EP 1027: Homework Assignment 2

IITH/Spring2019/Shubho Roy

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1. Show that if
$$\boldsymbol{\nabla} \times \mathbf{A} = 0,$$
this implies $\mathbf{A}$ can be expressed as a gradient of some potential,
$$\mathbf{A} = -\boldsymbol{\nabla}\Phi.$$
Hint: Use Stokes theorem to show the line integral of $\mathbf{A}$ is independent of the path connecting two points (i.e.depends only on the end points) and then recall the definition of gradient as a directional derivative,
$$d\mathbf{l} \cdot \boldsymbol{\nabla}\Phi = \Phi(\mathbf{x} + d\mathbf{x}) - \Phi(\mathbf{x}).$$
(10 points)

2. Calculate the ratio of the strengths of the electric and gravitational forces between an electron and proton placed some distance apart. You'll need to look up the mass and charge of the electron and proton and the value of the Newton's universal gravitational constant (in SI units). The ration should be independent of the separation distance or the system of units. (5 points)

3. In class we looked at the expression for the electric field at some location, $\mathbf{x}$ due to a **volume charge distribution,** $\rho(\mathbf{x}')$. What are the corresponding expression for a surface charge distribution, with surface charge density, $\sigma(\mathbf{x}')$ or linear charge density, $\lambda(\mathbf{x}')$ (5 points)

4. Use the integral version of Gauss law, i.e.
$$\oint\!\!\!\oint_S dS\, \hat{\mathbf{n}} \cdot \mathbf{E} = Q_{enclosed}$$
.
to prove *Newton's theorem*: The force on a point charge placed anywhere in the empty space within a spherical shell is zero. (5 points)

5. Prove Earnshaw's theorem, Mean-value theorem and Uniqueness theorem for the electrostatic scalar potential. (5+10+10=25 points)

6. A circular annulus of inner radius R_1 and outer radius R_2 has a uniform surface charge density σ and lying on the xy plane with its center at the origin. What is the electric field on the axis of the annulus (which is same as the z-axis) at a height z? (10 points)

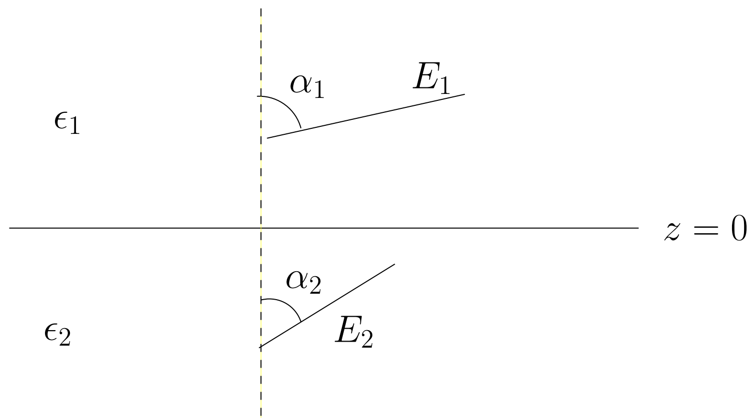

Figure 1: Electric field vectors across a flat interface

7. An amount of charge Q is uniformly distributed over a cylindrical volume of radius R and length L. Find the electric field and the potential at a point on the axis of the cylinder at a distance d from the center. (Hint: You will need to use cylindrical polar coordinate system) (10 points)

8. A spherically symmetric charge distribution has a density $\rho(\mathbf{x}) = k|\mathbf{x}|^n$, where k is a constant and $n > -3$.
   i. Find the electric field as a function of $\mathbf{x}$.
   ii. Find the potential difference between the points $|\mathbf{x}| = a$ and $|\mathbf{x}| = b$.
   (Hint: Use spherical polar coordinate system) (5+5=10 points)

