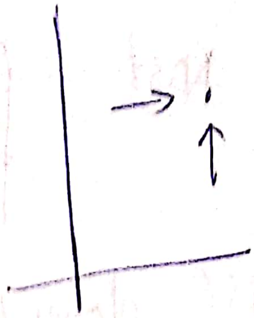
Complex Analysis



$\delta y = 0, \ \delta x \to 0$

$$\frac{df}{dz} = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} \quad -①$$

For δx & $\delta y \to 0$, we have

$$\frac{df}{dz} = \lim_{\delta z \to 0} \frac{\delta f}{\delta z}$$

$$= \lim_{\delta y \to 0} \frac{\delta u + i\,\delta v}{i\,\delta y}$$

$$= \lim_{\delta y \to 0} -i\frac{\delta u}{\delta y} + \frac{\delta v}{\delta y}$$

$$\frac{df}{dz} = -i\frac{\partial u}{\partial y} + \frac{\partial v}{\partial y} \quad -②$$

Equating real and imaginary parts of ① & ②.

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

These are Cauchy - Riemann conditions. These are necessary if derivative $\frac{df}{dz}$ exists.

Conversely, assume that Cauchy - Riemann Conditions are satisfied and that the partial derivatives of $u(x,y)$ & $v(x,y)$ are continuous. Then, we can prove that ~~~~ $\frac{df}{dz}$ exists.

We have

$$\delta f = \delta u + i\,\delta v$$

$$= \frac{\delta u}{\delta x}\delta x + \frac{\delta u}{\delta u}\delta y + i\left(\frac{\delta v}{\delta x}\delta x + \frac{\delta v}{\delta y}\delta y\right)$$

$$= \left(\frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}\right)\delta x + \left(\frac{\partial u}{\partial y} + i\frac{\partial v}{\partial y}\right)\delta y$$

Then,

$$\frac{\delta f}{\delta z} = \frac{\left(\frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}\right)\delta x + \left(\frac{\partial u}{\partial y} + i\frac{\partial v}{\partial y}\right)\delta y}{\delta x + i\delta y}$$

$\delta z \to 0$

$$= \frac{\frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} + \left(\frac{\partial u}{\partial y} + i\frac{\partial v}{\partial y}\right)\frac{\delta y}{\delta x}}{1 + \frac{\delta y}{\delta x}} \quad \longrightarrow \text{①}$$

Using cauchy Riemann conditions,

$$\frac{\partial u}{\partial y} + i\frac{\partial v}{\partial y} = -\frac{\partial u}{\partial x} + i\frac{\partial u}{\partial x}$$

$$= i\left(\frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}\right)$$

Then,

$$\frac{\delta f}{\delta z} = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}$$

Since the above relation doesn't depend on the direction of approach, derivative $\frac{df}{dz}$ exists.

A function $f(z)$ is analytic at $z = z_0$ if $f(z)$ is differentiable at $z = z_0$, otherwise, z_0 is a singular point.

Ex 1:- $f(z) = z^2 = u(x,y) + i\,v(x,y)$

$$u(x,y) = x^2 - y^2, \quad v(x,y) = 2xy$$

$$\frac{\partial u}{\partial x} = 2x = \frac{\partial v}{\partial y}, \quad \frac{\partial u}{\partial y} = -2y = -\frac{\partial u}{\partial x}$$

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$f(z) = z$ is analytic in the entire complex plane.

Ex 2:- $f(z) = z^*$

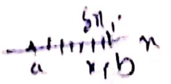
$u(x,y) = x \; , \; v(x,y) = -y$

Now,

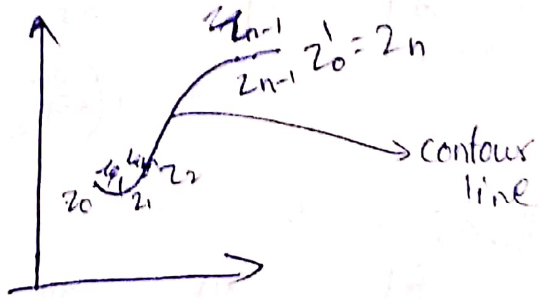
$$\frac{\partial u}{\partial x} = 1 \neq \frac{\partial v}{\partial y} = -1$$

$f(z) = z^*$ is not an analytic function.

Contour Integrals:-

$$\int_a^b f(x)\, dx$$ 

$$\int_{z_0}^{z_0'} f(z)\, dz$$



$$\sum_i f(x_i)\, \delta x_i \quad \underset{n \to \infty}{\delta x \to 0}$$

Let the points z_0 & z_0' are joined by a line which is called contour line. We divide the contour line into n segments by picking $n-1$ intermediate points $z_1, z_2, z_3 \ldots z_{n-1}$. ξ_j is a point on the contour line which lies between z_{j-1} & z_j. Now, consider the sum

$$S_i = \sum_{j=1}^{n} f(\xi_j)(z_j - z_{j-1})$$

In the limit $n \to \infty$ and $|z_j - z_{j-1}| \to 0$ if $\lim S_n$ exists and is independent of the details of choosing z_j and ζ_j then we write

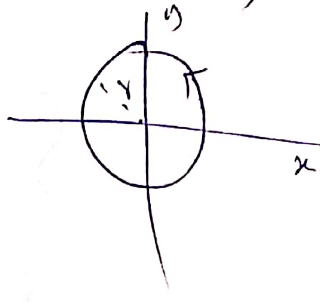
The integral as

$$\int_{z_0}^{z_0'} f(z)\,dz = \lim_{n\to\infty} \sum_{j=1}^{n} f(\zeta_j)(z_j - z_{j-1})$$

Consider the contour integral $\int_C z^n dz$, where C is a circle of radius $r>0$ around the origin $z=0$. the contour is in counter-clockwise direction.

Any point on the contour can be written as,

$$z = re^{i\theta}$$
$$dz = ire^{i\theta}d\theta$$



$$\int z^n dz = \int_0^{2\pi} r^n e^{in\theta} \cdot ire^{i\theta}\,d\theta$$
$$= i r^{n+1} \int_0^{2\pi} e^{i(n+1)\theta}\,d\theta$$

If $n \neq -1$,

$$\int_C z^n dz = i r^{n+1} \left[\frac{e^{i(n+1)\theta}}{i(n+1)}\right]_0^{2\pi} = \frac{r^{n+1}}{n+1}\left[e^{i(n+1)2\pi} - 1\right]$$
$$= 0$$

If $n=-1$,

$$\int_C z^n dz = i\int_0^{2\pi} d\theta = 2\pi i.$$

We have

$$\int_C \frac{dz}{z} = 2\pi i = \oint_C \frac{dz}{z}$$

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Cauchy's Integral Theorem:-

If $f(z)$ is analytic and its partial derivatives are continuous throughout the ~~the~~ some simply connected region R, then for every closed path 'C' in R its integral is

$$\oint_C f(z)\,dz = 0$$

This integral theorem can be proved with stokes theorem.

$$\oint_C \vec{A}\,d\vec{r} = \int_S (\vec{\nabla} \times \vec{A})\,d\vec{s}$$

For $\vec{A} = A_x i + A_y j$, $\vec{dr} = dx\hat{i} + dy\hat{j}$

we get

$$ds = dx\,dy\,\hat{k}$$

$$(\vec{\nabla} \times \vec{A})_3 = \frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y}$$

stokes theorem becomes

$$\oint A_x dx + A_y dy = \int \left(\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y}\right) dx\,dy$$

For complex function $f(z)$, we have (take),

$$f(z) = u(x,y) + i\,v(x,y) \qquad dz = dx + i\,dy$$

$$\oint_C f(z)\,dz = \oint (u+iv)(dx+idy)$$

$$= \oint \underset{A_x}{u\,dx} - \underset{A_y}{v\,dy} + i\oint v\,dx + u\,dy$$

$$= \int\left(-\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right)dx\,dy$$

$$+\, i\int\left(\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y}\right)dx\,dy$$

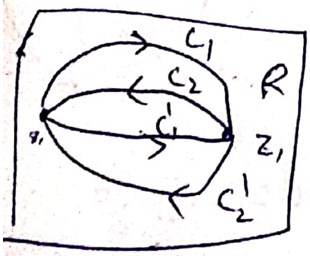
since $f(z)$ is analytic, we have,

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \quad \frac{\partial u}{\partial y} = -\frac{\partial u}{\partial x}$$

We get, $\oint f(z)\,dz = 0$

Cauchy's integral theorem is generalised by Goursay, where existence of partial derivatives of u & v are relaxed.

