Set of problems
MA1140: Elementary Linear Algebra
8th February - 8th March, 2019

1 Matrices, Linear equations and solvability

Q1. Solve (if solution exists) the following system of linear equations:

$$\begin{array}{lllll} u & +v & +w & +z & =6 \\ u & & +w & +z & =4 \\ u & & +w & & =2 \end{array}$$

What is the intersection if the fourth plane $u = -1$ is included? Find a fourth equation that leaves us with no solution.

Q2. Find two points on the line of intersection of the three planes $t = 0$, $z = 0$ and $x+y+z+t = 1$ in four-dimensional space.

Q3. Explain why the system

$$\begin{aligned} u + v + w &= 2 \\ u + 2v + 3w &= 1 \\ v + 2w &= 0 \end{aligned}$$

is *singular* (i.e., it does not have solutions at all). What value should replace the last zero on the right side to allow the system to have solutions, and what is one of the solutions?

Q4. Under what condition on x_1, x_2 and x_3 do the points $(0, x_1), (1, x_2)$ and $(2, x_3)$ lie on a straight line?

Q5. These equations are certain to have the solution $x = y = 0$. For which values of a is there a whole line of solutions?

$$\begin{aligned} ax + 2y &= 0 \\ 2x + ay &= 0 \end{aligned}$$

Q6. Are the following systems equivalent:

$$\begin{aligned} x - y &= 0 \\ 2x + y &= 0 \end{aligned}$$

and

$$\begin{aligned} 3x + y &= 0 \\ x + y &= 0 \end{aligned}$$

If so, then express each equation in each system as a linear combination of the equations in the other system.

Q7. Set $A = \begin{pmatrix} 6 & -4 & 0 \\ 4 & -2 & 0 \\ -1 & 0 & 3 \end{pmatrix}$ and $X = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$. Find all the solutions of $AX = 2X$, i.e., all X such that $AX = 2X$, where $2X$ is just componenntwise scalar multiplication.

Q8. Prove that the interchange of two rows of a matrix can be accomplished by a finite sequence of elementary row operations of the other two types.

Q9. Consider the system of equations $AX = 0$, where $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ is a matrix over $\mathbb{R}$. Prove the following:
(a) A is a zero matrix (i.e., all entries are zero) if and only if every pair (x_1, x_2) is a solution of $AX = 0$.
(b) $ad - bc \neq 0$ if and only if the system has only the trivial solution.
(c) $ad - bc = 0$ but A is a non-zero matrix (i.e., some entries are non-zero) if and only if there is $(y_1, y_2) \neq (0, 0)$ in $\mathbb{R}^2$ such that every solution of the system is given by $c(y_1, y_2)$ for some scalar c.

Q10. Prove that if two homogeneous systems each of two linear equations in two unknowns have the same solutions, then they are equivalent.

Q11. For the system

$$\begin{aligned} u + v + w &= 2 \\ 2u + 3v + 3w &= 0 \\ u + 3v + 5w &= 2, \end{aligned}$$

what is the triangular system after forward elimination, and what is the solution (by back substitution)? Also solve it by computing the equivalent system whose coefficient matrix is in row reduced echelon form. Verify whether both the solutions are same.

Q12. Describe explicitly all 2×2 row reduced echelon matrices.

Q13. Let $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ be a matrix over $\mathbb{R}$. Suppose that A is row reduced and also that $a + b + c + d = 0$. Prove that there are exactly three such matrices.

Q14. Let $A = \begin{pmatrix} 1 & -1 & 1 \\ 2 & 0 & 1 \\ 3 & 0 & 1 \end{pmatrix}$. Find some elementary matrices $E_1, E_2, \ldots, E_k$ such that $E_k \cdots E_2 E_1 A = I_3$, where I_3 is the 3×3 identity matrix. Deduce A^{-1}.

Hint. Apply elementary row operations on $(A \,|\, I_3)$ to get A^{-1}, and keep track of the row operations to get the corresponding $E_1, E_2, \ldots, E_k$.

2 Vector spaces

Throughout, V is a vector space over $\mathbb{R}$, the set of real numbers.

Q15. Let $0 \in V$ be the zero vector. Let $c \in \mathbb{R}$. Show that $c \cdot 0 = 0$.

