Essentials of Fourier Series and Gamma Function

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Lesson Plan

- Develop the basic theory of **Fourier** series

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- **Bessel's** inequality and **Parseval's** identity

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- Examples/applications of Fourier series assuming convergence
- The **Gamma** and **Beta** functions with an application

Periodic functions

- A real valued function f is said to be **periodic** if there is a $L \in \mathbb{R}$ such that

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- Generic examples are $\sin \omega x$ and $\cos \omega x$ having period $2\pi/\omega$
- Fourier philosophy – "Every reasonably behaved periodic function can be expressed in terms of sin and cos"

A useful integral identity

Lemma

Let f be a real valued $2L$-periodic integrable function defined on $\mathbb{R}$. Then for any $a \in \mathbb{R}$, we have

$$\int_{-L}^{L} f(x)dx = \int_{-L+a}^{L+a} f(x)dx.$$

Proof of the Lemma

- We have
$$\int_{-L}^{L} f(x)dx = \int_{-L}^{-L+a} f(x)dx + \int_{-L+a}^{L} f(x)dx$$

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$$\int_{-L}^{L} f(x)dx = \int_{-L}^{-L+a} f(x)dx + \int_{-L+a}^{L} f(x)dx$$
- Since $f(x+2L) = f(x)$, the first integral is
$$\begin{aligned}\int_{-L}^{-L+a} f(x)dx &= \int_{-L}^{-L+a} f(x+2L)dx \\ &= \int_{-L}^{-L+a} f(x+2L)d(x+2L) \\ &= \int_{L}^{L+a} f(x)dx\end{aligned}$$

Proof continued

- Therefore,

$$\begin{aligned}\int_{-L}^{L} f(x)dx &= \int_{-L}^{-L+a} f(x)dx + \int_{-L+a}^{L} f(x)dx \\ &= \int_{L}^{L+a} f(x)dx + \int_{-L+a}^{L} f(x)dx \\ &= \int_{-L+a}^{L+a} f(x)dx\end{aligned}$$

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- Proof is complete

Even and odd functions

- A function

$$f(x) \text{ is } \begin{cases} \text{even} & \text{if} \quad f(-x) = f(x) \\ \text{odd} & \text{if} \quad f(-x) = -f(x) \end{cases}$$

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- An easy yet useful fact about even/odd functions is that if $L > 0$ and the function f is integrable on $[-L, L]$, then

$$\int_{-L}^{L} f(x)dx = \begin{cases} 2\int_0^L f(x)dx & \text{if} \quad f \text{ is even} \\ 0 & \text{if} \quad f \text{ is odd} \end{cases}$$

Orthogonal functions

- Two real valued functions $f(x)$ and $g(x)$ are called **orthogonal** on $[a,b]$ if

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- A family $\{f_n\}$ of real valued functions is called an **orthonormal** family on $[a,b]$ if

$$\int_a^b f_m(x)f_n(x)dx = \begin{cases} 0 & \text{if} \quad m \neq n \\ 1 & \text{if} \quad m = n \end{cases}$$

Some orthonormal families

Proposition

Given a $L > 0$, *we have*

- *The family* $\mathscr{F}_{\sin} = \left\{ \frac{1}{\sqrt{L}} \sin\left(\frac{\pi n x}{L}\right) \right\}_{n=0}^{\infty}$ *is orthonormal on* $[-L, L]$
- *The family* $\mathscr{F}_{\cos} = \left\{ \frac{1}{\sqrt{e_n L}} \cos\left(\frac{\pi n x}{L}\right) \right\}_{n=0}^{\infty}$ *is orthonormal on* $[-L, L]$, *where* $e_0 = 2$ *and* $e_n = 1$ *for all* $n > 0$

Proof of the proposition

Really a high school level exercise!

