Set of problems
MA1140: Elementary Linear Algebra
8th February - 8th March, 2019

1 Matrices, Linear equations and solvability

(1) Solve (if solution exists) the following system of linear equations:

$$\begin{array}{lllll} u & +v & +w & +z & =6 \\ u & & +w & +z & =4 \\ u & & +w & & =2 \end{array}$$

What is the intersection if the fourth plane $u = -1$ is included? Find a fourth equation that leaves us with no solution.

(2) Find two points on the line of intersection of the three planes $t = 0$, $z = 0$ and $x + y + z + t = 1$ in four-dimensional space.

(3) Explain why the system

$$\begin{aligned} u + v + w &= 2 \\ u + 2v + 3w &= 1 \\ v + 2w &= 0 \end{aligned}$$

is *singular* (i.e., it does not have solutions at all). What value should replace the last zero on the right side to allow the system to have solutions, and what is one of the solutions?

(4) Under what condition on x_1, x_2 and x_3 do the points $(0, x_1), (1, x_2)$ and $(2, x_3)$ lie on a straight line?

(5) These equations are certain to have the solution $x = y = 0$. For which values of a is there a whole line of solutions?

$$\begin{aligned} ax + 2y &= 0 \\ 2x + ay &= 0 \end{aligned}$$

(6) Are the following systems equivalent:

$$\begin{aligned} x - y &= 0 \\ 2x + y &= 0 \end{aligned}$$

and

$$\begin{aligned} 3x + y &= 0 \\ x + y &= 0 \end{aligned}$$

If so, then express each equation in each system as a linear combination of the equations in the other system.

(7) Set $A = \begin{pmatrix} 6 & -4 & 0 \\ 4 & -2 & 0 \\ -1 & 0 & 3 \end{pmatrix}$ and $X = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$. Find all the solutions of $AX = 2X$, i.e., all X such that $AX = 2X$, where $2X$ is just componenntwise scalar multiplication.

(8) Prove that the interchange of two rows of a matrix can be accomplished by a finite sequence of elementary row operations of the other two types.

(9) Consider the system of equations $AX = 0$, where $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ is a matrix over $\mathbb{R}$. Prove the following:
(a) A is a zero matrix (i.e., all entries are zero) if and only if every pair (x_1, x_2) is a solution of $AX = 0$.
(b) $ad - bc \neq 0$ if and only if the system has only the trivial solution.
(c) $ad - bc = 0$ but A is a non-zero matrix (i.e., some entries are non-zero) if and only if there is $(y_1, y_2) \neq (0, 0)$ in $\mathbb{R}^2$ such that every solution of the system is given by $c(y_1, y_2)$ for some scalar c.

(10) Prove that if two homogeneous systems each of two linear equations in two unknowns have the same solutions, then they are equivalent.

(11) For the system

$$\begin{aligned} u + v + w &= 2 \\ 2u + 3v + 3w &= 0 \\ u + 3v + 5w &= 2, \end{aligned}$$

what is the triangular system after forward elimination, and what is the solution (by back substitution)? Also solve it by computing the equivalent system whose coefficient matrix is in row reduced echelon form. Verify whether both the solutions are same.

(12) Describe explicitly all 2×2 row reduced echelon matrices.

(13) Let $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ be a matrix over $\mathbb{R}$. Suppose that A is row reduced and also that $a + b + c + d = 0$. Prove that there are exactly three such matrices.

(14) Let $A = \begin{pmatrix} 1 & -1 & 1 \\ 2 & 0 & 1 \\ 3 & 0 & 1 \end{pmatrix}$. Find some elementary matrices $E_1, E_2, \ldots, E_k$ such that $E_k \cdots E_2 E_1 A = I_3$, where I_3 is the 3×3 identity matrix. Deduce A^{-1}.

Hint. Apply elementary row operations on $(A \,|\, I_3)$ to get A^{-1}, and keep track of the row operations to get the corresponding $E_1, E_2, \ldots, E_k$.

2 Vector spaces

Throughout, V is a vector space over $\mathbb{R}$, the set of real numbers.

(15) Let $0 \in V$ be the zero vector. Let $c \in \mathbb{R}$. Show that $c \cdot 0 = 0$.

(16) Let $v \in V$. Show that $0 \cdot v = 0$, where 0 in the right side is the zero vector, and 0 in the left side is the zero element of $\mathbb{R}$.

(17) Let W be a subspace of V. Show that (the zero vector) $0 \in W$.

(18) Let $V = \mathbb{R}^2$. Define

$$\begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} x_1 \\ y_1 \end{pmatrix} = \begin{pmatrix} x + x_1 \\ 0 \end{pmatrix} \quad \text{and} \quad c \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} cx \\ y \end{pmatrix}.$$

Is V, with these operations, a vector space over $\mathbb{R}$? What happens when we define the scalar multiplication as

$$c \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} cx \\ 0 \end{pmatrix}?$$

Definition. Let A be an $m \times n$ matrix over $\mathbb{R}$. The subspace of $\mathbb{R}^m$ generated by all columns (column vectors) of A is called the **column space** of A; while the subspace of $\mathbb{R}^n$ generated by all rows (row vectors) of A is called the **row space** of A.

(19) Let A be an $m \times n$ matrix over $\mathbb{R}$. Show that $\{X \in \mathbb{R}^n : AX = 0\}$ is a subspace of $\mathbb{R}^n$. It is called the **null space** of A.

(20) Let $A = \begin{pmatrix} 3 & -1 & 8 & 4 \\ 2 & 1 & 7 & 1 \\ 1 & -3 & 0 & 4 \end{pmatrix}$ and $X = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix}$. For which $Y = \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix}$ in $\mathbb{R}^3$ does the system $AX = Y$ have a solution? Describe the answer in terms of subspaces of $\mathbb{R}^3$. Use the following approaches, and verify whether you get the same answer.

Two approaches: (i) Apply elementary row eliminations on $(A \,|\, Y)$. (ii) Note that for every $X \in \mathbb{R}^4$, AX is nothing but a linear combination of the four column vectors of A:

$$AX = x_1 \begin{pmatrix} 3 \\ 2 \\ 1 \end{pmatrix} + x_2 \begin{pmatrix} -1 \\ 1 \\ -3 \end{pmatrix} + x_3 \begin{pmatrix} 8 \\ 7 \\ 0 \end{pmatrix} + x_4 \begin{pmatrix} 4 \\ 1 \\ 4 \end{pmatrix}.$$

So Y should belong into the column space of A.

(21)