CS 1310 : Discrete Structures 1
Exam 1

8 September 2017, Saturday

Marks : 75 **Duration** : 1.5 hours

"Logic is the beginning of wisdom, not the end" - Leonard Nimoy

The examination covers Logic and Proof techniques. Wish you good luck !

Q1) What Boolean search would you use to look for Web pages about beaches in New Jersey and it should not include results about beaches in the Isle of Jersey? (2.5 marks)

Ans :

Version 1: We could use New AND Jersey AND beaches to retrieve the web pages about beaches in New Jersey. These results do not include beaches in the isle of Jersey so no need to do anything.

Q2) Relate the below to inhabitants of the island of knights and knaves created by Smullyan, where knights always tell the truth and knaves always lie. You encounter two people,A and B. Determine, if possible, what A and B are if they address you in the ways described. If you cannot determine what these two people are, can you draw any conclusions? Give proper reasoning. (5 marks)

i. A says "At least one of us is a knave" and B says nothing.

ii. A says "The two of us are both knights" and B says "A is a knave."

Ans :

I. If *A* is a knight, then he is telling the truth, in which case *B* must be a knave. Since *B* said nothing, that is certainly possible. If A is a knave, then he is lying, which means that his statement that at least one of them is a knave is false; hence they are both knights. That is a contradiction. **So we can conclude that *A* is a knight and *B* is a knave.**

II. If A is knight then A is saying truth and B is also a knight but B says A is a knave so that is not possible. Thus A is a knave. B says A is a knave so B is a knight. **So we can conclude that A is a knave and B is a knight.**

Q3) Consider the following statements in a Puzzle by Lewis Carroll: (10 marks)

(a) No interesting poems are unpopular among people of real taste.

(b) No modern poetry is free from affectation.

(c) All *your* poems are on the subject of soap-bubbles.

(d) No affected poetry is popular among people of real taste.
(e) No ancient poem is on the subject of soap-bubbles.

The universe in this puzzle is the collection of all poems, while the five assertions are implications involving the simpler statements

I : it is interesting	,	P : it is popular among people of real taste
M : it is modern	,	A : it is affected
Y : it is your poem	,	S : it is on the subject of soap bubbles .

Write each statement symbolically, along with its contrapositive.
Also tell which of the following is true using rules of inference:

a) Your poetry is not interesting **Ans: True**
b) Modern poetry is not interesting **Ans: True**
c) Your poem is not affected **Ans: False**
d) Interesting poems are modern **Ans: False**

Again, we write each statement symbolically, along with its contrapositive:

(a) $I \rightarrow P$, $\sim P \rightarrow \sim I$

(b) $M \rightarrow A$, $\sim A \rightarrow \sim M$

(c) $Y \rightarrow S$, $\sim S \rightarrow \sim Y$

(d) $A \rightarrow \sim P$, $P \rightarrow \sim A$

(e) $\sim M \rightarrow \sim S$, $S \rightarrow M$.

You may have noticed in our previous two puzzles that the string of implications connecting all the statements begins with a letter occurring in only one of the assertions. In this puzzle the letters Y and I meet this criterion. If we begin with the letter I we produce the chain

$I \rightarrow P \rightarrow \sim A \rightarrow \sim M \rightarrow \sim S \rightarrow \sim Y$,

and if we begin with the letter Y we create the contrapositive chain,

$Y \rightarrow S \rightarrow M \rightarrow A \rightarrow \sim P \rightarrow \sim I$.

Thus the solution to the puzzle is $I \rightarrow \sim Y$, or the equivalent contrapositive $Y \rightarrow \sim I$. The simplest translation back into words is perhaps the cruel statement

"Your poetry is not interesting."

