CS6350: Topics in Combinatorics
Assignment 10

Abburi Venkata Sai Mahesh - CS18BTECH11001

November 26, 2020

This document is generated by LaTeX

1. Let A, B be two even-sized subsets of $\{1, ..., n\}$. The set B bisects the set A if $|B \cap A| = \frac{|A|}{2}$. A family $F = \{A_1, A_2, ..., A_m\}$ of even-sized subsets of $\{1, ..., n\}$ is bisection closed if for every $i, j \in \{1, ..., m\}, i \neq j$, either A_i bisects A_j or A_j bisects A_i. Prove that $m \leq n^2$.

A. Let $\Omega = \{0,1\}^n$ and $V_i \in \Omega$ be the incident vector of A_i.
For Example, if $n = 6, A_i = \{1, 2, 4, 5\}$ then $V_i = [1, 1, 0, 1, 1, 0]$.
Now Let us Consider the following function,

$$f_{A_i}(x) = \left(\langle x, V_i \rangle - \frac{\langle x, x \rangle}{2} \right) \left(\langle x, V_i \rangle - \frac{\langle V_i, V_i \rangle}{2} \right)$$

Now if we consider,

$$f_{A_i}(A_j) = \left(\langle V_j, V_i \rangle - \frac{\langle V_j, V_j \rangle}{2} \right) \left(\langle V_j, V_i \rangle - \frac{\langle V_i, V_i \rangle}{2} \right)$$

Here we can see that $\langle V_j, V_i \rangle = |A_j \cap A_i|$.
For Example, if n=6,

$$\begin{aligned} A_i = \{1, 2, 4, 5\} &\Rightarrow V_i = [1, 1, 0, 1, 1, 0] \\ A_j = \{5, 6\} &\Rightarrow V_j = [0, 0, 0, 0, 1, 1] \\ Then, A_i \cap A_j = \{5\} &\Rightarrow |A_i \cap A_j| = 1 \\ And, \langle V_j, V_i \rangle &= [1, 1, 0, 1, 1, 0].\, [0, 0, 0, 0, 1, 1] = 1 = |A_i \cap A_j| \end{aligned}$$

So, our equation will be,

$$f_{A_i}(A_j) = \left(|A_i \cap A_j| - \frac{|A_j|}{2} \right) \left(|A_i \cap A_j| - \frac{|A_i|}{2} \right)$$

After Applying the given conditions, we can observe that f is a bijection closed under

$$f_{A_i}(A_j) = \begin{cases} \neq 0, & \text{if } i = j \\ = 0, & \text{if } i \neq j \end{cases} \tag{1}$$

From the independent criterion, A_i is independent under vector space R^Ω. If we expand $f_{A_i}(A_j) = c_1x_1x_1 + c_2x_1x_2 + ... + c_{n^2}x_nx_n$, has n^2 vectors that span the space.
As, Number of linearly independent vectors $\leq$ Number of vectors spanning the space,

$$\boxed{\therefore m \leq n^2}$$

Hence Proved.