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The Local Lemma [Erdős, Lovasz, 1975]

$A_1, A_2, \ldots, A_n \longrightarrow$ 'bad' events in an arbitrary prob. space.

$$\Pr[A_1 \vee A_2 \vee \cdots \vee A_n] \leq \underbrace{\Pr[A_1] + \Pr[A_2] + \cdots + \Pr[A_n]}_{<1} \quad \text{(Union bound)}$$

lossy

Then show that

Or, it would imply

$$\Pr[\overline{A_1 \vee A_2 \vee \cdots \vee A_n}] > 0$$

i.e. $\Pr[\bar{A}_1 \wedge \bar{A}_2 \wedge \cdots \wedge \bar{A}_n] > 0.$

Suppose $\bar{A}_1, \bar{A}_2, \ldots, \bar{A}_n$ where mutually independent events.

$$\Pr[\bar{A}_i / \bar{A}_1 \bar{A}_2 \ldots] = \Pr[\bar{A}_i]$$

Then, $\Pr[\bar{A}_1 \wedge \bar{A}_2 \wedge \cdots \wedge \bar{A}_n] = \Pr[\bar{A}_1] \Pr[\bar{A}_2] \cdots \Pr[\bar{A}_n]$

Suppose $\Pr[A_0] < \ldots$ bad event

bad event

Suppose $\Pr[A_i] \leq p_i, \forall i$

$\Pr[\overline{A_n}]$

Local lemma helps when $\overline{A_1}, \dots, \overline{A_n}$ are not mutually indep. However, they form a "weakly dependent system."

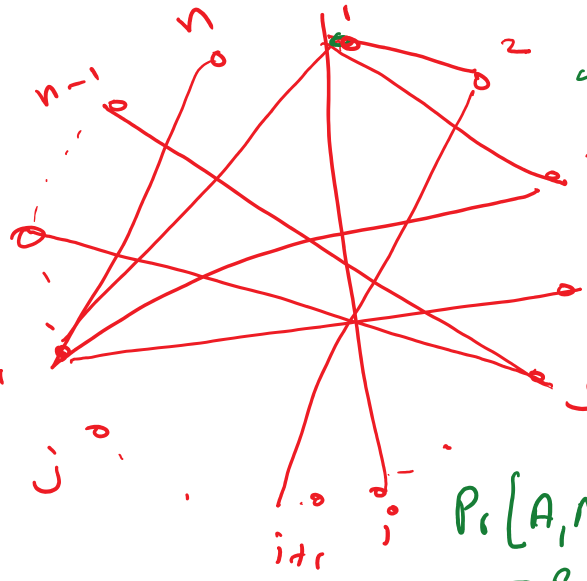
$$\leq \overbrace{(1-p_1)(1-p_2)\dots(1-p_n)}^{n \text{ times}}$$

$$= \prod_{i=1}^{n}(1-p_i)$$

$$> 0$$

$A_1 \quad A_2 \quad - \quad . \quad A_n \longrightarrow$ bad events

Dependency graph



~~A_1~~ is indep of A_4

~~A_1~~ is indep of $A_4 \wedge A_5$

A_1 is indep of $\{A_{n-1}, A_n, A_5, A_n\}$

$\Pr[A_1 \cap A_5 \cap A_n] = \Pr[A_1] \Pr[A_5] \Pr[A_n]$

A graph $D = (V, E)$ with $V = \{1, 2, \ldots, n\}$ is called a dependency graph for the events $A_1, \ldots, A_n$ if for each i, $1 \le i \le n$, the event A_i is mutually independent of all the events $\{A_j : (i,j) \notin E\}$.

degree of the dependency graph = max degree of the dependency graph

Lemma [The local lemma, general case] Let $A_1, A_2, \ldots, A_n$ be n events in an arbitrary probability space. Suppose $D=(V,E)$ is a dependency graph of the above events and suppose there are reals $x_1, x_2, \ldots, x_n$ such that $0 \le x_i \le 1$ and

$$\Pr[A_i] \le x_i \prod_{(i,j)\in E} (1-x_j) \text{ for all}$$

$1 \le i \le n$. Then,

$$\Pr\left[\bigwedge_{i=1}^{n} \overline{A_i}\right] \ge \prod_{i=1}^{n} (1-x_i)$$

Proof: Claim: For any $S \subseteq \{1,2,\ldots,n\}$, $i \notin S$,

$$\Pr\left[A_i \;\middle|\; \bigwedge_{j\in S} \overline{A_j}\right] \le x_i$$

Proof of Claim: By induction on $|S|$.

