Homework Assignments II

MA1130 VECTOR CALCULUS

January 14, 2019

Assume $\mathbf{r}(t), \mathbf{v}(t), \mathbf{a}(t)$ be the position vector, velocity vector and the acceleration vectors respectively whenever they arise.

1. Let
$$f(t) = (\frac{cost}{\sqrt{1+a^2t^2}}, \frac{sint}{\sqrt{1+a^2t^2}}, \frac{-at}{\sqrt{1+a^2t^2}}) \text{ with } a \neq 0$$
   (a) Show that $\|\mathbf{f}(t)\| = 1$ for all t.
   (b) Show directly that $\mathbf{f}'(t) \cdot \mathbf{f}(t) = 0$ for all t.

2. Show that

   A.
$$\frac{d}{dt}(\mathbf{r} \times (\mathbf{v} \times \mathbf{r})) = \|\mathbf{r}\|^2 \mathbf{a} + (\mathbf{r} \cdot \mathbf{v})\mathbf{v} - (\|v\|^2 + \mathbf{r} \cdot \mathbf{a})\mathbf{r}$$

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   B. Let $\mathbf{f}(t)$ be a differentiable curve such that $\mathbf{f} \neq 0$ for all t. Show that
$$\frac{d}{dt}\Big(\frac{\mathbf{f(t)}}{\|\mathbf{f(t)}\|}\Big) = \frac{\mathbf{f(t)} \times (\mathbf{f(t)}' \times \mathbf{f(t)})}{\|\mathbf{f(t)}\|^3}$$

3. Let $\mathbf{r}(t)$ be the position vector in $\mathbb{R}^3$ for a particle that moves with constant speed $c > 0$ in a circle of radius $a > 0$ in the xy-plane. Show that $\mathbf{a}(t)$ points in the opposite direction as $\mathbf{r}(t)$ for all t.

4. Give an example to show the Mean Value Theorem does not hold for vector-valued functions.

5. Parametrize the helix $\mathbf{f}(t) = (cost, sint, t)$, for t in $[0, 2\pi]$, by arc length.

6. Suppose that $r = r(t), \theta = \theta(t)$ and $z = z(t)$ are the cylindrical coordinates of a curve $f(t)$, for t in $[a, b]$. Then show that the arc length L of the curve over $[a, b]$ is

$$L = \int_a^b \sqrt{r'(t)^2 + r(t)^2\theta'(t)^2 + z'(t)^2}\, dt$$

7. Show that

   A.
   $$\Delta |\mathbf{r}|^n = n(n+1)\, |\mathbf{r}|^{n-2}$$

   where n is a constant.

   B.
   $$\Delta(\phi\psi) = (\Delta\phi)\psi + 2\nabla\phi\nabla\psi + \phi(\Delta\psi)$$

8. Show that if ω is a constant vector and $\mathbf{v} = \omega \times \mathbf{r}$ then $\nabla \cdot \mathbf{v} = 0$

9. Let $f : \mathbb{R}^3 \to \mathbb{R}$ is a radial function i.e. $f(\mathbf{r}) = f(|\mathbf{r}|)$. Then show that for $r = |\mathbf{r}|$ one has

$$\Delta f(r) = \frac{d^2 f}{dr^2} + \frac{2}{r}\frac{df}{dr}$$

   From the above find a example of a nontrivial radial function f such that $\Delta f = 0$

10. Find the equation of the tangent plane to the surface $x^2 + y^2 + z^2 = 9$ at the point $(2, 2, 1)$.