Lecture 11

Instructor: Subrahmanyam Kalyanasundaram

23rd September 2019

Plan

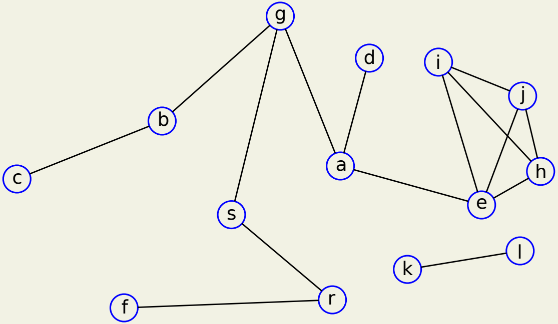
- Complete the proof of correctness of BFS
- After that, we see shortest path in weighted graphs

Algorithm 1 Breadth-first Search from vertex s

Color all vertices WHITE.
For all $u \in V$, $d[u] \leftarrow \infty$, $\pi[u] \leftarrow$ NIL.
$d[s] \leftarrow 0$, color$[s] \leftarrow$ GRAY.
Initialize queue $Q \leftarrow \emptyset$.
ENQUEUE(Q, s)
while $Q \neq \emptyset$ **do**
    $u \leftarrow$ DEQUEUE(Q)
    for each $v \in \mathcal{N}(u)$ **do**
        if color(v) =WHITE **then**
            color$[v] \leftarrow$ GRAY
            $d[v] \leftarrow d[u] + 1$
            $\pi[v] \leftarrow u$
            ENQUEUE(Q, v)
        end if
    end for
    color$[u] \leftarrow$ BLACK.
end while

