Sunflower lemma

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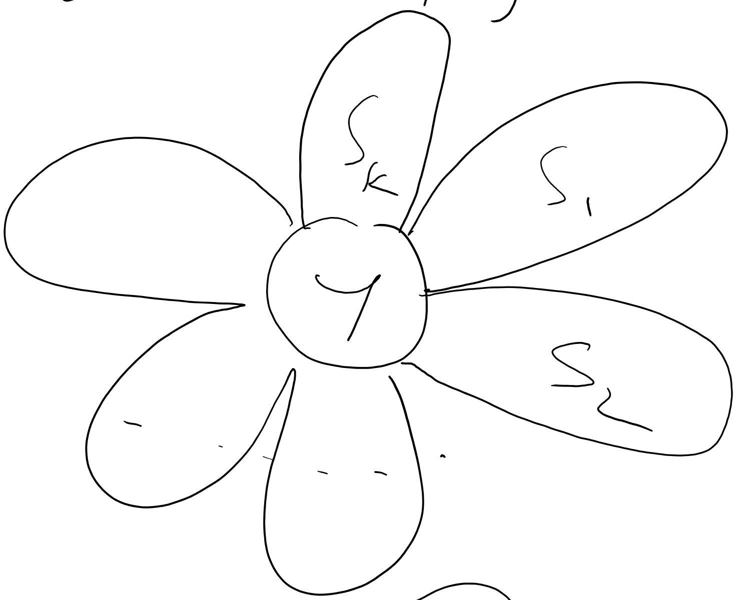
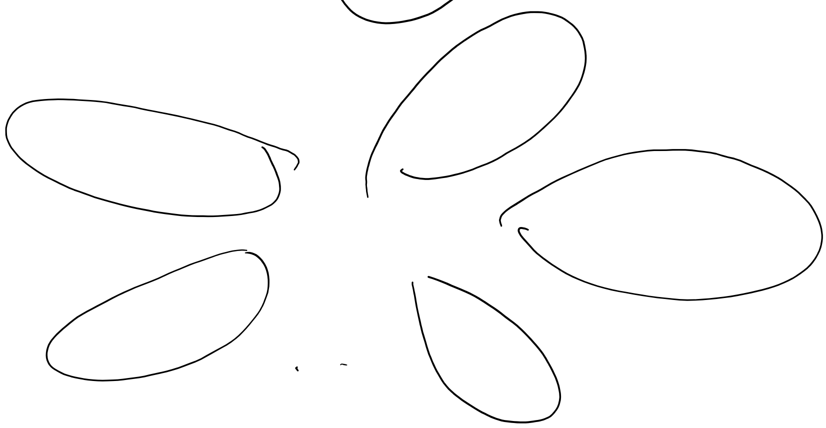
A collection of k sets $S_1, S_2, \dots, S_k$ is a sunflower with k petals and a <u>core Y</u> if

$\forall i, j \in [k], i \neq j, S_i \cap S_j = Y$.

$Y \rightarrow$ can be an empty set





w-uniform family of sets

Lemma [Erdös, Rado, 1960s].
Let $\mathcal{F}$ be a w-uniform family of sets. If $|\mathcal{F}| > w!(k-1)^w$, then $\mathcal{F}$ contains a sunflower with k petals.

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$> \omega!\,(k-1)^{\omega}$

Proof: induction on ω.

Base Case: $\omega = 1$. $|\mathcal{F}| > k-1$

Induction Step: Assume stmt is true for all $\omega \le r-1$, where $r > 1$.
Let $\omega = r$.

$|\mathcal{F}| > r!\,(k-1)^{r}$

If $\mathcal{F}$ contains k pairwise disjoint sets, then DONE.

else,

Let $S_1, S_2, \ldots, S_\ell$ be a maximal subcollection of $\mathcal{F}$ which ~~are~~ is pairwise disjoint.

$\ell < k$

$S_1, \dots, S_\ell$ } maximal pairwise disjoint subcollection of $\mathcal{F}$.

$\ell < k-1$.

Let $T = S_1 \cup S_2 \cup \dots \cup S_\ell$

(1) T is a hitting set for $\mathcal{F}$

Suppose T was not hitting a set S.

