Unbalancing lights

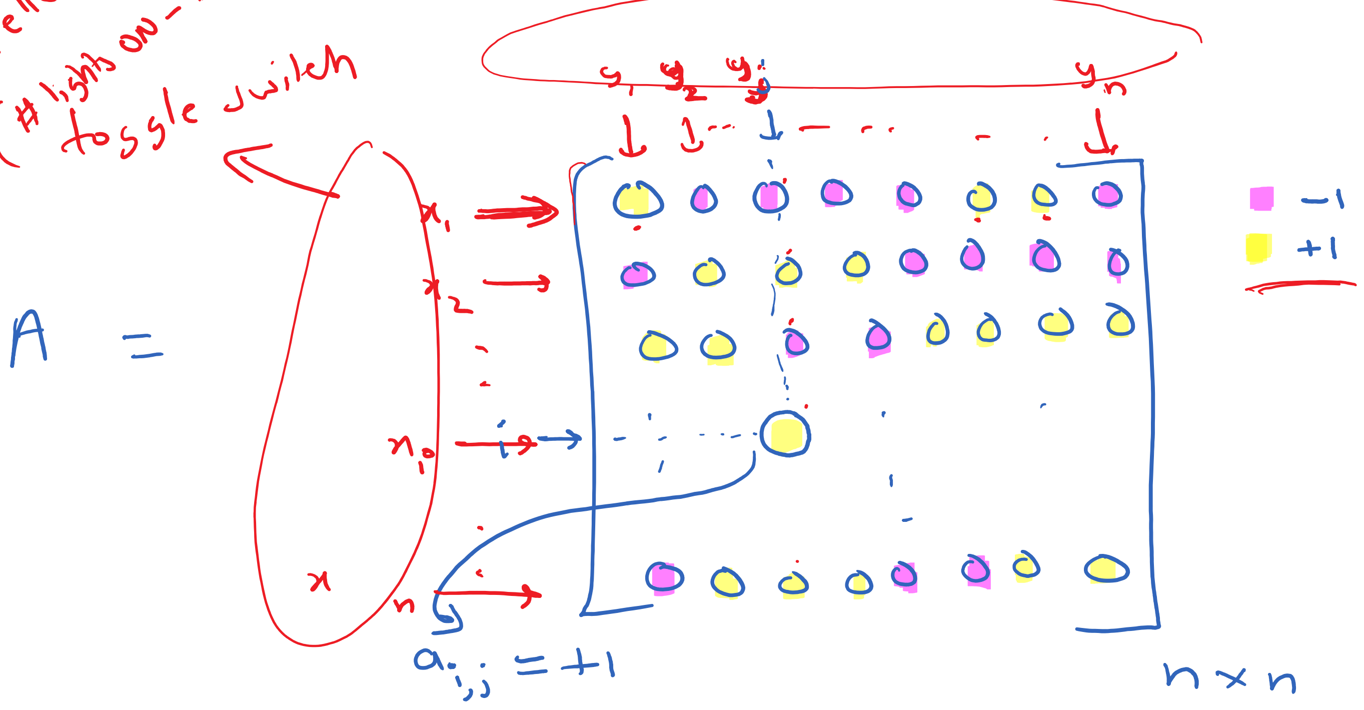
given

Theorem: Let $a_{i,j} = \pm 1$, for $1 \le i, j \le n$. Then, there exist $x_i, y_j = \pm 1$, $1 \le i, j \le n$, such that

$$\underbrace{\sum_{i=1}^{n} \sum_{j=1}^{n} a_{ij}\, x_i y_j}_{} \geq \left(\sqrt{\frac{2}{\pi}} + o(1)\right) n^{3/2}.$$

(Yellow − Purple)

(# lights ON − # lights OFF)



Proof: Fix all $\overbrace{x_1, x_2, \dots, x_n}^{\pm 1 \text{ variable}}$.

