# EP 1027: Homework Assignment 2

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1. Show that if
$$\boldsymbol{\nabla} \times \mathbf{A} = 0,$$
this implies $\mathbf{A}$ can be expressed as a gradient of some potential,
$$\mathbf{A} = -\boldsymbol{\nabla}\Phi.$$
Hint: Use Stokes theorem to show the line integral of $\mathbf{A}$ is independent of the path connecting two points (i.e.depends only on the end points) and then recall the definition of gradient as a directional derivative,
$$d\mathbf{l} \cdot \boldsymbol{\nabla}\Phi = \Phi(\mathbf{x} + d\mathbf{x}) - \Phi(\mathbf{x}).$$
(10 points)

**Solution:**
Applying Stokes' theorem,
$$\oint_C \mathbf{A} \cdot d\mathbf{l} = 0$$
for *any* closed path, $C$. This implies the line-integral of $\mathbf{A}$between two points, say 1 and 2 are same, i.e. it is *path-independent*,
$$\int_{\substack{1\\C_1}}^{2} \mathbf{A} \cdot d\mathbf{l} = \int_{\substack{1\\C_2}}^{2} \mathbf{A} \cdot d\mathbf{l}.$$
$C_1$ and $C_2$ are two distinct path (contour) connecting the points 1 and 2. This is only possible iff the line-integral depends on the beginning point and end-point of the integral contour,
$$\int_1^2 \mathbf{A} \cdot d\mathbf{l} = \Phi(1) - \Phi(2),$$
where $\Phi$ is some function of $\mathbf{x}$. The minus sign is because we must have,
$$\int_1^2 \mathbf{A} \cdot d\mathbf{l} = -\int_2^1 \mathbf{A} \cdot d\mathbf{l}$$
Now the fundamental theorem of (multi-variable) calculus says,
$$\Phi(1) - \Phi(2) = -\int_1^2 d\mathbf{l} \cdot \boldsymbol{\nabla}\Phi,$$
Thus we see,
$$\int_1^2 \mathbf{A} \cdot d\mathbf{l} = -\int_1^2 d\mathbf{l} \cdot \boldsymbol{\nabla}\Phi,$$
$$\int_1^2 d\mathbf{l} \cdot (\mathbf{A} + \boldsymbol{\nabla}\Phi) = 0.$$

This equation holds for arbitrary path, so the integrand must vanish, which means,

$$\mathbf{A} = -\boldsymbol{\nabla}\Phi.$$

2. Calculate the ratio of the strengths of the electric and gravitational forces between an electron and proton placed some distance apart. You'll need to look up the mass and charge of the electron and proton and the value of the Newton's universal gravitational constant (in SI units). The ration should be independent of the separation distance or the system of units. (5 points)

   **Solution:**
   The ratio is,

$$\frac{F_{Coulomb}}{F_{Newton}} = \frac{1}{4\pi\epsilon_0 G}\frac{e^2}{m_e m_p} \approx 10^{40}.$$

   Thus electromagnetic forces are overwhelmingly dominant over gravitational forces in the subatomic or condensed matter domain.

3. In class we looked at the expression for the electric field at some location, $\mathbf{x}$ due to a **volume charge distribution,** $\rho(\mathbf{x}')$. What are the corresponding expression for a surface charge distribution, with surface charge density, $\sigma(\mathbf{x}')$ or linear charge density, $\lambda(\mathbf{x}')$ (5 points)

   **Solution:**
   All that needs to be done is to replace,

$$\rho(\mathbf{x}')\, d^3\mathbf{x}' = \sigma(\mathbf{x}')\, dS = \lambda(\mathbf{x}')\, dl$$

   The expressions are,

$$\mathbf{E}(\mathbf{x}) = \frac{1}{4\pi\epsilon_0}\int dS\, \frac{\sigma(\mathbf{x}')}{|\mathbf{x}-\mathbf{x}'|^3}\left(\mathbf{x}-\mathbf{x}'\right),$$

   and,

$$\mathbf{E}(\mathbf{x}) = \frac{1}{4\pi\epsilon_0}\int dl\, \frac{\lambda(\mathbf{x}')}{|\mathbf{x}-\mathbf{x}'|^3}\left(\mathbf{x}-\mathbf{x}'\right).$$

4. Use the integral version of Gauss law, i.e.

$$\oint\!\!\!\oint_S dS\, \hat{\mathbf{n}}\cdot\mathbf{E} = Q_{enclosed}$$

   .

   to prove *Newton's theorem*: The force on a point charge placed anywhere in the empty space within a spherical shell is zero. (5 points)
   **Solution:**
   Consider a spherical Gaussian surface of radius, $r$ inside a spherical shell of radius, $R$ but both being concentric. By symmetry, the Electric field on the surface of the Gaussian surface can only be radially directed, and also it must be function of the radial distance, $r$, i.e.

$$\mathbf{E} = E(r)\hat{\mathbf{r}}$$

So, the Gauss law surface integral on the spherical Gaussian surface becomes,

$$\oiint_S dS\, \hat{\mathbf{n}} \cdot \mathbf{E} = \oiint_S dS\, \hat{\mathbf{r}} \cdot (E(r)\hat{\mathbf{r}}) = 4\pi r^2 E(r).$$

How the charge enclosed by this spherical Gaussian surface is zero,

$$Q_{enclosed} = 0,$$

so Gauss law tells,

$$E(r) = 0, r < R.$$

This is Newton's theorem.

