PH 1027 (Electromagnetism and Maxwell Equations): Final Exam

(Duration: 1.5 hours)

IITH/Spring17/Shubho Roy

May 1, 2017

1. a) If S^{ij} is a rank 2 symmetric tensor, and A^{ij} is a rank 2 antisymmetric tensor, prove that,

$$S^{ij}A^{ij} = 0.$$

b) Prove that,

$$\delta\left(f(x)\right) = \sum_i \frac{\delta(x - x_i)}{|f'(x_i)|}$$

where the $\{x^i\}$ are the zeros of the function, $f(x)$. (5+10=15 points)

SOLUTION:
a) By symmetry property of S and antisymmetry of A (2 points),

$$\begin{aligned} S^{ij}A^{ij} &= S^{ji}(-A^{ji}) \\ &= -S^{ji}A^{ji} \end{aligned} \tag{1}$$

Since i, j are dummy indices, we can relabel them, specifically, we relabel i as j and we relabel j as i in (1), (2 points)and we get,

$$S^{ij}A^{ij} = -S^{ji}A^{ji} = -S^{ij}A^{ij},$$

or, (1 point)

$$2S^{ij}A^{ij} = 0 \implies S^{ij}A^{ij} = 0.$$

□

b) 3 points First we show that for a constant, a

$$\delta(ax) = \frac{1}{|a|}\delta(x).$$

We know,

$$\int_{-\epsilon}^{\epsilon} \delta(y)\, dy = 1.$$

Now, we switch variables,

$$y = ax,$$

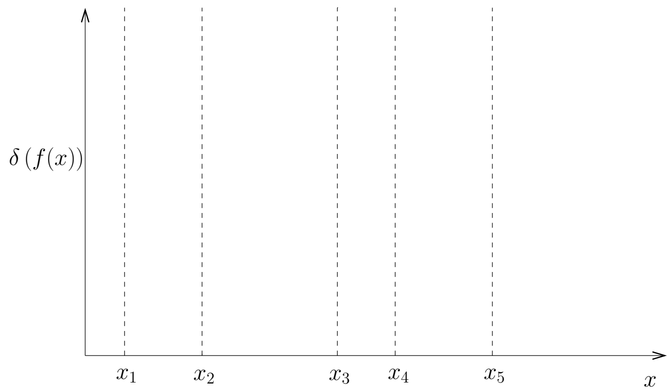

Figure 1: Graph of Dirac Delta of a function

then,

$$\int_{-\epsilon}^{\epsilon} dy = \int_{-\epsilon/a}^{\epsilon/a} a\, dx = \int_{-\epsilon}^{\epsilon} |a| dx,$$

and we have,

$$\begin{aligned}\int_{-\epsilon}^{\epsilon} \delta(y)\, dy = 1 \implies \int_{-\epsilon}^{\epsilon} |a|\, \delta(ax)\, dx &= 1 \\ &= \int_{-\epsilon}^{\epsilon} \delta(x)\, dx,\end{aligned}$$

which means,

$$|a|\, \delta(ax) = \delta(x),$$

or,

$$\delta(ax) = \frac{1}{|a|}\delta(x).$$

Next, let's consider the Dirac delta function, $\delta\left(f(x)\right)$ as a function of x. Here, the argument of the delta function vanishes at the location of the zeroes of the function, $f(x)$ which are denoted by $\{x_i\}, i = 1, 2., ...$ The plot of this would look something like a series of spikes,

and hence we can write, (3 points)

$$\delta\left(f(x)\right) = \sum_i C_i\, \delta(x - x_i), \tag{2}$$

as elsewhere it vanishes. C_i are yet undetermined constants. To determine the constant, C_i's we Taylor expand both sides to the leading order around x_i. We have,

$$f(x)|_{x \sim x_i} = f'(x_i)\,(x - x_i), \quad i \text{ is unsummed.}$$

and thus we have the lhs of (2) (2 points)

$$\begin{aligned}\delta\left(f(x)\right)|_{x\sim x_i} &= \delta\left(f'(x_i)\,(x - x_i)\right) \\ &= \frac{1}{|f'(x_i)|}\delta(x - x_i).\end{aligned}$$