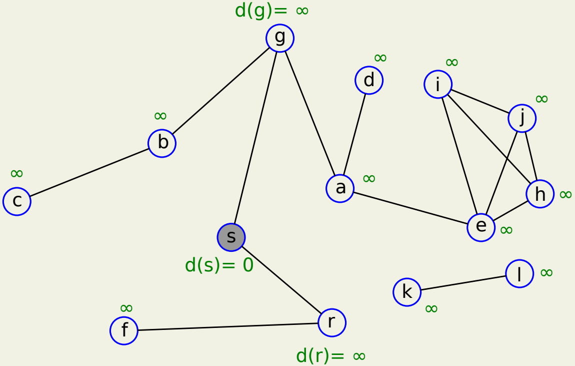
Breadth-first Search

Queue: $\emptyset$



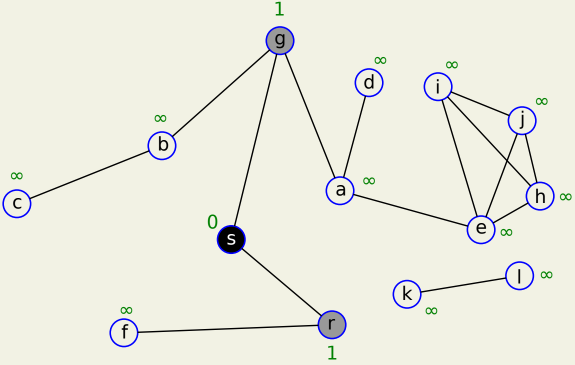
Breadth-first Search

Dequeued vertex: $\boxed{}$ Queue: $\boxed{s}$



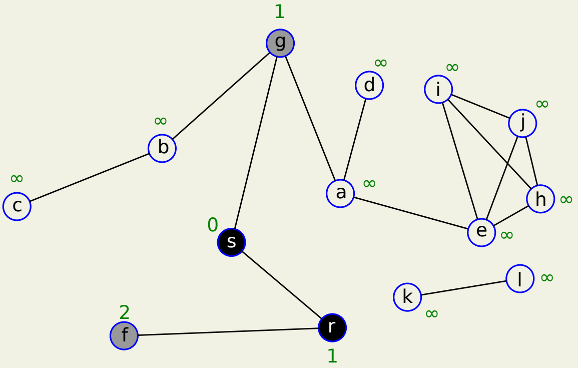
Breadth-first Search

Dequeued vertex: | s | Queue: | r | g |



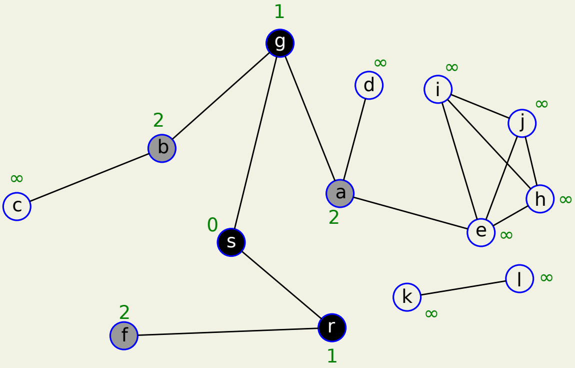
Breadth-first Search

Dequeued vertex: $\boxed{r}$ Queue: $\boxed{g \mid f}$



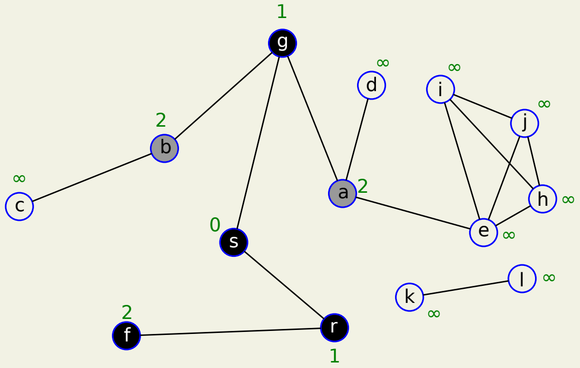
Breadth-first Search

Dequeued vertex: | g | Queue: | f | a | b |



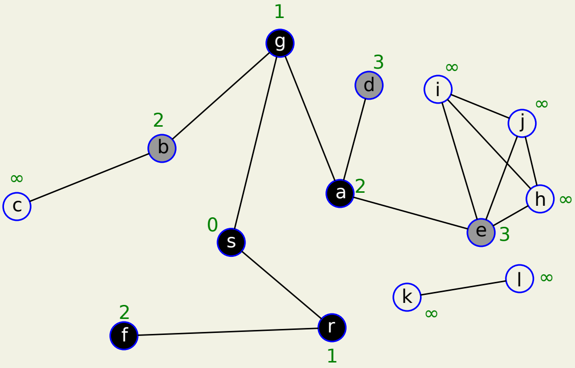
Breadth-first Search

Dequeued vertex: $\boxed{f}$ Queue: $\boxed{a \mid b}$



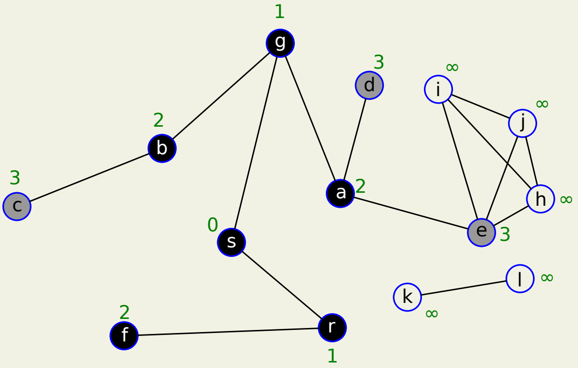
Breadth-first Search

Dequeued vertex: | a | Queue: | b | e | d |



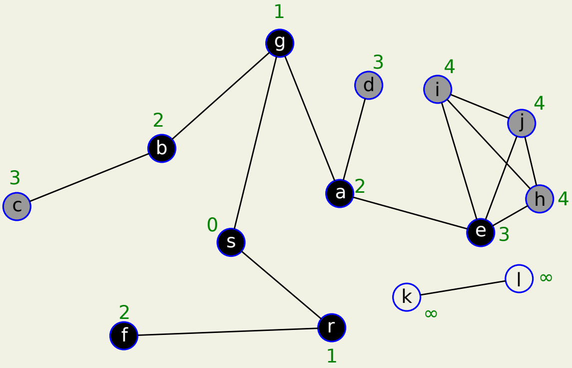
Breadth-first Search

Dequeued vertex: | b | Queue: | e | d | c |



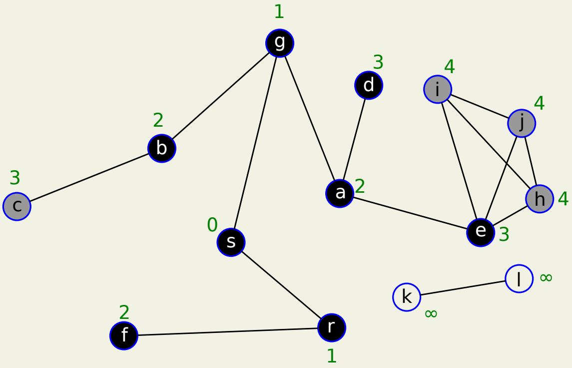
Breadth-first Search

Dequeued vertex: | *e* | Queue: | *d* | *c* | *j* | *h* | *i* |



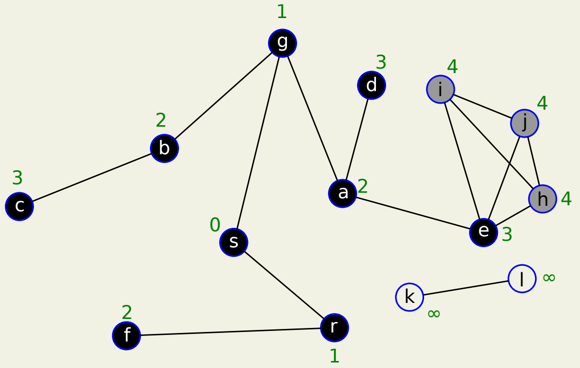
Breadth-first Search

