Code C

MA1140: Final Examination

Duration: 2 hour Date: 15/03/2019 Time: 7 - 9 am

Name: Roll no. _

1. For $A = \begin{pmatrix} 0 & 1 & 2 & 0 \\ 0 & 2 & 4 & 0 \end{pmatrix}$, fill in the following table.

 $[5 \times 1 = 5]$

Code C

Total marks: 50

(
A basis of the row space of A	$\{(0,1,2,0)\}$. Be careful as the answer is not unique.
A basis of the column space of A	$\left\{ \begin{pmatrix} 1 \\ 2 \end{pmatrix} \right\}$. The answer is not unique.
A basis of the null space of A	$\left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ -2 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} \right\}. $ The answer is not unique.
rank of A (i.e rank of a linear map)	1
nullity of A	3

2. Write T (for True) or F (for False) on the space provided.

 $[10 \times 1.5 = 15]$

- (a) Let A be a row reduced echelon matrix with m rows and n columns over \mathbb{R} , where m > n. Let r be the number of non-zero rows of A. Then r is less than or equal to n. \mathbf{T} **Reason.** In this case, r = row rank(A) = column rank(A). So $r \leq n$.
- (b) If u and v are eigenvectors of a matrix A, then u+v is also an eigenvector of A. \mathbf{F} **Reason.** One can construct a counterexample very easily by considering a 2×2 matrix. For example, one may take v = -u. In that case u + v = 0 (zero vector) which cannot be an eigenvector (by definition).
- (c) Let A be an $m \times n$ matrix over \mathbb{R} . Let $r \leq n$. Suppose the first r columns of A are linearly independent, and the last r columns span the whole column space. Then r is equal to the dimension of the column space of A. ___ \mathbf{T} ___
- (d) Let $T: \mathbb{R}^n \to \mathbb{R}^n$ be a linear map. If we know T(v) for n distinct non-zero vectors v in \mathbb{R}^n , then we know T(v) for every vector v in \mathbb{R}^n . \mathbf{F}
- (e) $\{(x,0,-x):x\in\mathbb{R}\}$ is a vector subspace of \mathbb{R}^3 . ____ \mathbf{T} ____

 $\{(x,0,-x):x\in\mathbb{R}\}$ is a vector subspace of \mathbb{R} . Reason. This is the set of all linear combinations of $\left\{\begin{pmatrix}1\\0\\-1\end{pmatrix}\right\}$ over \mathbb{R} . So it is a subspace.

(f) Consider $S = \{(x,y) \in \mathbb{R}^2 : x = 0 \text{ or } y = 2x\}$ with usual vector addition and scalar multiplication. Then S is a subspace of \mathbb{R}^2 . \mathbf{F}

Reason. Do not miss the word 'or'. Since it is 'x = 0' or 'y = 2x', S is a union of two subspaces x=0 (y-axis) and y=2x (a line). None of these two subspaces is contained in the other. So S is not a subspace. Also one can verify directly that S is not closed under vector addition, e.g.,

$$\begin{pmatrix} 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 2 \end{pmatrix} \in S$$
, but $\begin{pmatrix} 0 \\ 1 \end{pmatrix} + \begin{pmatrix} 1 \\ 2 \end{pmatrix} = \begin{pmatrix} 1 \\ 3 \end{pmatrix} \notin S$. Note that if you write 'and' in place of 'or', then S is a subspace, because in that case it is an intersection of two subspaces.