- For nonnegative integers m and n, define

$$I_{m,n} = \frac{1}{L}\int_{-L}^{L} \sin\left(\frac{\pi m x}{L}\right) \sin\left(\frac{\pi n x}{L}\right) dx$$

and

$$J_{m,n} = \frac{1}{\sqrt{e_m e_n} L}\int_{-L}^{L} \cos\left(\frac{\pi m x}{L}\right) \cos\left(\frac{\pi n x}{L}\right) dx$$

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- Then we have to show that

$$\text{both,}\quad I_{m,n} \text{ and } J_{m,n} = \begin{cases} 0 & \text{if} \quad m \neq n \\ 1 & \text{if} \quad m = n \end{cases}$$

Proof of the proposition

Make use of

- Let N be any integer, then

$$\int_{-L}^{L} \cos(N\pi x/L)dx = \begin{cases} 2L & \text{if} \quad N = 0 \\ 0 & \text{if} \quad N \neq 0 \end{cases}$$

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- Now use the trigonometric “product to sum” formulas

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- Now use the trigonometric "product to sum" formulas
- Also, note that $\int_{-L}^{L} \sin(N\pi x/L)dx = 0$

A complex orthonormal family

- From $e^{(\pi inx/L)} = \cos(\pi nx/L) + i\sin(\pi nx/L)$, we deduce that

$$\frac{1}{2L}\int_{-L}^{L} e^{(\pi inx/L)}dx = \begin{cases} 1 & \text{if} \quad n = 0 \\ 0 & \text{if} \quad n = \pm 1, \pm 2, \cdots \end{cases}$$

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- Thus, the family $\{e^{(\pi inx/L)}/\sqrt{2L}\}_{n=-\infty}^{\infty}$ is an orthonormal family of functions.
- If the normalising factor $1/\sqrt{2L}$ is removed, then we have an orthogonal family instead

Closing remarks

- For most practical scenarios, we take $L = \pi$ so that, our last result then takes the form

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} e^{inx} dx = \begin{cases} 1 & \text{if} \quad n = 0 \\ 0 & \text{if} \quad n = \pm 1, \pm 2, \cdots \end{cases}$$

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- It really helps to memorise the following version of the last identity

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} e^{imx}e^{-inx}dx = \begin{cases} 1 & \text{if} \quad m = n \\ 0 & \text{if} \quad m \neq n \end{cases}$$

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$$P(x) = Q(\sin x, \cos x)$$

- Thus, $P(x)$ is smooth with period 2π, i.e.,

$$\begin{aligned} P(x+2\pi) &= Q(\sin(x+2\pi), \cos(x+2\pi)) \\ &= Q(\sin x, \cos x) = P(x) \end{aligned}$$

Trigonometric Polynomials

- A particular TP that we will consider here is given by

$$P_N(x) = a_0 + \sum_{n=1}^{N} (a_n \cos nx + b_n \sin nx),$$

where a_n, b_n are *real*, and $x \in \mathbb{R}$

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Complex Version of TP

Each TP $P_N(x)$ can be expressed as

$$P_N(x) = \sum_{-N}^{N} c_n e^{inx}, \quad c_n \in \mathbb{C},\ x \in \mathbb{R}$$

Trigonometric Polynomials

Complex Version of TP

Each TP $P_N(x)$ can be expressed as

$$P_N(x) = \sum_{-N}^{N} c_n e^{inx}, \quad c_n \in \mathbb{C},\ x \in \mathbb{R}$$

- Write

$$\cos nx = \frac{e^{inx} + e^{-inx}}{2}, \quad \sin nx = \frac{-i(e^{inx} - e^{-inx})}{2}$$

Trigonometric Polynomials

- Then $P_N(x)$ can be expressed as

$$P_N(x)$$

$$= a_0 + \sum_{n=1}^{N} \left[\left(\frac{a_n - ib_n}{2} \right) e^{inx} + \left(\frac{a_n + ib_n}{2} \right) e^{-inx} \right]$$

$$= \sum_{-N}^{N} c_n e^{inx}, \qquad x \in \mathbb{R}$$

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- Moreover, we have that $c_n \in \mathbb{C}$ with

$$4|c_n|^2 = a_n^2 + b_n^2 \quad \text{for} \quad n > 0$$

An Integral Identity

A Key Integral Formula

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} e^{inx}e^{-imx}dx = \begin{cases} 1 & \text{if} \quad m = n \\ 0 & \text{if} \quad m \neq n \end{cases}$$