The rhs around $x = x_i$is of course,

$$C_i \delta(x - x_i),$$

so we identify (2 points) ,

$$C_i = \frac{1}{|f'(x_i)|}.$$

Thus,

$$\delta\left(f(x)\right) = \sum_i \frac{1}{|f'(x_i)|} \delta(x - x_i).$$

$\square$

2. In the first homework set you proved Green's theorem. Now use Green's theorem to write down the full exact solution for Poisson (or Laplace) equation for electrostatic potential at a point $\mathbf{x}$ in a finite region/volume V bounded by a closed surface, S is given, by

$$\Phi(\mathbf{x}) = \int d^3\mathbf{x}' \, \frac{\rho(\mathbf{x}')}{|\mathbf{x} - \mathbf{x}'|} + \oiint_S dS \, \left(\frac{1}{|\mathbf{x} - \mathbf{x}'|} \frac{\partial \Phi}{\partial n} - \Phi \frac{\partial}{\partial n} \frac{1}{|\mathbf{x} - \mathbf{x}'|} \right)$$

Since Poisson (or Laplace) equation is a second order differential equation you need to specify both boundary conditions i.e. potential and its normal derivative (These are called *Cauchy* type boundary conditions). Recall that Green's theorem states for two scalars, $\Phi(\mathbf{x})$ and $\psi(\mathbf{x})$, the following result holds,

$$\iiint_V d^3\mathbf{x} \, \left(\Psi \boldsymbol{\nabla}^2 \Phi - \Phi \boldsymbol{\nabla}^2 \Psi \right) = \oiint_S dS \, \hat{\mathbf{n}} \cdot \left(\Psi \boldsymbol{\nabla} \Phi - \Phi \boldsymbol{\nabla} \Psi \right).$$

for a region of volume V contained within the closed surface, S. (10 points)

SOLUTION:
We apply Green's theorem over the volume, V, points in V being denoted by primed coordinates, $\mathbf{x}'$,

$$\iiint_V d^3\mathbf{x}' \, \left(\Psi \boldsymbol{\nabla}'^2 \Phi - \Phi \boldsymbol{\nabla}'^2 \Psi \right) = \oiint_S dS \, \hat{\mathbf{n}} \cdot \left(\Psi \boldsymbol{\nabla}' \Phi - \Phi \boldsymbol{\nabla}' \Psi \right)$$

where we choose, (2 points)

$$\Psi = \frac{1}{|\mathbf{x} - \mathbf{x}'|}$$

and Φ to be the usual electrostatic potential, $\Phi(\mathbf{x}')$. This gives, (2 points)

$$\iiint_V d^3\mathbf{x}' \, \left(\frac{\boldsymbol{\nabla}'^2 \Phi}{|\mathbf{x} - \mathbf{x}'| \epsilon_0} - \Phi \boldsymbol{\nabla}'^2 \frac{1}{|\mathbf{x} - \mathbf{x}'|} \right) = \oiint_S dS \, \left(\frac{1}{|\mathbf{x} - \mathbf{x}'|} \hat{\mathbf{n}} \cdot \boldsymbol{\nabla}' \Phi - \Phi \hat{\mathbf{n}} \cdot \boldsymbol{\nabla}' \frac{1}{|\mathbf{x} - \mathbf{x}'|} \right).$$

