CS1340: DISCRETE STRUCTURES II

PRACTICE QUESTIONS III– ANSWERS

(1) Prove that for a bipartite graph G on n vertices the number of edges in G is at most $\frac{n^2}{4}$.
Proof: In a bipartite graph the n vertices can be partitioned into two subsets of size i and $(n-i)$ $0 \le i \le n$ and the edges are from vertices of different subsets, so you have a maximum of $i(n-i)$ edges if every member of one subset is connected to every member of the other subset.
$f(i) = i(n-i)$, $0 \le i \le n$ is maximized by $i = n/2$ which leads to $n/2 \cdot n/2 = n^4/4$ - being the maximum number of edges.

(2) A perfect matching on a graph is a matching containing $n/2$ edges, i.e. perfect matchings are possible only with an even number of vertices.
In a perfect matching every vertex of the graph is incident to exactly one edge of the matching. So it is a matching of a graph containing $n/2$ edges which means you can have perfect matchings only on even number of vertices.

(3) Every regular bipartite graph has a perfect matching. Prove.
Let G be a regular bipartite graph with bipartition (A, B) and degree k. Let $X \subseteq A$ and let t be the number of edges with one end in X. Since every vertex in X has degree k, this means $k|X| = t$. Similarly, every vertex in $N(X)$ has degree k, so $t \le k|N(X)|$, the neighbourhood of X. Thus $|X|$ is of at most the cardinality of $N(X)$. By Halls Theorem, this implies there is a complete matching from A to B. Analogously we can conclude that there is a complete matching from B to A. This implies there is a perfect matching from A to B.

(4) Every subgraph of a bipartite graph is bipartite.
The nodes of the graph are from the bipartition say V_1 and V_2. The subgraph selects some of the vertices from V_1 and some from V_2 and some of the edges. There will be no edges between the vertices in the same set V_i since a subgraph doesn't add new edges.

(5) Every simple graph $G = (V, E)$ has a bipartite graph with at least $|E|/2$ edges.
Consider the graph G and two sets V_1 and V_2 where we will partition the vertices of G into V_1 and V_2 by looking at each vertex of G one by one. Use this criterion to make the choice: If the vertex has more edges going from V_1

to V_2 then assign it to V_2, otherwise assign it to V_1. If you assign a vertex v to V_i color each edge from v to V_i as red and every edge from v to V_{3-i} blue. Then there are at least as many blue edges as there are red edges. When the process is finished, all edges will be colored, those within V_1 or V_2 will be red and those between V_1 and V_2 will be blue. 2-colorable implies bipartite.

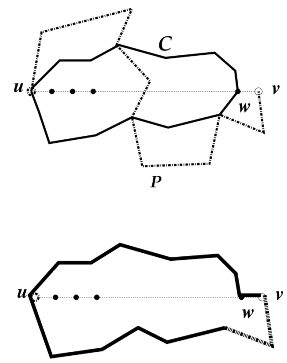
Proof by induction:
Let $P(n)$ be that every graph on n vertices has a bipartite subgraph with at least $|E(G)/2|$ edges. We need to show that $P(n)$ implies $P(n+1)$. For a single vertex it is trivial. So we assume for $P(n)$ and consider a graph G with $n+1$ vertices.

Pick a vertex v of G and let H be the subgraph obtained from G by deleting v and all edges of G incident at v. H has fewer vertices than G and therefore by induction hypothesis H has a bipartite subgraph B with at least $|E(H)|/2$ edges. If $d = deg(v)$, $|E(H)| = |E(G)| - d$. Since B is a bipartite subgraph we can assume V_1 and V_2 as the bipartition of B. We can assume that B keeps all the vertices of H (We just have to remove the edges.) Now consider $v \in G$. Let $d_i, i = 1, 2$ be the number of edges between v and V_i in G. Choose $i \in \{1, 2\}$ so that $d_i \geq d/2$. Depending on the choice of i, add v to V_i. This also helps decide which of the d edges that are incident at v should be kept in order to extend B to another bipartitite subgraph of G with at least $|E(G)|/2$ edges.

(6) When $G = (V, E)$ is a noncomplete connected graph with at least 3 vertices then vertex connectivity $\kappa(G) \leq min_{v \in V} deg(v)$ and edge connectivity $\lambda(G) \leq min_{v \in V} deg(v)$.
Let d be the minimum degree of a graph G. Then there is some vertex v with d neighbours. Provided that there are at least $d+2$ vertices in G, the removal of the d neighbours of v will disconnect v from the remainder of the graph. This will make G disconnected. Therefore there exists a vertex cut of size d , $\chi(G) \leq d$. If there are not at least $d+2$ vertices in G then there must be exactly $d+1$ vertices as otherwise the minimum degree of G cannot be d. Also we have $1 \leq \chi(G) \leq |G| - 2$ and $|G| = d + 1$, $\chi(G) \leq d - 1 \lneq d$ so $\chi(G) \leq d$.
Edge connectivity : Consider a vertex v of min degree, and denote this degree as d. By removing the d edges that are adjacent to v, we disconnect the graph.