Dequeued vertex: d Queue: | c | j | h | i |



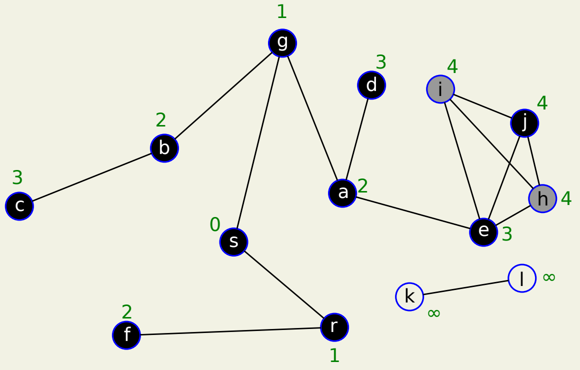
Breadth-first Search

Dequeued vertex: c Queue: j h i



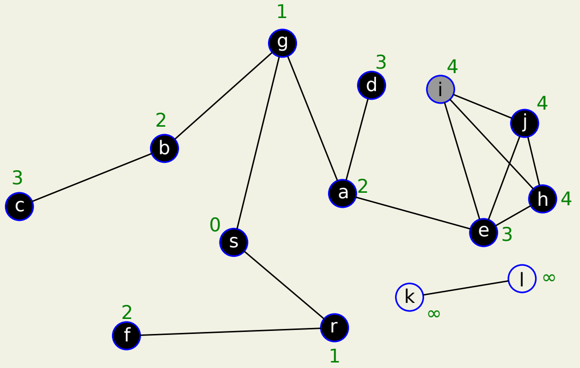
Breadth-first Search

Dequeued vertex: $\boxed{j}$ Queue: $\boxed{h \mid i}$



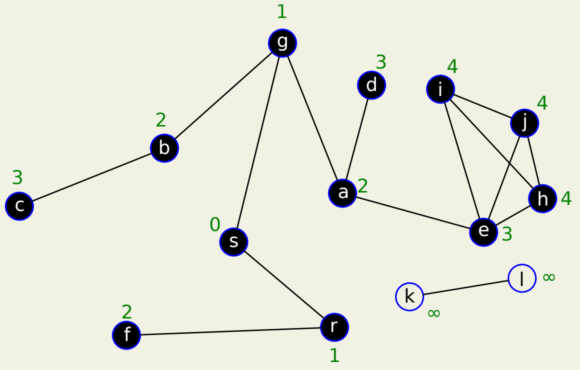
Breadth-first Search

Dequeued vertex: $\boxed{h}$ Queue: $\boxed{i}$



Breadth-first Search

Dequeued vertex: $\boxed{i}$ Queue: $\emptyset$



Algorithm 2 Breadth-first Search from vertex s

Color all vertices WHITE.
For all $u \in V$, $d[u] \leftarrow \infty$, $\pi[u] \leftarrow$ NIL.
$d[s] \leftarrow 0$, color$[s] \leftarrow$ GRAY.
Initialize queue $Q \leftarrow \emptyset$.
ENQUEUE(Q, s)
while $Q \neq \emptyset$ **do**
    $u \leftarrow$ DEQUEUE(Q)
    for each $v \in \mathcal{N}(u)$ **do**
        if color(v) =WHITE **then**
            color$[v] \leftarrow$ GRAY
            $d[v] \leftarrow d[u] + 1$
            $\pi[v] \leftarrow u$
            ENQUEUE(Q, v)
        end if
    end for
    color$[u] \leftarrow$ BLACK.
end while