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- Integrate both sides of $P_N(x)e^{-imx}$ on $[-\pi, \pi]$ to get

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} P_N(x)e^{-imx} = \sum_{-N}^{N} c_n \left(\frac{1}{2\pi}\int_{-\pi}^{\pi} e^{i(n-m)x}dx \right)$$

An Integral Identity

- Which yields

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} P_N(x)e^{-imx}dx = \begin{cases} c_m & \text{if} \quad |m| \leq N \\ 0 & \text{if} \quad |m| > N \end{cases}$$

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- Setting $c_m = 0$ for $|m| > N$, we thus have

$$P_N(x) = \sum_{-\infty}^{\infty} c_n e^{inx}$$

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- Where the coefficients are given by

$$c_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} P_N(x)e^{-inx}dx$$

Trigonometric Series

- In the same spirit, define a **trigonometric series**

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- Impose the condition $c_{-n} = \overline{c_n}$
- Writing $a_0 = c_0$, and $c_n = (a_n - ib_n)/2$ for $n > 0$, we get the real trigonometric series

$$a_0 + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx)$$

Fourier Series

- Given a 2π-periodic, integrable function f on $[-\pi, \pi]$, for $n \in \mathbb{Z}$, we set

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- Now, define the **Fourier series** associated with f as

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- Numbers a_n, b_n and c_n are called the Fourier coefficients of $f(x)$

Remarks

- Note that for $n > 0$ the real Fourier coefficients a_n and b_n can be recovered from c_n as

$$
\begin{aligned}
c_n &= \frac{a_n}{2} - i\frac{b_n}{2} \\
&= \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)e^{-inx}dx \\
&= \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)(\cos nx - i\sin nx)dx \\
&= \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\cos nx dx - \frac{i}{2\pi}\int_{-\pi}^{\pi} f(x)\sin nx dx
\end{aligned}
$$

Remarks

- Thus, we find that

$$a_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)dx, \quad a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx dx$$

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- The series $\sum_{-\infty}^{\infty} c_n e^{inx}$ converges?
- Even if does, whether the series converges to $f(x)$?

Partial Sums of a Fourier Series

- Given $f(x) \sim \sum_{-\infty}^{\infty} c_n e^{inx}$, define the N-th partial sum $S_N(f,x) := S_N(x)$ to be the TP

$$S_N(x) = \sum_{-N}^{N} c_n e^{inx} = a_0 + \sum_{n=1}^{N} (a_n \cos nx + b_n \sin nx),$$

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- Where, for $n \leq N$, we have

$$\frac{1}{2\pi} \int_{-\pi}^{\pi} f(x) e^{-inx} dx = c_n = \frac{1}{2\pi} \int_{-\pi}^{\pi} S_N(x) e^{-inx} dx$$

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$$\frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)e^{-inx}dx = c_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} S_N(x)e^{-inx}dx$$

- Thus,

$$\sum_{-\infty}^{\infty} c_n e^{inx} = \lim_{N\to\infty} S_N(x)$$

Partial Sum Identities

Proposition (Integral Identities for $S_N(x)$)

Let $f(x) \sim \sum_{-\infty}^{\infty} c_n e^{inx}$, and let $S_N(x) = \sum_{-N}^{N} c_n e^{inx}$ be the N-th partial sum. Then

$$(i) \quad \frac{1}{2\pi} \int_{-\pi}^{\pi} |S_N(x)|^2 dx = \sum_{-N}^{N} |c_n|^2$$

$$(ii) \quad \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x) S_N(x) dx = \sum_{-N}^{N} |c_n|^2.$$

Proof of the Proposition

(i) follows from the following 2 observations:

- One has

$$|S_N(x)|^2 = S_N(x)\overline{S_N(x)} = \sum_{n,m=-N}^{N} c_n\overline{c_m}e^{i(n-m)x}$$

Proof of the Proposition

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- One has

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- And

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} c_n\overline{c_m}e^{i(n-m)x}dx = \begin{cases} |c_n|^2 & \text{if} \quad m = n \\ 0 & \text{if} \quad m \neq n \end{cases}$$

