CS1340: DISCRETE STRUCTURES II

EXAM I

Instructions

- Answer all the questions
- Total Marks : 30 marks Max time : 1 hour 25 minutes.

(1) Prove that the set of all irrational numbers is uncountable.

(5 marks)

Proof: The set of real numbers is the union of the set of rational numbers and the set of irrational numbers. We have already seen that the set of rational numbers is countable and the set of real numbers is uncountable. We also saw that the union of countable sets is countable. Lets assume that the set of all irrational numbers is countable but then the union of these two countable sets, rational numbers and irrational numbers, will be countable. The union is the set of real numbers which is uncountable and therefore a contradiction. This implies the set of irrational numbers is uncountable.

(2) Consider the following two cases:

(a) How many ways are there to split a dozen people into 3 teams, where one team has 2 people and the other two teams have 5 people each?

(b) How many tuples of integers (x_1, x_2, x_3, x_4) are there such that $0 \leq x_1 \leq x_2 \leq x_3 \leq x_4 \leq 10$?

(5 marks)

Answer:

(a) First we pick team of 2, then two teams of 5. But to avoid overcounting of the two teams of 5 we have to divide by 2. Therefore the answer is $C(12, 2)C(10, 5)/2$. If you clearly mention "labelled" teams or distinguishable teams then the answer is $C(12, 2)C(10, 5)$.

(b) Imagine each x_i as a "choice" of an integer. The problem is then to choose 4 integers from 11 possibilities with repetition allowed. The number of ways is $C(11 + 4 - 1, 11 - 1) = C(14, 10) = 1001$.

(3) Prove the following identity:

Let n, k, l be nonnegative integers with k not exceeding n and l not exceeding either n or k, then $C(n, k)C(k, l) = C(n, l)C(n - l, k - l)$.

(5 marks)

Proof: Let X be an n-set. Count the number of pairs (Y, Z) such that $Z \subseteq Y \subseteq X$ and $|Y| = k$, $|Z| = l$. There are $C(n, k)$ choices for Y, and then $C(k, l)$ of an l-subset Z of Y. Or we could count: There are $C(n, l)$ choices of Z, then we obtain Y by choosing $k - l$ points from $X \setminus Z$ and add to Z. Since $|X \setminus Z = n - l$, this can be done in $C(n - l, k - l)$ ways.

(4) Solve the recurrence $a_n = 6a_{n-1} - 9a_{n-2}$ with $a_0 = 1$ and $a_1 = 6$.

(5 marks)

Proof: Find its characteristic equation $x^2-6x+9=0$, which is $(x-3)^2=0$. Therefore 3 is a root with multiplicity 2. So we have $a_n = (\alpha_1+\alpha_2 n)3^n$ is a solution. Now we find constants using boundary conditions. We have, $\alpha_0 = a_0 = 1$. $a_1 = 3\alpha_0 + 3\alpha_1 = 6$, $\alpha_1 = 1$. Therefore, $a_n = 3^n + n3^n$ is a solution.

(5) Solve by first finding a recurrence relation. The bank charges 18% per month interest on the outstanding balance on your credit card. If you have a balance of 1 rupee and you forgot to pay it for it was such a small balance, how much will you owe the bank in 5 years?

(5 marks)

Proof: $P_n = P_{n-1} + .18P_{n-1}$, where P_n is the balance at the n th month. Therefore, the recurrence relation is $P_n - 1.18P_{n-1} = 0$. Also, $P_0 = 1$. Consider the recurrence relation, $x = 1.18$. So we have $P_n = \alpha_1(1.18)^n$. We have, $P_0 = 1 = \alpha_1$. Therefore the final recurrence is $P_n = 1.18^n$. There are $5 * 12 = 60$ months and therefore the balance after 60 months is $(1.18)^{60}$. (This value is approximately $20,555$ rupees but you don't have to simplify that much.)

(6) You are climbing a stair case. It takes n steps to reach to the top. Each time you can either climb 1 or 2 steps. In how many distinct ways can you climb to the top?

(5 marks)

Proof: Let F_n be the number of ways to climb n stairs taking only 1 or 2 steps. We can see that $F_0 = 0$ and $F_1 = 1$. If $n \geq 3$, the final step will be of size 1 or 2 so $F_n = F_{n-1} + F_{n-2}$. Thus we have the Fibonacci recurrence and how to solve this was discussed in class.

The corresponding characteristic equation is $x^2 = x+1$. We have $x = \frac{1 \pm \sqrt{5}}{2}$.

From the first condition, we know:

$$f(0) = c_1\left(\frac{1+\sqrt{5}}{2}\right)^0 + c_2\left(\frac{1-\sqrt{5}}{2}\right)^0 = c_1 + c_2 = 0.$$

From the second boundary condition, we have:

$$f(1) = c_1\left(\frac{1+\sqrt{5}}{2}\right)^1 + c_2\left(\frac{1-\sqrt{5}}{2}\right)^1 = 1.$$

From this we have $c_1 = 1/\sqrt{5}$ and $c_2 = -1/\sqrt{5}$.

We have a complete solution to the Fibonacci recurrence with boundary conditions:

$$f(n) = \frac{1}{\sqrt{5}}\left(\frac{1+\sqrt{5}}{2}\right)^n - \frac{1}{\sqrt{5}}\left(\frac{1-\sqrt{5}}{2}\right)^n$$