Time Complexity of BFS

- Each enqueue/dequeue takes $O(1)$ time.
- Total queue operations take $O(|V|)$ time.
- Each list in the adj. list is scanned once. This requires total $\Theta(|E|)$. This is assuming the graph is provided using adjacency list.
- Initialization required $\Theta(|V|)$.
- Total running time is $O(|V| + |E|)$.

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- Initialization required $\Theta(|V|)$.
- Total running time is $O(|V| + |E|)$.

- **Note:** The colors can be omitted. Instead, check if $d[v] = \infty$

Correctness of BFS

Notation: Let $\delta(s, v)$ denote the minimum number of edges on a path from s to v.

Theorem

Let $G = (V, E)$ be a graph. When BFS is run on G from vertex $s \in V$:

1. Every vertex that is reachable from s gets discovered.
2. On termination, $d[v] = \delta(s, v)$ for all v.

We will first show (2).

Proof of correctness

Proof

Suppose, for the sake of contradiction, (2) does not hold.
Let v be the vertex with smallest $\delta(s, v)$ such that $d[v] \neq \delta(s, v)$.
Claim 1: $d[v] \geq \delta(s, v)$
Choose a *shortest* path from s to v.
Let u be the vertex immmediately preceding v.
Then $\delta(s, v) = \delta(s, u) + 1 = d[u] + 1$.
So we have:

$$d[v] > \delta(s, v) = \delta(s, u) + 1 = d[u] + 1$$

Proof of correctness

Proof cont...

We have:

$$d[v] > \delta(s, v) = \delta(s, u) + 1 = d[u] + 1$$

Consider the time step when u is dequeued.

- Case 1: v was white.
  The algo sets $d[v] = d[u] + 1$.
  This contradicts the eq above.
- Case 2: v is black.
  Then, v was dequeued before u.
  Claim 2: If v was dequeued before u, then $d[v] \leq d[u]$.

Proof of correctness

Proof cont...

- Case 3: v was gray.
  Vertex v was colored gray after dequeuing some vertex w earlier.
  So $d[v] = d[w] + 1$.
  By Claim 2, $d[w] \leq d[u]$ since w was dequeued before u.
  This gives: $d[v] = d[w] + 1 \leq d[u] + 1$.

Exercise

Show (1) using (2). That is, given that $d[v] = \delta(s, v)$, show that every vertex reachable from s gets discovered.

Proof of correctness

Claim 3

Let $(u, v) \in E$. Then we have:

$$\delta(s, v) \leq \delta(s, u) + 1$$

Proof

If u is reachable from s, then:
Take the shortest path from s to u. Then take the edge (u, v).
This gives a path from s to v.
The shortest path from s to v can only be shorter than the above path.

$\square$

Proof of correctness

Claim 1

$\forall v \in V, d[v] \geq \delta(s, v)$

Proof

Induction on the number of enqueue operations.
Hypothesis: same as claim.
Base case: The time when the first vertex enqueued.
The first vertex enqueued is s. At this time we have:

- $\forall v \in V \setminus \{s\}, d[v] = \infty$
- $d[s] = \delta(s, s) = 0.$

Hence the claim holds for the base case.

Proof of correctness

Proof

Hypothesis: $\forall v \in V, d[v] \geq \delta(s, v)$
Step: A white (undiscovered) vertex v gets discovered while we are visiting a vertex u with $(u, v) \in E$.
From induction, we have: $d[u] \geq \delta(s, u)$.
The algorithm assigns $d[v] \leftarrow d[u] + 1$. So:

$$\begin{aligned} d[v] &= d[u] + 1 \\ &\geq \delta(s, u) + 1 \\ &\geq \delta(s, v) \end{aligned}$$

Last inequality follows from Claim 3.