Q16. Let $v \in V$. Show that $0 \cdot v = 0$, where 0 in the right side is the zero vector, and 0 in the left side is the zero element of $\mathbb{R}$.

Q17. Let W be a subspace of V. Show that (the zero vector) $0 \in W$.

Q18. Let $V = \mathbb{R}^2$. Define

$$\begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} x_1 \\ y_1 \end{pmatrix} = \begin{pmatrix} x + x_1 \\ 0 \end{pmatrix} \quad \text{and} \quad c \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} cx \\ y \end{pmatrix}.$$

Is V, with these operations, a vector space over $\mathbb{R}$?

What happens when we define the scalar multiplication as

$$c\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} cx \\ 0 \end{pmatrix}?$$

Q19. Which of the following sets of vectors $X = (x_1, \ldots, x_n)$ in $\mathbb{R}^n$ are subspaces of $\mathbb{R}^n$ $(n \geqslant 3)$?

(i) all X such that $x_1 \geqslant 0$;

(ii) all X such that $x_1 + 2x_2 = 3x_3$;

(iii) all X such that $x_1 = x_2^2$;

(iv) all X such that $x_1 x_2 = 0$;

(v) all X such that x_1 is rational.

Q20. Let A be an $m \times n$ matrix over $\mathbb{R}$. Show that $\{X \in \mathbb{R}^n : AX = 0\}$ is a subspace of $\mathbb{R}^n$. It is called the **null space** of A. The dimension of the null space of A is called **nullity** of A.

Q21. Let W_1 and W_2 be subspaces of V such that the set-theoretic union $W_1 \cup W_2$ is also a subspace of V. Prove that one of the subspaces W_1 and W_2 is contained in the other.

Q22. Let W_1 and W_2 be subspaces of V such that $W_1 + W_2 = V$ and $W_1 \cap W_2 = 0$. Prove that for every vector $v \in V$, there are unique vectors $w_1 \in W_1$ and $w_2 \in W_2$ such that $v = w_1 + w_2$.

In this case, we write $V = W_1 \oplus W_2$, and call this as **direct sum** of W_1 and W_2.

Q23. Prove that all the subspaces of $\mathbb{R}^1$ are 0 and $\mathbb{R}^1$.

Q24. Prove that a subspace of $\mathbb{R}^2$ is either 0, or $\mathbb{R}^2$, or a subspace consisting of all scalar multiplies of some fixed non-zero vector in $\mathbb{R}^2$ (which is intuitively a straight line through the origin).

Q25. (i) Let A be an $m \times n$ matrix. Suppose B is obtained from A by applying an elementary row operation. Prove that row space(A) = row space(B).

(ii) Deduce from (i) that if any two $m \times n$ matrices A and B are row equivalent, then row space(A) = row space(B).

(iii) Let $B = \begin{bmatrix} R_1 \\ \vdots \\ R_r \\ 0 \\ \vdots \\ 0 \end{bmatrix}$ be an $m \times n$ row reduced echelon matrix with the non-zero rows $R_1, \ldots, R_r \in \mathbb{R}^n$ and the last $(m - r)$ zero rows. Prove that $\{R_1, \ldots, R_r\}$ is a basis of the row space of B.

(iv) Let A be an $m \times n$ matrix. Let A be reduced to a row reduced echelon matrix B. Then deduce from (ii) and (iii) that the non-zero rows of B gives a basis of the row space of A. Hence the row rank of A is same as the number of non-zero rows of B.

Q26. Let A be an $m \times n$ matrix. By applying elementary row operations, how can you find a basis of the column space of A?

Solution: Note that the column space of A is same as the row space of A^t (transpose of A). Then apply elementary row operations on A^t to get a row reduced echelon matrix, say B. Then, by Q25(iv), the non-zero rows of B gives a basis of the row space of A^t, which is same as the column space of A.

Q27. Consider some column vectors $v_1, \ldots, v_n \in \mathbb{R}^m$. By applying elementary row operations, how can you find a basis of the subspace $\mathrm{Span}(\{v_1, \ldots, v_n\})$ of $\mathbb{R}^m$?

Solution: Set $A := \begin{bmatrix} v_1 & v_2 & \cdots & v_n \end{bmatrix}$ an $m \times n$ matrix with the columns $v_1, \ldots, v_n \in \mathbb{R}^m$. Then the subspace $\mathrm{Span}(\{v_1, \ldots, v_n\})$ is same as the column space of A. Now follow the solution of Q26.

