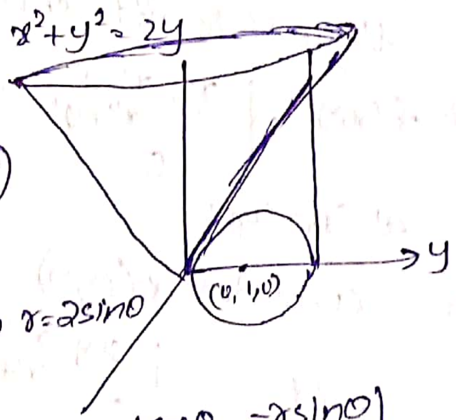
$$= \int_0^2 \left[\frac{12-12y+3y^2}{2}\right] dy$$

$$= \left[\frac{12y-6y^2+y^3}{2}\right]_0^2$$

$$= \frac{24-24+8}{2} = 4$$

$\therefore$ volume of tetrahedron = 4 units^3

Q. Find the volume below the surface $z=x^2+y^2$ above the plane $z=0$ & inside the cylinder $x^2+y^2=2y$



Sol consider an arbitrary point on surface $z=x^2+y^2$ as $(r\cos\theta, r\sin\theta, r^2)$

$$(r\cos\theta)^2+(r\sin\theta)^2 = 2(r\sin\theta)$$

$$r^2 = 2r\sin\theta \Rightarrow r=0 \text{ to } r=2\sin\theta$$

$$\therefore V = \int_{\theta=0}^{\pi}\int_{r=0}^{2\sin\theta} r^2 .\, r .\, dr\, d\theta$$

$$= \int_{\theta=0}^{\pi} \left[\frac{r^4}{4}\right]_0^{2\sin\theta} d\theta$$

$$= 4\int_{\theta=0}^{\pi} \sin^4\theta\, d\theta$$

$$J_n = \begin{vmatrix}\cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta\end{vmatrix} = r(\cos^2\theta+\sin^2\theta) = r$$

change of variable:

$x = X(u,v)$

$y = Y(u,v)$

$$\iint_R f(x,y)\,dx\,dy = \iint f(X,Y)\,|J_x(u,v)|\,du\,dv$$

Proofs:

1. Let $f: \mathbb{R}^2 \to \mathbb{R}$ $f(x,y) = x^2+y^2$. show that f is differentiable at ;

A. if f is differentiable $\exists \vec{\alpha}$ unique $\vec{\alpha}$

$$\underset{\vec{x}\to\vec{x_0}}{Lt} \frac{|f(x) - (f(x_0) + \langle\vec{\alpha}, \vec{x}-\vec{x_0}\rangle)|}{\|\vec{x}-\vec{x_0}\|} = 0$$

Now consider $\vec{\alpha} = (2,2)$ let $x-1 = a$ & $y-1 = b$

$$\underset{(a,b)\to(0,0)}{Lt} \frac{|f(a+1,b+1) - (f(1,1) + (2,2)\cdot(a\ b))|}{\|a\cdot b\|}$$

$$\underset{(a,b)\to(0,0)}{Lt} \frac{(a+1)^2+(b+1)^2 - (2+2a+2b)}{\sqrt{a^2+b^2}}$$

$$\underset{(a,b)\to(0,0)}{Lt} \frac{a^2+b^2}{\sqrt{a^2+b^2}} = \underset{(a,b)\to(0,0)}{Lt} \sqrt{a^2+b^2} = 0$$

Hence f is differentiable. and $f'(1,1) = (2,2)$

2. $f(x,y) = \begin{cases} xy\,\frac{x^2-y^2}{x^2+y^2} & (x,y)\neq(0,0) \\ 0 & (x,y)=(0,0) \end{cases}$. show that f is differentiable

at $\vec{x_0} = (0,0)$

A. consider $\alpha = (0,0)$

$$\underset{(x,y)\to(0,0)}{Lt} \frac{\left| xy\,\frac{x^2-y^2}{x^2+y^2} - (0 + (0,0)\cdot(x\ y))\right|}{\|x\ y\|}$$

$$\underset{(x,y)\to(0,0)}{Lt} \frac{|x||y|\left|\frac{x^2-y^2}{x^2+y^2}\right|}{\|x\ y\|} \leq \underset{(x,y)\to(0,0)}{Lt} \frac{\sqrt{x^2+y^2}\cdot\cancel{\sqrt{x^2+y^2}}}{\cancel{\sqrt{x^2+y^2}}}$$

≤ 0

i.e. f is diff at $(0,0)$ and $f'(0,0) = (0,0)$

3. If $f(x,y) = \begin{cases} x^2 \sin\frac{1}{y} + y^2 \sin\frac{1}{x} & (x,y) \neq (0,0) \\ 0 & (x,y) = (0,0) \end{cases}$. Show that $f'((0,0)) = (0,0)$