□

Proof of correctness

Claim 2

If v was dequeued before u, then $d[v] \leq d[u]$.

We will show a stronger claim:

Claim 4

If at some point, the queue contained $v_1, v_2, \ldots, v_r$ where v_1 was the head. Then:

(a) $d[v_1] \leq d[v_2] \leq \cdots \leq d[v_r]$

(b) $d[v_r] \leq d[v_1] + 1$

Proof of Claim 2:

Write down vertices in the order they went through the queue.

By claim 4 (a), the calculated d values for them are non-decreasing.

Vertex v will appear before u in this order.

Hence claim 2 follows. $\square$

Proof of correctness

Claim 4

If queue contains $v_1, v_2, \ldots, v_r$ where v_1 is the head. Then:

(a) $d[v_1] \leq d[v_2] \leq \cdots \leq d[v_r]$

(b) $d[v_r] \leq d[v_1] + 1$

Proof

Induction on number of queue operations.

Hypothesis: Same as claim. We show that the claim holds after every enqueue and dequeue.

Base case: The first queue operation - enqueuing s.

The claim trivially holds.

Proof of correctness

Claim 4

If queue contains $v_1, v_2, \ldots, v_r$ where v_1 is the head. Then:

(a) $d[v_1] \leq d[v_2] \leq \cdots \leq d[v_r]$

(b) $d[v_r] \leq d[v_1] + 1$

Proof

Step:

- **Dequeue:** After v_1 is dequeued, v_2 is the new head.
  Part (a): From induction,
  $d[v_1] \leq d[v_2] \leq d[v_3] \leq \cdots \leq d[v_r]$.
  Hence (a) holds.
  Part (b): From induction, $d[v_r] \leq d[v_1] + 1$. And so:

$$\begin{aligned} d[v_r] &\leq d[v_1] + 1 \\ &\leq d[v_2] + 1 \end{aligned}$$

Proof of correctness

Proof

- **Enqueue:** When a vertex v is enqueued:
  It was enqueued because:
  - it was undiscovered so far.
  - it was present in the adjacency list of a vertex u that was just dequeued.

  Since u was the previous head of the list, from induction we have:
  - $d[u] \leq d[v_1] \leq d[v_2] \leq \cdots \leq d[v_r]$.
  - $d[v_r] \leq d[u] + 1$.

  We assign $d[v] \leftarrow d[u] + 1$ and then enqueue v. Hence, we have:
  - $d[v_r] \leq d[u] + 1 = d[v]$
  - $d[v_1] \leq d[v_2] \leq \cdots \leq d[v_r] \leq d[v]$.

$\square$

Loop Invariant

Claim 4

If queue contains $v_1, v_2, \ldots, v_r$ where v_1 is the head. Then:

(a) $d[v_1] \leq d[v_2] \leq \cdots \leq d[v_r]$

(b) $d[v_r] \leq d[v_1] + 1$

Claim 4 is actually a loop invariant!

Another loop invariant

The queue Q consists of the set of GRAY vertices.

Weighted Graphs

A weighted graph is a graph $G = (V, E)$ with a weight function:

$$w : E \to \mathbb{Z}$$

The weight of an edge $(u, v) \in E$ is $w((u, v))$.
For this lecture, we look at directed weighted graphs with weight function $w : E \to \mathbb{Z}^+$.

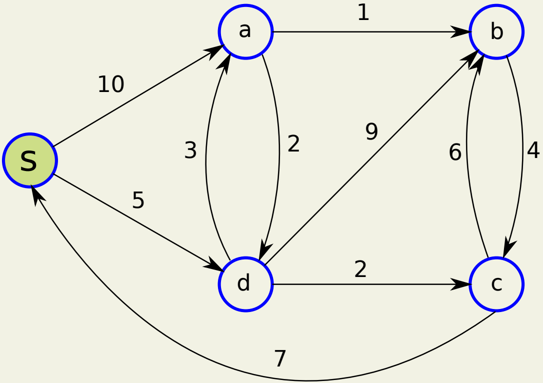
Shortest path in weighted graphs

Input:

- Graph $G = (V, E)$
- Weight function $w : E \to \mathbb{Z}^+$
- Source vertex $s \in V$.

Goal: Compute the shortest path from s to all reachable vertices.