Now we use in the LHS we use Poisson's equation (1 point)

$$\boldsymbol{\nabla}'^2 \Phi = -\frac{\rho(\mathbf{x}')}{\epsilon_0},$$

and recall the identity (2 points)

$$\boldsymbol{\nabla}'^2 \frac{1}{|\mathbf{x} - \mathbf{x}'|} = -4\pi \delta^{(3)} \left(\mathbf{x} - \mathbf{x}' \right),$$

while in the RHS, we use the following notation for the normal derivative on the surface S (1 point)

$$\hat{\mathbf{n}} \cdot \boldsymbol{\nabla}' = \frac{\partial}{\partial n},$$

to obtain, (1 point)

$$\iiint_V d^3\mathbf{x}' \left(-\frac{\rho(\mathbf{x}')}{|\mathbf{x}-\mathbf{x}'|\epsilon_0} + \Phi(\mathbf{x}')4\pi\delta^{(3)}\left(\mathbf{x}-\mathbf{x}'\right) \right) = \oiint_S dS \left(\frac{1}{|\mathbf{x}-\mathbf{x}'|}\frac{\partial \Phi}{\partial n} - \Phi \frac{\partial}{\partial n}\frac{1}{|\mathbf{x}-\mathbf{x}'|} \right)$$

which upon performing the delta function integral and rearranging terms gives, (1 point)

$$\Phi(\mathbf{x}) = \frac{1}{4\pi\epsilon_0}\iiint_V d^3\mathbf{x}' \frac{\rho(\mathbf{x}')}{|\mathbf{x}-\mathbf{x}'|} + \frac{1}{4\pi}\oiint_S dS \left(\frac{1}{|\mathbf{x}-\mathbf{x}'|}\frac{\partial \Phi}{\partial n} - \Phi \frac{\partial}{\partial n}\frac{1}{|\mathbf{x}-\mathbf{x}'|} \right).$$

□

3. State the Maxwell Equations for electric and magnetic fields in vacuum (with sources) and in media (with free charges), along with the boundary conditions at the interface of two media.(10 points)

SOLUTION:
Maxwell Equations in vacuum (2 points),

$$\boldsymbol{\nabla} \cdot \mathbf{E} = \frac{\rho}{\epsilon_0}, \qquad \boldsymbol{\nabla} \cdot \mathbf{B} = 0,$$

$$\boldsymbol{\nabla} \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}, \qquad \boldsymbol{\nabla} \times \mathbf{B} = \mu_0 \mathbf{j} + \frac{1}{c^2}\frac{\partial \mathbf{E}}{\partial t},$$

where $\rho, \mathbf{j}$ are the source charge density and current density respectively.
Maxwell Equations in media (2 points).

$$\boldsymbol{\nabla} \cdot \mathbf{D} = \rho_f, \qquad \boldsymbol{\nabla} \cdot \mathbf{B} = 0,$$

$$\boldsymbol{\nabla} \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}, \qquad \boldsymbol{\nabla} \times \mathbf{H} = \mathbf{j}_f + \frac{\partial \mathbf{D}}{\partial t},$$

where $\rho_f, \mathbf{j}_f$ are free charge density and free current density respectively. In addition, we need to specify constitutive relations, $\mathbf{D} = \mathbf{D}(\mathbf{E})$ and $\mathbf{H} = \mathbf{H}(\mathbf{B})$ to bring down the number of independent variables equal to the number of equations (both equal to six). In particular, for **linear isotropic media** the constitutive relations are, (2 points)

$$\begin{aligned} \mathbf{D}(\mathbf{E}) &= \epsilon\, \mathbf{E}, \\ \mathbf{H}(\mathbf{B}) &= \mu\, \mathbf{B}. \end{aligned}$$

The boundary conditions to be satisfied at the interface of two media (denoted by the subscripts 1 and 2) are (2 points),

$$D_1^{\perp} - D_2^{\perp} = \sigma_f, \qquad B_1^{\perp} = B_2^{\perp},$$

$$\mathbf{E}_1^{||} = \mathbf{E}_2^{||}, \qquad \mathbf{H}_1^{||} - \mathbf{H}_2^{||} = \mathbf{K}_f \times \hat{\mathbf{n}},$$

where $\hat{\mathbf{n}}$ is the unit normal at the interface (pointing towards media 1) and $\sigma_f, \mathbf{K}_f$ are sufrace density of free charge and free current respectively at the interface. If both media are **linear and isotropic**, with permittivity and permeability values, (ϵ_1, μ_1) and (ϵ_2, μ_2), then the boundary conditions are (2 points),

$$\epsilon_1 E_1^{\perp} - \epsilon_2 E_2^{\perp} = \sigma_f, \qquad B_1^{\perp} = B_2^{\perp},$$

$$\mathbf{E}_1^{\parallel} = \mathbf{E}_2^{\parallel}, \qquad \frac{1}{\mu_2}\mathbf{B}_1^{\parallel} - \frac{1}{\mu_2}\mathbf{B}_2^{\parallel} = \mathbf{K}_f \times \hat{\mathbf{n}}.$$

