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Linearity of Expectation

$X_1, X_2, \ldots, X_n \longrightarrow$ R.V.

$c_1, c_2, \ldots, c_n \longrightarrow$ constants

$$X = c_1X_1 + c_2X_2 + \ldots + c_nX_n$$

$$E[X] = E[c_1X_1 + c_2X_2 + \ldots + c_nX_n]$$

$$= c_1E[X_1] + c_2E[X_2] + \ldots + c_nE[X_n]$$

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$$\begin{array}{lcccccccc} & 1 & 2 & 3 & \cdots & i & \cdots & n \\ \sigma: & 3 & 2 & 5 & 1 & j & \cdots & k \end{array}$$

fixes 2.

σ fixes a $j \in [n]$ if $\sigma(j) = j$.

$\sigma(1) = 3$

$\sigma(2) = 2$

$\sigma(3) = 5$

$\sigma(4) = 1$

$\vdots$

A random variable

$X(\sigma)$: no. of elements that are fixed in σ.

a permutation of $[n]$ chosen uniformly at random from the set of all $n!$ permutations.

$$X(\sigma) = X_1 + X_2 + \cdots + X_n,$$

when $X_i = \begin{cases} 1, & \text{if } i \text{ is fixed in } \sigma \\ 0, & \text{otherwise} \end{cases}$

$$\Pr[X_i = 1] = \frac{1}{n}.$$

$$\Pr[X_i = 0] = 1 - \frac{1}{n}$$

$$E[X_i] = \frac{1}{n} \cdot 1 + \left(1 - \frac{1}{n}\right) \cdot 0$$

$$= \frac{1}{n}$$

$$E[X(\sigma)] = E\left[\sum_{i=1}^{n} X_i\right]$$

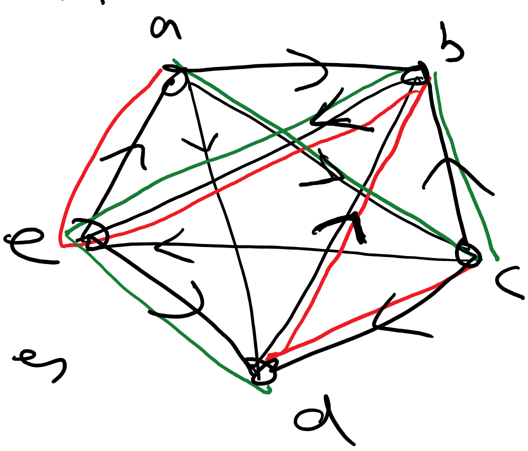
$$= \sum_{i=1}^{n} E[X_i]$$ (by lin of expectation)

$$= n \cdot \frac{1}{n}$$

$$= 1$$

Hamiltonian paths in tournaments

A tournament T on n vertices is a complete graph on n vertices where every edge is oriented.



Hamiltonian path

→ A path that goes through all the vertices.

Hamiltonian path. → c d b e a

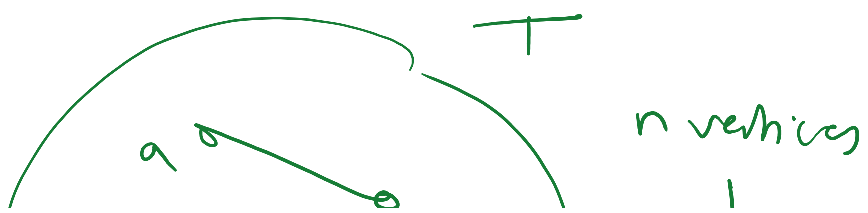
a c b e d

a c d b e

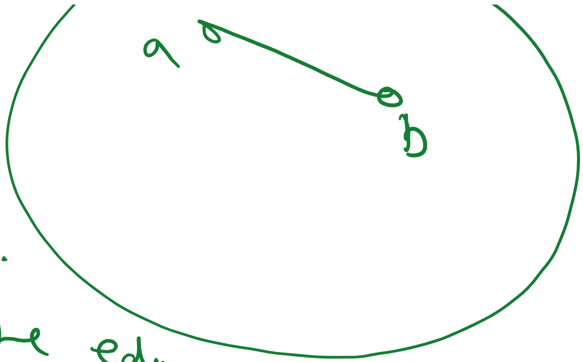
Theorem: There is a tournament T with n players and at least $\frac{n!}{2^{n-1}}$ Hamiltonian paths.