5. Prove Earnshaw's theorem, Mean-value theorem and Uniqueness theorem for the electrostatic scalar potential. (5+10+10=25 points)
**Solution:**
**Earnshaw's theorem:** ***A collection of point charges cannot be maintained in a stable stationary equilibrium configuration solely by the electrostatic interaction of the charges.***
**Proof:** First thing we will prove is that in a charge-free region the electrostatic potential cannot have an extremum. In a charge free region, one has Laplace equation for the potential,

$$\frac{\partial^2 \Phi}{\partial x^2} + \frac{\partial^2 \Phi}{\partial y^2} + \frac{\partial^2 \Phi}{\partial y^2} = 0$$

For an extremum (in all three directions),

$$\frac{\partial \Phi}{\partial x} = \frac{\partial \Phi}{\partial y} = \frac{\partial \Phi}{\partial z} = 0,$$

$$\frac{\partial^2 \Phi}{\partial x^2}, \frac{\partial^2 \Phi}{\partial y^2}, \frac{\partial^2 \Phi}{\partial y^2} > 0 \text{ (minimum)}$$

or,

$$\frac{\partial^2 \Phi}{\partial x^2}, \frac{\partial^2 \Phi}{\partial y^2}, \frac{\partial^2 \Phi}{\partial y^2} < 0 \text{ (maximum)}.$$

However all three second derivatives cannot all at once be negative or position since their sum must be equal to zero according to Laplace equation. Thus the extremum condition can never be satisfied in a charge free region. Now, let's look at a collection of point charges. Each of the charges experiences the electric field and the electric potential of the other charges; thus, each of the charges can be regarded as being located in a charge-free region. Since the potential in such a region cannot have an extremum, stable equilibrium is impossible.

**Mean-value theorem** for source-free regions of space : ***In sourcefree regions of space, the electrostatic potential at any point (P) in that region is equal to the average value of the potential over a sphere centered around P.***
**Proof :** Let's imagine a region of space bounded by a 2-sphere of radius $a$. And let's also place the origin of our coordinate system at the center of the sphere which is our point $P$. Then average value of the gravitational potential over this sphere of radius $a$, and centered at $P$:

$$\langle \Phi \rangle_a = \frac{\oiint a^2\, d\Omega\, \Phi(\mathbf{x})}{4\pi a^2},$$

where $\mathbf{x}$ is the position vector of a point on the surface of the sphere and $d\Omega$ is an infinitesimal solid angle from the origin to an infinitesimal area on the 2-sphere around the point, $\mathbf{x}$. Since $a$ is a constant it can be pulled out of the integral and canceled get canceled out by the factor of $a^2$ in the denominator, thus giving,

$$\langle \Phi \rangle_a = \frac{1}{4\pi} \oiint d\Omega\, \Phi(\mathbf{x}).$$

This is an equation which at first might look independent of the radius $a$, but in fact the information about the radius size, $a$ is contained in the position vector, $\mathbf{x}$, to wit, $|\mathbf{x}| = a$. Now we compute the derivative of this averaged potential wrt $a$,

$$\begin{aligned} \frac{d\langle\Phi\rangle_a}{da} &= \frac{1}{4\pi}\oiint d\Omega\ \frac{d\Phi(\mathbf{x})}{da} \\ &= \frac{1}{4\pi}\oiint d\Omega\ \left.\frac{\partial\Phi(\mathbf{x})}{\partial r}\right|_{r=a} \\ &= \frac{1}{4\pi}\oiint d\Omega\ \hat{\mathbf{n}}\cdot\boldsymbol{\nabla}\Phi, \end{aligned}$$

where $\hat{\mathbf{n}}$ is the unit normal vector on the surface of the sphere which is radially directed. Further convert this integral over solid angle into a surface integral and then we use Gauss' theorem to convert it into a volume term,

$$\begin{aligned} \frac{d\langle\Phi\rangle_a}{da} &= \frac{1}{4\pi}\oiint d\Omega\ \hat{\mathbf{n}}\cdot\boldsymbol{\nabla}\Phi \\ &= \frac{1}{4\pi a^2}\oiint dS\hat{\mathbf{n}}\cdot\boldsymbol{\nabla}\Phi, \quad dS = a^2 d\Omega \\ &= \frac{1}{4\pi a^2}\iiint d^3\mathbf{x}\ \underbrace{\boldsymbol{\nabla}^2\Phi}_{=0} \end{aligned}$$

where we have used the "source free" condition, i.e. $\rho = 0$, whereby the potential satisfies Laplace equation, $\boldsymbol{\nabla}^2\Phi = 0$. Thus we have the result that the mean value over the sphere is independent of the radius,

$$\frac{d\langle\Phi\rangle_a}{da} = 0.$$

i.e. the mean value is identical for distinct values of the radius, say $a$ and $\epsilon$,

$$\langle\Phi\rangle_\epsilon = \langle\Phi\rangle_a.$$

In particular when $\epsilon \to 0$, this mean value turns into the value at the point, $p$

$$\langle\Phi\rangle_\epsilon = \Phi(P).$$

Thus we have arrived at the mean value theorem,

$$\Phi(P) = \langle\Phi\rangle_a.$$

$\square$

**Uniqueness theorem**: ***The electrostatic potential in a region of space is unique once the value or potential or its normal derivative is specified on the boundary of that region.***
**Proof:** The gravitational potential satisfies the Poisson's equation,

$$\boldsymbol{\nabla}^2\Phi = 4\pi G_N\rho.$$

Let's assume that the solution to this in a region of space is non-unique and that there are two distinct solutions, $\Phi_1$ and $\Phi_2$ which satisfy the same boundary conditions at the boundary of this region,

$$\Phi_1|_{boundary} = \Phi_2|_{boundary},$$

$$\mathbf{n}\cdot\boldsymbol{\nabla}\Phi_1|_{boundary} = \mathbf{n}\cdot\boldsymbol{\nabla}\Phi_2|_{boundary}.$$

