EP 1027: Maxwell's Equations and Electromagnetic Waves

Instructor: Shubho Roy[1]
(Dept. of Physics)

Lecture 8

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[1]Email: sroy@iith.ac.in, Office: C 313 D, Office hrs: Walk in/ by email appointment

Agenda

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- Recap of Maxwell's Eqns (in vacuum), Potential formulation, Gauge freedom

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- Magentic fields in media: the "H field" and new Ampere's law

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- Energy Conservation law for EM forces: Poynting vector
- Force on charge distribution: Maxwell Stress tensor
- Linear momentum and Angular momentum contained in EM fields

References/Readings

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- Griffiths, D.J., **Introduction to Electrodynamics, Ch. 6-8**

Maxwell's equations in vacuum

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- **Maxwell's Equations**

$$\nabla \cdot \mathbf{E} = \frac{\rho}{\epsilon_0}, \text{(Gauss law)}$$

$$\nabla \times \mathbf{E} + \frac{\partial \mathbf{B}}{\partial t} = 0, \text{(Faraday-Lenz law)}$$

$$\nabla \cdot \mathbf{B} = 0, \text{ ("No source or sink" law)}$$

$$\nabla \times \mathbf{B} - \frac{1}{c^2}\frac{\partial \mathbf{E}}{\partial t} = \mu_0\, \mathbf{j}. \text{(Ampere law)}$$

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- The first and the last Maxwell's equation are truly equations of motion as their RHS contain the sources $(\rho, \mathbf{j})$. They tell you, how or what kind of field you produce for a given source.
- The second and third Maxwell's equations have no sources, i.e. they must hold for all situations. So, these are not equations of motion, rather they are *constraints* that must hold for all bona fide solutions of the equation of motion. (Bianchi Identities)

Maxwell's equations in Potential formulation

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$$\mathbf{B} = \nabla \times \mathbf{A}.$$

Plug this in Maxwell Eq. 2, aka Faraday's law, we get:.

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-
$$\nabla \times \left(\mathbf{E} + \frac{\partial \mathbf{A}}{\partial t} \right) = 0,$$
$$\implies \mathbf{E} + \frac{\partial \mathbf{A}}{\partial t} = -\nabla \Phi$$
$$\implies \mathbf{E} = -\nabla \Phi - \frac{\partial \mathbf{A}}{\partial t}.$$

  Φ is the new Electric potential.

Potential formulation of Maxwell's Equations

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- Plugging, $\mathbf{E} = -\nabla\boldsymbol{\Phi} - \frac{\partial \mathbf{A}}{\partial \mathbf{t}}$, $\mathbf{B} = \nabla \times \mathbf{A}$ in Gauss law and Ampere law (i.e. the Maxwell Eq.s 1 and 4),

$$\boxed{\begin{gathered} \nabla^2\Phi + \frac{\partial\left(\nabla\cdot\mathbf{A}\right)}{\partial t} = -\frac{\rho}{\epsilon_0}, \\ \left(\nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right)\mathbf{A} - \nabla\left(\nabla\cdot\mathbf{A} + \frac{1}{c^2}\frac{\partial\Phi}{\partial t}\right) = -\mu_0\mathbf{j} \end{gathered}}$$

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- Recall, $\mathbf{A}$ is ambiguous upto addition of a gradient of a scalar, χ,
$$\mathbf{A}' = \mathbf{A} + \nabla\chi$$
$$\nabla \times \mathbf{A}' = \nabla \times \mathbf{A} = \mathbf{B}.$$

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- Recall, $\mathbf{A}$ is ambiguous upto addition of a gradient of a scalar, χ,

$$\mathbf{A}' = \mathbf{A} + \boldsymbol{\nabla}\chi$$

$$\boldsymbol{\nabla} \times \mathbf{A}' = \boldsymbol{\nabla} \times \mathbf{A} = \mathbf{B}.$$

- This implies, Φ is ambiguous as well,

$$\Phi' = \Phi - \frac{\partial\chi}{\partial t}$$

$$\mathbf{E} = -\boldsymbol{\nabla}\Phi - \frac{\partial\mathbf{A}}{\partial t} = -\boldsymbol{\nabla}\Phi' - \frac{\partial\mathbf{A}'}{\partial t}.$$

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- Freedom in redefining potentials,

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Called *Gauge Freedom/ Gauge Symmetry*.

