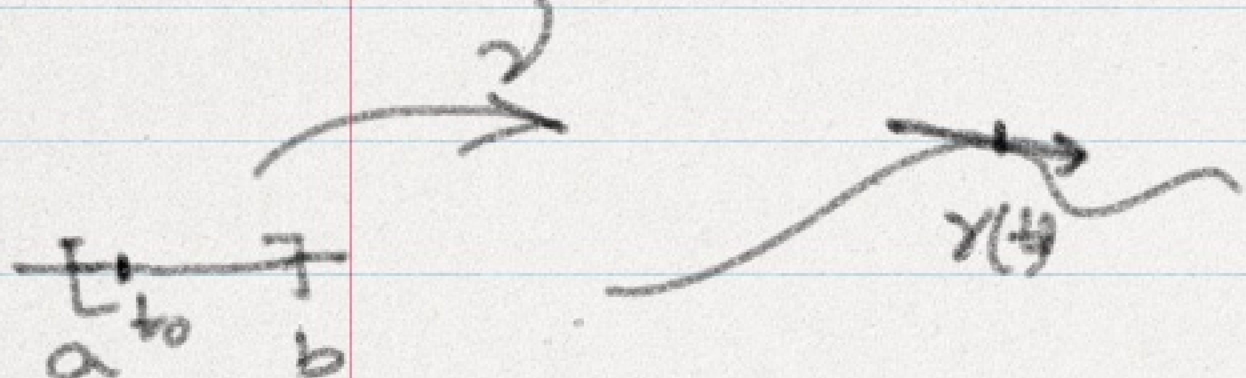
Let $\gamma : [a,b] \longrightarrow \mathbb{R}^n$ be a smooth curve.

$\gamma'(t)$

Along the curve γ at t_0 $:= \dfrac{\gamma'(t_0)}{\|\gamma'(t)\|}$.



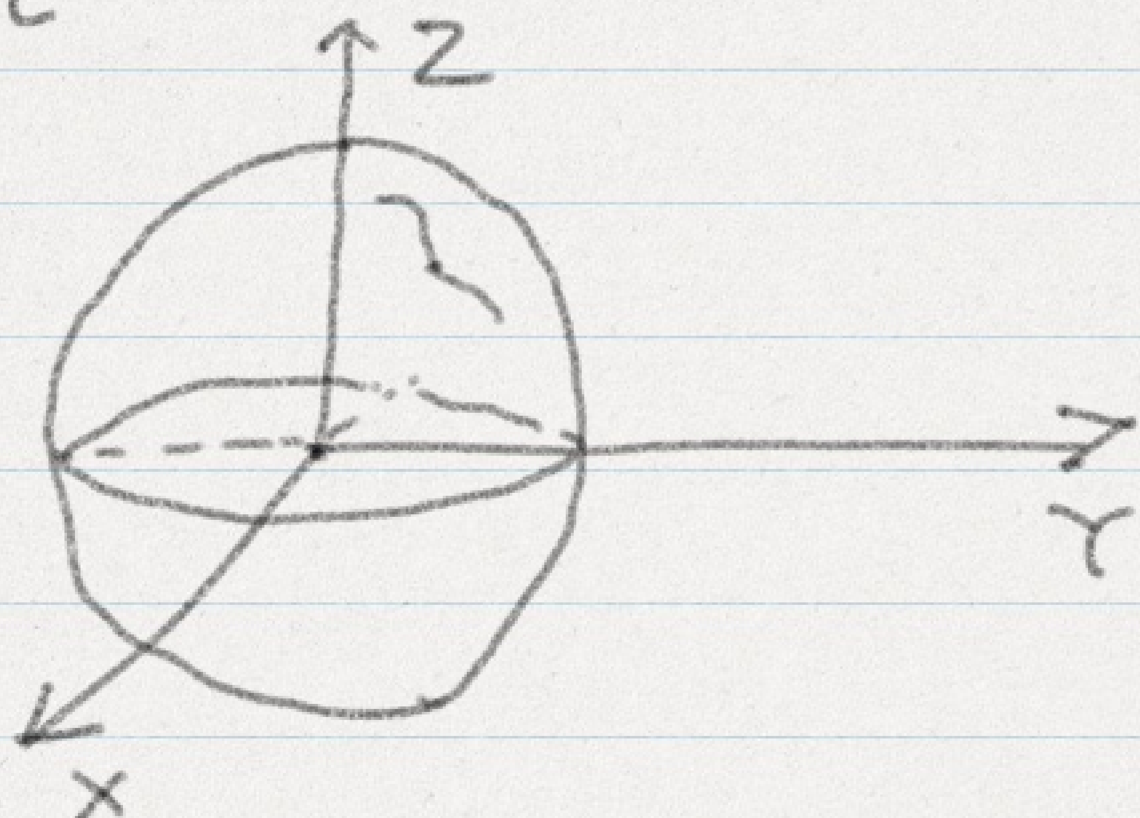
Let $f : \mathbb{R}^3 \longrightarrow \mathbb{R}$ be a diff scalar field.