According to Cauchy-Goursat theorem if $f(z)$ is analytic in a region R, then for any closed path c in it, the integral is $\oint_c f(z)\,dz = 0$



$$\Rightarrow \oint_C f(z)\,dz = 0 \Rightarrow \int_{C_1} f(z)\,dz + \int_{C_2} f(z)\,dz = 0$$

or, also,

$$\int_{C_1'} f(z)\,dz + \int_{C_2'} f(z)\,dz = 0$$

Since the relation is satisfied for any closed contour, joining z_1 & z_2, we can write

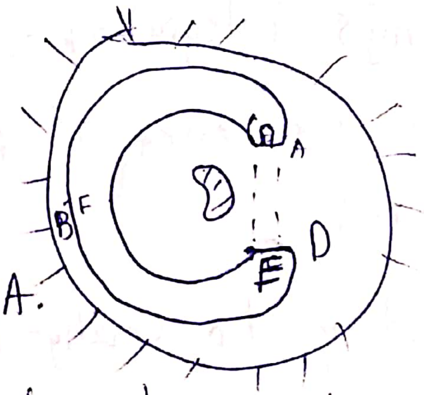
$$\int_{C_1} f(z)\,dz = \int_{z_1}^{z_2} f(z)\,dz = g(z_2) - g(z_1)$$

$$\int_{C_2} f(z)\,dz = \int_{z_2}^{z_1} f(z)\,dz = g(z_1) - g(z_2)$$

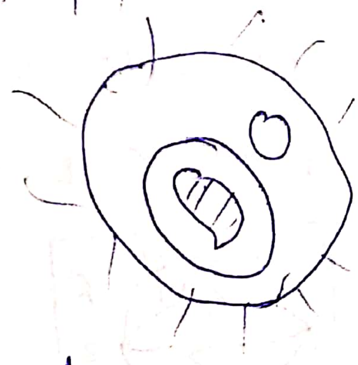
g(z) is some function

Multiply Connected Regions

Consider the following region in the complex plane, where $f(z)$ is analytic in the unshaded region.



In the unshaded region, let us consider a closed contour c' which is ABDEFGA.



ince $f(z)$ is analytic within and on c',
ɔy Cauchy's integral theorem,

$$\oint_{C'} f(z)\,dz = 0$$

$$\Rightarrow \int_{ABD} f(z)\,dz + \int_{DE} f(z)\,dz + \int_{EFG} f(z)\,dz + \int_{GA} f(z)\,dz = 0$$

the limit that the lines DE and GA are nciding each other, we have,

$$\int_{DE} f(z)\,dz = \int_{D}^{E} f(z)\,dz = -\int_{G}^{A} f(z)\,dz = -\int_{GA} f(z)\,dz$$

In the limit that $A \to D$ & $G \to E$, we can represent ABD with contour C_1 & EFG with $-C_2$

$$\oint_{C_1} f(z)\,dz + \oint_{-C_2} f(z)\,dz = 0$$

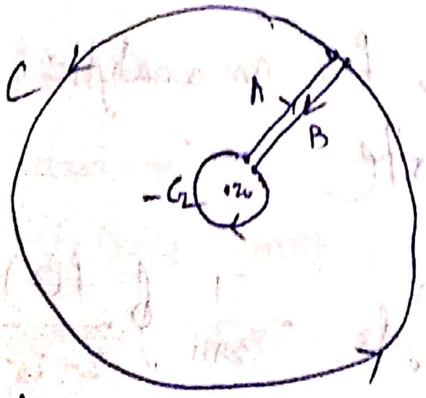
$$\Rightarrow \oint_{C_1} f(z)\,dz = \oint_{C_2} f(z)\,dz$$

Cauchy's Integral Formula

If $f(z)$ is analytic on a closed contour C and within the interior region bounded by C then

$$\frac{1}{2\pi i} \oint_{C} \frac{f(z)}{z - z_0}\,dz = f(z_0)$$

where z_0 is some interior point bounded by C



$$\underset{\text{(deformed)}}{C'} = C + (-C_2) + A + B$$

We can deform the contour C as above C_2 is a circular contour of radius r. In the limit that line A approaching line B, we can write

$$\oint_C \frac{f(z)}{z-z_0}\,dz = \oint_{C_2} \frac{f(z)}{z-z_0}\,dz$$

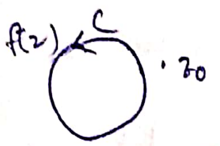
For points on contour C_2, we can write $z = z_0 + re^{i\theta}$

$$\oint_C \frac{f(z)}{z-z_0}\,dz = \int_0^{2\pi} \frac{f(z_0+re^{i\theta})}{re^{i\theta}}\, i r e^{i\theta}\, d\theta$$

In the limit $r \to 0$, we get

$$\oint_C \frac{f(z)}{z-z_0}\,dz = \int_0^{2\pi} f(z_0)\, i\, d\theta = 2\pi i f(z_0)$$

For any point z_0, we can write



$$\frac{1}{2\pi i}\oint_C \frac{f(z)}{z-z_0}\,dz = \begin{cases} f(z_0), & \text{if } z_0 \text{ is interior of } C \\ 0, & \text{if } z_0 \text{ is exterior of } C. \end{cases}$$

<u>Derivatives</u>

Using Cauchy's Integral formula, for an analytic function $f(z)$, we can write

$$\frac{f(z_0+\delta z_0)-f(z_0)}{\delta z_0} = \frac{-1}{\delta z_0}\left[\frac{1}{2\pi i}\oint \frac{f(z)}{z-z_0}\,dz - \frac{1}{2\pi i}\oint \frac{f(z)}{z-z_0-\delta z_0}\,dz\right]$$

$$= \frac{1}{2\pi i\,\delta z_0}\oint\left(\frac{f(z)}{z-z_0-\delta z_0} - \frac{f(z)}{z-z_0}\right)dz$$

$$= \frac{1}{2\pi i \delta z_0} \oint \frac{\delta z_0 f(z)}{(z - z_0 - \delta z_0)(z - z_0)} dz$$

$$\frac{f(z_0 + \delta z_0) - f(z_0)}{\delta z_0} = \frac{1}{2\pi i \delta z_0} \oint \frac{\delta z_0 f(z)}{(z - z_0 - \delta z_0)(z - z_0)} dz$$

The derivative of $f(z)$ at z_0 is

$$f'(z_0) = \lim_{\delta z_0 \to 0} \frac{f(z_0 + \delta z_0) - f(z_0)}{\delta z_0}$$

$$= \frac{1}{2\pi i} \oint \frac{f(z)}{(z - z_0)^2} dz$$

second derivative of $f(z)$ at z_0 is

$$f^{(2)}(z_0) = \lim_{\delta z_0 \to 0} \frac{f'(z_0 + \delta z_0) - f'(z_0)}{\delta z_0}$$

$$= \frac{2}{2\pi i} \oint \frac{f(z)}{(z - z_0)^3} dz$$

By method of induction, for any positive integer n, we can show that

$$f^{(n)}(z_0) = \frac{n!}{2\pi i} \oint \frac{f(z)}{(z - z_0)^{n+1}} dz$$

If $f(z)$ is analytic on and within a closed contour c then derivatives of all orders of $f(z)$ can exist at any interior ~~part~~ point z_0. Moreover all these derivatives are analytic.

Morera's Theorem:-

If a function $f(z)$ is continuous in a simply connected region R and $\oint f(z)\,dz = 0$ for every closed contour C within R, then $f(z)$ is analytic throughout R.

since, $\oint_C f(z)\,dz = 0$ for every closed contour C within R, we can state that the integral $\int_{z_1}^{z_2} f(z)\,dz$ depends only on the end points z_1 & z_2

$$\int_{z_1}^{z_2} f(z)\,dz = g(z_2) - g(z_1)$$

where $g(z)$ is some continuous function.