Now consider cooking up a new field, $\Phi \equiv \Phi_1 - \Phi_2$. This field of course satisfies the Laplace equation instead of the Poisson equation,

$$\begin{aligned} \boldsymbol{\nabla}^2\Phi &= \boldsymbol{\nabla}^2\Phi_1 - \boldsymbol{\nabla}^2\Phi_2 \\ &= 4\pi G_N\rho - 4\pi G_N\rho \\ &= 0. \end{aligned}$$

Now let's look at the vector, $\mathbf{V} = \Phi\boldsymbol{\nabla}\Phi$ and apply Gauss's divergence theorem to this vector,

$$\int d^3\mathbf{x}\, \boldsymbol{\nabla}\cdot\mathbf{V} = \oiint dS\, \mathbf{n}\cdot\mathbf{V}.$$

The LHS

$$\begin{aligned} \int d^3\mathbf{x}\, \boldsymbol{\nabla}\cdot\mathbf{V} &= \int d^3\mathbf{x}\, \boldsymbol{\nabla}\cdot(\Phi\boldsymbol{\nabla}\Phi) \\ &= \int d^3\mathbf{x}\, \left(\boldsymbol{\nabla}\Phi\cdot\boldsymbol{\nabla}\Phi + \Phi\, \underbrace{\boldsymbol{\nabla}^2\Phi}_{=0}\right) \\ &= \int d^3\mathbf{x}\, \left(\boldsymbol{\nabla}\Phi\right)^2, \end{aligned}$$

where we have used Laplace equation to get rid of the second term. Note that this term being a square quantity is manifestly positive semi-definite.
The RHS is,

$$\begin{aligned} \oiint dS\, \mathbf{n}\cdot\mathbf{V} &= \oiint dS\, \mathbf{n}\cdot(\Phi\boldsymbol{\nabla}\Phi) \\ &= \oiint dS\, \Phi\, \mathbf{n}\cdot\boldsymbol{\nabla}\Phi \\ &= 0, \end{aligned}$$

because either $\Phi = 0$ on the boundary (Dirichlet bc) or the $\mathbf{n}\cdot\boldsymbol{\nabla}\Phi = 0$ (Neumann bc).
So Gauss theorem gives us,

$$\int d^3\mathbf{x}\, \left(\boldsymbol{\nabla}\Phi\right)^2 = 0.$$

Since the integrand is a square quantity and which is manifestly positive semi-definite while the RHS is zero. Since the integral is actually a "Riemann sum" then with the integrand being a square quantity this means the LHS is a (Riemann) sum of squares. Therefore, the only solution is if each square term in the sum is zero,

$$\boldsymbol{\nabla}\Phi(x) = 0,$$

or,

$$\boldsymbol{\nabla}\Phi_1(x) = \boldsymbol{\nabla}\Phi_2(x),$$

or, integrating both sides,

$$\Phi_1(x) = \Phi_2(x) + c,$$

where $c$ is a constant. However if this is valid for all points including, $x$ on the boundary, then the values are different by an amount, $c$ on the boundary, which is not true by assumption. So $c = 0$ and we have

$$\Phi_1 = \Phi_2,$$

i.e. there can't be two distinct solutions. $\square$

6. A circular annulus of inner radius $R_1$ and outer radius $R_2$ has a uniform surface charge density $\sigma$ and lying on the $xy$ plane with its center at the origin. What is the electric field on the axis of the annulus (which is same as the $z$-axis) at a height $z$? (10 points)

**Solution:**

We use cylindrical polar coordinate system, with the origin of coordinates located at the center of the annulus. The axial symmetry of the situation dictates, the electric field on anywhere on the $z$-axis can only be directed along the axis and be a function of $z$,

$$\mathbf{E}(\rho = 0, \phi, z) = E(\mathbf{z})\hat{\mathbf{z}}.$$

We need to compute the integral,

$$\mathbf{E}(\rho = 0, \phi, z) = \frac{1}{4\pi\epsilon_0} \int_{Annulus} dS\, \sigma(\mathbf{x}')\, \frac{(\mathbf{x} - \mathbf{x}')}{|\mathbf{x} - \mathbf{x}'|^3},$$

where since the field point is located on the $z$-axis and the source charge is located on the $xy$-plane i.e. $z = 0$ plane, we have

$$\mathbf{x} = z\hat{\mathbf{z}},\ \mathbf{x}' = \rho'\hat{\boldsymbol{\rho}} + \phi'\hat{\boldsymbol{\phi}}.$$