Base Case: $|S|=0$. Trivial.

Induction Step: Assume the claim is

true when $|S| < s$. Let $|S| = s$.

Then, we partition $S = S_1 \uplus S_2$,

where $S_1 = \{ j \in S : (i,j) \in E \}$,

$S_2 = S \setminus S_1$.

We have,

$$\text{L.H.S.} = \Pr\left[A_i \,\middle|\, \bigwedge_{j \in S} \bar{A}_j\right]$$

$$= \Pr\left[A_i \,\middle|\, \left(\bigwedge_{j \in S_1} \bar{A}_j\right)\left(\bigwedge_{j \in S_2} \bar{A}_j\right)\right]$$

$$= \frac{\Pr\left[\left(A_i \wedge \left(\bigwedge_{j \in S_1} \bar{A}_j\right)\right) \,\middle|\, \bigwedge_{l \in S_2} \bar{A}_l\right]}{\Pr\left[\bigwedge_{j \in S_1} \bar{A}_j \,\middle|\, \bigwedge_{l \in S_2} \bar{A}_l\right]} \longrightarrow \text{(1)}$$

$$\Pr[X \mid (Y \wedge Z)]$$

$$= \frac{\Pr[X \wedge Y \wedge Z]}{\Pr[Y \wedge Z]}$$

$$= \frac{\Pr[X \wedge Y \wedge Z]\, \cancel{\Pr[Z]}}{\Pr[Y \mid Z]\, \cancel{\Pr[Z]}}$$

$$= \frac{\Pr[(X \wedge Y) \mid Z]}{\Pr[Y \mid Z]}$$

$$\overline{\Pr[Y \mid Z]\, \cancel{\Pr[Z]}}$$

$$= \frac{\Pr[(X \wedge Y) \mid Z]}{\Pr[Y \mid Z]}$$

Numerator of 1 $= \Pr\left[\left(A_i \wedge \cancel{\left(\bigwedge_{j \in S_1} \overline{A_j}\right)}\right) \,\middle|\, \bigwedge_{l \in S_2} \overline{A_l}\right]$

$$\leq \Pr\left[A_i \,\middle|\, \bigwedge_{l \in S_2} \overline{A_l}\right]$$

$$= \Pr[A_i]$$

$$\leq x_i \prod_{(i,j) \in E} (1 - x_j) \quad \text{--- } (2)$$

Denominator of 1 $= \Pr\left[\bigwedge_{j \in S_1} \overline{A_j} \,\middle|\, \bigwedge_{l \in S_2} \overline{A_l}\right]$

$$\Pr[\overline{X} \mid (\overline{Y} \wedge \overline{Z})]$$

$$= 1 - \Pr[X \mid (\overline{Y} \wedge \overline{Z})]$$

Let $S_1 = \{j_1, j_2, \ldots, j_r\}$

$$= \left(1 - \Pr\left[A_{j_1} \,\middle|\, \bigwedge_{\ell \in S_2} \bar{A}_\ell\right]\right)\left(1 - \Pr\left[A_{j_2} \,\middle|\, \bar{A}_{j_1} \wedge \left(\bigwedge_{\ell \in S_2} \bar{A}_\ell\right)\right]\right) \cdots \left(1 - \Pr\left[A_{j_r} \,\middle|\, \bar{A}_{j_1} \wedge \bar{A}_{j_2} \wedge \cdots \wedge \bar{A}_{j_{r-1}} \left(\bigwedge_{\ell \in S_2} \bar{A}_\ell\right)\right]\right)$$

$$\Pr[(X \wedge Y) / Z] = \Pr[X/Z] \Pr[Y/(X \wedge Z)]$$

$$= (1 - x_{j_1})(1 - x_{j_2}) \cdots (1 - x_{j_r})$$

[by Induction hypothesis]

$$= \prod_{j \in S_1} (1 - x_j)$$

$$\geq \prod_{(i,j) \in E} (1 - x_j) \quad \text{—— } (3)$$

From (1), (2), and (3), we have

$$\Pr\left[A_i \,\middle|\, \bigwedge_{j \in S} \bar{A}_j\right] \leq \frac{x_i \cancel{\prod_{(i,j) \in E} (1 - x_j)}}{\cancel{\prod_{(i,j) \in E} (1 - x_j)}}$$

$$= x_i$$

This proves the claim.

