CS1340: DISCRETE STRUCTURES II

PRACTICE QUESTIONS I - SOLUTIONS

(1) Given any sequence of $mn+1$ real numbers, some subsequence of $(m+1)$ numbers is increasing or some subsequence of $(n+1)$ numbers is decreasing.

Proof: Assume that the statement is not true. For each number x in the sequence, form the ordered pair (i,j), where i is the length of the longest increasing subsequence beginning with x, and j is the length of the longest decreasing subsequence ending with x. Then since the result is false, $1 \leq i \leq m$ and $1 \leq j \leq n$. Thus we have $mn+1$ ordered pairs of which at most mn are distinct. Hence, two members of the sequence say a and b are associated with the same ordered pair (s,t). W.l.o.g. let us assume a precedes b in the sequence.

If $a \lneq b$ then a together with the longest increasing subsequence beginning with b is an increasing subsequence beginning with b, is an increasing subsequence of length $(s+1)$, contradicting the fact that that s is the length of the longest increasing subsequence beginning with a. Hence $a \geq b$. But then, b, together with the longest decreasing subsequence ending with a, is a subsequence of length $(t+1)$, contradicting that the longest decreasing subsequence ending with b is of length t. There is no way out; our assumption is false, and the result is therefore true.

(2) Let $(x_i, y_i), i = 1,2,3,4,5$ be a set of five distinct points with integer coordinates in the xy-plane. Show that the midpoint of the line joining at least one pair of these points has integer coordinates.

Proof: The midpoint of the segment whose endpoints are (a,b) and (c,d) is $(a+c/2, b+d/2)$.We are concerned only with integer values of the original coordinates. Clearly the coordinates of these fractions will be integers as well if and only if a and c have the same parity - both odd or both even and b and d have the same parity. Thus what matters in this problem is the parities of the coordinates. There are four possible pairs of parities: (odd, odd), (odd, even), (even, odd), and (even, even). Since we are given five points, the pigeonhole principle guarantees that at least two of them will have the same pair of parities. The midpoint of the segment joining these two points will therefore have integer coordinates.

(3) How many ordered pairs of integers (a,b) are needed to guarantee that there are two ordered pairs (a_1,b_1) and (a_2,b_2) such that $a_1 \bmod 5 = a_2 \bmod 5$ and $b_1 \bmod 5 = b_2 \bmod 5$.

Proof: This is similar in spirit to the previous example. Working modulo 5 there are 25 pairs : $(0,0),(0,1),\ldots,(4,4)$. Then we have 25 ordered pairs of integers (a,b) such that no two of them are equal when reduced modulo 5.

The pigeonhole principle, however, guarantees that if we have 26 such pairs, then at least two of them will have the same coordinates.

(4) Show that a non-empty set **(of finite size!!)** has an equal number of even subsets (that is, subsets with an even number of elements) and odd subsets.

It's false for the empty set (and meaningless for infinite sets).

Proof: Let n be a positive integer. Then,

$$\sum_{k=0}^{n} (-1)^k C(n,k) = 0.$$

The above corollary also implies that $C(n,0) + C(n,2) + C(n,4) + \cdots = C(n,1) + C(n,3) + C(n,5) + \cdots$.

Or:

For every subset with an even number of elements, there is a corresponding set with an odd number of elements, that corresponds in this way:

- If 1 is a member of the set with an even number of elements, then delete 1 from the set to get a set with an odd number of elements.
- If 1 is not a member of the set with an even number of elements, then add 1 to the set to get a set with an odd number of elements.

(5) Show that in a group of five people (where any two people are either friends or enemies), there are not necessarily three mutual friends or three mutual enemies. The only thing there really is to do here is try things out. In particular, you find that one arrangement that works is as follows: The following people are friends: $(A, B), (B, C), (C, D), (D, E), (E, A)$. The rest are all enemies.

1. Other Questions

(1) Twenty five boys and twenty five girls sit around a table. Prove that it is always possible to find a person both of whose neighbors are girls.

Proof:

We again assume that there is a sitting arrangement such that there is no one sitting between two girls. We denote the position $a_0, a_1, \ldots, a_{50}$ so that position a_{50} is next to a_1. Now we split the group into and odd and even groups : $(a_1, a_3, \ldots, a_{49})$ and $(a_2, \ldots, a_{50})$. As per our assumption no girls are next to each other at either table. So at each table there are at most 12 girls for a total of at most 24 girls. A contradiction. Therefore our assumption was wrong and it is always possible to find someone sitting between two girls.

(2) Given the set of natural numbers, we saw that the power set of $\mathbb{N}$ is uncountable. What about the set of all finite subsets of the natural numbers, is it countable?

Answer: Yes.

Let X be the following set $X = \{A \subseteq \mathbb{N} : A \text{ is finite }\}$. T.S.T X is countably infinite. We need to build a bijective map from X to $\mathbb{N}$. (We assume that $\mathbb{N}$ contains 0.) For any set $A \subseteq \mathbb{N}$ we can encode a bit string $b_A = (b_0, b_1, b_2, \ldots)$ where $b_i = 1$ if $i \in A$ and 0 otherwise, for all $i \in \mathbb{N}$.

For every A in X the bit string will be finite. Consider the mapping $f : X \to \mathbb{N}$, $f(A) = \sum_{k=0}^{\infty} b_k 2^k$ where $A \in X$. All that remains to be shown is this is indeed bijective and well-defined. That is an exercise. It comes from the fact that every natural number has a unique binary representation.

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