This gives,

$$|\mathbf{x} - \mathbf{x}'| = \sqrt{\rho'^2 + z^2}$$

We also have the area element in cylindrical polar coordinates,

$$dS = \rho'\, d\rho'\, d\phi'.$$

Putting all these together and keeping only the $z$-component,

$$\begin{aligned}
\mathbf{E}(\rho = 0, z) &= \frac{1}{4\pi\epsilon_0} \int_0^{2\pi} d\phi' \int_{R_1}^{R_2} d\rho'\, \rho'\, \sigma\, \frac{z}{(\rho'^2 + z^2)^{3/2}}\hat{\mathbf{z}} \\
&= \frac{\sigma z}{2\epsilon_0} \int_{R_1}^{R_2} d\rho'\, \frac{\rho'}{(\rho'^2 + z^2)^{3/2}} \\
&= -\frac{\sigma z}{2\epsilon_0}\, \frac{1}{\sqrt{\rho'^2 + z^2}}\Bigg|_{R_1}^{R_2} \\
&= \frac{\sigma z}{2\epsilon_0} \left( \frac{1}{\sqrt{R_1^2 + z^2}} - \frac{1}{\sqrt{{R_2}^2 + z^2}} \right).
\end{aligned}$$

7. An amount of charge $Q$ is uniformly distributed over a cylindrical volume of radius $R$ and length $L$. Find the electric field and the potential at a point on the axis of the cylinder at a distance $d$ from the center. (Hint: You will need to use cylindrical polar coordinate system) (10 points)

**Solution:**

Again as before, axial/cylindrical symmetry dictates that a) the electric field on the axis can only be a function of the $z$-coordinate, and b) it is directed along the $z$-axis,

$$\begin{aligned}
\mathbf{E}(\rho=0,z=d) &= \frac{1}{4\pi\epsilon_0}\frac{Q}{\pi R^2 L}\int_{z'=-L/2}^{L/2}\int_{\rho'=0}^{R} 2\pi\rho' d\rho' dz' \frac{(d-z')}{\left[(d-z')^2+\rho'^2\right]^{3/2}} \\
&= \frac{Q}{2\pi R^2 L\epsilon_0}\int_0^R d\rho'\,\rho' \int_{z'=-L/2}^{L/2} \frac{dz'\,(d-z')}{\left[(d-z')^2+\rho'^2\right]^{3/2}} \\
&= \frac{Q}{2\pi R^2 L\epsilon_0}\int_0^R d\rho'\,\rho' \left(\frac{1}{\sqrt{(d-L/2)^2+\rho'^2}}-\frac{1}{\sqrt{(d+L/2)^2+\rho'^2}}\right) \\
&= \frac{Q}{2\pi R^2 L\epsilon_0}\left[\sqrt{(d-L/2)^2+R^2}-|d-L/2|-\sqrt{(d+L/2)^2+R^2}+(d+L/2)\right].
\end{aligned}$$

8. A spherically symmetric charge distribution has a density $\rho(\mathbf{x}) = k|\mathbf{x}|^n$, where $k$ is a constant and $n > -3$.
i. Find the electric field as a function of $\mathbf{x}$.
ii. Find the potential difference between the points $|\mathbf{x}| = a$ and $|\mathbf{x}| = b$.
(Hint: Use spherical polar coordinate system) (5+5=10 points)
**Solution:**
i. To find the electric field, we use a spherical Gaussian surface with the origin being the center. By the isotropy, i.e. spherical symmetry of the problem, one can say that the electric field would only be a function of the radius (distance from the origin) and radially directed:

$$\mathbf{E}(\mathbf{x}) = E(r)\,\hat{\mathbf{r}}.$$

Now the flux of this field over the spherical Gaussian surface of radius say $R$ is,

$$\oiint dS\,\hat{\mathbf{n}}\cdot\mathbf{E} = 4\pi R^2\,E(R).$$

The enclosed charge i.e. contained in the volume bounded by this Gaussian surface is,

$$\begin{aligned}
Q_{enclosed} &= \int d^3\mathbf{x}\,\rho(\mathbf{x}) \\
&= \int d^3\mathbf{x}\,k|\mathbf{x}|^n \\
&= \int_{r<R,\theta=[0,\pi],\phi=[0,2\pi]} r^2\sin\theta d\theta d\phi\,kr^n \\
&= 4\pi k\int_0^R dr\,r^{n+2} \\
&= \frac{4\pi k\,R^{n+3}}{n+3}.
\end{aligned}$$

Thus Gauss law gives,

$$\begin{aligned}
4\pi R^2\,E(R) &= \frac{4\pi k\,R^{n+3}}{(n+3)\,\epsilon_0}, \\
\implies E(R) &= \frac{k}{(n+3)\,\epsilon_0}\,R^{n+1}.
\end{aligned}$$

So the electric field as a function of $\mathbf{x}$ is,

$$\mathbf{E}(\mathbf{x}) = \frac{k}{(n+3)\,\epsilon_0}\,|\mathbf{x}|^n\,\mathbf{x}.$$

ii. The potential difference between two points at radius $a$ and $b$ respectively is given by line integrating the electric field along a radially directed path

$$
\begin{aligned}
\Phi(a) - \Phi(b) &= -\int_{|\mathbf{x}|=a}^{|\mathbf{x}|=b} \mathbf{E} \cdot d\mathbf{l} \\
&= -\int_{r=a}^{r=b} \mathbf{E}(r)\, dr \\
&= -\frac{k}{(n+3)\,\epsilon_0} \int_a^b dr\, r^{n+1} \\
&= -\frac{k}{(n+3)\,(n+2)\epsilon_0} \left(b^{n+2} - a^{n+2}\right).
\end{aligned}
$$

