PH 1027 (Maxwell Equations and EM waves)
Final Exam

IITH/Spring 2018/Shubho Roy

(Duration: 3 hours, Total Marks: 80)

May 8, 2018

1. **Maxwell Equations:** State the Maxwell Equations for electric and magnetic fields in vacuum (with sources) and in media (with free charges), the boundary conditions at the interface of two media as well as any other condition(s) or relations between the fields needed to make the system soluble. (10 points)

SOLUTION:
Maxwell Equations in vacuum (2 points),

$$\boldsymbol{\nabla}\cdot\mathbf{E}=\frac{\rho}{\epsilon_0},\qquad \boldsymbol{\nabla}\cdot\mathbf{B}=0,$$

$$\boldsymbol{\nabla}\times\mathbf{E}=-\frac{\partial\mathbf{B}}{\partial t},\qquad \boldsymbol{\nabla}\times\mathbf{B}=\mu_0\mathbf{j}+\frac{1}{c^2}\frac{\partial\mathbf{E}}{\partial t},$$

where $\rho,\mathbf{j}$ are the source charge density and current density respectively.
Maxwell Equations in media (2 points).

$$\boldsymbol{\nabla}\cdot\mathbf{D}=\rho_f,\qquad \boldsymbol{\nabla}\cdot\mathbf{B}=0,$$

$$\boldsymbol{\nabla}\times\mathbf{E}=-\frac{\partial\mathbf{B}}{\partial t},\qquad \boldsymbol{\nabla}\times\mathbf{H}=\mathbf{j}_f+\frac{\partial\mathbf{D}}{\partial t},$$

where $\rho_f,\mathbf{j}_f$ are free charge density and free current density respectively. In addition, we need to specify constitutive relations, $\mathbf{D}=\mathbf{D}(\mathbf{E})$ and $\mathbf{H}=\mathbf{H}(\mathbf{B})$ to bring down the number of independent variables equal to the number of equations (both equal to six). In particular, for **linear isotropic media** the constitutive relations are, (2 points)

$$\begin{aligned}\mathbf{D}(\mathbf{E}) &= \epsilon\,\mathbf{E},\\ \mathbf{H}(\mathbf{B}) &= \frac{1}{\mu}\mathbf{B}.\end{aligned}$$

The boundary conditions to be satisfied at the interface of two media (denoted by the subscripts 1 and 2) are (2 points),

$$D_1^{\perp}-D_2^{\perp}=\sigma_f,\qquad B_1^{\perp}=B_2^{\perp},$$

$$\mathbf{E}_1^{||}=\mathbf{E}_2^{||},\qquad \mathbf{H}_1^{||}-\mathbf{H}_2^{||}=\mathbf{K}_f\times\hat{\mathbf{n}},$$

where $\hat{\mathbf{n}}$ is the unit normal at the interface (pointing towards media 1) and $\sigma_f,\mathbf{K}_f$ are surface density of free charge and free current respectively at the interface. If both media are **linear and isotropic**, with permittivity and permeability values, (ϵ_1,μ_1) and (ϵ_2,μ_2), then the boundary conditions are (2 points),

$$\epsilon_1E_1^{\perp}-\epsilon_2E_2^{\perp}=\sigma_f,\qquad B_1^{\perp}=B_2^{\perp},$$

$$\mathbf{E}_1^{\parallel} = \mathbf{E}_2^{\parallel}, \qquad \frac{1}{\mu_2}\mathbf{B}_1^{\parallel} - \frac{1}{\mu_2}\mathbf{B}_2^{\parallel} = \mathbf{K}_f \times \hat{\mathbf{n}}.$$

2. Prove the identities:

$$\text{(A) } \boldsymbol{\nabla} \times (\mathbf{A} \times \mathbf{B}) = \mathbf{A}\,(\boldsymbol{\nabla} \cdot \mathbf{B}) - \mathbf{B}\,(\boldsymbol{\nabla} \cdot \mathbf{A}) + (\mathbf{B} \cdot \boldsymbol{\nabla})\,\mathbf{A} - (\mathbf{A} \cdot \boldsymbol{\nabla})\,\mathbf{B}$$

$$\text{(B) } \boldsymbol{\nabla}\,(\mathbf{A} \cdot \mathbf{B}) = (\mathbf{A} \cdot \boldsymbol{\nabla})\,\mathbf{B} + (\mathbf{B} \cdot \boldsymbol{\nabla})\,\mathbf{A} + \mathbf{A} \times (\boldsymbol{\nabla} \times \mathbf{B}) + \mathbf{B} \times (\boldsymbol{\nabla} \times \mathbf{A})$$

(5 + 5 = 10 points)

(Hint: Compare the i-th component of both sides.)

SOLUTION:
(A) The i-th component of the LHS,

$$\begin{aligned}
[\boldsymbol{\nabla} \times (\mathbf{A} \times \mathbf{B})]_i &= \epsilon_{ijk}\partial_j\,(\mathbf{A} \times \mathbf{B})_k \\
&= \epsilon_{ijk}\partial_j\,(\epsilon_{klm}A_l B_m) \\
&= \epsilon_{ijk}\epsilon_{klm}\partial_j\,(A_l B_m) \\
&= (\delta_{il}\delta_{jm} - \delta_{im}\delta_{jl})\,\partial_j\,(A_l B_m) \\
&= \partial_j\,(A_i B_j) - \partial_j\,(A_j B_i) \\
&= A_i\partial_j B_j + B_j\partial_j A_i - B_i\partial_j A_j - A_j\partial_j B_i \\
&= A_i\,(\boldsymbol{\nabla} \cdot \mathbf{B}) + (\mathbf{B} \cdot \boldsymbol{\nabla})\,A_i - B_i\,(\boldsymbol{\nabla} \cdot \mathbf{A}) - (\mathbf{A} \cdot \boldsymbol{\nabla})\,B_i
\end{aligned}$$

which is nothing but the i-th component of the RHS.