4. Show that for a uniform magnetic field, $\mathbf{B}$ the magnetic vector potential everywhere can be given by,

$$\mathbf{A}(\mathbf{x}) = \frac{1}{2}\mathbf{B} \times \mathbf{x}.$$

(10 points)

SOLUTION:
To prove that,

$$\mathbf{A}(\mathbf{x}) = \frac{1}{2}\mathbf{B} \times \mathbf{x}$$

can be a vector potential for a constant/uniform magnetic field, we take curl of the rhs and consider the k-th component of that curl,

$$\begin{aligned}
\left[\boldsymbol{\nabla} \times \left(\frac{1}{2}\mathbf{B} \times \mathbf{x}\right)\right]^k &= \epsilon^{ijk}\nabla^i \left(\frac{1}{2}\mathbf{B} \times \mathbf{x}\right)^j \\
&= \epsilon^{ijk}\frac{\partial}{\partial x^i}\left(\frac{1}{2}\mathbf{B} \times \mathbf{x}\right)^j \\
&= \epsilon^{ijk}\frac{\partial}{\partial x^i}\left(\frac{1}{2}\epsilon^{lmj}B^l x^m\right) \\
&= \frac{1}{2}\epsilon^{lmj}\epsilon^{ijk}B^l\frac{\partial x^m}{\partial x^i} \\
&= \frac{1}{2}\epsilon^{lmj}\epsilon^{kij}B^l\delta^{mi} \\
&= \frac{1}{2}\epsilon^{lij}\epsilon^{kij}B^l \\
&= \frac{1}{2}\left(\delta^{lk}\underbrace{\delta^{ii}}_{=3} - \delta^{li}\delta^{ik}\right)B^l \\
&= \frac{1}{2}\left(3\delta^{lk} - \delta^{lk}\right)B^l \\
&= B^k.
\end{aligned}$$

□

5. In the third homework set you found out the magnetic field produced by a uniformly moving point electric charge by doing a (Lorentz) transformation from the rest frame of the point particle (i.e.

the frame attached to particle and in which the particle is at rest always) to the lab frame (where the particle is moving with uniform velocity). Here you find the electric field due to a uniformly moving point charge in the limit of low velocity ($|\mathbf{v}| \ll c$). The electric field in the lab frame, $\mathbf{E}(\mathbf{x}, t)$ is given in terms of the rest frame, $\mathbf{E}'(\mathbf{x}', t')$ by the relation,

$$E'^i = \left[\left(L^{-1}\right)^0{}_0 \left(L^{-1}\right)^i{}_j - \left(L^{-1}\right)^0{}_j \left(L^{-1}\right)^i{}_0 \right] E'^j(x'^i, t')$$

(Hint: The matrix, L^{-1} is the inverse of the forward boost, $\mathbf{v}$ should be the same as doing a boost in the reverse direction, i.e. boost with velocity, $-\mathbf{v}$, namely, $L^{-1}(\mathbf{v}) = L(-\mathbf{v})$. The forward Lorentz transformation equations are

$$\begin{aligned} \mathbf{x}' &= \mathbf{x} - \gamma \mathbf{v} t + + \left(\frac{\gamma - 1}{\mathbf{v}^2} \right) (\mathbf{v} \cdot \mathbf{x}) \mathbf{v}, \\ t' &= \gamma t - \frac{\gamma}{c^2} \mathbf{v} \cdot \mathbf{x}, \end{aligned} \tag{3}$$

