Lecture 3

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Last Class

We saw Binary Search Trees and the following:

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1. SEARCH
2. INSERT
3. SUCC (also PRED)

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We saw Binary Search Trees and the following:

1. Search
2. Insert
3. Succ (also Pred)

4. Today we see, Delete

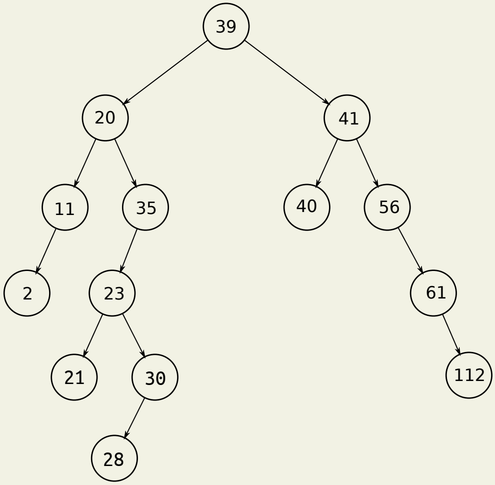
Successor

The SUCC(VAL) procedure is as follows:

- Find the node which stores *val*. Refer to this node as "*node*".
- Two cases:

Case 1: *node* has a right child.
Case 2: *node* does not have a right child.

Example



Delete

The Delete(val) procedure is as follows:
Find the node that has value *val*.

Delete

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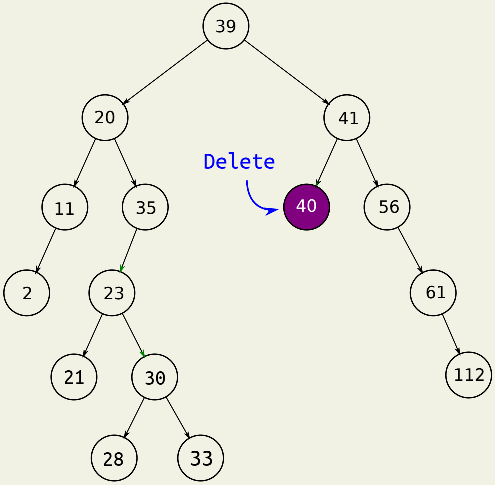
Find the node that has value *val*.
Three cases:

1. *node* has 0 children. (trivial)
2. *node* has 1 child. (splice)
3. *node* has 2 children:
    - Find successor node X with value x.
    - Splice X out of the tree.
    - Replace *val* with x.
    - Delete node X.

Example



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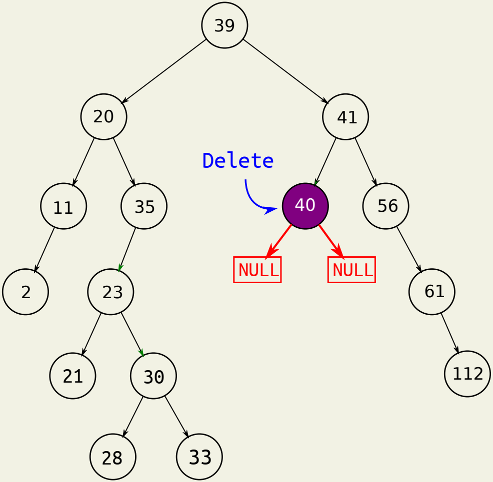

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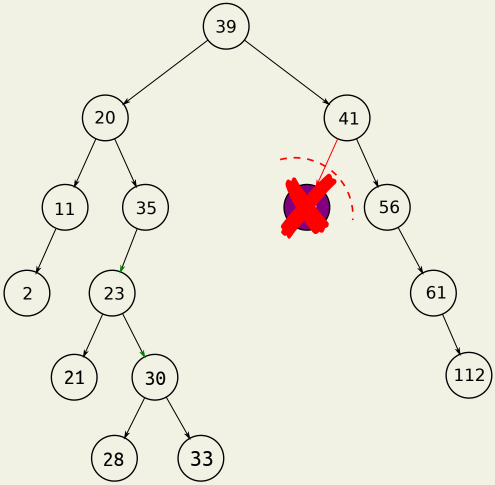