Now consider

$$\frac{g(z_2)-g(z_1)}{z_2-z_1} - f(z_1) = \frac{1}{z_2-z_1}\int_{z_1}^{z_2} f(z')\,dz'$$

$$= \frac{1}{z_2-z_1} = \int_{z_1}^{z_2} f(z_1)\,dz'$$

$$\frac{g(z_2)-g(z_1)}{z_2-z_1} - f(z_1) = \frac{1}{z_2-z_1}\int_{z_1}^{z_2} \left(f(z')-f(z_1)\right) dz'$$

$z_2 \to z_1$
we can show

$$\left|\int_C f(z)\,dz\right| \leq |f|_{max}\, L.$$

where $|f|_{max}$ is the max value of $|f(z)|$ on the contour C and L is the length of the contour now.

$$\left|\frac{\int_{z_1}^{z_2} f(z') - f(z_1)\, dz'}{z_2 - z_1}\right| \leq \frac{|f(z') - f(z_1)|_{max}}{|z_2 - z_1|} L(z_2)$$

In the limit $z_2 \to z_1$, $|f(z') - f(z_1)|_{max} \to |f(z_1) - f(z_1)| = 0$

and $\frac{L(z_2)}{|z_2 - z_1|} \simeq 1$

$\therefore$ We get

$$\lim_{z_2 \to z_1} \frac{\int_{z_1}^{z_2} (f(z') - f(z_1))\, dz'}{z_2 - z_1} = 0$$

From eq(1),

$$\lim_{z_2 \to z_1} \frac{g(z_2) - g(z_1)}{z_2 - z_1} = f(z_1)$$

$$g'(z)\Big|_{z = z_1} = f(z_1)$$

$g(z)$ is analytic at $z = z_1$

Since z_1 is any point in the region R, $g(z)$ is analytic in R.

Using Cauchy's integral formula, we can state that its first derivative $g'(z)$ is also analytic in R.

$\therefore$ $f(z)$ is analytic in R.

Taylor expansion:-

Consider a function $f(z)$ which is analytic at $z=z_0$. Suppose we know that z_1 is a nearest point to z_0 at which $f(z)$ is not analytic. Then for any point in the region $|z-z_0|<|z_1-z_0|$, we can expand $f(z)$ as

$$f(z)=\sum_{n=0}^{\infty}(z-z_0)^n \frac{f^{(n)}(z_0)}{n!}.$$

Using Cauchy's Integral formula, for any point z within C, we can write

$$f(z)=\frac{1}{2\pi i}\oint_C \frac{f(z')}{z'-z}dz'$$

$$=\frac{1}{2\pi i}\oint_C \frac{f(z')}{z'-z_0-(z-z_0)}dz'$$

$$=\frac{1}{2\pi i}\oint_C \frac{f(z')\,dz'}{(z'-z_0)\left[1-\frac{z-z_0}{z'-z_0}\right]} \quad -(1)$$

We can prove that

$$\frac{1}{1-t}=1+t+t^2+\ldots=\sum_{n=0}^{\infty}t^n$$

The above series converges for $|t|<1$,

Now, we have

$$\left|\frac{z-z_0}{z'-z_0}\right|<1$$

$$\frac{1}{1-\frac{z-z_0}{z'-z_0}} = \sum_{n=0}^{\infty}\left(\frac{z-z_0}{z'-z_0}\right)^n$$

substituting this in eq(1),

$$f(z) = \frac{1}{2\pi i}\oint_C \sum_{n=0}^{\infty} \frac{(z-z_0)^n f(z')\,dz'}{(z'-z_0)^{n+1}}$$

$$= \frac{1}{2\pi i}\oint \sum_{n=0}^{\infty}(z-z_0)^n \oint_C \frac{f(z')}{(z'-z_0)^{n+1}}\,dz'$$

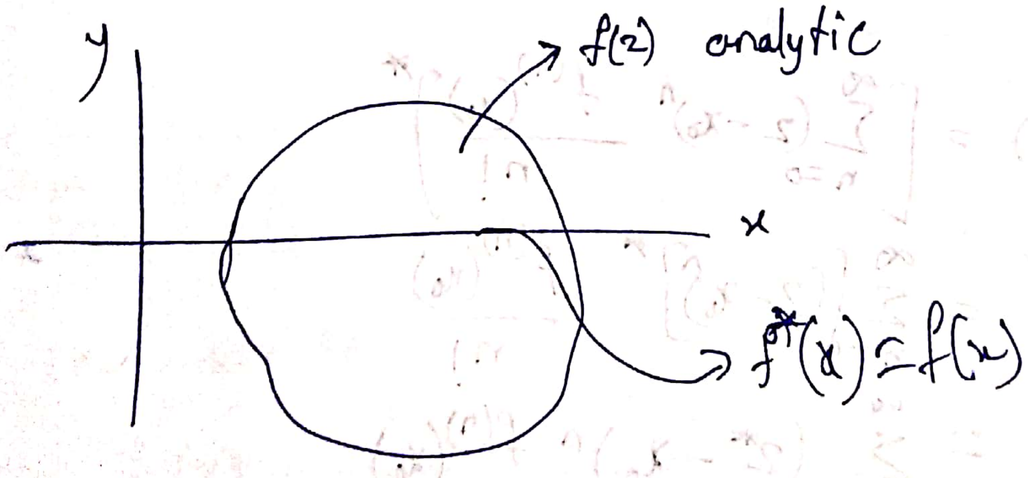
$$f(z) = \sum_{n=0}^{\infty}(z-z_0)^n \frac{f^{(n)}(z_0)}{n!}$$

Hence proved!!!!

Schwarz reflection principle:-

The function $f(z)$ is (1) analytic over some region including the real axis and (2) real when z is real, then,

$$f^*(z) = f(z^*)$$



For some point x_0 on the real axis, we can make Taylor expansion for $f(z)$ as

$$f(z) = \sum_{n=0}^{\infty} (z-x_0)^n \frac{f^{(n)}(x_0)}{n!}$$

Since $f(x_0)$ is real, we can argue that $f^{(n)}(x_0)$ must be real for all n.

$$f^{(1)}(x_0) = \lim_{\delta z_0 \to 0} \frac{f(x+\delta z_0) - f(x_0)}{\delta z_0}$$

$$= \lim_{\delta x_0 \to 0} \frac{f(x_0+\delta x_0) - f(x_0)}{\delta x_0} \quad ; \; \delta y_0 = 0$$

$$\Rightarrow f'(x_0) \text{ is real}$$

$$f^{(2)}(x_0) = \lim_{\delta z_0 \to 0} \frac{f'(x_0+\delta z_0) - f'(x_0)}{\delta z_0}$$

Following this, one can show that $f^{(n)}(x_0)$ is real.

Now,

$$f^*(z) = \left[\sum_{n=0}^{\infty} (z-x_0)^n \frac{f^{(n)}(x_0)}{n!}\right]^*$$

$$= \sum_{n=0}^{\infty} \left[(z-x_0)^n\right]^* \frac{f^{(n)}(x_0)}{n!}$$

$$= \sum_{n=0}^{\infty} (z^*-x_0)^n \frac{f^{(n)}(x_0)}{n!}$$

$$= f(z^*)$$

Analytic Continuation :-

Consider the following function

$$f(z) = \frac{1}{1+z}$$

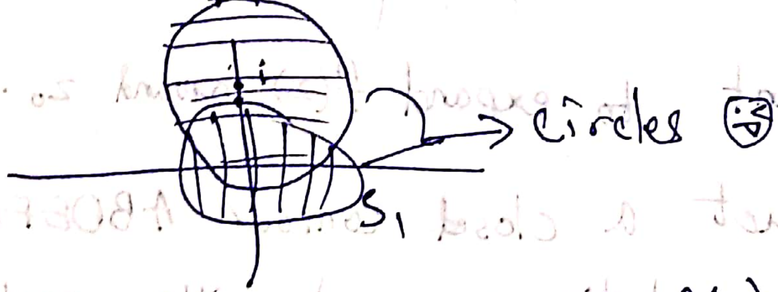
This function has singularity at $z=-1$

For $|z|<1$

we can expand $f(z)$ as

$$f(z) = \frac{1}{1-(-z)} = \sum_{n=0}^{\infty} (-z)^n \qquad \text{--- (1)}$$

The series expansion is valid in the region S_1



Suppose we want to expand $f(z)$ around $z=i$, then we can write

$$f(z) = \frac{1}{1+z} = \frac{1}{1+i+(z-i)}$$

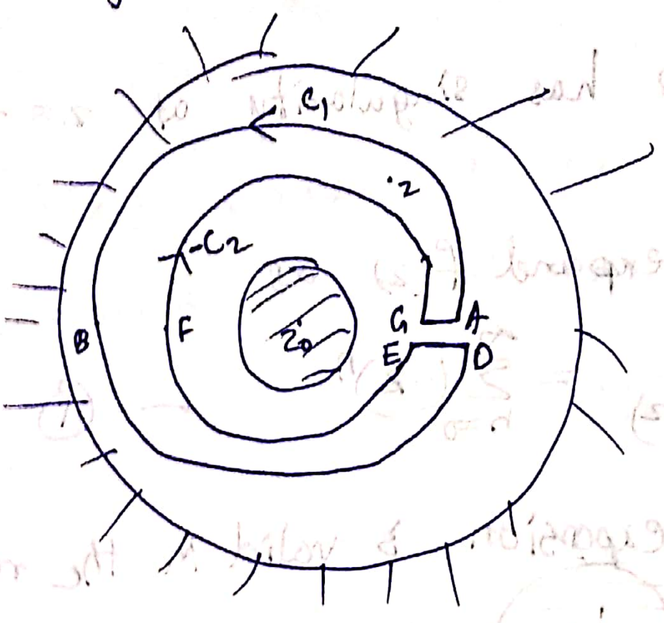
$$= \frac{1}{1+i} \times \frac{1}{1+\frac{z-i}{1+i}}$$

For $\left|\frac{z-i}{1+i}\right| < 1$ i.e, $|z-i| < |1+i| = \sqrt{2}$, we can write

$$f(z) = \frac{1}{1+i} \sum_{n=0}^{\infty} (-1)^n \left(\frac{z-i}{1+i}\right)^n \qquad \text{--- (2)}$$

Laurent Series

Suppose a function $f(z)$ is analytic in the annular region



We want to expand $f(z)$ around z_0.

