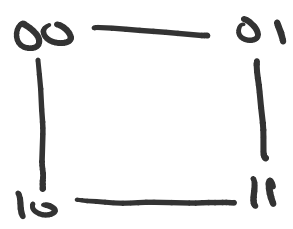
07 December 2020 13:39

More applications of combinatorial nullstellensatz

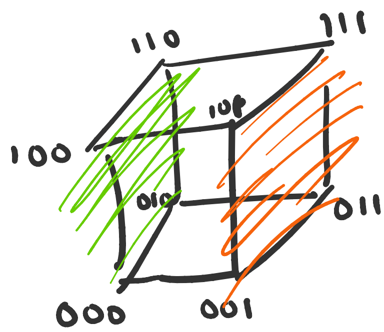
Covering the n-dimensional Hamming cube with hyperplanes.

1-dim Hamming Cube

2-dim " "



3-dim Hamming Cube



Hyperplane

$$H(\vec{a}, b) = \{ x \in \mathbb{R}^n : \langle a, x \rangle = b \},$$

where $a \in \mathbb{R}^n$, $b \in \mathbb{R}$, and $a \neq \vec{0}$.

Let $\vec{e}_1 = \overbrace{(0,0,0,0,\ldots,0,1)}^{n \text{ bits}}$

$\vec{e}_2 = (0,0,0\ldots,0,1,0)$

$\vdots$

$\vec{e}_n = (1,0,0,\ldots,0)$

$$\vec{e}_n = (1, 0, 0, \ldots, 0)$$

$H(\vec{e}_1, 0)$ and $H(\vec{e}_2, 0)$ cover all the vertices of an n-dim Hamming cube.

all $x = (x_n, x_{n-1}, \ldots, x_1) \in \mathbb{R}^n$ when $x_1 = 0$

$x = (x_n, x_{n-1}, \ldots x_2, x_1) \in \mathbb{R}^n$ where $x_2 = 0$.

What if we change the problem to:

"cover every point/vertex of the n-dim Hamming cube except the point $(0, 0, \ldots, 0)$."

Soln: Take n hyperplanes

$$H(\vec{e}_1, 1), \; H(\vec{e}_2, 1), \; \ldots \ldots, \; H(\vec{e}_n, 1)$$

$\in$ $(0, 0, \ldots, 0, 1)$ $(0, 0, \ldots, 1, 0)$ $(1, 0, 0, \ldots, 0)$

Theorem: Consider the 0-1 n-dim Hamming cube. In order to cover every point of the Hamming cube but not the $(0, 0, \ldots, 0)$ point, one needs at least n hyperplanes.

Proof: Suppose not. Let $H(a^1, b_1)$, $H(a^2, b_2)$, ..., $H(a^m, b_m)$ be m hyperplanes that do the job, where $m < n$, and $a^i \in \mathbb{R}^n$, $b_i \in \mathbb{R}$.

$a^2 \in \mathbb{R}^n$, $b_2 \in \mathbb{R}$, $a^m \in \mathbb{R}^n$, $b_m \in \mathbb{R}$

$\langle x, a^2 \rangle = b_2$

Multiplying every a^i with $\frac{1}{b_i}$, we can get the following hyperplanes which do the same job:

$H(a^1, 1)$, $H(a^2, 1)$, ..., $H(a^i, 1)$, $H(a^m, 1)$, where $m < n$.

$\underbrace{(0, 0, \overset{n-2}{\ldots}, 1, 0, 0, \ldots, 0)}_{x =}$

$\langle x, a^i \rangle = 1$

From this, we write the following polynomial:

$$f(x) = \prod_{i=1}^{m} \left(1 - \langle a^i, x \rangle\right) - \prod_{i=1}^{n} (1 - x_i)$$

$i = n-2$

$1 - x_{n-2} = 1 - 1 = 0$

Properties of $f(x)$

if $x = (0, 0, \ldots, 1, 0)$

① $f(x) = \begin{cases} 0, & \text{when } x \in \{0,1\}^n \end{cases}$

② $\deg(f) = n$, since $m < n$ and the monomial $x_1' x_2' x_3' \ldots x_{n-1}' x_n'$ has a

The monomial $x_1 x_2' x_3' \cdots x_{n-1}' x_n'$ has a non-zero coefficient.

coefficient is $(-1)^{n+1} \neq 0$.

Applying combinatorial nullstellensatz, we have $x \in \underbrace{\{0,1\}}_{S_1} \times \underbrace{\{0,1\}}_{S_2} \times \cdots \times \underbrace{\{0,1\}}_{S_n}$ such that $f(x) \neq 0$, which is a contradiction to Property II of $f(x)$. Thus, our assumption that $m < n$ is false.

Hence proved.

$\square$