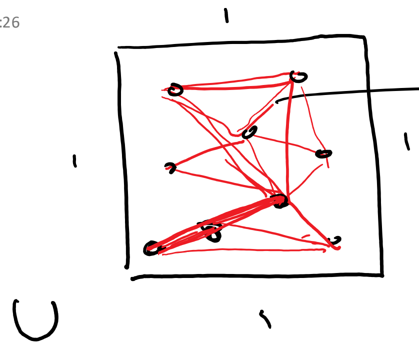
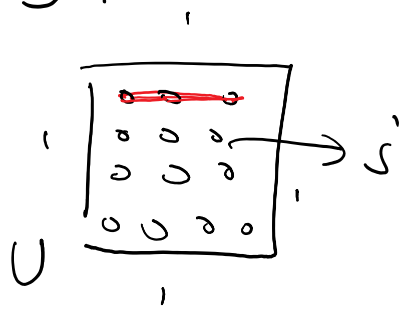


S: a set of n points in U.

$T(S)$ is the min of area of a Δ whose vertices are 3 distinct points of S.

$$T(n) = \max_{\substack{S:\, S \text{ is} \\ \text{a set of } n \\ \text{pts in } U.}} T(S).$$



Conjecture (Heilbronn): $T(n) = O\left(\frac{1}{n^2}\right)$.

Disproved by Komlós, Pintz and Szemerédi in 1982. $\longrightarrow$ Which means they showed the existence of a point set S in U where $T(S) = \Omega\left(\frac{\log n}{n^2}\right)$

Theorem: There is a set S of n points in the unit square U such that

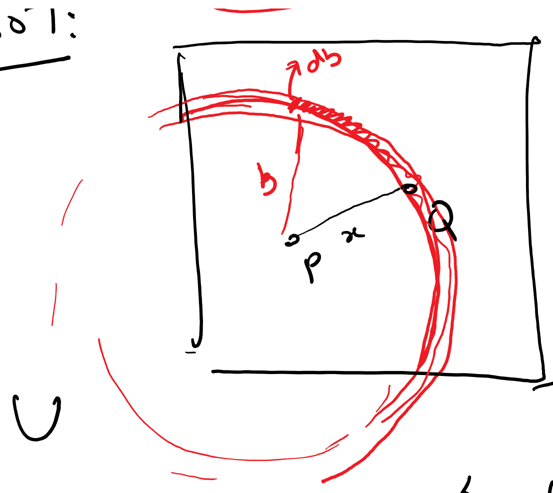
$$T(S) \geq \frac{1}{100 n^2}.$$

Proof: P and Q are chosen indep and uniformly at random from U.

Let x denote the distance between P and Q.

What is the prob that

Proof:

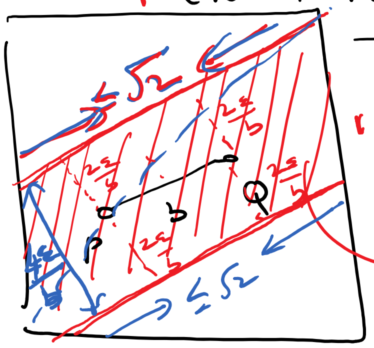


Let x … between P and Q. What is the prob that $x = b$?

$$P_r\left[b \le x \le b + db\right] \le 2\pi b\, db \quad \text{(1)}$$

To FIND: Prob that area of a Δ formed by 3 pts P, Q, R chosen indep and uniformly at random is $\le \varepsilon$

Given $dist(P,Q) = b$, for area of ΔPQR to be $\le \varepsilon$, we want R to be at a height $\le \frac{2\varepsilon}{b}$.



If R is outside this shaded region, then area of $\Delta PQR > \varepsilon$.

$$\left.\begin{array}{l}\text{Area of shaded}\\ \text{region}\end{array}\right\} \le \frac{4\sqrt{2}\,\varepsilon}{b}.$$

Given $dist(P,Q) = b$, Prob that $\left(\text{area of } \Delta PQR \le \varepsilon\right) \le \frac{4\sqrt{2}\,\varepsilon}{b}$.

$$\text{Prob}\left[\text{area of } \Delta PQR \le \varepsilon\right] \le \int_0^{\sqrt{2}} \underbrace{(2\pi b)}_{\text{Prob}[dist(P,Q)\approx b]} \underbrace{\left(\frac{4\sqrt{2}\,\varepsilon}{b}\right)}_{\text{Prob}\left[\substack{\text{area of}\\ \Delta PQR} \le \varepsilon \,\middle|\, dist(P,Q)\approx b\right]} db$$

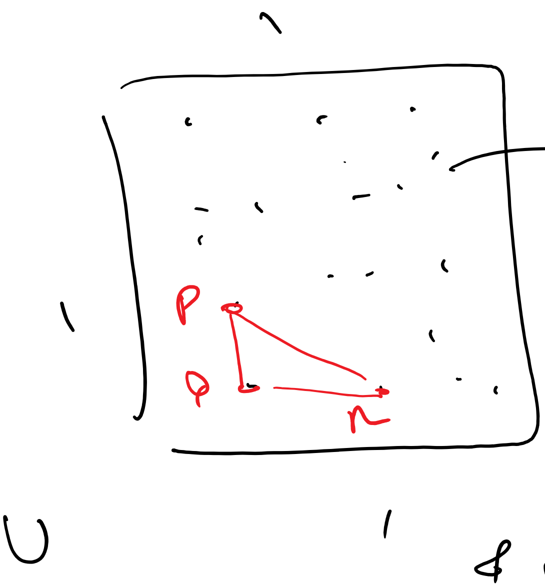
(P, Q, R chosen indep & uniformly at random from U)

$$= 2\pi \cdot 4\sqrt{2}\,\varepsilon \cdot (\sqrt{2} - 0)$$

$$= \underline{\underline{16\pi\varepsilon}} \quad \text{———} \; (2)$$

When $\varepsilon = \dfrac{1}{100n^2}$,

$$\Pr\left[(\text{area of } \Delta PQR) \le \frac{1}{100n^2}\right] \le \frac{16\pi}{100n^2} < \frac{0.6}{n^2} \quad \text{——} \; (3)$$



S' : is a set of $2n$ points, where each point is chosen independently & uniformly at random from U.

$$\text{Prob}\left[\text{area of } PQR \le \frac{1}{100n^2}\right] \le \frac{0.6}{n^2}.$$

$$X_{PQR} = \begin{cases} 1, & \text{if area of } \Delta PQR \le \frac{1}{100n^2} \\ 0, & \text{o/w} \end{cases}$$

$$E[X_{PQR}] < \frac{0.6}{n^2} \;//$$

R.U.

X : denote the no. of Δs whos

$$\text{area} \leq \frac{1}{100n^2}$$

$$X = \sum_{\substack{P,Q,R \in S',\\ P,Q,R \text{ are distinct}}} X_{PQR}$$

By Linearity of Expectation,

$$E[X] = \sum_{\substack{P,Q,R \in S',\\ P,Q,R \text{ are distinct}}} E[X_{PQR}]$$

$$< \binom{2n}{3} \frac{0.6}{100n^3}$$

$$< n$$

That is, there is a way of of choosing $2n$ points in U such that there are less than n triangles (formed by any 3 of these points) whose area is $\leq \frac{1}{100n^2}$.

Each such bad Δ (means Δ whose area $\leq \frac{1}{100n^2}$) can be killed by removing any one of its 3 endpoints.

Let S be the point set that you get from S' by removing one point corresponding to each bad triangle.

What do we have! We have S which is a set of at least n points in U such that

$$T(S) \geq \frac{1}{100n^2}.$$

$\square$