(B) The i-th component of the RHS

$$\begin{aligned}
&(\mathbf{A} \cdot \boldsymbol{\nabla})\,B_i + (\mathbf{B} \cdot \boldsymbol{\nabla})\,A_i + [\mathbf{A} \times (\boldsymbol{\nabla} \times \mathbf{B})]_i + [\mathbf{B} \times (\boldsymbol{\nabla} \times \mathbf{A})]_i \\
=&\,(A_j\partial_j)\,B_i + (B_j\partial_j)\,A_i + \epsilon_{ijk}A_j\,(\boldsymbol{\nabla} \times \mathbf{B})_k + \epsilon_{ijk}B_j\,(\boldsymbol{\nabla} \times \mathbf{A})_k \\
=&A_j\partial_j B_i + B_j\partial_j A_i + \epsilon_{ijk}\epsilon_{klm}A_j\partial_l B_m + \epsilon_{ijk}\epsilon_{klm}B_j\partial_l A_m \\
=&A_j\partial_j B_i + B_j\partial_j A_i + \epsilon_{ijk}\epsilon_{klm}\,(A_j\partial_l B_m + B_j\partial_l A_m) \\
=&A_j\partial_j B_i + B_j\partial_j A_i + (\delta_{il}\delta_{jm} - \delta_{im}\delta_{jl})\,(A_j\partial_l B_m + B_j\partial_l A_m) \\
=&A_j\partial_j B_i + B_j\partial_j A_i + (A_j\partial_i B_j + B_j\partial_i A_j) - (A_j\partial_j B_i + B_j\partial_j A_i) \\
=&A_j\partial_i B_j + B_j\partial_i A_j \\
=&\partial_i\,(A_j B_j) \\
=&\partial_i\,(\mathbf{A} \cdot \mathbf{B})
\end{aligned}$$

which is the i-th component of the LHS.

3. **Point charge:** As discussed in the class a point charge q moving along a trajectory $\mathbf{x}'(t)$ is described by the charge and current density,

$$\begin{aligned}
\rho(\mathbf{x}, t) &= q\,\delta^3(\mathbf{x} - \mathbf{x}'(t)), \\
\mathbf{j}(\mathbf{x}, t) &= \rho\,\mathbf{v} = q\,\mathbf{v}\,\delta^3(\mathbf{x} - \mathbf{x}'(t)),
\end{aligned}$$

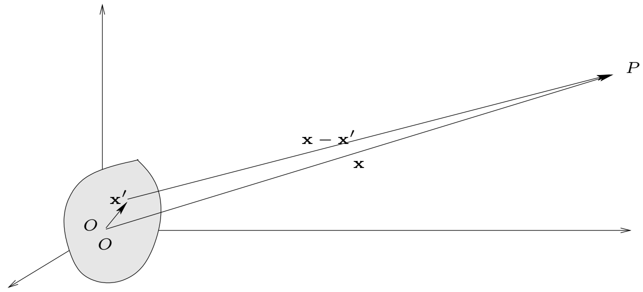

Figure 1: Multipole Expansion of Scalar Potential.

where the $\mathbf{v}(t) = \frac{d\mathbf{x}'(t)}{dt}$ is the instantaneous velocity of the point charge at time t. (For a point charge density is zero everywhere and infinite at the location of the point charge, hence the appearance of the Dirac delta function). Show that these expressions satisfy the continuity equation (local conservation of charge),

$$\frac{\partial \rho}{\partial t} + \boldsymbol{\nabla} \cdot \mathbf{j} = 0.$$

(5 points)

(Hint: Use chain rule to take the time derivative).

SOLUTION:

$$\begin{aligned}
\frac{\partial \rho}{\partial t} &= q\,\frac{\partial}{\partial t}\delta^3\left(\mathbf{x} - \mathbf{x}'(t)\right) \\
&= q\,\frac{\partial x'_j}{\partial t}\frac{\partial}{x'_j}\delta^3\left(\mathbf{x} - \mathbf{x}'\right) \\
&= q\,v_j\frac{\partial}{x'_j(t)}\delta^3\left(\mathbf{x} - \mathbf{x}'\right) \\
&= -q\,v_j\frac{\partial}{x_j}\delta^3\left(\mathbf{x} - \mathbf{x}'\right) \\
&= -q\mathbf{v} \cdot \boldsymbol{\nabla}\delta^3\left(\mathbf{x} - \mathbf{x}'\right) \\
&= -\boldsymbol{\nabla} \cdot \left(q\,\mathbf{v}\,\delta^3\left(\mathbf{x} - \mathbf{x}'\right)\right) \\
&= -\boldsymbol{\nabla} \cdot \mathbf{j}
\end{aligned}$$

$$\Longrightarrow \frac{\partial \rho}{\partial t} + \boldsymbol{\nabla} \cdot \mathbf{j} = 0.$$

4. **Multipole Expansion of scalar potential:** Consider the situation when the observer is a located at $\mathbf{x}$, far from an extended charge distribution described by a charge density function $\rho(\mathbf{x}')$. In this far limit, $\mathbf{x} - \mathbf{x}' \approx \mathbf{x}$ for all points inside the extended source such as $\mathbf{x}'$. Show that in the far limit, the electrostatic scalar potential can be approximated as,

$$\Phi(\mathbf{x}) = \frac{1}{4\pi\varepsilon_0}\left(\frac{Q}{|\mathbf{x}|} + \frac{\mathbf{x}\cdot\mathbf{P}}{|\mathbf{x}|^3} + \frac{1}{2}\frac{\mathbf{x}\overleftrightarrow{\mathbf{Q}}\mathbf{x}}{|\mathbf{x}|^5} + \dots\right)$$

where $Q = \int d^3\mathbf{x}'\,\rho(\mathbf{x}')$ is the total charge of the distribution, $P_i = \int d^3\mathbf{x}'\,x'_i\,\rho(\mathbf{x}')$ is the total (net) dipole moment of the distribution, and

$$Q_{ij} = \int d^3\mathbf{x}'\ \left(3x_i'x_j' - \delta_{ij}\mathbf{x}'\cdot\mathbf{x}'\right)\ \rho(\mathbf{x}')$$

is the total quadrupole moment of the charge distribution.

