Fourier Integrals

$\longrightarrow \quad \longleftarrow$

$g : \mathbb{R} \to \mathbb{R}$

$g(x+2L) = g(x), \forall x \in \mathbb{R}$.

$$g(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{L} + b_n \sin\frac{n\pi x}{L}$$

$$a_0 = \frac{1}{2L}\int_{-L}^{L} g_L(u)\,du$$

$$a_n = \frac{1}{L}\int_{-L}^{L} g_L(u)\cos\frac{n\pi x}{L}\,dx$$

$$b_n = \frac{1}{L}\int_{-L}^{L} g_L(u)\sin\frac{n\pi x}{L}\,dx.$$

what happens when $L \to \infty$

$$w_n = \frac{n\pi}{L}.$$

$$g_L(x) = \frac{1}{2L}\int_{-L}^{L} g_L(x)\,dx + \sum_{n=1}^{\infty}\left(\frac{1}{L}\int_{-L}^{L} g_L(x)\cos(w_n u)\,du\right)\cos w_n x.$$

$$+ \sum_{n=1}^{\infty}\frac{1}{L}\int_{-L}^{L}\left(g_L(x)\sin(w_n u)\,du\right)\sin w_n x.$$

$$\Delta w_n = w_{n+1} - w_n = \frac{(n+1)\pi}{L} - \frac{n\pi}{L}$$

$$= \frac{\pi}{L}$$

$$\Rightarrow \frac{1}{L} = \frac{\Delta w_n}{\pi}$$

$$g_L(x) = \frac{1}{2L}\int_{-L}^{L} g_L(u)\,du + \sum_{n=1}^{\infty}\frac{\Delta w_n}{\pi}\left(\int_{-L}^{L}\left(g_L(u)\cos w_n u\,du\right)\cos w_n x\right)$$

$$+ \sum_{n=1}^{\infty}\frac{\Delta w_n}{\pi}\left(\int_{-L}^{L}\left(g_L(u)\sin w_n u\,du\right)\sin w_n x\right).$$

Write,

$$h(w_n) = \int_{-L}^{L} g_L(u) \cos w_n u \, du$$

$$\Rightarrow k(w_n) = \int_{-L}^{L} g_L(u) \sin w_n u \, du.$$

$$g_L(x) = \frac{1}{2L}\int_{-L}^{L} g_L(u)\,du + \frac{1}{\pi}\sum_{n=1}^{\infty} h(w_n)\cos w_n x \, \Delta w_n +$$

$$\frac{1}{\pi}\sum_{n=1}^{\infty} k(w_n)\sin w_n x \,(\Delta w_n).$$

Assume that,

1) $g_L(u) \longrightarrow g(u) \quad \forall\, u \in \mathbb{R}.$

2) $\int_{-\infty}^{\infty} |g(u)|\,du < \infty$ (Absolutely integrable function).

Equivalent to:

$$\lim_{b\to\infty} \int_{(-b)}^{0} |g(x)|\,dx + \lim_{a\to\infty}\int_{0}^{a} |g(x)|\,dx < \infty.$$

$$g(x) = \frac{1}{\pi}\int_0^{\infty} h(w)\cos wx \, dw + \frac{1}{\pi}\int_0^{\infty} k(w)\sin wx \, dw$$

$$h(w) = \int_{-\infty}^{\infty} g(u)\cos wu \, du.$$

$$k(w) = \int_{-\infty}^{\infty} g(u)\sin wu \, du$$

$$\Rightarrow \quad g(x) = \int_0^{\infty} \left(A(w)\cos wx + B(w)\sin wx\right) dx.$$

where,

$$\left.\begin{aligned} A(w) &= \frac{1}{\pi} h(w) = \frac{1}{\pi}\int_{-\infty}^{\infty} g(u)\cos wu \, du \\ B(w) &= \frac{1}{\pi} k(w) = \frac{1}{\pi}\int_{-\infty}^{\infty} g(u)\sin wu \, du \end{aligned}\right\} \text{fourier integral of } g.$$

$$f(x) = \sum_{n=0}^{\infty} \left(a_n \cos w_n x + b_n \sin w_n x\right).$$

* If $g : \mathbb{R} \longrightarrow \mathbb{R}$ such that.