Proof of the Proposition

For (ii), observe that

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)S_N(x)dx = \sum_{-N}^{N} c_n\left(\frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)e^{inx}dx\right)$$

$$= \sum_{-N}^{N} c_n c_{-n} = \sum_{-N}^{N} c_n\overline{c_n}$$

$$= \sum_{-N}^{N} |c_n|^2$$

Bessel's Inequality & Riemann - Lebesgue Lemma

Proposition

Let f be 2π-periodic and integrable real valued function, and let $f(x) \sim \sum_{-\infty}^{\infty} c_n e^{inx}$, then

$$(i)\ \sum_{-\infty}^{\infty} |c_n|^2 \leq \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)^2 dx \quad (\textit{Bessel's Inequality})$$

$$(ii)\ \lim_{|n|\to\infty} c_n = 0 \quad (\textit{Riemann - Lebesgue Lemma})$$

Other forms of Riemann - Lebesgue Lemma

Versions of Riemann - Lebesgue Lemma

(ii) is equivalent to

$$(1)\ \lim_{|n|\to\infty}\left(\int_{-\pi}^{\pi} f(x)e^{-inx}dx\right)=0$$

$$(2)\ \lim_{n\to\infty}\left(\int_{-\pi}^{\pi} f(x)\cos nx dx\right)=0$$

$$(3)\ \lim_{n\to\infty}\left(\int_{-\pi}^{\pi} f(x)\sin nx dx\right)=0$$

Proof of the Proposition

For (i), observe that

$$0 \le \frac{1}{2\pi}\int_{-\pi}^{\pi} |f(x) - S_N(x)|^2 dx$$

$$= \frac{1}{2\pi}\int_{-\pi}^{\pi} (f(x) - S_N(x))(\overline{f(x) - S_N(x)})dx$$

$$= \frac{1}{2\pi}\int_{-\pi}^{\pi} (|f(x)|^2 - f(x)\overline{S_N(x)} - \overline{f(x)}S_N(x) + |S_N(x)|^2)dx$$

Proof of the Proposition

At this point note that f and S_N are real valued, so that the last expression becomes

$$
\begin{aligned}
0 &\leq \frac{1}{2\pi}\int_{-\pi}^{\pi}(f(x)^2 - 2f(x)S_N(x) + S_N(x)^2)dx \\
&= \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)^2 dx - 2\sum_{-N}^{N}|c_n|^2 + \sum_{-N}^{N}|c_n|^2 \\
&= \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)^2 dx - \sum_{-N}^{N}|c_n|^2
\end{aligned}
$$

Proof of the Proposition

- Deduce that

$$\sum_{-N}^{N} |c_n|^2 \leq \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x)^2 dx \quad \forall \quad N$$

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- Deduce that
$$\sum_{-N}^{N} |c_n|^2 \leq \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)^2 dx \quad \forall \quad N$$
- Taking $N \to \infty$ on both sides, we have (i)

Proof of the Proposition

- Deduce that

$$\sum_{-N}^{N} |c_n|^2 \leq \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)^2 dx \quad \forall \quad N$$

- Taking $N \to \infty$ on both sides, we have (i)
- For (ii), notice that from $c_{-n} = \overline{c_n}$, and

$$\sum_{-\infty}^{\infty} |c_n|^2 \leq \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)^2 dx < \infty$$

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- One has that

$$c_0^2 + 2\sum_{n=1}^{\infty} |c_n|^2 < \infty$$

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- Therefore, we necessarily have

$$\lim_{n\to\infty} |c_n|^2 = 0 \quad \Longrightarrow \quad \lim_{n\to\infty} c_n = 0 \quad (\textbf{Ex.})$$

- Consequently, we have

$$0 = \lim_{n\to\infty} \overline{c_n} = \lim_{n\to\infty} c_{-n} = \lim_{n\to-\infty} c_n$$

Parseval's Identity

Theorem (Parseval's Theorem)

