EP 1027: Maxwell's Equations and Electromagnetic Waves

Instructor: Shubho Roy¹ (Dept. of Physics)

Lecture 7 (Make up lecture)

April 8, 2019

¹Email: sroy@iith.ac.in, Office: C 313 D, Office hrs.: Walk in/Email

▶ Short Review of last lecture (7)

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► Charges in motion: Magnetic fields

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► Steady Currents in conductors: Ohm's law

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▶ Magnetic force on point charge and current element

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Steady Currents in conductors: Ohm's law

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▶ Magnetic force on point charge and current element

Maxwell's equations

References/Readings

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► Griffiths, D.J., Introduction to Electrodynamics, Ch. **5,7**

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For Low speeds., $|\mathbf{v}| \ll c$, $\gamma \sim 1$, we get back familiar Galilean transformation

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$$\mathsf{E}(\mathsf{x},t) pprox rac{q}{4\pi\epsilon_0} rac{\mathsf{x}}{|\mathsf{x} - \mathsf{v}t|^3},$$

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- How do magnetic field affect charges, say a charge of Q moving with velocity V in a region with magnetic field B^2 ?
- Answer provided by Lorentz force law,

$$F = Q \mathbf{V} \times \mathbf{B}$$
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▶ $\rho = 0$, but $\mathbf{E} \neq 0$ for steady currents in conductors.



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Ampere Law

$$\nabla \times \mathbf{B} = \mu_0 \,\mathbf{j}. \tag{2}$$

$$\oint_C \mathbf{B} \cdot d\mathbf{l} = \mu_0 \int_S dS \,\mathbf{j} \cdot \hat{\mathbf{n}}.$$

Poisson (like) Equation for the vector potential,

$$\nabla^2 \mathbf{A} - \nabla (\nabla \cdot \mathbf{A}) = -\mu_0 \mathbf{i}$$
.



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- Force on a current element

$$\mathbf{F} = dq \, \mathbf{v} \times \mathbf{B} = dq \, \frac{d\mathbf{x}}{dt} \times \mathbf{B} = \frac{dq}{dt} \, d\mathbf{x} \times \mathbf{B} = I \, d\mathbf{x} \times \mathbf{B}.$$



► Faraday's law:

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- This is not true in general. In general,

$$\mathbf{\nabla} \cdot \mathbf{j} = -\frac{\partial \rho}{\partial t}$$

So Ampere's law needs to be fixed as well!

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Maxwell's insight: Need to add a "displacement current" term to the RHS of Ampere's law,

$$\nabla \times \mathbf{B} = \mu_0 \left(\mathbf{j} + \epsilon_0 \frac{\partial \mathbf{E}}{\partial t} \right).$$



Maxwell's Equations

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- ▶ The first and the last Maxwell's equation are truly equations of motion as their RHS contain the sources (ρ, \mathbf{j}) . They tell you, how or what kind of field you produce for a given source.
- The second and third Maxwell's equations have no sources, i.e. they must hold for all situations. So, these are not equations of motion, rather they are *constraints* that must hold for all bona fide solutions of the equation of motion.

 (Ripschi Identities)

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 Φ is the new Electric potential.

▶ Plugging, $\mathbf{E} = -\nabla \mathbf{\Phi} - \frac{\partial \mathbf{A}}{\partial \mathbf{t}}$, $\mathbf{B} = \nabla \times \mathbf{A}$ in Gauss law and Ampere law (i.e. the Maxwell Eq.s 1 and 4),

$$\begin{split} \boldsymbol{\nabla}^2 \boldsymbol{\Phi} + \frac{\partial \left(\boldsymbol{\nabla} \cdot \mathbf{A} \right)}{\partial t} &= -\frac{\rho}{\epsilon_0}, \\ \left(\boldsymbol{\nabla}^2 - \frac{1}{c^2} \frac{\partial^2}{\partial t^2} \right) \mathbf{A} - \boldsymbol{\nabla} \left(\boldsymbol{\nabla} \cdot \mathbf{A} + \frac{1}{c^2} \frac{\partial \boldsymbol{\Phi}}{\partial t} \right) &= -\mu_0 \mathbf{j} \end{split}$$

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$$\nabla^{2}\Phi + \frac{\partial\left(\nabla \cdot \mathbf{A}\right)}{\partial t} = -\frac{\rho}{\epsilon_{0}},$$

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 \blacktriangleright Recall, **A**is ambiguous upto addition of a gradient of a scalar, χ ,

$$\mathbf{A}' = \mathbf{A} + \mathbf{\nabla} \chi$$
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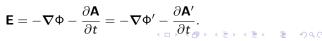
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This implies, Φ is ambiguous as well,

$$\Phi' = \Phi - rac{\partial \chi}{\partial t}$$



Freedom in redefining potentials,

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• Use this freedom to choose χ so that following condition (Lorenz gauge condition)

$$\mathbf{\nabla}\cdot\mathbf{A}+\frac{1}{c^2}\frac{\partial\Phi}{\partial t}=0.$$

Freedom in redefining potentials,

$$\mathbf{A}' = \mathbf{A} + \mathbf{\nabla} \chi, \quad \Phi' = \Phi - \frac{\partial \chi}{\partial t}.$$

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Maxwell's Equations look like,

$$\left(\nabla^2 - \frac{1}{c^2} \frac{\partial^2}{\partial t^2}\right) \Phi = -\frac{\rho}{\epsilon_0},$$

$$\left(\nabla^2 - \frac{1}{c^2} \frac{\partial^2}{\partial t^2}\right) \mathbf{A} = -\mu_0 \mathbf{j}.$$

The operator, $\Box \equiv \nabla^2 - \frac{1}{c^2} \frac{\partial^2}{\partial t^2}$ is called the **D'Alembertian** operator or the wave-operator



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- Why are there 8 Maxwell's equations when written in terms of the electric and magnetic fields (E and B) while there are just 4 Maxwell's equation when expressed in terms of the potentials Φ and A?
- Answer: When one introduces the potentials, one is already solving the homogeneous (sourcless) Maxwell's equations thus getting rid of 4 equations. All that remains are the inhomogeneous Maxwell's equations i.e. Maxwell equations with sources, which are 4 in number.