1) g is p.c on every finite interval.

2) g is absolutely integrable.

3) g has left and right hand limits at every point of finite interval,

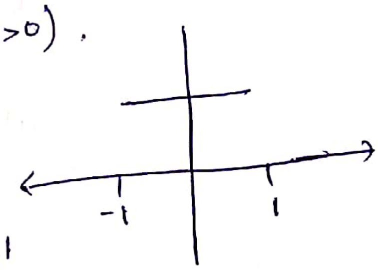
If g is discontinuous at x_0,

then $$g(x_0) = \frac{g(x_0^+) + g(x_0^-)}{2}.$$

<u>Example:-</u>

Find the fourier integral representation of.

$$f(x) = \begin{cases} k & -1 < x < 1 \\ 0 & \text{else} \end{cases} \qquad (k > 0).$$



$$A(w) = \frac{k}{\pi} \int_{-1}^{1} \cos wu \, du = \frac{k}{\pi} \left[\frac{\sin wu}{w}\right]_{-1}^{1}$$

$$= \frac{2k}{\pi w} \sin w$$

$$B(w) = \frac{k}{\pi} \int_{-1}^{1} \sin wu \, du = 0;$$

$$f(x) = \int_{0}^{\infty} \frac{2k}{\pi w} \sin w \cos wx \, dw.$$

$$\int_{0}^{\infty} \frac{\sin w}{w} \cos wx \, dw = \frac{\pi}{2k} f(x) = \begin{cases} \pi/2 & 0 < x < 1 \\ \pi/4 & x = 1 \\ 0 & \text{else} \end{cases}$$

when $x = 0$.

$$\int_0^{\infty} \frac{\sin w}{w}\, dw = \pi/2,$$

$$\lim_{u \to \infty} \int_0^{u} \frac{\sin w}{w}\, dw \quad \text{(since Integral)}$$

$$\Rightarrow \quad f(x) = \int_0^{\infty} \left(A(w) \cos wx + B(w) \sin wx\right) dw .$$

Suppose, f is even i.e. $f(x) = f(-x)$.

$$A(w) = \frac{1}{\pi} \int_{-\infty}^{\infty} f(x) \cos wu \, du$$

$$= \frac{1}{\pi} \int_{-\infty}^{\infty} f(-u) \cos wu \, du$$

write $v = -u$

$$\Rightarrow \quad A(w) = \frac{1}{\pi} \int_{-\infty}^{\infty} f(v) \cos wv \, dv .$$

$$= \frac{1}{\pi} \int_{-\infty}^{\infty} f(u) \cos v$$

$$B(w) = \frac{1}{\pi} \int_{-\infty}^{\infty} f(u) \sin wu \, du$$

$$= \frac{1}{\pi} \int_{-\infty}^{\infty} f(-u) \sin wu \, du$$

$$B(w) = \frac{1}{\pi} \int$$

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$f : [-\frac{L}{\cancel{\pi}}, \frac{L}{\cancel{\pi}}] \longrightarrow \mathbb{R}$

$$f(x+2\pi) = f(x).$$

$$f(x) = \sum_{n=0}^{\infty} \left(a_n \cos \tfrac{n\pi}{L} x + b_n \sin \tfrac{n\pi}{L} x\right)$$

