MA 1140: Lecture 8
Eigenvalues and eigenvectors

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Eigenvalues and eigenvectors (of linear operators)

Let $T : V \to V$ be a linear map, which we call linear operator.

Definition

A non-zero vector $v \in V$ is called an **eigenvector** of T if $T(v) = \lambda v$ for some $\lambda \in \mathbb{R}$. This λ is called the **eigenvalue** of T associated with the eigenvector v.

- Geometrically, an eigenvector, corresponding to an eigenvalue, points in a direction that is stretched by the transformation, and the eigenvalue is the factor by which it is stretched. If the eigenvalue is negative, the direction is reversed.
- An eigenvalue can be positive, negative or zero.

Since there is a one to one correspondence between the set of all linear operators from V ($\cong \mathbb{R}^n$) to itself and the collection of all $n \times n$ matrices over $\mathbb{R}$, it is equivalent to define eigenvalues and eigenvectors of $n \times n$ matrices.

Definition

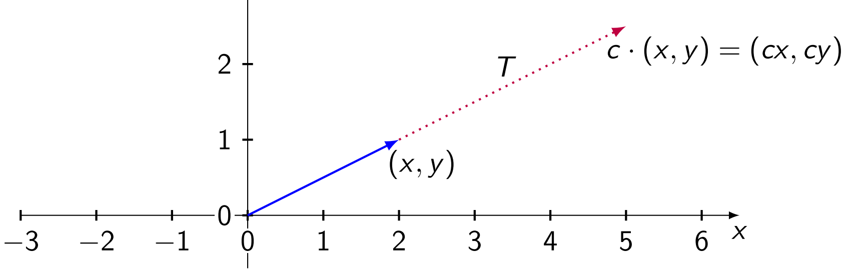
Let A be an $n \times n$ matrix over $\mathbb{R}$. A non-zero column vector $v \in \mathbb{R}^n$ is called an **eigenvector** of A if $Av = \lambda v$ for some $\lambda \in \mathbb{R}$. This λ is called the **eigenvalue** of A associated with the eigenvector v.

Example 1: eigenvalues and eigenvectors of stretching

Let $c \in \mathbb{R}$. Define $T : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ by

$$T \begin{pmatrix} x \\ y \end{pmatrix} = c \begin{pmatrix} x \\ y \end{pmatrix} \text{ for every } \begin{pmatrix} x \\ y \end{pmatrix} \in \mathbb{R}^2.$$

The matrix representation: $[T] = \begin{pmatrix} c & 0 \\ 0 & c \end{pmatrix}$

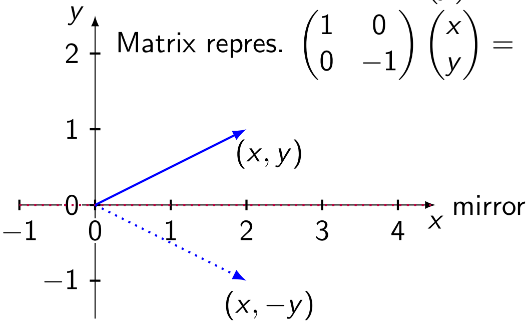


Every $v\,(\neq 0) \in \mathbb{R}^2$ is an eigenvector of T with the eigenvalue c.

Example 2: eigenvalues and eigenvectors of reflection

$T : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ is defined by $T\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x \\ -y \end{pmatrix}$.

Matrix repres. $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x \\ -y \end{pmatrix}$



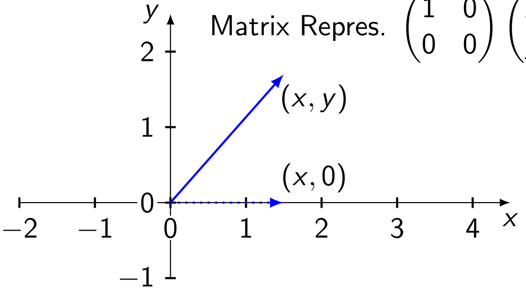
For $x \neq 0$, $\begin{pmatrix} x \\ 0 \end{pmatrix}$ is an eigenvector of T with eigenvalue 1.

For $y \neq 0$, $\begin{pmatrix} 0 \\ y \end{pmatrix}$ is an eigenvector of T with eigenvalue -1.

These are ALL the eigenvectors of T. (Verify it!)