We construct a closed contour ABDEFGA, where $f(z)$ is analytic on and within contour.

Applying Cauchy's integral formula and in the limit DE approaching GA, we get

$$f(z) = \frac{1}{2\pi i} \oint_{C_1} \frac{f(z')}{z'-z} dz' - \frac{1}{2\pi i} \oint_{C_2} \frac{f(z')}{z'-z} dz'$$

z' on contour

On C_1 we have $|z'-z_0| > |z-z_0|$. On C_2 we have $|z'-z_0| < |z-z_0|$.

$$f(z) = \frac{1}{2\pi i} \oint_{C_1} \frac{f(z')\, dz'}{(z'-z_0)\left[1 - \frac{z-z_0}{z'-z_0}\right]} + \frac{1}{2\pi i} \oint_{C_2} \frac{f(z')\, dz'}{(z-z_0)\left[1 - \frac{z'-z_0}{z-z_0}\right]}$$

$$= \frac{1}{2\pi i} \oint_{C_1} \sum_{n=0}^{\infty} \frac{f(z')}{z'-z_0} \left(\frac{z-z_0}{z'-z_0}\right)^n dz'$$

$$+ \frac{1}{2\pi i} \oint_{C_2} \sum_{m=0}^{\infty} \frac{f(z')}{z-z_0} \left(\frac{z'-z_0}{z-z_0}\right)^m dz'$$

$$= \frac{1}{2\pi i} \sum_{n=0}^{\infty} (z-z_0)^n \oint_{C_1} \frac{f(z')}{(z'-z_0)^{n+1}} dz'$$

$$+ \frac{1}{2\pi i} \sum_{m=0}^{\infty} (z-z_0)^{-(m+1)} \oint_{C_2} f(z')(z'-z_0)^m dz'$$

$$= S_1 + S_2 \quad \text{—(1)}$$

Now,

$$S_2 = \frac{1}{2\pi i} \sum_{m=0}^{\infty} (z-z_0)^{-(m+1)} \oint_{C_2} f(z')(z'-z_0)^m dz'$$

put $m+1 = l$

$$S_2 = \frac{1}{2\pi i} \sum_{l=1}^{\infty} (z-z_0)^{-l} \oint_{C_2} f(z')(z'-z_0)^{l-1} dz'$$

put $l = -n$,

$$S_2 = \frac{1}{2\pi i} \sum_{n=-\infty}^{-1} (z-z_0)^n \oint_{C_2} \frac{f(z')}{(z'-z_0)^{n+1}} dz'$$

substituting this in eq (1),

$$f(z) = \frac{1}{2\pi i} \sum_{n=0}^{\infty} (z-z_0)^n \oint_{C_1} \frac{f(z')}{(z'-z_0)^{n+1}} dz'$$

$$+ \frac{1}{2\pi i} \sum_{n=-\infty}^{-1} (z-z_0)^n \oint_{C_2} \frac{f(z')}{(z'-z_0)^{n+1}} dz'$$

$$= \sum_{n=-\infty}^{\infty} a_n (z-z_0)^n \; ; \quad a_n = \frac{1}{2\pi i} \oint_C \frac{f(z')}{(z'-z_0)^{n+1}} dz' \neq f^{(n)}(z_0)$$

This is *Laurent Series* of $f(z)$ around z_0.

Poles:- Suppose $f(z)$ is not analytic at z_0

We can express $f(z)$ as

$$f(z) = \sum_{n=-\infty}^{\infty} a_n (z-z_0)^n$$

In this expansion if $a_n = 0$ for $n < -m < 0$, then z_0 is called a pole of order m.

If the first non-vanishing term of $f(z)$ is $n = -1$ then z_0 is called 'a simple pole.

The coefficient a_{-1} in the expansion of $f(z)$ is called residue of $f(z)$ at $z = z_0$.

If, in the expansion of $f(z)$, the non-vanishing terms are upto $n = -\infty$ then z_0 is called essential singularity

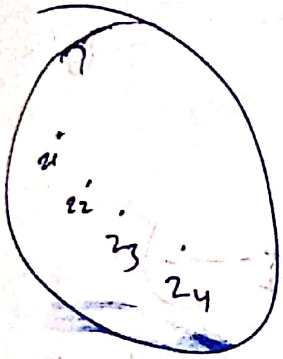
$$f(z) = \frac{a_{-m}}{(z-z_0)^m} + \cdots + \frac{a_{-1}}{z-z_0} + a_0 + a_1(z-z_0) + \cdots$$

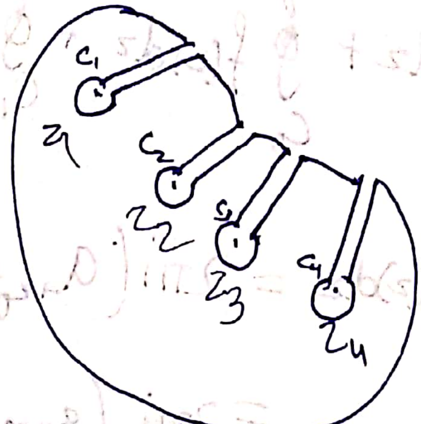
$(z-z_0)^m f(z) \longrightarrow$ no singularity

Residue Theorem:-

Suppose $f(z)$ has some isolated singularities in the complex plane. For a contour encircling these singularities, the integral is,

$$\oint_C f(z)\,dz = 2\pi i \text{ (Sum of enclosed residues)}$$





We deform the original contour so the each isolated singular point is closed by an infinitesimally small circular contours which is shown above.

Around isolated singular point z_i we can expand $f(z)$ as

$$f(z) = \sum_{n=-\infty}^{\infty} a_n (z-z_i)^n$$

For a circular contour c centered at z_i, we have

$$\oint_C (z-z_i)^n \, dz = \begin{cases} 0 & \text{if } n \neq -1 \\ 2\pi i & \text{if } n = -1 \end{cases}$$

For contours C_i,

$$\oint f(z)dz = \cancel{-2\pi i \, a_{-1 z_i}} \quad i = 0, 1, \ldots$$
$$-2\pi i \, a_{-1 z_i}$$

Here $a_{-1 z_i}$ is residue of $f(z)$ at $z = z_i$

Applying Cauchy's integral theorem to the deformed contour and in the limit straight lines approaching, we get

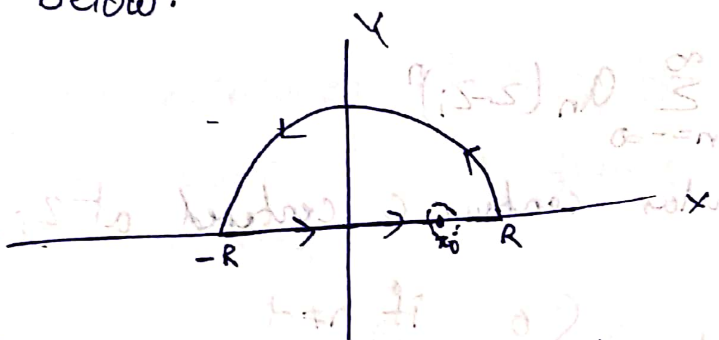
$$\oint_C f(z)dz + \oint_{C_0} f(z)dz + \oint_{C_1} f(z)dz + \cdots = 0$$

$$\Rightarrow \oint_C f(z)dz = 2\pi i\left(a_{-1z_0} + a_{-1z_1} + a_{-1z_2} + \cdots\right)$$

$$= 2\pi i \text{ (sum of enclosed singularities)}$$

Cauchy Principle Value :-

Suppose $f(z)$ has isolated singular point x_0 (simple point) lying on the contour as shown below.