Bessel's Inequality states that if f is 2π-periodic and integrable on $-\pi,\pi]$, then

$$\sum_{-\infty}^{\infty} |c_n|^2 \leq \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)^2 dx.$$

Parseval's result asserts something stronger, that for any f as above, we in fact have

$$\sum_{-\infty}^{\infty} |c_n|^2 = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)^2 dx.$$

Convergence Questions for FS

- We have seen that the FS associated with a 2π-periodic function f is in some sense square convergent -

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$$\sum_{-\infty}^{\infty} |c_n|^2 < \infty$$

- Naturally, we would like to know when does

$$\sum_{-\infty}^{\infty} c_n e^{inx} \quad \text{or} \quad \sum_{-\infty}^{\infty} |c_n|$$

converge?

Convergence Questions for FS

- If $\sum_{-\infty}^{\infty} c_n e^{inx} \searrow$ on $[-\pi, \pi]$, then does it converge to $f(x)$?

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Convergence Questions for FS

- If $\sum_{-\infty}^{\infty} c_n e^{inx} \searrow$ on $[-\pi, \pi]$, then does it converge to $f(x)$?
- If $\sum_{-\infty}^{\infty} c_n e^{inx} = f(x)$ on $[-\pi, \pi]$, does it converge uniformly?
- For a piecewise continuous function f and a $x \in [-\pi, \pi]$, let us define

$$f(x \pm 0) = \lim_{\delta \to 0} f(x \pm \delta)$$

Convergence Questions for FS

- Define $g(x)$ on $[-\pi, \pi]$ as

$$g(x) = \frac{f(x+0) + f(x-0)}{2}$$

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$$g(x) = \frac{f(x+0) + f(x-0)}{2}$$

- Then g is well defined and $g(t) = f(t)$ if f is continuous at $x = t$
- If f has a discontinuity at $x = t$, then g is also discontinuous at $x = t$

Piecewise Smooth Functions

A real valued $f(x)$ is piecewise smooth on $[a,b]$ if

- f is piecewise continuous on $[a,b]$, and if f is continuous on an interval $I \subset [a,b]$, then f is also smooth on I

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- This, in particular, means that $f'(x)$ is piecewise continuous on $[a,b]$
- Piecewise smooth functions are one class of functions on which Fourier theory yields rich results

Convergence Results for FS

Theorem

Let a real valued function f satisfies:

(i) f *is* 2π*-periodic*

(ii) f *is piecewise smooth on* $[-\pi, \pi]$.

Then the Fourier series at associated with f *converges pointwise to* $g(x) = \frac{f(x+0)+f(x-0)}{2}$ *for all* $x \in [-\pi, \pi]$.

Convergence Results for FS

Theorem

Let a real valued function f satisfies:

(i) *f is 2π-periodic*

(ii) *f is smooth on $[-\pi, \pi]$.*

Then the Fourier series associated with f converges uniformly to $f(x)$ for all $x \in [-\pi, \pi]$.

Examples and Applications

Problem 1.

Discuss the Fourier series expansion of

$$f(x) = \begin{cases} -1 & -\pi \leq x < 0 \\ 1 & 0 \leq x < \pi. \end{cases}$$

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- We compute the coefficients c_n, a_n and b_n
- We have

$$a_0 = c_0 = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x)dx = 0$$

Solution Contd. ..

for $n > 0$, c_n is given by

$$\begin{aligned} c_n &= \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)e^{-inx}dx \\ &= \frac{1}{2\pi}\int_{-\pi}^{0} f(x)e^{-inx}dx + \frac{1}{2\pi}\int_{0}^{\pi} f(x)e^{-inx}dx \\ &= \frac{1}{2\pi}\int_{-\pi}^{0} -e^{-inx}dx + \frac{1}{2\pi}\int_{0}^{\pi} e^{-inx}dx \\ &= \frac{-i(1-\cos n\pi)}{n\pi} = \frac{-i(1-(-1)^n)}{n\pi} \end{aligned}$$

Solution Contd. ..