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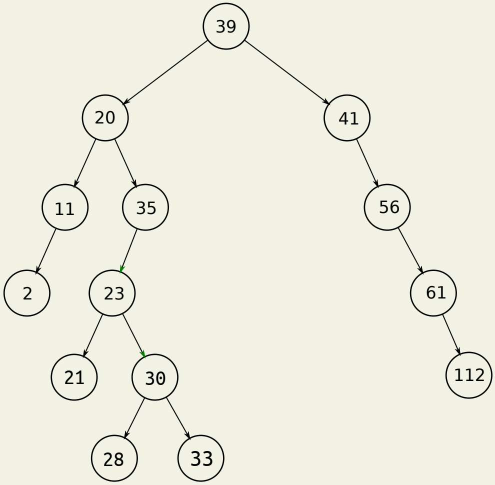

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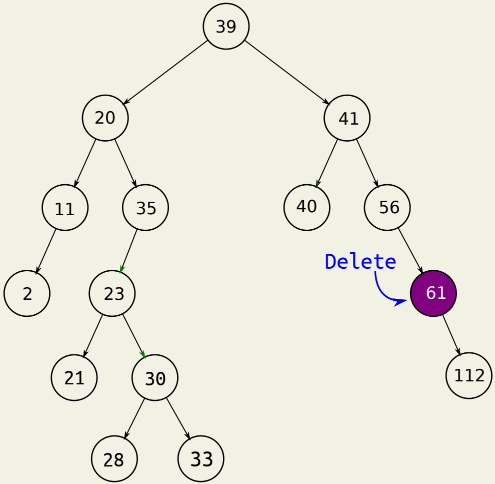

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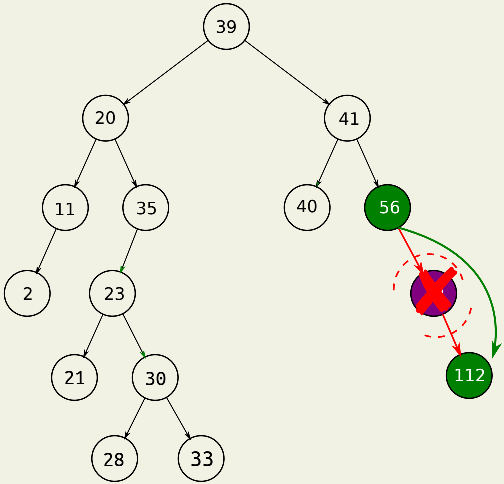

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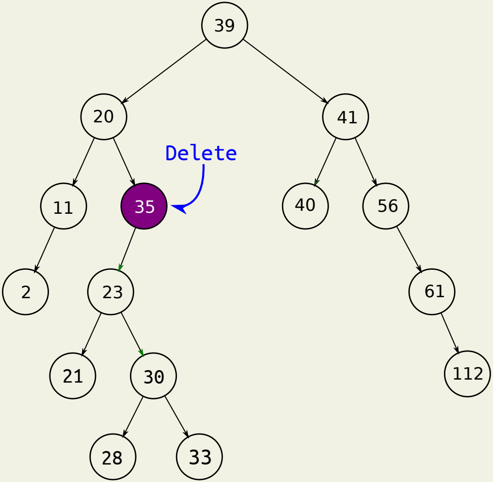

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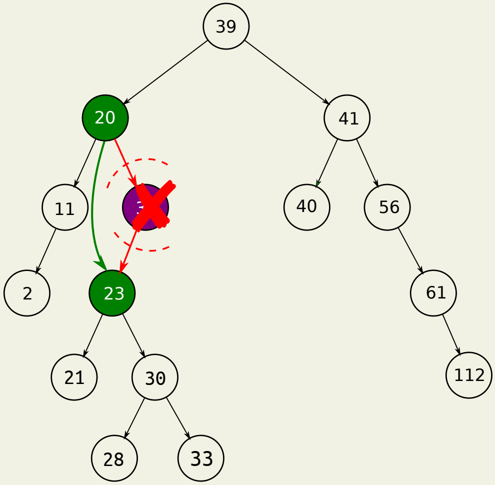