$$a_0 = \frac{1}{2L} \int_{-L}^{L} f(x)\, dx.$$

$$a_n = \frac{1}{L} \int_{-L}^{L} f(x) \cos \frac{n\pi x}{L}\, dx$$

$$b_n = \frac{1}{L} \int_{-L}^{L} f(x) \sin \frac{n\pi x}{L}\, dx$$

$f : \mathbb{R} \longrightarrow \mathbb{R}$

$$f(x) = \int_0^{\infty} \left(A(w) \cos wx + B(w) \sin wx\right) dw$$

$$A(w) = \frac{1}{\pi} \int_{-\infty}^{\infty} f(u) \cos wu\, du$$

$$B(w) = \frac{1}{\pi} \int_{-\infty}^{\infty} f(u) \sin wu\, du$$

Fourier Cosine Integral:-

$f : \mathbb{R} \longrightarrow \mathbb{R}$

$$f(x) = \int_0^{\infty} \tilde{A}(w) \cos wx\, dw$$

$$\tilde{A}(w) = \frac{2}{\pi} \int_0^{\infty} f(u) \cos wu\, du.$$

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Fourier sine Integral representation of f:-

$f: \mathbb{R} \to \mathbb{R}$.

$$f(x) = \int_0^\infty \tilde{B}(w) \sin wx \, dw$$

$$\tilde{B}(w) = \frac{2}{\pi} \int_0^\infty f(u) \sin wu \, du.$$

Example:-

Find the fourier cosine and sine Integral rep of

$$f(x) = e^{-kx} \; ; \quad x > 0, \; (k > 0 \text{ fixed})$$

Sol:- 1) f is continuous on $(0, \infty)$

2) $$\int_0^\infty |f(x)| \, dx = \int_0^\infty e^{-kx} dx = \left[\frac{e^{-kx}}{-k}\right]_0^\infty$$

$$= \frac{1}{k} < \infty.$$

$\Rightarrow f$ is absolutely integrable.

Fourier cosine Integral:-

$$\tilde{A}(w) = \frac{2}{\pi} \int_0^\infty f(u) \cos wu \, du$$

$$= \frac{2}{\pi} \int_0^\infty e^{-ku} \cos wu \, du \qquad \text{I LATE}$$

$$= \frac{2}{\pi} \left[e^{-ku} \cancel{\frac{\sin wu}{w}} \right]_0^\infty - \cancel{\int_0^\infty e^{-ku}}$$

$$\tilde{A}(w) = \frac{2}{\pi} \frac{k}{w^2 + k^2}$$

$$f(x) = \int_0^\infty \tilde{A}(w) \cos wx \, dw$$

$$e^{-kx} = \int_0^\infty \frac{2}{\pi} \frac{k}{k^2+w^2} \cos wx \, dw$$

$$\int_0^\infty \frac{\cos wx}{k^2+w^2} \, dw = \frac{\pi}{2k} e^{-kx}$$

Fourier sine integral.

$$\tilde{B}(w) = \frac{2}{\pi} \int_0^\infty f(u) \sin wu \, du$$

$$= \frac{2}{\pi} \int_0^\infty e^{-ku} \sin wu \, du$$

$$= \frac{2}{\pi} \frac{w}{w^2+k^2}$$

$$f(x) = \int_0^\infty \tilde{B}(w) \cos wx \, dw$$

$$e^{-kx} = \int_0^\infty \frac{2}{\pi} \frac{w}{k^2+w^2} \sin wx \, dw$$

$$\int_0^\infty \frac{w \sin wx}{k^2+w^2} \, dw = \frac{\pi}{2} e^{-kx}.$$

$f : \mathbb{R} \longrightarrow \mathbb{R}$

$$f(x) = \int_0^\infty \tilde{A}(w) \cos wx \, dw$$

$$\tilde{A}(w) = \frac{2}{\pi} \int_0^\infty f(u) \cos wu \, du$$

$$= \sqrt{\frac{2}{\pi}} \cdot \sqrt{\frac{2}{\pi}} \int_0^\infty f(u) \cos wu \, du$$

$$= \sqrt{\frac{2}{\pi}} \hat{f}_c(w)$$

$$\hat{f}_c(w) = \sqrt{\frac{2}{\pi}} \int_0^\infty f(u) \cos wu \, du$$

- Fourier cosine transform of f.