(10 points)

Hint: Start from the expression of the electrostatic scalar potential created by the extended source at $\mathbf{x}$, namely,

$$\Phi(\mathbf{x}) = \frac{1}{4\pi\varepsilon_0}\int d^3\mathbf{x}'\frac{\rho(\mathbf{x}')}{|\mathbf{x}-\mathbf{x}'|}$$

and then Taylor expand $\frac{1}{|\mathbf{x}-\mathbf{x}'|} \equiv [(\mathbf{x}-\mathbf{x}')\cdot(\mathbf{x}-\mathbf{x}')]^{-1/2}$ around $\mathbf{x}'=0$ which holds good in the far limit up to the first three terms.

SOLUTION:
The Taylor expansion is,

$$\begin{aligned}\frac{1}{|\mathbf{x}-\mathbf{x}'|} &= \left[\left(\mathbf{x}-\mathbf{x}'\right)\cdot\left(\mathbf{x}-\mathbf{x}'\right)\right]^{-\frac{1}{2}} \\ &= \left[\left(\mathbf{x}-\mathbf{x}'\right)\cdot\left(\mathbf{x}-\mathbf{x}'\right)\right]^{-\frac{1}{2}}\Big|_{\mathbf{x}'=0} + \left(\partial_i'\left[\left(\mathbf{x}-\mathbf{x}'\right)\cdot\left(\mathbf{x}-\mathbf{x}'\right)\right]^{-\frac{1}{2}}\Big|_{\mathbf{x}'=0}\right)x_i' \\ &\qquad + \frac{1}{2}\left(\partial_j'\,\partial_i'\left[\left(\mathbf{x}-\mathbf{x}'\right)\cdot\left(\mathbf{x}-\mathbf{x}'\right)\right]^{-\frac{1}{2}}\Big|_{\mathbf{x}'=0}\right)x_i'\,x_j' + \ldots\end{aligned}$$

The first term is ,

$$\left[\left(\mathbf{x}-\mathbf{x}'\right)\cdot\left(\mathbf{x}-\mathbf{x}'\right)\right]^{-\frac{1}{2}}\Big|_{\mathbf{x}'=0} = \frac{1}{|\mathbf{x}|}.$$

The second term is,

$$\begin{aligned}\left(\partial_i'\left[\left(\mathbf{x}-\mathbf{x}'\right)\cdot\left(\mathbf{x}-\mathbf{x}'\right)\right]^{-\frac{1}{2}}\Big|_{\mathbf{x}'=0}\right)x_i' &= \left[-\frac{1}{2}\left[\left(\mathbf{x}-\mathbf{x}'\right)\cdot\left(\mathbf{x}-\mathbf{x}'\right)\right]^{-\frac{3}{2}}2\left(x_i'-x_i\right)\right]_{\mathbf{x}'=0}x_i' \\ &= \frac{x_ix_i'}{|\mathbf{x}|^3}.\end{aligned}$$

The third term is,

$$\begin{aligned}\left(\partial_j'\,\partial_i'\left[\left(\mathbf{x}-\mathbf{x}'\right)\cdot\left(\mathbf{x}-\mathbf{x}'\right)\right]^{-\frac{1}{2}}\Big|_{\mathbf{x}'=0}\right)x_i'\,x_j' &= \left[\partial_j'\left(-\frac{1}{2}\left[\left(\mathbf{x}-\mathbf{x}'\right)\cdot\left(\mathbf{x}-\mathbf{x}'\right)\right]^{-\frac{3}{2}}2\left(x_i'-x_i\right)\right)\right]_{\mathbf{x}'=0}x_i'\,x_j' \\ &= \left[-\delta_{ij}\left[\left(\mathbf{x}-\mathbf{x}'\right)\cdot\left(\mathbf{x}-\mathbf{x}'\right)\right]^{-\frac{3}{2}} + \frac{1}{2}.\frac{3}{2}\left[\left(\mathbf{x}-\mathbf{x}'\right)\cdot\left(\mathbf{x}-\mathbf{x}'\right)\right]^{-\frac{5}{2}}2\left(x_j'-\right.\right. \\ &= \left(-\frac{\delta_{ij}}{|\mathbf{x}|^3} + 3\frac{x_ix_j}{|\mathbf{x}|^5}\right)x_i'\,x_j'.\end{aligned}$$

So the Taylor expansion is,

$$\begin{aligned}\frac{1}{|\mathbf{x}-\mathbf{x}'|} &= \frac{1}{|\mathbf{x}|}+\frac{x_i x_i'}{|\mathbf{x}|^3}+\frac{1}{2}\left(3\frac{x_i x_j}{|\mathbf{x}|^5}-\frac{\delta_{ij}}{|\mathbf{x}|^3}\right)x_i'\,x_j'+\dots\\ &= \frac{1}{|\mathbf{x}|}+\frac{x_i x_i'}{|\mathbf{x}|^3}+\frac{1}{2}\left(3\frac{x_i x_j x_i'\,x_j'}{|\mathbf{x}|^5}-\frac{\delta_{ij}x_i'\,x_j'}{|\mathbf{x}|^3}\right)+\dots\\ &= \frac{1}{|\mathbf{x}|}+\frac{x_i x_i'}{|\mathbf{x}|^3}+\frac{1}{2}\left(3\frac{x_i x_j x_i'\,x_j'}{|\mathbf{x}|^5}-\frac{(\mathbf{x}'\cdot\mathbf{x}')\,(\mathbf{x}\cdot\mathbf{x})}{|\mathbf{x}|^5}\right)+\dots\\ &= \frac{1}{|\mathbf{x}|}+\frac{x_i x_i'}{|\mathbf{x}|^3}+\frac{1}{2}\frac{x_i x_j}{|\mathbf{x}|^5}\left(3x_i'\,x_j'-\delta_{ij}\mathbf{x}'\cdot\mathbf{x}'\right)+\dots\end{aligned}$$

