EP 1027: Maxwell Equations and Electromagnetic waves
Homework Set 3

IITH/Spring19/Shubho Roy

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1. In the last homework set you were asked to compute the field due to an electric dipole. Here you will see/show that a tiny current loop behaves like a magnetic dipole, with a dipole moment given by

$$\mathbf{m} = \frac{I}{2} \oint \mathbf{x}' \times d\mathbf{x}',$$

where I is the current running in the loop and $\mathbf{x}'$ is the position vector of a general point on the loop (see Figure 1).

To arrive at this conclusion you need to start from the Biot-Savant law for the magnetic field produced by the loop at a far point, $\mathbf{x}$,

$$\mathbf{B}(\mathbf{x}) = \frac{\mu_0 I}{4\pi} \oint d\mathbf{x}' \times \frac{\mathbf{x} - \mathbf{x}'}{|\mathbf{x} - \mathbf{x}'|^3},$$

and show that when the loop is small, i.e. $\mathbf{x} - \mathbf{x}' \approx \mathbf{x}$, the magnetic field looks like the electric field of an electric dipole at the origin, namely,

$$\mathbf{B}(\mathbf{x}) = \frac{3\,(\mathbf{m} \cdot \mathbf{x})\,\mathbf{x}}{|\mathbf{x}|^5} - \frac{\mathbf{m}}{|\mathbf{x}|^3}.$$

(**Hint**: Expand $\frac{1}{|\mathbf{x}-\mathbf{x}'|^3}$ in a Taylor series about $\mathbf{x}' = 0$ since it is a very small loop and retain only terms up to first power/linear in $\mathbf{x}'$). (10 points)

2. **d'Alembert's solution to the wave equation in one dimension**: Prove that the general solution to the wave equation in one dimension, namely

$$\left(\frac{\partial^2}{\partial x^2} - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right) f(x,t) = 0 \qquad (1)$$

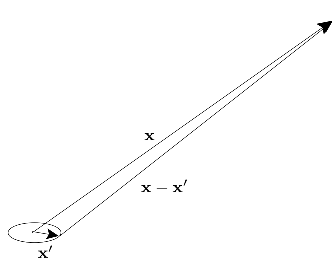

Figure 1: Magnetic field due to a small current loop

is,

$$f(x,t) = g(x+ct) + h(x-ct),$$

where g and h are some arbitrary functions. (**Hint**: Switch to new variables, $\zeta = x+ct,\ \eta = x-ct$.) (5 points)

3. **Bernoulli solution to the wave equation in one dimension:** Prove that the general solution (which is finite/regular everywhere in space and time) to the wave equation in one dimension (1) is given by,

$$\begin{aligned} f(x,t) &= \sum_{k\in\mathbb{R}^+} \left(A_k \sin kx + B_k \cos kx\right)\,\left(C_k \cos\omega t + D_k \sin\omega t\right), \omega = k\, c. \\ &= \sum_{k\in\mathbb{R}^+} A_k \cos\left(kx - \omega t + \Delta_k\right) + C_k \cos(kx + \omega t + \tilde{\Delta}_k) \end{aligned}$$

(**Hint:** Take an ansatz of the form, $f(x,t) = X(x)T(t)$, where $X(x)$ and $T(t)$ are purely functions of x and t respectively and show that after plugging this ansatz in the wave equation (1), you get an equation of the form, $X''/X = \ddot{T}/T$ where dot and prime are respectively the time and space derivatives. Now this equation has a function purely of x on one side and purely a function of t on the right side, and hence they can be equal if and only if they are individually equal to a constant. This method is called *Separation of variables method* to solve partial differential equations by converting them into ordinary differential equations.) (5 points)

4. Show that the Fresnel's equations for off-plane polarized EM wave is,

$$\frac{|\tilde{\mathbf{E}}_{0R}|}{|\tilde{\mathbf{E}}_{0I}|} = \frac{1-\alpha\beta}{1+\alpha\beta}, \quad \frac{|\tilde{\mathbf{E}}_{0T}|}{|\tilde{\mathbf{E}}_{0I}|} = \frac{2}{1+\alpha\beta}$$

where α and β have been defined in the class. Draw the diagram. (Hint: For concreteness consider the same diagram as in the class, i.e. the interface is the y-axis running from up to down and the normal to the interface between the media is the z-axis, the left half i.e. the negative z-axis is in media 1 while the positive z-axis is in the 2nd media. The electric field is perpendicular to the plane of incidence i..e. the yz-plane, i.e. entirely along x-axis, $\mathbf{E}_{I,R,T} = E^1_{I,R,T}\hat{\mathbf{x}}$). (10 points)

5. In class I talked about EM waves which are not of infinite extent but are confined in a cavity or a pipe made out of a conducting walls (a waveguide). I mentioned that the boundary conditions for EM waves in the interior of such waveguides are,

$$\mathbf{E}^{\parallel} = 0, \qquad B^{\perp} = 0$$

at the inner boundary of the waveguide. Show that there are two other boundary conditions which are,

$$E^{\perp} = \frac{\sigma}{\epsilon_0}, \mathbf{B}^{\parallel} = \mu_0 \hat{\mathbf{n}} \times \mathbf{K},$$

where σ and $\mathbf{K}$ are induced free surface charge density at the conducting walls of the wave-guide. (5 points)

6. **Derivation of Biot-Savart law for steady current configurations:** In this problem you will derive the very well known Biot-Savart law for the magnetic field produced by a steady current distribution using (**??**). (Hint: For a steady current distribution, there is always a fixed/time-independent current density at a given location in the charge/current distribution) (5 points)