9. Find the expression for the electric field strength at a point, $\mathbf{x}$, due to an ideal/point dipole located at $\mathbf{x}'$. Hint: The potential due to the dipole is given by,

$$\Phi(\mathbf{x}) = \frac{1}{4\pi\epsilon_0} \frac{\mathbf{p} \cdot (\mathbf{x} - \mathbf{x}')}{|\mathbf{x} - \mathbf{x}'|^3}$$

and the electric field is, $\mathbf{E}(\mathbf{x}) = -\boldsymbol{\nabla}\Phi(\mathbf{x}))$. (5 points)
**Solution:**

$$
\begin{aligned}
E^i(\mathbf{x}) &= -\frac{1}{4\pi\epsilon_0} \frac{\partial}{\partial x^i} \left( \frac{p^j\left(x^j - x'^j\right)}{|\mathbf{x} - \mathbf{x}'|^3} \right) \\
&= -\frac{1}{4\pi\epsilon_0} \left[ \frac{p^j}{|\mathbf{x} - \mathbf{x}'|^3} \frac{\partial}{\partial x^i} \left(x^j - x'^j\right) + p^j\left(x^j - x'^j\right) \frac{\partial}{\partial x^i} \left( \frac{1}{|\mathbf{x} - \mathbf{x}'|^3} \right) \right] \\
&= -\frac{1}{4\pi\epsilon_0} \left[ \frac{p^j}{|\mathbf{x} - \mathbf{x}'|^3} \delta^{ij} + p^j\left(x^j - x'^j\right) \left( \frac{-3}{|\mathbf{x} - \mathbf{x}'|^4} \frac{\partial |\mathbf{x} - \mathbf{x}'|}{\partial x^i} \right) \right] \\
&= -\frac{1}{4\pi\epsilon_0} \left[ \frac{p^i}{|\mathbf{x} - \mathbf{x}'|^3} + \mathbf{p} \cdot (\mathbf{x} - \mathbf{x}') \left( \frac{-3}{|\mathbf{x} - \mathbf{x}'|^4} \frac{(x - x')^i}{|\mathbf{x} - \mathbf{x}'|} \right) \right], \\
\mathbf{E}(\mathbf{x}) &= \frac{1}{4\pi\epsilon_0} \left[ \frac{3\mathbf{p} \cdot (\mathbf{x} - \mathbf{x}')}{|\mathbf{x} - \mathbf{x}'|^5} (\mathbf{x} - \mathbf{x}') - \frac{\mathbf{p}}{|\mathbf{x} - \mathbf{x}'|^3} \right].
\end{aligned}
$$

10. Starting with the expression for the dielectric bound charge volume density, $\rho_{bound} = -\boldsymbol{\nabla} \cdot \mathbf{P}$ and the dielectric surface charge density, $\sigma_{bound} = \mathbf{P} \cdot \hat{\mathbf{n}}$, show the sum of the bound volume charge and total surface charge is zero. This shows that the dielectric is as a whole neutral, although the charge separation causes local imbalance of charge of one sign or the other. (5 points)

$$Q_{total} = -\iiint_V d^3\mathbf{x}\, \boldsymbol{\nabla}.\mathbf{P} + \oiint_S dS\, \hat{\mathbf{n}} \cdot \mathbf{P},$$

where the integration domains are respectively the volume of the dielectric$V$, and the surface of the dielectric, $S$. Now applying Gauss' divergence theorem,

$$\iiint_V d^3\mathbf{x}\, \boldsymbol{\nabla}.\mathbf{P} = -\oiint_S dS\, \hat{\mathbf{n}} \cdot \mathbf{P},$$

Thus we see that,

$$Q_{total} = 0.$$

11. Two very long, conducting cylinders of sheet metal have radii $R$ and $3R$, respectively. The space between them is filled with a gas of dielectric constant $\epsilon$. If the potential difference between the cylinders is $V_0$, find the $E$ and $D$ fields in the region between the cylinders using the Gauss law for dielectrics. (10 points)

**Solution:**
From the axial/cylindrical symmetry of the problem, the Electric field and hence the Displacement field should both point in radial direction, and be a function of the distance from the axis,

$$\begin{aligned} \mathbf{E} &= E(\rho)\,\hat{\boldsymbol{\rho}}, \\ \mathbf{D} &= D(\rho)\,\hat{\boldsymbol{\rho}} \end{aligned}$$

Gauss law for dielectrics, $\boldsymbol{\nabla}\cdot\mathbf{D} = \rho_{free}$, applied over a cylindrical Gaussian surface of radius, $\rho$ gives,

$$2\pi\rho D(\rho) = (2\pi R)\sigma_{free},$$

where $\sigma_{free}$ is the free charge on the inner conducting cylinder. This gives,

$$D(\rho) = \frac{R}{\rho}\sigma_{free}$$

and further,

$$E(\rho) = \frac{D(\rho)}{\epsilon} = \frac{\sigma_{free}R}{\epsilon\rho}$$

Since the potential difference is, $V_0$ we have,

$$-\int_R^{3R} \mathbf{E}\cdot d\mathbf{l} = V_0$$

where we take, $d\mathbf{l}$ to be along the radial direction, i.e. $d\mathbf{l} = d\rho\hat{\boldsymbol{\rho}}$. This gives,

$$-\frac{\sigma_{free}R}{\epsilon}\ln 3 = V_0,$$

or,

$$\sigma_{free} = -\frac{\epsilon V_0}{R\,\ln 3}.$$

Thus, we have,

$$\mathbf{D}(\mathbf{x}) = -\frac{\epsilon V_0}{\ln 3}\frac{1}{\rho}\hat{\boldsymbol{\rho}}, \mathbf{E}(\mathbf{x}) = -\frac{V_0}{\ln 3}\frac{1}{\rho}\hat{\boldsymbol{\rho}}.$$