We deform the original contour by making a semicircle around x_0.

$f(z)$ around x_0 can be written as ~~f(z)=~~

$$f(z) = \frac{a_{-1}}{z-x_0} + a_0 + a_1(z-x_0) + \cdots$$

On C_{x_0}, $z = x_0 + \delta e^{i\phi}$

If C_{x_0} is clockwise,

$$\int_{C_{x_0}} f(z)dz = a_{-1}\int_{C_{x_0}} \frac{dz}{z-x_0} + a_0\int_{C_{x_0}} dz + \cdots$$

$$= a_{-1}\int_{\pi}^{0} \frac{\delta e^{i\phi}\, i\, d\phi}{\delta e^{i\phi}} + a_0\int_{\pi}^{0} \delta e^{i\phi}\, i\, d\phi + \cdots$$

In the limit $\delta \to 0$,

$$\oint_{C_{x_0}} f(z)\, dz = -a_{-1}\pi i$$

If C_{x_0} is counter clockwise

$$\oint_{C_{x_0}} f(z) dz = a_{-1}\int_{C_{x_0}} \frac{dz}{z-x_0} = a_{-1}\int_{\pi}^{2\pi} \frac{\delta e^{i\theta} i d\theta}{\delta e^{i\theta}} = a_{-1}\pi i$$

In the limit $R \to \infty$, from reside theorem,

$$\oint f(z) dz = 2\pi i \sum \text{enclosed residue}$$

$$\Rightarrow \int_{\substack{\text{infinite}\\ \text{semicircle}}} f(z) dz + \int_{-\infty}^{x_0-\delta} f(x) dx + \int_{C_{x_0}} f(z) dz + \int_{x_0+\delta}^{\infty} f(x) dx$$
$$= 2\pi i \left(\sum \text{enclosed residue}\right)$$

The cauchy principle value of $\int_{-\infty}^{\infty} f(x) dx$ is defind as

$$P\int_{-\infty}^{\infty} f(x) dx = \lim_{\delta \to 0} \int_{-\infty}^{x_0-\delta} f(x) dx + \int_{x_0+\delta}^{\infty} f(x) dx$$

If C_{x_0} is clockwise,

$$\int_{\substack{\text{infinite}\\ \text{semicircle}}} f(z) dz + P\int_{-\infty}^{\infty} f(x) dx - a_{-1}\pi i = 0$$

If C_{x_0} is counter-clockwise

$$\int_{\substack{\text{infinite}\\ \text{semicircle}}} f(z) dz + P\int_{-\infty}^{\infty} f(x) dx + a_{-1}\pi i = 2\pi i a_{-1}$$

* Integrals of the form $\int_0^{2\pi} f(\sin\theta, \cos\theta)\, d\theta$

1) Consider $I = \int_0^{2\pi} f(\sin\theta, \cos\theta)\, d\theta$

For $z = e^{i\theta}$, $dz = ie^{i\theta}d\theta = iz\,d\theta$

$$d\theta = -i\frac{dz}{z}$$

$$\sin\theta = \frac{e^{i\theta} - e^{-i\theta}}{2i} = \frac{z - z^{-1}}{2i}$$

$$\cos\theta = \frac{z + z^{-1}}{2}$$

Integral becomes,

$$I = -i \oint_{\text{unit circle}} f\left(\frac{z - z^{-1}}{2i}, \frac{z + z^{-1}}{2}\right) \frac{dz}{z}$$

By residue theorem,

$$I = (-i)(2\pi i) \sum \text{residue with unit circle.}$$

Example:- $I = \int_0^{2\pi} \frac{d\theta}{1 + \epsilon\cos\theta}$ $|\epsilon| < 1$

Using previous method, we can write

$$I = \oint_{\text{unit circle}} \frac{1}{1 + \epsilon\left(\frac{z + z^{-1}}{2}\right)} \left(-i\frac{dz}{z}\right)$$

$$= \oint_{\text{unit circle}} \frac{-2i}{\epsilon z^2 + 2z + \epsilon}\, dz$$

$z^2 + \frac{2}{\epsilon}z + 1 = 0$

$\Rightarrow z_- = \frac{-1}{\epsilon} - \frac{1}{\epsilon}\sqrt{1-\epsilon^2}$, $z_+ = \frac{-1}{\epsilon} + \frac{1}{\epsilon}\sqrt{1-\epsilon^2}$

For $|\epsilon|<1$, z_- lies outside the circle & z_+ lies inside the circle.

$$f(z) = \frac{1}{z^2+\frac{2}{\epsilon}z+1} = \frac{1}{(z-z_-)(z-z_+)}$$

$$= \frac{1}{z-z_+}\left[\frac{1}{(z-z_+)-(z_--z_+)}\right]$$

$$= \frac{1}{(z-z_+)(z_--z_+)} \cdot \frac{1}{\left[1+\frac{z-z_+}{z_+-z_-}\right]}$$

$$= \frac{1}{(z-z_+)(z_--z_+)}\left[1-\left(\frac{z-z_+}{z_+-z_-}\right)+\left(\quad\right)^2+\cdots\right]$$

The residue of $f(z)$ at $z=z_+$ is (coeff of $(z-z_+)^{-1}$)

$$\frac{1}{z_+-z_-}.$$

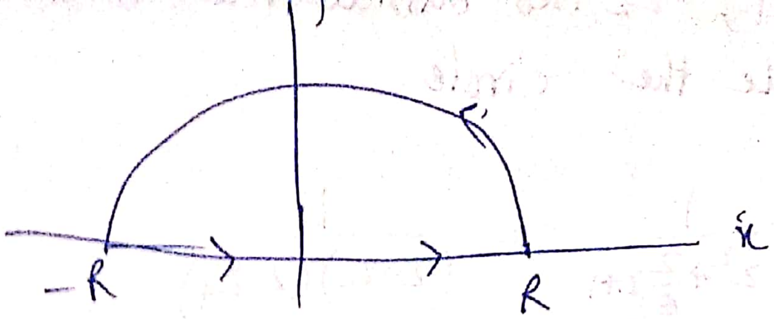
$$I = -i\frac{2}{\epsilon}(2\pi i)\left(\frac{1}{z_+-z_-}\right) = \frac{2\pi}{\sqrt{1-\epsilon^2}}$$

2) Consider $I = \int_{-\infty}^{\infty} f(x)\,dx$ where the function satisfies the two conditions:

i) $f(z)$ has finite number of poles

ii) $f(z)$ vanishes as $\frac{1}{z^2}$ for $|z|\to\infty$, $0 \le \arg(z) \le \pi$

Let's choose a semicircular contour in the upper half plane.



From residue theorem,

$$\oint f(z)\,dz = 2\pi i \sum \text{residue in upper half plane.}$$

$$\Rightarrow \lim_{R\to\infty} \int_0^{\pi} f(Re^{i\theta})\, Re^{i\theta}\, d\theta + \lim_{R\to\infty} \int_{-R}^{R} f(x)\,dx$$

$$= 2\pi i \sum \text{residues in upper half plane}$$

for $R \to \infty$,

$$f(Re^{i\theta})R \to \frac{1}{R^2}\cdot R \to 0.$$

we get,

$$\int_{-\infty}^{\infty} f(x)\,dx = 2\pi \sum \text{residues in upper half plane}$$

<u>Example</u>:- $I = \displaystyle\int_{-\infty}^{\infty} \frac{dx}{1+x^2}$

Consider $f(z) = \dfrac{1}{1+z^2} \approx \dfrac{1}{z^2}$ as $z \to \infty$

$$f(z) = \frac{1}{z^2 - i^2} = \frac{1}{(z-i)(z+i)}$$

we can show $f(z)$ has simple poles at $i, -i$ with residues $\frac{1}{2i}$, $\frac{-1}{2i}$

