EP 1027: Supplementary material for lecture 14

April 25, 2019

1 Radiation fields

In class I mentioned that we need the expressions for $\frac{\partial t'}{\partial t}$ and $\boldsymbol{\nabla} t'$ to find the expressions for the radiation part of $\mathbf{E}$ and $\mathbf{B}$. Here I work these out. First consider $\frac{\partial t'}{\partial t}$,

$$\begin{aligned}
\frac{\partial t'}{\partial t} &= \frac{\partial}{\partial t}\left(t - \frac{|\mathbf{x}-\mathbf{x}'|}{c}\right)_{\mathbf{x}=\boldsymbol{\zeta}(t')} \\
&= \frac{\partial t}{\partial t} - \frac{1}{c}\left.\frac{\partial|\mathbf{x}-\mathbf{x}'|}{\partial t}\right|_{\mathbf{x}=\boldsymbol{\zeta}(t')} \\
&= 1 - \frac{1}{c}\left(\boldsymbol{\nabla}'|\mathbf{x}-\mathbf{x}'|\right)\cdot\left.\frac{\partial \mathbf{x}'}{\partial t}\right|_{\mathbf{x}=\boldsymbol{\zeta}(t')} \\
&= 1 - \frac{1}{c}\frac{\mathbf{x}'-\mathbf{x}}{|\mathbf{x}-\mathbf{x}'|}\cdot\left.\frac{\partial \mathbf{x}'}{\partial t'}\right|_{\mathbf{x}=\boldsymbol{\zeta}(t')}\frac{\partial t'}{\partial t} \\
&= 1 + \frac{1}{c}\hat{\mathbf{n}}\cdot\mathbf{v}(t')\frac{\partial t'}{\partial t}.
\end{aligned} \tag{1}$$

Here in going from the second to third line we have applied chain rule, $\frac{\partial}{\partial t} = \frac{\partial \mathbf{x}'}{\partial t}\cdot\boldsymbol{\nabla}$ and in going from the third to fourth line we have again applied a chain rule, $\frac{\partial}{\partial t} = \frac{\partial t'}{\partial t}\frac{\partial}{\partial t'}$. In the final step we have just substituted,

$$\hat{\mathbf{n}} = \left[\frac{\mathbf{x}-\mathbf{x}'}{|\mathbf{x}-\mathbf{x}'|}\right]_{\mathbf{x}=\boldsymbol{\zeta}(t')},$$

and,

$$\left.\frac{\partial \mathbf{x}'}{\partial t'}\right|_{\mathbf{x}=\boldsymbol{\zeta}(t')} = \frac{d\boldsymbol{\zeta}(t')}{dt'} = \mathbf{v}(t').$$

Thus from (1) it is easy to see,

$$\frac{\partial t'}{\partial t} = \frac{1}{1-\frac{\hat{\mathbf{n}}\cdot\mathbf{v}(t')}{c}}. \tag{2}$$

Next consider $\boldsymbol{\nabla} t'$,

$$\boldsymbol{\nabla} t' = \boldsymbol{\nabla}\left(t - \frac{|\mathbf{x}-\mathbf{x}'|_{\mathbf{x}=\boldsymbol{\zeta}(t')}}{c}\right) = -\frac{1}{c}\boldsymbol{\nabla}|\mathbf{x}-\mathbf{x}'|_{\mathbf{x}=\boldsymbol{\zeta}(t')} \tag{3}$$