12. At the horizontal interface between two semi-infinite dielectrics (of constants $\epsilon_1$ on top and $\epsilon_2$ under, respectively), the electric field in the first dielectric has magnitude $E_1$ and makes an angle $\alpha_1$ with the vertical/normal to the interface (see). What is the angle, $\alpha_2$, the electric field makes in the second dielectric (as a function of $E_1$ and $\alpha_1$), and what is the magnitude of this electric field? (10 points)

**Solution:**
From the figure we identify for a field vector making an angle $\alpha$ with the vertical,

$$E_{\parallel} = E\sin\alpha, \quad E_{\perp} = E\cos\alpha.$$

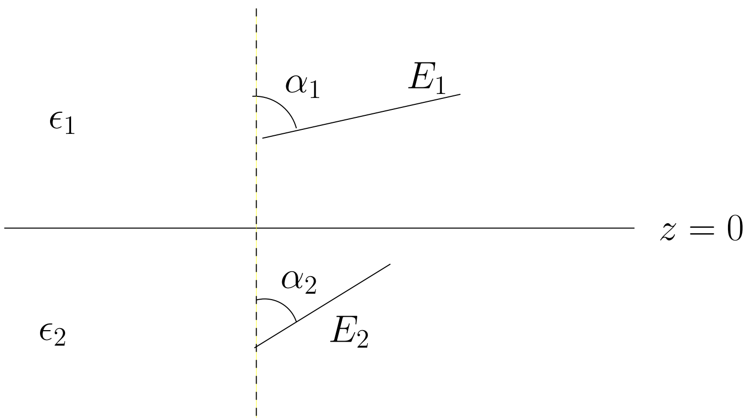


Figure 1: Electric field vectors across a flat interface

(To denote the two media, I will use superscripts $1, 2$). The boundary conditions for dielectrics are the continuity of the tangential component (parallel to the boundary) of $\mathbf{E}$,

$$E^1_{\parallel} = E^2_{\parallel},$$

i.e.,

$$E^1 \sin\alpha_1 = E^2 \sin\alpha_2 \tag{1}$$

and then the (dis)continuity of the normal component of $\mathbf{D}$,

$$D^1_{\perp} = D^2_{\perp},$$

or,

$$\epsilon^1 E^1_{\perp} = \epsilon^2 E^2_{\perp},$$

i.e.,

$$\epsilon^1 E^1 \cos\alpha_1 = \epsilon^2 E^2 \cos\alpha_2, \tag{2}$$

Solving (1) and (2), we get,

$$\tan\alpha_2 = \left(\frac{\epsilon^2}{\epsilon^1}\tan\alpha_1\right),$$

$$E^2 = E^1 \sqrt{\sin^2\alpha_1 + \left(\frac{\epsilon^1}{\epsilon^2}\cos\alpha_1\right)^2}.$$

13. A dielectric sphere of radius $R$ with a dielectric constant $\epsilon$ has a free charge $Q$ uniformly distributed over its volume. The sphere is surrounded by empty space. Find $\mathbf{E}$ and $\mathbf{D}$ inside and outside the sphere. Find the polarization volume charge density and surface charge density. Evaluate the net polarization volume charge and surface charge. (5+5+5+5=20 points)

**Solution:**

Due to the spherical symmetry/isotropy of the problem, we realize that both $\mathbf{D}$ and $\mathbf{E}$ are functions of the radial distance (from the center of the sphere) and radially directed as well,

$$\mathbf{D} = D(r)\hat{\mathbf{r}},\ \mathbf{E} = E(r)\,\hat{\mathbf{r}}.$$

The uniform free charge volume density is,

$$\rho_{free} = \frac{3Q}{4\pi R^3}.$$

First let's compute the displacement field inside. For this we consider a Gaussian surface of radius $r$, where, $r < R$ and apply the integral version of Gauss law for the displacement field in dielectric, i.e.

$$
\begin{aligned}
\oiint dS\,\hat{\mathbf{n}}\cdot\mathbf{D} &= \int d^3\mathbf{x}\,\rho_{free}, \\
4\pi r^2 D(r) &= Q\frac{r^3}{R^3}, \\
D(r) &= \frac{Q}{4\pi R^3}r,\ r < R.
\end{aligned}
$$

Now let's compute the displacement field outside, i.e. for $r > R$. Again we apply Gauss' law to get,

$$
D(r) = \begin{cases} \frac{1}{4\pi}\frac{Q\,r}{R^3}, r < R, \\ \frac{1}{4\pi}\frac{Q}{r^2}, r > R. \end{cases} \tag{3}
$$

The expression for the electric field can be obtained by using the constitutive relation,

$$
\begin{aligned}
\mathbf{D} &= \epsilon\epsilon_0\mathbf{E}, r < R \\
&= \epsilon_0\mathbf{E}, r > R
\end{aligned}
$$

which gives,

$$
E(r) = \begin{cases} \frac{1}{4\pi\epsilon_0\epsilon}\frac{Q\,r}{R^3}, r < R, \\ \frac{1}{4\pi\epsilon_0}\frac{Q}{r^2}, r > R. \end{cases} \tag{4}
$$