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- Maxwell's Equations look like,

$$\Box\,\Phi = -\frac{\rho}{\epsilon_0},$$

$$\Box\,\mathbf{A} = -\mu_0\mathbf{j}.$$

  The operator,

$$\Box \equiv \nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}$$

  is called the **D'Alembertian (or wave operator)**

Current carrying loop: Magnetic dipole moment

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- Vector Potential due to a current loop

$$\mathbf{A}(\mathbf{x}) = \frac{\mu_0}{4\pi} \mathbf{m}(\mathbf{x}') \times \frac{\mathbf{x} - \mathbf{x}'}{|\mathbf{x} - \mathbf{x}'|^3}$$

Paramagnetic materials: Magnetization

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- Develop Magnetization: Moment per unit volume, **M**
- Magentic vector potential:

$$\begin{aligned}
\mathbf{A}(\mathbf{x}) &= \frac{\mu_0}{4\pi}\int d^3\mathbf{x}'\left(\mathbf{M}(\mathbf{x}')\times\frac{\mathbf{x}-\mathbf{x}'}{|\mathbf{x}-\mathbf{x}'|^3}\right)\\
&= \frac{\mu_0}{4\pi}\int d^3\mathbf{x}'\,\mathbf{M}(\mathbf{x}')\times\boldsymbol{\nabla}'\frac{1}{|\mathbf{x}-\mathbf{x}'|}\\
&= -\frac{\mu_0}{4\pi}\int d^3\mathbf{x}'\,\boldsymbol{\nabla}'\times\left(\frac{\mathbf{M}(\mathbf{x}')}{|\mathbf{x}-\mathbf{x}'|}\right)+\frac{\mu_0}{4\pi}\int \mathbf{d}^3\mathbf{x}'\,\frac{\boldsymbol{\nabla}'\times\mathbf{M}(\mathbf{x}')}{|\mathbf{x}-\mathbf{x}'|}\\
&= \frac{\mu_0}{4\pi}\int dS'\,\frac{\mathbf{M}(\mathbf{x}')\times\hat{\mathbf{n}}}{|\mathbf{x}-\mathbf{x}'|}+\frac{\mu_0}{4\pi}\int \mathbf{d}^3\mathbf{x}'\,\frac{\boldsymbol{\nabla}'\times\mathbf{M}(\mathbf{x}')}{|\mathbf{x}-\mathbf{x}'|}\\
&= \frac{\mu_0}{4\pi}\int dS\,\frac{\mathbf{k}_{bound}(\mathbf{x}')}{|\mathbf{x}-\mathbf{x}'|}+\mu_0\int d^3\mathbf{x}\,\frac{\mathbf{j}_{bound}(\mathbf{x}')}{|\mathbf{x}-\mathbf{x}'|}.
\end{aligned}$$

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\end{aligned}$$

- Ampere's law in media: The "*H* field" Maxwell equation

$$\begin{aligned}
\nabla\times\mathbf{B} &= \mu_0\left(\mathbf{j}_{free}+\mathbf{j}_{bound}\right) \\
&= \mu_0\left(\mathbf{j}_{free}+\boldsymbol{\nabla}\times\mathbf{M}\right),
\end{aligned}$$

Maxwell eqn. for Magnetic Field in Media: Boundary conditions

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$$\Longrightarrow \; H_{\parallel}^{over} - H_{\parallel}^{under} = \hat{\mathbf{n}} \times K_{free}.$$

Maxwell eqn. for Magnetic Field in Media: Boundary conditions

- **Normal component of B continuous,**

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- **Potential continuous, normal derivative discontinous**

$$A^{over} = A^{under},$$

$$\frac{\partial A^{over}}{\partial n} - \frac{\partial A^{under}}{\partial n} = -\mu_0 K$$

Maxwell Equations in Media

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- $$\begin{aligned} \nabla \cdot \mathbf{D} &= \rho_{free}, \\ \nabla \times \mathbf{E} &= -\frac{\partial \mathbf{B}}{\partial t}, \\ \nabla \cdot \mathbf{B} &= 0, \\ \nabla \times \mathbf{H} &= \mathbf{j}_{free} + \frac{\partial \mathbf{D}}{\partial t}. \end{aligned}$$

  Now the name "Displacement current" is obvious.

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- Constitutive relations:

$$\begin{aligned} \mathbf{D} &= \mathbf{D}(\mathbf{E}) = \epsilon\,\mathbf{E}, \\ \mathbf{H} &= \mathbf{H}(\mathbf{B}) = \frac{1}{\mu}\mathbf{B}. \end{aligned}$$

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- Boundary conditions...