To proceed further we go to component form,

$$\begin{aligned}
\left(\boldsymbol{\nabla}|\mathbf{x}-\mathbf{x}'|_{\mathbf{x}=\boldsymbol{\zeta}(t')}\right)_i &= \partial_i|\mathbf{x}-\boldsymbol{\zeta}(t')| \\
&= \frac{x_j-\zeta_j(t')}{|\mathbf{x}-\boldsymbol{\zeta}(t')|}\left(\partial_i x_j - \partial_i\zeta_j(t')\right) \\
&= \frac{x_j-\zeta_j(t')}{|\mathbf{x}-\boldsymbol{\zeta}(t')|}\left(\delta_{ij} - \partial_i t'\frac{\partial\zeta_j(t')}{\partial t'}\right) \\
&= \frac{x_i-\zeta_i(t')}{|\mathbf{x}-\boldsymbol{\zeta}(t')|} - \left(\frac{x_j-\zeta_j(t')}{|\mathbf{x}-\boldsymbol{\zeta}(t')|}\frac{\partial\zeta_j(t')}{\partial t'}\right)\partial_i t' \\
&= \frac{x_i-x_i'}{|\mathbf{x}-\mathbf{x}'|} - \left(\frac{x_j-x_j'}{|\mathbf{x}-\mathbf{x}'|}v_j(t')\right)\partial_i t' \\
\Rightarrow \boldsymbol{\nabla}|\mathbf{x}-\mathbf{x}'|_{\mathbf{x}=\boldsymbol{\zeta}(t')} &= \frac{\mathbf{x}-\mathbf{x}'}{|\mathbf{x}-\mathbf{x}'|} - \left(\frac{\mathbf{x}-\mathbf{x}'}{|\mathbf{x}-\mathbf{x}'|}\cdot\mathbf{v}(t')\right)\boldsymbol{\nabla}t' \\
&= \hat{\mathbf{n}} - \left(\hat{\mathbf{n}}\cdot\mathbf{v}(t)\right)\boldsymbol{\nabla}t'.
\end{aligned}$$

Substituting this back in (3) we get,

$$\begin{aligned}
\boldsymbol{\nabla}t' &= -\frac{\hat{\mathbf{n}}}{c} + \left(\frac{\hat{\mathbf{n}}\cdot\mathbf{v}(t)}{c}\right)\boldsymbol{\nabla}t' \\
\Rightarrow \boldsymbol{\nabla}t' &= -\frac{\hat{\mathbf{n}}/c}{1-\frac{\hat{\mathbf{n}}\cdot\mathbf{v}(t)}{c}}.
\end{aligned} \tag{4}$$

2 Radiation fields

2.1 Far zone approximation Taylor expansion

In class I used the Taylor expansion of $|\mathbf{x}-\mathbf{x}'|$ around $\mathbf{x}'=0$ for the far zone approximation i.e. when $|\mathbf{x}| \ll |\mathbf{x}'|$. This is a Taylor expansion of a function in 3 variables, namely $x_1'-0, x_2'=0, x_3'=0$. Here I perform the Taylor expansion to first order. First I Taylor expand in powers of x_1

$$f(x_1',x_2',x_3') = f(0,x_2',x_3') + \left.\frac{\partial f(x_1',x_2',x_3')}{\partial x_1'}\right|_{x_1'=0} x_1' + \frac{1}{2!}\left.\frac{\partial^2 f(x_1',x_2',x_3')}{\partial {x_1'}^2}\right|_{x_1'=0} {x_1'}^2 + \ldots \tag{5}$$

Next we Taylor expand the RHS around $x_2'=0$. We will be content to just evaluate up to the second order. The first term on the RHS of (5) is,

$$f(0,x_2',x_3') = f(0,0,x_3') + \left.\frac{\partial f(0,x_2',x_3')}{\partial x_2'}\right|_{x_2'=0} x_2' + \frac{1}{2!}\left.\frac{\partial^2 f}{\partial {x_2'}^2}\right|_{x_2'=0} {x_2'}^2 + \ldots.$$

The second term in the RHS of (5) is,

$$\left.\frac{\partial f(x_1',x_2',x_3')}{\partial x_1'}\right|_{x_1'=0} = \left.\frac{\partial f(x_1',x_2',x_3')}{\partial x_1'}\right|_{x_1'=x_2'=0} + \left.\frac{\partial^2 f(x_1',x_2',x_3')}{\partial x_1'\partial x_2'}\right|_{x_1'=x_2'=0} x_2' + \ldots.$$

The third term in the RHS of the (5) is,

$$\frac{1}{2!}\left.\frac{\partial^2 f(x_1',x_2',x_3')}{\partial {x_1'}^2}\right|_{x_1'=0} {x_1'}^2 = \frac{1}{2!}\left.\frac{\partial^2 f(x_1',x_2',x_3')}{\partial {x_1'}^2}\right|_{x_1'=x_2'=0} {x_1'}^2 + \ldots.$$

