MA 1140: Elementary Linear Algebra

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Welcome!

- Welcome to my course MA 1140.
- We'll study Elementary Linear Algebra in the next four weeks.
- I will follow the text books:
  "Linear Algebra" written by K. Hoffman and R. Kunze
  "Linear Algebra and Its Applications" by Gilbert Strang.
- I will make the lecture notes. So you can follow that also.
- In case you need any further assistance, please get in touch with me or one of your course associates.
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What is Linear Algebra?

- Linear algebra is the branch of mathematics concerning linear equations such as

$$a_1x_1 + \cdots + a_nx_n = b$$

linear functions such as

$$(x_1, \ldots, x_n) \mapsto a_1x_1 + \ldots + a_nx_n$$

and their representations through matrices and vector spaces.
- It is central to almost all areas of mathematics.
- For instance, linear algebra is fundamental in modern presentations of geometry: for describing basic objects such as lines, planes and rotations.
- It is also used in most sciences and engineering areas, because it allows modeling many natural phenomena, and efficiently computing with such models. I will get back to this point later.

Vector Space

- MA 1140 is the study of ‘vector spaces’ and the ‘maps’ between them.
- For now, we keep $\mathbb{R}^n$ as an example of a vector space.
- Essentially, a vector space means a collection of objects, we call them vectors, where we can add two vectors, and what we get is a vector; we can multiply a vector by a scalar, and what we get is a vector.

Definition of Vector Space

A set V of objects (called vectors) along with vector addition '$+$' and scalar multiplication '$\cdot$' is said to be a vector space over a field $\mathbb{F}$ (say, $\mathbb{F} = \mathbb{R}$, the set of real numbers) if the following hold:

1. V is closed under '$+$', i.e. $x + y \in V$ for all $x, y \in V$.
2. Addition is commutative, i.e. $x + y = y + x$ for all $x, y \in V$.
3. Addition is associative, i.e. $(x + y) + z = x + (y + z)$ for all $x, y, z \in V$.
4. Additive identity, i.e. there is $0 \in V$ such that $x + 0 = x$ for all $x \in V$.
5. Additive inverse, i.e. for every $x \in V$, there is $-x \in V$ such that $x + (-x) = 0$.
6. V is closed under '$\cdot$', i.e. $c \cdot x \in V$ for all $c \in \mathbb{F}$ and $x \in V$.
7. $1 \cdot x = x$ for all $x \in V$.
8. $a \cdot (b \cdot x) = (ab) \cdot x$ for all $a, b \in \mathbb{R}$ and $x \in V$.
9. $a \cdot (x + y) = a \cdot x + a \cdot y$ for all $a \in \mathbb{R}$ and $x, y \in V$.
10. $(a + b) \cdot x = a \cdot x + b \cdot x$ for all $a, b \in \mathbb{R}$ and $x \in V$.

Examples of vector spaces

(1) The n-tuple space, $V = \mathbb{R}^n$.

$$\begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} + \begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{pmatrix} = \begin{pmatrix} x_1 + y_1 \\ x_2 + y_2 \\ \vdots \\ x_n + y_n \end{pmatrix} \quad \text{and} \quad c \cdot \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} cx_1 \\ cx_2 \\ \vdots \\ cx_n \end{pmatrix}$$

(2) The space $\mathbb{R}^{m \times n}$ of all $m \times n$ matrices

$$\begin{pmatrix} x_{11} & x_{12} & \cdots & x_{1n} \\ x_{21} & x_{22} & \cdots & x_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ x_{m1} & x_{m2} & \cdots & x_{mn} \end{pmatrix} \quad \text{where } x_{ij} \in \mathbb{R}.$$

Component wise addition and component wise scalar multiplication.

(3) Let S be any non-empty set. Let V be the set of all functions from S into $\mathbb{R}$. The sum $f+g$ of two vectors f and g in V is defined to be

$$(f+g)(s) := f(s) + g(s) \text{ for all } s \in S.$$

The scalar multiplication $c \cdot f$ (for $c \in \mathbb{R}$) is defined by

$$(c \cdot f)(s) := c\, f(s) \text{ for all } s \in S.$$

Is V a vector space? Answer: Yes.

(4) The set $\mathbb{R}[x]$ of all polynomials $a_0 + a_1 x + \cdots + a_m x^m$, where $a_i \in \mathbb{R}$, x is an indeterminate and m varies over non-negative integers. The vector addition and scalar multiplication are defined in obvious way. Then $\mathbb{R}[x]$ is a vector space over $\mathbb{R}$.

Examples

(5) The set $\mathbb{R}^{n \times n}$ of all $n \times n$ matrices with vector addition:

$$\begin{pmatrix} x_{11} & x_{12} & \cdots & x_{1n} \\ x_{21} & x_{22} & \cdots & x_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ x_{n1} & x_{n2} & \cdots & x_{nn} \end{pmatrix} \times \begin{pmatrix} y_{11} & y_{12} & \cdots & y_{1n} \\ y_{21} & y_{22} & \cdots & y_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ y_{n1} & y_{n2} & \cdots & y_{nn} \end{pmatrix}$$

$$= \begin{pmatrix} -- & -- & -- \\ -- & (\sum_{k=1}^{n} x_{ik} y_{kj}) & -- \\ -- & -- & -- \end{pmatrix} \quad \text{(matrix multiplication)}$$

and scalar multiplication as before.

