Matrices, Linear equations and solvability

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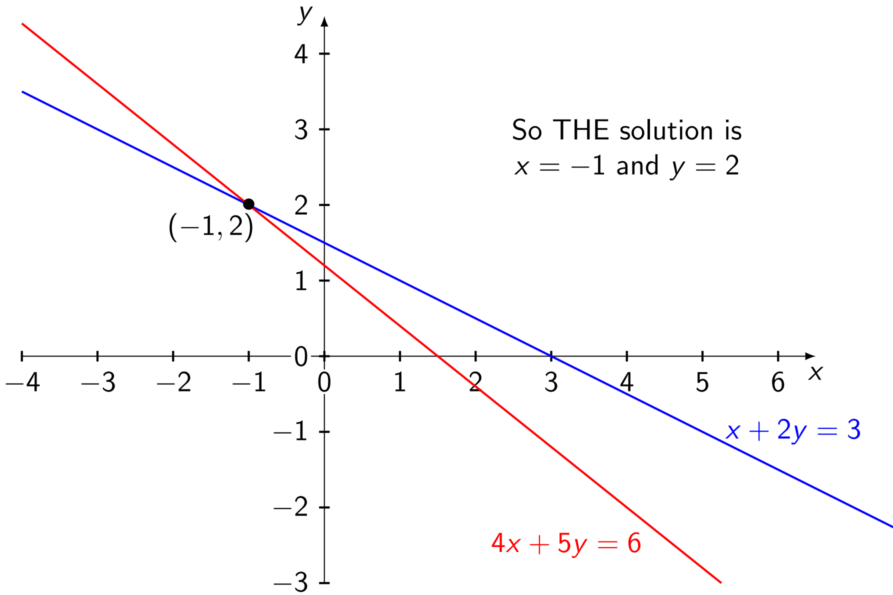
Solving linear equations

- One of the central problem of linear algebra is '**solving linear equations**'.
- Consider the following **system of linear equations:**

$$\begin{aligned} x + 2y &= 3 \qquad \text{(1st equation)} \\ 4x + 5y &= 6 \qquad \text{(2nd equation)}. \end{aligned}$$

- Here x and y are the unknowns. We want to solve this system, i.e., we want to find the values of x and y in $\mathbb{R}$ such that the equations are satisfied.

What does it mean geometrically?



How can we solve the system?

- We can solve the system by **Gaussian Elimination**. The original system is

$$\begin{aligned} x+2y &= 3 \qquad \text{(1st equation)} \\ 4x+5y &= 6 \qquad \text{(2nd equation)}. \end{aligned} \tag{1}$$

- We want to change it into an equivalent system, which is comparatively easy to solve.
- Eliminating x from the 2nd equation, we obtain a **triangulated** system:

$$\begin{aligned} x+2y &= 3 \qquad \text{(equation1)} \\ -3y &= -6 \qquad \text{(equation 2) - 4(equation 1)}. \end{aligned} \tag{2}$$

- Both the systems have same solutions. We can solve the 2nd system by **Back-substitution**. What is it?
- In this case, the solution is $y=2,\ x=-1$.

Another method to solve the system: Cramer's Rule

- The system can be written as

$$\begin{aligned} x + 2y &= 3 \\ 4x + 5y &= 6 \end{aligned} \quad \text{or} \quad \begin{pmatrix} 1 & 2 \\ 4 & 5 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 3 \\ 6 \end{pmatrix}.$$

- The solution depends completely on those six numbers in the equations. There must be a formula for x and y in terms of those six numbers. Cramer's Rule provides the formula:

$$x = \frac{\begin{vmatrix} 3 & 2 \\ 6 & 5 \end{vmatrix}}{\begin{vmatrix} 1 & 2 \\ 4 & 5 \end{vmatrix}} = \frac{3 \cdot 5 - 6 \cdot 2}{1 \cdot 5 - 4 \cdot 2} = \frac{3}{-3} = -1$$

$$y = \frac{\begin{vmatrix} 1 & 3 \\ 4 & 6 \end{vmatrix}}{\begin{vmatrix} 1 & 2 \\ 4 & 5 \end{vmatrix}} = \frac{1 \cdot 6 - 4 \cdot 3}{1 \cdot 5 - 4 \cdot 2} = \frac{-6}{-3} = 2.$$

Which approach is better?

- The direct use of the **determinant formula** for large number of equations and variables would be very difficult.
- So the better method is **Gaussian Elimination**. Let's study it systematically.
- We understand the Gaussian Elimination method by examples.

How many solutions do exist for a given system?

