PH 1027: Maxwell's Equations and Electromagnetic Waves

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Lecture 5

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- Conductors

References/Readings

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- Griffiths, D.J., **Introduction to Electrodynamics, Ch. 2, 4**

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$$\mathbf{F}_1 = \lambda_e \frac{Q_1 Q_2}{|\mathbf{x}_1 - \mathbf{x}_2|^3} (\mathbf{x}_1 - \mathbf{x}_2) \,,$$

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- λ_e depends on the system of units. In Gaussian units (CGS), its just 1. In MKS, $\lambda = 9 \times 10^9 Nm^2/C^2$.

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- Region containing charges. Electric field (plot)

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- **Permittivity of free space (ϵ_0):** Rename, $4\pi\lambda_e = \frac{1}{\epsilon_0}$,

$$\boldsymbol{\nabla}\cdot\mathbf{E}(\mathbf{x}) = \frac{\rho(\mathbf{x})}{\epsilon_0}.$$

The first Maxwell's equation.

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- **Solution:**

$$\Phi(\mathbf{x}) = -\int_{\mathbf{x_0}}^{\mathbf{x}} \mathbf{E}.d\mathbf{x}, \qquad \Phi(\mathbf{x_0}) = 0.$$

$$\Phi(\mathbf{x}) = \sum_{i=1}^{N} \frac{1}{4\pi\epsilon_0} \frac{Q_i}{|\mathbf{x} - \mathbf{x}_i'|} = \frac{1}{4\pi\epsilon_0} \iiint \frac{\rho(\mathbf{x}')d^3\mathbf{x}'}{|\mathbf{x} - \mathbf{x}'|}.$$

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- **Earnshaw's Theorem:** System of point-like charges *cannot* be in stable, static equilibrium under the influence of purely electric forces.

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- **Mean-Value Theorem**: If S is the surface of a sphere whose interior contains no charge, then the potential at the center is equal to the average potential over the surface S.

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- **Boundary conditions (BC's) for Poisson/Laplace** :

$$\text{Dirichlet : Specify } \Phi(\mathbf{x}) \text{ on the boundary}$$

$$\text{Neumann : Specify } \hat{\mathbf{n}} \cdot \mathbf{E}(\mathbf{x}) = \frac{\partial \Phi}{\partial n} \text{ on the boundary}$$

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- **Uniqueness Theorem**: If $\Phi_1(\mathbf{x})$ and $\Phi_2(\mathbf{x})$ are two soln.s of Poisson Eq. with the *same* charge density and the *same* BC's, then they differ *at most* by an additive constant.

$$\Phi_1(\mathbf{x}) = \Phi_2(\mathbf{x}) + \Phi_0 .$$

Field and Potential of an electric dipole

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- Scalar potential at a point, $\mathbf{x}$ due to an ideal dipole, located at $\mathbf{x}'$,

$$\begin{aligned}
\Phi(\mathbf{x}) &= \lim_{Q\to\infty;\mathbf{x}_+\to\mathbf{x}_-} \frac{Q}{4\pi\epsilon_0}\left(\frac{1}{|\mathbf{x}-\mathbf{x}_+|} - \frac{1}{|\mathbf{x}-\mathbf{x}_-|}\right) \\
&= \frac{1}{4\pi\epsilon_0}\frac{\mathbf{p}\cdot(\mathbf{x}-\mathbf{x}')}{|\mathbf{x}-\mathbf{x}'|^3}, \quad \mathbf{x}' = \frac{\mathbf{x}_+ + \mathbf{x}_-}{2} \\
&= -\frac{1}{4\pi\epsilon_0}\mathbf{p}|_{\mathbf{x}'}\cdot\boldsymbol{\nabla}\frac{1}{|\mathbf{x}-\mathbf{x}'|} = \frac{1}{4\pi\epsilon_0}\mathbf{p}|_{\mathbf{x}'}\cdot\boldsymbol{\nabla}'\frac{1}{|\mathbf{x}-\mathbf{x}'|}.
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- **Potential** produced by dipoles

$$\Phi_{\mathbf{P}}(\mathbf{x}) = \frac{1}{4\pi\epsilon_0} \int d^3\mathbf{x}' \, \mathbf{P}(\mathbf{x}') \cdot \boldsymbol{\nabla}' \frac{1}{|\mathbf{x} - \mathbf{x}'|}$$

- **Induced volume and surface charge:**

$$\rho_{bound} = -\boldsymbol{\nabla} \cdot \mathbf{P}, \ \sigma_{bound} = \mathbf{P} \cdot \mathbf{n}$$

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- **Gauss law gets modified:**

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- **Linear dielectrics**

$$\mathbf{P} = \epsilon_0 \boldsymbol{\chi} \cdot \mathbf{E}, \; P_i = \epsilon_0 \chi_{ij} E_j,$$

$\boldsymbol{\chi}$ is the **Electric Susceptibility tensor.**

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- Boundary value problem **inside Dielectrics:**

$$\epsilon_{ij} \partial_i \partial_j \Phi = -\rho_{free}.$$

- **Linear Isotropic dielectric:** $\chi_{ij} = \chi \delta_{ij}$

$$\nabla^2 \Phi = -\epsilon^{-1} \rho_{free}, \epsilon = \epsilon_0 \left(1 + \chi\right)$$

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- **In terms of potential:**

$$-\epsilon^{over}\partial_n \Phi^{over} + \epsilon^{under}\partial_n \Phi^{under} = \sigma_f \,,$$

$$\partial_t \Phi^{over} = \partial_t \Phi^{over} \,.$$

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- **Electrostatic self energy**

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- **Inside Dielectrics:**

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- **Electric field is normal at the boundary,**

$$\mathbf{E}^{out} = \frac{\sigma}{\epsilon_0}\hat{\mathbf{n}}.$$

- **Capacitance,**

$$\Delta Q = \mathbf{C}\,\Delta V$$

C measures the quantity of charge need to increase the potential by 1V. The better the conducor, the higher this capacity is.

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- **in terms of scalar potential (linear isotropic)**

$$\nabla^2\Phi=-\epsilon^{-1}\rho.$$

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- In the frame attached to the charge aka the REST FRAME: Only observe an electric field,

$$\mathbf{E}'(\mathbf{x}') = \frac{1}{4\pi\epsilon_0}\frac{q}{|\mathbf{x}'|^3}\mathbf{x}'$$

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- But in the Lab frame: Observe both Electric *and* Magnetic field!,

$$\mathbf{E}(\mathbf{x}) = \frac{\gamma}{4\pi\epsilon_0}\frac{q}{|\mathbf{x}'|^3}\mathbf{x},$$

$$\mathbf{B}(\mathbf{x}) = \gamma\,\frac{\mathbf{v}}{c^2}\times\mathbf{E}'(\mathbf{x}'),\ \gamma = \sqrt{1-\mathbf{v}^2/c^2}$$

Magnetic fields: Produced by charges in motion and affects charges in motion

- Consider a point charge, q moving in the Lab frame at a velocity, $\mathbf{v}$. What fields does it produce?
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- For Low speeds., $|\mathbf{v}| \ll c$, $\gamma \sim 1$, we get back familiar Galilean transformation

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