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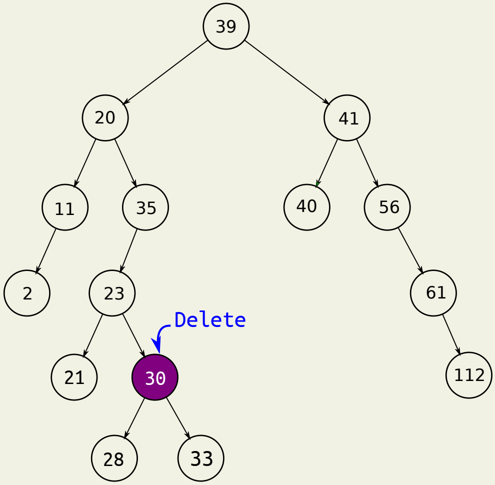

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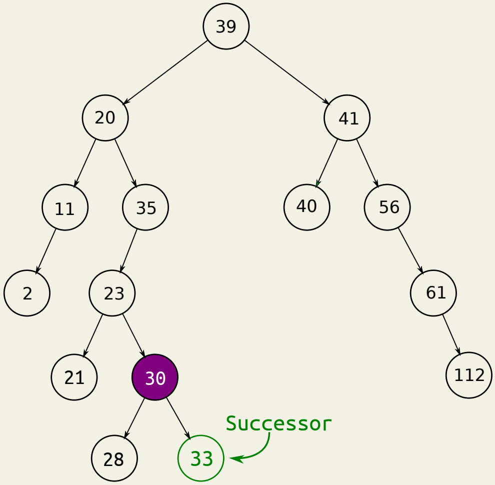

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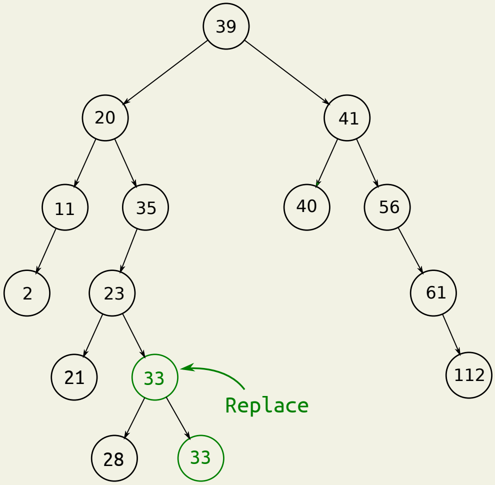

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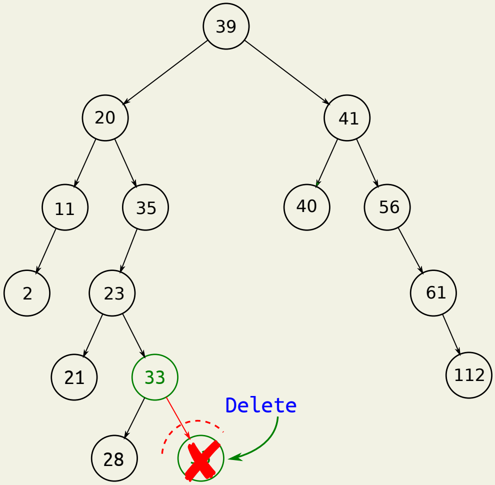

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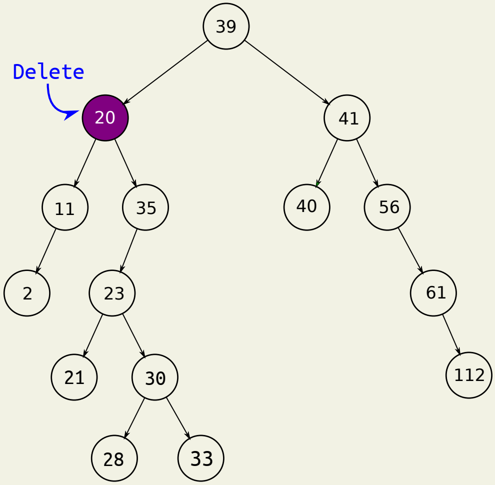