Q4) Determine whether the given compound proposition is satisfiable
$(p \lor \neg q) \land (q \lor \neg r) \land (r \lor \neg p) \land (p \lor q \lor r) \land (\neg p \lor \neg q \lor \neg r)$ (2.5 marks)

Ans: for simplicity replace $\lor$, $\land$ and $\neg$ with +, . and ' respectively then ,

$(p \lor \neg q) \land (q \lor \neg r) \land (r \lor \neg p) \land (p \lor q \lor r) \land (\neg p \lor \neg q \lor \neg r)$

$= (p+q')\,(q+r')\,(r+p')\,(p+q+r)\,(p'+q'+r')$

$= (p+q')\,(p+(q+r))\,(q+r')(r+p')(p'+q'+r')$ Associative law

$= (p+(q'(q+r)))\,(q+r')(r+p')(p'+q'+r')$ Distributive law

$= (p+ q'q+ q'r)\,(q+r')(r+p')(p'+q'+r')$ Distributive law

$= (p+q'r)\,(q+r')(r+p')(p'+q'+r')$ complement law

$= (p+q'r)\,(q+r')\,(p'+r)\,(p'+(q'+r'))$ Associative law

$= (p+q'r)\,(q+r')\,(p'+(rq'+rr'))$ Distributive law

$= (p+q'r)\,(q+r')\,(p'+rq')$ complement law

$= (q+r')\,(p+rq')\,(p'+rq')$ Associative law

$= (q+r')\,(pp'+rq')$ Distributive law

$= (q+r')\,(rq')$ complement law

$= qrq' + r'rq'$ Distributive law

$= rqq'+ rr'q'$ Associative law

$= F$ complement law

So, the given proposition is a contradiction therefore it is not satisfiable.

Q5) The following sentence is taken from the specification of a telephone system: "If the directory database is opened, then the monitor is put in a closed state, if the system is not in its initial state." This specification is hard to understand because it involves two conditional statements. Find an equivalent, easier-to-understand specification that involves disjunctions and negations but not conditional statements. (2.5 marks)

Ans:: let, p = the directory database is opened,
q = the monitor is put in a closed state,
r = the system is in its initial state

The logical notation of the statement "If the directory database is opened, then the monitor is put in a closed state, if the system is not in its initial state." will be,

$\neg r \rightarrow (p\rightarrow q)$

$= r+ (p\rightarrow q)$ Implication rule

$= r + (p'+q)$ Implication rule

In words we can say that the above expression is always be true that either the system is in its initial state, or the directory database is not opened, or the monitor is put in a closed state.

OR

If the directory database is opened then either the monitor is put in a closed state, or the system is in its initial state.

Q6) Check whether the preposition $[(p \lor q) \land \neg p] \rightarrow q$ and q are logically equivalent using logical equivalence laws. (5 marks)

Ans: :

$[(p \lor q) \land \neg p] \rightarrow q$	LHS
$[\neg p \land (p \lor q)] \rightarrow q$	Commutative
$[(\neg p \land p) \lor (\neg p \land q)] \rightarrow q$	Distributivity Laws
$[F \lor (\neg p \land q)] \rightarrow q$	Negation Laws
$[\neg p \land q] \rightarrow q$	Identity Laws
$(\neg[\neg p \land q]) \lor q$	Implication Equivalence
$(p \lor \neg q) \lor q$	De Morgan's Laws
$p \lor (\neg q \lor q)$	Associative Laws
$p \lor T$	Negation Laws
T	Domination Laws

LHS is a tautology. But RHS is not. So, they aren't logically equivalent.

Q7) *The Labyrinth Guardians.* (10 marks)

Freedom is behind the **green door!**

You are walking in a labyrinth and all of a sudden you find yourself in front of three possible roads: the road on your left is paved with gold, the one in front of you is paved with marble, while the one on your right is made of small stones. Each street is protected by a guardian. You talk to the guardians and this is what they tell you:

- *The guardian of the gold street:* "This road will bring you straight to the center. Moreover, if the stones take you to the center, then also the marble takes you to the center."
- *The guardian of the marble street:* "Neither the gold nor the stones will take you to the center."
- *The guardian of the stone street:* "Follow the gold and you'll reach the center, follow the marble and you will be lost."

Given that you know that all the guardians are liars, can you choose a road being sure that it will lead you to the center of the labyrinth? If this is the case, which road you choose?