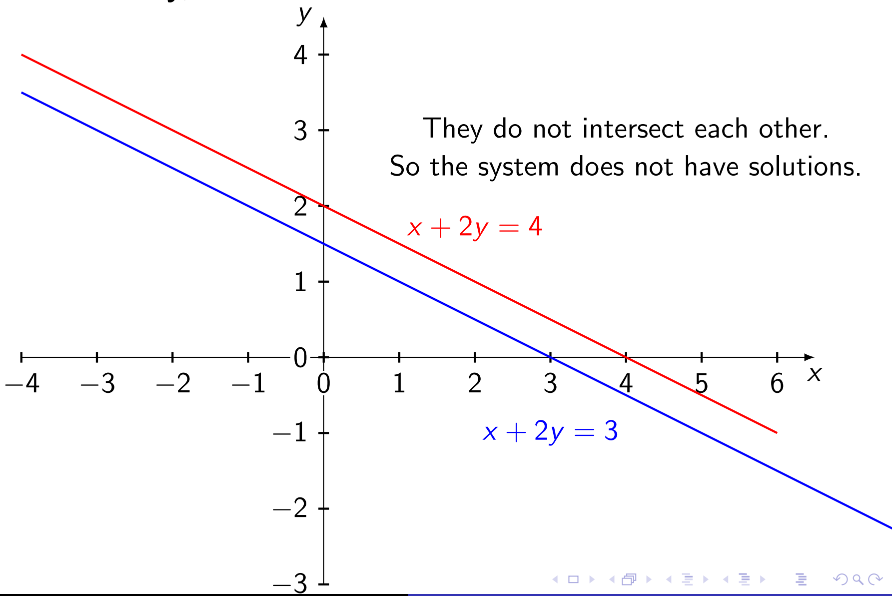
- A system may have only ONE solution. For example the system which we have already discussed.
- A system may NOT have a solution at all. For example

$$\left.\begin{array}{l} x+2y=3 \\ x+2y=4 \end{array}\right\}. \quad \text{After Gaussian Elimination} \quad \begin{array}{ll} x+2y & =3 \\ \mathbf{0} & \mathbf{=1} \end{array}$$

- This is absurd. So the system does not have solutions.

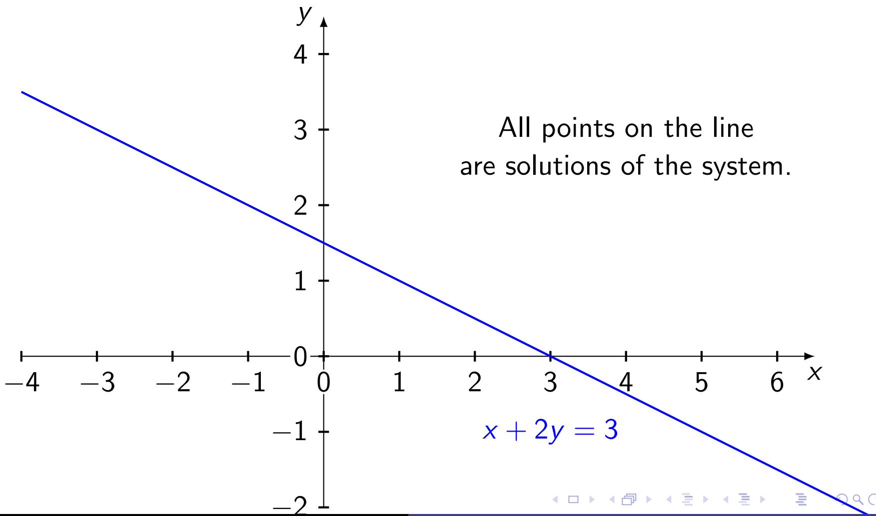
A system may NOT have a solution at all

Geometrically,



A system may have infinitely many solutions

$$\left.\begin{aligned} x + 2y &= 3 \\ 2x + 4y &= 6 \end{aligned}\right\}$$ equivalent to the system $x + 2y = 3$



An example to understand Gaussian Elimination

- Consider the system:

$$
\begin{aligned}
v + w &= 5 \\
4u - 6v &= -2 \\
-2u + 7v + 2w &= 9
\end{aligned}
$$

- There is no harm to **interchange the positions of two equations**. So the original system is equivalent to the following system.

$$
\begin{aligned}
4u - 6v &= -2 \\
v + w &= 5 \\
-2u + 7v + 2w &= 9
\end{aligned}
$$

- What was the aim? To change the system so that the coefficient of u in the 1st equation becomes non-zero.

An example to understand Gaussian Elimination contd...