- Thus, on $[-\pi, \pi]$, we have

$$f(x) \sim \sum_{k=-\infty}^{\infty} \frac{-2i}{(2k+1)\pi} e^{((2k+1)ix)}$$

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- Note that '$\sim$' can be replaced by '$=$' as long as $x \notin \{0, \pm\pi\}$

Solution Contd. ..

- From the periodicity property of the integrals of periodic functions, we find for any $a \in \mathbb{R}$ that

$$c_n = \int_{-\pi}^{\pi} f(x)dx = \int_{-\pi+a}^{\pi+a} f(x)dx$$

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$$c_n = \int_{-\pi}^{\pi} f(x)dx = \int_{-\pi+a}^{\pi+a} f(x)dx$$

- Therefore, the series formula for $f(x)$ is valid for any $x \in \mathbb{R}/\{n\pi : n \in \mathbb{Z}\}$

Examples and Applications

Problem 2.

Find the Fourier expansion of $f(x) = x$ in $[0, 2\pi)$, and derive that

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}.$$

- Note that f is CD, hence the FS of f converges to f everywhere on $[0, 2\pi]$

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- Note that f is CD, hence the FS of f converges to f everywhere on $[0, 2\pi]$
- As before, we compute

$$a_0 = c_0 = \frac{1}{2\pi} \int_0^{2\pi} x dx = \pi$$

Solution Contd. ..

for $n > 0$, c_n is given by (integrating by parts)

$$
\begin{aligned}
c_n &= \frac{1}{2\pi}\int_0^{2\pi} xe^{-inx}dx \\
&= \frac{1}{2\pi}\left(x\int e^{-inx}dx\right)\Big|_0^{2\pi} - \frac{1}{2\pi}\int_0^{2\pi}\left(\int e^{-inx}dx\right)dx \\
&= \frac{1}{2\pi}\frac{xie^{-inx}}{n}\Big|_0^{2\pi} + \frac{1}{2ni\pi}\int_0^{2\pi} e^{-inx}dx \\
&= \frac{i}{n} + 0
\end{aligned}
$$

Solution Contd. ..

- Thus,

$$x = \pi + \sum_{\substack{-\infty \\ n \neq 0}}^{\infty} \frac{i}{n} e^{inx}$$

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$$x = \pi + \sum_{n=1}^{\infty} \frac{-2}{n} \sin nx$$

- Parseval's Identity, now yields

$$\frac{1}{2\pi} \int_0^{2\pi} x^2 dx = \pi^2 + \sum_{\substack{-\infty \\ n \neq 0}}^{\infty} \frac{1}{n^2}$$

Solution Contd. ..

- Therefore,

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- Upon rearranging, we have

$$\sum_{n=1}^{\infty}\frac{1}{n^2} = \frac{\pi^2}{6}$$

Fourier Series - A worked out Example

Problem.

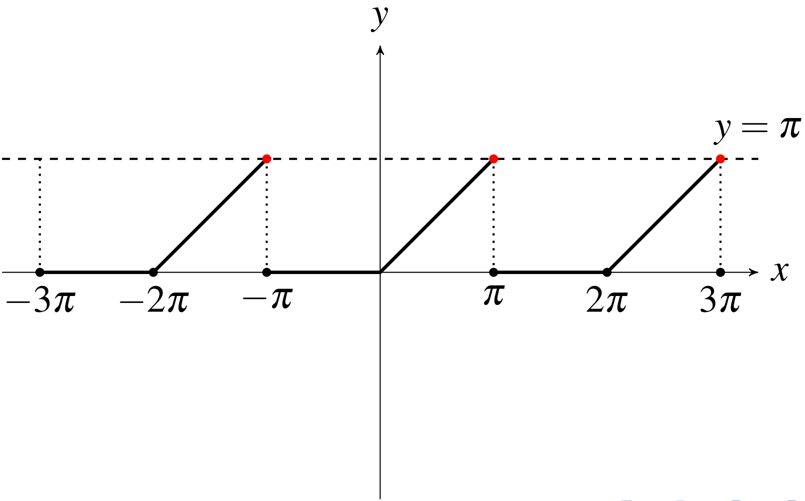
Consider the 2π-*periodic* function f given by $f(x) = 0$ on $[-\pi, 0)$; and $f(x) = x$ on $[0, \pi)$.

