CS6350: Topics in Combinatorics - Assignment 3

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1. **A round-robin tournament of 2n teams lasted for (2n - 1) days, as follows: On each day, every team played one game against another team, with one team winning and one team losing in each of the n games. Over the course of the tournament, each team played every other team exactly once. Can one necessarily choose one winning team from each day without choosing any team more than once?**

A. Let us consider a Bipartite Graph $G = (V_1, V_2, E)$ with partities as Days(V_1) and Teams(V_2).

$$\implies V_1 = \{1, 2, 3, ..., 2n - 1\}$$
$$\implies V_2 = \{1, 2, 3, ..., 2n\}$$

Let us define the Edge E from V_1 to V_2 as

$$E = (x, y) \quad where\ x\ \in\ V_1,\ y\ \in\ V_2\ and\ Team\ y\ won\ on\ day\ x$$

Now we have to check whether there exits atleast one unique winning day for every team. That is, we have to check whether there exists a matching of size 2n-1(Perfect Matching) for the above Bipartite Graph or not.

Halls Theorem: A bipartite graph $G = (V_1, V_2, E)$, has a perfect matching (every vertex in V_1 is matched) iff $\forall S \subseteq V_1,\ |N(S)|\ \geq\ |S|$ where N(S) denotes the neighborhood of S.

From Hall Marraige Theorem, it is enough if we can check that

$$\forall S \subseteq V_1,\ |N(S)|\ \geq\ |S|$$

Let us by contradiction consider that $\exists S \subseteq V_1\ such\ that\ |N(S)|\ <\ |S|$ Then there should exists atleast one team $t \in V_2 \setminus N(S)$ which has lost on every day in S. Since no pair of teams play more than once, there must be at least $|S|$ teams that won against t, which implies $|N(S)| \geq |S|$.

So we can conclude that, one can always choose one winning team from each day without choosing any team more than once.