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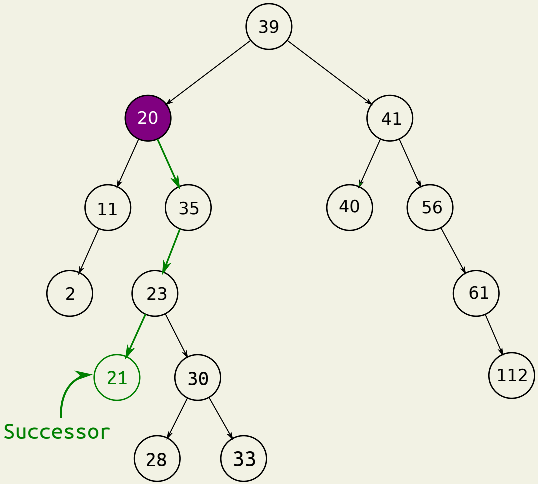

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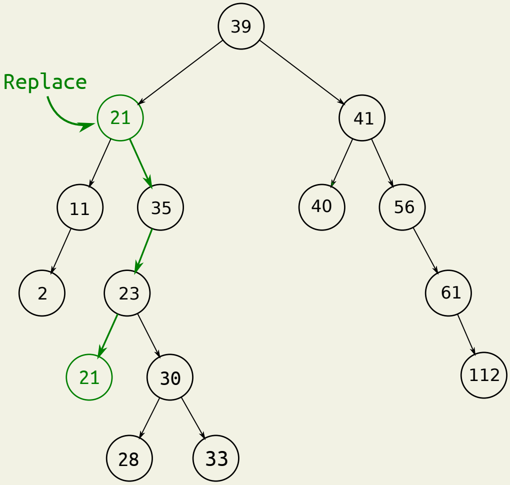

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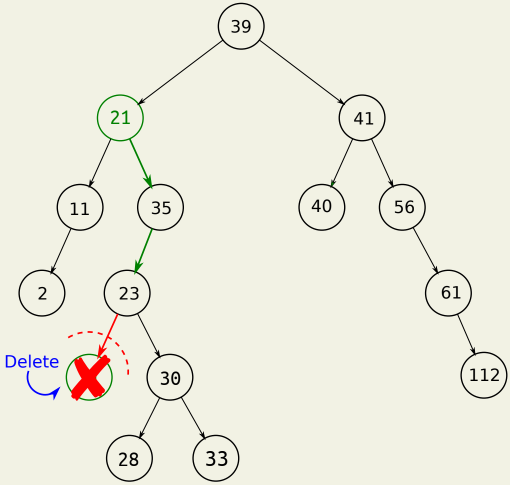

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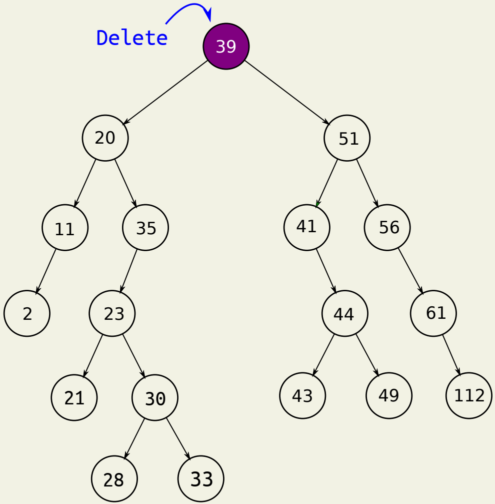

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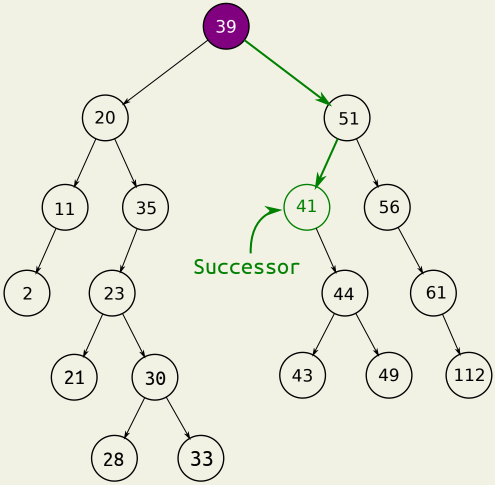