9. Find the expression for the electric field strength at a point, $\mathbf{x}$, due to an ideal/point dipole located at $\mathbf{x}'$. Hint: The potential due to the dipole is given by,

$$\Phi(\mathbf{x}) = \frac{1}{4\pi\epsilon_0} \frac{\mathbf{p} \cdot (\mathbf{x} - \mathbf{x}')}{|\mathbf{x} - \mathbf{x}'|^3}$$

   and the electric field is, $\mathbf{E}(\mathbf{x}) = -\boldsymbol{\nabla}\Phi(\mathbf{x})$). (5 points)

10. Starting with the expression for the dielectric bound charge volume density, $\rho_{bound} = -\boldsymbol{\nabla} \cdot \mathbf{P}$ and the dielectric surface charge density, $\sigma_{bound} = \mathbf{P} \cdot \hat{\mathbf{n}}$, show the sum of the bound volume charge and total surface charge is zero. This shows that the dielectric is as a whole neutral, although the charge separation causes local imbalance of charge of one sign or the other. (5 points)

11. Two very long, conducting cylinders of sheet metal have radii R and $3R$, respectively. The space between them is filled with a gas of dielectric constant ϵ. If the potential difference between the cylinders is V_0, find the E and D fields in the region between the cylinders using the Gauss law for dielectrics. (10 points)

12. At the horizontal interface between two semi-infinite dielectrics (of constants ϵ_1 on top and ϵ_2 under, respectively), the electric field in the first dielectric has magnitude E_1 and makes an angle α_1 with the vertical/normal to the interface (see). What is the angle, α_2, the electric field makes in the second dielectric (as a function of E_1 and α_1), and what is the magnitude of this electric field? (10 points)

13. A dielectric sphere of radius R with a dielectric constant ϵ has a free charge Q uniformly distributed over its volume. The sphere is surrounded by empty space. Find $\mathbf{E}$ and $\mathbf{D}$ inside and outside the sphere. Find the polarization volume charge density and surface charge density. Evaluate the net polarization volume charge and surface charge. (5+5+5+5=20 points)

14. Energy in a dielectric. Show that the energy contained per unit volume in a dielectric is

$$u = \frac{1}{2}\mathbf{D} \cdot \mathbf{E}.$$

(Hint: The energy inside a dielectric comes from two sources. One is to set up a field, $\mathbf{E}$ inside, and this field carries energy (density) $\frac{\epsilon_0}{2}\mathbf{E}^2$, and two the the energy stored by stretching the molecules/dipoles to increase the dipole moment or polarization. Try to find out the infinitesimal/incremental work done, ΔU to increase the electric field inside a dielectric from $\mathbf{E}$ to $\mathbf{E} + \mathbf{\Delta E}$ and work done to increase the polarization (by stretching the molecular dipoles) from $\mathbf{P}$ to $\mathbf{P} + \mathbf{\Delta P}$. Show that the sum of these two incremental work done turns out to be ΔU=$\int d^3\mathbf{x}\, \Delta \left(\frac{1}{2}\mathbf{D} \cdot \mathbf{E}\right)$). (10 points)

15. Prove Green's reciprocity theorem: Suppose we have an arrangement of n conductors so that the potentials on the conductors are $\Phi_1, \Phi_2, \ldots, \Phi_n$when the charges on them are respectively $Q_1, Q_2, ..., Q_n$. And suppose that the potentials change to $\Phi'_1, \Phi'_2, \ldots, \Phi'_n$ when the charges are changed to $Q'_1, Q'_2, ..., Q'_n$;. Then,

$$\sum_{l=1}^{n} Q_l\, \Phi'_l = \sum_{l=1}^{n} Q'_l\, \Phi_l.$$

Corollary: For a capacitor system consisting of n separate conductors, we define coefficients of capacitance as follows: Suppose that the conductors carry charges $Q_1, Q_2, ..., Q_n$ and are at potentials $\Phi_1, \Phi_2, \ldots, \Phi_n$. Then, we define capacitance in a slightly different manner compared to a single conductor case. We can define a set of *coefficients of capacitance*, $\{c_{kl}\}$ by the relation:

$$Q_k = \sum_l c_{kl}\Phi_l$$

where k and l both take values from 1 to n. Use Green's reciprocity theorem to prove the follow symmetry relation

$$c_{kl} = c_{lk}.$$

(10+5=15 points)