To find L's these should be expressed in the form,

$$x'^0 = L^0{}_0 x^0 + L^0{}_i x^i, \quad x'^i = L^i{}_0 x^0 + L^i{}_j x^j$$

where $x^0 = ct$.) (10 points)

SOLUTION:
Writing, the Lorentz transformation (3) in index/component notation, we get,

$$L^0{}_0 = \gamma,\ L^0{}_i = -\frac{\gamma v^i}{c},\ L^i{}_0 = -\frac{\gamma v^i}{c},\ L^i{}_j = \delta^{ij} + \left(\frac{\gamma - 1}{\mathbf{v}^2} \right) v^i v^j,$$

As was indicated in the homework, the inverse transformations are given by the forward transformation for boost/velocity in reverse direction, i.e. $L^{-1}(\mathbf{v}) = L(-\mathbf{v})$. This immediately gives the inverse transformation elements,

$$\left(L^{-1}\right)^0{}_0 = \gamma,\ \left(L^{-1}\right)^0{}_i = \frac{\gamma v^i}{c},\ \left(L^{-1}\right)^i{}_0 = \frac{\gamma v^i}{c},\ \left(L^{-1}\right)^i{}_j = \delta^{ij} + \left(\frac{\gamma - 1}{\mathbf{v}^2} \right) v^i v^j.$$

In the non-relativistic limit, i.e. $|\mathbf{v}| \ll c$, we will only retain terms which are linear in v/c. In particular, we can set, $\gamma \approx 1$. In this limit, hese inverse transformation elements simplify,

$$\left(L^{-1}\right)^0{}_0 = 1,\ \left(L^{-1}\right)^0{}_i = \frac{v^i}{c},\ \left(L^{-1}\right)^i{}_0 = \frac{v^i}{c},\ \left(L^{-1}\right)^i{}_j = \delta^{ij}.$$

We substitute this in the formula provided for the electric field for a uniformly moving point charge,

$$\begin{aligned} E'^i(x, t) &= \left[\left(L^{-1}\right)^0{}_0 \left(L^{-1}\right)^i{}_j - \left(L^{-1}\right)^0{}_j \left(L^{-1}\right)^i{}_0 \right] E'^j(x'^i, t') \\ &= \left(\delta^{ij} - \underbrace{\frac{v^j v^i}{c^2}}_{=0} \right) E'^j(x'^i, t') \\ &= E'^i(x', t'). \end{aligned}$$

Equivalently, in component free notation,

$$\mathbf{E}(\mathbf{x}, t) = \mathbf{E}'(\mathbf{x}', t').$$

In the rest frame of the point charge, the electric field is of the Coulomb electrostatic form,

$$\mathbf{E}'(\mathbf{x}', t') = \frac{1}{4\pi\epsilon_0}\frac{q}{|\mathbf{x}'|^3}\mathbf{x}'.$$

Thus,

$$\begin{aligned}\mathbf{E}(\mathbf{x}, t) &= \frac{1}{4\pi\epsilon_0}\frac{q}{|\mathbf{x}'|^3}\mathbf{x}' \\ &= \frac{1}{4\pi\epsilon_0}\frac{q}{|\mathbf{x}-\mathbf{v}t|^3}\left(\mathbf{x}-\mathbf{v}t\right),\end{aligned}$$

where in the last line we have used the non-relativistic limit of the Lorentz transformation (Galilean transformation),

$$\mathbf{x}' = \mathbf{x} - \mathbf{v}t.$$

6. i. Check whether or not $\mathbf{E} = 4y\hat{\mathbf{x}} - 2x\hat{\mathbf{y}} + \hat{\mathbf{z}}$ is an allowed electrostatic field.
ii. The electric field in a region is given by, $\mathbf{E} = 2a\,x\hat{\mathbf{x}} + b\,y\hat{\mathbf{y}}$, where a, b are constants. Find the charge density which created this field. (5+5=10 points)

SOLUTION:
i. Electrostatic field must have a vanishing curl (conservative). So let's compute the curl.