Conservation of Energy: Poynting Vector

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- Energy Conservation: Work done

$$
\begin{aligned}
dW &= q\mathbf{E} \cdot d\mathbf{x} + q\mathbf{v} \times \mathbf{B} \cdot d\mathbf{x} \\
&= \int d^3\mathbf{x}\, \mathbf{E} \cdot \mathbf{j} dt
\end{aligned}
$$

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- Vector identities simplify

$$\mathbf{E}\cdot\mathbf{j} = -\frac{\partial}{\partial t}\left(\underbrace{\frac{1}{2}\epsilon_0 E^2 + \frac{1}{2}\frac{1}{\mu_0}B^2}_{u_{EM}}\right) - \boldsymbol{\nabla}\cdot\underbrace{\left(\frac{1}{\mu_0}\mathbf{E}\times\mathbf{B}\right)}_{\mathbf{S}}$$

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- Total Energy Conservation

$$\frac{dW}{dt} = -\frac{dU_{EM}}{dt} - \frac{1}{\mu_0}\left(\oiint_\infty dS\,\hat{\mathbf{n}}.\mathbf{S}\right), \qquad U_{EM} = \int d^3\mathbf{x}\,u_{EM}$$

- Work energy theorem,

$$\frac{dW}{dt} = \frac{dU_{kinetic}}{dt} = \frac{d}{dt}\int d^3\mathbf{x}\,u_{kinetic},$$

- Local energy conservation:

$$\frac{\partial u}{\partial t} + \nabla\cdot\mathbf{S} = 0,\, u = u_{kinetic} + u_{EM}$$

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- **Energy density in Electric and magnetic fields** ,

$$u_{EM} = \frac{\epsilon_0}{2}\mathbf{E}^2 + \frac{1}{2\mu_0}\mathbf{B}^2$$

- **"Energy flux per unit time" in Electric and Magnetic Fields**,

$$\mathbf{S} = \frac{1}{\mu_0}(\mathbf{E} \times \mathbf{B})$$

Force on charge distribution: Maxwell Stress tensor

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- Force on a charge-current distribution,:

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- Using Maxwell's Eq.s and vector identities:

$$\begin{aligned}\mathbf{f} &= \epsilon_0\left(\mathbf{E}\nabla\cdot\mathbf{E} + \mathbf{E}\cdot\nabla\mathbf{E}\right) + \frac{1}{\mu_0}\left(\mathbf{B}\nabla\cdot\mathbf{B} + \mathbf{B}\cdot\nabla\mathbf{B}\right) \\ &\qquad -\nabla u_{EM} - \frac{1}{c^2}\frac{\partial\mathbf{S}}{\partial t},\end{aligned}$$

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- To Simplify: Introduce the Maxwell stress tensor,

$$T_{ij} = \epsilon_0 \left(E_i E_j - \frac{1}{2}\delta_{ij}\mathbf{E}^2\right) + \frac{1}{\mu_0}\left(B_i B_j - \frac{1}{2}\delta_{ij}\mathbf{B}^2\right)$$

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- To Simplify: Introduce the Maxwell stress tensor,

$$T_{ij} = \epsilon_0\left(E_iE_j - \frac{1}{2}\delta_{ij}\mathbf{E}^2\right) + \frac{1}{\mu_0}\left(B_iB_j - \frac{1}{2}\delta_{ij}\mathbf{B}^2\right)$$

- Nice result:

$$\mathbf{f} = \boldsymbol{\nabla}\cdot\mathbf{T} - \frac{1}{c^2}\frac{\partial\mathbf{S}}{\partial t}, \quad \boldsymbol{\nabla}\cdot\mathbf{T} = \partial_i T_{ij}.$$

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- Net Force on a charge-current distribution,:

$$\mathbf{F} = \int dS\, \hat{\mathbf{n}} \cdot \mathbf{T} - \frac{d}{dt}\left(\int d^3\mathbf{x}\, \frac{\mathbf{S}}{c^2}\right)$$

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- Recall,

$$\mathbf{f} = \frac{\partial \mathbf{p}_{charges}}{\partial t}$$

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- Recall,

$$\mathbf{f} = \frac{\partial \mathbf{p}_{charges}}{\partial t}$$

- Then, local conservation of momentum

$$\frac{\partial}{\partial t}\left(\mathbf{p}_{charges} + \underbrace{\frac{1}{c^2}\mathbf{S}}_{\boldsymbol{\pi}_{EM}}\right) + \boldsymbol{\nabla} \cdot (-\mathbf{T}) = 0.$$

$\mathbf{T}$ is a the momentum flux per unit area per unit time (Momentum current density).

Conservation of Angular Momentum

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- Energy density stored in EM fields

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- Linear Momentum stored in EM fields:

$$\boldsymbol{\pi}_{EM} = \frac{1}{c^2}\mathbf{S} = \epsilon_0\left(\mathbf{E} \times \mathbf{B}\right).$$

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$$u_{EM} = \frac{\epsilon_0}{2}\mathbf{E}^2 + \frac{1}{2\mu_0}\mathbf{B}^2.$$

- Linear Momentum stored in EM fields:

$$\boldsymbol{\pi}_{EM} = \frac{1}{c^2}\mathbf{S} = \epsilon_0\left(\mathbf{E}\times\mathbf{B}\right).$$

- What about Angular momentum (per unit volume in the field)?.

$$\boldsymbol{l}_{EM} \stackrel{?}{=} \mathbf{x}\times\boldsymbol{\pi}_{EM}$$