Now, the polarization is given by,

$$
\mathbf{P} = \mathbf{D} - \epsilon_0\mathbf{E}.
$$

So this means $\mathbf{P}$, just like $\mathbf{E}$and $\mathbf{D}$, is radially directed and a function of radius,

$$
\mathbf{P} = P(r)\hat{\mathbf{r}}.
$$

From (3) and (4), we get,

$$
P(r) = \begin{cases} \left(1 - \frac{1}{\epsilon}\right)\frac{1}{4\pi}\frac{Q\,r}{R^3}, r < R, \\ 0, \quad r > R. \end{cases}
$$

which makes sense because outside the dielectric there is nothing to polarize.
The induced volume charge density due to polarization is,

$$
\begin{aligned}
\rho_{bound} &= -\boldsymbol{\nabla}\cdot\mathbf{P} \\
&= \begin{cases} -\left(1 - \frac{1}{\epsilon}\right)\frac{1}{4\pi}\frac{3Q}{R^3}, r < R, \\ 0, \quad r > R. \end{cases}
\end{aligned}
$$

So the net polarization volume charge is,

$$
\begin{aligned}
Q_{vol} &= \int_{r<R} d^3\mathbf{x}\,\rho(\mathbf{x}) \\
&= -\int_{r<R} d^3\mathbf{x}\,\boldsymbol{\nabla}\cdot\mathbf{P} \\
&= -\left(1 - \frac{1}{\epsilon}\right)\frac{1}{4\pi}\frac{3Q}{R^3}\int_{r<R} d^3\mathbf{x} \\
&= -\left(1 - \frac{1}{\epsilon}\right)Q.
\end{aligned}
$$

The polarization surface charge density is,

$$\begin{aligned}\sigma_{bound} &= \mathbf{P}\cdot\hat{\mathbf{n}}|_{r=R} \\ &= \left(1-\frac{1}{\epsilon}\right)\frac{1}{4\pi}\frac{Q}{R^2}.\end{aligned}$$

So the net polarization surface charge is,

$$\begin{aligned}Q_{Surface} &= \int_{r=R} dS\,\sigma_{bound} \\ &= \left(1-\frac{1}{\epsilon}\right)\frac{1}{4\pi}\frac{Q}{R^2}\underbrace{\left(\int_{r=R} dS\right)}_{4\pi R^2} \\ &= \left(1-\frac{1}{\epsilon}\right)Q.\end{aligned}$$

14. Energy in a dielectric. Show that the energy contained per unit volume in a dielectric is

$$u = \frac{1}{2}\mathbf{D}\cdot\mathbf{E}.$$

(Hint: The energy inside a dielectric comes from two sources. One is to set up a field, $\mathbf{E}$ inside, and this field carries energy (density) $\frac{\epsilon_0}{2}\mathbf{E}^2$, and two the the energy stored by stretching the molecules/dipoles to increase the dipole moment or polarization. Try to find out the infinitesimal/incremental work done, $\Delta U$ to increase the electric field inside a dielectric from $\mathbf{E}$ to $\mathbf{E}+\mathbf{\Delta E}$ and work done to increase the polarization (by stretching the molecular dipoles) from $\mathbf{P}$ to $\mathbf{P}+\mathbf{\Delta P}$. Show that the sum of these two incremental work done turns out to be $\Delta U{=}\int d^3\mathbf{x}\,\Delta\left(\frac{1}{2}\mathbf{D}\cdot\mathbf{E}\right)$). (10 points)

**Solution:**

The potential energy of a dipole placed in an electric field is,

$$U_{dipole} = -\mathbf{p}\cdot\mathbf{E}$$

When a dipole is stretched it leads to an increase in dipole moment by $\Delta\mathbf{p}$, and the work done by the electric field is,

$$\Delta_{strech} = -\Delta U_{dipole} = -\Delta\left(-\mathbf{p}\cdot\mathbf{E}\right) = \Delta\mathbf{p}\cdot\mathbf{E}.$$

Now for a dielectric we know the dipole moment of an elementary chunk of the dielectric is

$$\mathbf{p}(\mathbf{x}) = d^3\mathbf{x}\,\mathbf{P}(\mathbf{x}).$$

So the work done to polarize the entire dielectric little bit is,

$$\Delta W = \int d^3\mathbf{x}\,\Delta\mathbf{P}(\mathbf{x})\cdot\mathbf{E}(\mathbf{x}).$$

Therefore the total work done,

$$\begin{aligned}\Delta U &= \int d^3\mathbf{x}\,\Delta\left(\frac{\epsilon_0}{2}\mathbf{E}\cdot\mathbf{E}\right) + \int d^3\mathbf{x}\,\Delta\mathbf{P}\cdot\mathbf{E} \\ &= \int d^3\mathbf{x}\,\left(\epsilon_0\Delta\mathbf{E}\cdot\mathbf{E}\right) + \int d^3\mathbf{x}\,\Delta\mathbf{P}\cdot\mathbf{E} \\ &= \int d^3\mathbf{x}\,\left(\epsilon_0\Delta\mathbf{E}\cdot\mathbf{E}\right) + \int d^3\mathbf{x}\,\Delta\mathbf{P}\cdot\mathbf{E} \\ &= \int d^3\mathbf{x}\,\Delta\left(\epsilon_0\mathbf{E}+\mathbf{P}\right)\cdot\mathbf{E} \\ &= \int d^3\mathbf{x}\,\Delta\mathbf{D}\cdot\mathbf{E},\end{aligned}$$

where we have used in the last step, $\mathbf{D} = \epsilon_0 \mathbf{E} + \mathbf{P}$. Thus this is valid for *any* kind of dielectric. But for a linear dielectric, we further have,