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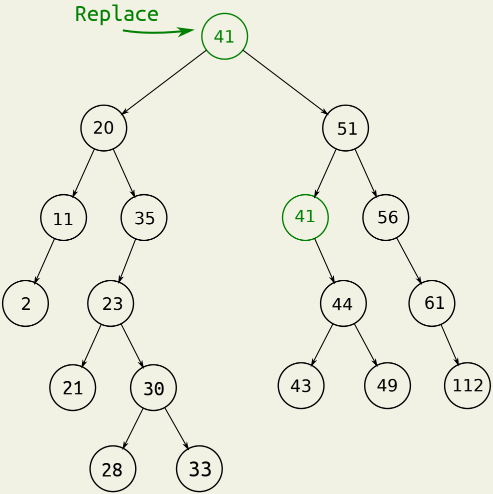

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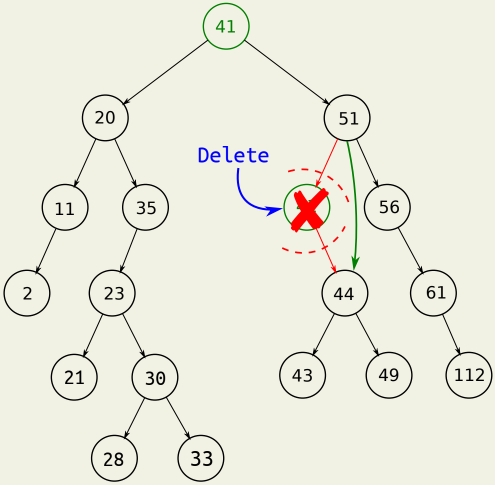

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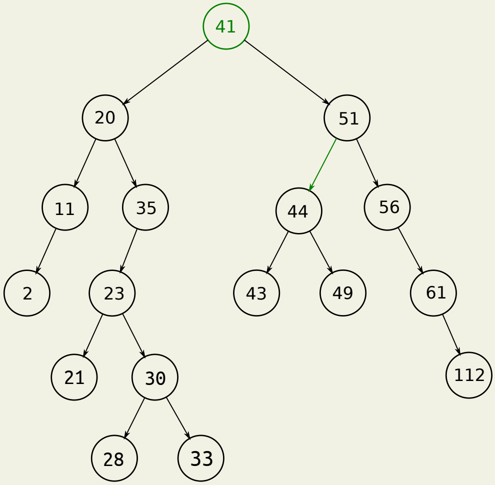

Running Time

Worst case running times for a BST of height h:

- Insert – $O(h)$.
- Succ – $O(h)$.
- Search – $O(h)$.
- Delete – $O(h)$.

The height of a BST depends on the input sequence and can be n after inserting n elements in the worst case.

Balancing a BST

The biggest drawback of BSTs are that they can be quite “unbalanced”.

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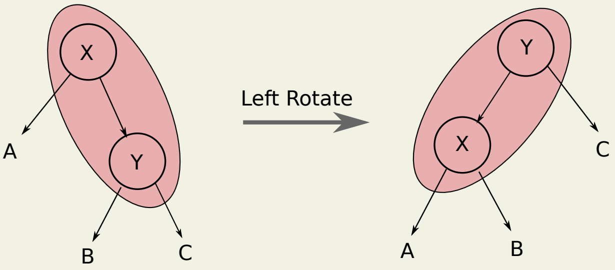
We want to make sure this difference does not get too large.

Balancing a BST

Balancing a BST is done by making structural changes to the underlying tree.
Rotations are operations on nodes of a BST. They are of two variants:

1. Left Rotate.
2. Right Rotate.

Left Rotate



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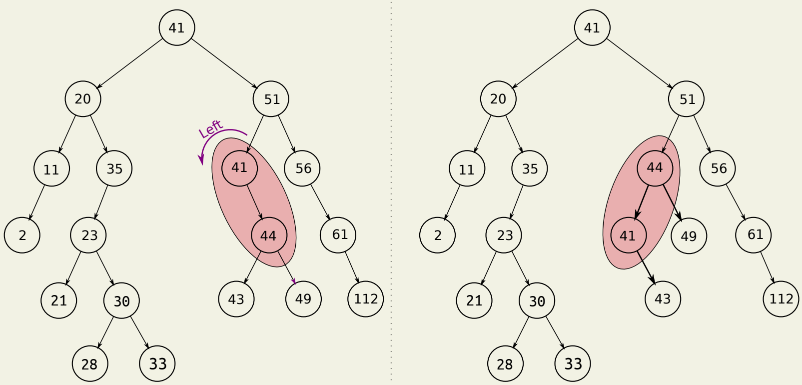