Plugging these in the RHS of (5),

$$
\begin{aligned}
f(x_1', x_2', x_3') = f(0,0,x_3') &+ \left.\frac{\partial f}{\partial x_1'}\right|_{x_1'=x_2'=0} x_1' + \left.\frac{\partial f}{\partial x_2'}\right|_{x_1'=x_2'=0} x_2' \\
&+ \frac{1}{2!}\left.\frac{\partial^2 f}{\partial {x_1'}^2}\right|_{x_1'=x_2'=0} {x_1'}^2 + \left.\frac{\partial^2 f}{\partial x_1' \partial x_2'}\right|_{x_1'=x_2'=0} x_2' x_1' + \frac{1}{2!}\left.\frac{\partial^2 f}{\partial {x_2'}^2}\right|_{x_1'=x_2'=0} {x_2'}^2 \ldots .
\end{aligned}
$$

Finally we can Taylor expand all the terms in the RHS around $x_3' = 0$ to second order to obtain,

$$
\begin{aligned}
f(x_1', x_2', x_3') = f(0,0,0) &+ \left.\frac{\partial f}{\partial x_1'}\right|_{x_1'=x_2'=x_3'=0} x_1' + \left.\frac{\partial f}{\partial x_2'}\right|_{x_1'=x_2'=x_3'=0} x_2' + \left.\frac{\partial f}{\partial x_3'}\right|_{x_1'=x_2'=x_3'=0} x_3' \\
&+ \frac{1}{2!}\left.\frac{\partial^2 f}{\partial {x_1'}^2}\right|_{x_1'=x_2'=x_3'=0} {x_1'}^2 + \frac{1}{2!}\left.\frac{\partial^2 f}{\partial {x_2'}^2}\right|_{x_1'=x_2'=x_3'=0} {x_2'}^2 + \frac{1}{2!}\left.\frac{\partial^2 f}{\partial {x_3'}^2}\right|_{x_1'=x_2'=x_3'=0} {x_3'}^2 \\
&+ \left.\frac{\partial^2 f}{\partial x_1' \partial x_2'}\right|_{x_1'=x_2'=x_3'=0} x_1' x_2' + \left.\frac{\partial^2 f}{\partial x_2' \partial x_3'}\right|_{x_1'=x_2'=x_3'=0} x_2' x_3' + \left.\frac{\partial^2 f}{\partial x_3' \partial x_1'}\right|_{x_1'=x_2'=x_3'=0} x_3' x_1' + \ldots
\end{aligned}
$$

It is easy to write the above expression in terms of $\boldsymbol{\nabla}$ operator,

$$
f(x_1', x_2', x_3') = f(0,0,0) + \mathbf{x}' \cdot \boldsymbol{\nabla}' f\big|_{\mathbf{x}'=0} + \frac{1}{2} x_i' x_j' \left.\frac{\partial^2 f}{\partial x_i' \partial x_j}\right|_{\mathbf{x}'=0} + \ldots . \tag{6}
$$

For the case at hand, $f = |\mathbf{x} - \mathbf{x}'|$. Plugging this in the multi-variable Taylor expansion (6) to first order,

$$
\begin{aligned}
|\mathbf{x} - \mathbf{x}'| &= |\mathbf{x}| + \mathbf{x}' \cdot \left(\boldsymbol{\nabla}' |\mathbf{x} - \mathbf{x}'|\right)\big|_{\mathbf{x}'=0} + \ldots \\
&= |\mathbf{x}| + \mathbf{x}' \cdot \left.\left(\frac{\mathbf{x}' - \mathbf{x}}{|\mathbf{x} - \mathbf{x}'|}\right)\right|_{\mathbf{x}'=0} + \ldots \\
&= |\mathbf{x}| - \mathbf{x}' \cdot \frac{\mathbf{x}}{|\mathbf{x}|} + \ldots \\
&\approx |\mathbf{x}| - \mathbf{x}' \cdot \hat{\mathbf{x}}.
\end{aligned}
$$

