# Topics in Combinatorics

Exam I (out of 10 marks)
(Date: 12 Oct 2020. Timing: 12:00 to 13:05 hours)

**Notation.** Let $G$ be a simple, undirected, finite graph. We use $V(G)$ to denote its vertex set and $E(G)$ to denote its edge set. For each vertex $v \in V(G)$, let $N(v) = \{u \in V(G) \ : \ \{u, v\} \in E(G)\}$ denote the *open neighborhood* of $v$. Let $N[v] = \{v\} \cup N(v)$ denote the *closed neighborhood* of $v$. The *degree* of $v$, denoted by $deg(v)$, is the number of vertices in the open neighborhood of $v$. That is, $deg(v) = |N(v)|$. The *maximum degree* of a graph $G$ is $\max\{deg(v) \ : \ v \in V(G)\}$.

**Definition.** Let $G$ be a simple, undirected, finite graph. Let $\sigma$ be a linear order (or total order or simple order) of the vertices of $G$. For any $u, v \in V(G)$ with $\{u, v\} \notin E(G)$, we say the ordered pair $(u, v)$ is **separated** in $\sigma$ if $v$ succeeds every vertex in the set $N[u]$ in the linear order $\sigma$.
**Example**: Let $G$ be a graph with $V(G) = \{v_1, v_2, v_3, v_4, v_5, v_6\}$,
$E(G) = \{\{v_1, v_2\}, \{v_2, v_3\}, \{v_3, v_4\}, \{v_4, v_5\}, \{v_5, v_6\}, \{v_6, v_1\}, \{v_4, v_5\}\}$. Let $\sigma \ : \ v_2, v_3, v_1, v_6, v_4, v_5$. Note that $\sigma$ *separates* the ordered pair $(v_1, v_4)$ as $v_4$ succeeds every vertex in $N[v_1] = \{v_1, v_2, v_6\}$ in $\sigma$. However, $\sigma$ does not separate the ordered pair $(v_3, v_6)$ as $v_6$ precedes $v_4$ which is a neighbor of $v_3$.

## Problem

Let $G$ be a simple, undirected, finite graph on $n$ vertices. Let $\Delta$ denote the maximum degree of $G$. Show that there is a collection $\mathcal{F} = \{\sigma_1, \sigma_2, \ldots\}$ of $O(\Delta \log n)$ linear orders of the vertices of $G$ such that for every distinct $u, v \in V(G)$ with $\{u, v\} \notin E(G)$, there exists some $\sigma_i \in \mathcal{F}$ such that the ordered pair $(u, v)$ is separated in $\sigma_i$. **10 marks**