$$y_j = \begin{cases} +1, & \text{with prob } 1/2 \\ -1, & '' \quad '' \quad '' \end{cases}$$

$$E[y_j] = \frac{1}{2}(+1) + \frac{1}{2}(-1) = 0 \,//$$

$$E[X_{ij}] = x_i a_{ij} E[y_j] = 0 \,//$$

Let X_{ij} be a RV that denotes the value of $(x_i y_j a_{ij})$

value of $x_i y_j a_{ij} = c_j$ – j^{th} column

$$\begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$$

Let $C_j = \sum_{i=1}^{n} X_{ij}$

$$E[C_j] = \sum_{i=1}^{n} E[X_{ij}]$$

$$= \sum_{i=1}^{n} 0$$

$$= 0 \,//$$

R.V. $X = \sum_{j=1}^{n} C_j$

C_j → # lights ON in j^{th} column − # lights OFF in j^{th} column

X → denote # lights ON − # lights OFF

$$E[X] = \sum_{j=1}^{n} E[C_j]$$

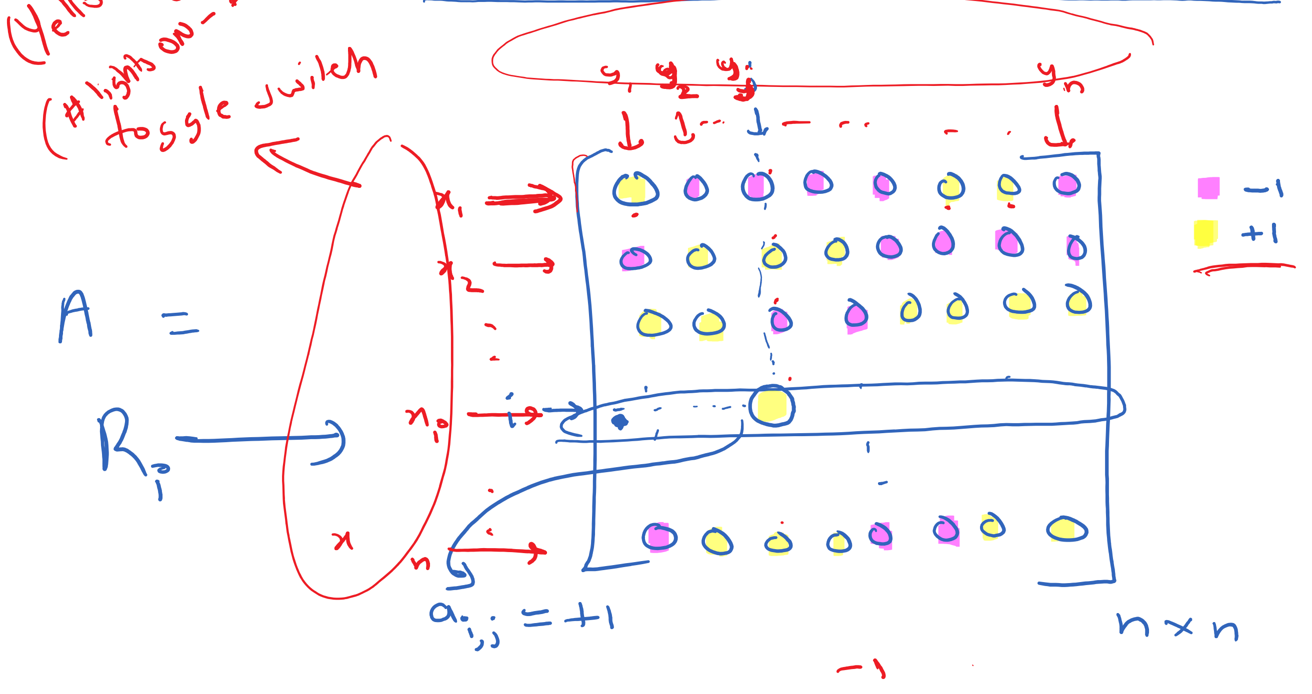
$$= 0 \,//$$

Theorem. Let $a_{i,j} = \pm 1$, for $1 \le i,j \le n$. Then, there exist $x_i, y_j = \pm 1$, $1 \le i,j \le n$, such that

$$\sum_{i=1}^{n}\sum_{j=1}^{n} a_{ij}\, x_i y_j \ge \left(\sqrt{\frac{2}{\pi}} + o(1)\right) n^{3/2}.$$

(Yellow – Purple)

(# lights ON – # lights OFF)



Proof: Fix all x_i's.

