EP 1027: Maxwell's Equations and Electromagnetic Waves

Instructor: Shubho Roy[1]

(Dept. of Physics)

Lecture 4
April 2, 2019

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Agenda

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- Recap of Lecture 3: Geometric interpretation of gradient, divergence, curl, Gauss & Stokes theorem, Continuity equation

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- General Orthogonal Curvilinear Coordinate Systems
- Laplacian of $1/|\mathbf{x}| \Rightarrow$ Dirac's delta "function", $\delta(\mathbf{x})$

Final Exam

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- Date: May 1, Time: 2.00 - 5.00 PM, Venue: LH 1, Auditorium

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- Review of grades (tentatively on May 3)

References/Readings

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- Griffiths, D.J., **Introduction to Electrodynamics, Ch. 1**
- Spiegel, M.R., **Schaum's Outline of Vector Analysis, Ch. 7**
- Boas, M.L., **Mathematical Methods..., Ch. 10**

Geometric meaning of gradient, divergence, curl

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- **Gradient**: Consider a volume element, $\Delta V = \Delta x \Delta y \Delta z$, around point, $\mathbf{x}$
$$\lim_{\Delta V \to 0} \frac{\oiint dS\, \hat{\mathbf{n}}\, \Phi(x)}{\Delta V} = \boldsymbol{\nabla}\Phi,$$
$\hat{\mathbf{n}}$ is the unit outward normal vector on the surface S.

Geometric meaning of gradient, divergence, curl

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- **Divergence:** Consider a volume element, $\Delta V = \Delta x \Delta y \Delta z$, around $\mathbf{x}$,

$$\lim_{\Delta V \to 0} \frac{\oiint_S dS \, \hat{\mathbf{n}} \cdot \mathbf{A}}{\Delta V} = \boldsymbol{\nabla} \cdot \mathbf{A}$$

  $\hat{\mathbf{n}}$ is the unit outward normal vector on the surface S. Thus, **Divergence $=$ flux over an infinitesimal closed surface per unit volume enclosed by the surface.**

Geometric meaning of gradient, divergence, curl

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$\hat{\mathbf{n}}$ is the unit outward normal vector on the surface S. Thus, **Divergence = flux over an infinitesimal closed surface per unit volume enclosed by the surface.**

- **Curl:** Consider an area element, $\Delta S_{yz} = \Delta y \Delta z$, around $\mathbf{x}$,

$$\lim_{\Delta S_{yz} \to 0} \frac{\oint \mathbf{A} \cdot d\mathbf{l}}{\Delta S_{yz}} = (\boldsymbol{\nabla} \times \mathbf{A})_x\,,$$

$d\mathbf{l}$ is the (tangential) line element. So, **Curl = Anticlockwise circulation in an infinitesimal loop per unit normal area bounded by the loop.**

Tensor fields: Integration

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- **Gauss Divergence theorem**: If S is a closed surface enclosing a volume, V

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- **Stokes Curl theorem:** If S is an open surface, with a boundary, C (closed curve)

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- Best worked out in not Cartesian, but **Curvilinear** coordinates $(dS, \mathbf{n}, d\mathbf{l})$

Application of Gauss theorem: Continuity Equation

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- Consider a closed surface, S enclosing a volume, V containing a fluid of mass density (or electric charge density), ρ. The total mass/charge inside is then,

$$\iiint_V d^3\mathbf{x}\,\rho$$

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- The amount of mass/charge coming out of the volume V by crossing the surface, S per unit time = outward flux per unit time thru the entire surface, S

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$$-\frac{d}{dt}\left(\iiint_V d^3\mathbf{x}\,\rho\right)$$

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- The rate of decrease of mass/charge inside V is,

$$-\frac{d}{dt}\left(\iiint_V d^3\mathbf{x}\, \rho\right)$$

- Since there are no sources or sinks; the amount of mass (or charge) escaped by crossing the surface, S = Amount of mass (or charge) decreased in the volume, V

$$\oiint_S dS\, \hat{\mathbf{n}} \cdot \mathbf{j} = -\frac{d}{dt}\left(\iiint_V d^3\mathbf{x}\, \rho\right)$$

