Chains & Antichains

Let $P = (X, \leq)$ be a tuple where X is a set and $\leq$ is a binary relation on X.

$\leq \; \subseteq X \times X$

We say P is a partially ordered set (or poset) if

$\leq$ is a partial order relation on X

means, $\leq$ is a

* Reflexive relation
  $\forall x \in X, \quad x \leq x$
* Antisymmetric relation
  For every $x, y \in X$ $[((x \leq y)$ and $(y \leq x)] \Rightarrow$
* Transitive

...ve relation $x = y$.

For every $x, y, z \in X$,

if $x \leq y$ and $y \leq z$, then

$x \leq z$.

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Examples

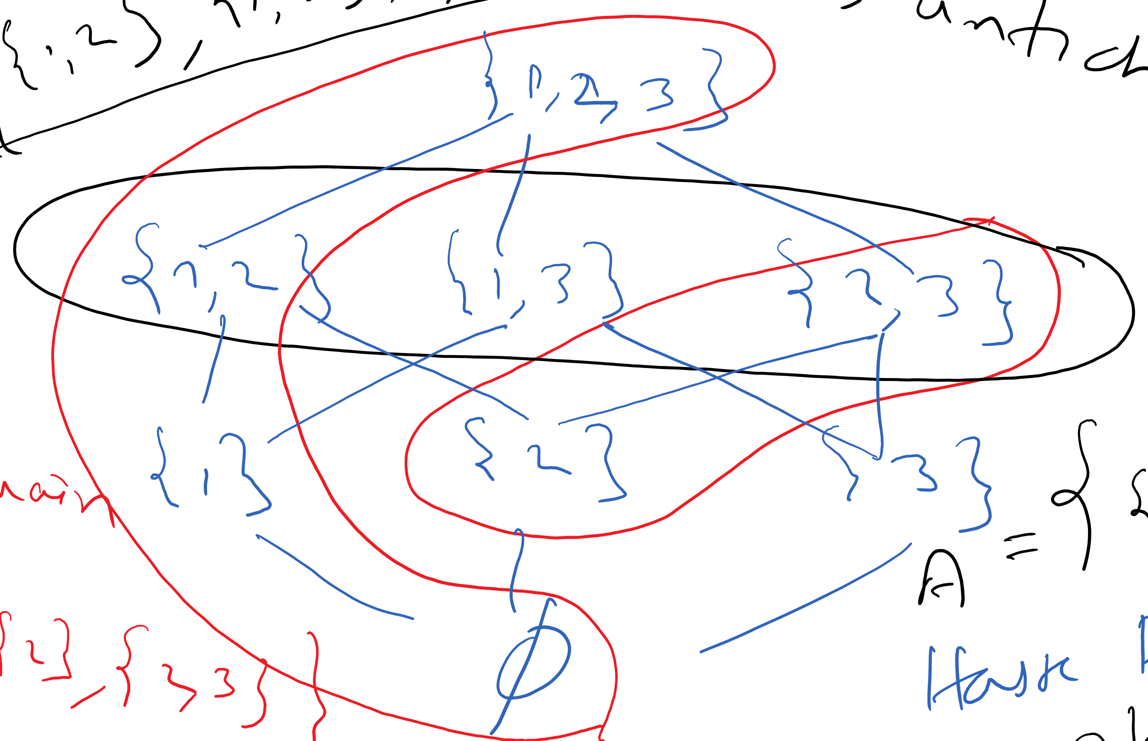
Let $A = \{1, 2, 3\}$

$X = \text{Power set}(A)$

$= \{\{1\}, \{2\}, \{3\}, \{1,2\}, \{1,3\}, \{2,3\}, \{1,2,3\}, \phi\}$

$P = (X, \subseteq)$ → subset relation or containment relation

$Y = \{\{1,2\}, \{1,3\}, \{2,3\}\}$ → antichain



→ chain

$X' = \{\{2\}, \{2,3\}\}$

$A = \{\{3\}\}$

Hasse Diagram

chain ← $X' = \{\phi, \{1\}, \{1,2\}, \{1,2,3\}\}$ → length 4

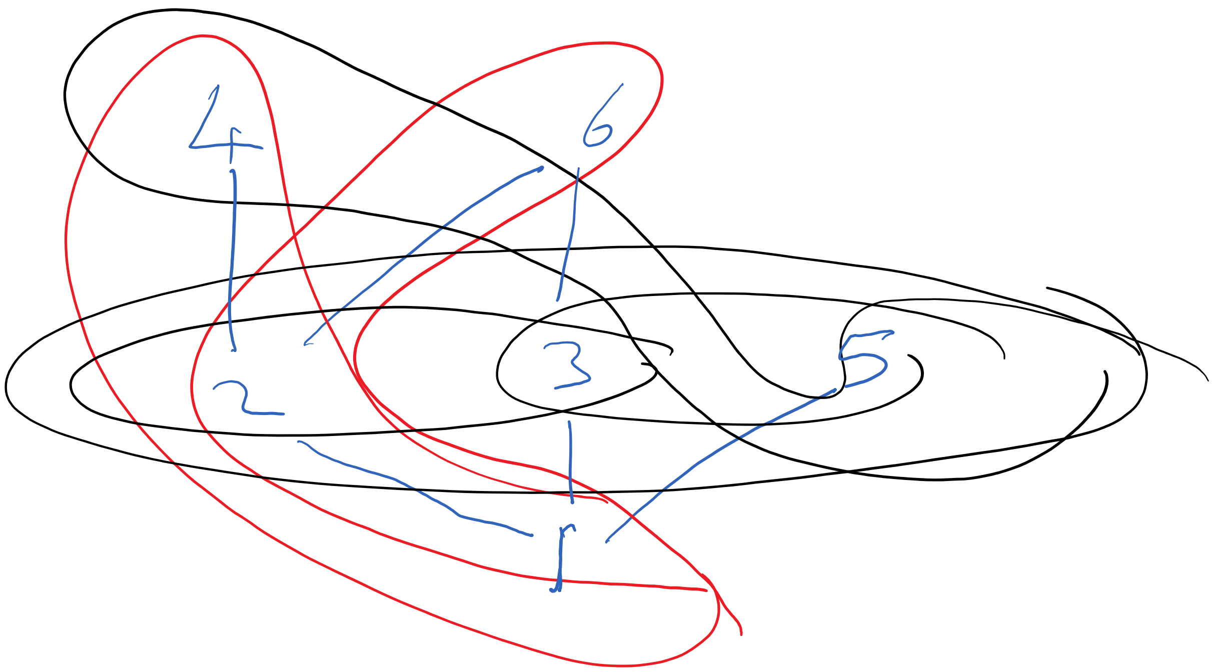
Example 2

$X = \{1\ 2\ 3\ 4 \dots$

$X = \{1, 2, 3, 4, \overline{5, 6}\}$ $\{\{1, 2, 3\}\}$

$P = (X, \mid)$ "divides by"