Now to prove the lemma,

$$\Pr\left[\bigwedge_{i=1}^{n} \overline{A_i}\right] = \left(1 - \Pr(A_1)\right)\left(1 - \Pr\left[A_2 \mid \overline{A_1}\right]\right)\left(1 - \Pr\left[A_3 \mid \left(\overline{A_1} \cap \overline{A_2}\right)\right]\right) \cdots$$

$$\geq (1 - x_1)(1 - x_2)(1 - x_3) \cdots \quad \text{(from Claim)}$$

$$= \prod_{i=1}^{n} (1 - x_i)$$

$\square$

Corollary [The local lemma, symmetric case]

Let $A_1, A_2, \ldots, A_n$ be events in an arbitrary probability space. Suppose that each event A_i is mutually independent of all but at most d other events and $\Pr[A_i] \le p$, $\forall i \in [n]$.

If $ep(d+1) \le 1$, then

$$\Pr\left[\bigwedge_{i=1}^{n} \overline{A_i}\right] > 0.$$

Proof:

Lemma The local lemma, general case. Let $A_1, A_2, \ldots, A_n$ be n events in an arbitrary probability space. Suppose $D = (V, E)$ is a dependency graph of the above events and suppose there are reals $x_1, x_2, \ldots, x_n$ such that $0 \le x_i \le 1$ and

$\Pr[A_i] \le x_i \prod (1 - x_j)$ for all

$$\Pr[A_i] \le x_i \prod_{(i,j)\in E}(1-x_j)$$ for all $1 \le i \le n$. Then,

$$\Pr\left[\bigwedge_{i=1}^{n} \bar{A}_i\right] \ge \prod_{i=1}^{n}(1-x_i)$$

Put $x_i = \frac{1}{d+1}$, $\forall i \in [n]$.

Then, for any $i \in [n]$,

$$x_i \prod_{(i,j)\in E}(1-x_j) = \frac{1}{d+1}\prod_{(i,j)\in E}\left(1-\frac{1}{d+1}\right)$$

$$\ge \frac{1}{d+1}\left(1-\frac{1}{d+1}\right)^d$$

$$\ge \frac{1}{e(d+1)}$$

$$\ge p$$

$$\ge \Pr[A_i]$$

Given that $ep(d+1) \le 1$

To show $\left(1-\frac{1}{d+1}\right)^d \ge \frac{1}{e}$

Enough to show, $\left(\frac{d}{d+1}\right)^d \ge \frac{1}{e}$

or $e \ge \left(\frac{d+1}{d}\right)^d$

or $e \ge \left(1+\frac{1}{d}\right)^d$

We know $(1+x) \le e^x$

We know $(1+x) \le e^x$

So $\left(1+\frac{1}{d}\right)^d \le e^{\frac{d}{d}} = e$

Then, by the lemma,

$$\Pr\left[\bigwedge_{i=1}^{n} \overline{A_i}\right] \ge \prod_{i=1}^{n}(1-x_i)$$