Continuity Equation

Continuity Equation

- Conservation of mass or electric charge

$$\oiint_S dS\,\hat{\mathbf{n}}\cdot\mathbf{j} = -\frac{d}{dt}\left(\iiint_V d^3\mathbf{x}\,\rho\right) \tag{1}$$

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$$\oiint_S dS\, \hat{\mathbf{n}} \cdot \mathbf{j} = -\frac{d}{dt} \left(\iiint_V d^3\mathbf{x}\, \rho \right) \tag{1}$$

- One can convert the surface integral on the LHS into a volume integral using Gauss theorem,

$$\oiint_S dS\, \hat{\mathbf{n}} \cdot \mathbf{j} = \iiint_V d^3\mathbf{x}\, \boldsymbol{\nabla} \cdot \mathbf{j},$$

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$$\oiint_S dS\, \hat{\mathbf{n}} \cdot \mathbf{j} = \iiint_V d^3\mathbf{x}\, \boldsymbol{\nabla} \cdot \mathbf{j},$$

- And in the RHS one can take the time-derivative from outside the volume integral to inside the volume integral,

$$-\frac{d}{dt}\left(\iiint_V d^3\mathbf{x}\, \rho\right) = \iiint_V d^3\mathbf{x}\left(-\frac{\partial \rho}{\partial t}\right).$$

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- Thus, the conservation equation, (1), becomes,

$$\iiint_V d^3\mathbf{x}\, \boldsymbol{\nabla} \cdot \mathbf{j} = \iiint_V d^3\mathbf{x}\left(-\frac{\partial \rho}{\partial t}\right),$$

or,

$$\begin{aligned}
\iiint_V d^3\mathbf{x}\left(\frac{\partial \rho}{\partial t} + \boldsymbol{\nabla} \cdot \mathbf{j}\right) &= 0, \\
\Rightarrow \frac{\partial \rho}{\partial t} + \boldsymbol{\nabla} \cdot \mathbf{j} &= 0.
\end{aligned}$$

Cylindrical Polar Coordinates

Cylindrical Polar Coordinates

- A point in $\mathbb{R}^3$ is specified by, (ρ, ϕ, z)

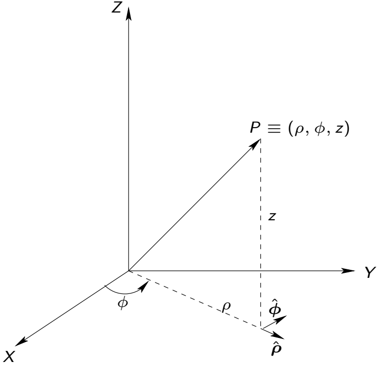

Figure: Cylindrical Polar Coordinates

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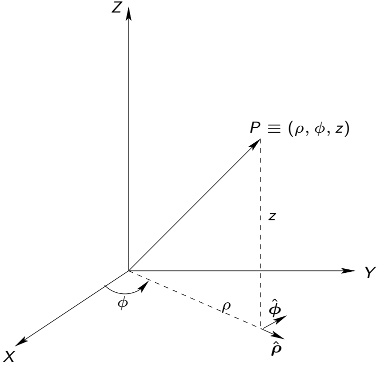

Figure: Cylindrical Polar Coordinates

- Relation to Cartesian coordinates

$$x = \rho\cos\phi,\ y = \rho\sin\phi,\ z = z,$$

$$\rho = \sqrt{x^2 + y^2},\ \tan\phi = y/x,\ z = z.$$

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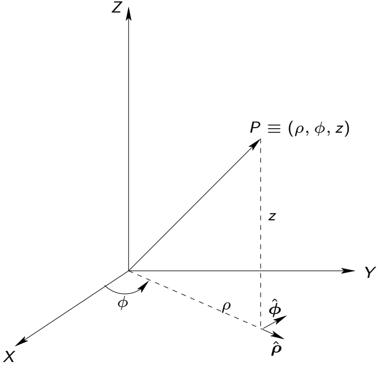