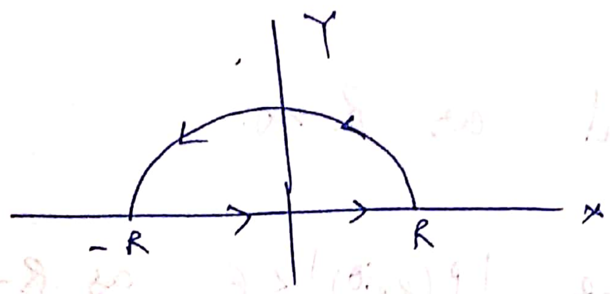
Applying previous method,

$$\int_{-\infty}^{\infty} \frac{dx}{1+x^2} = 2\pi i \left(\frac{1}{2i}\right)$$

3) Consider $I = \int_{-\infty}^{\infty} f(x)\, e^{iax}\, dx$ with a real and positive, the function $f(z)$ satisfies

i) $f(z)$ has finite no. of poles

ii) $\lim_{|z|\to\infty} f(z) = 0 \qquad 0 \le \arg(z) \le \pi$



$$\oint f(z)\, e^{iaz}\, dz = 2\pi i \sum \text{residues}$$

$$\lim_{R\to\infty} \int_{-R}^{R} f(x)\, e^{iax}\, dx + \lim_{R\to\infty} \int_{0}^{\pi} f(Re^{i\theta})\, e^{ia(R\cos\theta + Ri\sin\theta)}\, Re^{i\theta} d\theta$$

$$= 2\pi i \sum \text{residues}.$$

$$* \int_{-\infty}^{\infty} f(x)\, e^{iax}\, dx$$

$$\oint f(z)\, e^{iaz}\, dz = \left(\sum \text{residues}\right) 2\pi i$$

$$\Rightarrow \lim_{R\to\infty} \int_{-\infty}^{\infty} f(x)\, e^{iax}\, dx + \lim_{R\to\infty} \int_{0}^{\pi} f(Re^{i\theta})\, e^{iaR(\cos\theta + i\sin\theta)}\, Re^{i\theta}\, i\, d\theta$$

$$= 2\pi i \left(\sum \text{residues}\right)$$

* $I_R = \int_0^{\pi} f(Re^{i\theta})\, e^{iaR\cos\theta - aR\sin\theta}\, Re^{i\theta}\, d\theta$

Using,

$$\left| \int f(\theta)\, d\theta \right| \leq \int |f(\theta)|\, d\theta$$

$$|I_R| \leq \int_0^{\pi} |f(Re^{i\theta})|\, e^{-aR\sin\theta}\, d\theta$$

Since,

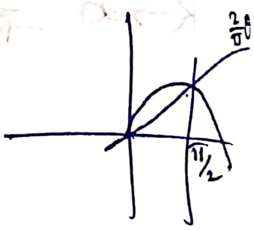
$\lim_{|z|\to\infty} f(z) = 0$, we can expect $|f(z)|$

be bounded as $R \to \infty$.

Say, ~~$\int f$~~ $|f(Re^{i\theta})| < \epsilon$ as $R \to \infty$

$$\Rightarrow |I_R| \leq \epsilon R \int_0^{\pi} e^{-aR\sin\theta}\, d\theta$$

$$\leq 2\epsilon R \int_0^{\pi/2} e^{-aR\sin\theta}\, d\theta$$



for $\theta \in [0, \frac{\pi}{2}]$, $\frac{2\theta}{\pi} \leq \sin\theta$

$$|I_R| \leq 2\epsilon R \int_0^{\pi/2} e^{-a \cdot 2\theta/\pi}\, d\theta$$

$$\leq 2\epsilon R \left(\frac{1 - e^{-aR}}{\frac{aR^2}{\pi}} \right)$$

$\therefore \lim_{R\to\infty} |I_R| \le \frac{\pi}{a}\epsilon \to 0$

$$\lim_{|z|\to\infty} \oint_{\text{semicircle}} f(z)\, e^{iaz}\, dz = 0$$

This is called Jordan's lemma.

Using this, we get,

$$\int_{-\infty}^{\infty} f(x)\, e^{iax}\, dx = 2\pi i \sum \text{residues in upper halfplane}$$

Example:- $I = \int_0^{\infty} \frac{\sin x}{x}\, dx = \frac{1}{2}\int_{-\infty}^{\infty} \frac{\sin x}{x}\, dx$

Consider $\int \frac{e^{iz}}{z}\, dz$ as the below contour

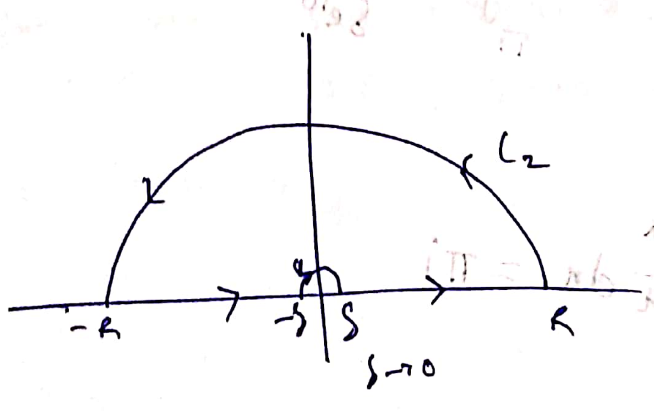
$f(z)\, e^{iaz}$

$\hookrightarrow a = 1$

$f(z) = \frac{1}{z}$

as $z \to \infty$

$f(z) \to 0$



We have,

$$\oint \frac{e^{iz}}{z}\, dz = 0$$

$$\Rightarrow \int_{C_2} \frac{e^{iz}}{z}\, dz + \int_{-R}^{-\delta} \frac{e^{ix}}{x}\, dx + \int_{C_2} \frac{e^{iz}}{z}\, dz + \int_{\delta}^{R} \frac{e^{ix}}{x}\, dx = 0$$

Using Jordan's lemma with $R \to \infty$

$$\int_{C_2} \frac{e^{iz}}{z}\, dz = 0$$

with $R \to \infty$, $\delta \to 0$

$$\int_{-R}^{-\delta} \frac{e^{ix}}{x}\, dx + \int_{\delta}^{R} \frac{e^{ix}}{x}\, dx = P\int_{-\infty}^{\infty} \frac{e^{ix}}{x}\, dx$$

~~On~~ C_1,

$\cancel{z = \delta e^{i\theta}}$

$$\int_{C_1} \frac{e^{iz}}{z}\, dz = \int_{C_1} \frac{1 + iz + (iz)^2 + \cdots}{z}\, dz$$

$dz \propto \delta$

On C_1, $z = \delta e^{i\theta}$, In the limit $\delta \to 0$,

$$\int_{C_1} \frac{e^{iz}}{z}\, dz = \int_{\pi}^{0} \frac{\delta e^{i\theta}\, i\, d\theta}{\delta e^{i\theta}} = -\pi i$$

We get

$$\int_{-\infty}^{\infty} \frac{e^{ix}}{x}\, dx = \pi i$$

Equating imaginary parts,

$$\int_{-\infty}^{\infty} \frac{\sin x}{x}\, dx = \pi.$$

Example: In Quantum Mechanics, we can have $I(\sigma) = \int_{-\infty}^{\infty} \frac{x \sin x}{x^2 - \sigma^2}\, dx$, σ is real

Consider

$$\underline{I} = \int \frac{z \sin z}{z^2 - \sigma^2}\, dz$$

$$I = \frac{1}{2i} \int \frac{z e^{iz}}{z^2 - \sigma^2}\, dz - \frac{1}{2i} \int \frac{z e^{-iz}}{z^2 - \sigma^2}\, dz$$

$$= I_1 + I_2$$

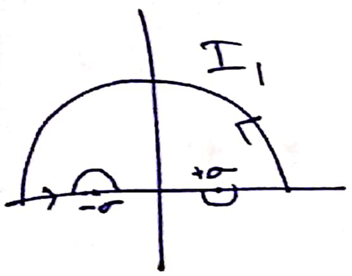
these are of the form

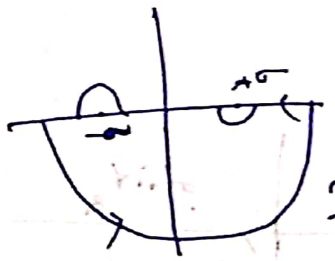
$f(z)\, e^{iaz}$

$\downarrow$

$\frac{z}{z^2 - \sigma^2} \to 0$

as $|z| \to \infty$





We can evalute this integral also by taking $\sigma \Rightarrow \sigma + i\gamma$; $\gamma \to 0$; $\gamma \in \mathbb{R}$

$$I(\sigma) = \int_{-\infty}^{\infty} \frac{x \sin x}{x^2 - \sigma^2}\, dx \quad , \sigma \in \mathbb{R}$$

