Dilworth Theorem 2

Let $P = (X, \leq)$ be a poset and let r be the length of a largest antichain in P. Then the elements of X can be partitioned into r chains.

Proof: Induction on $|X|$.

Base Case: $|X| = 1$.

Induction Hypo: Assume the stmt is true for all posets defined on X with $|X| \leq n-1$

Induction Step: Let $P = (X, \leq)$ be a poset where $|X| = n$.

Let $x \in X$ be a maximal element of $P = (X, \preceq)$.

it means there is no element $y \in X$, $y \neq x$ such that $x \preceq y$.

Let $P' = (X \setminus \{x\}, \preceq)$ be a subposet of P.

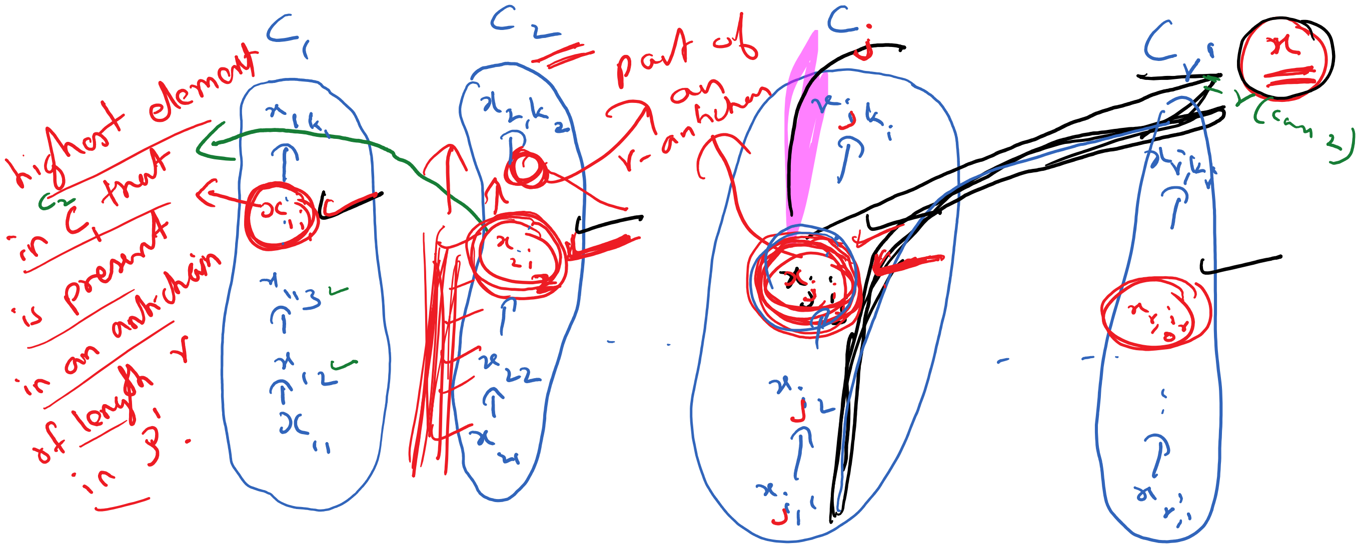
set minus

Let r' be the length of a largest antichain in P'.

By Induction hypothesis, the element of X' can be partitioned into r' chains.

Let these chains be $C_1, C_2, \ldots, C_{r'}$.

$P' = (X', \preceq)$ $r' \to$ length of largest antichain in P'.



$X' = C_1 \uplus C_2 \uplus \cdots \uplus C_{r'}$ $C'_j = \{x_{j1}, \ldots, x_{ji_j}, x\}$

~~Case 1~~: $r' < r$. Then $r' = r - 1$. $D'_j = \{x_{j i_{j+1}}, \ldots, x_{jk_j}\}$

$C_r = x$

Case 2: $r' = r$.

$X' = C_1 \uplus C_2 \uplus \cdots \uplus C_{r=r'}$

~~Claim~~ Claim $A' = \{x_{1i_1}, x_{2i_2}, \ldots, x_{ri_r}\}$ is an antichain of length r in P'.

Q. Is $A = A' \cup \{x\}$ an antichain in P?

NO

Let $P'' = (X \setminus C'_j, \preceq)$.

Let $P'' = (X \setminus C_j', \leq)$.

Claim: Length of largest antichain in P'' is $\leq r-1$.

By Ind hypothesis, $X \setminus C_j'$ can be partitioned into $r-1$ chains.

$\Downarrow$

X can be partitioned into r chains.

$\hookrightarrow$ add C_j'

$\square$