$$\mathbf{D} = \epsilon\,\mathbf{E},$$

and this enables us to write,

$$\Delta U = \int d^3\mathbf{x}\,\Delta\mathbf{D}\cdot\mathbf{E} = \epsilon\int d^3\mathbf{x}\,\Delta\mathbf{E}\cdot\mathbf{E} = \frac{1}{2}\int d^3\mathbf{x}\Delta\left(\mathbf{E}\cdot\mathbf{E}\right) = \int d^3\mathbf{x}\,\Delta\left(\epsilon\mathbf{E}\cdot\mathbf{E}\right) = \Delta\left(\int d^3\mathbf{x}\,\frac{1}{2}\mathbf{D}\cdot\mathbf{E}\right),$$

so,

$$u_{dielectric} = \frac{1}{2}\mathbf{D}\cdot\mathbf{E}.$$

15. Prove Green's reciprocity theorem: Suppose we have an arrangement of $n$ conductors so that the potentials on the conductors are $\Phi_1, \Phi_2, \ldots, \Phi_n$when the charges on them are respectively $Q_1, Q_2, ..., Q_n$. And suppose that the potentials change to $\Phi'_1, \Phi'_2, \ldots, \Phi'_n$ when the charges are changed to $Q'_1, Q'_2, ..., Q'_n$;. Then,

$$\sum_{l=1}^{n} Q_l\,\Phi'_l = \sum_{l=1}^{n} Q'_l\,\Phi_l.$$

Corollary: For a capacitor system consisting of $n$ separate conductors, we define coefficients of capacitance as follows: Suppose that the conductors carry charges $Q_1, Q_2, ..., Q_n$ and are at potentials $\Phi_1, \Phi_2, \ldots, \Phi_n$. Then, we define capacitance in a slightly different manner compared to a single conductor case. We can define a set of *coefficients of capacitance*, $\{c_{kl}\}$ by the relation:

$$Q_k = \sum_l c_{kl}\Phi_l$$

where $k$ and $l$ both take values from 1 to $n$. Use Green's reciprocity theorem to prove the follow symmetry relation

$$c_{kl} = c_{lk}.$$

(10+5=15 points)

**Solution:**
To prove this theorem, we subdivide each of the charges $Q_k$ on the conductors into small charge elements of magnitudes $q_k = Q_k/N$. If the positions of these charge elements are$\mathbf{x}_1, , \mathbf{x}_2, \ldots, \mathbf{x}_{\mathbf{nN}}$. Then the potential at the position of one of these charge elements is

$$\Phi_k = \sum_{l\neq k}^{nN} \frac{q_l}{|\mathbf{x}_k - \mathbf{x}_l|}.$$

In this expression, the contribution that the charge element $q_k$ makes to the potential at its own position has been neglected for obvious reasons. Similarly, we can write,

$$\Phi'_k = \sum_{l\neq k}^{nN} \frac{q'_l}{|\mathbf{x}_k - \mathbf{x}_l|}.$$

Thus, we have,

$$\sum_{k=1}^{nN} q_k\Phi'_k = \sum_{k=1}^{nN}\sum_{l\neq k}^{nN} \frac{q_k q'_l}{|\mathbf{x}_k - \mathbf{x}_l|} = \sum_{l=1}^{nN}\sum_{k\neq l}^{nN} \frac{q'_l q_k}{|\mathbf{x}_k - \mathbf{x}_l|} = \sum_{l=1}^{nN} q'_l\Phi_l,$$

where in going from the second to third expression we realize that we can perform the double sum omitting the $l = k$ term in two different orders/sequence. Now this expression holds for charge elements, but we know that the conductor is an equipotential, so $\Phi_k$ (or $\Phi_k$) is same for elements on the same conductor and we can write,

$$\sum_{k=1}^{nN} q_k \Phi'_k = \sum_l \left( \sum_{l\text{-th conductor}}^{k=1,N} q_k \Phi'_k \right) = \sum_l \Phi'_l \underbrace{\left( \sum_{l\text{-th conductor}}^{k=1,N} q_k \right)}_{Q_l} = \sum_{l=1}^{n} Q_l \, \Phi'_l$$

and likewise,

$$\sum_{l=1}^{nN} q'_l \Phi_l = \sum_{l=1}^{n} Q'_l \, \Phi_l.$$

Thus finally we have the reciprocity theorem,

$$\sum_{l=1}^{n} Q_l \, \Phi'_l = \sum_{l=1}^{n} Q'_l \, \Phi_l. \tag{5}$$

**Corollary:**
The defining equation for coefficients of capacitance is,

$$Q_l = \sum_{k=1}^{n} c_{lk} \Phi_k.$$

Plugging this in place of $Q_k$ (and the corresponding substitution for $Q'_k$) in the reciprocity theorem expression, (5), we get,

$$\begin{aligned} \sum_{l=1}^{n} \sum_{k=1}^{n} c_{lk} \Phi_k \Phi'_l &= \sum_{l=1}^{n} \sum_{k=1}^{n} c_{lk} \Phi'_k \Phi_l \\ &= \sum_{k=1}^{n} \sum_{l=1}^{n} c_{kl} \Phi'_l \Phi_k, \end{aligned}$$

where in going from the first line rhs to the second line rhs, we exploited the fact that $k, l$ are both dummy (repeated/summed) indices and hence can be relabeled, so $k \to l$ and $l \to k$. Since both sides must be same for arbitrary values of $\Phi_k$ and $\Phi'_l$, we must have,

$$c_{lk} = c_{kl}.$$

□