CS1340: DISCRETE STRUCTURES II

QUIZ II

Instructions

- Answer all the questions
- Total Marks : 15 marks Max time : 45 minutes.

(1) Given the Fibonacci sequence $\langle F_n \rangle$, where $F_0 = 0, F_1 = 1$ and $F_{n+2} = F_n + F_{n+1}$.

(a) Prove that the generating function $G(x)$ corresponding to $\langle F_n \rangle$ is $\frac{x}{1-x-x^2}$.

(5 marks)

Proof:
We have the g.f of the Fibonacci sequence $G(x) = \sum_{n\geq 0} F_n x^n = x + 1x^2 + 2x^3 + 3x^4 + \cdots$.
We have by definition of the Fibonacci sequence,

$$F_{n+2} = F_n + F_{n+1}.$$

Multiplying both sides by x^{n+2} we get,

$$F_{n+2}x^{n+2} = F_n x^{n+2} + F_{n+2}x^{n+2}.$$

Summing from $n = 0$ to ∞ we have,

$$\sum_{n=0}^{\infty} F_{n+2}x^{n+2} = \sum_{n=0}^{\infty} F_n x^{n+2} + \sum_{n=0}^{\infty} F_{n+1}x^{n+2}.$$

Substituting using the definition of $G(x)$,

$$G(x) - x = x^2G(x) + xG(x)$$

$$G(x)(1 - x^2 - x) = x$$

$$G(x) = \frac{x}{1 - x - x^2}.$$

(b) Let $\phi = \frac{(1+\sqrt{5})}{2}$ and $\hat{\phi} = \frac{(1-\sqrt{5})}{2}$. We have, $\phi + \hat{\phi} = 1$, $\phi - \hat{\phi} = \sqrt{5}$ and $\phi \cdot \hat{\phi} = -1$.
Therefore, $1 - x - x^2 = (1 - \phi x)(1 - \hat{\phi}x)$. This implies

$$G(x) = \frac{x}{1 - x - x^2} = \frac{x}{(1 - \phi x)(1 - \hat{\phi}x)} = \frac{1}{\sqrt{5}(1 - \phi x)} - \frac{1}{\sqrt{5}(1 - \hat{\phi}x)}.$$

(i) Show that the Fibonacci convolution $\sum_{k=0}^{n} F_k F_{n-k}$ is the coefficient of x^n in

$$\frac{1}{5}\sum_{n\geq 0}(n+1)\phi^n x^n - \frac{2}{5}\sum_{n\geq 0} F_{n+1}x^n + \frac{1}{5}\sum_{n\geq 0}(n+1)\hat{\phi}^n x^n.$$

(5 marks)

Proof:
$\sum_{k=0}^{n} F_k F_{n-k}$ is the coefficient of x^n in $(G(x))^2$ from the product theorem we learned in class. We have,

$$G(x) = \frac{1}{\sqrt{5}(1-\phi x)} - \frac{1}{\sqrt{5}(1-\hat{\phi} x)}.$$

Therefore,

$$\begin{aligned}(G(x))^2 &= \frac{1}{5}[\frac{1}{(1-\phi x)^2} - \frac{2}{(1-\phi x)(1-\hat{\phi}x)} + (1-\hat{\phi}x)^2]\\ &= \frac{1}{5}\sum_{n\geq 0}(n+1)\phi^n x^n - \frac{2}{5}\sum_{n\geq 0} F_{n+1}x^n + \frac{1}{5}\sum_{n\geq 0}(n+1)\hat{\phi}^n x^n\end{aligned}$$

since $\frac{1}{(1-ax)^2} = \sum_{n\geq 0}(n+1)a^n x^n$ and $\frac{1}{(1-\phi x)(1-\hat{\phi}x)} = \frac{G(x)}{x}$.

(ii) Simplify the above result to obtain the following closed form solution,

$$\sum_{k=0}^{n} F_k F_{n-k} = \frac{2nF_{n+1} - (n+1)F_n}{5}.$$

(5 marks)

Proof:
We have from formal power series expansion,

$$\begin{aligned}\phi^n + \hat{\phi}^n &= \text{ coefficient of } x^n \text{ in } [\frac{1}{1-\phi x} + \frac{1}{1-\hat{\phi}x}]\\ &= [x^n]\frac{2-x}{1-x-x^2}\\ &= 2F_{n+1} - F_n\end{aligned}$$

from Q 1(a), using the fact that the g.f of F_n is $\frac{1}{1-x-x^2}$.

Replacing in $2(b)(i)$ we get,

$$\begin{aligned}\sum_{k=0}^{n} F_k F_{n-k} &= [x^n][\frac{1}{5}\sum_{n\geq 0}(n+1)(2F_{n+1}-F_n)x^n - \frac{2}{5}\sum_{n\geq 0}F_{n+1}x^n] \\ &= [x^n][\frac{1}{5}[\sum_{n\geq 0} 2nF_{n+1}-(n+1)F_n]x^n]\end{aligned}$$

($[x^n]$ is used to represent the coefficient of x^n in the expansion).
Therefore, we have $\sum_{k=0}^{n} F_k F_{n-k} = \frac{2nF_{n+1}-(n+1)F_n}{5}$.