2.2 Radiation fields in the far zone limit

In class I showed that the retarded potential in the far zone limit is,

$$
A^\mu(t, \mathbf{x}) \approx \frac{1}{4\pi\varepsilon_0 c^2} \frac{1}{|\mathbf{x}|} \int d^3\mathbf{x}' j^\mu(t - \frac{|\mathbf{x}|}{c} + \frac{\hat{\mathbf{x}} \cdot \mathbf{x}'}{c}, \mathbf{x}')
$$

and then I showed that the $\mathbf{B}$-field is given by,

$$
\mathbf{B} = \frac{\int d^3\mathbf{x}' \, \boldsymbol{\nabla} \times \mathbf{j}(t - \frac{|\mathbf{x}|}{c} + \frac{\hat{\mathbf{x}} \cdot \mathbf{x}'}{c}, \mathbf{x}')}{4\pi\varepsilon_0 |\mathbf{x}|} + \text{non-radiation pieces.}
$$

Then I used the chain rule to turn spatial derivatives into of time derivatives,

$$
\frac{\partial}{\partial x}[\mathbf{j}(t - \frac{|\mathbf{x}|}{c} + \frac{\hat{\mathbf{x}} \cdot \mathbf{x}'}{c}, \mathbf{x}')] = \frac{1}{c} \frac{\partial \mathbf{j}}{\partial t} \frac{\partial \left(|\mathbf{x}| - \hat{x} \cdot \mathbf{x}'\right)}{\partial x},
$$

thus leading to,

$$\begin{aligned}
\mathbf{B} &= \frac{\int d^3\mathbf{x}'\ \boldsymbol{\nabla} \times \mathbf{j}(t - \frac{|\mathbf{x}|}{c} + \frac{\hat{\mathbf{x}}\cdot\mathbf{x}'}{c}, \mathbf{x}')}{4\pi\varepsilon_0 |\mathbf{x}|} + O\left(\frac{1}{|\mathbf{x}|^2}\right) \\
&= \frac{\int d^3\mathbf{x}'\ \frac{\partial \mathbf{j}}{\partial t} \times \boldsymbol{\nabla}\left(|\mathbf{x}| - \hat{x}\cdot\mathbf{x}'\right)}{4\pi\varepsilon_0 c|\mathbf{x}|} + O\left(\frac{1}{|\mathbf{x}|^2}\right) \\
&= \frac{\int d^3\mathbf{x}'\ \frac{\partial \mathbf{j}}{\partial t} \times \hat{\mathbf{x}}}{4\pi\varepsilon_0 c|\mathbf{x}|} + O\left(\frac{1}{|\mathbf{x}|^2}\right) \\
&= -\frac{\hat{\mathbf{x}}}{c} \times \frac{\partial}{\partial t}\left(\int d^3\mathbf{x}'\ \mathbf{j}(t - \frac{|\mathbf{x}|}{c} + \frac{\hat{\mathbf{x}}\cdot\mathbf{x}'}{c}, \mathbf{x}')\right) + O\left(\frac{1}{|\mathbf{x}|^2}\right) \\
&= -\frac{\hat{\mathbf{x}}}{c} \times \frac{\partial \mathbf{A}}{\partial t} + O\left(\frac{1}{|\mathbf{x}|^2}\right)
\end{aligned}$$

Thus,

$$\mathbf{B}_{rad} = -\frac{\hat{\mathbf{x}}}{c} \times \frac{\partial \mathbf{A}}{\partial t}. \tag{7}$$

This was presented without proof in class.

Next, let's construct the electric field $\mathbf{E} = -\nabla\Phi - \frac{\partial \mathbf{A}}{\partial t}$. The first term,