Proof:

For each edge



For each edge independently toss an unbiased coin.



n vertices $\downarrow$ $[n]$

Head $\longrightarrow$ direct the edge from left to right

Tail $\longrightarrow$ direct " " " right to left.

For each permutation σ of $[n]$,

$$X_\sigma = \begin{cases} 1, & \text{if } \sigma \text{ corresponds to a Hamiltonian path in } T \\ 0, & \text{otherwise} \end{cases}$$

Let X be the R.V. that denote the no. of Hamiltonian paths in T.

$$X = \sum_{\substack{\sigma:\ \sigma \text{ is a permutation} \\ \text{of } [n].}} X_\sigma$$

$$\Pr[X_\sigma = 1] = \frac{1}{2^{n-1}}$$

$\sigma = 1 \to 2 \to 3 \to 4 \to 5 \to \cdots \to n$

$$E[X_\sigma] = \frac{1}{2^{n-1}} \cdot 1 + \left(1 - \frac{1}{2^{n-1}}\right) \cdot 0$$

$$= \frac{1}{2^{n-1}} \quad \text{—— (A)}$$

By linearity of expectation,

$$E[X] = \sum_{\substack{\sigma:\ \sigma \text{ is a} \\ \text{permutation of} \\ [n]}} E[X_\sigma]$$

$$= \sum \frac{1}{\ } \quad (\text{using}$$

$$= \sum_{\substack{\sigma : \sigma \text{ a permutation} \\ \text{of } [n]}} \frac{1}{2^{n-1}} \quad (\text{from } A)$$

$$= \frac{n!}{2^{n-1}}$$

$\square$

Ramsey Theory

Theorem: There is a way of coloring the edges of a K_n with 2 colors such that there are at most

$$\frac{\binom{n}{a}}{2^{\binom{a}{2}-1}}$$

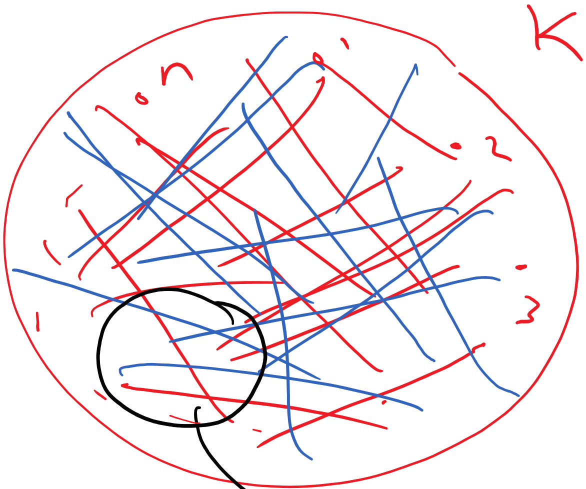
monochromatic a-cliques.

complete graph on n vertices

K_a

Red, Blue

K_n



$|S| = a$

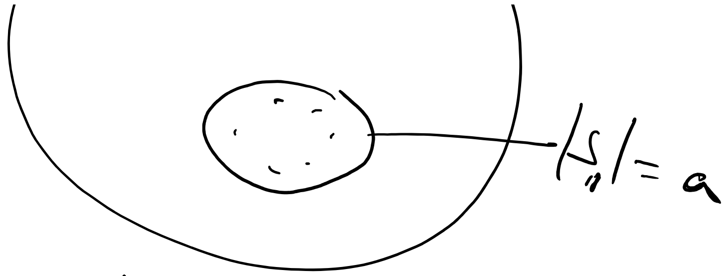
K_a is monochromatic if all the $\binom{a}{2}$ edges get the same color.

Proof: $a \le n$.