Notations :- $\mathcal{F}_c(f)$, $\hat{f}_c(w)$

Fourier sine transform of f

$$\hat{f}_s(w) = \sqrt{\frac{2}{\pi}} \int_0^\infty f(u) \sin wu \, du$$

$$\Rightarrow \quad f(x) = \int_0^\infty \tilde{A}(w) \cos wx \, dw$$

$$= \sqrt{\frac{2}{\pi}} \int_0^\infty \hat{f}_c(w) \cos wx \, dw$$

- Inverse fourier cosine transform.

$$\Rightarrow \quad f(x) = \int_0^\infty \tilde{B}(w) \sin wx \, dw$$

$$= \sqrt{\frac{2}{\pi}} \int_0^\infty \hat{f}_s(w) \sin wx \, dw$$

- Inverse fourier sine transform

Example :-

1) Let $k>0$ and $a>0$

Define $f(x) = \begin{cases} k & 0<x<a \\ 0 & \text{else} \end{cases}$

Find the fourier cosine and sine transform of f.

$$\Rightarrow \quad \hat{f}_c(w) = \sqrt{\frac{2}{\pi}} \int_0^\infty k \cdot \cos wu \, du$$

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$$= \sqrt{\frac{2}{\pi}}\, k \cdot \frac{\sin wu}{w}\Bigg]_0^a$$

$$\hat{f}_c(w) = \sqrt{\frac{2}{\pi}}\, k\, \frac{\sin aw}{w}.$$

$$\Rightarrow \quad \hat{f}_s(w) = \sqrt{\frac{2}{\pi}} \int_0^a k \cdot \sin wu \, du = \sqrt{\frac{2}{\pi}}\, k \, \frac{-\cos wu}{w}\Bigg]_0^a$$

$$\hat{f}_s(w) = \sqrt{\frac{2}{\pi}}\, k \left[\frac{1}{w} - \frac{\cos aw}{w}\right]$$

2) Let,

$$f(x) = \begin{cases} e^{-x}, & x > 0 \\ 0 & x \leq 0. \end{cases}$$

Find fourier cosine and sine transform of f.

$$\Rightarrow \quad \hat{f}_c(w) = \sqrt{\frac{2}{\pi}} \int_0^\infty e^{-x} \cos wx \, dx$$

$$= \sqrt{\frac{2}{\pi}} \left(\frac{1}{w^2+1}\right)$$

e^{-st}

$$\Rightarrow \quad \hat{f}_s(w) = \sqrt{\frac{2}{\pi}} \int_0^\infty e^{-x} \sin wx \, dx$$

$$= \sqrt{\frac{2}{\pi}} \left(\frac{w}{w^2+1}\right)$$

$$F_c(f) = \sqrt{\frac{2}{\pi}} \int_0^\infty f(u) \cos wu \, du$$

$$F_c(f') = \sqrt{\frac{2}{\pi}} \int_0^\infty f'(u) \cos wu \, du$$

$$= \sqrt{\frac{2}{\pi}} \left\{ f(u) \cos wu \Big]_0^\infty + w \int_0^\infty f(u) \sin wu \, du \right\}$$

Assume that:-

$$f(u) \to 0 \text{ as } |u| \to \infty \ (u \to \infty)$$

$$F_c(f') = -\sqrt{\frac{2}{\pi}}\, f(0) + \omega F_s(f)$$

$$\Rightarrow \quad F_s(f') = \sqrt{\frac{2}{\pi}} \int_0^\infty f'(u) \sin \omega u \, du$$

$$= -\omega F_c(f)$$

when $f(u) \to 0$ as $u \to \infty$.

$$F_c(f'') = -\sqrt{\frac{2}{\pi}}\, f'(0) + \omega F_s(f')$$

$$= -\sqrt{\frac{2}{\pi}}\, f'(0) + \omega\left(-\omega F_c(f)\right)$$

$$= -\sqrt{\frac{2}{\pi}}\, f'(0) - \omega^2 F_c(f).$$

$$F_s(f'') = -\omega F_c(f')$$

$$= -\omega\left(-\sqrt{\frac{2}{\pi}}\, f(0) + \omega F_s(f)\right) = \sqrt{\frac{2}{\pi}}\, \omega f(0) - \omega^2 F_s(f)$$