$$\begin{aligned}
\boldsymbol{\nabla}\Phi &= \boldsymbol{\nabla}\frac{1}{4\pi\varepsilon_0}\frac{1}{|\mathbf{x}|}\int d^3\mathbf{x}'\ \rho(t - \frac{|\mathbf{x}|}{c} + \frac{\hat{\mathbf{x}}\cdot\mathbf{x}'}{c}, \mathbf{x}') \\
&= \frac{1}{4\pi\varepsilon_0}\left[\left(\boldsymbol{\nabla}\frac{1}{|\mathbf{x}|}\right)\int d^3\mathbf{x}'\ \rho + \frac{1}{|\mathbf{x}|}\int d^3\mathbf{x}'\ \nabla\rho\right] \\
&= \frac{1}{4\pi\varepsilon_0}\frac{1}{|\mathbf{x}|}\int d^3\mathbf{x}'\ \nabla\rho + O\left(\frac{1}{|\mathbf{x}^2|}\right) \\
&= \frac{1}{4\pi\varepsilon_0}\frac{1}{|\mathbf{x}|}\int d^3\mathbf{x}'\ \frac{1}{c}\frac{\partial\rho}{\partial t}\boldsymbol{\nabla}\left(|\mathbf{x}| - \hat{x}\cdot\mathbf{x}'\right) + O\left(\frac{1}{|\mathbf{x}^2|}\right) \\
&= \frac{1}{4\pi\varepsilon_0 c}\frac{\hat{\mathbf{x}}}{|\mathbf{x}|}\int d^3\mathbf{x}'\ \frac{\partial\rho}{\partial t} + O\left(\frac{1}{|\mathbf{x}^2|}\right) \\
&= \frac{\hat{\mathbf{x}}}{c}\frac{\partial}{\partial t}\left(\frac{1}{4\pi\varepsilon_0}\frac{1}{|\mathbf{x}|}\int d^3\mathbf{x}'\ \rho(t - \frac{|\mathbf{x}|}{c} + \frac{\hat{\mathbf{x}}\cdot\mathbf{x}'}{c}, \mathbf{x}')\right) + O\left(\frac{1}{|\mathbf{x}^2|}\right) \\
&= \frac{\hat{\mathbf{x}}}{c}\frac{\partial\Phi}{\partial t} + O\left(\frac{1}{|\mathbf{x}^2|}\right) \\
&= -c\hat{\mathbf{x}}\boldsymbol{\nabla}\cdot\mathbf{A} + O\left(\frac{1}{|\mathbf{x}^2|}\right).
\end{aligned}$$

In this last line I have used the Lorenz gauge condition, $\frac{1}{c^2}\frac{\partial\Phi}{\partial t} + \boldsymbol{\nabla}\cdot\mathbf{A} = 0$. But since we can convert spatial derivatives into time derivatives, $\boldsymbol{\nabla}\cdot\mathbf{A} = \frac{1}{c}\hat{\mathbf{x}}\cdot\frac{\partial \mathbf{A}}{\partial t}$,and we have,

$$\begin{aligned}
\boldsymbol{\nabla}\Phi &= -c\hat{\mathbf{x}}\boldsymbol{\nabla}\cdot\mathbf{A} + O\left(\frac{1}{|\mathbf{x}^2|}\right) \\
&= -\hat{\mathbf{x}}\left(\hat{\mathbf{x}}\cdot\frac{\partial \mathbf{A}}{\partial t}\right) + O\left(\frac{1}{|\mathbf{x}^2|}\right).
\end{aligned}$$

Thus the full E-field is,

$$\begin{aligned}
\mathbf{E_{rad}} &= \hat{\mathbf{x}}\left(\hat{\mathbf{x}}\cdot\frac{\partial \mathbf{A}}{\partial t}\right) - \frac{\partial \mathbf{A}}{\partial t} \\
&= \hat{\mathbf{x}}\times\left(\hat{\mathbf{x}}\times\frac{\partial \mathbf{A}}{\partial t}\right).
\end{aligned} \tag{8}$$

Hence proved.

3 Half wave antenna

3.1 Retarded potential for half wave antenna

First we recall the expression for the four dimensional potential in the far-zone approximation,

$$A^\mu(t,\mathbf{x}) \approx \frac{1}{4\pi\varepsilon_0 c^2}\frac{1}{|\mathbf{x}|}\int d^3\mathbf{x}' j^\mu(t - \frac{|\mathbf{x}|}{c} + \frac{\hat{\mathbf{x}}\cdot\mathbf{x}'}{c}, \mathbf{x}'). \tag{9}$$

For a *one-dimensional* current distribution along z-direction one has the location of the current element to be $\mathbf{x}' = z\hat{\mathbf{z}}$ and one has to replace,

$$d^3x'\, j^3(t', \mathbf{x}') = dz'\, I(t', z').$$

Plugging this in (9), with $\mathbf{x}' = z\hat{\mathbf{z}}$

$$A^3(t,\mathbf{x}) \approx \frac{1}{4\pi\varepsilon_0 c^2}\frac{1}{|\mathbf{x}|}\int dz'\, I\left(t - \frac{|\mathbf{x}|}{c} + \frac{z'\cos\theta}{c}, z'\right) \tag{10}$$

