CS6350 - Topics in Combinatorics

Assignment 2

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- CS18BTECH11001

Q. Prove Hall's theorem using Dilworth's Theorem.

Hall's marriage theorem:

Let $G = (V, E)$ be a bipartite graph with V_1 and V_2 as bipartite sets. Then a perfect matching exists iff for every subset $S \subseteq V_1$,

$$|S| \leq |\Gamma(S)| \text{ where } \Gamma(S) \text{ denotes Neighbourhood of } S$$

Dilworth's Theorem: In any finite partially ordered set (poset), the longest antichain has the same size as the number of chains in the poset.

Hall's theorem using Dilworth's theorem

Let $S_1, S_2, S_3, \dots S_n$ be the Neighbourhood sets and the elements of their union be $x_1, x_2, \dots x_m$.

Now let us consider a set Q

$$Q = \{x_1, x_2, \dots x_m, S_1, S_2, \dots S_n\}$$

and define a relation $\leq$ as

$$x_i \leq S_j \text{ iff } x_i \in S_j$$

else not comparable.

We can observe that Q is a poset with the binary relation $\leq$

Case-1:- if $|S| \leq |\Gamma(W)|$, we show that there is a perfect mapping.

Claim: The Maximum length of antichain that can be formed from poset Q is m (No. of x's)

Proof: We can observe that an antichain that can be formed using all x's contains m elements.

i.e., $\{x_1, x_2, \dots x_m\}$ is an antichain with m elements.

So the Maximimum length of antichain should be $\geq m$.

Now let us consider an anti-chain with a ~~x~~ elements and b 's' elements

i.e No. of x elements $= a$ ($< m$)

No. of s elements $= b$ ($< n$)

$P = \{x_1, x_2, \dots x_a, s_1, s_2, \dots s_b\}$

As these b 's' should contain only x's that are not in the antichain P which is ~~less than~~ atmost $m-y$. i.e.

i.e., $x \leq m-y$

$x+y \leq m$.

i.e length of antichain $\leq m$ and $\geq m$ (from before)

∴ The length of antichain that can be formed for poset Q is m. Hence proved.

From Dilworth's theorem

No. of chains = Maximum length of anti-chain.

No. of chains $= m$

There exists m covers with exactly one x_i and the chains over the cover are either $\{x_i\}$ or $\{x_i, X_j\}$

Since there are only m chains, all the 'S''s must appear alongside some or all of r's ($\because m \geq n$)
Hence, all the n ~~se~~ subsets have a match which determines that all the vertices of V_1 are matched
Therefore a perfect matching always exists.

case 2: if there exists a perfect mapping, then prove that $|T(s)| \geq |s|$

Let us consider an arbitary subset S of V_1.
As given that there exists a perfect matching, all the vertices in V_1 and $S(\subseteq V)$ must be matched.
And for every vertex in set S, there should be a uniquely matched vertex in set V_2, else two edges of the matching would share a vertex which leads to a contradiction.
From this we can able to say that there exists at least one unique neighbour for every vertex in S.

$\therefore |T(S)| \geq |S| \; \forall \; S \subseteq V_1$

Hence proved.