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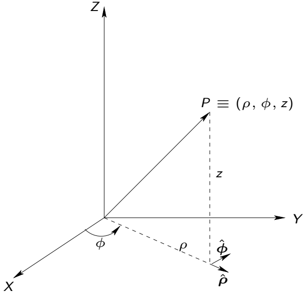
- 

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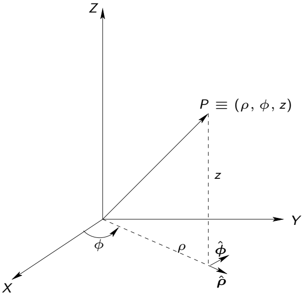
- 

- **Unit Vectors**,

$$\hat{\boldsymbol{\rho}} = \frac{x\hat{\mathbf{x}} + y\hat{\mathbf{y}}}{\sqrt{x^2 + y^2}}, \ \hat{\boldsymbol{\phi}} = \frac{-y\hat{\mathbf{x}} + x\hat{\mathbf{y}}}{\sqrt{x^2 + y^2}}, \ \hat{\mathbf{z}} = \hat{\mathbf{z}},$$

$$\hat{\mathbf{x}} = \cos\phi\hat{\boldsymbol{\rho}} - \sin\phi\hat{\boldsymbol{\phi}}, \hat{\mathbf{y}} = \sin\phi\hat{\boldsymbol{\rho}} + \cos\phi\hat{\boldsymbol{\phi}}, \hat{\mathbf{z}} = \hat{\mathbf{z}}.$$

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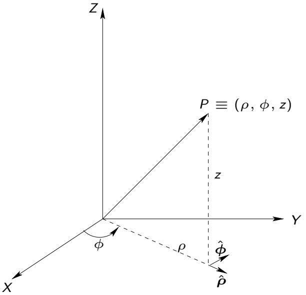
- 

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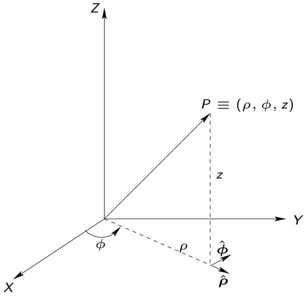
- **Components:**

$$\mathbf{A} = A_1\hat{\mathbf{x}} + A_2\hat{\mathbf{y}} + A_3\hat{\mathbf{z}} = A_\rho\hat{\boldsymbol{\rho}} + A_\phi\hat{\boldsymbol{\phi}} + A_z\,\hat{\mathbf{z}}.$$

Cylindrical Polar Coordinates

- 

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- **Components:**

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- **Line element:**

$$d\mathbf{x} = dx\,\hat{\mathbf{x}} + dy\,\hat{\mathbf{y}} + dz\,\hat{\mathbf{z}} = d\rho\,\hat{\boldsymbol{\rho}} + \rho d\phi\,\hat{\boldsymbol{\phi}} + dz\,\hat{\mathbf{z}}.$$

Cylindrical Polar Coordinates

[2] Derivations for the expressions of grad, div and curl were given in the supplementary notes provided

Cylindrical Polar Coordinates

- **Gradient Expression**: From before

$$d\Phi \equiv \Phi(\mathbf{x} + d\mathbf{x}) - \Phi(\mathbf{x}) = d\mathbf{x} \cdot \nabla\Phi(\mathbf{x}),$$

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- But also, from multi-variable calculus

$$d\Phi = \frac{\partial \Phi}{\partial \rho}\, d\rho + \frac{\partial \Phi}{\partial \phi}\, d\phi + \frac{\partial \Phi}{\partial z}\, dz$$

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- **Gradient**

$$\boldsymbol{\nabla} = \hat{\boldsymbol{\rho}}\, \frac{\partial}{\partial \rho} + \hat{\boldsymbol{\phi}}\, \frac{1}{\rho}\frac{\partial}{\partial \phi} + \hat{\mathbf{z}}\, \frac{\partial}{\partial z}$$

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$$d\Phi = \frac{\partial \Phi}{\partial \rho}\, d\rho + \frac{\partial \Phi}{\partial \phi}\, d\phi + \frac{\partial \Phi}{\partial z}\, dz$$

