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Two distance sets

Euclidean space $\mathbb{R}^n$

Given: d_1, d_2.

Definition: A point set in $\mathbb{R}^n$ is a two-distance set if the Euclidean distance between any points in the set is either d_1 or d_2.

Q. How many points can be present in a 2-distance set in $\mathbb{R}^n$?

Theorem [Larman et al.] Every 2-distance set in $\mathbb{R}^n$ has at most $\binom{n}{2} + 3n + 2$ points.

$$\sum_{i=1}^{n-1} i = \frac{n(n-1)}{2} + n + (n+1) + n+1 = \binom{n+2}{2} + n + 1$$

Proof:

Example $x = (x_1, x_2, \ldots, x_n)$

$$S = \left\{ x \in \{0,1\}^n : \sum_{i=1}^{n} x_i = 2 \right\}$$

$(0, 0, \ldots, 1, 1)$

$(0, 0, \ldots, 1, 0, 1)$

$\vdots$

$\binom{n}{2}$

dist is $\sqrt{2}$

$d_1 = \sqrt{2}$

$d_2 = 2$

$f_{i,j}$

dist

2

$(1,1, 0,0,0, \dots, 0)$

a_1

$d_2 = 2$

$= (a_{11}, a_{12}, \dots, a_{1n})$

Let $a_1, a_2, \dots, a_m$ be a two distance set of points in $\mathbb{R}^n$ s.t. $\|a_i - a_j\| = d_1$ or $\|a_i - a_j\| = d_2$, $\forall i, j \in [m]$ if $i \neq j$.

We define functions $f_1, f_2, \dots, f_m$,

$$f_i : \mathbb{R}^n \to \mathbb{R}$$

$$x = (x_1, x_2, \dots, x_n)$$

$$f_i(x) = \left(\|x - a_i\|^2 - d_1^2\right)\left(\|x - a_i\|^2 - d_2^2\right) \quad (1)$$

<u>Note:</u> $f_i(a_i) = d_1^2 d_2^2 \neq 0$

$j \neq i$, $f_i(a_j) = 0$

Therefore, by independence criterion, $f_1, f_2, \dots, f_m$ are L.I. in the vector space $\mathbb{R}^{\mathbb{R}^n}$ over $\mathbb{R}$.

Stmt A

Expanding Eqn (1)

Expanding Eqn 1

$$f_i(x) = \left(\|x - a_i\|^2 - d_1^2\right)\left(\|x - a_i\|^2 - d_2^2\right)$$

$$= \left(\langle x - a_i, x - a_i\rangle - d_1^2\right)\left(\langle x - a_i, x - a_i\rangle - d_2^2\right)$$

$$= \left(\|x\|^2 + \|a_i\|^2 - 2\langle x, a_i\rangle - d_1^2\right)$$
$$\left(\|x\|^2 + \|a_i\|^2 - 2\langle x, a_i\rangle - d_2^2\right)$$

$$= \underline{\text{expand it}}$$

Claim 1: Every f_i can be obtained as a Linear Combination of the following polynomials:

$$\underbrace{\overbrace{\left(\sum_{i=1}^{n} x_i^2\right)^2}^{\|x\|^4}}_{1}, \quad \underbrace{\overbrace{\left(\sum_{i=1}^{n} x_i^2\right)}^{\|x\|^2} x_j, \; 1 \le j \le n}_{n}, \quad \underbrace{x_i x_j, \; 1 \le i \le j \le n}_{\binom{n}{2} + n}, \quad \underbrace{x_i, \; 1 \le i \le n}_{n}, \quad \underbrace{1}_{1}$$

$$\binom{n}{2} + 3n + 2$$

Combining Stmt 1 and Claim 1

Combining Stmt (1) and Claim (1),
we get

$$m \leq \binom{n}{2} + 3n + 2.$$

$\square$

Swallowing Trick ..

Terminology: For any $I \subseteq [n]$, we shall use x_I to denote $\prod_{i \in I} x_i$.

When $I = \phi$, $x_I = 1$.

For any $I \subseteq [n)$, we use v_I to denote the 0-1 n-bit incidence vector of the set I.

Let $I, J \subseteq [n]$.

Let $x_I(v_J)$ denote evaluating x_I over the incidence vector v_J of J.

Example: Let $I = \{1, 3, 4\} \subseteq \{1, 2, \ldots, n\}$

$$x_I = x_1 x_3 x_4$$

Let $J = \{2, 3, 4\}$, Then

$$\begin{array}{cccccccccc} & 1 & 2 & 3 & 4 & 5 & 6 & \ldots & & n \\ v_J = (& 0, & 1, & 1, & 1, & 0, & 0, & \ldots & & 0) \end{array}$$

$x_I(v_J)$

$$x_I(v_J) = 0 \cdot 1 \cdot 1$$
$$= 0 \,/\!/$$

Note,
$$x_I(v_J) = \begin{cases} 1, & \text{if } I \subseteq J \\ 0, & \text{otherwise} \end{cases}$$

<u>Lemma</u> Let $\Omega = \{0,1\}^n$. Let $f \in \mathbb{R}^\Omega$.
Assume $f(v_I) \neq 0$, for every $|I| \leq r$.
Then, the set $\{x_I f : |I| \leq r\} \subseteq \mathbb{R}^\Omega$
is Linearly Independent in the v.s.
$\mathbb{R}^\Omega$ over $\mathbb{R}$.

<u>Proof:</u> Arrange every I in a linear order,
where $|I| \leq r$, such that $I \prec J$
if $|I| \leq |J|$.

Suppose,
$$\sum_{I : |I| \leq r} \lambda_I x_I f = 0 \longrightarrow \text{Ⓐ}$$
has a
non-trivial solution for λ_I s.

non-trivial solution for λ_I s.

Observe, for any $I, J \subseteq [n]$, with $|I| \leq r$, $|J| \leq r$, we have

$$x_I f(v_J) = x_I(v_J) f(v_J) = \begin{cases} f(v_J) \neq 0, & \text{when } I = J \\ 0, & \text{if } J < I \end{cases}$$

Find the first set $I = I_0$ in the linear order such that $\lambda_I \neq 0$.

Evaluate both sides of (A) at v_{I_0}

LHS

$$= \sum_{I : |I| \leq r} \lambda_I x_I f(v_{I_0}) = \lambda_{I_0}$$

$$\text{RHS} = 0(v_{I_0}) = 0$$

This implies $\lambda_{I_0} = 0$ which is a contradiction to the assumption that $\lambda_{I_0} \neq 0$.

Hence, our assumption that $\{x_I f : |I| \leq r\}$ is Linearly Dependent is false.