$$\begin{aligned}\boldsymbol{\nabla}\times(4y\hat{\mathbf{x}} - 2x\hat{\mathbf{y}} + \hat{\mathbf{z}}) &= \begin{vmatrix}\hat{\mathbf{x}} & \hat{\mathbf{y}} & \hat{\mathbf{z}} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 4y & -2x & 1\end{vmatrix} \\ &= \left(\frac{\partial(1)}{\partial y} - \frac{\partial(-2x)}{\partial z}\right)\hat{\mathbf{x}} + \left(\frac{\partial(4y)}{\partial z} - \frac{\partial(1)}{\partial x}\right)\hat{\mathbf{y}} + \left(\frac{\partial(-2x)}{\partial x} - \frac{\partial(4y)}{\partial y}\right)\hat{\mathbf{z}} \\ &= -6\hat{\mathbf{z}} \neq 0.\end{aligned}$$

Thus, the field, $\mathbf{E} = 4y\hat{\mathbf{x}} - 2x\hat{\mathbf{y}} + \hat{\mathbf{z}}$ cannot be an electric field.
ii. We use Gauss law,

$$\begin{aligned}\rho(\mathbf{x}) &= \epsilon_0\boldsymbol{\nabla}\cdot\mathbf{E} \\ &= \epsilon_0\boldsymbol{\nabla}\cdot(2a\,x\hat{\mathbf{x}} + b\,y\hat{\mathbf{y}}) \\ &= \epsilon_0\left(\frac{\partial(2a\,x)}{\partial x} + \frac{\partial(b\,y)}{\partial y}\right) \\ &= \epsilon_0\,(2a + b)\,.\end{aligned}$$

Thus this field was created by an uniform charge density all over space.

7. Magnetic field in some region has the form, $\mathbf{B} = k\hat{\mathbf{x}}$, where k is some constant. Find the force on a square loop of side "a" lying on the yz-plane with the centroid coinciding with the origin. The loop carries a current of I in the counterclockwise direction as you look down on the x-axis.(5 points)

SOLUTION:
Force on a current element placed in a magnetic field is,

$$d\mathbf{F} = I\, d\mathbf{x} \times \mathbf{B}.$$

The force on the total circuit, (5 points)

$$\mathbf{F} = I \oint d\mathbf{x} \times \mathbf{B}.$$

In the present case, $\mathbf{B} = k\hat{\mathbf{x}}$, i.e. uniform, and the circuit has four straight segments, so the net force is, (5 points)

$$\begin{aligned} \mathbf{F} &= \int_{-a/2}^{a/2} (dz\,\hat{\mathbf{z}} \times k\hat{\mathbf{x}}) + \int_{-a/2}^{a/2} (-dy\hat{\mathbf{y}} \times k\hat{\mathbf{x}}) + \int_{-a/2}^{a/2} (-dz\,\hat{\mathbf{z}} \times k\hat{\mathbf{x}}) + \int_{-a/2}^{a/2} (dy\hat{\mathbf{y}} \times k\hat{\mathbf{x}}) \\ &= 0. \end{aligned}$$

In general, in an uniform magnetic field,

$$\mathbf{F} = I \oint d\mathbf{x} \times \mathbf{B} = I \left(\underbrace{\oint d\mathbf{x}}_{=0} \right) \times \mathbf{B} = 0,$$

i.e. the magnetic force on a circuit vanishes

8. Write down the expression for conservation of energy and linear momentum in Electromagnetic fields and charges/currents in the form of continuity equations identifying each term.What does Poynting vector measure. (10 points)

SOLUTION:
Conservation of energy in the presence of EM fields,

$$\frac{\partial u}{\partial t} + \boldsymbol{\nabla} \cdot \mathbf{S} = 0,$$

where u is the total energy density i.e. the density of kinetic energy of the charged particles (sources) plus the energy density stored in EM fields,

$$u = u_{kinetic} + u_{EM},$$

$$u_{kinetic} = \mathbf{E} \cdot \mathbf{j}, \quad u_{EM} = \frac{\epsilon_0}{2}\mathbf{E}^2 + \frac{1}{2\mu_0}\mathbf{B}^2,$$

and the energy current density

$$\mathbf{S} = \frac{1}{\mu_0}\mathbf{E} \times \mathbf{B}.$$

This quantity $\mathbf{S}$measures the Poynting vector. It measures the rate of flow of energy per unit cross-sectional area per unit time.

