EP 1027: Maxwell's Equations and Electromagnetic Waves

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Lecture 3

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Office hrs. - Walk in or by Email appointment

Agenda

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- Recap of Lecture 2

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- Useful Vector identities

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- Geometric meaning of Gradient, Divergence, Curl
- Integration of vector fields: Gauss' and Stokes' Theorem
- Application of Gauss divergence theorem: Continuity equation for conservation of mass or charge

References/Readings

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- Griffiths, D.J., **Introduction to Electrodynamics, Ch. 1**
- Spiegel M.R., **Schaum's Outline of Vector Analysis**
- Boas, M. L., **Mathematical Methods in the Physical Sciences Ch. 6**
- Arfken, G. B., **Mathematical Methods for Physicists Ch.3**

Cartesian Vectors: General treatment

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- Rank 2 tensor: Under coordinate axes rotation,
$$T_{ij}' = O_{il} O_{jm} T_{lm},$$
e.g., Outer product of two vectors- $a^i b^j$; Kronecker delta- δ_{ij}; Moment of Inertia tensor, $I_{ij} = m\left(\delta_{ij} x_k x_k - x_i x_j\right)$.

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- Check: Cross-product of two tensors give a vector:

$$(\mathbf{a} \times \mathbf{b})_k = \epsilon_{ijk} a_i b_j$$

Scalar field Differentiation: Gradient

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\phi(\mathbf{x}+d\mathbf{x})-\phi(\mathbf{x}) &= dx_1\,\frac{\partial\phi}{\partial x_1}+dx_2\,\frac{\partial\phi}{\partial x_2}+dx_3\,\frac{\partial\phi}{\partial x_3} \\
&= dx_i\,\frac{\partial\phi}{\partial x_i} \\
&= d\mathbf{x}\cdot(\boldsymbol{\nabla}\phi)\,,
\end{aligned}$$

  where we have introduced the **Gradient operator**, $\boldsymbol{\nabla}$,

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- $$\boldsymbol{\nabla}\equiv\hat{\mathbf{e}}_1\frac{\partial}{\partial x_1}+\hat{\mathbf{e}}_2\frac{\partial}{\partial x_2}+\hat{\mathbf{e}}_3\frac{\partial}{\partial x_3}=\hat{\mathbf{e}}_k\frac{\partial}{\partial x_k}.$$

Vector differentiation: Divergence and Curl

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- Since, $\boldsymbol{\nabla}$ acts like a vector, we can construct a scalar by taking the inner product, with a vector field, $\mathbf{A}(\mathbf{x})$,

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- Call it **Divergence of the vector field**,

$$\boldsymbol{\nabla} \cdot \mathbf{A} = \frac{\partial A_1}{\partial x_1} + \frac{\partial A_2}{\partial x_2} + \frac{\partial A_3}{\partial x_3}$$

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$$\boldsymbol{\nabla} \times \mathbf{A} = (\boldsymbol{\nabla} \times \mathbf{A})_k \hat{\mathbf{e}}_k,$$

$$\begin{aligned}
(\boldsymbol{\nabla} \times \mathbf{A})_k &= \epsilon_{ijk}\, \partial_i\, A_j \\
&= \epsilon_{kij}\, \partial_i\, A_j.
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- Can define a double derivative thru the inner product, the **Laplacian**

$$\boldsymbol{\nabla}^2 \equiv \boldsymbol{\nabla} \cdot \boldsymbol{\nabla} = \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} + \frac{\partial^2}{\partial x_3^2}.$$

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- Use it to derive $\mathbf{A} \times (\mathbf{B} \times \mathbf{C}) = \mathbf{B}\,(\mathbf{A}.\mathbf{C}) - \mathbf{C}(\mathbf{A}.\mathbf{B})$ i.e. the "BAC minus CAB" rule for vector triple products.

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$$\begin{aligned}
[\mathbf{A} \times (\mathbf{B} \times \mathbf{C})]_k &= \epsilon_{ijk} A_i \underbrace{(\mathbf{B} \times \mathbf{C})_j} \\
&= \epsilon_{ijk} A_i (\epsilon_{lmj} B_l C_m) \\
&= \epsilon_{ijk}\epsilon_{lmj} A_i B_l C_m \\
&= \epsilon_{kij}\epsilon_{lmj} A_i B_l C_m \\
&= (\delta_{kl}\delta_{im} - \delta_{km}\delta_{il})\, A_i B_l C_m \\
&= (\delta_{kl} B_l)\, A_i\, (\delta_{im} C_m) - (\delta_{km} C_m)\, A_i\, (\delta_{il} B_l) \\
&= B_k A_i C_i - C_k A_i B_i = B_k (\mathbf{A} \cdot \mathbf{C}) - C_k\, (\mathbf{A} \cdot \mathbf{B})
\end{aligned}$$

where in the fourth line we have used the fact that the ϵ-tensor is unchanged under cyclic permutation of its indices $\epsilon_{ijk} = \epsilon_{kij}$. In the 5th line we have used the identity (1) and in going from 7th to 8th line we have used,

$$\delta_{kl} B_l = B_k,\ \delta_{im} C_m = C_i,\ \delta_{km} C_m = C_k,\ \delta_{il} B_l = B_i.$$

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because ϵ_{ijk} is antisymmetric in i, j while $\frac{\partial^2\Phi}{\partial x_i \partial x_j}$ is symmetric in i, j, as order of derivatives do not matter. The product of symmetric and antisymmetric objects in i, j vanishes.

