CS6350: Topics in Combinatorics
Assignment 8

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1. **Definition:** Given a graph G, a collection of permutations (or total order) of the vertices of G is said to be 3-mixing if for every pair of edges, say ab, bc, in G that share a vertex (b in this case), there is a permutation in the collection in which the shared vertex b appears between the other two vertices.
For example, if the graph G is a 4-cycle on vertices a, b, c, d having edges ab, bc, cd, and da, then C = $\{\sigma 1, \sigma 2\}$ is a 3-mixing family of permutations of the vertices of G where $\sigma 1$: a, b, c, d and $\sigma 2$: c, d, a, b.
Let $\beta(G)$ denote the cardinality of a smallest family of permutations of V(G) that is 3-mixing.
Problem Let G be a graph where every vertex has at most Δ neighbours (that is, the degree of every vertex is at most Δ). Show that, $\beta(G) \in O(\log \Delta)$.

A. **Local Lemma; Symmetrical Case:** Let $A_1, A_2, A_3, ..., A_k$ be a sequence of events with probability of each event is atmost p and each event is independent of other except for atmost d events, if

$$4pd \leq 1$$

then the probability that none of the events occur is non-zero.
Now consider the edges (a,b) and (b,c). The probability that b is between a and c is

$$Pr(b \textit{ is between } a \textit{ and } c \textit{ in } \sigma_i) \ = \ \frac{1}{3} \quad (\because \ a, b, c \textit{ are equally likely})$$

Let E((a,b),(b,c),σ_i) be the event that b is not in between a,c in σ_i. Then

$$\begin{aligned} Pr(E((a,b),(b,c),\sigma_i)) &= 1 - Pr(b \textit{ is between } a \textit{ and } c \textit{ in } \sigma_i) \\ &= 1 - \frac{1}{3} \\ &= \frac{2}{3} \end{aligned}$$

Probability that b is not in between a,c in any $\sigma_1, \sigma_2, ..., \sigma_r$ is

$$\begin{aligned} Pr(E((a,b),(b,c))) &= Pr(E((a,b),(b,c),\sigma_1) \wedge E((a,b),(b,c),\sigma_2) \wedge ... \wedge E((a,b),(b,c),\sigma_r)) \\ &= \Pi_{i=1}^{r} Pr(E((a,b),(b,c),\sigma_i))) \\ &= \Pi_{i=1}^{r} \left(\frac{2}{3}\right) \\ &= \left(\frac{2}{3}\right)^r \end{aligned}$$

The pair X = ((a,b),(b,c)) is dependent on the other pair Y, if Y contains any of x $\in \{a, b, c\}$. Then Y would be of the form

$$Y = ((x, p), (p, q)) \ or \ ((p, x), (x, q))$$

So Number of possibilities of X dependent on Y is

$$\begin{aligned}
d &= \sum_{x \in a,b,c} ((((x, p) \in G) * ((p, q) \in G))) + (((p, x) \in G) * ((x, q) \in G)))) \\
&< \sum_{x \in \{a,b,c\}} ((\Delta * \Delta) + (\Delta * \Delta)) \\
&= \sum_{x \in \{a,b,c\}} (2\Delta^2) \\
&= 6\Delta^2
\end{aligned}$$

So now appling Local Lemma Theorem, we get

$$\begin{aligned}
4pd &\le 1 \\
4 * \left(\frac{2}{3}\right)^r * (6\Delta^2) &\le 1 \\
\frac{3}{2}^r &\ge 24\Delta^2 \\
r &\ge 2 \log_{3/2}^{\Delta} \\
\therefore \boxed{r \in O(\log \Delta)}
\end{aligned}$$

Hence Proved