A. $\alpha = (0,0)$

$$\underset{(x,y)\to(0,0)}{Lt} \frac{\left| x^2 \sin\frac{1}{y} + y^2 \sin\frac{1}{x} - (0 + (0,0)\cdot(x,y)) \right|}{\|x \;\; y\|}$$

$$\leq \underset{(x,y)\to(0,0)}{Lt} \frac{|x^2||\sin\frac{1}{y}| + |y^2||\sin\frac{1}{x}|}{\sqrt{x^2+y^2}}$$

$$\leq \underset{(x,y)\to(0,0)}{Lt} \frac{2(x^2+y^2)}{\sqrt{x^2+y^2}}$$

$$= \underset{(x,y)\to(0,0)}{Lt} 2\left(\sqrt{x^2+y^2}\right)$$

$$= 0$$

i.e $f'(0,0) = (0,0)$

4. Let f be diff at $\vec{x_0}$. Then prove that $D_{\vec{v}} f(\vec{x_0}) = \langle f'(\vec{x_0}), \vec{v} \rangle$

A. By definition $D_v f(\vec{x_0}) = \underset{t\to 0}{Lt} \frac{f(\vec{x_0} + t\vec{v}) - f(\vec{x_0})}{t}$

$$= \underset{t\to 0}{Lt} \left[\frac{f(\vec{x_0}+t\vec{v}) - f(\vec{x_0})}{t} - \langle f'(\vec{x_0}), \vec{v} \rangle \right] + \langle f'(\vec{x_0}), \vec{v} \rangle$$

$$= \underset{t\to 0}{Lt} \left[\frac{f(\vec{x_0}+t\vec{v}) - f(\vec{x_0}) - \langle f'(\vec{x_0}), t\vec{v} \rangle}{t} \right] + \langle f'(\vec{x_0}), \vec{v} \rangle$$

$$= 0 + \langle f'(x_0), \vec{v} \rangle$$

$$= \langle f'(x_0), \vec{v} \rangle$$

Hence proved.

5. prove that $D_{\vec{v}} f(\vec{x_0}) = \frac{d}{dt} f(\vec{x_0} + t\vec{v}) \Big|_{t=0}$

A. consider $g(t) = f(\vec{x_0} + t\vec{v})$

from definition $D_{\vec{v}} f(\vec{x_0}) = \lim_{t \to 0} \frac{f(\vec{x_0} + t\vec{v}) - f(\vec{x_0})}{t}$

$$= \lim_{t \to 0} \frac{g(t) - g(0)}{t}$$

$$= \lim_{t \to 0} \frac{g(0+t) - g(0)}{(0+t) - (0)}$$

$$= g'(0)$$

$$= \frac{d}{dt} f(\vec{x_0} + t\vec{v}) \Big|_{t=0}$$

6. prove that $D_{\vec{v}} f(\vec{x_0}) = \begin{pmatrix} \frac{\delta f_1}{\delta x_1}(x_0) & \cdots & \frac{\delta f_1}{\delta x_n}(x_0) \\ \frac{\delta f_2}{\delta x_2}(x_0) & \cdots & \frac{\delta f_2}{\delta x_n}(x_0) \\ \vdots & & \vdots \\ \frac{\delta f_m}{\delta x_1}(x_0) & \cdots & \frac{\delta f_m}{\delta x_n}(x_0) \end{pmatrix}_{m \times n} \begin{pmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{pmatrix}_{n \times 1}$

A. The $D_{\vec{v}} f(\vec{x_0}) = \begin{pmatrix} D_v f_1(\vec{x_0}) \\ \vdots \\ D_v f_m(x_0) \end{pmatrix}$

if we consider $D_v f_i(\vec{x_0}) = \langle f_i'(\vec{x_0}), \vec{v} \rangle$

But $\vec{v}$ can be represented as $\vec{v} = \sum_{j=1}^{n} v_j \hat{e_j}$

then $D_v f_i(\vec{x_0}) = \sum_{j=1}^{n} v_j \langle f_i'(\vec{x_0}), \hat{e_j} \rangle$

$$= \sum_{j=1}^{n} v_j \frac{\delta f_i}{\delta x_j}(\vec{x_0})$$

$$= \begin{pmatrix} \frac{\delta f_i}{\delta x_1}(x_0) & \frac{\delta f_i}{\delta x_2}(x_0) & \cdots & \frac{\delta f_i}{\delta x_n}(x_0) \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{pmatrix}$$

$\therefore D_{\vec{v}} f(\vec{x_0}) = \begin{pmatrix} \frac{\delta f_1}{\delta x_1}(x_0) & \frac{\delta f_1}{\delta x_2}(x_0) & \cdots & \frac{\delta f_1}{\delta x_n}(x_0) \\ \vdots & & & \vdots \\ \frac{\delta f_m}{\delta x_1}(x_0) & \frac{\delta f_m}{\delta x_2}(x_0) & \cdots & \frac{\delta f_m}{\delta x_n}(x_0) \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{pmatrix}$ Hence proved

Note: if $f: R^n \to R^m$ then $f'(x_0) = (a_{ij})_{m\times n}$, $D_{\vec{v}} f(\vec{x_0}) = (a_{ij})_{m\times n}(v_i)_{n\times 1}$

7. Derive Mean value theorem for scalar field

A. MVT states that if $f: R^n \to R$ be diff and $\vec{a}, \vec{b} \in R^n$ then $\exists$ $\vec{z} \in [\vec{a}, \vec{b}]$ s.t $f(\vec{b}) - f(\vec{a}) = D_{\vec{v}} f(\vec{z})$ where $\vec{v} = \dfrac{\vec{b}-\vec{a}}{\|\vec{b}-\vec{a}\|}$