Given Language

• g: "the gold road brings to the center"

• m: "the marble road brings to the center"

• s*:* “the stone road brings to the center”

Provide a propositional language for the set of axioms (statements) given below that formalize the problem and find a road being sure it will lead to the center using truth table.

Axioms

1. *“The guardian of the gold street is a liar”*
   Formulation*:* $\neg(g \wedge (s \rightarrow m))$, $\neg g \vee (s \wedge \neg m)$

2. *“The guardian of the marble street is a liar”*
   Formulation*:* $\neg(\neg g \wedge \neg s)$, $g \vee s$

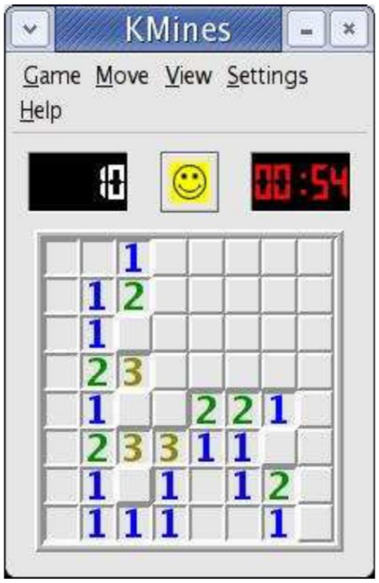
3. *“The guardian of the stone street is a liar”*
   Formulation*:* $\neg(g \wedge \neg m)$, $\neg g \vee m$

   Using Truth table on AND of the formulations of all axioms:
   We have two possible interpretations that satisfy the axioms, and in both of them the stone street brings to the center.
   Thus we can choose the **stone street** being sure that it leads to the center.

Q8) Minesweeper is a single-player computer game invented by Robert Donner in 1989. The object of the game is to clear a minefield without detonating a mine. The game screen consists of a rectangular field of squares. Each square can be cleared, or uncovered, by clicking on it. If a square that contains a mine is clicked, the game is over. If the square does not contain a mine, one of two things can happen: (1) A number between 1 and 8 appears indicating the amount of adjacent (including diagonally-adjacent) squares containing mines, or (2) no number appears; in which case there are no mines in the adjacent cells. An example of game situation is provided in the following figure: Provide predicate logic that allows to formalize the knowledge of a player in a game state. In such a language you should be able to formalize the following knowledge: (10 marks)

Axioms:
1. there are exactly n *mines in the minefield*
2. if a cell contains the number 1, then there is exactly one mine in the adjacent cells.

Show by means of deduction that there must be a mine in the position (3,3) of the game state of picture given below.



Suggestion: define the predicate Adj(x, y) *to formalize the fact that two cells* x *and* y *are adjacent.*

Language

1. *A unary predicate mine, where mine(x) means that the cell x contains a mine*
2. *A binary predicate adj, where adj(x, y) means that the cell x is adjacent to the cell y*
3. *A binary predicate contains, where contains(x,n) means that the cell x contains the number n*

Solution:

Axioms

1. *There are exactly n mines in the game.*

$$\exists x_1,..\exists x_n\left(\bigwedge_{i=1}^{n}\mathsf{mine}(x_i)\wedge\forall y\left(\mathsf{mine}(y)\rightarrow\bigvee_{i=1}^{n}y=x_i\right)\right)$$

First Order Logic

2. *If a cell contains the number 1, then there is exactly one mine in the adjacent cells.*

$$\forall x.\,(\mathsf{contains}(x,1)\rightarrow\exists z.\,(\mathsf{adj}(x,z)\wedge\mathsf{mine}(z)\wedge\forall y.\,(\mathsf{adj}(x,y)\wedge\mathsf{mine}(y)\rightarrow y=z)))$$