Plugging this expansion in the expression for the potential is,

$$\begin{aligned}\Phi(\mathbf{x}) &= \frac{1}{4\pi\varepsilon_0}\int d^3\mathbf{x}'\,\frac{\rho(\mathbf{x}')}{|\mathbf{x}-\mathbf{x}'|}\\ &= \frac{1}{4\pi\varepsilon_0}\int d^3\mathbf{x}'\rho(\mathbf{x}')\left[\frac{1}{|\mathbf{x}|}+\frac{x_i x_i'}{|\mathbf{x}|^3}+\frac{1}{2}\frac{x_i x_j}{|\mathbf{x}|^5}\left(3x_i'\,x_j'-\delta_{ij}\mathbf{x}'\cdot\mathbf{x}'\right)+\dots\right]\\ &= \frac{1}{4\pi\varepsilon_0}\left(\frac{\underbrace{\int d^3\mathbf{x}'\rho(\mathbf{x}')}_{=Q}}{|\mathbf{x}|}+\frac{x_i\underbrace{\int d^3\mathbf{x}'\rho(\mathbf{x}')\,x_i'}_{P_i}}{|\mathbf{x}|^3}+\frac{1}{2}\frac{x_i x_j}{|\mathbf{x}|^5}\underbrace{\left[\int d^3\mathbf{x}'\rho(\mathbf{x}')\,\left(3x_i'\,x_j'-\delta_{ij}\mathbf{x}'\cdot\mathbf{x}'\right)\right]}_{Q_{ij}}+\dots\right)\\ &= \frac{1}{4\pi\varepsilon_0}\left(\frac{Q}{|\mathbf{x}|}+\frac{\mathbf{x}\cdot\mathbf{P}}{|\mathbf{x}|^3}+\frac{1}{2}\frac{\mathbf{x}\overleftrightarrow{\mathbf{Q}}\mathbf{x}}{|\mathbf{x}|^5}+\dots\right).\end{aligned}$$

5. Show that for a uniform magnetic field, $\mathbf{B}$ the magnetic vector potential everywhere can be given by,

$$\mathbf{A}(\mathbf{x})=\frac{1}{2}\mathbf{B}\times\mathbf{x}.$$

(5 points)

SOLUTION:
To prove that,

$$\mathbf{A}(\mathbf{x})=\frac{1}{2}\mathbf{B}\times\mathbf{x}$$

can be a vector potential for a constant/uniform magnetic field, we take curl of the rhs and consider the k-th component of that curl,

$$
\begin{aligned}
\left[\boldsymbol{\nabla}\times\left(\frac{1}{2}\mathbf{B}\times\mathbf{x}\right)\right]^k &= \epsilon^{ijk}\nabla^i\left(\frac{1}{2}\mathbf{B}\times\mathbf{x}\right)^j \\
&= \epsilon^{ijk}\frac{\partial}{\partial x^i}\left(\frac{1}{2}\mathbf{B}\times\mathbf{x}\right)^j \\
&= \epsilon^{ijk}\frac{\partial}{\partial x^i}\left(\frac{1}{2}\epsilon^{lmj}B^l x^m\right) \\
&= \frac{1}{2}\epsilon^{lmj}\epsilon^{ijk}B^l\frac{\partial x^m}{\partial x^i} \\
&= \frac{1}{2}\epsilon^{lmj}\epsilon^{kij}B^l\delta^{mi} \\
&= \frac{1}{2}\epsilon^{lij}\epsilon^{kij}B^l \\
&= \frac{1}{2}\left(\delta^{lk}\underbrace{\delta^{ii}}_{=3}-\delta^{li}\delta^{ik}\right)B^l \\
&= \frac{1}{2}\left(3\delta^{lk}-\delta^{lk}\right)B^l \\
&= B^k.
\end{aligned}
$$

6. **Energy density and Poynting vector for media:** Starting from the expression for work done (per unit volume) on free charges by the electric and magnetic fields in a dielectric medium, show that the energy density stored in EM fields in linear dielectric medium is

$$u_{EM} = \frac{1}{2}\left(\mathbf{E}\cdot\mathbf{D}+\mathbf{B}\cdot\mathbf{H}\right)$$

and the Poynting vector (energy current density) in a general dielectric is,

$$\mathbf{S} = \mathbf{E}\times\mathbf{H}.$$

(10 points)

SOLUTION:
The force on a free charge element, $dq_{free} = \rho_{free}(\mathbf{x})\,d^3\mathbf{x}$ moving with velocity $\mathbf{v}$ due to electric and magnetic fields is,

$$\mathbf{F} = dq_{free}\,\mathbf{E}(\mathbf{x}) + dq_{free}\,\mathbf{v}\times\mathbf{B}(\mathbf{x})$$

and the work done is,

$$
\begin{aligned}
dW = \mathbf{F}\cdot d\mathbf{x} = \mathbf{F}\cdot\mathbf{v}dt &= dq_{free}\,\mathbf{E}(\mathbf{x})\,\mathbf{v}dt \\
&= d^3\mathbf{x}\,\rho_{free}(\mathbf{x})\,\mathbf{v}\cdot\mathbf{E}(\mathbf{x})\,dt \\
&= d^3\mathbf{x}\,\mathbf{j}_{free}(\mathbf{x})\cdot\mathbf{E}(\mathbf{x})\,dt
\end{aligned}
$$