Example graph

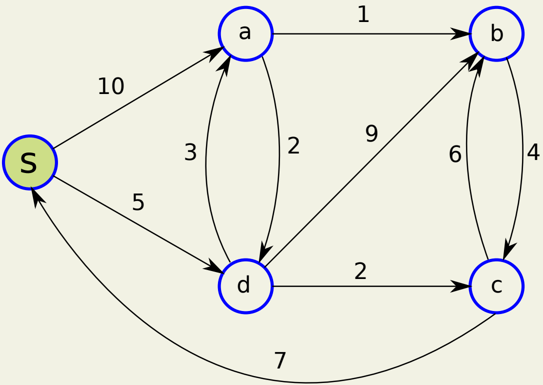


Dijkstra's Algorithm Pseudocode

Algorithm 3 Dijkstra's algorithm

```
For all u ∈ V, d[u] ← ∞, π[u] ← NIL
d[s] ← 0
Initialize min-priority queue Q ← V
S ← ∅
while Q ≠ ∅ do
    u ← EXTRACT-MIN(Q)
    S ← S ∪ {u}
    for each v ∈ 𝒩(u) do
        if d[u] + w(u, v) < d[v] then
            d[v] ← d[u] + w(u, v)
            DECREASE-KEY(v, d[v]).
            π[v] ← u
        end if
    end for
end while
```

Example graph



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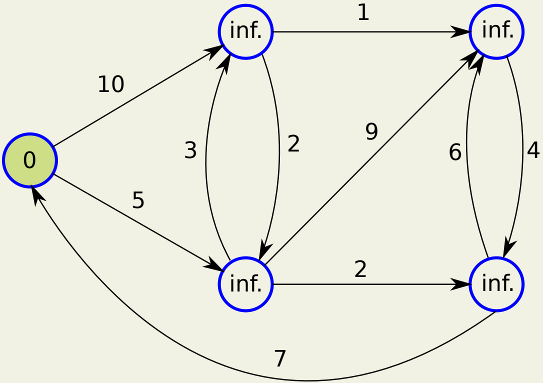

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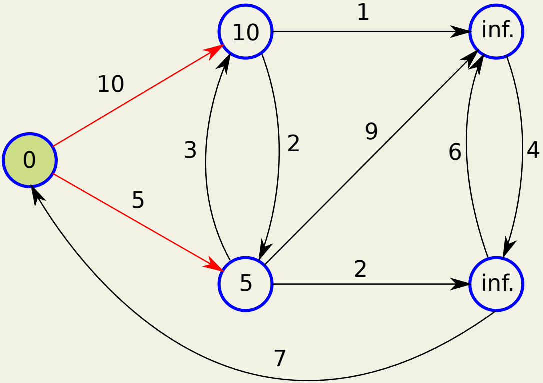

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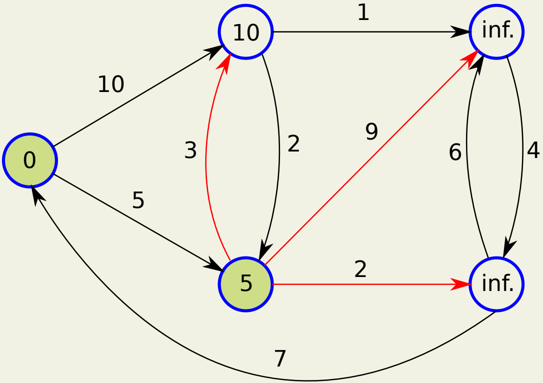

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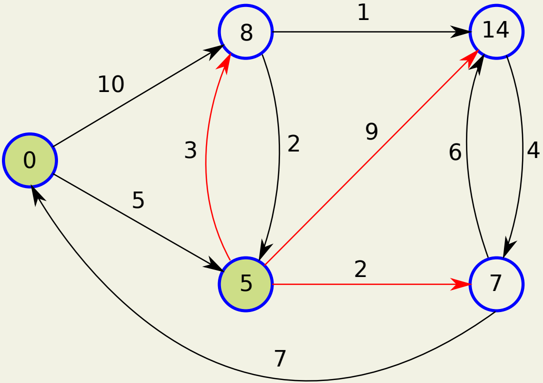