(7) The existence of a simple circuit of a particular length is a graph invariant.
Suppose $G = \langle V_G, E_G \rangle$ and $H = \langle V_H, E_H \rangle$ are isomorphic graphs and suppose that G has a simple circuit of length m. Since G and H are isomorphic there is a bijection $h : V_G \to V_H$ s.t for each $u, v \in V_G$, $\{u, v\} \in E_G$ iff $\{h(u), h(v)\} \in E_H$. Let $\{\mathbf{v}_1, v_2, \ldots, v_m\}$ be the vertices of a simple circuit of size m in G s.t $\{v_k, v_{k+1}\} \in E_G$ for $k = 1, \ldots, m-1$ and $\{v_m, v_1\} \in E_G$. Then $\{h(v_k), h(v_{k+1})\} \in E_H$, for $k = 1, \ldots, m-1$, and $\{h(v_m), h(v_1)\} \in E_H$ and $h(v_1), \ldots, h(v_m)$ are distinct so they are the vertices of a simple m-cycle in H.

FIGURE 0.1. 2-connectivity implies cycle



(8) Show that K_n has a Hamilton circuit whenever $n \geq 3$.
Follows from Dirac's theorem that says if G is a simple graph with n vertices $n \geq 3$ s.t. the degree of every vertex is at least $n/2$ then G has a Hamilton circuit.

(9) If G is a connected planar simple graph then G has a vertex of degree not exceeding 5.
If G has one or two vertices the result is true. If G has at least three vertices then $e \leq 3v - 6$, so $2e \leq 6v - 12$ (Result stated in class). If the degree of every vertex were at least 6 then by handshaking theorem, $2e = \sum_{v \in V} deg(v)$, that is $2e \geq 6v$. But this contradicts the inequality $2e \leq 6v - 12$. It follows that there must be a vertex with degree no greater than 5.

(10) A graph with at least 3 vertices is 2-connected iff every pair of vertices lie in a cycle.
A connected graph is called 2-connected if for every vertex $x \in V(G)$, $G - x$ is connected.

Sufficient condition: If every two vertices belong to a cycle, no removal of one vertex can disconnect the graph.

Necessary condition that needs to proved : If G is 2-connected every two vertices belong to a cycle.

We will prove it by induction on the distance $dist(u, v)$ between two vertices in the graph.

Base case: Since the vertices are distinct, the smallest distance is 1. This means u and v are adjacent. Let z be any vertex in G other than u and v. Because of the removal of u (or v) does not disconnect G. There is a path P_1 (or P_2) that connects u (or v) with z and that does not contain v (or u).

The cycle containing u and v consists of the edge (u, v) and a path from u to v obtained from the walk from v to z using P_2 and the reverse of P_1 from z to u.

Inductive step: Let the proposition be true for all pairs of vertices on the distance $\leq k$ and let $dist(u,v) = k+1$. Consider the shortest path from u to v and let w be the vertex on the path which is adjacent to v. Since $dist(u,w) = k$ there is a cycle C containing u and w. Since the removal of w does not disconnect u from v there is a path P that connects u and v that does not contain w. A cycle containing u and v can be constructed from C and P and edge between w and v. Look at Figure 0.1 for details.

(11) If G_1 and G_2 are two connected subgraphs of G having at least one vertex in common then $G_1 \cup G_2$ is connected.
Proof: Let $v \in V(G_1) \cap V(G_2)$. Let $a \in V(G_1)$ and $b \in V(G_2)$ but $a, b \notin V(G_1) \cap V(G_2)$ Then there is a path a to v P_1 in G_1. Let $P_1 : a = x_0, x_1, \cdots, x_k = v$. Let i be the smallest such that $x_i \in G_2$. $i \geq 1$. Let Q be the path from x_i to b in G_2. Then $x_0, x_1, x_{i-1}Q$ is a path from a to b in $G_1 \cup G_2$ as no x_j can occur in Q for $j < i$.

(12) The **complementary graph** $\hat{G}$ of a simple graph G has the same vertices as G, two vertices are adjacent in $\hat{G}$ if and only if they are not adjacent in G. If a graph G is not connected, prove that its complement graph is connected.

Let $G_1, \cdots, G_k$ be the connected components of G. Let $\hat{G}$ be the complement graph of G. As there is no edge in G between a vertex in G_i and a vertex in G_j, there is an edge between any vertexin G_i and any vertex in G_j.

Lets consider an edge in G, such as the edge $\{v, w\}$. They are in the same component of G. Since G is disconnected, we can find a vertex u in a different component such that neither uw nor uv are edges of G. Then vuw is a path from v to w in $\hat{G}$. Thus, $\hat{G}$ is connected.

(13) Show that the property that a graph is bipartite is an isomorphic invariant.
If G and H are isomorphic and G is a bipartite graph, we show that H is also a bipartite graph.

Since G is bipartite graph, there is a bipartition (V_1, V_2). Let f be the isomorphism between G and H. Then let $W_1 = f(V_1)$ and $W_2 = f(V_2)$. As f is a bijective function, W_1 and W_2 are disjoint since V_1 and V_2 are. Also the union of W_1 and W_2 gives the vertex set of H.

We only need to verify that every edge in H has an endpoint in W_1 and the other one in W_2. As G and H are isomorphic then for every distinct vertices a and b in G, they are adjacent iff $f(a)$ and $f(b)$ are adjacent. Therefore, for any edge $e = \{a, b\}$ in G we can find a corresponding one $e^{'} = \{f(a), f(b)\}$ in H. As G is bipartite one of the vertices is in V_1 and the other one is in V_2 meaning one of $f(a)$ or $f(b)$ is in W_1 and the other is in W_2. Therefore, H is bipartite.

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