Let us define $g(t) = f(\vec{a}+t(\vec{b}-\vec{a}))$ $g: [0,1] \to R$

Now we apply Mean value theorem of single variable calculus for $g(t)$ from $[0,1]$. then $\exists$ $c \in [0,1]$ such that

$$g'(c) = \frac{g(1)-g(0)}{1-0}$$

But $g'(t) = f'(\vec{a}+t(\vec{b}-\vec{a}))\cdot(\vec{b}-\vec{a})$

$= D_{\vec{v}} f(\vec{a}+t(\vec{b}-\vec{a}))$

$D_{\vec{v}} f(\vec{a}+t(\vec{b}-\vec{a})) = f(\vec{b}) - f(\vec{a})$

$D_{\vec{v}} f(\vec{z}) = f(\vec{b}) - f(\vec{a})$

as $\vec{a}+t(\vec{b}-\vec{a}) \in [\vec{a}, \vec{b}]$

let $\vec{z} = \vec{a}+t(\vec{b}-\vec{a})$

8. Prove that MVT is not appilicable for vector field.

A. Let us prove this by taking counter example

Counter example: $g: [0, 2\pi] \to R^2$, $g(t) = (\cos t, \sin t)$

$g(2\pi) - g(0) = (1,0) - (1,0) = (0,0)$

$g'(t) = (-\sin t, \cos t)$ so there doesn't exist any t such that $\sin t$ & $\cos t$ are 0 simultaneously

so $g(2\pi) - g(0) \neq 2\pi g'(c)$ for any $c \in (0, 2\pi)$

Hence MVT doesn't satisfies for vector field

9. ~~Prove th~~ Derive the equation of tangent plane for a surface

A. Let the surface S be defined as

$S = \{(x,y,z) : f(x,y,z) = K\}$ where K is a constant.

Let $\gamma: [a,b] \to R^3$ such that $\gamma([a,b]) \subset S$ where γ is a smooth curve.

as $\gamma([a,b]) \subset S$ $\quad f(\gamma(t)) = K \quad \forall t \in [a,b]$

so $f'(\gamma(t)) \cdot \gamma'(t) = 0 \quad \forall t \in [a,b]$

$\nabla f(\gamma(t)) \cdot \gamma'(t) = 0 \quad \forall t \in [a,b]$

i.e, If γ is a smooth curve lying on surface S, then $\nabla f(\gamma(t))$ is orthogonal to the tangent to the curve at the point $\gamma(t)$

~~Now~~ Hence the equation of the tangent plane at P to the surface is $(\vec{x}-\vec{x_0})\cdot\nabla f(\vec{x_0})=0$

Taylor's theorem:

if f has derivatives of order n at x_0. Then

$$f(x_0+h)=f(x_0)+hf'(x_0)+\frac{h^2}{2!}f''(x_0)+\cdots+\frac{h^n}{n!}f^{(n)}(x_0+\theta h)$$

for some $0<\theta<1$

for a multivariable function $f: E(\subset R^2)\to R$. Let $(a,b)\in E$ and $(h,k)\in R^2$ such that $(a+h,b+k)\in E$ then

$$f(a+h,b+k)=f(a,b)+\left(h\frac{\delta}{\delta x}+k\frac{\delta}{\delta y}\right)f(a,b)+\frac{1}{2!}\left(h^2\frac{\delta^2}{\delta x^2}+2hk\frac{\delta^2}{\delta x\delta y}+k^2\frac{\delta^2}{\delta y^2}\right)f''(a,b)+\cdots+\frac{1}{n!}\left(h\frac{\delta}{\delta x}+k\frac{\delta}{\delta y}\right)^n f^{(n)}(a+\theta h,b+\theta k)$$

for some $0<\theta<1$

$$\Rightarrow \boxed{f(a+h,b+k)=f(a,b)+\left(h\frac{\delta}{\delta x}+k\frac{\delta}{\delta y}\right)f'(a,b)+\frac{1}{2!}\left(h\frac{\delta}{\delta x}+k\frac{\delta}{\delta y}\right)^2 f''(a,b)+\cdots+\frac{1}{n!}\left(h\frac{\delta}{\delta x}+k\frac{\delta}{\delta y}\right)^n f^{(n)}(a+\theta h,b+\theta k)}$$

for some $0<\theta<1$

Proof: consider ~~g(t)~~ $f(a+th,b+tk)$ be a function $\exists\, g:[0,1]\to R$

Then by taylor's theorem for one variable

$$g'(t)=\frac{\delta f}{\delta x}h+\frac{\delta f}{\delta y}k$$

$$g''(t)=\frac{d}{dt}\left(h\frac{\delta f}{\delta x}+k\frac{\delta f}{\delta y}\right)$$

$$=h^2\frac{\delta^2 f}{\delta x^2}+2hk\frac{\delta^2 f}{\delta x\delta y}+k^2\frac{\delta^2 f}{\delta y^2}$$

$$=\left(h\frac{\delta}{\delta x}+\frac{\delta}{\delta y}k\right)^2 f$$

$$\vdots$$

$$g^{(n)}(t)=\left(h\frac{\delta}{\delta x}+k\frac{\delta}{\delta y}\right)^n f$$

$$\text{so } g(0+1)=g(0)+1\cdot g'(0)+\frac{1}{2!}g''(0)+\cdots+\frac{1}{n!}g^{(n)}(0+\theta h)$$

$$f(a+h,b+k)=f(a,b)+\left(h\frac{\delta f}{\delta x}+k\frac{\delta f}{\delta y}\right)+\frac{1}{2!}\left(h\frac{\delta}{\delta x}+k\frac{\delta}{\delta y}\right)^2 f+\cdots+\frac{1}{n!}\left(h\frac{\delta}{\delta x}+k\frac{\delta}{\delta y}\right)^n f(a+\theta h,b+\theta k).$$