- **Gradient**

$$\boldsymbol{\nabla} = \hat{\boldsymbol{\rho}}\, \frac{\partial}{\partial \rho} + \hat{\boldsymbol{\phi}}\, \frac{1}{\rho}\frac{\partial}{\partial \phi} + \hat{\mathbf{z}}\, \frac{\partial}{\partial z}$$

- **Divergence:**[2]

$$\boldsymbol{\nabla} \cdot \mathbf{A} = \frac{1}{\rho}\frac{\partial}{\partial \rho}\left(A_\rho\, \rho\right) + \frac{1}{\rho}\frac{\partial}{\partial \phi} A_\phi + \frac{\partial}{\partial z} A_z .$$

---

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- **Curl Expression**:

$$(\boldsymbol{\nabla} \times \mathbf{A})_\rho = \left(\frac{1}{\rho} \frac{\partial A_z}{\partial \phi} - \frac{\partial A_\phi}{\partial z} \right)$$

$$(\boldsymbol{\nabla} \times \mathbf{A})_\phi = \left(\frac{\partial A_\rho}{\partial z} - \frac{\partial A_z}{\partial \rho} \right)$$

$$(\boldsymbol{\nabla} \times \mathbf{A})_z = \frac{1}{\rho} \left(\frac{\partial}{\partial \rho} (\rho A_\phi) - \frac{\partial A_\rho}{\partial \phi} \right)$$

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$$(\nabla \times \mathbf{A})_z = \frac{1}{\rho} \left(\frac{\partial}{\partial \rho} (\rho A_\phi) - \frac{\partial A_\rho}{\partial \phi} \right)$$

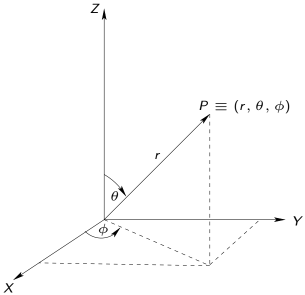
- **Laplacian**

$$\nabla \cdot \nabla = \frac{1}{\rho} \frac{\partial}{\partial \rho} \left(\rho \frac{\partial}{\partial \rho} \right) + \frac{1}{\rho^2} \frac{\partial^2}{\partial \phi^2} + \frac{\partial^2}{\partial z^2}$$

Spherical Polar Coordinates

Spherical Polar Coordinates

- **Label** a point by, (r, θ, ϕ)



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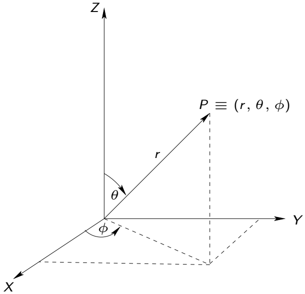

- **Relation to Cartesian:**

$$x = r\sin\theta\cos\phi, y = r\sin\theta\sin\phi, z = r\cos\theta,$$

$$r = \sqrt{x^2 + y^2 + z^2}, \theta = \cos^{-1}\left(\frac{z}{\sqrt{x^2+y^2+z^2}}\right), \phi = \tan^{-1}\left(\frac{y}{x}\right)$$

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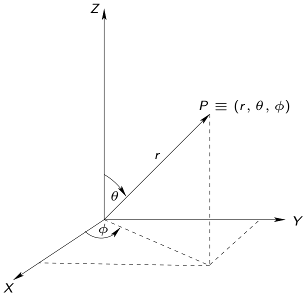

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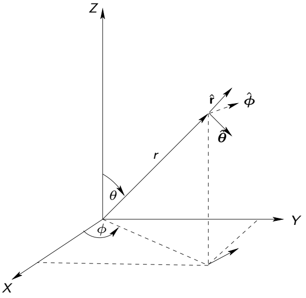
▶ 

Unit Vectors:

$$\hat{\mathbf{r}} = \frac{x\hat{\mathbf{x}} + y\hat{\mathbf{y}} + z\hat{\mathbf{z}}}{\sqrt{x^2+y^2+z^2}},\ \hat{\boldsymbol{\theta}} = \ \ldots, \hat{\boldsymbol{\phi}} = \frac{-y\hat{\mathbf{x}} + x\hat{\mathbf{y}}}{\sqrt{x^2+y^2}}$$

$$\hat{\mathbf{x}} = \sin\theta\cos\phi\,\hat{\mathbf{r}} + \cos\theta\cos\phi\,\hat{\boldsymbol{\theta}} - \sin\phi\,\hat{\boldsymbol{\phi}}, \hat{\mathbf{y}} = \ldots, \hat{\mathbf{z}} = \cos\theta\,\hat{\mathbf{r}} - \sin\theta\,\hat{\boldsymbol{\theta}}.$$