Using Ampere-Maxwell equation, $\mathbf{j}_{free} = \boldsymbol{\nabla}\times\mathbf{H} - \frac{\partial\mathbf{D}}{\partial t}$, we get,

$$
\begin{aligned}
dW &= d^3\mathbf{x}\left(\boldsymbol{\nabla}\times\mathbf{H}-\frac{\partial\mathbf{D}}{\partial t}\right)\cdot\mathbf{E}\,dt \\
&= d^3\mathbf{x}\,\left(\boldsymbol{\nabla}\times\mathbf{H}\right)\cdot\mathbf{E} - \frac{\partial\mathbf{D}}{\partial t}\cdot\mathbf{E}\,dt
\end{aligned}
$$

Next we use the identity,

$$\boldsymbol{\nabla}\cdot(\mathbf{E}\times\mathbf{H}) = (\boldsymbol{\nabla}\times\mathbf{E})\cdot\mathbf{H} - \mathbf{E}\cdot(\boldsymbol{\nabla}\times\mathbf{H})$$

to get,

$$dW = d^3\mathbf{x}\,\left[(\boldsymbol{\nabla}\times\mathbf{E})\cdot\mathbf{H} - \boldsymbol{\nabla}\cdot(\mathbf{E}\times\mathbf{H}) - \frac{\partial\mathbf{D}}{\partial t}\cdot\mathbf{E}\right]\,dt$$

Using Faraday's law, $\boldsymbol{\nabla}\times\mathbf{E} = -\frac{\partial\mathbf{B}}{\partial t}$ we get,

$$dW = d^3\mathbf{x}\,\left[-\boldsymbol{\nabla}\cdot(\mathbf{E}\times\mathbf{H}) - \left(\frac{\partial\mathbf{B}}{\partial t}\cdot\mathbf{H} + \frac{\partial\mathbf{D}}{\partial t}\cdot\mathbf{E}\right)\right]\,dt$$

By work energy theorem, the work done on free charges is same as their increase in kinetic energy, dT, so we have, the density of increase of kinetic energy,

$$du_{kin} = \frac{dT}{d^3\mathbf{x}} = \frac{dW}{d^3\mathbf{x}} = \left[-\boldsymbol{\nabla}\cdot(\mathbf{E}\times\mathbf{H}) - \left(\frac{\partial\mathbf{B}}{\partial t}\cdot\mathbf{H} + \frac{\partial\mathbf{D}}{\partial t}\cdot\mathbf{E}\right)\right]\,dt$$

or, the rate of change of kinetic energy density,

$$\frac{\partial u_{kin}}{\partial t} = -\boldsymbol{\nabla}\cdot(\mathbf{E}\times\mathbf{H}) - \left(\frac{\partial\mathbf{B}}{\partial t}\cdot\mathbf{H} + \frac{\partial\mathbf{D}}{\partial t}\cdot\mathbf{E}\right) \tag{1}$$

For linear media,

$$\mathbf{B} = \mu\mathbf{H},\ \mathbf{D} = \varepsilon\mathbf{E},$$

and hence,

$$\frac{\partial\mathbf{B}}{\partial t}\cdot\mathbf{H} = \mu\frac{\partial\mathbf{H}}{\partial t}\cdot\mathbf{H} = \frac{\partial\mathbf{H}}{\partial t}\cdot\mathbf{B} = \frac{1}{2}\left(\frac{\partial\mathbf{B}}{\partial t}\cdot\mathbf{H} + \frac{\partial\mathbf{H}}{\partial t}\cdot\mathbf{B}\right) = \frac{\partial}{\partial t}\left(\frac{1}{2}\mathbf{B}\cdot\mathbf{H}\right)$$

and similarly,

$$\frac{\partial\mathbf{D}}{\partial t}\cdot\mathbf{E} = \frac{\partial}{\partial t}\left(\frac{1}{2}\mathbf{E}\cdot\mathbf{D}\right).$$

Plugging these in (1), we get,

$$\frac{\partial u_{kin}}{\partial t} = -\boldsymbol{\nabla}\cdot(\mathbf{E}\times\mathbf{H}) - \frac{\partial}{\partial t}\left(\frac{1}{2}\mathbf{E}\cdot\mathbf{D} + \frac{1}{2}\mathbf{B}\cdot\mathbf{H}\right),$$

which can be arranged in the form,

$$\frac{\partial u}{\partial t} + \boldsymbol{\nabla}\cdot\mathbf{S} = 0$$

where, $u = u_{kin} + u_{EM}$ and u_{EM} is the energy density contained in EM waves in a linear media,

$$u_{EM} = \frac{1}{2}\left(\mathbf{E}\cdot\mathbf{D} + \mathbf{B}\cdot\mathbf{H}\right)$$

and,

$$\mathbf{S} = \mathbf{E}\times\mathbf{H}$$

is the Poynting's vector.

7. **Momentum density and Maxwell stress tensor for media:** Starting from the expression for the total electric and magnetic force acting on free charges per unit volume inside a (dielectric paradigmatic) medium, show that the (linear) momentum contained in the EM fields is given by,

$$\boldsymbol{\pi}_{EM} = \mathbf{D} \times \mathbf{B}$$

and the stress tensor for a linear media is,

$$T_{ij} = \frac{1}{2}\delta_{ij}\left(\mathbf{B}\cdot\mathbf{H} + \mathbf{D}\cdot\mathbf{E}\right) - E_i D_j - H_i B_j.$$

(3 + 7 = 10 points)

Hint: Use identity in problem 2(**B**) to simplify the Maxwell Stress tensor piece.