- So the system becomes

$$\begin{aligned} \mathbf{4}u - 6v &= -2 \\ v + w &= 5 \\ -2u + 7v + 2w &= 9 \end{aligned}$$

- We call the coefficient 4 as the **first pivot**.
- There is no harm if we multiply an equation by a non-zero constant. So we can always make the pivot element 1.
- We now eliminate u from the 3rd equation.
- Adding (1/2) times the 1st equation to the 3rd equation,

$$\begin{aligned} \mathbf{4}u - 6v &= -2 \\ \mathbf{1} \cdot v + w &= 5 \\ 4v + 2w &= 8 \end{aligned}$$

- We already got the 2nd pivot. In the last stage, we eliminate v from the 3rd equation. Apply (3rd eqn) $-$ 4 (2nd eqn).

Triangular system and back-substitution

- After the elimination process, we obtain a **triangular system**:

$$
\begin{aligned}
\mathbf{4}u - 6v + 0w &= -2 \\
\mathbf{1} \cdot v + w &= 5 \\
-\mathbf{2}w &= -12
\end{aligned}
$$

- Now the system can be solved by **backward substitution**, bottom to top. The red colored coefficients are pivots.
- The last equation gives $w = 6$.
- Substituting $w = 6$ into the 2nd equation, we find $v = -1$.
- Substituting $w = 6$ and $v = -1$ into the 1st equation, we get $u = -2$.

Gaussian Elimination process in short

$$
\begin{aligned}
&\text{Original System}\\
&\quad \Downarrow \ \text{Forward Elimination}\\
&\text{Triangular System}\\
&\quad \Downarrow \ \text{Backward Substitution}\\
&\text{Solution}
\end{aligned}
$$

Augmented matrix of the system

- Consider the system:

$$\begin{aligned} v + w &= 5 \\ 4u - 6v &= -2 \\ -2u + 7v + 2w &= 9 \end{aligned}$$

- The **coefficient matrix** of the system is given by

$$\begin{bmatrix} 0 & 1 & 1 \\ 4 & -6 & 0 \\ -2 & 7 & 2 \end{bmatrix}$$

- The **augmented matrix** of the system is given by

$$\begin{bmatrix} 0 & 1 & 1 & 5 \\ 4 & -6 & 0 & -2 \\ -2 & 7 & 2 & 9 \end{bmatrix}$$

The forward elimination steps

- The forward elimination steps can be described as follows.

$$\begin{bmatrix} 0 & 1 & 1 & 5 \\ 4 & -6 & 0 & -2 \\ -2 & 7 & 2 & 9 \end{bmatrix} \xRightarrow{R1 \leftrightarrow R2} \begin{bmatrix} 4 & -6 & 0 & -2 \\ 0 & 1 & 1 & 5 \\ -2 & 7 & 2 & 9 \end{bmatrix} \xRightarrow{R3 \to R3+(1/2)R1}$$

$$\begin{bmatrix} 4 & -6 & 0 & -2 \\ 0 & 1 & 1 & 5 \\ 0 & 4 & 2 & 8 \end{bmatrix} \xRightarrow{R3 \to R3-4\cdot R2} \begin{bmatrix} 4 & -6 & 0 & -2 \\ 0 & 1 & 1 & 5 \\ 0 & 0 & -2 & -12 \end{bmatrix}$$

- Now one can solve the corresponding system by back substitution.
- In this case, where we have a full set of 3 pivots, there is only one solution.

When we have less pivots than 3

- When we have less pivots than 3, i.e., if a zero appears in a pivot position, then the system may not have solution at all, or it can have infinitely many solutions.
- For example, if the augmented matrix corresponding to a system has the form

$$\begin{bmatrix} 1 & 1 & 1 & * \\ 2 & 2 & 5 & * \\ 4 & 4 & 8 & * \end{bmatrix} \Longrightarrow \begin{bmatrix} 1 & 1 & 1 & * \\ 0 & \mathbf{0} & 3 & * \\ 0 & 0 & 4 & * \end{bmatrix}.$$

- Now consider some particular values of $*$.

When we have less pivots than 3 contd...

- $\begin{bmatrix} 1 & 1 & 1 & * \\ 0 & 0 & 3 & * \\ 0 & 0 & 4 & * \end{bmatrix}$. Considering some particular values of $*$,
- Example 1: $\begin{bmatrix} 1 & 1 & 1 & * \\ 0 & 0 & 3 & 6 \\ 0 & 0 & 4 & 7 \end{bmatrix} \xrightarrow{R3-(4/3)R2} \begin{bmatrix} 1 & 1 & 1 & * \\ 0 & 0 & 3 & 6 \\ 0 & 0 & 0 & -1 \end{bmatrix}$.
- The corresponding system is

$$\begin{aligned} u + v + w &= * \\ 3w &= 6 \\ 0 &= -1 \end{aligned}$$

- This system does not have solution.