* Find fourier cosine and sine transforms of

$$f(x) = e^{-ax}, \quad x > 0$$
$$a > 0$$

$f: \mathbb{R} \to \mathbb{R}$

$$f(x) = \int_0^\infty \left(A(\omega)\cos\omega x + B(\omega)\sin\omega x\right) d\omega$$

$$= \int_0^\infty \left(\frac{1}{\pi}\int_{-\infty}^{\infty} f(u)\cos\omega u \, du\right)\cos\omega x \, d\omega + \int_0^\infty \left(\frac{1}{\pi}\int_{-\infty}^{\infty} f(u)\sin\omega u \, du\right)\sin\omega x \, d\omega$$

$$= \frac{1}{\pi}\int_0^\infty \int_{-\infty}^{\infty} f(u)\left[\cos\omega u \cos\omega x + \sin\omega u \sin\omega x\right] du \, d\omega$$

$$= \frac{1}{\pi}\int_0^\infty \underbrace{\boxed{\int_{-\infty}^{\infty} f(u)\cos\omega(u-x)\, du}}_{h(\omega)} \, d\omega$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(u)\cos\omega(u-x)\, du\, d\omega \to (1)$$

Check that,

$$\frac{1}{2\pi}\int_{-\infty}^{\infty}\left(\int_{-\infty}^{\infty} f(u)\sin\omega(u-x)\, du\right) d\omega = 0 \longrightarrow (2)$$

$$f(x) = (1) + i(2)$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(u)\, e^{i\omega(u-x)}\, du\, d\omega$$

$$= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\left(\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(u)\, e^{i\omega u}\, du\right) e^{-i\omega x}\, d\omega \longrightarrow (3)$$

$$\hat{f}(\omega) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(u)\, e^{-i\omega u}\, du \longrightarrow (4)$$

- Fourier tranform of f.

From ③, ④

$$f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \hat{f}(w) e^{iwx} dw$$

- The inverse F.T of f.

Examples:-

1) Let $f(x) = \begin{cases} k & 0 < x < a \\ 0 & x \geq a \end{cases}$ $a>0, k>0$.

$$\hat{f}(w) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(u) e^{-iwu} du$$

$$= \frac{1}{\sqrt{2\pi}} \int_{0}^{a} k \cdot e^{-iwu} du = \frac{k}{\sqrt{2\pi}} \left[\frac{e^{-iwu}}{-iw} \right]_0^a$$

$$\hat{f}(w) = \frac{k}{\sqrt{2\pi}} \left(\frac{1 - e^{-iwa}}{iw} \right)$$

2) Let $a>0$. Define.

$f(x) = e^{-ax^2}$, $x \in (-\infty, \infty)$. Find $\hat{f}(w)$?

$$\hat{f}(w) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-au^2} \cdot e^{-iwu} du.$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-(au^2 + iwu)} du.$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\left[(\sqrt{a}u)^2 + 2\left(\frac{iw}{2\sqrt{a}}\right)(\sqrt{a}u) + \left(\frac{iw}{2\sqrt{a}}\right)^2 - \left(\frac{iw}{2\sqrt{a}}\right)^2\right]} du$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\left(\sqrt{a}u + \frac{iw}{2\sqrt{a}}\right)^2} \cdot e^{-\frac{w^2}{4a}} du.$$