(i) Sketch the graph of $f(x)$ in $[-3\pi, 3\pi]$.

(ii) Find the Fourier Series of f.

(iii) Pick appropriate values of x to show that
$$\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots, \text{ and.}$$
$$\frac{\pi^2}{8} = 1 + \frac{1}{3^2} + \frac{1}{5^2} + \frac{1}{7^2} + \cdots.$$

Solution - The Graph of f



Solution

For part (ii), we compute the Fourier coefficients, c_n, a_n and b_n.

$$\begin{aligned} a_0 = c_0 &= \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)dx \\ &= \frac{1}{2\pi}\int_{0}^{\pi} xdx \\ &= \frac{\pi}{4}. \end{aligned}$$

Solution

For $n \neq 0$,

$$c_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)e^{-inx}dx = \frac{1}{2\pi}\int_0^{\pi} xe^{-inx}dx$$

$$= \frac{1}{2\pi}\left[x \cdot \frac{e^{-inx}}{-in}\right]_0^{\pi} - \frac{1}{2\pi}\int_0^{\pi} \frac{e^{-inx}}{-in}dx$$

$$= \frac{i\cos n\pi}{2n} - \frac{1 - \cos n\pi}{2\pi n^2}.$$

Solution

- Thus, for $n > 0$, we have that

$$a_n = \begin{cases} 0, & n - \text{even} \\ -\dfrac{2}{\pi n^2}, & n - \text{odd}, \end{cases} \qquad b_n = \frac{(-1)^{n+1}}{n}$$

Solution

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- Therefore,

$$f(x) \sim \frac{\pi}{4} - \frac{2}{\pi}\sum_{k=0}^{\infty}\frac{\cos(2k+1)x}{(2k+1)^2} + \sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}\sin nx$$

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- Deduce that for $x \neq (2m+1)\pi$, one has

$$f(x) = \frac{\pi}{4} - \frac{2}{\pi}\left(\cos x + \frac{\cos 3x}{3^2} + \frac{\cos 5x}{5^2} + \cdots\right) + \left(\sin x - \frac{\sin 2x}{2} + \frac{\sin 3x}{3} - \frac{\sin 4x}{4} + \cdots\right)$$

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- For part (iii), set $x = \pi/2$ and $x = 0$, respectively

Practice Problems

Problems.

Repeat (i) and (ii) of the last problem, and pick x suitably to find sums of interesting series.

1. $f(x) = x^2$ for $-\pi \leq x \leq \pi$
2. f is given by

$$\begin{cases} \pi - x, & 0 \leq x < \pi \\ 0, & \pi \leq x < 2\pi. \end{cases}$$

The Gamma Function - $\Gamma(x)$

Gamma Function

For $0 < x < \infty$, define the **Gamma** function $\Gamma(x)$ is defined as

$$\Gamma(x) = \int_0^\infty t^{x-1} e^{-t} dt.$$

The integral converges for all x in the indicated range.

Convergence of $\Gamma(x)$

If $0 < x \leq 1$, then

$$\begin{aligned}\Gamma(x) &= \int_0^\infty t^{x-1}e^{-t}dt \\ &= \int_0^1 t^{x-1}e^{-t}dt + \int_1^\infty t^{x-1}e^{-t}dt \\ &\leq \int_0^1 t^{x-1}dt + \int_1^\infty e^{-t}dt \\ &= \frac{1}{x} + \frac{1}{e} < \infty.\end{aligned}$$

Convergence of $\Gamma(x)$

If $x > 1$, then t^{x-1} is *dominated* by $e^{t/2}$ in $[0, \infty)$, meaning that there is a constant $B := B(x) < \infty$, depending on x such that

$$t^{x-1} < e^{t/2} \quad \forall \quad x \geq B$$

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$$t^{x-1} < e^{t/2} \quad \forall \quad x \geq B$$