Define $T : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ by $T\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x \\ 0 \end{pmatrix}$

Matrix Repres. $\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x \\ 0 \end{pmatrix}$



projection on x-axis

For $x \neq 0$, $\begin{pmatrix} x \\ 0 \end{pmatrix}$ is an eigenvector of T with eigenvalue 1.

For $y \neq 0$, $\begin{pmatrix} 0 \\ y \end{pmatrix}$ is an eigenvector of T with eigenvalue 0.

These are ALL the eigenvectors of T. (Verify it!)

Example 4: A may not have eigenvalues and eigenvectors

- Consider the matrix $A = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ over $\mathbb{R}$.
- Does A have eigenvalues and eigenvectors over $\mathbb{R}$?
- If yes, then there are $\begin{pmatrix} x \\ y \end{pmatrix} \in \mathbb{R}^2 \smallsetminus \{\begin{pmatrix} 0 \\ 0 \end{pmatrix}\}$ and $\lambda \in \mathbb{R}$ such that

$$\begin{aligned} & \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \lambda \begin{pmatrix} x \\ y \end{pmatrix} \\ \Longrightarrow \quad & \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} \\ \Longrightarrow \quad & \begin{pmatrix} -\lambda & 1 \\ -1 & -\lambda \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \end{aligned}$$

  Since $\begin{pmatrix} x \\ y \end{pmatrix} \neq \begin{pmatrix} 0 \\ 0 \end{pmatrix}$, $\det \begin{pmatrix} -\lambda & 1 \\ -1 & -\lambda \end{pmatrix} = 0$. Hence $\lambda^2 + 1 = 0$.
  But no such λ exists in $\mathbb{R}$.
- So A does not have eigenvalues and eigenvectors over $\mathbb{R}$.

The existence of eigenvalues and eigenvectors

- Consider the matrix $A = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ over $\mathbb{C}$, the set of complex numbers.
- Does A have eigenvalues and eigenvectors over $\mathbb{C}$? Ans. Yes.
- Note that $\lambda^2 + 1$ has solutions: $\pm i \in \mathbb{C}$.
- Then, for each $\lambda = \pm i$, in view of the previous slide, one should solve the system
$$\begin{pmatrix} -\lambda & 1 \\ -1 & -\lambda \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$
to get
$$\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} i \\ -1 \end{pmatrix} = i \begin{pmatrix} i \\ -1 \end{pmatrix} \text{ and } \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} i \\ 1 \end{pmatrix} = -i \begin{pmatrix} i \\ 1 \end{pmatrix}$$
- **Conclusion:** The existence of eigenvalues and eigenvectors of a given matrix depends on the base field.

Similarity of matrices

Definition

Two $n \times n$ matrices A and B are called **similar** if there exists an invertible $n \times n$ matrix P such that $B = P^{-1}AP$.

Some statements (without proof) about importance of similarity of matrices:

- Two matrices are similar if and only if they represent the same linear operator with respect to (possibly) different bases.
- Two similar matrices A and B share many properties:
  - $\mathrm{rank}(A) = \mathrm{rank}(B)$ as operators from $\mathbb{R}^n$ to itself.
  - $\det(A) = \det(B)$; $\mathrm{tr}(A) = \mathrm{tr}(B)$ (sum of all diagonal entries).
  - A and B have same characteristic polynomial, $\det(xI_n - A)$.
  - Minimal polynomials of A and B are same. A monic polynomial $p(X) \in \mathbb{R}[X]$ is said to be a minimal polynomial of A if $p(A) = 0$ (zero matriz) and p has minimal possible degree.
  - Jordan canonical forms of A and B are same.

Motivation: For a matrix, eigenvalues and eigenvectors can be used to decompose the matrix, for example by diagonalizing it.

Definition

A matrix A is said to be **diagonalizable** if A is similar to a diagonal matrix D, i.e., if there is an invertible matrix P such that

$$P^{-1}AP = \begin{pmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_n \end{pmatrix} \quad \text{(a diagonal matrix).}$$

The set of eigenvectors helps us to test whether a matrix is diagonalizable or not.

