Fourier Series & Fourier Transforms

1) Let $f(x)$ be a function of period 2π s.t $f(x) = \begin{cases} 1 & -\pi < x < 0 \\ 0 & 0 < x < \pi \end{cases}$

a) Find the fourier series of $f(x)$ in the interval $-\pi < x < \pi$

Ans:) $\frac{1}{2} - \frac{2}{\pi}\left[\sin x + \frac{1}{3}\sin 3x + \frac{1}{5}\sin 5x + \cdots\right]$

b) Also prove that

$$\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots$$

Ans:) put $x = \pi/2$ in (a).

2) Find the fourier transform of $f(t) = e^{-|t|}$ Ans:) $\sqrt{\frac{2}{\pi}}\left(\frac{1}{1+x^2}\right)$

(a) Also deduce that $\int_0^\infty \frac{dx}{1+x^2} = \frac{\pi}{2}$ (by inversion formula)

(b) Deduce that $\int_0^\infty \frac{x \sin(xt)}{1+x^2}\,dx = \frac{\pi e^{-t}}{2}, \quad t > 0$

(differentiating in the inversion formula)

3) Find the fourier sine transform of x^{n-1}

Ans) $\frac{\Gamma n}{s^n}\sin\left(\frac{n\pi}{2}\right)$

4) Find the fourier cosine transform of x^{n-1}

Ans) $\frac{\Gamma n}{s^n}\cos\left(\frac{n\pi}{2}\right)$

5) Find the fourier sine transform of $2e^{-3x} + 3e^{-2x}$

Ans) $\sqrt{\frac{2}{\pi}}\left[\frac{5s^3 + 35s}{(9+s^2)(4+s^2)}\right]$

6) Find the fourier sine and cosine transform of $f(x) = e^{-x}$ and show that

$$\int_0^\infty \frac{\cos mx}{1+x^2}\,dx = \frac{\pi}{2}e^{-m} = \int_0^\infty \frac{x \sin mx}{1+x^2}\,dx$$

7) Find the fourier transform of $f(x) = \begin{cases} 1-x^2 & |x| \le 1 \\ 0 & |x| > 1 \end{cases}$ and show that

$$\int_0^\infty \left(\frac{x\cos x - \sin x}{x^3}\right)\cos\frac{x}{2}\,dx = -\frac{3\pi}{16}$$

Ans) $\frac{2\sqrt{2}}{\sqrt{\pi}}\left(\frac{\sin s - s\cos s}{s^3}\right)$

8) Find the fourier cosine transform of $f(x) = \frac{1}{1+x^2}$

Ans) $\sqrt{\frac{\pi}{2}}\,e^{-s}$

9) Find the fourier sine transform of $f(x) = \frac{e^{-ax}}{x}$ and use it to evaluate $\int_0^\infty \tan^{-1}\left(\frac{x}{a}\right)\sin x\,dx$

Ans) $\sqrt{\frac{2}{\pi}}\tan^{-1}\left(\frac{s}{a}\right)$

10) Given $t>0$ show that $\int_0^\infty \frac{\cos st}{s^2+a^2}\,ds = \frac{\pi}{2a}e^{-at}$ $(a>0)$

11) Find the fourier cosine transform of e^{-x^2}

Ans) $\frac{1}{\sqrt{2}}e^{-s^2/4}$

12) Find the fourier transform of xe^{-ax^2}, $a>0$

Ans) $\frac{ise^{-s^2/4a}}{2a\sqrt{2a}}$

13) Find the function $f(x)$ given its fourier cosine transforms.

(i) $\frac{\sin as}{s}$ Ans) $f(x) = \begin{cases} 1 & x<a \\ 0 & x>a \end{cases}$

Ans) $\frac{\sin^2 ax}{\pi x^2}$ ← (ii) $\begin{cases} \frac{1}{\sqrt{2\pi}}\left(a-\frac{s}{2}\right) & s<2a \\ 0 & s\ge 2a \end{cases}$

14) Find the function $f(x)$, given its fourier sine transform is e^{-ad} Ans) $\frac{2}{\pi} \frac{x}{a^2+x^2}$.

15) Find Fourier inverse of $\frac{1}{(a+\lambda^2)(a+\lambda^2)}$

Ans) $\frac{-1}{30} e^{-3|x|} + \frac{1}{20} e^{-2|x|}$