$$= \left(1-\frac{1}{d+1}\right)^n$$

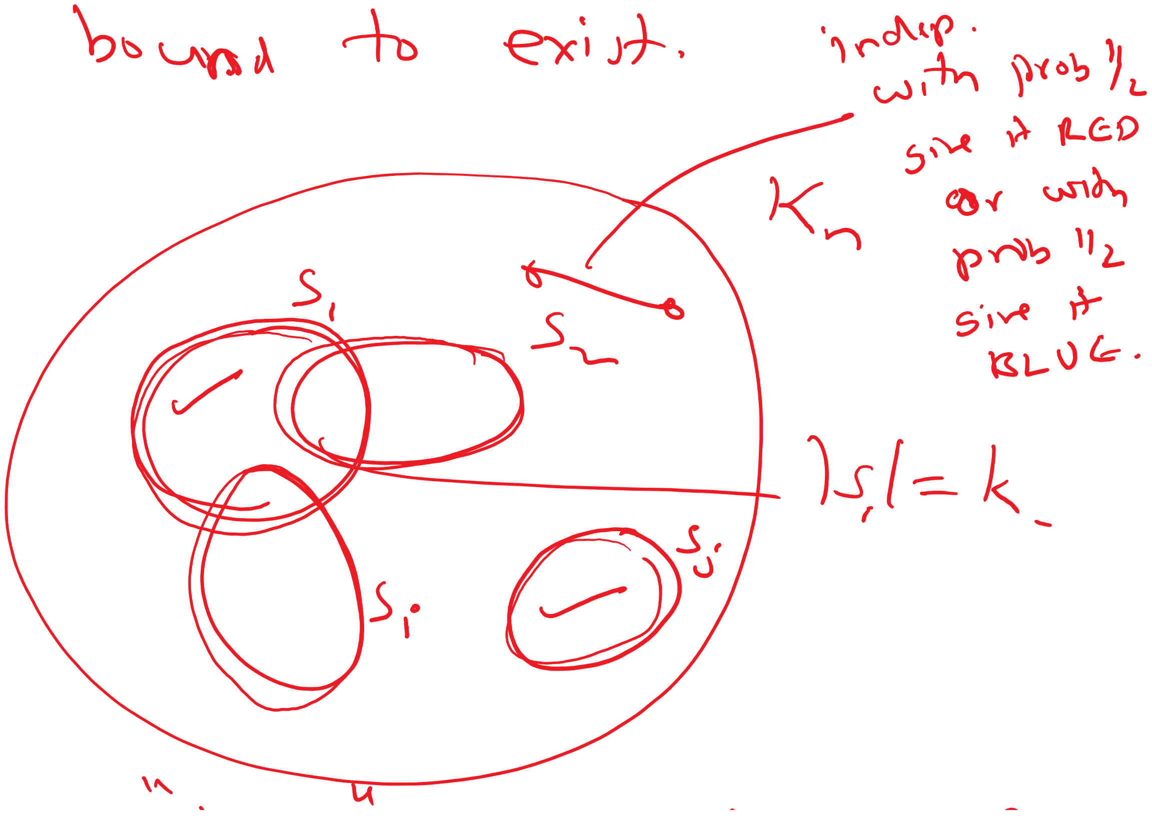
$$> 0$$

$\square$

Applications of Local Lemma

Theorem Every k-regular digraph has a collection of $\left\lfloor \frac{k}{3\ln k} \right\rfloor$ vertex disjoint cycles.

$R(k,k)$: min n such that no matter how one bicolors the edges of a K_n with red and blue, a monochromatic k-clique is bound to exist.