Let $y_1, y_2, \ldots, y_n = \pm 1$ be selected independently and uniformly at random.

$$y_j = \begin{cases} +1, & \text{w.p. } 1/2 \\ -1, & \text{w.p. } 1/2 \end{cases}$$

R.V. → $R_i = \sum_{j=1}^{n} a_{ij}\, y_j$ → (# lights ON in i^{th} row

– (# lights OFF in i^{th}

$$\underline{\underline{\qquad j=1 \qquad}}$$

(# lights OFF in i^{th} row)

R.V. $\rightarrow$ $$R = \sum_{i=1}^{n} |R_i|$$

In order to find $E[R]$, let us compute $E[\,|R_i|\,]$, $\forall i$.

R_i :

$$R_i = (n-1)(+1) + 1(-1)$$
$$= \underline{\underline{n-2}}$$

R_i :

$$R_i = (n-1)(-1) + 1(+1)$$
$$= -(n-2)$$

$$|R_i| = \underline{\underline{n-2}}$$

$$E[\,|R_i|\,] = \frac{\binom{n}{0} n}{2^{n-1}} + \frac{\binom{n}{1}\underline{\underline{(n-2)}}}{2^{n-1}} + \frac{\binom{n}{2}(n-4)}{2^{n-1}}$$

$\rightarrow = \frac{2}{2^n} = \frac{1}{2^{n-1}}$

$$+ \binom{n}{3}(n-6) + \cdots + \binom{n}{\frac{n-1}{2}})^2$$

It is known Sterling's approximation

n

It is (by Sterling's ap

$$\binom{n}{n/2} \approx \frac{2^n}{\sqrt{n}} \cdot \underset{\text{constant}}{c}$$

$$+ \frac{\binom{n}{3}(n-6)}{2^{n-1}} + \dots + \frac{\binom{n}{\lfloor\frac{n-1}{2}\rfloor}^2}{2^{n-1}}$$

$$= \frac{n}{2^{n-1}} \binom{n-1}{\lfloor\frac{n-1}{2}\rfloor} \longrightarrow \approx \frac{2^{n-1}}{\sqrt{n}} \cdot c$$

$$= \left(\sqrt{\frac{2}{\pi}} + o(1)\right)\sqrt{n} \qquad \text{—— ①}$$

$$R = \sum_{i=1}^{n} |R_i|$$

$$E[R] = E\left[\sum_{i=1}^{n} |R_i|\right]$$

$$= \sum_{i=1}^{n} E[|R_i|] \qquad \text{(by lin of expectation)}$$

$$= \left(\sqrt{\frac{2}{\pi}} + o(1)\right) n^{3/2} \qquad \text{(from ①)} \qquad ②$$

We want to show that $\exists\, x_i, y_j = \pm 1$

$$\sum_{i=1}^{n} \sum_{j=1}^{n} a_{ij} x_i y_j \geq \left(\sqrt{\frac{2}{\pi}} + o(1)\right) n^{3/2}$$

Pick x_i's with the same sign as R_i's.

$$\sum^{n} \sum^{n} a_{ij} x_i y_j = \sum^{n} x_i \left(\sum^{n} a_{ij} y_j\right) \to R_i$$

$$\sum_{i=1}^{n}\sum_{j=1}^{n} a_{ij} x_i y_j = \sum_{i=1}^{n} x_i \boxed{\sum_{j=1}^{n} a_{ij} y_j} \rightarrow R_i$$

$$= \sum_{i=1}^{n} x_i R_i$$

$$= \sum_{i=1}^{n} |R_i|$$

(because we chose x_i having the same sign as R_i)

$$= R$$

$$\geq \underline{\underline{\left(\sqrt{\frac{2}{\pi}} + o(1)\right) n^{3/2}}}$$

(from ②)

□