10. prove $\underline{\int_a^b} f(x)dx \leq \overline{\int_a^b} f(x)dx$

A. Let P_1, P_2 be any two partitions of $[a,b]$ then

Let $P = P_1 \cup P_2$, then P is a new partition of $[a,b]$ then

$$L(P,f) \geq L(P_1,f) \ \& \ L(P_2,f)$$

$$U(P,f) \leq U(P_1,f) \ \& \ U(P_2,f)$$

so $L(P_1,f) \leq L(P,f) \leq U(P,f) \leq U(P_2,f)$

Hence $\underline{\int_a^b} f(x) \leq \overline{\int_a^b} f(x)dx$

11. prove Fundamental theorem of calculus (I)

A. $F = \int_a^x f(t)dt$

Case-1: Consider $x_1, x_2 \in [a,b]$, $x_1 < x_2$

then $|F(x_1) - F(x_2)| = \left|\int_{x_1}^{x_2} f(t)dt\right|$

$$\leq \int_{x_1}^{x_2} |f(t)|dt$$

$$\leq \int_{x_1}^{x_2} M dt$$

$$= M(x_1 - x_2)$$

$$\leq M|x_1 - x_2|$$

i.e F is continous on $[a,b]$

Case-2: when f is continous

then $\left|\frac{F(x)-F(x_0)}{x-x_0} - f(x_0)\right|$

$$= \left|\frac{1}{x-x_0}\int_{x_0}^{x} f(t)dt - f(x_0)\right|$$

$$= \left|\frac{1}{x-x_0}\int_{x_0}^{x} (f(t)-f(x_0))dt\right|$$

$$< \epsilon$$

finally $\lim_{x \to x_0} \frac{F(x)-F(x_0)}{x-x_0} = f(x_0)$

Hence F is differentiable at x_0

12. Let $f: [0,1] \to R$ be a cont function, show that $\lim_{n\to\infty} \int_0^1 x^n f(x) = 0$

A. $\lim_{n\to\infty} \int_0^1 x^n f(x)dx = \int_0^1 (\lim_{n\to\infty} x^n) f(x)\,dx$ [$\because$ n is independent of x]

$$= \int_0^1 0 \cdot f(x)\,dx \quad [\because 0<x<1, \ \lim_{n\to\infty} x^n \to 0$$

$$= 0$$

13. $f: [0,1] \to R$ be cont function. S.T $\lim_{n\to\infty} \int_0^1 f(x^n)dx = f(0)$

A. $\lim_{n\to\infty} \int_0^1 f(x^n)dx = \int_0^1 \lim_{n\to\infty} f(x^n)dx$

$$= \int_0^1 f(\lim_{n\to\infty} x^n)dx$$

$$= \int_0^1 f(0)\,dx = f(0)\cdot\int_0^1 1\,dx = f(0)\cdot 1 = f(0)$$

13. find the volume of a sphere of radius r

$f(x,y,z) = x^2+y^2+z^2 = r^2$

$$V = \int_{-r}^{r} \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} \int_{-\sqrt{1-x^2-y^2}}^{\sqrt{1-x^2-y^2}} 1 \cdot dz \cdot dy \cdot dx$$

$x = r\sin\phi\cos\theta$

$y = r\sin\phi\sin\theta$

$z = r\cos\phi$

$$= \int_{r=0}^{r} \int_{\theta=0}^{2\pi} \int_{\phi=0}^{\pi} 1 \cdot \underbrace{(r^2\sin\phi)}_{\text{Jacobian value}}\, dr\, d\theta\, d\phi$$

$$= \int_{r=0}^{r} r^2 dr \cdot \int_{0=0}^{2\pi} \sin\phi\, d\phi \int_{0=0}^{2\pi} d\theta$$

$$= \frac{r^3}{3} \cdot \left[\cos\phi\right]_0^{\pi} \cdot 2\pi$$

$$= \frac{2\pi r^3}{3} \cdot 2$$

$$= \frac{4}{3}\pi r^3$$

14. Let $f(x,y)$ be defined in $S = \{(x,y) \in R^2 : a<x<b,\ c<y<d\}$. Suppose that the partial derivatives of f exist and are bounded in S. then show that f is continous in S.

A. Given f is bounded $|f_x(x,y)| \le M$ & $|f_y(x,y)| \le M$

Now consider

$$f(x+h, y+k) - f(x,y) = f(x+h, y+k) - f(x+h, y) + f(x+h, y) - f(x,y)$$

$$= k \cdot f_x(x+h, y+\theta k) + h\, f_y(x+\theta h, y)$$

$$\le M(|h| + |k|)$$

$$\le 2M\sqrt{h^2+k^2}$$

Choose $\delta = \frac{\epsilon}{2M}$.

Pointwise convergence and uniform convergence:

Def: Let D be a subset of R and let $\{f_n\}$ be a sequence of functions defined on D. We say that $\{f_n\}$ converges pointwise on D if

$$\underset{n\to\infty}{Lt}\, f_n(x) \text{ exist for each point } x \text{ in } D$$

This means that $\underset{n\to\infty}{Lt}\, f(x)$ is a real number $(R\in(-\infty,\infty))$ that depends only on x.

if $\{f_n\}$ is pointwise convergent then the function defined by $f_n(x) = \underset{n\to\infty}{Lt}\, f_n(x)$ $\forall x \in D$, is called the pointwise limit of the sequence $\{f_n\}$.