Let, $\sqrt{a}u + \frac{iw}{2\sqrt{a}} = y$

$$= \frac{e^{-\frac{w^2}{4a}}}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-y^2} \cdot \frac{1}{\sqrt{a}}\, dy.$$

$$= \frac{e^{-\frac{w^2}{4a}}}{\sqrt{2a}} \qquad \left[\because \int_{-\infty}^{\infty} e^{-y^2} \cdot dy = \sqrt{\pi} \right]$$

$$\hat{f}^{-1}\left(\frac{e^{-\frac{w^2}{4a}}}{\sqrt{2a}} \right) = e^{-ax^2}$$

$$\hat{f}^{-1}\left| e^{-\frac{w^2}{4a}} \right| = \sqrt{2a}\, e^{-ax^2}$$

Let, $a = 1/2$

$$f(x) = e^{-\frac{x^2}{2}}, \quad x \in \mathbb{R}.$$

$$\hat{f}(w) = \frac{1}{\sqrt{2 \cdot 1/2}} \cdot e^{-\frac{w^2}{4 \cdot 1/2}}.$$

$$= e^{-\frac{w^2}{2}}.$$

$$\hat{f}(w) = f(w)$$

$\Rightarrow F(f) = \hat{f}(w)$

1. $F(f+g) = F(f) + F(g)$

2. $F(\alpha f) = \alpha F(f)$

3. $\hat{f}(w) = f(w), \ \forall\, w \in \mathbb{R}.$

4. $\hat{f}(w) \longrightarrow 0$ as $|w| \longrightarrow \infty.$

If $F(f')$ exists, then what is the relation between $F(f')$ and $F(f)$

$$\Rightarrow \quad F(f') = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f'(u) \cdot e^{-i\omega u}\, du$$

$$= \frac{1}{\sqrt{2\pi}} \left[\left[f(u)\, e^{-i\omega u} \right]_{-\infty}^{\infty} - (-i\omega) \int_{-\infty}^{\infty} f(u) e^{-i\omega u}\, du \right]$$

Assume that $|f(u)| \longrightarrow 0$ as $|u| \to \infty$.

$$\Rightarrow \quad \boxed{F(f') = i\omega F(f)}$$

$\Rightarrow$ If f'' exists and $F(f'')$ exists.

$$F(f'') = i\omega F(f')$$

$$F(f'') = -\omega^2 F(f).$$

$\Rightarrow$ Example:-

Find the fourier transform of

$$f(x) = -\cancel{\frac{1}{2}}\, x\, e^{-x^2/2}, \quad x \in \mathbb{R}.$$

Sol:- Let, $g(x) = e^{-x^2/2}$

then $g'(x) = -x \cdot e^{-x^2/2} = f(x)$.

$$\Rightarrow \quad F[g'] = F(f)$$

$$i\omega F(g) = F(f)$$

$$\Rightarrow \quad i\omega e^{-\omega^2/2} = F(f).$$

$\Rightarrow$ $f, g : \mathbb{R} \longrightarrow \mathbb{R}$.

$$(f * g)(t) = \int_{-\infty}^{\infty} f(u)\, g(t-u)\, du$$

If $\hat{f}$ and $\hat{g}$ exists, then $\mathcal{F}(f*g)$ exists and

$$\mathcal{F}(f*g) = \sqrt{2\pi}\, \mathcal{F}(f) \cdot \mathcal{F}(g)$$

Let, $a, b > 0$. Find

$$e^{-ax^2} * e^{-bx^2}$$

$$\mathcal{F}\left(e^{-ax^2} * e^{-bx^2}\right) = \sqrt{2\pi} \cdot \mathcal{F}\left(e^{-ax^2}\right) \cdot \mathcal{F}\left(e^{-bx^2}\right)$$

$$= \sqrt{2\pi} \cdot \frac{1}{\sqrt{2a}} e^{\frac{-\omega^2}{4a}} \cdot \frac{1}{\sqrt{2b}} e^{\frac{-\omega^2}{4b}}$$

$$= \frac{\sqrt{\pi}}{\sqrt{2ab}} e^{-\omega^2\left(\frac{1}{4a}+\frac{1}{4b}\right)}$$

$$e^{-ax^2} * e^{-bx^2} = \sqrt{\frac{\pi}{2ab}}\, \mathcal{F}^{-1}\left(e^{-\omega^2\left(\frac{1}{4a}+\frac{1}{4b}\right)}\right)$$