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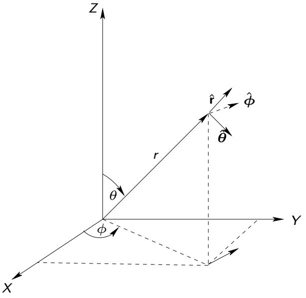
- 

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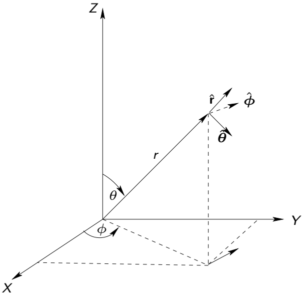
- **Components:**

$$\mathbf{A} = A_1\hat{\mathbf{x}} + A_2\hat{\mathbf{y}} + A_3\hat{\mathbf{z}} = A_r\hat{\mathbf{r}} + A_\theta\hat{\boldsymbol{\theta}} + A_\phi\hat{\boldsymbol{\phi}}.$$

Spherical Polar Coordinates

- 

Unit Vectors:

$$\hat{\mathbf{r}} = \frac{x\hat{\mathbf{x}} + y\hat{\mathbf{y}} + z\hat{\mathbf{z}}}{\sqrt{x^2+y^2+z^2}},\ \hat{\boldsymbol{\theta}} = \ \dots, \hat{\boldsymbol{\phi}} = \frac{-y\hat{\mathbf{x}} + x\hat{\mathbf{y}}}{\sqrt{x^2+y^2}}$$

$$\hat{\mathbf{x}} = \sin\theta\cos\phi\,\hat{\mathbf{r}} + \cos\theta\cos\phi\,\hat{\boldsymbol{\theta}} - \sin\phi\,\hat{\boldsymbol{\phi}}, \hat{\mathbf{y}} = \dots, \hat{\mathbf{z}} = \cos\theta\,\hat{\mathbf{r}} - \sin\theta\,\hat{\boldsymbol{\theta}}.$$

- **Components:**

$$\mathbf{A} = A_1\hat{\mathbf{x}} + A_2\hat{\mathbf{y}} + A_3\hat{\mathbf{z}} = A_r\hat{\boldsymbol{r}} + A_\theta\hat{\boldsymbol{\theta}} + A_\phi\hat{\boldsymbol{\phi}}.$$

- **Line element:**

$$d\mathbf{x} = dx\,\hat{\mathbf{x}} + dy\,\hat{\mathbf{y}} + dz\,\hat{\mathbf{z}} = dr\,\hat{\boldsymbol{r}} + rd\theta\,\hat{\boldsymbol{\theta}} + r\sin\theta\,\hat{\boldsymbol{\phi}},$$

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- **Exact Definition (1D delta function) Part a**: An object depending on a single variable x, in a way that

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- **Exact Definition Part b)** If the spike is **inside** the range of integration, answer is non-zero and equal to 1,

$$\begin{aligned} \int_{-a}^{b} dx\, \delta(x) &= 1, \forall a, b > 0, \\ \int_{-a}^{b} dx\, \delta(x)\, f(x) &= f(0) \forall a, b > 0, \end{aligned}$$

i.e. if spike is **outside** range of integration, then the integral vanishes.

$$\int_{-a}^{-b} dx\, \delta(x) = \int_{a}^{b} dx\, \delta(x) = 0, \forall a, b > 0.$$

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for an arbitrary normal function, $f(x)$.

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- Including the origin, $\mathbf{x} = 0$, the result can be expressed in terms of the $\delta(\mathbf{x})$,

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- If $\nabla \cdot \left(\frac{\mathbf{x}}{|\mathbf{x}|^3} \right) = 4\pi\delta(\mathbf{x})$, plugging in the LHS,

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