3. *Show by means of deduction that there must be a mine in the position (3,3)*

   from Picture 3.1 we have:

   a. $\mathsf{contains}((2,2),1)$

   b. $\neg\mathsf{mine}((1,1))\wedge\neg\mathsf{mine}((1,2))\wedge\neg\mathsf{mine}((1,3))$

   c. $\neg\mathsf{mine}((2,1))\wedge\neg\mathsf{mine}((2,2))\wedge\neg\mathsf{mine}((2,3))$

   d. $\neg\mathsf{mine}((3,1))\wedge\neg\mathsf{mine}((3,2))$

   we can deduce:

   e. $\exists z.\,(\mathsf{adj}((2,2),z)\wedge\mathsf{mine}(z)\wedge\forall y.\,(\mathsf{adj}((2,2),y)\wedge\mathsf{mine}(y)\rightarrow y=z))$ $\rightsquigarrow$ *from a. and axiom 2*

   f. $\mathsf{mine}((1,1))\vee\mathsf{mine}((1,2))\vee\mathsf{mine}((1,3))\vee\mathsf{mine}((2,1))\vee\mathsf{mine}((2,2))\vee\mathsf{mine}((2,3))\vee\mathsf{mine}((3,1))\vee\mathsf{mine}((3,2))\vee\mathsf{mine}((3,3))$ $\rightsquigarrow$ *from e.*

   g. $\mathsf{mine}((3,3))$ $\rightsquigarrow$ *from b.,c.,d. and f.*

Q9) Let Loves(x,y) mean "x loves y", Traveler(x) mean "x is a traveler", City(x) mean "x is a city", Lives(x,y)mean "x lives in y." (2.5+2.5+5=10 marks)

A. $\exists x\ \forall y\ \forall z(City(x) \land Traveler(y) \land Lives(z,x)) \rightarrow (Loves(y,x) \land \neg Loves(z,x))$

Translate the following proposition into the most natural equivalent statement in English. Try to make the sentence as simple and as natural as possible.

Answer: A few of the translations that work here: "There is a city that all travelers love but everyone who lives there doesn't love." Also: "Some cities are loved by all travelers but no person who lives there." Many other translations within that context work well.

B. Translate the following statement into predicate logic. "No traveler loves the city they live in."

Answer: Either of these two translations worked well:

$\forall x \forall y((Traveler(x) \land City(y) \land Live(x,y)) \rightarrow \neg Love(x,y))$

$\forall x \exists y((Traveler(x) \land City(y) \land Live(x,y)) \rightarrow \neg Love(x,y))$

C. Translate the following statement into predicate logic. "Travelers love atmost two cities."

Answer:

Explanation: The important thing in this translation is to make sure that the traveler can love 0, 1, or 2 cities, but no more. Given that no solution accounted for 0, we are not penalizing solutions that just state 1 or 2, but not 3. Solutions that translated something other than that, but within the ballpark of 2 cities were given 5 points. Solutions not quantifying the amount of cities were given 4 or less.

Thisisonewaytoallowfor0,or1or2andnot3. Wecansaythatforanyone, if they are a traveler, then one of two things must be true. First, it could be the case that the traveler loves no cities. Second, if there are two cities that the traveler loves (and there's no restriction on whether those two cities are the same or not, so they could be the same or different - hence accounting for one or two), then for all other cities, if he loves them, then that city must be one of the first two. I simplify the predicates to T(x) for "x is a traveler," C(x) for "x is a city," and L(x,y) for "x loves y."

Final Answer: $\forall x(T(x) \rightarrow$
$(\forall d(C(d) \rightarrow \neg L(x,d)) \lor (\exists a \exists b \forall c(C(a) \land L(x,a) \land C(b) \land L(x,b) \land C(c) \land L(x,c)) \rightarrow ((a=c) \lor (b$
$=c)))))$

Q10) Use rules of inference to show that if $\forall x(P(x) \vee Q(x))$, $\forall x(\neg Q(x) \vee S(x))$, $\forall x(R(x) \rightarrow \neg S(x))$, and $\exists x \neg P(x)$are true, then $\exists x \neg R(x)$ is true. (2.5 marks)

We can set this up in two-column format. The proof is rather long but straightforward if we go one step at a time.