(2) $|T| = \ell \cdot r \leq (k-1)r$

T — hitting set for $\mathcal{F}$
— $|T| \leq (k-1)r$

Let $T = \{x_1, x_2, x_3, x_4, \ldots, x_{|T|}\}$

$\mathcal{F} = \{S_1, S_2, S_3, \ldots S_l, S_{l+1}, \ldots \ldots S_{>r!(k-1)^r}\}$

$\exists x \in T$ such that x is present in $> \frac{r!(k-1)^r}{r(k-1)} = (r-1)!(k-1)^{r-1}$ sets of $\mathcal{F}$.

Let $\mathcal{F}_x = \{S \in \mathcal{F} : x \in S\}$

$|\mathcal{F}_x| > (r-1)!(k-1)^{r-1}$

$|\mathcal{F}_x| \geq (r-1)!\,(k-1)^{r-1}$

$\vee\!\!\vee$

$\mathcal{F}_x'$ → after removing x from every set in $\mathcal{F}_x$.

→ $(r-1)$-uniform family

By Ind hypo, $\mathcal{F}_x'$ contains a k-sunflower.

$\square$

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Let $f(k,\omega)$ denote the min no of ω-sized sets required to ensure the presence of a k-sunflower.

$$(k-1)^{\omega} < f(k,\omega) \leq \omega!(k-1)^{\omega}+1$$

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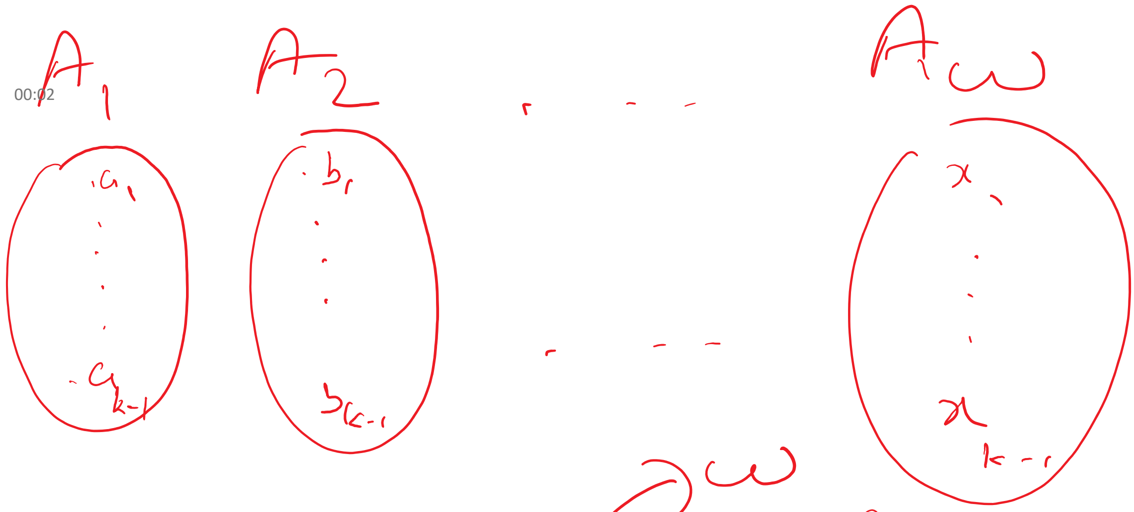

$$\mathcal{F} = \Big\{ \underbrace{\{a_3, b_2, \ldots, x_7\}}_{\omega}, \{a_1, b_1, \ldots, x_1\}, \ldots, \{a_{k-1}, b_{k-1}, \ldots, x_{k-1}\} \Big\}$$

$$|\mathcal{F}| = (k-1)^\omega$$

Sunflower Conjecture

1960s

For a fixed k,
$$f(k,w) \le w!\,(k-1)^w + 1$$

$$f(k,w) < c^w$$ where $c = c(k)$.

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$$f(k,w) \le (\log w)^{w(1+o(1))}$$

$\Rightarrow$ depends on k.

$$w^{w(1+o(1))}$$