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Balancing Vectors

Euclidian norm

Theorem Let $v_1, v_2, \dots, v_n \in \mathbb{R}^n$ where $|v_i| = 1$, $\forall i \in [n]$. (i) Then, there exist $\varepsilon_1, \varepsilon_2, \dots, \varepsilon_n$, where every $\varepsilon_i \in \{-1, +1\}$ such that

$$\left| \overbrace{\varepsilon_1 v_1 + \varepsilon_2 v_2 + \dots + \varepsilon_n v_n}^{v} \right| \leq \sqrt{n}$$

(ii) Similarly there exist $\varepsilon_1, \varepsilon_2, \dots, \varepsilon_n$, where every $\varepsilon_i \in \{-1, +1\}$ such that

$$\left| \varepsilon_1 v_1 + \varepsilon_2 v_2 + \dots + \varepsilon_n v_n \right| \geq \sqrt{n}.$$

Proof:

$$|v_i| = \sqrt{v_i \cdot v_i}$$

$$v_i = (v_{i1}, v_{i2}, \dots, v_{in}) \in \mathbb{R}^n$$

$$v_j = (v_{j1}, v_{j2}, \dots, v_{jn}) \in \mathbb{R}^n$$

$$v_i \cdot v_j = v_{i1} v_{j1} + v_{i2} v_{j2} + \dots + v_{in} v_{jn}$$

For each i, independently and uniformly at random assign a value from the set $\{-1, +1\}$ to ε_i.

scalars

think as R.V.

$$\varepsilon_i = \begin{cases} +1, & \text{with prob } 1/2 \\ -1, & \text{with prob } 1/2 \end{cases}$$

$$X = \Big| \underbrace{\varepsilon_1 v_1 + \varepsilon_2 v_2 + \dots + \varepsilon_n v_n}_{v} \Big|^2$$

In order to prove the theorem, it is enough to show that

$$E[X] = n$$

$$\langle \alpha_1 u + \alpha_2 v, \beta_1 w, \beta_2 z \rangle \in \mathbb{R}^n$$
$$= \alpha_1 \beta_1 \langle u, w \rangle + \alpha_1 \beta_2 \langle u, z \rangle + \alpha_2 \beta_1 \langle v, w \rangle + \alpha_2 \beta_2 \langle v, z \rangle$$

$$X = \Big| \overbrace{\varepsilon_1 v_1 + \varepsilon_2 v_2 + \dots + \varepsilon_n v_n}^{v} \Big|^2$$

$$= (\varepsilon_1 v_1 + \varepsilon_2 v_2 + \dots + \varepsilon_n v_n) \cdot (\varepsilon_1 v_1 + \varepsilon_2 v_2 + \dots + \varepsilon_n v_n)$$

$$= \sum_{i=1}^{n} \sum_{j=1}^{n} \varepsilon_i \varepsilon_j v_i \cdot v_j$$

new random variables

scalar

$$E[X] = E\left[\sum_{i=1}^{n} \sum_{j=1}^{n} \varepsilon_i \varepsilon_j \, v_i \cdot v_j \right]$$

$$= \sum_{i=1}^{n} \sum_{j=1}^{n} \underline{\underline{v_i \cdot v_j}} \, E[\varepsilon_i \varepsilon_j] \quad \text{(by Linearity of Expectation)}$$

$$\sum_{i=1} \sum_{j=1} \quad \ldots \quad \text{expectation)}$$

$$= \sum_{i=1}^{n} v_i \cdot v_i \, E[\varepsilon_i^2] + \sum_{i=1}^{n} \sum_{\substack{j \in \{1,2,..n\} \\ j \neq i}} v_i \cdot v_j \, E[\varepsilon_i \varepsilon_j]$$

$$E[\varepsilon_i^2] = 1$$

$$E[\varepsilon_i \varepsilon_j] = E[\varepsilon_i] \, E[\varepsilon_j] = 0 \, /\!/$$

(because ε_i and ε_j are independent R.V.)

$$E[\varepsilon_i] = \frac{1}{2} \cdot 1 + \frac{1}{2}(-1) = 0 \, /\!/$$

$$= \sum_{i=1}^{n} v_i \cdot v_i \cdot 1$$

$$v_i \cdot v_i = 1 \text{ (given)}$$

$$= \sum_{i=1}^{n} 1$$

$$= n \, /\!/$$

$\square$