Comservaton of momentum in the presence of EM field is expressed in differential form as a continuity equation:

$$\frac{\partial}{\partial t}\left(\mathbf{p}_{charges} + \underbrace{\frac{1}{c^2}\mathbf{S}}_{\boldsymbol{\pi}_{EM}}\right) + \boldsymbol{\nabla}\cdot(-\mathbf{T}) = 0,$$

where, $\mathbf{p}_{charges}$ is the mechanical linear momentum of the charges (sources), the linear momentum density contained in the EM fields,

$$\boldsymbol{\pi}_{EM} = \frac{1}{c^2}\mathbf{S},$$

and,

$$T_{ij} = \epsilon_0\left(E_iE_j - \frac{1}{2}\delta_{ij}\mathbf{E}^2\right) + \frac{1}{\mu_0}\left(B_iB_j - \frac{1}{2}\delta_{ij}\mathbf{B}^2\right)$$

is the Maxwell stress tensor. So we see that the Poynting vector is also propotional to the linear momentum density contained in the EM fields.

9. In one sentence each, explain a) what is Brewster angle? b) what is skin depth of a conductor? Give expressions for each. (5+5=10 points)

SOLUTION:
Brewster angle: The angle of incidence of an "in-plane" (electric field) polarized electromagnetic wave on an interface of two media, for which the wave is fully transmitted into the second media, i.e. without any reflection. In slides for lecture 11, the Brewster angle is given by the condition,

$$\frac{\cos\theta_T}{\cos\theta_B} = \frac{\mu_1 n_2}{\mu_2 n_1}.$$

Using Snell's law,

$$\frac{\sin\theta_B}{\sin\theta_T} = \frac{n_2}{n_1}$$

,we get,

$$\sin\theta_B = \frac{\sqrt{\beta^2 - 1}}{\sqrt{\beta^2 - \left(\frac{n_1}{n_2}\right)^2}},\ \beta = \frac{\mu_1 n_2}{\mu_2 n_1}.$$

Skin depth of a conductor: An EM wave incident on the surface of a conductor is unable to penetrate/propagate into the conductor, it decays rapidly (exponentially) within a short distance from the surface. The characteristic scale of this penetration depth is called "Skin Depth". It is a function of frequency, and is given by the expression,

$$d \sim \frac{1}{\omega}\sqrt{\frac{2}{\mu\epsilon}}\left(\sqrt{1 + \left(\frac{\sigma}{\epsilon\omega}\right)^2} - 1\right)^{-1/2} \approx \sqrt{\frac{2}{\sigma\mu\omega}}.$$

10. i. What is the difference between a rationalized vs a non-rationalized system of units in Electromagnetism
ii. Which properties of vacuum do the constants ϵ_0 and μ_0 measure. (5+5=10 points)

SOLUTION:
i. A rationalized system of units in electromagnetism is one where factors of 4π are present in denominators of Coulomb law for eletric field and Biot-Savart law for magnetic fields while there are no factors of 4π in Maxwell Equation. Example. SI units.
An non-rationalized system of units is one where factors of 4π are present in numerators RHS of Maxwell Equation (Gauss law and Ampere law) while there are no factors of 4π in the expression of Coulomb law and Biot-Savart law. Example: Gaussian/CGS.

ii. The constants, ϵ_0 and μ_0, dubbed as the “permittivity” and the “permeability” of vacuum actually do not characterize any physical property of vacuum, there are just constants of proportionality which appear as a result of introducing an unit for electric charge (or equivalently current) such as Coulomb (or equivalently Ampere) in SI system.