For each edge, toss an

K_n

For each edge, toss an unbiased coin. If it is Head, color the edge RED. Otherwise, color it BLUE.



$$\Pr[S_1 \text{ is monochromatic}] = \Pr\left[\begin{array}{l}\text{all edges inside}\\ S_1 \text{ got RED color}\end{array}\right] + \Pr\left[\begin{array}{l}\text{all edge inside } S_1\\ \text{got BLUE color}\end{array}\right]$$

$$= \frac{1}{2^{\binom{a}{2}}} + \frac{1}{2^{\binom{a}{2}}}$$

$$= \frac{1}{2^{\binom{a}{2}-1}} \quad \text{— A}$$

$$X_1 = \begin{cases} 1, & \text{if } S_1 \text{ is monochromatic} \\ 0, & \text{o/w.} \end{cases}$$

$$E[X_1] = \frac{1}{2^{\binom{a}{2}-1}} \cdot 1 + \left(1 - \frac{1}{2^{\binom{a}{2}-1}}\right) 0$$

$$= \frac{1}{2^{\binom{a}{2}-1}} \quad \text{— B}$$

$\binom{n}{a}$ a-sized subsets $S_1, S_2, \ldots$

$S_{\binom{n}{a}}$

$X_1, X_2, \ldots X_i \ldots X_{\binom{n}{a}}$

$X_i = $ 1, if S_i is monochromatic
0, otherwise.

X : R.V. that denotes the no. of monochromatic a-cliques in K_n.

$$X = \sum_{i=1}^{\binom{n}{a}} X_i$$

$$E[X] = E\left[\sum_{i=1}^{\binom{n}{a}} X_i\right]$$

$$= \sum_{i=1}^{\binom{n}{a}} E[X_i]$$, by linearity of expectation

$$= \frac{\binom{n}{a}}{2^{\binom{a}{2}-1}}$$ [from (B)]

$\square$ $\square$

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Splitting graphs.

Theorem: Let G be a graph on n vertices and m edges. Then G contains a bipartite subgraph with at least $\frac{m}{2}$ edges.

Proof:

G

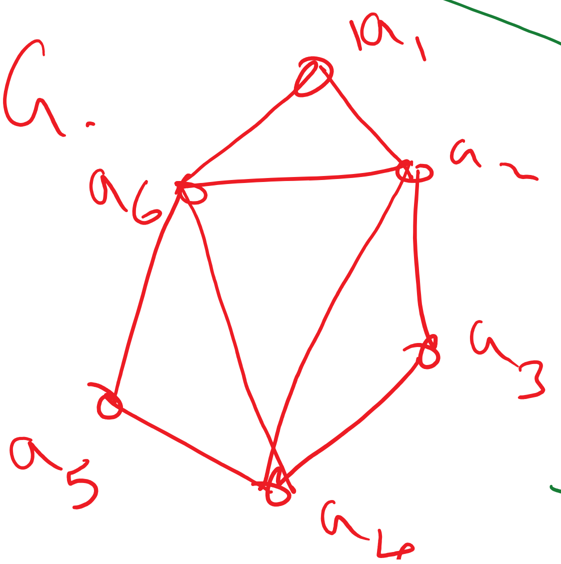
$E(G) \to$ edge set

$V(G) \to$ vertex set

$E(G) \subseteq V \times V$.

H is a subgraph of G if $V(H) \subseteq V(G)$ and $E(H) \subseteq E(G)$.

G.

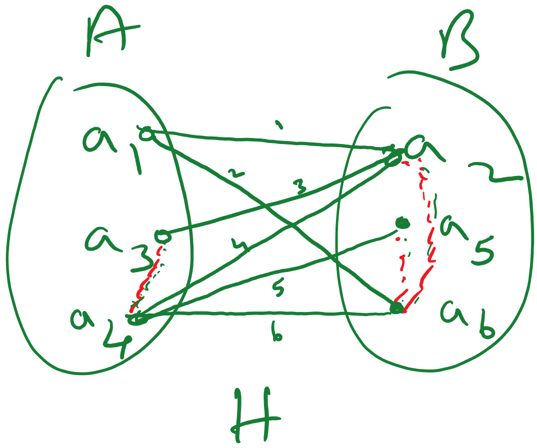


$|E(G)| = 9$

We want a bipartite subgraph H of G with

~ 5 ~~a_4~~ a_4

Subgraph H of G with $\geq \frac{9}{2}$ edges.