Here we have used, $\hat{\mathbf{x}}\cdot\mathbf{x}' = \hat{\mathbf{x}}\cdot z'\hat{\mathbf{z}} = z'\cos\theta$. Now recall from the lecture that for the half-wave emitter/antenna, the current distribution in the antenna is that of a standing wave pattern,

$$I(t,z) = I_0\cos\left(\frac{2\pi z}{\lambda}\right)\cos\omega t,\ \lambda = \frac{2\pi c}{\omega}$$

while the range of integration $z \in \left(-\frac{\lambda}{2}, \frac{\lambda}{2}\right)$. Substituting this form of the current in (10) we get the retarded potential for the half-wave antenna in the far-zone approximation,

$$\begin{aligned}
A^3(t,\mathbf{x}) &\approx \frac{1}{4\pi\varepsilon_0 c^2}\frac{I_0}{|\mathbf{x}|}\int_{-\frac{\lambda}{4}}^{\frac{\lambda}{4}} dz'\ \cos\left(\frac{2\pi z'}{\lambda}\right)\cos\omega\left(t - \frac{|\mathbf{x}|}{c} + \frac{z'\cos\theta}{c}\right)\\
&= \frac{1}{4\pi\varepsilon_0 c^2}\frac{I_0}{|\mathbf{x}|}\int_{-\frac{\lambda}{4}}^{\frac{\lambda}{4}} dz'\ \cos\left(\frac{2\pi z'}{\lambda}\right)\cos\omega\left(t - \frac{|\mathbf{x}|}{c} + \frac{z'\cos\theta}{c}\right)\\
&= \frac{1}{8\pi\varepsilon_0 c^2}\frac{I_0}{|\mathbf{x}|}\int_{-\frac{\lambda}{4}}^{\frac{\lambda}{4}} dz'\left[\cos\omega\left(t - \frac{|\mathbf{x}|}{c} + \frac{z'(1+\cos\theta)}{c}\right) + \cos\omega\left(t - \frac{|\mathbf{x}|}{c} - \frac{z'(1-\cos\theta)}{c}\right)\right]\\
&= \frac{1}{8\pi\varepsilon_0 \omega c}\frac{I_0}{|\mathbf{x}|}\left[\frac{\sin\omega\left(t - \frac{|\mathbf{x}|}{c} + \frac{z'(1+\cos\theta)}{c}\right)}{1+\cos\theta} - \frac{\sin\omega\left(t - \frac{|\mathbf{x}|}{c} - \frac{z'(1-\cos\theta)}{c}\right)}{1-\cos\theta}\right]_{-\lambda/4}^{\lambda/4}\\
&= \frac{1}{8\pi\varepsilon_0 \omega c}\frac{I_0}{|\mathbf{x}|}\left[\left\{\frac{\sin\omega\left(t - \frac{|\mathbf{x}|}{c} + \frac{\lambda(1+\cos\theta)}{4c}\right) - \sin\omega\left(t - \frac{|\mathbf{x}|}{c} - \frac{\lambda(1+\cos\theta)}{4c}\right)}{1+\cos\theta}\right\}\right.\\
&\qquad\left. - \frac{\sin\omega\left(t - \frac{|\mathbf{x}|}{c} - \frac{\lambda(1-\cos\theta)}{4c}\right) - \sin\omega\left(t - \frac{|\mathbf{x}|}{c} + \frac{\lambda(1-\cos\theta)}{4c}\right)}{1-\cos\theta}\right]\\
&= \frac{1}{4\pi\varepsilon_0 \omega c}\frac{I_0}{|\mathbf{x}|}\left[\frac{\cos\left(\omega t - \omega\frac{|\mathbf{x}|}{c}\right)\sin\frac{\pi(1+\cos\theta)}{2}}{1+\cos\theta} + \frac{\cos\left(\omega t - \omega\frac{|\mathbf{x}|}{c}\right)\sin\frac{\pi(1-\cos\theta)}{2}}{1-\cos\theta}\right]\\
&= \frac{1}{4\pi\varepsilon_0 \omega c}\frac{I_0\cos\left(\omega t - \omega\frac{|\mathbf{x}|}{c}\right)}{|\mathbf{x}|}\left[\frac{\cos\left(\frac{\pi\cos\theta}{2}\right)}{1+\cos\theta} + \frac{\cos\left(\frac{\pi\cos\theta}{2}\right)}{1-\cos\theta}\right]\\
&= \frac{I_0}{2\pi\varepsilon_0 \omega c}\frac{\cos\left(\omega t - \omega\frac{|\mathbf{x}|}{c}\right)}{|\mathbf{x}|}\frac{\cos\left(\frac{\pi\cos\theta}{2}\right)}{\sin^2\theta}.
\end{aligned}$$

This was the expression presented in the lecture slides in the class.