<u>Defn</u>: By a surface in $\mathbb{R}^3$ we mean a set of the form

$$S = \{(x,y,z) \in \mathbb{R}^3 : f(x,y,z) = c\}$$

for some $c \in \mathbb{R}$.

Let $f(x,y,z) = x^2+y^2+z^2$
and
$S = \{(x,y,z) : f(x,y,z) = 1\}$



Let $\gamma : [a,b] \longrightarrow \mathbb{R}^3$ s.t
$\gamma([a,b]) \subset S$ where γ is a
smooth curve.

$\gamma : [a,b] \longrightarrow \mathbb{R}^3 \xrightarrow{f} \mathbb{R}$

$f \circ \gamma : [a,b] \longrightarrow \mathbb{R}$

$$(f \circ \gamma)'(t) = 0 \quad \forall\, t \in [a,b]$$

$$f'(\gamma(t)) \cdot \gamma'(t) = 0 \quad \forall\, t \in [a,b]$$

$$\nabla f(\gamma(t)) \cdot \gamma'(t) = 0 \quad \forall\, t \in [a,b]$$

Hence the vector $\nabla f(\gamma(t))$ is orthogonal to $\gamma'(t)$

— True for all $t \in [a,b]$.

Hence we can conclude that.

If γ is a smooth curve lying on the surface $S = \{\vec{x} \in \mathbb{R}^n : f(\vec{x}) = c\}$ Then $\nabla f(\gamma(t))$ is orthogonal to the tangent to the curve at t.