Is V a vector space? Answer: No.

Reasons:

- The operation '$\times$' is not **commutative**, because

$$\begin{pmatrix}1 & 0\\ 0 & 0\end{pmatrix}\times\begin{pmatrix}0 & 1\\ 0 & 0\end{pmatrix}=\begin{pmatrix}0 & 1\\ 0 & 0\end{pmatrix}\neq\begin{pmatrix}0 & 0\\ 0 & 0\end{pmatrix}=\begin{pmatrix}0 & 1\\ 0 & 0\end{pmatrix}\times\begin{pmatrix}1 & 0\\ 0 & 0\end{pmatrix}$$

- Moreover, every matrix does not necessarily have multiplicative inverse. For example, $\begin{pmatrix}1 & 0\\ 0 & 0\end{pmatrix}$ does not have multiplicative inverse, because it is not possible to find a matrix A such that

$$\begin{pmatrix}1 & 0\\ 0 & 0\end{pmatrix}\times A=\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}.$$

Linear Combination

Definition

A vector β in V is said to be a **linear combination** of vectors α_1, α_2 and α_r in V if

$$\beta = c_1\alpha_1 + c_2\alpha_2 + \cdots + c_r\alpha_r \text{ for some } c_1, c_2, \ldots, c_r \in \mathbb{R}.$$

Example

In $\mathbb{R}^2$,

$$\begin{pmatrix} 1 \\ 2 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ 2 \end{pmatrix} = 2\begin{pmatrix} 1 \\ 1 \end{pmatrix} + (-1)\begin{pmatrix} 1 \\ 0 \end{pmatrix}.$$

Definition

Let V be a vector space over $\mathbb{R}$. A subspace of V is a subset W of V which is itself a vector space over $\mathbb{R}$ with the same operations of vector addition and scalar multiplication on V.

Theorem

Let W be a non-empty subset of a vector space V over $\mathbb{R}$. Then W is a subspace of V

if and only if

for each pair of vectors $\alpha, \beta \in W$ and each scalar $c \in \mathbb{R}$, the vector $c\alpha + \beta$ belongs to W.

Examples

- The subset W consisting of the zero vector of V is a subspace of V.
- In $\mathbb{R}^n$, the set of n-tuples $(x_1, \ldots, x_n)$ with $x_1 = 0$ is a subspace; while the set of n-tuples with $x_1 = 1$ is NOT a subspace.
- The set of all 'symmetric matrices' forms a subspace of the space of all $n \times n$ matrices. Recall that an $n \times n$ square matrix A is said to be symmetric if $A_{ij} = A_{ji}$ for each i and j.

Definition

Let S be a set of vectors in a vector space V. The **subspace spanned** by (or generated by) S is defined to be the smallest subspace of V containing S.

Theorem

Let S be a set of vectors in a vector space V. The following subspaces are equal.

1. *The intersection of all subspaces of V containing S.*
2. *The set of all linear combinations of vectors in S, i.e.*

$$\{c_1 v_1 + \cdots + c_r v_r : c_i \in \mathbb{R},\ v_i \in S\}.$$

3. *The subspace spanned by S, i.e. the smallest subspace of V containing S.*

Subspace spanned by a set

Theorem

Let S be a subset of V. The following subspaces are equal.

1. *The intersection of all subspaces of V containing S.*
2. *The set of all linear combinations of vectors in S, i.e.* $\{c_1 v_1 + \cdots + c_r v_r : c_i \in \mathbb{R},\ v_i \in S\}$.
3. *The subspace spanned by S, i.e. the smallest subspace of V containing S.*

Proof. Let W_1, W_2 and W_3 be the subspaces described as in (1), (2) and (3) respectively. Clearly, W_1 is contained in any subspace of V containing S. Since W_1 is a subspace, W_1 is the smallest subspace of V containing S, i.e. $W_1 = W_3$.
Note that W_2 is a subspace containing S. So $W_1 \subseteq W_2$. Notice that any subspace of V containing S also contains all linear combinations of vectors in S. Hence it follows that $W_2 \subseteq W_1$. Therefore $W_1 = W_2$. Thus $W_1 = W_2 = W_3$. ∎

Examples

1. Let $V = \mathbb{R}^{2\times 2}$ be the space of all 2×2 matrices over $\mathbb{R}$. Set

$$S := \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \right\}.$$

   The subspace spanned by S is the subspace of all 2×2 symmetric matrices over $\mathbb{R}$.

2. Let $V = \mathbb{R}[x]$ (set of all polynomials). Set

$$S := \{f_n(x) = x^n : n = 0, 1, 2, \ldots\}.$$

   Then the subspace spanned by S is V.

Thank You!