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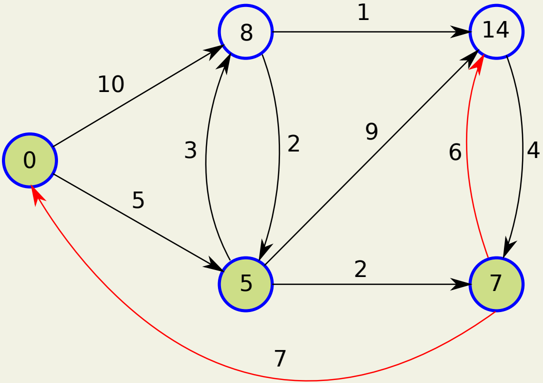

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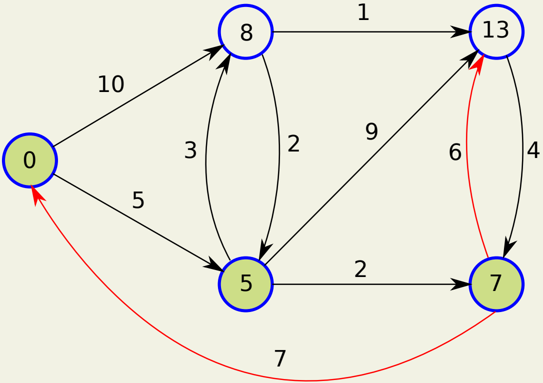

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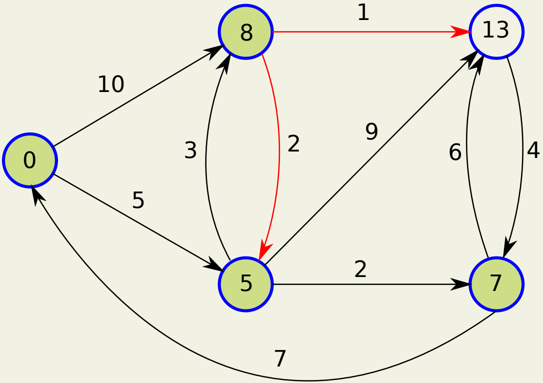

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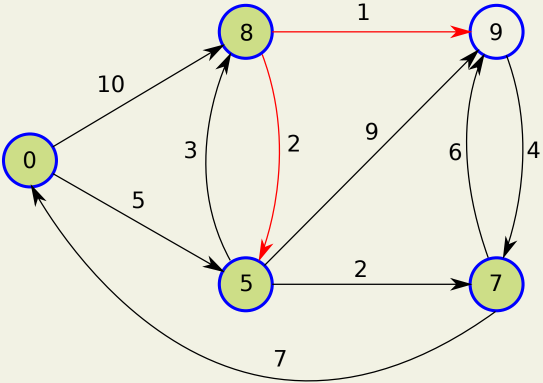

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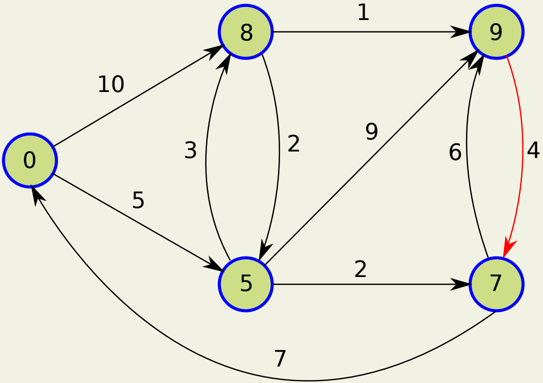

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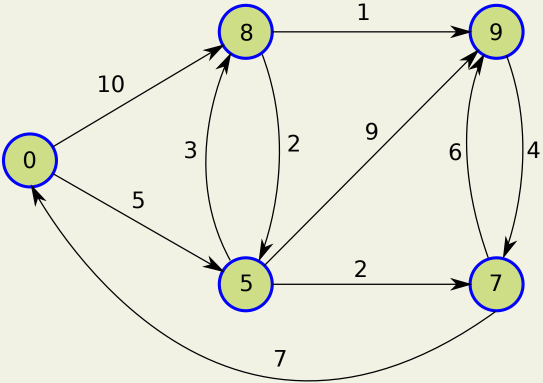

Dijkstra's algorithm

"It is the algorithm for the shortest path, which I designed in about twenty minutes. One morning I was shopping in Amsterdam with my young fiancée, and tired, we sat down on the café terrace to drink a cup of coffee and I was just thinking about whether I could do this, and I then designed the algorithm for the shortest path. As I said, it was a twenty-minute invention."