SOLUTION: The force on free charges and free currents is,

$$\begin{aligned}
\mathbf{F} &= \int d^3\mathbf{x}\left[\rho_{free}(\mathbf{x})\,\mathbf{E}(\mathbf{x}) + \rho_{free}(\mathbf{x})\,\mathbf{v}\times\mathbf{B}(\mathbf{x})\right] \\
&= \int d^3\mathbf{x}\left[\rho_{free}(\mathbf{x})\,\mathbf{E}(\mathbf{x}) + \mathbf{j}_{free}(\mathbf{x})\times\mathbf{B}(\mathbf{x})\right]
\end{aligned}$$

So the force per unit volume,

$$\mathbf{f} = \rho_{free}\,\mathbf{E} + \mathbf{j}_{free}\times\mathbf{B}$$

Using Maxwell equations to replace,

$$\rho_{free} = \boldsymbol{\nabla}\cdot\mathbf{D}, \quad \mathbf{j}_{free} = \boldsymbol{\nabla}\times\mathbf{H} - \frac{\partial\mathbf{D}}{\partial t}$$

we get,

$$\begin{aligned}
\mathbf{f} &= \rho_{free}\,\mathbf{E} + \mathbf{j}_{free}\times\mathbf{B} \\
&= \mathbf{E}\left(\boldsymbol{\nabla}\cdot\mathbf{D}\right) + \left(\boldsymbol{\nabla}\times\mathbf{H} - \frac{\partial\mathbf{D}}{\partial t}\right)\times\mathbf{B} \\
&= \mathbf{E}\left(\boldsymbol{\nabla}\cdot\mathbf{D}\right) + \left(\boldsymbol{\nabla}\times\mathbf{H}\right)\times\mathbf{B} - \frac{\partial\mathbf{D}}{\partial t}\times\mathbf{B} \\
&= \mathbf{E}\left(\boldsymbol{\nabla}\cdot\mathbf{D}\right) + \left(\boldsymbol{\nabla}\times\mathbf{H}\right)\times\mathbf{B} - \frac{\partial\left(\mathbf{D}\times\mathbf{B}\right)}{\partial t} + \mathbf{D}\times\frac{\partial\mathbf{B}}{\partial t}
\end{aligned}$$

Using Faraday-Lenz law, i.e $\frac{\partial\mathbf{B}}{\partial t} = -\boldsymbol{\nabla}\times\mathbf{E}$, we get,

$$\begin{aligned}
\mathbf{f} &= \mathbf{E}\left(\boldsymbol{\nabla}\cdot\mathbf{D}\right) - \mathbf{B}\times\left(\boldsymbol{\nabla}\times\mathbf{H}\right) - \mathbf{D}\times\left(\boldsymbol{\nabla}\times\mathbf{E}\right) - \frac{\partial\left(\mathbf{D}\times\mathbf{B}\right)}{\partial t} \\
&= \mathbf{E}\left(\boldsymbol{\nabla}\cdot\mathbf{D}\right) + \mathbf{H}\left(\boldsymbol{\nabla}\cdot\mathbf{B}\right) - \mathbf{B}\times\left(\boldsymbol{\nabla}\times\mathbf{H}\right) - \mathbf{D}\times\left(\boldsymbol{\nabla}\times\mathbf{E}\right) - \frac{\partial\left(\mathbf{D}\times\mathbf{B}\right)}{\partial t}
\end{aligned}$$

Now from Newton's second law, force on charges is same as the rate of change of linear momentum of the charges, call it $\boldsymbol{\pi}_{\mathbf{charges}}$ i.e.,

$$\mathbf{f} = \frac{\partial\boldsymbol{\pi}_{charges}}{\partial t}$$

Thus we have,

$$\frac{\partial\boldsymbol{\pi}_{charges}}{\partial t} = \mathbf{E}\left(\boldsymbol{\nabla}\cdot\mathbf{D}\right) + \mathbf{H}\left(\boldsymbol{\nabla}\cdot\mathbf{B}\right) - \mathbf{B}\times\left(\boldsymbol{\nabla}\times\mathbf{H}\right) - \mathbf{D}\times\left(\boldsymbol{\nabla}\times\mathbf{E}\right) - \frac{\partial\left(\mathbf{D}\times\mathbf{B}\right)}{\partial t},$$

or, equivalently,

$$\frac{\partial}{\partial t}\left(\boldsymbol{\pi}_{charges}+\mathbf{D}\times\mathbf{B}\right)+\mathbf{B}\times\left(\boldsymbol{\nabla}\times\mathbf{H}\right)+\mathbf{D}\times\left(\boldsymbol{\nabla}\times\mathbf{E}\right)-\mathbf{E}\left(\boldsymbol{\nabla}\cdot\mathbf{D}\right)-\mathbf{H}\left(\boldsymbol{\nabla}\cdot\mathbf{B}\right)=0$$

For linear media,

$$\mathbf{B}=\mu\mathbf{H},\ \mathbf{D}=\varepsilon\mathbf{E},$$

hence, one can write,

$$\mathbf{B}\times\left(\boldsymbol{\nabla}\times\mathbf{H}\right)=\frac{1}{2}\left[\mathbf{B}\times\left(\boldsymbol{\nabla}\times\mathbf{H}\right)+\mathbf{H}\times\left(\boldsymbol{\nabla}\times\mathbf{B}\right)\right],$$