Step	**Reason**
1. $\exists x \neg P(x)$	1. Premise
2. $\neg P(c)$	2. Existential instantiation using (1)
3. $\forall x(P(x) \vee Q(x))$	3. Premise
4. $P(c) \vee Q(c)$	4. Universal instantiation using (3)
5. $Q(c)$	5. Disjunctive syllogism using (4) and (2)
6. $\forall x(\neg Q(x) \vee S(x))$	6. Premise
7. $\neg Q(c) \vee S(c)$	7. Universal instantiation using (6)
8. $S(c)$	8. Disjunctive syllogism using (5) and (7), since $\neg\neg Q(c) = Q(c)$
9. $\forall x(R(x) \rightarrow \neg S(x))$	9. Premise
10. $R(c) \rightarrow \neg S(c)$	10. Universal instantiation using (9)
11. $\neg R(c)$	11. Modus tollens using (8) and (10), since $\neg\neg S(c) = S(c)$
12. $\exists x \neg R(x)$	12. Existential generalization using (11)

Q11) Determine whether this argument is valid using rules of inference. Provide rules of inference used in each step (5 marks)

If Superman were able and willing to prevent evil, he would do so. If Superman were unable to prevent evil, he would be impotent; if he were unwilling to prevent evil, he would be malevolent. Superman does not prevent evil. If Superman exists, he is neither impotent nor malevolent. Therefore, Superman does not exist.

Ans: Let us take these clauses.

W : Superman is willing to prevent evil.
A : Superman is able to prevent evil.
P : Superman prevents evil.
I : Superman is impotent.
M : Superman is malevolent.
E : Superman exists

1. Able and willing to prevent evil, he would do so $(A \land W) \rightarrow P$
2. Unable to prevent evil, he would be impotent $\neg A \rightarrow I$
3. Unwilling to prevent evil, he would be malevolent $\neg W \rightarrow M$
4. Superman does not prevent evil $\neg P$
5. If Superman exists, he is neither impotent nor malevolent $E \rightarrow \neg I \land \neg M$

Taking *contrapositive* of 1., $\neg P \rightarrow (\neg A \lor \neg W)$

i.e., if Superman does not prevent evil then he would either be unable or unwilling.

By *implication* from 2. and 3., $\neg P \rightarrow I \lor M$

Taking *contrapositive* of RHS, $\neg P \rightarrow \neg(\neg I \land \neg M)$

Contradiction with 5., $E \rightarrow \neg I \land \neg M$ is true only if E is false.

I.e., if $\neg P$ is true, then E is false.

Hence, Superman doesn't exist (given if the provided inferences 1-5 are true).

"What is the point of being clever if you can't prove it?" - Sherlock Holmes

Q12) Prove that √ 2 is irrational by giving a proof by contradiction. (5 marks)

Ans: Let √ 2 be rational.

Then, we can write √ 2 in the form of a/b where a,b are coprime integers.

$\sqrt{2} = a/b$

Squaring both sides,

$2 = a^2/b^2$

$a^2 = 2b^2$

As a and b are coprime, the cases where either one of a,b is a multiple of 2 or neither are.

If a is not a multiple of 2 (b may or may not be), then the above equation results in a contradiction. If a is a multiple of 2, then a can be written as

$a = 2^n c$ where c is odd and n>0

Then, $2^{2n}c^n = 2b^2$

$2^{2n-1}c^n = b^n$

Which results in a contradiction as b cannot be a multiple of 2 when a is.

Hence, our assumption that √ 2 is rational is incorrect.

Q13) Prove that if n is an integer and n^3 + 5 is odd, then n is even using: (2.5 + 2.5 = 5 marks)

a) a proof by contraposition.
b) a proof by contradiction.

Ans :

a) Contrapositive -> If n is odd, then n^3 + 5 is even.

If n is odd then n^3 is also odd.
If n^3 is odd, then n^3 + 5 is even (odd + odd = even).
Hence, contrapositive is true => statement must be true.

b) Contradiction -> n is odd.
If n is odd then n^3 is also odd.
If n^3 is odd, then n^3 + 5 is even. (from above part)
Contradicts original statement => our assumption is wrong.
Hence, n must be even.

b)pow(n,3)+5 is odd implies pow(n,3) is even implies n is even contradicting the assumption that n is odd. So, n is even.