Now using this expression for the vector potential, we can go ahead and compute the magnetic field $\mathbf{B}$ in the far zone approximation,

$$
\begin{aligned}
\mathbf{B} &= -\frac{\hat{\mathbf{x}}}{c} \times \frac{\partial \mathbf{A}}{\partial t} \\
\Rightarrow |\mathbf{B}| &= \frac{\sin\theta}{c} \left| \frac{\partial A^3}{\partial t} \right| \\
&= \frac{I_0}{2\pi\varepsilon_0 c} \frac{\sin\left(\omega t - \omega \frac{|\mathbf{x}|}{c}\right)}{|\mathbf{x}|} \frac{\cos\left(\frac{\pi\cos\theta}{2}\right)}{\sin\theta}.
\end{aligned} \tag{11}
$$

3.2 Poynting vector and power emitted by the half-wave antenna

Since $\hat{\mathbf{x}}, \mathbf{E}$ and $\mathbf{B}$ form an orthogonal triad, the Poynting vector is,

$$
\mathbf{S} = \frac{1}{\mu_0}(\mathbf{E} \times \mathbf{B}) = \frac{|\mathbf{B}|^2}{\mu_0 c}\hat{\mathbf{x}} = \frac{I_0^2}{4\pi^2\varepsilon_0 c} \frac{\sin^2\left(\omega t - \omega \frac{|\mathbf{x}|}{c}\right)}{|\mathbf{x}|^2} \frac{\cos^2\left(\frac{\pi\cos\theta}{2}\right)}{\sin^2\theta}\hat{\mathbf{x}}.
$$

The time-averaged value of $\langle \sin^2\left(\omega t - \omega\frac{|\mathbf{x}|}{c}\right)\rangle = \frac{1}{2}$. Hence the time-averaged Poynting vector is,

$$
\langle |\mathbf{S}| \rangle = \frac{I_0^2}{8\pi^2\varepsilon_0 c} \frac{1}{|\mathbf{x}|^2} \frac{\cos^2\left(\frac{\pi\cos\theta}{2}\right)}{\sin^2\theta}.
$$

Since Poynting vector is the power emitted per unit normal area, we get the power radiated by multiplying with the area element, $dA = |\mathbf{x}|^2 d\Omega$ where Ω is the solid angle,

$$
\begin{aligned}
dP &= \langle |\mathbf{S}| \rangle |\mathbf{x}|^2 \, d\Omega \\
\Rightarrow \frac{dP}{d\Omega} &= \langle |\mathbf{S}| \rangle |\mathbf{x}|^2 \\
&= \frac{I_0^2}{8\pi^2\varepsilon_0 c} \frac{\cos^2\left(\frac{\pi\cos\theta}{2}\right)}{\sin^2\theta}.
\end{aligned} \tag{12}
$$

This is the power distribution as a function of the polar angle θ as presented in class.

Finally, the full(integrated) power emitted by the half-wave emitter can be computed by integrating over all solid angle. In polar coordinates, $d\Omega = \sin\theta \, d\theta \, d\phi$.,

$$
\begin{aligned}
P &= \int_{\theta=0}^{\pi} \int_{\phi=}^{2\pi} \frac{I_0^2}{8\pi^2\varepsilon_0 c} \frac{\cos^2\left(\frac{\pi\cos\theta}{2}\right)}{\sin^2\theta} \sin\theta d\theta d\phi \\
&= \frac{I_0^2}{4\pi\varepsilon_0 c} \left(\int_0^{\pi} \frac{\cos^2\left(\frac{\pi\cos\theta}{2}\right)}{\sin\theta} d\theta \right) \\
&\approx 1.22 \, \frac{I_0^2}{4\pi\varepsilon_0 c}.
\end{aligned}
$$