The use of eigenvalues and eigenvectors on diagonalization

Theorem

Let A be an $n \times n$ matrix over $\mathbb{R}$. The following are equivalent:

1. *A is diagonalizable.*
2. *The eigenvectors of A form a basis, or equivalently, A has n linearly independent eigenvectors $v_1, \ldots, v_n$ with associated eigenvalues $\lambda_1, \ldots, \lambda_n$ (which need not be distinct).*

Proof. (1) $\Rightarrow$ (2): There is an $n \times n$ invertible matrix P such that

$$P^{-1}AP = \begin{pmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_n \end{pmatrix} \quad \text{for some } \lambda_1, \ldots, \lambda_n \in \mathbb{R}.$$

Hence $AP = P \begin{pmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \end{pmatrix}$.

Proof. (1) $\Rightarrow$ (2): ... Thus $AP = P \begin{pmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_n \end{pmatrix}$.

Write $P = \begin{bmatrix} v_1 & v_2 & \cdots & v_n \end{bmatrix}$ for some $v_1, \dots, v_n \in \mathbb{R}^n$.
Then $AP = \begin{bmatrix} Av_1 & Av_2 & \cdots & Av_n \end{bmatrix}$ and

$$P \begin{pmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_n \end{pmatrix} = \left[P \begin{pmatrix} \lambda_1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} \quad P \begin{pmatrix} 0 \\ \lambda_2 \\ \vdots \\ 0 \end{pmatrix} \quad \cdots \quad P \begin{pmatrix} 0 \\ 0 \\ \vdots \\ \lambda_n \end{pmatrix} \right]$$

$= \begin{bmatrix} \lambda_1 v_1 & \lambda_2 v_2 & \cdots & \lambda_n v_n \end{bmatrix}$.
Therefore $Av_i = \lambda_i v_i$ for every $1 \leq i \leq n$.
Note that $v_1, \dots, v_n$ are linearly independent, since P is invertible.

Proof. (2) $\Rightarrow$ (1): A has n linearly independent eigenvectors $v_1, \ldots, v_n \in \mathbb{R}^n$ with associated eigenvalues $\lambda_1, \ldots, \lambda_n \in \mathbb{R}$. Set $P := \begin{bmatrix} v_1 & v_2 & \cdots & v_n \end{bmatrix}$. Clearly P is an $n \times n$ matrix. Since $v_1, \ldots, v_n$ are linearly independent, P is invertible. Moreover

$$AP = \begin{bmatrix} Av_1 & Av_2 & \cdots & Av_n \end{bmatrix} = \begin{bmatrix} \lambda_1 v_1 & \lambda_2 v_2 & \cdots & \lambda_n v_n \end{bmatrix}$$
$$= P \begin{pmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_n \end{pmatrix}.$$

Therefore

$$P^{-1}AP = \begin{pmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_n \end{pmatrix}.$$

Eigenspace associated with an eigenvalue

Let A be an $n \times n$ matrix over $\mathbb{R}$. Denote the identity matrix by I_n.

- The following are equivalent:
  - $v \in \mathbb{R}^n$ is an eigenvector of A with the corr. eigenvalue λ.
  - $v \in \mathbb{R}^n$ is a non-trivial solution of the system $(A - \lambda I_n)X = 0$, i.e., $v \in \mathbb{R}^n \smallsetminus \{0\}$ lies in $\mathrm{Null}(A - \lambda I_n)$.

  Proof. Note that $Av = \lambda v$ if and only if $(A - \lambda I_n)v = 0$.
- The set of all eigenvectors of A corresponding to an eigenvalue λ, together with the zero vector, is called the **eigenspace** of A associated with λ.
  It is nothing but $\mathrm{Null}(A - \lambda I_n)$.

Characteristic polynomial of a matrix

Let A be an $n \times n$ matrix over $\mathbb{R}$. Denote the identity matrix by I_n.

- The following are equivalent:
  - $\lambda \in \mathbb{R}$ is an eigenvalue of A.
  - $\det(\lambda I_n - A) = 0$.

  Proof. Note that λ is an eigenvalue of A if and only if there is $v \neq 0$ in $\mathbb{R}^n$ such that $Av = \lambda v$ if and only if $(A - \lambda I_n)v = 0$ for some $v \neq 0$ in $\mathbb{R}^n$ if and only if $(A - \lambda I_n)X = 0$ has a non-trivial solution if and only if $\det(A - \lambda I_n) = 0$ if and only if $\det(\lambda I_n - A) = 0$.

Definition

The characteristic polynomial of A, denoted by $p_A(x)$, is the polynomial over $\mathbb{R}$ defined by $p_A(x) := \det(xI_n - A)$.