1. Consider the sequence $\{f_n\}$ of functions defined by $f_n(x) = x^n.n^2$ for $0 \le x \le 1$. Determine whether $\{f_n\}$ is pointwise convergent or not.

A. first of all, observe that $f_n(0) = 0$ for every n in N. so $\{f_n(0)\}$ converges to zero. Now suppose $0<x<1$ then $n^2x^n = n^2 e^{n\ln x}$. But $\ln x < 0$ when $0<x<1$, it follows that

$$\underset{n\to\infty}{Lt}\, f_n(x) = 0 \text{ for } 0<x<1$$

Finally, $f_n(1) = n^2$ $\forall n$, So $\underset{n\to\infty}{Lt}\, f(1) = \infty$. Therefore, $\{f_n\}$ is not pointwise convergent on $[0,1]$

2. consider the sequence $\{f_n\}$ of functions defined by

$$f_n(x) = \frac{\sin(nx+3)}{\sqrt{n+1}} \quad \forall x \in R.$$

Show that $\{f_n\}$ converges pointwise

A. for every $x \in R$, we have

$$\frac{-1}{\sqrt{n+1}} \le \frac{\sin(nx+3)}{\sqrt{n+1}} \le \frac{1}{\sqrt{n+1}}$$

Appling the squeeze theorem for sequences, we obtain that

$$\underset{n\to\infty}{Lt}\, \frac{-1}{\sqrt{n+1}} \le \underset{n\to\infty}{Lt}\, \frac{\sin(nx+3)}{\sqrt{n+1}} \le \underset{n\to\infty}{Lt}\, \frac{1}{\sqrt{n+1}}$$

$$0 \le \underset{n\to\infty}{Lt}\, \frac{\sin(nx+3)}{\sqrt{n+1}} \le 0$$

$$\therefore \underset{n\to\infty}{Lt}\, \frac{\sin(nx+3)}{\sqrt{n+1}} = 0 \quad \forall x \in R$$

$\therefore$ $\{f_n\}$ converges pointwise to the function $f \equiv 0$ on R.

Def: Let D be a subset of R and let $\{f_n\}$ be a sequence of real valued functions defined on D. Then $\{f_n\}$ converges uniformly to f if given $\epsilon>0$, there exists a natural number $N=N(\epsilon)$ such that $|f_n(x)-f(x)|<\epsilon$ for every $n>N$ & $\forall x\in D$

Note: In the above definition the natural number N depends only on ϵ. Therefore, uniform convergence implies pointwise convergence.

1. Let $\{f_n\}$ be the sequence of functions on $(0,\infty)$ defined by $f_n(x)=\frac{nx}{1+n^2x^2}$. Determine whether $\{f_n\}$ is uniformly convergent or not

A. Consider $f_n(x)=\frac{nx}{1+n^2x^2}$

$$\lim_{n\to\infty}\frac{nx}{1+n^2x^2}=\lim_{n\to\infty}\frac{\cancel{x}}{\ldots n x^{\cancel{2}}}=\frac{1}{x}\lim_{n\to\infty}\frac{1}{n}=0\ \forall x\in(0,\infty)$$

This function converges pointwise to zero.

But for any $\epsilon>\frac{1}{2}$, we have

$$\left|f_n\left(\tfrac{1}{n}\right)-f\left(\tfrac{1}{n}\right)\right|=\frac{1}{2}-0>\epsilon$$

Hence $\{f_n\}$ is not uniformly convergent.

Sequential criterion for Non-uniform convergence:

$f_n(x)\not\rightrightarrows f(x)$ if for given $\epsilon_0>0$ $\exists$ a subsequence f_{n_k} of f_n and x_k in A such that $|f_{n_k}(x_k)-f(x_k)|\geq\epsilon_0$ where $k, n_k\in N$

Eg: Consider $f_n(x)=\frac{x^2+nx}{n}$

$$\lim_{n\to\infty}f_n(x)=\lim_{n\to\infty}\left(\frac{x^2}{n}+x\right)=x\ \forall x\in R$$

Let Consider $n_k=k$ and $x_k=-k$

$$|f_{n_k}(x_k)-f(x_k)|=\left|\frac{k^2-k^2}{k}-k\right|=|-k|=k>\epsilon_0\ [\because k\in N]$$

$\therefore\{f_n\}$ is non-uniform convergence

Note: Pointwise convergence is denoted by $\longrightarrow$ & uniform convergence is denoted by $\Longrightarrow$.

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Uniform Norm: Let ϕ be a bounded function over $[a,b]$

$$||\phi|| = \sup\{|\phi(x)|, x \in [a,b]\}$$

Uniform Convergence Test or M_N-Test (weierstrauss test)

Let $\{f_n(x)\}$ be a sequence such that $f_n(x) \longrightarrow f(x)$, then define

$$M_n = ||f_n(x) - f(x)|| = \max|f_n(x) - f(x)| \ \forall x \in A.$$

if $\lim_{n\to\infty} M_n = 0$ then $\{f_n(x)\}$ is uniformly convergent

else $\{f_n(x)\}$ is non-uniform convergent.