41
20
51
Left
11
35
41
56
2
23
44
61
21
30
43
49
112
28
33
41
20
51
11
35
44
56
2
23
41
49
61
21
30
43
112
28
33

Set

maintains a set of elements from the universe

A set has the following functions:

- INSERT(x) – Insert x into the set.
- SEARCH(x) – Return True if x is an element of the set.
- SUCC(x) – Returns the smallest value larger than x.
- PRED(x) – Returns the largest value smaller than x.
- GETMAX() – Returns the largest value in the set.
- GETMIN() – Returns the smallest value in the set.
- ISEMPTY() – Returns True if and only if the set is empty.
- DELETE(x) – Remove x from the set.

Data Structures for Set

Many choices of data structure to implement set:

- Array
- Sorted Array
- Heap
- Binary Search Tree
- Balanced Binary Search Trees

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We now study a balanced Binary Search Tree called Red-Black Trees.

Red-Black Trees

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Red-Black Trees (RBT) are Binary Search Trees that balance themselves! RBTs have the following properties:

1. All nodes are colored either Red or Black.
2. The root node is black.
3. The leaf nodes (NIL) are black.
4. Both children of a red node are black.
5. For any node, all paths from the node to the descendant leaves have the same number of black nodes.

Example

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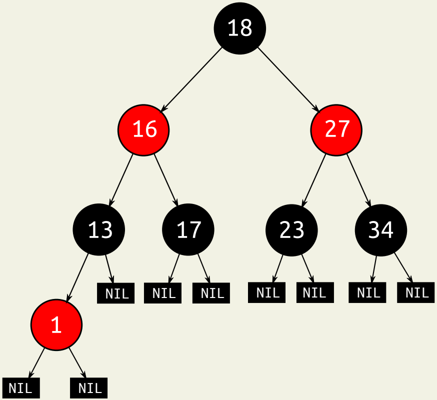

A Red-Black Tree supports all procedures of a BST:

- INSERT(*val*) – Inserts *val* into the RBT rooted at *node*.
- SEARCH(*val*) – Returns True of *val* exists in the BST rooted at *node*. False otherwise.
- SUCC(*val*) – Returns the smallest element greater than *val* in the RBT.
- PRED(*val*) – Returns the largest element lesser than *val* in the RBT.
- DELETE(*val*) – Deletes *val* from the RBT.

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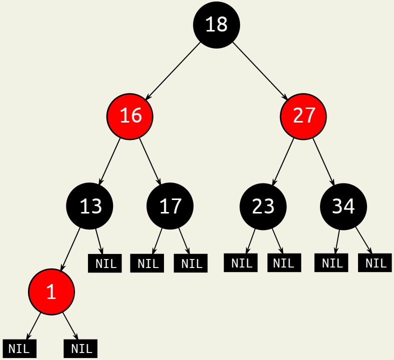
The procedures in green are implemented exactly like in a BST.

Black-Height

The black-height of a node X is the number of black colored nodes encountered on a path starting from X to any leaf (excluding X itself).



The black-height of the node with value 13 is 1.
The black-height of the root node is 2.
The black height of an red-black tree is the black height of its root.

Observations

Claim

A red-black tree with black-height β has height at most 2β.

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Proof sketch:

- Try to construct the longest possible path with at most β many black nodes.
- Property 4 will force you to color every alternate node black.

Observations

Theorem

A red-black tree with n *internal* nodes has height at most $2\log(n+1)$.

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Proof sketch

- Show that for any node X with black-height β, the number of *internal* nodes in the subtree rooted at X is at least $2^\beta - 1$.
- Conclude that with n internal nodes, the black-height must be at most $\log(n+1)$.
- Use previous claim that height is at most twice the black-height to conclude the Theorem.