When we have less pivots than 3 contd...

- $\begin{bmatrix} 1 & 1 & 1 & * \\ 0 & 0 & 3 & * \\ 0 & 0 & 4 & * \end{bmatrix}$. Considering some particular values of $*$,
- Example 2: $\begin{bmatrix} 1 & 1 & 1 & * \\ 0 & 0 & 3 & 6 \\ 0 & 0 & 4 & 8 \end{bmatrix} \xrightarrow{R3-(4/3)R2} \begin{bmatrix} 1 & 1 & 1 & * \\ 0 & 0 & 3 & 6 \\ 0 & 0 & 0 & 0 \end{bmatrix}$.
- The corresponding system is

$$\begin{aligned} u + \mathbf{v} + w &= * \\ 3w &= 6 \end{aligned}$$

- This system has infinitely many solutions.
- From the last equation, we get $w = 2$.
- Substituting $w = 2$ to the 1st equation, we have $u + v = *$, which has infinitely many solutions. We call v a free variable.

System of linear equations (in general)

- Consider a system of m linear equations in n variables $x_1, \dots, x_n$.

$$\begin{aligned} A_{11}x_1 + A_{12}x_2 + \cdots + A_{1n}x_n &= b_1 \\ A_{21}x_1 + A_{22}x_2 + \cdots + A_{2n}x_n &= b_2 \\ &\vdots \\ A_{m1}x_1 + A_{m2}x_2 + \cdots + A_{mn}x_n &= b_m \end{aligned}$$

- Here $A_{ij}, b_i \in \mathbb{R}$, and $x_1, \dots, x_n$ are unknown. We try to find the values of $x_1, \dots, x_n$ in $\mathbb{R}$ satisfied by the system.
- Any n tuple $(x_1, \dots, x_n)$ of elements of $\mathbb{R}$ which satisfies the system (i.e., which satisfies every equation of the system) is called a **solution** of the system.
- If $b_1 = \cdots = b_m = 0$, then it is called a **homogeneous system**.
- Every homogeneous system has a trivial solution $x_1 = \cdots = x_n = 0$. What about non-homogeneous system?

Linear combination of linear equations in a system

- Linear combination of equations of the previous system yields another equation, e.g., (2nd eqn) + c(1st eqn):

$$(A_{21} + cA_{21})x_1 + (A_{22} + cA_{12})x_2 + \cdots + (A_{2n} + cA_{1n})x_n = (b_2 + cb_1$$

- Note that if $(x_1, \ldots, x_n) \in \mathbb{R}^n$ satisfies the original system, then it satisfies any such linear combination also.

Equivalent systems of linear equations

- Consider the original system:

$$\begin{aligned} A_{11}x_1 + A_{12}x_2 + \cdots + A_{1n}x_n &= b_1 \\ A_{21}x_1 + A_{22}x_2 + \cdots + A_{2n}x_n &= b_2 \\ &\vdots \\ A_{m1}x_1 + A_{m2}x_2 + \cdots + A_{mn}x_n &= b_m \end{aligned}$$

- Suppose we have another system:

$$\begin{aligned} B_{11}x_1 + B_{12}x_2 + \cdots + B_{1n}x_n &= b'_1 \\ B_{21}x_1 + B_{22}x_2 + \cdots + B_{2n}x_n &= b'_2 \\ &\vdots \\ B_{m1}x_1 + B_{m2}x_2 + \cdots + B_{mn}x_n &= b'_m \end{aligned}$$

- Suppose every equation in the 2nd system is a linear combination of the equations in the 1st system, then every solution of the 1st system is also a solution of the 2nd one.

Equivalent systems of linear equations

- Consider the original system:

$$\begin{aligned} A_{11}x_1 + A_{12}x_2 + \cdots + A_{1n}x_n &= b_1 \\ A_{21}x_1 + A_{22}x_2 + \cdots + A_{2n}x_n &= b_2 \\ &\vdots \\ A_{m1}x_1 + A_{m2}x_2 + \cdots + A_{mn}x_n &= b_m \end{aligned}$$

- Suppose we have another system:

$$\begin{aligned} B_{11}x_1 + B_{12}x_2 + \cdots + B_{1n}x_n &= b'_1 \\ B_{21}x_1 + B_{22}x_2 + \cdots + B_{2n}x_n &= b'_2 \\ &\vdots \\ B_{m1}x_1 + B_{m2}x_2 + \cdots + B_{mn}x_n &= b'_m \end{aligned}$$

- Suppose every equation in the 2nd system is a linear combination of the equations in the 1st system, vice versa.
- Then we call that such systems are **equivalent**.

Thank You!