Eg: consider $f_n(x) = x^n(1-x) \quad -1 < x \leq 1$

$$\lim_{n\to\infty} f_n(x) = \lim_{n\to\infty} x^n(1-x) = \begin{cases} 0 & -1 < x < 1 \\ 0 & x = 1 \end{cases} = 0 \quad \forall x \in (-1,1]$$

$$M_n = ||x^n(1-x) - 0|| = \max|x^n(1-x)|$$

consider $g(x) = x^n(1-x)$

$$g'(x) = 0 \Rightarrow nx^{n-1}(1-x) + x^n(-1) = 0$$

$$n(1-x) = x$$

$$\boxed{x = \frac{n}{1+n}}$$

$$M_n = \left|\left(\frac{n}{1+n}\right)^n \cdot \frac{1}{1+n}\right| = \left(\frac{n}{1+n}\right)^n \cdot \frac{1}{1+n}$$

$$\lim_{n\to\infty} M_n = \lim_{n\to\infty}\left(\frac{1}{1+\frac{1}{n}}\right)^n \cdot \frac{1}{1+n}$$

$$= \left(\lim_{n\to\infty}\left(\frac{1}{1+\frac{1}{n}}\right)^n\right) \cdot \left(\lim_{n\to\infty}\frac{1}{1+n}\right)$$

$$= 1 \cdot 0$$

$$\lim_{n\to\infty} M_n = 0$$

$\therefore \{f_n\}$ is uniformly convergent. ($f_n \rightrightarrows f(x)$) where $f(x) = 0 \ \forall x \in (-1,1]$

Remark: if $\{f_n\}, \{g_n\}$ are two bounded sequence of functions such that $f_n(x) \rightrightarrows f$, $g_n(x) \rightrightarrows g$ then $f_n(x)\cdot g_n(x) \rightrightarrows f\cdot g$

Corollary: if $\{f_n(x)\}$ such that $f_n(x) \rightrightarrows f$ and $|f_n(x)| \leq M$ $\forall x \in \mathbb{R}, n \in \mathbb{N}$ and $g: [-M,M] \to \mathbb{R}$ is a continous function then $g \circ f_n(x) \rightrightarrows g \circ f(x)$

Theorem: Let D be a subset of $\mathbb{R}$ and let $\{f_n\}$ be a sequence of continous functions on D which converges uniform to f on D. Then f is continous on D

Remark: if $f_n(x) \rightrightarrows f$ and if f is discontinous then convergence must be non-uniform where $f_n(x)$ is a sequence of continous functions

Theorem: Let D be a subset of $\mathbb{R}$ and Let $\{f_n\}$ be a sequence of differentiable functions on D such that $\{f_n(x)\}$ converges uniformly to f and $\{f_n'(x)\}$ converges uniformly to g, then

$$f'(x) = g(x).$$

Theorem: Let D be a subset of $\mathbb{R}$ and let $\{f_n\}$ be a sequence of Riemann integrable functions on D such that $\{f_n(x)\}$ converges uniformly to f then

$$\lim_{n\to\infty} \int f_n(x)\,dx = \int f(x)\,dx$$

Theorem: if $\{f_n\}$ be a sequence of Monotonic bounded continous function & if $f_n(x) \to f(x)$ then $f_n(x) \rightrightarrows f(x)$

stone-weierstrauss theorem:

Let f be a continous function on $[a,b]$ then $\exists$ a seq^n of polynomials $(P_n)_{n=1}^{\infty}$ such that (P_n) converges to f uniformly.

Bernstein polynomial: Let f be continous on $[a,b]$. Define

$$B_n(x) = \sum_{k=0}^{n} \binom{n}{k} f\left(\frac{k}{n}\right) x^k (1-x)^{n-k}$$

Then $(B_n(x))_{n=1}^{\infty}$ converges to f uniformly on $[a,b]$

1. Find the region in $\mathbb{R}$ where the series $\sum_{n=1}^{\infty} \frac{1}{1+x^n}$ $(x \neq -1)$ converges pointwise

A. Consider ratio test

$$\lim_{n\to\infty} \left| \frac{1+x^n}{1+x^{n+1}} \right|$$

if $|x|>1$

$$\lim_{n\to\infty} \left| \frac{1+\frac{1}{x^n}}{x+\frac{1}{x^n}} \right| = \frac{1}{x} < 1$$

$\therefore$ converges

if $0 \le |x| < 1$ the n^{th} term

$$\lim_{n\to\infty} \frac{1}{1+x^n} = 1 \neq 0$$

$\therefore$ diverges

if $x=1$ the n^{th} term

$$\sum_{n=1}^{\infty} \frac{1}{1+1} = \infty$$

$\therefore$ diverges

$\therefore$ region is $|x|>1$ or $R-[-1,1]$

2. study the uniform convergence of the following series on

A. $\sum_n x^2(1-x^2)^{n-1}$ on $A=[-1,1]$

case-1: $|x|=1$

$\sum_n 1(0)^{n-1}=0$

$\therefore$ converges to 0

case-2: $|x|=0$

$\sum_n 0(1-0)^{n-1}=0$

$\therefore$ converges to 0

case-3: $0<|x|<1$

consider root test

$\lim_{n\to\infty} |x^2(1-x^2)^{n-1}|^{1/n}$

$=\lim_{n\to\infty} x^{2/n}(1-x^2)^{n-1/n}$

$=\lim_{\frac{1}{n}\to 0} x^{2/n}(1-x^2)^{1-1/n}$

$=x^0(1-x^2)^{1-0}$

$=1-x^2$

<1

$\therefore$ converges.