Q28. Let $A = \begin{pmatrix} 3 & -1 & 8 & 4 \\ 2 & 1 & 7 & 1 \\ 1 & -3 & 0 & 4 \end{pmatrix}$ and $X = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix}$. For which $Y = \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix}$ in $\mathbb{R}^3$ does the system $AX = Y$ have a solution? Describe the answer in terms of subspaces of $\mathbb{R}^3$. Use the following approaches, and verify whether you get the same answer.

Two approaches: (i) Apply elementary row eliminations on $(A \,|\, Y)$, conclude when the system $AX = Y$ has solutions. (ii) Note that for every $X \in \mathbb{R}^4$, AX is nothing but a linear combination of the four column vectors of A:

$$AX = x_1 \begin{pmatrix} 3 \\ 2 \\ 1 \end{pmatrix} + x_2 \begin{pmatrix} -1 \\ 1 \\ -3 \end{pmatrix} + x_3 \begin{pmatrix} 8 \\ 7 \\ 0 \end{pmatrix} + x_4 \begin{pmatrix} 4 \\ 1 \\ 4 \end{pmatrix}.$$

So Y should belong into the column space of A. Furthermore, you may try to find a basis of the column space of A. To do that follow Q26.

Q29. Let $S = \{v_1, \ldots, v_r\}$ be a collection of r vectors of a vector space V. Then show that S is linearly independent if and only if $\dim(\mathrm{Span}(S)) = r$. (See Corollary 2.26 in the notes)

Q30. Check whether the following vectors in $\mathbb{R}^4$ are linearly independent:

$$v_1 = \begin{pmatrix} 1 \\ 1 \\ 2 \\ 4 \end{pmatrix}, v_2 = \begin{pmatrix} 2 \\ -1 \\ -5 \\ 2 \end{pmatrix}, v_3 = \begin{pmatrix} 1 \\ -1 \\ -4 \\ 0 \end{pmatrix} \text{ and } v_4 = \begin{pmatrix} 2 \\ 1 \\ 1 \\ 6 \end{pmatrix}.$$

Two approaches: (i) Consider a relation $x_1v_1 + x_2v_2 + x_3v_3 + x_4v_4 = 0$. It yields a homogeneous system of linear equations:

$$\begin{pmatrix} 1 & 2 & 1 & 2 \\ 1 & -1 & -1 & 1 \\ 2 & -5 & -4 & 1 \\ 4 & 2 & 0 & 6 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}.$$

Observe that this system has a non-trivial solution if and only if $\{v_1, v_2, v_3, v_4\}$ is linearly dependent.

(ii) Consider the matrix A whose rows are the vectors v_1, v_2, v_3, v_4. By Q29, $\{v_1, v_2, v_3, v_4\}$ is linearly independent if and only if $\dim(\mathrm{Span}(\{v_1, v_2, v_3, v_4\})) = 4$. Since $\mathrm{Span}(\{v_1, v_2, v_3, v_4\})$ is nothing but the row space of A, we just have to compute row rank of A. So follow Q25(iv).

Q31. Let V be the vector space of all $m \times n$ matrices over $\mathbb{R}$ with usual vector addition and scalar multiplication. Show that $\dim(V) = mn$.

Hint: Consider the standard basis $\{A^{ij} : 1 \leqslant i \leqslant m, 1 \leqslant j \leqslant n\}$, where A^{ij} is the $m \times n$ matrix with (i, j) entry 1 and all other entries 0.

Q32. Let V be the vector space of all $n \times n$ matrices over $\mathbb{R}$ with usual vector addition and scalar multiplication. Show that the following are subspaces of V.

(i) The subset of V consisting of all symmetric matrices.

(ii) The subset of V consisting of all skew-symmetric (or anti-symmetric) matrices.

(iii) The subset of V consisting of all upper triangular matrices (i.e., $A_{ij} = 0$ for all $i > j$).

What is the dimension of each of these subspaces?

Show that the following are not subspaces of V.

(iv) The subset of V consisting of all invertible matrices.

(v) The subset of V consisting of all non-invertible matrices.

(vi) The subset of V consisting of all matrices A such that $A^2 = A$.