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Corollary of the [Skew Version of Bollobas Thm] Let $(A_1, A_2, \dots, A_m)$ and $(B_1, B_2, \dots, B_m)$ be two sequences of sets such that $\forall i, j \in [m]$, (i) $A_i \cap B_i = \emptyset$, and, (ii) $|A_i \cap B_j| \neq \emptyset$, if $i < j$. Further, $\forall i \in (m)$, $|A_i| \leq a$, $|B_i| \leq b$. Then,

$$m \leq \binom{a+b}{a}.$$

System of distinct representatives

A system of distinct representatives of the sets $S_1, S_2, \ldots, S_k$ is the k-tuple $(x_1, x_2, \ldots, x_k)$ such that (i) $\forall i \in [k]$, $x_i \in S_i$ and (ii) $\forall i, j \in [k]$ with $i \neq j$, we have $x_i \neq x_j$.

distinct

Further, $(x_1, x_2, \ldots, x_k)$ is a strong system of distinct representatives if (i), (ii) and (iii) $\forall i, j \in [k]$ $i \neq j$, $x_i \notin S_j$.

A <u>strong system of distinct representatives</u> for the sets $S_1, S_2, \ldots, S_k$ is a k-tuple $(x_1, x_2, \ldots, x_k)$ where

not equal

(i) $\forall i \in [k]$, $x_i \in S_i$, and

~~(ii)~~ $\forall i, j \in [k]$, $i \neq j$, $x_i \notin S_j$.

$S_1 = \{3, 4, 5, 9\}$, $S_2 = \{1, 2, 4, 8\}$

$S_3 = \{1, 4, 7, 5\}$

$(3, 2, 7)$ is a strong syst. of dist. rep. for S_1, S_2, S_3.

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Julkuna ← $k+2$??

Theorem [Furedi, Tuza, 1985]. Let $\mathcal{F}$ be a family of size greater than $\binom{r+k}{k}$. Further, every set in $\mathcal{F}$ is of size at most r. Then, there exist some $k+1$ sets in $\mathcal{F}$ that have a strong system of distinct representatives.

Proof: Arrange the sets in $\mathcal{F}$ in non-decreasing order of their sizes.

Let $\mathcal{F} = \{S_1, \ldots, S_m\}$ $m > \binom{r+k}{k}$

$|S_1| \le |S_2| \le \ldots \le |S_m|$

Assume, for the sake of contradiction, that no $k+1$ sets in $\mathcal{F}$ have a strong system of dist. rep.

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~~ASSUMING~~: no $k+1$ subcollection has strong sys of dist rep.

GIVEN: ~~(a) $m >$~~ $\binom{r+k}{k}$

(b) $|S_i| \le r$.

$S_1 \le S_2 \le \ldots S_i \le \ldots S_j \le \ldots \le S_m$

$T_1 \quad T_2 \quad T_i \ldots T_j \ldots T_m$

$\{\ldots, e, \ldots\}$

T_j: a minimal set that intersects

$S_j \setminus S_1, \; S_j \setminus S_2, \ldots, S_j \setminus S_i \ldots S_j \setminus S_{j-1}$

all are non-empty.

(a) ~~S_j~~$\cap T_j = \phi$

(b) $|T_j| \le k$

c) $|S_j| \le r$

d) $S_i \cap T_j \ne \phi$, $i < j$.

~~Property~~ For every element $e \in T_j$, $\exists$ set $S \in \{S_1, S_2, \ldots, S_{j-1}\}$ such that $T_j \cap S = \{e\}$.

~~Claim~~ $\forall j, \; |T_j| \le k$.

Proof of claim: Suppose not.

Suppose $\exists j, \; |T_j| \ge k+1$.

$T_j = \{e_1, e_2, \ldots, e_{k+1} \ldots\}$

$T \cap S_1 = \{e_1\} \quad S_2 \quad S_{k+1}$

$S_1, S_2, \ldots, S_{k+1}$ have $(e_1, e_2, \ldots, e_{k+1})$ as a strong syst of distinct repr.

By Skew version of Bollobas Thm,

$$m \le \binom{r+k}{k}.$$

But, this contradicts the fact that $m > \binom{r+k}{k}$.

Hence, our assumption that no subcollection of $\mathcal{F}$ has a strong system of distinct rep is false

$\square$