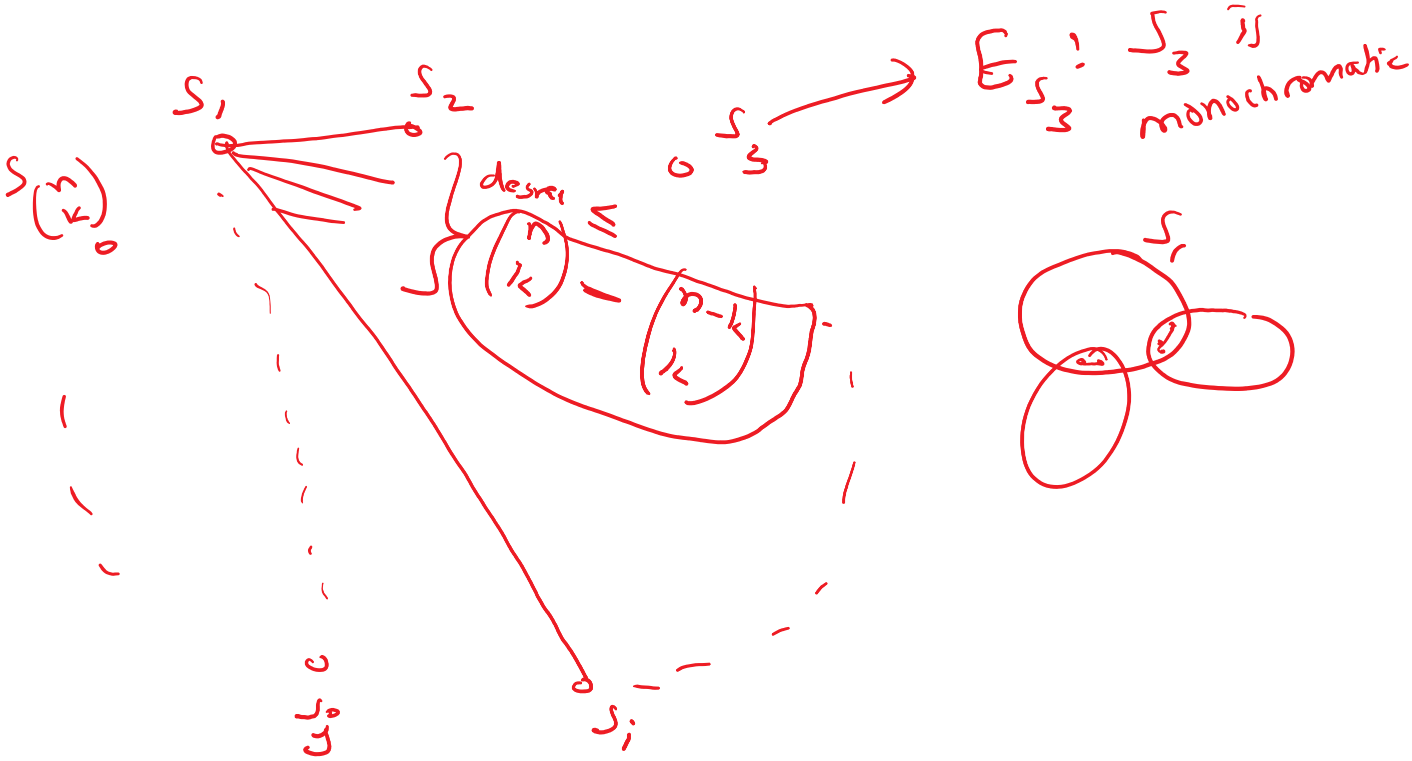


E_{S_1} : bad event that S_1 is a monochromatic k-clique

$$\Pr[E_{S_1}] = \frac{1}{2^{\binom{k}{2}}} + \frac{1}{2^{\binom{k}{2}}}$$

$$= \frac{1}{2^{\binom{k}{2}-1}}$$

Let $p := \frac{1}{2^{\binom{k}{2}-1}}$



$$d = \binom{n}{k} - \binom{n-k}{k}$$

From the symmetric case of the local

lemma, none of the bad events

$E_{S_1}, E_{S_2}, \ldots, E_{S_{\binom{n}{k}}}$ occur if $\Pr\left[\bigwedge_{i=1}^{\binom{n}{k}} \overline{E}_{S_i}\right]$

$$e\,p(d+1) \le 1$$

Or

$$e \frac{1}{2^{\binom{k}{2}-1}} \left(\binom{n}{k} - \binom{n-k}{k} + 1\right) \le 1 \quad B$$

solve it.

Instead, if we were using the union bound,

$$\Pr\left[E_{S_1} \vee E_{S_2} \vee \cdots \vee E_{S_{\binom{n}{k}}}\right] \le \sum_{i=1}^{\binom{n}{k}} \Pr\left[E_{S_i}\right]$$

$$= \frac{\binom{n}{k}}{2^{\binom{k}{2}-1}} \to A$$

We want $A < 1$, to

We want $\overline{A} < 1$, to

Justify that $\Pr\left[\bigwedge_{i=1}^{\binom{n}{k}} \overline{E_{S_i}}\right] > 0$.

From B, we get

$$R(k,k) > \frac{\sqrt{2}}{e}(1+o(1))\,k\,2^{k/2}$$

Whereas, from 1, all we get is

$$R(k,k) > \frac{1}{e\sqrt{2}}(1+o(1))\,k\,2^{k/2}$$