-Edsger Dijkstra

Source: Wikipedia and "An Interview with Edsger W. Dijkstra". Communications of the ACM

Dijkstra's Algorithm Pseudocode

Algorithm 4 Dijkstra's algorithm

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while Q ≠ ∅ do
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    for each v ∈ N(u) do
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            d[v] ← d[u] + w(u, v)
            DECREASE-KEY(v, d[v]).
            π[v] ← u
        end if
    end for
end while
```

Time Complexity of Dijkstra's

- Initialization: $O(|V|)$
- We need to do $|V|$ Extract-Min's and $|E|$ Decrease-Key's
- Depends on the implementation of the priority queue.

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- Array: Extract-Min takes $O(|V|)$ and Decrease-Key takes $O(1)$
- Heap: Extract-Min and Decrease-Key both take $O(\log |V|)$
- Fibonacci Heap: Decrease-Key takes $O(1)$ amortized time

Proof of Correctness

Theorem

At the end of Dijkstra's algorithm, we have:

$$\forall u \in V, d[u] = \delta(s, u)$$

Proof

Loop Invariant:
At the start of each iteration, we have $\forall v \in S, d[v] = \delta(s, v)$.
Init: At the start of the first iteration, $S = \emptyset$.
Maintenance: Let $u \in V$ be the first vertex for which $d[u] \neq \delta(s, u)$.
If u is not reachable from s, then $d[u] = \delta(s, u) = \infty$, so u must be reachable. Why?
If $u = s$, then the claim holds. So assume $u \neq s$.

Proof of Correctness

Take a shortest path σ from s to u.
Let y be the first vertex on σ that is outside S.
Let $x \in S$ be the vertex on σ just before y.
So the path σ looks like:

$$s \overset{\sigma_1}{\rightsquigarrow} x \to y \overset{\sigma_2}{\rightsquigarrow} u$$

Claim 1: $d[y] = \delta(s, y)$.

Proof of Correctness

$$\sigma = s \overset{\sigma_1}{\rightsquigarrow} x \to y \overset{\sigma_2}{\rightsquigarrow} u$$

Claim 1: $d[y] = \delta(s, y)$.
Since y appears before u in σ, we have $\delta(s, y) \leq \delta(s, u)$.
Claim 2: $d[u] \geq \delta(s, u)$.
Thus:

$$d[y] = \delta(s, y) \leq \delta(s, u) \leq d[u]$$

Although y and u were in $V \setminus S$, Extract-Min returned u.
This means $d[u] \leq d[y]$. Hence:

$$d[y] = \delta(s, y) = \delta(s, u) = d[u]$$

$\square$

Proof of Correctness

Claim 1

$$\sigma = s \overset{\sigma_1}{\rightsquigarrow} x \to y \overset{\sigma_2}{\rightsquigarrow} u$$

We have $d[y] = \delta(s, y)$

Proof

From loop invariant, for all vertices that were added to S before u, we computed the correct shortest distance.
So $d[x] = \delta(s, x)$.
We updated $d[y]$ when we added x to S.
Now we note a *convergence* property:
Let $s \rightsquigarrow x \to y$ be a shortest path, and $d[x] = \delta(s, x)$.
Then, relaxing the edge (x, y) sets $d[y] = \delta(s, y)$.

$\square$

Proof of Correctness

Claim 2

$d[u] \geq \delta(s, u)$

Proof

Induction on number of times d is updated after initialization.

Base case: Immediately after init, $\forall v$, $d[v] = \infty$ except $d[s] = 0$. So the claim holds.

Step: Assume claim for up to k many updates on d.

The value of $d[u]$ is updated when:

- We visit a vertex v and there exists edge (v, u).
- $d[u] > d[v] + w((v, u))$.

Proof of Correctness

Claim 2

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The value of $d[u]$ is updated when:

- We visit a vertex v and there exists edge (v, u).
- $d[u] > d[v] + w((v, u))$.

The new $d[u] = d[v] + w((v, u))$.

The hypothesis holds for vertex v: $d[v] \geq \delta(s, v)$. So:

$$d[u] = d[v] + w((u, v)) \geq \delta(s, v) + w((u, v)) \geq \delta(s, u)$$