

Corollary of the
[Sierpinski version of Bollobas Thm] Let (A_1, A_2, \dots, A_m)
and (B_1, B_2, \dots, B_m) be two sequences of
sets such that $\forall i, j \in [m]$, (i) $A_i \cap B_j = \emptyset$,
and (ii) $|A_i \cap B_j| \neq \emptyset$ if $i < j$. Further,
 $\forall i \in [m]$, $|A_i| \leq a$, $|B_i| \leq b$. Then,

$$m \leq \binom{a+b}{a}.$$

System of distinct representatives

A system of distinct representatives of the sets S_1, S_2, \dots, S_k is the k -tuple (x_1, x_2, \dots, x_k) such that

- (i) $\forall i \in [k], x_i \in S_i$ and (ii) $\forall i, j \in [k]$
with $i \neq j$, we have $x_i \neq x_j$.
distinct

Further, (x_1, x_2, \dots, x_k) is a strong system of distinct representatives

if (i), (ii) and (iii)
 $\forall i, j \in [k], i \neq j, x_i \notin S_j$.

A strong system of distinct representatives for the sets S_1, S_2, \dots, S_k is a k -tuple (x_1, x_2, \dots, x_k) where

(i) $\forall i \in [k], x_i \in S_i$, and

(ii) $\forall i, j \in [k], i \neq j, x_i \notin S_j$.

$$S_1 = \{3, 4, 5, 9\}, S_2 = \{1, 2, 4, 8\}$$

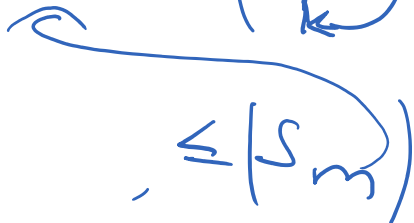
$$S_3 = \{1, 4, 7, 5\}$$

$(3, 2, 7)$ is a strong syst.
of dist. rep. for S_1, S_2, S_3 .

Theorem [Furedi, Tazac, 1985]. Let \mathcal{F} be a family of size greater than $\binom{r+k}{k}$. Further, every set in \mathcal{F} is of size at most r . Then, there exist some $k+1$ sets in \mathcal{F} that have a strong system of distinct representatives.

Julner \leftarrow $k+2$??

Proof? Arrange the sets in \mathcal{F}
 in non-decreasing order of their
 sizes.
 Let $\mathcal{F} = \{S_1, \dots, S_m\}$
 $|S_1| \leq |S_2| \leq \dots \leq |S_m|$

$m \geq \binom{r+k}{k}$


Assume, for the sake of contradiction,
 that no $k+1$ sets in \mathcal{F} have
 a strong system of dist. rep.

ASSUMING: no $k+1$ subcollection has strong sys of dist rep. GIVEN: (a) $m > \binom{r+k}{k}$ (b) $|S_i| \leq r$

$$S_1 \leq S_2 \leq \dots \leq S_j \leq \dots \leq S_m$$

$$T_1 \quad T_2 \quad T_i \quad \dots \quad T_j \quad \dots \quad T_m$$

a minimal set that intersects

$$S_j \setminus S_1, S_j \setminus S_2, \dots, S_j \setminus S_{i-1}, S_j \setminus S_{i+1}, \dots, S_j \setminus S_m$$

all are non-empty.

Property

For every element $e \in T_j$, \exists set $S \in \{S_1, S_2, \dots, S_m\}$ such that $T_j \cap S = \{e\}$.

Claim

$$\forall j, |T_j| \leq k.$$

Proof of claim Suppose not.

$$\text{Suppose } \exists j, |T_j| \geq k+1.$$

$$T_j = \{e_1, e_2, \dots, e_{k+1}, \dots\}$$

$$T \cap S_1 = \{e_1\}$$

$$S_2$$

$$S_{k+1}$$

s_1, s_2, \dots, s_{k+1} have
 $(e_1, e_2, \dots, e_{k+1})$ as a strong
system of distinct repr.

By skew version of Bollobas Thm,

$$m \leq \binom{r+k}{k}.$$

But, this contradicts the fact
 that $m > \binom{r+k}{k}$.

Hence, our assumption that no
subcollection of \mathcal{F} has a strong
system of distinct rep is false