Thus,

$$\begin{aligned}\Gamma(x) &= \int_0^\infty t^{x-1}e^{-t}dt \\ &= \int_0^B t^{x-1}e^{-t}dt + \int_B^\infty t^{x-1}e^{-t}dt \\ &\leq \text{a finite no.} + \int_B^\infty e^{-t/2}dt < \infty\end{aligned}$$

Properties of $\Gamma(x)$

Theorem

We have

(a) $\Gamma(x+1) = x\Gamma(x) \quad \forall \quad 0 < x < \infty.$

(b) $\Gamma(n+1) = n!$ *for all* $n = 1,\ 2,\ 3,\ \cdots.$

(c) $\Gamma(x)$ *is differentiable on* $(0, \infty)$ *(proving this is beyond the scope of this course).*

Proof of the Theorem

For (a), integrate by parts, taking t^x as the first function, we have

$$\begin{aligned}\Gamma(x+1) &= \int_0^\infty t^x e^{-t} dt \\ &= \left[-\frac{t^x}{e^t}\right]_0^\infty + x\int_0^\infty t^{x-1}e^{-t}dt \\ &= x\Gamma(x).\end{aligned}$$

For part (b), use induction on n and then part (a).

The Beta function - $\beta(x,y)$

Beta Function

For $x > 0$ and $y > 0$, the *Beta* function $\beta(x,y)$ is defined as

$$\beta(x,y) = \int_0^1 t^{x-1}(1-t)^{y-1}dt.$$

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Theorem

For $x > 0$ and $y > 0$, one has

$$\beta(x,y) = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}.$$

Proof of the Theorem

Note that by invoking Fubini's Theorem, one has

$$\Gamma(x)\Gamma(y) = \left(\int_0^\infty t^{x-1}e^{-t}dt\right)\left(\int_0^\infty s^{y-1}e^{-s}ds\right)$$

$$= \int_0^\infty \int_0^\infty t^{x-1}s^{y-1}e^{-(s+t)}dtds$$

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$$= \int_0^\infty \int_0^\infty t^{x-1}s^{y-1}e^{-(s+t)}dtds$$

Set $t = uv$, and $s = u(1-v)$, so that the Jacobian $\partial(s,t)/\partial(u,v)$ is given by

$$\frac{\partial(s,t)}{\partial(u,v)} = \begin{vmatrix} \partial s/\partial u & \partial s/\partial v \\ \partial t/\partial u & \partial t/\partial u \end{vmatrix} = \begin{vmatrix} 1-v & -u \\ v & u \end{vmatrix} = u$$

Proof of the Theorem

Thus, by invoking Fubini again, we have

$$\begin{aligned}\Gamma(x)\Gamma(y) &= \int_0^1\int_0^\infty (uv)^{x-1}(u(1-v))^{y-1}e^{-u}u\,du\,dv \\ &= \int_0^1\int_0^\infty v^{x-1}(1-v)^{y-1}u^{x+y-1}e^{-u}du\,dv \\ &= \left(\int_0^\infty u^{x+y-1}e^{-u}du\right)\left(\int_0^1 v^{x-1}(1-v)^{y-1}dv\right) \\ &= \Gamma(x+y)\beta(x,y).\end{aligned}$$

Consequences of $\Gamma - \beta$ formula

- Observe that

$$\beta(x,y) = 2\int_0^{\pi/2} (\sin\theta)^{2x-1}(\cos\theta)^{2y-1}\,d\theta$$

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- The last formula gives for $x = y = 1/2$

$$\frac{\Gamma(1/2)\Gamma(1/2)}{\Gamma(1)} = \beta(1/2,1/2) = 2\int_0^{\pi/2} d\theta = \pi$$

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- Since $\Gamma(x)$ is nonnegative, deduce that

$$\Gamma(1/2) = \sqrt{\pi}$$

Consequences of $\Gamma - \beta$ formula

- Now, substitute $t = s^2$ in the formula for $\Gamma(x)$ to get

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- Thus

$$\int_{-\infty}^\infty e^{-s^2} ds = 2\int_0^\infty e^{-s^2} ds = \sqrt{\pi}$$

Good Luck!