Therefore the set of all eigenvalues of A in $\mathbb{R}$ is given by all the real roots of the characteristic polynomial $p_A(x)$.

Example: Characteristic polynomial and eigenvalues

- Consider $A = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$.
- The characteristic polynomial of A is given by
$$\begin{aligned} p_A(x) = \det(xI_2 - A) &= \det \begin{pmatrix} x-1 & -2 \\ -3 & x-4 \end{pmatrix} \\ &= (x-1)(x-4) - 6 = x^2 - 5x - 2. \end{aligned}$$
- The roots of $p_A(x)$ are $\dfrac{5 \pm \sqrt{33}}{2}$.
- Therefore the eigenvalues of A are $\dfrac{5 \pm \sqrt{33}}{2}$.

How to compute eigenvalues and eigenvectors

- First compute the characteristic polynomial $p_A(x) = \det(xI_n - A)$ of A.
- Next compute the roots of $p_A(x)$ by factorizing it into linear factors. Which gives the eigenvalues.
- Then, for each eigenvalue λ, solve the homogeneous system:
$$(A - \lambda I_n)X = 0$$
to get eigenspace of A associated with λ.
- Recall that in order to solve a linear system, you may apply elementary row operations to make it into a system with row reduced echelon coefficient matrix.

- Let A be an $n \times n$ matrix over $\mathbb{R}$.
- Write A^r for the matrix multiplication of r many copies of A.
- For $c \in \mathbb{R}$, cA is just component wise scalar multiplication.
- If $f(x) = a_r x^r + \cdots + a_2 x^2 + a_1 x + a_0 \in \mathbb{R}[x]$, then $f(A) = a_r A^r + \cdots + a_2 A^2 + a_1 A + a_0 I_n$ is an $n \times n$ matrix/$\mathbb{R}$.

Theorem (Cayley-Hamilton)

Consider the characteristic polynomial $p_A(x) := \det(xI_n - A)$. Then $p_A(A) = 0$ (zero matrix of order $n \times n$).

Warning: $p_A(A) \neq \det(AI_n - A)$. LHS is a matrix; RHS is a scalar.

Example

If $A = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$, then $p_A(x) = x^2 - 5x - 2$. The Cayley-Hamilton Theorem says that $A^2 - 5A - 2I_2 = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$.

Similar matrices have same characteristic polynomial

Theorem

Let A and B be similar, i.e., $B = P^{-1}AP$ for some invertible matrix P. Then $\det(xI_n - A) = \det(xI_n - B)$.

Proof. $\det(xI_n - B) = \det(xP^{-1}I_nP - P^{-1}AP)$
$= \det(P^{-1}(xI_n - A)P) = (1/\det(P))\det(xI_n - A)\det(P)$
$= \det(xI_n - A)$. $\square$

Theorem

Let A and B be similar, i.e., $B = P^{-1}AP$. For a polynomial $f(x) \in \mathbb{R}[x]$, $f(A) = 0$ if and only if $f(B) = 0$ (zero matrix).

Proof. Note: $P^{-1}A^rP = (P^{-1}AP)(P^{-1}AP)\cdots(P^{-1}AP) = B^r$,
and $P^{-1}(c_1D_1 + c_2D_2)P = c_1(P^{-1}D_1P) + c_2(P^{-1}D_2P)$.
Verify that $P^{-1}f(A)P = f(B)$ and $P\,f(B)\,P^{-1} = f(A)$.
Hence the theorem follows. $\square$

Proof of Cayley-Hamilton Thm for diagonalizable matrix

Let A be a diagonalizable matrix, i.e., there is P such that

$$P^{-1}AP = \begin{pmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_n \end{pmatrix} = B \text{ (say)}.$$

Note that $\det(xI_n - A) = \det(xI_n - B) = (x - \lambda_1) \cdots (x - \lambda_n)$.
By induction on n, one can verify that

$$(B - \lambda_1 I_n)(B - \lambda_2 I_n) \cdots (B - \lambda_n I_n) = 0.$$

Hence, multiplying P on left and P^{-1} on right, we get

$$(A - \lambda_1 I_n)(A - \lambda_2 I_n) \cdots (A - \lambda_n I_n) = 0$$

i.e., $p_A(A) = 0$ (zero matrix). □

Thank You!