And then using identity in problem 2(**B**),

$$\begin{aligned}\mathbf{B}\times\left(\boldsymbol{\nabla}\times\mathbf{H}\right)&=\frac{1}{2}\left[\mathbf{B}\times\left(\boldsymbol{\nabla}\times\mathbf{H}\right)+\mathbf{H}\times\left(\boldsymbol{\nabla}\times\mathbf{B}\right)\right]\\&=\frac{1}{2}\boldsymbol{\nabla}\left(\mathbf{B}\cdot\mathbf{H}\right)-\frac{1}{2}\left(\mathbf{H}\cdot\boldsymbol{\nabla}\right)\mathbf{B}-\frac{1}{2}\left(\mathbf{B}\cdot\boldsymbol{\nabla}\right)\mathbf{H}\\&=\frac{1}{2}\boldsymbol{\nabla}\left(\mathbf{B}\cdot\mathbf{H}\right)-\left(\mathbf{B}\cdot\boldsymbol{\nabla}\right)\mathbf{H}\end{aligned}$$

Same goes for,

$$\mathbf{D}\times\left(\boldsymbol{\nabla}\times\mathbf{E}\right)=\frac{1}{2}\boldsymbol{\nabla}\left(\mathbf{D}\cdot\mathbf{E}\right)-\left(\mathbf{D}\cdot\boldsymbol{\nabla}\right)\mathbf{E}$$

So we have the equation,

$$\frac{\partial}{\partial t}\left(\boldsymbol{\pi}_{charges}+\mathbf{D}\times\mathbf{B}\right)+\frac{1}{2}\boldsymbol{\nabla}\left(\mathbf{B}\cdot\mathbf{H}+\mathbf{D}\cdot\mathbf{E}\right)-\left(\mathbf{D}\cdot\boldsymbol{\nabla}\right)\mathbf{E}-\mathbf{E}\left(\boldsymbol{\nabla}\cdot\mathbf{D}\right)-\left(\mathbf{B}\cdot\boldsymbol{\nabla}\right)\mathbf{H}-\mathbf{H}\left(\boldsymbol{\nabla}\cdot\mathbf{B}\right)=0,$$

which can be rewritten in index notation as,

$$\frac{\partial\pi_i}{\partial t}+\partial_j T_{ji}=0$$

where

$$T_{ij}=\frac{1}{2}\delta_{ij}\left(\mathbf{B}\cdot\mathbf{H}+\mathbf{D}\cdot\mathbf{E}\right)-E_iD_j-H_iB_j$$

8. In one sentence each, explain a) what is Brewster angle? b) what is skin depth of a conductor? Provide the mathematical expressions for each. Simplify the expression for skin depth for poor conductor and good conductor

(4 + 6 = 10 points)

SOLUTION:
Brewster angle: The angle of incidence of an “in-plane” (electric field) polarized electromagnetic wave on an interface of two media, for which the wave is fully transmitted into the second media, i.e. without any reflection. In slides for lecture 11, the Fresnel equation for reflection of in-plane polarization is,

$$\frac{E_{OR}}{E_{OI}}=\frac{\alpha-\beta}{\alpha+\beta}$$

where,

$$\alpha=\frac{\cos\theta_T}{\cos\theta_I},\beta=\frac{\mu_1 n_2}{\mu_2 n_1}$$

Brewster angle is given by the condition,

$$\frac{E_{OR}}{E_{OI}} = 0 \implies \alpha = \beta$$

i.e.,

$$\frac{\cos\theta_T}{\cos\theta_B} = \frac{\mu_1 n_2}{\mu_2 n_1}.$$

Using Snell's law,

$$\frac{\sin\theta_B}{\sin\theta_T} = \frac{n_2}{n_1}$$

,we get,

$$\sin\theta_B = \frac{\sqrt{\beta^2 - 1}}{\sqrt{\beta^2 - \left(\frac{n_1}{n_2}\right)^2}},\ \beta = \frac{\mu_1 n_2}{\mu_2 n_1}.$$

Skin depth of a conductor: An EM wave incident on the surface of a conductor is unable to penetrate/propagate into the conductor, it decays rapidly (exponentially) within a short distance from the surface. The characteristic scale of this penetration depth is called "Skin Depth". It is a function of frequency, and is given by the expression,

$$d \sim \frac{1}{\omega}\sqrt{\frac{2}{\mu\epsilon}}\left(\sqrt{1 + \left(\frac{\sigma}{\epsilon\omega}\right)^2} - 1\right)^{-1/2}.$$

For the good conductor limit, $\sigma \gg \epsilon\omega$, and then, dropping the 1's in comparison to $\frac{\sigma}{\epsilon\omega}$,

$$d \approx \sqrt{\frac{2}{\sigma\mu\omega}}$$

.

For the poor conductor, $\sigma \ll \epsilon\omega$, and then we can expand binomially $\sqrt{1 + \left(\frac{\sigma}{\epsilon\omega}\right)^2} = 1 + \frac{1}{2}\left(\frac{\sigma}{\epsilon\omega}\right)^2$and get,

$$d \sim \frac{1}{\omega}\sqrt{\frac{2}{\mu\epsilon}}\left(\sqrt{1 + \left(\frac{\sigma}{\epsilon\omega}\right)^2} - 1\right)^{-1/2} = \frac{1}{\omega}\sqrt{\frac{2}{\mu\epsilon}}\left(1 + \frac{1}{2}\left(\frac{\sigma}{\epsilon\omega}\right)^2 - 1\right)^{-1/2} = \frac{2}{\sigma}\sqrt{\frac{\epsilon}{\mu}}.$$

9. (**A**). Check whether or not

$$\mathbf{B} = \frac{\mu_0}{4\pi}\frac{1}{r^3}\left[3\left(\mathbf{m}.\hat{\mathbf{r}}\right)\hat{\mathbf{r}} - \mathbf{m}\right]$$

is an allowed magnetic field.
(**B**). The electric field in a region is given by, $\mathbf{E} = 2a\, x\hat{\mathbf{x}} + b\, y\hat{\mathbf{y}}$, where a, b are constants. Find the charge density which created this field.