Let $t \in [a,b]$, then $\gamma(t) \in \mathbb{R}^n$

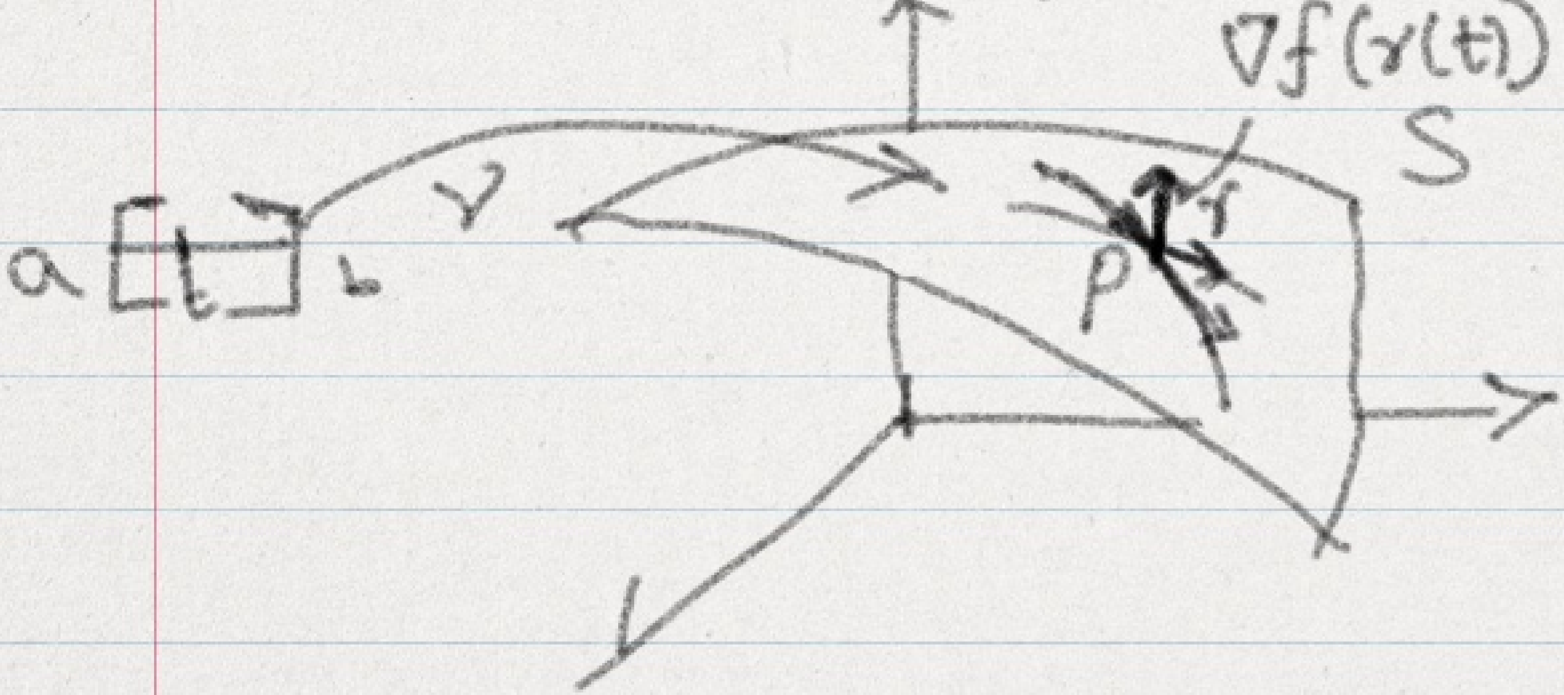
$\dfrac{\gamma'(t)}{\|\gamma'(t)\|} \in \mathbb{R}^n$ be a unit vector.

Now at $\gamma(t)$ the directional derivative of f at $\gamma(t)$ is zero.

Because. $D_{\vec{v}} f(x_0) = f'(x_0) \cdot \vec{v}$

$\vec{v} = \dfrac{\gamma'(t)}{\|\gamma'(t)\|}$, $\vec{x}_0 = \gamma(t)$

Let $S = \{(x,y,z) \in \mathbb{R}^3 ; f(x,y,z) = c\}$ be a smooth surface in $\mathbb{R}^3$.



The vector $\nabla f(P)$ is orthogonal

to the tangent to the curve passing through the point P and lying on the surface S.

Hence equation of the tangent plane at P to the surface S is

$$(\vec{x} - \vec{x_0}) \cdot \nabla f(\vec{x_0}) = 0$$

Ex: Find the cartesian equation of the tangent plane to the surface $xyz = a^3$ at (x_0, y_0, z_0).

Ex: Let $S = \{\vec{x} \in \mathbb{R}^n : \|\vec{x}\| = 1\}$

Let $\vec{x_0} \in S$, find $\nabla \|\cdot\| (\vec{x_0})$.

Ans: $\nabla \|\cdot\| (\vec{x_0}) = \vec{x_0}$.

If $f : \mathbb{R}^n \longrightarrow \mathbb{R}$ then

$\frac{\partial f}{\partial x_i} : \mathbb{R}^n \longrightarrow \mathbb{R}$

Let $\frac{\partial}{\partial x_j}\left(\frac{\partial f}{\partial x_i}(x)\right) = \frac{\partial^2 f}{\partial x_j \partial x_i}(\vec{x_0})$.

Clairaut' Thm: Let $f : \mathbb{R}^n \longrightarrow \mathbb{R}$ be continuously diff. at $\vec{x_0}$ then all 2nd order mixed partial derivatives are same.
Ref: Apostol.

Maxima / minima