$\therefore$ converges pointwise

Now consider $0<|x|<1$

$$\sum_n x^2(1-x^2)^{n-1} = x^2\underbrace{\sum_{n=1}^{\infty}(1-x^2)^{n-1}}_{\text{infinte G.P}} \qquad 1-x^2<1$$

$$= x^2\cdot\frac{1}{1-(1-x^2)} = 1 \quad (\therefore \text{converges to } 1)$$

$\therefore$ not uniformly convergent.

Note: suppose $\sum_{n=1}^{\infty} f_n(x)$ converges uniformly to some f on $[-1,1]$

then $\sum_{n=1}^{\infty} f_n(x)$ also converges pointwise to f.

Absolute convergence:

Defn: A series $\sum u_n$ is said to be absolutely convergent if the series $\sum |u_n|$ converges

Theorem: Every absolutely convergent series is a convergent series.

Theorem: Assume that the functions $f_n, g_n : A \to R$ satisfy the following conditions.

1) series $\sum_{n=1}^{\infty} |f_{n+1}(x) - f_n(x)|$ is uniform on A.

2) $\sup_{x \in A} |f_n(x)| \to 0$ as $n \to \infty$

3) The sequence $(G_n(x))_{n=1}^{n}$, $G_n(x) = \sum_{n=1}^{\infty} g_n(x)$ is uniformly bounded on A.

Then $\sum_n f_n g_n$ converges uniformly on A.

Eg: study the uniform convergence of $\sum_{n=1}^{\infty} \frac{1}{n} 2^n (3x-1)^n$ on $x \in [\frac{1}{6}, \frac{1}{3}]$

A. $\sum_{n=1}^{\infty} \frac{1}{n} (6x-2)^n \quad x \in [\frac{1}{6}, \frac{1}{3}]$

$\sum_{n=1}^{\infty} \frac{(-1)^n}{n} (2-6x)^n \quad x \in [\frac{1}{6}, \frac{1}{3}]$

$\sum_{n=1}^{\infty} \frac{(-1)^n}{n} \cdot t^n \quad t \in [0,1]$

Now consider $f_n(x) = \frac{1}{n}$, $g_n(x) = (-1)^n t^n$, $A = [0,1]$

1. $\sum_{n=1}^{\infty} |f_{n+1}(x) - f_n(x)| = \sum_{n=1}^{\infty} |\frac{1}{n+1} - \frac{1}{n}| = \sum_{n=1}^{\infty} (\frac{1}{n} - \frac{1}{n+1})$

$= 1 - \cancel{\frac{1}{2}} + \cancel{\frac{1}{2}} - \cancel{\frac{1}{3}} \ldots \quad 0$

$= 1$

$\therefore$ uniformly converges to 1 on A.

2. $\sup_{x \in A} |f_n(x)| = \sup_{x \in A} |\frac{1}{n}| = \frac{1}{n} = 0$ as $n \to \infty$

3. $\sum_{n=1}^{\infty} (-1)^n t^n \le \sum_{n=1}^{\infty} (-1)^n \cdot 1^n \quad \underbrace{\{0, -1\}}_{\text{bounded}}$

i.e $-1 \le \sum_{n=1}^{\infty} (-1)^n t^n \le 0$ (Bounded)

$\therefore \sum_n \frac{1}{n} (-1)^n t^n$ is uniformly convergent on $t \in [0,1]$

$\therefore \sum_{n=1}^{\infty} \frac{1}{n} 2^n (3x-1)^n$ is uniformly convergent on $x \in [\frac{1}{6}, \frac{1}{3}]$

Abel's test: Assume that the functions $f_n, g_n : A \to R$ satisfy the following conditions

1) for each $x \in A$ $(f_n(x))$ is monotonic
2) seq (f_n) is uniformly bounded on A
3) $\sum_{n=1}^{\infty} g_n(x)$ converges on A.

The $\sum_{n=1}^{\infty} f_n(x) \cdot g_n(x)$ converges uniformly on A.

Eg: for previous example consider $f_n(x) = t^n$ and $g_n(x) = \frac{(-1)^n}{n}$

Taylor series:

Let $f : [a,b] \to R$, f has derivative of all orders and they are uniformly bounded. then for any $x, x_0 \in [a,b]$

$$f(x) = f(x_0) + (x - x_0) f'(x_0) + \cdots + \frac{(x-x_0)^n}{n!} f^{(n)}(x_0) + \cdots + \infty$$

$$= \sum_{n=0}^{\infty} a_n (x - x_0)^n \quad \text{[power series representation]}$$

Define R such that

$R = 0$ if $\overline{\lim_{n\to\infty}} |a_n|^{1/n} = \infty$

$R = \infty$ if $\overline{Lt}_{n\to\infty} |a_n|^{1/n} = 0$

$\frac{1}{R} = \overline{Lt}_{n\to\infty} |a_n|^{1/n}$ otherwise

we call R as radius of convergence and $f_n(x)$ is said to be uniformly convergent if $|x - x_0| < R$

Remark: suppose $Lt_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right|$ exists then $\frac{1}{R} = Lt_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right|$.