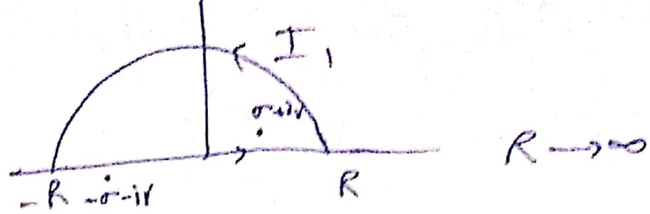
$$\underline{I} = \frac{1}{2i} \int \frac{z e^{iz}}{z^2 - \sigma^2}\, dz - \frac{1}{2i} \int \frac{z e^{-iz}}{z^2 - \sigma^2}\, dz$$

substitute $\sigma \to \sigma + i\gamma$ with $\gamma \to 0$ & $\gamma \in \mathbb{R}$

$$I = \frac{1}{2i} \int \frac{z e^{iz}\, dz}{z^2 - (\sigma + i\gamma)^2} - \frac{1}{2i} \int \frac{z e^{-iz}}{z^2 - (\sigma + i\gamma)^2}\, dz$$

$$= I_1 + I_2$$

For I_1



Using residue theorem

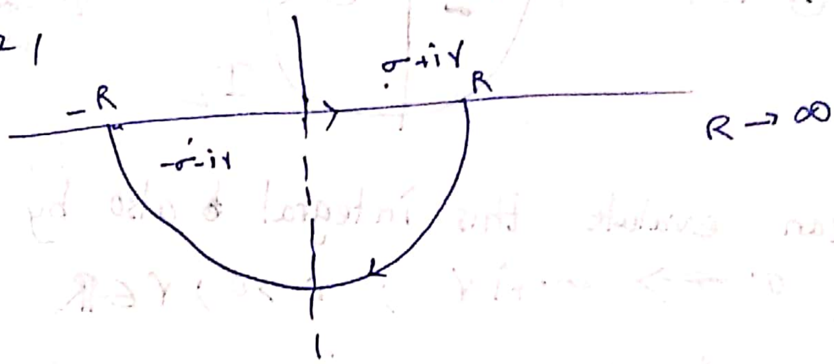
$$\frac{1}{2i}\int_{\substack{\text{infinite}\\ \text{semicircle}}} \frac{z e^{iz}}{z^2-(\sigma+ir)^2}\,dz + \frac{1}{2i}\int_{-\infty}^{\infty}\frac{x e^{ix}}{x^2-(\sigma+ir)^2}\,dx$$

$$= \frac{1}{2i}\,2\pi i\,\underbrace{\frac{e^{i(\sigma+ir)}}{2}}_{\text{residue}}$$

with $r\to 0$

$$I_1 = \frac{1}{2i}\int_{-\infty}^{\infty}\frac{x e^{ix}}{x^2-\sigma^2}\,dx = \frac{\pi}{2}e^{i\sigma}$$

For I_2,



Using Residue Theorem,

$$\frac{1}{2i}\int_{\substack{\text{infinite}\\ \text{semicircle}}} \frac{z e^{-iz}}{z-(\sigma+ir)^2}\,dz + \frac{1}{2i}\int_{-\infty}^{\infty}\frac{x e^{-ix}}{x-(\sigma+ir)^2}\,dx = \frac{1}{2i}(-2\pi i)\frac{e^{i(\sigma+ir)}}{2}$$

With $r\to 0$

$$I_2 = -\frac{1}{2i}\int_{-\infty}^{\infty}\frac{x e^{-ix}}{x^2-\sigma^2}\,dx = \frac{\pi}{2}e^{i\sigma}$$

$$I = I_1 + I_2 = \pi e^{i\sigma}$$

$\sigma \to \sigma + i\gamma \longrightarrow I = \pi e^{i\sigma}$

$\sigma \to \sigma - i\gamma \longrightarrow I = \pi e^{-i\sigma}$

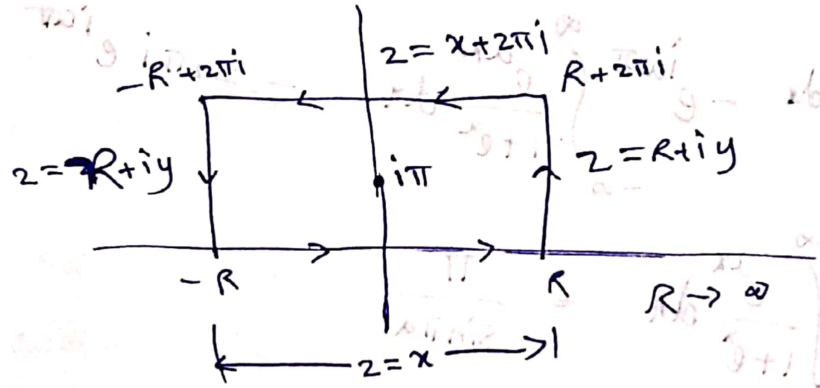
Example:- $I = \int_{-\infty}^{\infty} \frac{e^{ax}}{1+e^{x}}\, dx$; $0 < a < 1$ ~~det~~

Let $f(z) = \frac{e^{az}}{1+e^{z}}$

For singularities, $e^{z} = -1 \Rightarrow z = \log_e(-1) = \ln e^{i(2n+1)\pi}$

$z = i(2n+1)\pi$; $n \in \mathbb{Z}$

Let us choose



Residue of $f(z)$ at $i\pi$ is !

$$f(z) = \frac{e^{a(z-i\pi)}\, e^{ai\pi}}{1 + e^{z-i\pi}\, e^{i\pi}}$$

$$= \frac{e^{ia\pi}\left[1 + a(z-i\pi) + \cdots\right]}{1 - \left[1 + (z-i\pi) + \frac{(z-i\pi)^2}{2} + \cdots\right]}$$

$$= \frac{-e^{ia\pi}\left[1 + a(z-i\pi) + \cdots\right]}{(z-i\pi)\left[1 + \frac{z-i\pi}{2} + \cdots\right]} \longrightarrow 1-t \text{ (can be written)}$$

$$= \frac{-e^{ia\pi}}{z-i\pi} + \cdots$$

we look only at leading term

$\frac{1}{1-t} = 1+t+t^2+\dots$

From residue theorem,

$$\oint f(z)\,dz = 2\pi i\left(-e^{ia\pi}\right)$$

$$\Rightarrow \lim_{R\to\infty}\left[\int_{-R}^{R}\frac{e^{ax}}{1+e^{x}}\,dx + \int_{R}^{-R}\frac{e^{a(x+2\pi i)}}{1+e^{x+2\pi i}}\,dx + \int_{0}^{2\pi}\frac{e^{a(R+iy)}}{1+e^{R+iy}}\,i\,dy + \int_{2\pi}^{0}\frac{e^{(-R+iy)}}{1+e^{-R+iy}}\,i\,dy\right]$$

$$= 2\pi i\left(-e^{ia\pi}\right)$$

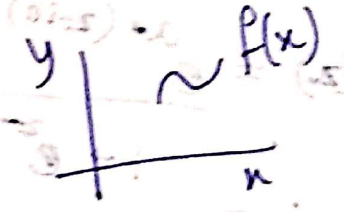
$$\Rightarrow \int_{-\infty}^{\infty}\frac{e^{ax}}{1+e^{x}}\,dx - e^{ia2\pi}\int_{-\infty}^{\infty}\frac{e^{ax}}{1+e^{x}}\,dx = -2\pi i\,e^{ia\pi}$$

$$\Rightarrow \int_{-\infty}^{\infty}\frac{e^{ax}}{1+e^{x}}\,dx = \frac{\pi}{\sin \pi a}$$

Mapping:-

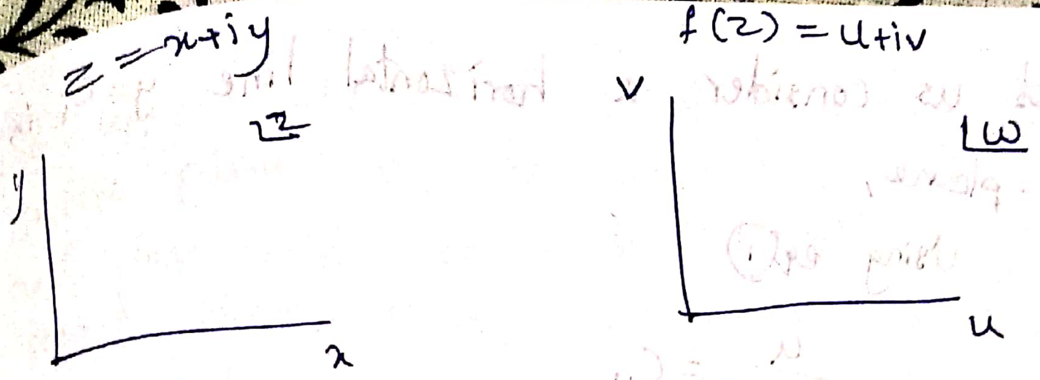
$y = f(x)$

Cartesian $\rightarrow$



We have in $\mathbb{C}$,

$f(z) = u(x,y) + i\,v(x,y)$



Let $f(z) = \frac{1}{z}$

$$u+iv = \frac{1}{x+iy}$$

$$= \frac{x-iy}{x^2+y^2}$$

$$u = \frac{x}{x^2+y^2} \qquad v = \frac{-y}{x^2+y^2}$$

$$\Rightarrow \quad x = \frac{u}{u^2+v^2} \qquad y = \frac{v}{u^2+v^2} \quad -(1)$$