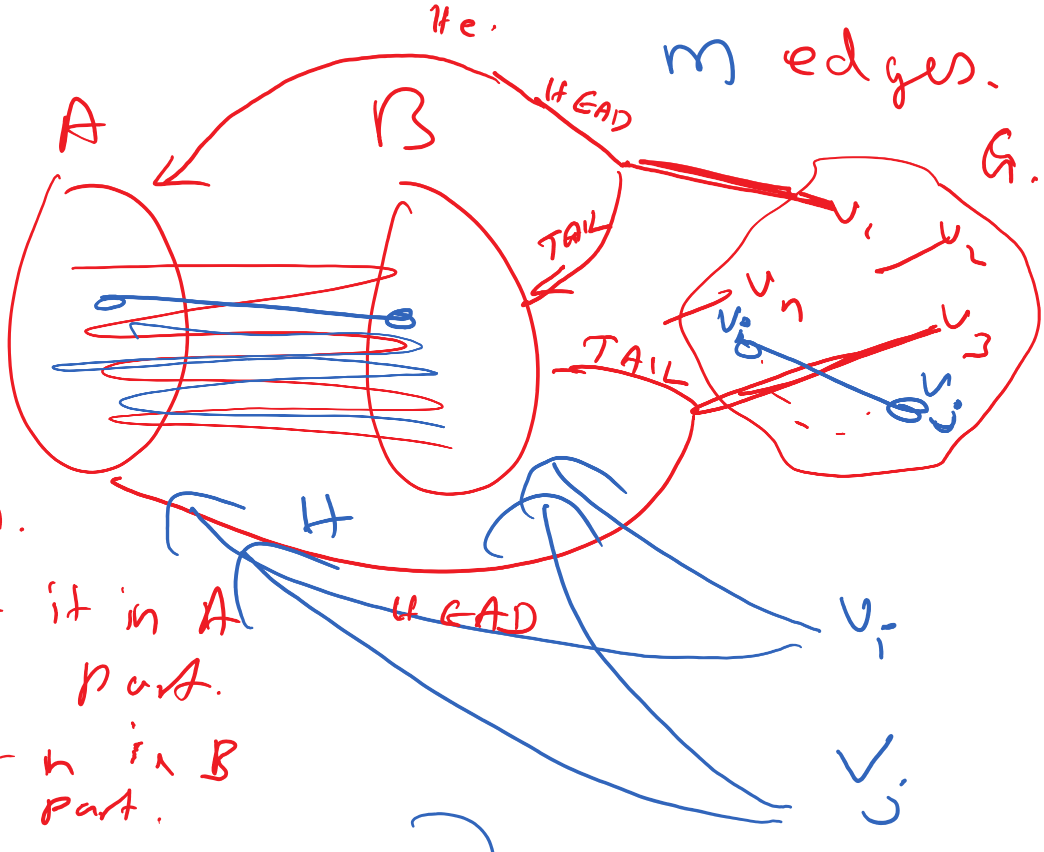
H is a bipartite subgraph of G with 6 edges.

Proof: Given G with n vertices m edges.

For each $v \in V(G)$, independently toss an unbiased coin.

Head $\to$ put it in A part.

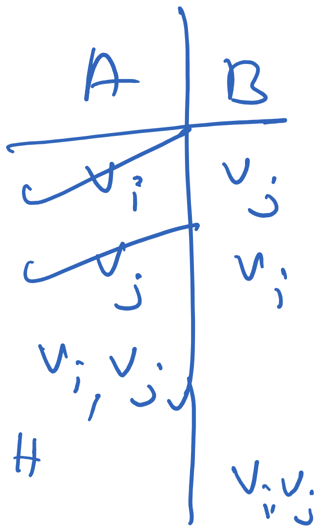
Tail $\to$ put it in B part.



$$\Pr\left[\begin{array}{l} v_i v_j \text{ appears as} \\ \text{a } \underline{\text{cross-edge}} \text{ in} \\ H \end{array}\right] = \frac{1}{2} \quad \text{— (A)}$$

$A \mid B$

H



For each edge e in G,

$$X_e = \begin{cases} 1, & \text{if } e \text{ appears as a cross-edge in } H \\ 0, & \text{otherwise} \end{cases}$$

$$E[X_e] = Pr(X_e = 1) \cdot 1 + Pr(X_e = 0) \cdot 0$$

$$= \frac{1}{2} \quad (\text{from } A)$$

$$= \quad B$$

X : A R.V. that denotes the no. of cross-edges in H.

$$X = \sum_{e \in E(G)} X_e$$

$$E(X) = E\left[\sum_{e \in E(G)} X_e\right]$$

$$= \sum_{e \in E(G)} E[X_e] \quad (\text{by linearity of ex...})$$

$e \in E(G)$ $\qquad$ of expectation)

$$= m \cdot \frac{1}{2}$$

$$= \frac{m}{2}$$

$\square$