Eg: $\sum_{n=1}^{\infty} \frac{2^n}{n!} \cdot x^n$, Find the radius of convergence

$$\frac{1}{R} = Lt_{n\to\infty} \left| \frac{2^{n+1}}{(n+1)!} \cdot \frac{n!}{2^n} \right|$$

$$\frac{1}{R} = Lt_{n\to\infty} \left| \frac{2}{n+1} \right|$$

$$\Rightarrow \boxed{R = \infty}$$

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Eg: $\sum_{n=1}^{\infty} 2^n x^{n^2}$. find radius of convergence

$$\frac{1}{R} = \lim_{n\to\infty} |2^n|^{1/n^2} = \lim_{n\to\infty} |2|^{1/n} = 1$$

Eg: Find the domain of convergence of the following series

a) $\sum_{n=1}^{\infty} \frac{(x-1)^{2n}}{2^n n^3}$ b) $\sum_{n=1}^{\infty}\left(\frac{n}{n+1}\right)\left(\frac{2x+1}{x}\right)^n$

a) Let $(x-1)^2 = y$ then

$$\sum_{n=1}^{\infty} \frac{y^n}{2^n n^3} \qquad a_n = \frac{1}{2^n n^3}$$

$$\frac{1}{R} = \lim_{n\to\infty}\left|\frac{1}{2^{n+1}(n+1)^3}\cdot\frac{2^n \cdot n^3}{1}\right|$$

$$= \lim_{n\to\infty}\frac{1}{2}\left|\frac{n^3}{(n+1)^3}\right| = \frac{1}{2}$$

$\boxed{R = 2}$

$|(x-1)^2| \leq 2$ — ②

for $|y| = 2$

$y = 2$: $\sum_{n=1}^{\infty}\frac{\cancel{2^n}}{\cancel{2^n}\cdot n^3}$, $\sum_{n=1}^{\infty}\frac{1}{n^3}$ convergent

$y = -2$: $\sum_{n=1}^{\infty}\frac{(-2)^n}{2^n n^3}$, $\sum_{n=1}^{\infty}\frac{(-1)^n}{n^3}$ convergent as it is uniformly convergent

$1-\sqrt{2} \leq x \leq 1+\sqrt{2}$

$\therefore \sum_{n=1}^{\infty} f_n(x)$ is convergent if $x \in [1-\sqrt{2}, 1+\sqrt{2}]$.

b) $\sum_{n=1}^{\infty}\left(\frac{n}{n+1}\right)\left(\frac{2x+1}{x}\right)^n$

$$\sum_{n=1}^{\infty}\left(\frac{n}{n+1}\right)\left(2+\frac{1}{x}\right)^n$$

Let $y = 2+\frac{1}{x}$

$$\sum_{n=1}^{\infty}\left(\frac{n}{n+1}\right)(y)^n \qquad a_n = \frac{n}{n+1}$$

$$\frac{1}{R} = \lim_{n\to\infty}\left|\frac{n+1}{n+2}\cdot\frac{n+1}{n}\right| = 1$$

$\boxed{R = 1}$ $\quad \left|2+\frac{1}{x}\right| < 1$

$-1 \leq 2+\frac{1}{x} < 1 \Rightarrow \boxed{x \in \left(-1, \frac{-1}{3}\right)}$

$y = 1$: $\sum_{n=1}^{\infty}\frac{n}{n+1}$, $\sum_{n=1}^{\infty}\left(1-\frac{1}{n+1}\right)$, $\sum_{n=1}^{\infty} 1 - \sum_{n=1}^{\infty}\frac{1}{n+1}$, ∞ diverges, ~~converges~~

$y = -1$ diverges

Theorem: Let R_1, R_2 be the radius of convergence of the P.S $\sum_{n=0}^{\infty} a_n x^n$ and $\sum_{n=0}^{\infty} b_n x^n$ respectively, then

1. The radius of convergence of P.S $\sum_{n=0}^{\infty} (a_n + b_n) x^n$ is $\min\{R_1, R_2\}$

2. The radius of conv of $\sum_{n=0}^{\infty} a_n b_n x^n$ satisfies $R \geq R_1 R_2$

Eg: Suppose that the radius of convg of P.S $\sum_{n=0}^{\infty} a_n x^n$ is R and $0 < R < \infty$. Evaluate radius of conv of

1. $\sum_{n=0}^{\infty} 2^n a_n x^n$

$\frac{1}{R'} = \lim_{n\to\infty} |2^n a_n|^{1/n}$

$= 2 \cdot \frac{1}{R}$

$\boxed{R' = R/2}$

2. $\sum_{n=0}^{\infty} n^n a_n x^n$

$\frac{1}{R'} = \lim_{n\to\infty} (n^n a_n)^{1/n}$

$= \lim_{n\to\infty} n \cdot |a_n|^{1/n}$

$= \infty$

$\boxed{R' = 0}$

3. $\sum_{n=0}^{\infty} a_n^2 x^n$

$\frac{1}{R'} = \lim_{n\to\infty} |a_n|^{2/n}$

$= \frac{1}{R^2}$

$\boxed{R' = R^2}$

Remark: Let $\sum_{n=0}^{\infty} a_n x^n$ & $\sum_{n=1}^{\infty} b_n x^n$ be a P.S having a common region of convergence $(-r, r)$ and are represented by $f_n(x)$ & $g_n(x)$ respectively, then the power series representation of $f(x) \cdot g(x) = \sum_{n=0}^{\infty} c_n x^n$ where

$c_n = \sum_{m=0} a_n b_{m-n}$