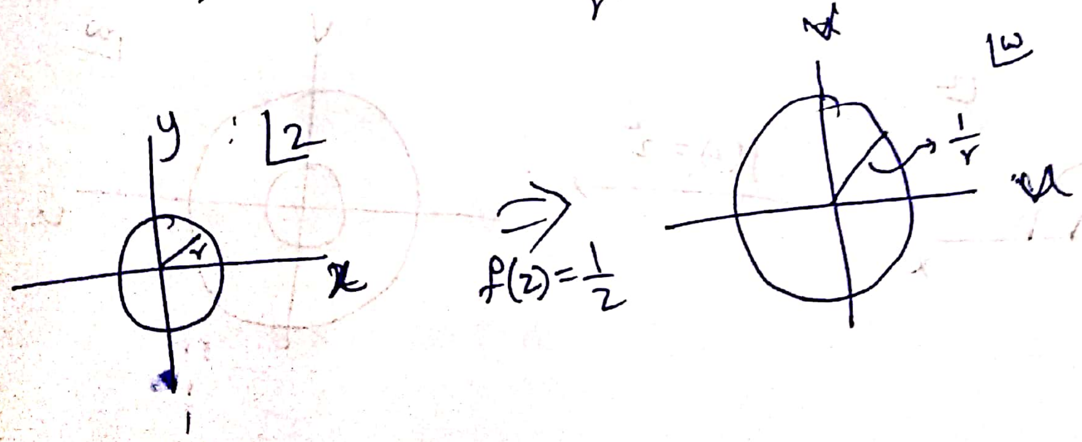
Let us consider a circle in z-plane centered at origin, then,

$$x^2+y^2 = r^2$$

Using eq(1),

$$\frac{u^2}{(u^2+v^2)^2} + \frac{v^2}{(u^2+v^2)^2} = r^2$$

$$\Rightarrow \quad u^2+v^2 = \frac{1}{r^2}$$



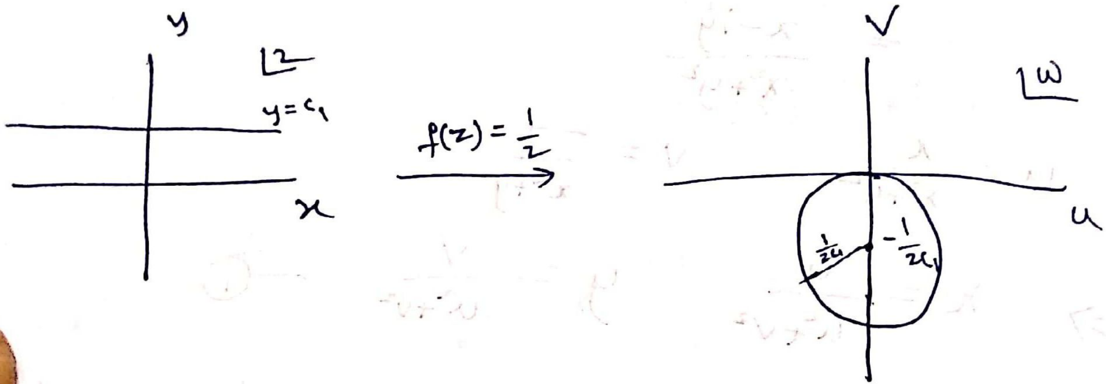
Let us consider a horizontal line $y=c_1$ in z-plane,

Using eq (1)

$$\frac{-u}{u^2+v^2} = c_1$$

$$u^2+v^2+\frac{1}{c_1}v = 0$$

$$u^2+\left(v+\frac{1}{2c_1}\right)^2 = \left(\frac{1}{2c_1}\right)^2$$



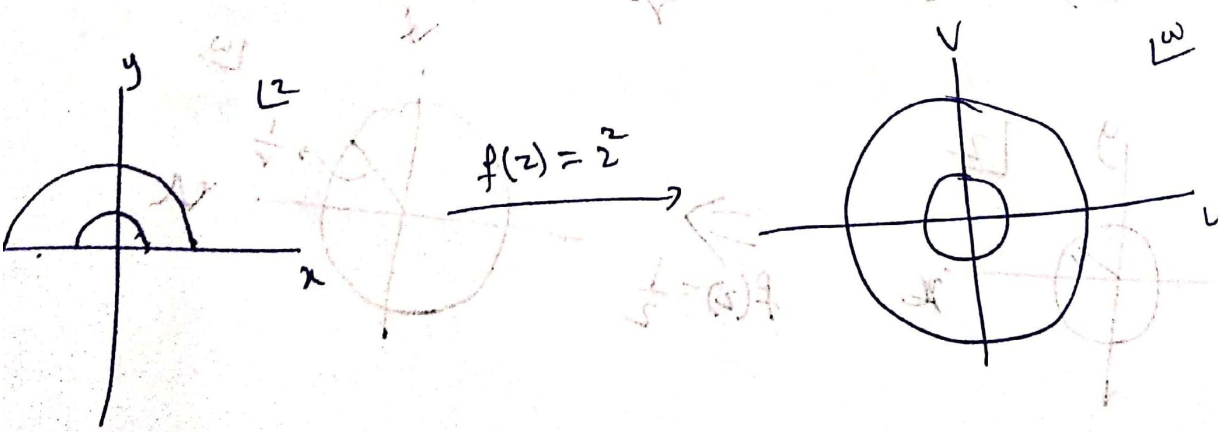
Brach points and Multivalent functions:-

Consider

$$f(z) = z^2$$

$$\rho e^{i\phi} = (re^{i\theta})^2$$

$$\rho = r^2, \quad \phi = 2\theta$$



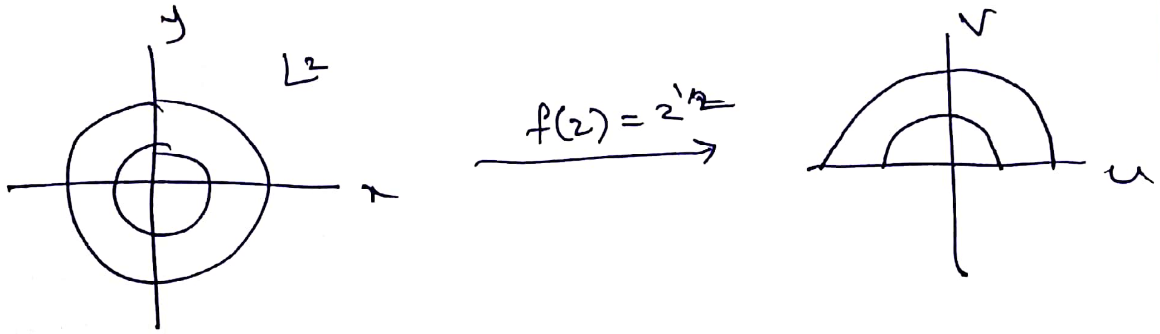
The upper half-plane in z-plane maps to entire plane in the w-plane. The lower half plane will cover the w-plane for a second time. Here, two points in z-plane are mapped to one point in w-plane.

Now considers

$$f(z) = z^{1/2}$$

$$\rho e^{i\phi} = (re^{i\theta})^{1/2}$$

$$\rho = r^{1/2} \qquad \phi = \theta/2$$



Here, one point in z-plane is mapped to two different points in the w-plane.

Points in z-plane under $f(z) = z^{1/2}$ are mapped as

$0 \le r < \infty,\ 0 \le \theta \le 2\pi \rightarrow$ upper half-plane in w-plane

$0 \le r < \infty,\ 2\pi \le \theta \le 4\pi \longrightarrow$ lower half-plane in w-plane

To make $f(z) = z^{1/2}$ we define a cut line which is a straight line from $z=0$ to infinity.

For example positive x-axis can be a cut lin