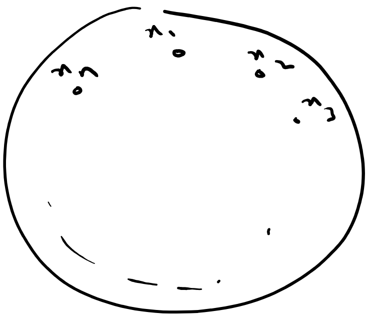
04 November 2020 23:54

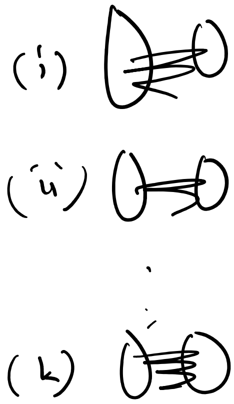
Graham-Pollak Theorem [Jukna's book]

Theorem: The edges of a complete graph K_n on n vertices cannot be decomposed or partitioned into fewer than $n-1$ edge disjoint complete bipartite graphs.

Proof [Tverberg, 1980]:

K_n





What is the smallest +ve integer t such that

$$S(x) := \sum_{1 \le i < j \le n} x_i x_j$$

all the edges of K_n

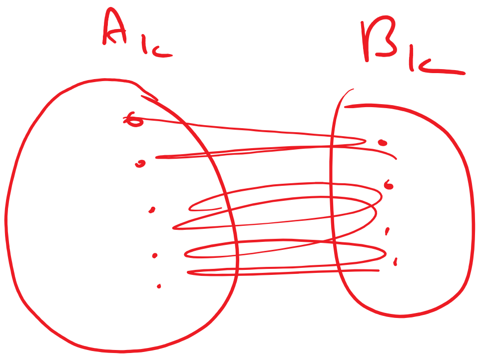
where $x = (x_1, x_2, x_3, \dots, x_n)$ can be written as

$$S(x) = \sum_{k=1}^{t} \left(\sum_{i \in A_k} x_i \right) \cdot \left(\sum_{j \in B_k} x_j \right) \quad (1)$$

, where

$\forall k \in [t]$, $A_k \subseteq [n]$, $B_k \subseteq [n]$, and

$A_k \cap B_k = \emptyset$.



$$\left(\sum_{i=1}^{n} x_i\right)^2 = \sum_{i=1}^{n} x_i^2 + 2 \sum_{1\le i<j\le n} x_i x_j$$

$$= \sum_{i=1}^{n} x_i^2 + 2S(x)$$

That is,

$$\sum_{i=1}^{n} x_i^2 = \left(\sum_{i=1}^{n} x_i\right)^2 - 2S(x)$$

evaluate this eqn on the non-trivial soln.

i.e. $$\sum_{i=1}^{n} x_i^2 = \underbrace{\left(\sum_{i=1}^{n} x_i\right)^2}_{\text{last eqn} \;=0} - 2\underbrace{\sum_{k=1}^{t} \overbrace{\left(\sum_{i\in A_k} x_i\right)}^{=0}\left(\sum_{j\in B_k} x_j\right)}_{\text{first } t \text{ eqns} \;=0} \qquad (2)$$

When evaluated at that point LHS > 0

Assume, for contradiction, that $t \le n-2$. Let $x = (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n$

$$\begin{cases} \sum_{i\in A_1} x_i = 0 \\ \sum_{i\in A_2} x_i = 0 \\ \vdots \end{cases}$$

n-1 equations

$\leq n-1$ equations

variables is n.

$$\left\{ \begin{array}{l} \vdots \\ \displaystyle\sum_{i \in A_t} x_i = 0 \\ \displaystyle\sum_{i=1}^{n} x_i = 0 \end{array} \right.$$

Then, we know that there is a non-trivial solution to this system of linear equations.

Evaluating Eqn (2) at the above non-trivial solution gives a contradiction.

So our assumption that $t \leq n-2$ is false. This completes the proof.

$\square$

Independence criterion / Triangular criterion

[Jukna's Book] also present in Babai–Frankl

Lemma: Let Ω be a set and let $\mathbb{F}$ be a field. For all $i \in [m]$, let $f_i : \Omega \to \mathbb{F}$ be functions and $v_i \in \Omega$ elements such that

(a) $f_i(v_i) \neq 0, \ \forall\, 1 \leq i \leq m$, and

(b) $f_i(v_j) = 0, \ \forall\, 1 \leq j < i \leq m$.

Then, $f_1, f_2, \ldots, f_m$ are linearly independent in the vector space $\mathbb{F}^{\Omega}$ over $\mathbb{F}$.

set of all functions from Ω to $\mathbb{F}$.

Proof:

Suppose $f_1, f_2, \ldots, f_m$ were

linearly dependent in the vector space $\mathbb{F}^n$ over $\mathbb{F}$. Then, $\exists\, \alpha_1, \alpha_2, \ldots, \alpha_m \in \mathbb{F}$, not all of them zeroes, such that

$$\alpha_1 f_1 + \alpha_2 f_2 + \cdots + \alpha_m f_m = 0 \longrightarrow (A)$$

Evaluate both sides on v_1

$$\text{LHS} = (\alpha_1 f_1 + \alpha_2 f_2 + \cdots + \alpha_m f_m)(v_1)$$

$$= \alpha_1 f_1(v_1) + \alpha_2 f_2(v_1) + \cdots + \alpha_m f_m(v_1)$$

$$= \alpha_1 f_1(v_1) + 0 + 0 \cdots + 0$$

$$= \alpha_1 f_1(v_1) \quad \text{— (1)}$$

$$\text{RHS} = 0(v_1)$$

$$= 0 \quad \text{— (2)}$$

So we have, from (1), (2), and (A),

$$\alpha_1 f_1(v_1) = 0$$

Since $f_1(v_1) \neq 0$, this implies that

$$\alpha_1 = 0. \quad \text{— (i)}$$

So eqn (A) becomes

$$\alpha_2 f_2 + \alpha_3 f_3 + \cdots + \alpha_m f_m = 0$$

Evaluate both sides on v_2,

$$\text{LHS} = \alpha_2 f_2(v_2)$$

$$\text{RHS} = 0$$

Since $f_2(v_2) \neq 0$, we get

$$\alpha_2 = 0. \quad \text{——} \quad (ii)$$

Continuing in this fashion, we can show that

$$\alpha_1 = \alpha_2 = \alpha_3 = \cdots = \alpha_m = 0.$$

Thus, $f_1, f_2, \ldots, f_m$ are L.I. in the vector space $\mathbb{F}^n$ over $\mathbb{F}$.

$\square$