- (g) Let A be an $n \times n$ matrix such that for every $b \in \mathbb{R}^n$, AX = b has at least one solution. There may exist some b such that AX = b has more than two solutions. ____ \mathbf{F} ___ Reason. A can be thought as a linear map from \mathbb{R}^n to itself. Using the Rank-Nullity Theorem, we have A is surjective if and only if A is injective.
- (h) Let A be a non-invertible square matrix over \mathbb{F} , where \mathbb{F} is \mathbb{R} or \mathbb{C} . Depending on \mathbb{F} , the matrix A may not have an eigenvalue. F **Reason.** Many students missed the term 'non-invertible'. Since A is non-invertible, AX = 0 has a non-trivial solution. Thus there is a non-zero vector v such that $Av = 0 \cdot v$. Hence 0 is an eigenvalue of A irrespective of the base field \mathbb{F} . Do not forget that $\mathbb{R} \subset \mathbb{C}$, i.e., all real numbers are contained in the set of complex numbers.
- (i) Let u, v ∈ R³ be such that u ≠ cv for any c ∈ R. Then there are only finitely many subspaces of R³ containing the vectors u, v. ___ T ___
 Reason. By the given condition, u, v are linearly independent vectors. So dim(Span{u, v}) = 2. Comparing the dimension, one obtains that there are only two subspaces of R³ containing u, v which are Span{u, v} and R³.
- (j) Let A and B be row equivalent matrices. Then column $\operatorname{rank}(A) = \operatorname{column\ rank}(B)$. $_$ \mathbf{T} $_$ Reason. Since A and B are row equivalent, row $\operatorname{space}(A) = \operatorname{row\ space}(B)$. This yields that $\operatorname{row\ rank}(A) = \operatorname{row\ rank}(B)$, hence $\operatorname{column\ rank}(A) = \operatorname{column\ rank}(B)$.
- **3.** Write only the answers to the following questions: $[2 \times 1.5 = 3]$
 - (a) The dimension of the vector space of $n \times n$ diagonal matrices with usual operations? $\underline{\hspace{1cm}} n \underline{\hspace{1cm}}$

Notes. Many students have written 0 as an answer. Note that the rank of T is 0 is and only if T is the trivial map, i.e., the zero map. But it is given that T is a non-zero map.

4. Tick all the matrices which are elementary (otherwise cross) from the following: $\begin{bmatrix} 4 \times 0.5 = 2 \end{bmatrix}$

(a)
$$\begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$
 \checkmark (b) $\begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$ \checkmark (c) $\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}$ \mathbf{X} (d) $\begin{pmatrix} 0 & 0 & 1 \\ 0 & 2 & 0 \\ 1 & 0 & 0 \end{pmatrix}$ \mathbf{X}

5. Let $\mathcal{C}(A)$ and $\mathcal{N}(A)$ denote the column and null spaces of an $n \times n$ matrix A over \mathbb{R} respectively. Then $\mathbb{R}^n = \mathcal{C}(A) + \mathcal{N}(A)$. [1+2=3]

Write
$$(T/F)$$
: F

Justification: You should give a counterexample. Consider
$$\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$$
. [1]

For this matrix, both $C(A) = \operatorname{Span}\left\{\begin{pmatrix} 1 \\ 0 \end{pmatrix}\right\}$ and $\mathcal{N}(A) = \operatorname{Span}\left\{\begin{pmatrix} 1 \\ 0 \end{pmatrix}\right\}$.

Therefore
$$C(A) + \mathcal{N}(A) = \operatorname{Span}\left\{ \begin{pmatrix} 1 \\ 0 \end{pmatrix} \right\} \neq \mathbb{R}^n$$
. [1]

6. Let A be an $n \times n$ matrix over \mathbb{R} . Then $\operatorname{nullity}(A) = \operatorname{nullity}(A^t)$. [1 + 2 = 3]

Write
$$(T/F)$$
: T

Justification: Considering A as a linear map,

$$rank(A) = column \ rank(A)$$
 [observation]
= $row \ rank(A)$ [by a theorem proved in the class]

Note that

 $row space(A) = column space(A^t)$

$$\Longrightarrow$$
 row rank (A) = column rank (A^t)

7. Let $T: \mathbb{R}^3 \to \mathbb{R}^2$ be a map defined by $T \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} x_1 - x_3 \\ x_2 + x_3 \end{pmatrix}$. Write the matrix representation A of T with respect to the ordered bases $\left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right\}$ and $\left\{ \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right\}$ of \mathbb{R}^3 and \mathbb{R}^2

respectively. [3]

Answer (only):
$$A = \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & 1 \end{pmatrix}$$
 (For each entry, 0.5 marks.)

8. Set $C^{\infty}(\mathbb{R}) = \{f : \mathbb{R} \to \mathbb{R} \mid f \text{ is infinitely differentiable function}\}$. Consider $T : C^{\infty}(\mathbb{R}) \to C^{\infty}(\mathbb{R})$ defined by T(f) = f' (first derivative). Then any real number is an eigenvalue for T. [1 + 2 = 3]

Write
$$(T/F)$$
: T

Justification: Consider the non-zero element (vector) $e^{\lambda x}$ of $C^{\infty}(\mathbb{R})$ for every $\lambda \in \mathbb{R}$. [1] Since $T(e^{\lambda x}) = \lambda e^{\lambda x}$, $\lambda \in \mathbb{R}$ is an eigenvalue of T (with the corresponding eigenvector $e^{\lambda x}$). Thus any real number is an eigenvalue for T.