(8 + 2 = 10 points)

SOLUTION:
(**A**). Magnetic field must have a vanishing divergence (magnetic field has no sources) so we need to check the divergence of the given candidate magnetic field:

$$\mathbf{B} = \frac{\mu_0}{4\pi}\frac{1}{r^3}\left[3\left(\mathbf{m}.\hat{\mathbf{r}}\right)\hat{\mathbf{r}} - \mathbf{m}\right].$$

First we rewrite it a bit by replacing, $\hat{\mathbf{r}} = \frac{\mathbf{r}}{r}$,

$$\mathbf{B} = \frac{\mu_0}{4\pi}\left[3\frac{(\mathbf{m}.\mathbf{r})\,\mathbf{r}}{r^5} - \frac{\mathbf{m}}{r^3}\right].$$

Then we compute the divergence,

$$\boldsymbol{\nabla}\cdot\mathbf{B} = \frac{\mu_0}{4\pi}\left[3\boldsymbol{\nabla}\cdot\left(\frac{(\mathbf{m}.\mathbf{r})\,\mathbf{r}}{r^5}\right) - \boldsymbol{\nabla}\cdot\left(\frac{\mathbf{m}}{r^3}\right)\right].$$

Now, recalling that $r = \sqrt{\mathbf{r}\cdot\mathbf{r}} = (x_k\, x_k)^{1/2}$, and $\mathbf{m}$ is a constant vector, we compute the divergence of each term. The second one,

$$\begin{aligned}
\boldsymbol{\nabla}\cdot\left(\frac{\mathbf{m}}{r^3}\right) = \partial_i\left(\frac{m_i}{r^3}\right) &= \partial_i\left(\frac{m_i}{(x_k\, x_k)^{3/2}}\right) \\
&= m_i\partial_i\left[(x_k\, x_k)^{-3/2}\right] \\
&= m_i\left(-\frac{3}{2}\right)(x_k\, x_k)^{-5/2}\; 2\, x_l\left(\underbrace{\partial_i x_l}_{\delta_{il}}\right) \\
&= -3\frac{m_i x_l \delta_{il}}{r^5} \\
&= -3\frac{\mathbf{m}\cdot\mathbf{r}}{r^5}.
\end{aligned}$$

Note: If you are using curvilinear coordinates, then recall that in curvilinear coordinates, the basis vectors are not constant all over space, but varies from location to location. As a result, a constant vector when expanded in such non-constant basis will have non-constant coefficients. For example, if you use spherical polar coordinates, $m_r = \mathbf{m}\cdot\hat{\mathbf{r}}$ is not a constant, because $\hat{\mathbf{r}}$ changes from point to point. (4 points for the above)

The first term,

$$
\begin{aligned}
3\boldsymbol{\nabla}\cdot\left(\frac{(\mathbf{m}.\mathbf{r})\,\mathbf{r}}{r^5}\right) = 3\partial_i\left(\frac{m_j x_j x_i}{r^5}\right) &= 3\partial_i\left(\frac{m_j x_j x_i}{(x_k\,x_k)^{5/2}}\right)\\
&= 3m_j\,\partial_i\left(\frac{x_j x_i}{(x_k\,x_k)^{5/2}}\right)\\
&= 3m_j\left(\underbrace{\partial_i x_j}_{=\delta_{ij}}\right)\frac{x_i}{(x_k\,x_k)^{5/2}} + 3m_j\left(\underbrace{\partial_i x_i}_{=3}\right)\frac{x_j}{(x_k\,x_k)^{5/2}} + 3m_j x_j x_i \partial_i\left[(x_k\,x_k)^{-5/2}\right.\\
&= 3\frac{m_j\delta_{ij}x_i}{(x_k\,x_k)^{5/2}} + 9\frac{m_j x_j}{(x_k\,x_k)^{5/2}} + 3m_j x_j x_i\left(-\frac{5}{2}\right)(x_k\,x_k)^{-7/2}\;2\,x_l\left(\underbrace{\partial_i x_l}_{\delta_{il}}\right)\\
&= 3\frac{\mathbf{m}\cdot\mathbf{r}}{r^5} + 9\frac{\mathbf{m}\cdot\mathbf{r}}{r^5} - 15\frac{\mathbf{m}\cdot\mathbf{r}}{r^5}\\
&= -3\frac{\mathbf{m}\cdot\mathbf{r}}{r^5}.
\end{aligned}
$$

4 points for the above)

Thus we have,

$$
\boldsymbol{\nabla}\cdot\mathbf{B} = \frac{\mu_0}{4\pi}\left[3\boldsymbol{\nabla}\cdot\left(\frac{(\mathbf{m}.\mathbf{r})\,\mathbf{r}}{r^5}\right) - \boldsymbol{\nabla}\cdot\left(\frac{\mathbf{m}}{r^3}\right)\right] = \frac{\mu_0}{4\pi}\left[-3\frac{\mathbf{m}\cdot\mathbf{r}}{r^5} - \left(-3\frac{\mathbf{m}\cdot\mathbf{r}}{r^5}\right)\right] = 0.
$$

and the given expression does qualify to be a magnetic field.
(**B**). We use Gauss law,

$$
\begin{aligned}
\rho(\mathbf{x}) &= \epsilon_0\boldsymbol{\nabla}\cdot\mathbf{E}\\
&= \epsilon_0\boldsymbol{\nabla}\cdot(2a\,x\hat{\mathbf{x}} + b\,y\hat{\mathbf{y}})\\
&= \epsilon_0\left(\frac{\partial(2a\,x)}{\partial x} + \frac{\partial(b\,y)}{\partial y}\right)\\
&= \epsilon_0\,(2a+b)\,.
\end{aligned}
$$

Thus this field was created by an uniform charge density all over space.