$2 \mid 6$, $2 \mid 4$



$\underbrace{P = (X, \preceq)}_{\text{POSET}}$

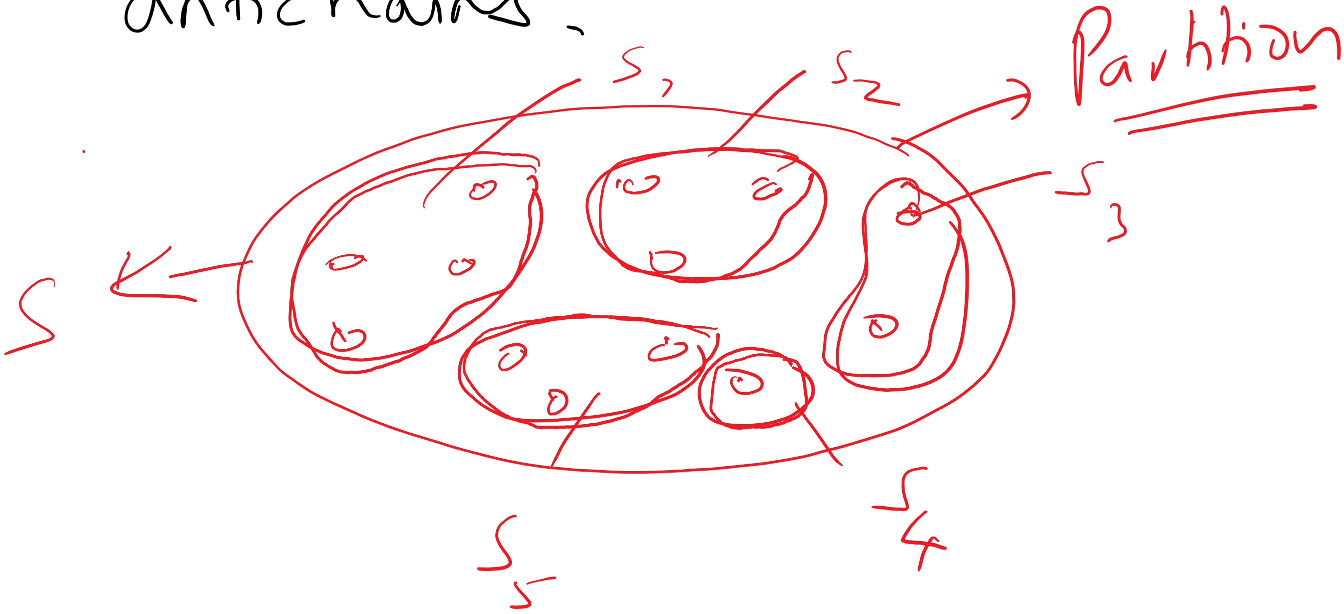
For any $a, b \in X$, we say a and b are comparable if $a \preceq b$ or $b \preceq a$.

Otherwise, we say a and b are incomparable.

Chain $X' \subseteq X$ is a chain if every two elements in X' are comparable with each other.

Antichain $Y' \subseteq X$ is an antichain if no two elements of Y' are comparable with each other.

<u>Theorem 1</u> [Dilworth] Let $P = (X, \leq)$ be a poset. If the length of the longest chain in P is r, then the elements of X can be partitioned into r antichains.



$$S = S_1 \uplus S_2 \uplus \dots \uplus S_5$$

$A_4 = \{\{1,2,3\}, \{1,2,4\}, \{1,3,4\}, \{2,3,4\}\}$

$X = \text{Powerset}(\{1,2,3,4\})$

~~Min~~ no. of antichains into which one can partition the elmts of X $= \gamma$ $\geq$ longest chain length $= \gamma$

Suppose $\gamma - 1$ antichains

$A_1 \to A_2 \to \cdots \to A_{\gamma-1}$

$$P = (X, \leq)$$

Proof:

For any $1 \leq i \leq r$, we define

$A_i = \{ x \in X$: the length of a longest chain terminating at x is $i \}$

Claim: For every $1 \leq i \leq r$, A_i is an antichain.

Suppose not.

$\implies \exists\, x, y \in A_i$ s.t. $x \leq y$.

But then, length of a longest chain ending at y should be $\geq i+1$.

But this contradicts the fact that $y \in A_i$.

So our assumption that $x \preceq y$ is false. This proves the claim.

$\square$

Dilworth Theorem 2

Let $\mathcal{P} = (X, \leq)$ be a poset. Let r be the length of a largest antichain in $\mathcal{P}$. Then, the elements of X can be partitioned into r chains.

Can you partition into $r-1$ chains?

NO.