Notes. By definition, a scalar $\lambda \in \mathbb{R}$ is an eigenvalue of T if there EXISTS a non-zero vector v such that $T(v) = \lambda v$. Note that if λ is an eigenvalue, then every non-zero vector is not necessarily an eigenvector of T corresponding to λ , but there is at least one non-zero vector v such that $T(v) = \lambda v$. Thus, in order to show that every $\lambda \in \mathbb{R}$ is an eigenvalue of T, you just have to find at least one non-zero vector v_{λ} for every particular λ (i.e., v_{λ} is depending on λ) such that $T(v_{\lambda}) = \lambda v_{\lambda}$. Read carefully the definition of eigenvectors and eigenvalues discussed in class.

9. Let V be the vector space of all 2×2 matrices over \mathbb{R} with usual operations. Let W be the subspace consisting of all matrices $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ such that a+b=c+d. Extend the set $\left\{ \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \right\}$ basis of W.

Extended basis:
$$\left\{ \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix} \right\}$$
. Note that the answer is not unique. [2]

Justification:

You should check two things:

There are many other ways also to check whether a subset is a basis.

10. Let V be the space of all 3×3 real matrices with usual operations. Consider $A = \begin{pmatrix} 1 & -3 & 0 \\ 0 & 2 & 0 \\ 3 & -2 & 1 \end{pmatrix}$. Is $A^3 \in \text{Span}\{A^2, A, I_3\}$, where I_3 is the identity matrix? Ans. $(Y/N) \longrightarrow Y \longrightarrow$. Justify your answer in the space below. Are A^3 , A^2 , A, I_3 linearly independent? Ans. $(Y/N) \longrightarrow N \longrightarrow$.

Justification:

Step 1. Compute the characteristic polynomial
$$p_A(x) = \det(xI_3 - A)$$
. In this case, $p_A(x) = (x-1)^2(x-2) = x^3 - 4x^2 + 5x - 2$.

[1+2+1=4]

Step 2. Now by Caylay-Hamilton Theorem, $A^3 - 4A^2 + 5A - 2I_3 = 0$ (zero matrix). It shows that A^3, A^2, A, I_3 are linearly dependent. Moreover $A^3 = 4A^2 - 5A + 2I_3$. [1]

Notes: One can verify the non-trivial relation $A^3 - 4A^2 + 5A - 2I_3 = 0$ directly. But that would be painful. First of all, they have to compute A^2 and A^3 , and then they have to either guess or find out the non-trivial relation, and ultimately they should verify that relation.

11. Let A be a 2×2 matrix over \mathbb{R} . Suppose A has two eigenvalues λ_1 and λ_2 in \mathbb{R} such that $\lambda_1 \neq \lambda_2$. Prove or disprove that A is diagonalizable. [1 + 4 = 5]

Proof/disproof:

Step 1. Let v_1, v_2 be two eigenvectors corresponding to the eigenvalues λ_1, λ_2 respectively. [1]

Step 2. We claim that v_1 and v_2 are linearly independent. Let $c_1v_1 + c_2v_2 = 0$. Apply A to obtain that $c_1\lambda_1v_1 + c_2\lambda_2v_2 = 0$. [1] Step 3. We have $(c_1\lambda_1v_1 + c_2\lambda_2v_2) - \lambda_1(c_1v_1 + c_2v_2) = 0$, which yields that $c_2(\lambda_2 - \lambda_1)v_2 = 0$. Since $\lambda_1 \neq \lambda_2$ and $v_2 \neq 0$ (by definition), we get that $c_2 = 0$. It follows that $c_1v_1 = 0$. Thus, since $v_1 \neq 0$, $c_1 = 0$. Therefore v_1 and v_2 are linearly independent.

Step 4. Since \mathbb{R}^2 has a basis $\{v_1, v_2\}$ consisting of eigenvectors, one can conclude directly (by using the **diagonalizability criteria** proved in the class) that A is diagonalizable. (Or one can prove this by taking $P = \begin{bmatrix} v_1 & v_2 \end{bmatrix}$ with v_1 and v_2 as the 1st and 2nd columns respectively.) [1]

Notes. There are many other ways also to show diagonalizability. If your argument is complete, you will get full marks.