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Irrotational and Solenoidal Vector fields: Scalar and Vector potentials

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then $\mathbf{C}$ can be expressed as the gradient of a scalar, say, Φ
$$\mathbf{C} = -\boldsymbol{\nabla}\Phi.$$

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- $\mathbf{A}$ is non-unique, $\mathbf{A}' \sim \mathbf{A} + \boldsymbol{\nabla}\chi$.

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- Only true for 3 dim space!

Geometric meaning of gradient, divergence, curl

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- **Gradient**: Consider a volume element, $\Delta V = \Delta x \Delta y \Delta z$, around point, $\mathbf{x}$

$$\lim_{\Delta V \to 0} \frac{\oiint dS\, \hat{\mathbf{n}}\, \Phi(x)}{\Delta V} = \boldsymbol{\nabla}\Phi,$$

$\hat{\mathbf{n}}$ is the unit outward normal vector on the surface S.

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- **Curl:** Consider an area element, $\Delta S_{yz} = \Delta y \Delta z$, around $\mathbf{x}$,

  $$\lim_{\Delta S_{yz} \to 0} \frac{\oint \mathbf{A} \cdot d\mathbf{l}}{\Delta S_{yz}} = \left(\boldsymbol{\nabla} \times \mathbf{A}\right)_x,$$

  $d\mathbf{l}$ is the (tangential) line element. So, **Curl = Anticlockwise circulation in an infinitesimal loop per unit normal area bounded by the loop.**

Integration of vector fields

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- **Gauss Divergence theorem**: If S is a closed surface enclosing a volume, V

$$\iiint_V d^3\mathbf{x}\ \boldsymbol{\nabla}\cdot\mathbf{A} = \oint\!\!\!\oint_S dS\ \hat{\mathbf{n}}\cdot\mathbf{A},$$

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- **Stokes Curl theorem:** If S is an open surface, with a boundary, C (closed curve)

$$\iint_S dS\ \hat{\mathbf{n}} \cdot (\nabla \times \mathbf{A}) = \oint_C d\mathbf{l} \cdot \mathbf{A}$$

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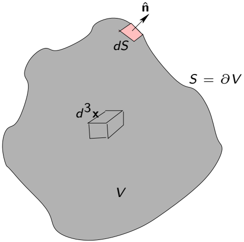
- Should be thought of as vector generalizations of *Fundamental theorem* of single variable calculus:

$$\int_a^b dx\, \frac{df(x)}{dx} = f(b) - f(a)$$

Gauss and Stokes theorem



$$\iiint_V d^3\mathbf{x}\ \boldsymbol{\nabla}\cdot\mathbf{A} = \oiint_{\mathbf{S}} \mathbf{dS}\,\hat{\mathbf{n}}\cdot\mathbf{A}$$

$$\iint_S dS\ \boldsymbol{\nabla}\times\mathbf{A} = \oint \mathbf{dl}\cdot\mathbf{A}$$

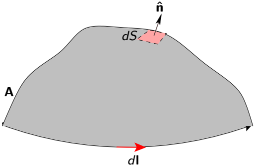


Figure: Pictorial representation of Gauss and Stokes Theorems.

Application of Gauss' theorem: Continuity Equation

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- Consider a closed surface, S enclosing a volume, V containing a fluid of mass density (or electric charge density), ρ. The total mass/charge inside is then,

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- Since there are no sinks (or sources) where the fluid can disappear to (or appear from),
Amount of mass (or charge) escaped by crossing the surface, S = Amount of mass (or charge) decreased in the volume, V

Continuity Equation

Continuity Equation

- Conservation of mass or electric charge

$$\oiint_S dS\, \hat{\mathbf{n}} \cdot \mathbf{j} = -\frac{d}{dt}\left(\iiint_V d^3\mathbf{x}\, \rho\right) \tag{4}$$

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- And in the RHS one can take the time-derivative from outside the volume integral to inside the volume integral,

$$-\frac{d}{dt}\left(\iiint_V d^3\mathbf{x}\, \rho\right) = \iiint_V d^3\mathbf{x}\left(-\frac{\partial \rho}{\partial t}\right).$$

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- Thus, the conservation equation, (4), becomes,

$$\iiint_V d^3\mathbf{x} = \iiint_V d^3\mathbf{x}\left(-\frac{\partial \rho}{\partial t}\right),$$

or,

$$\begin{aligned} \iiint_V d^3\mathbf{x}\left(\frac{\partial \rho}{\partial t} + \boldsymbol{\nabla} \cdot \mathbf{j}\right) &= 0, \\ \implies \frac{\partial \rho}{\partial t} + \boldsymbol{\nabla} \cdot \mathbf{j} &= 0. \end{aligned}$$