09 November 2020 15:32

Independence criterion/Triangular criterion

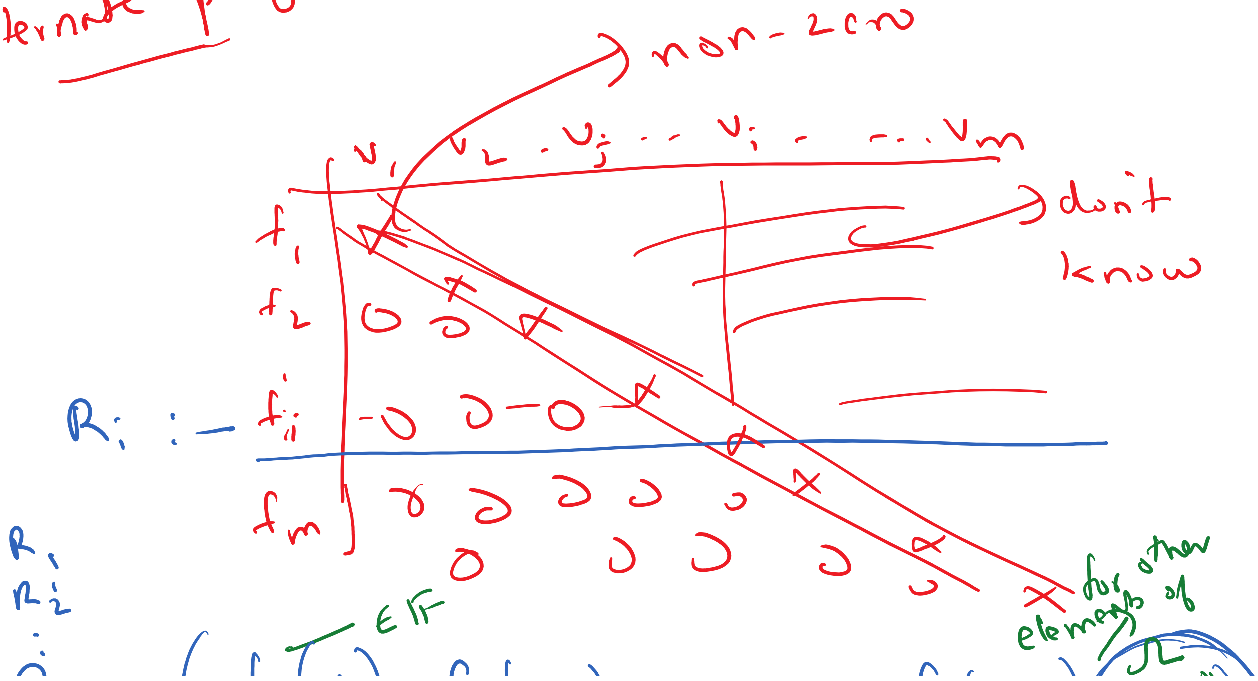
Lemma: Let Ω be a set and let $\mathbb{F}$ be a field. Let $f_1, f_2, \ldots, f_m$ be functions from Ω to $\mathbb{F}$ such that for some elements $v_1, v_2, \ldots, v_m \in \Omega$, we have

(i) $\forall i \in [m]$, $f_i(v_i) \neq 0$, and

(ii) $\forall i, j \in [m]$, $j < i$, $f_i(v_j) = 0$.

Then, the functions $f_1, f_2, \ldots, f_m$ are L.I. in V.S. $\mathbb{F}^{\Omega}$ over $\mathbb{F}$.

Alternate proof:



R_2

$\vdots$

$R_i \quad (f_i(v_1), f_i(v_2), \ldots, f_i(v_m)) \in \mathbb{F}$

for elements of Ω

$\vdots$

$R_m \quad (\ldots \ldots \ldots, \ldots)$

09 November 2020 15:41

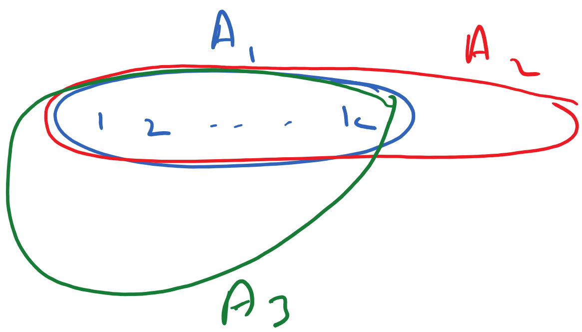
Fisher's Inequality

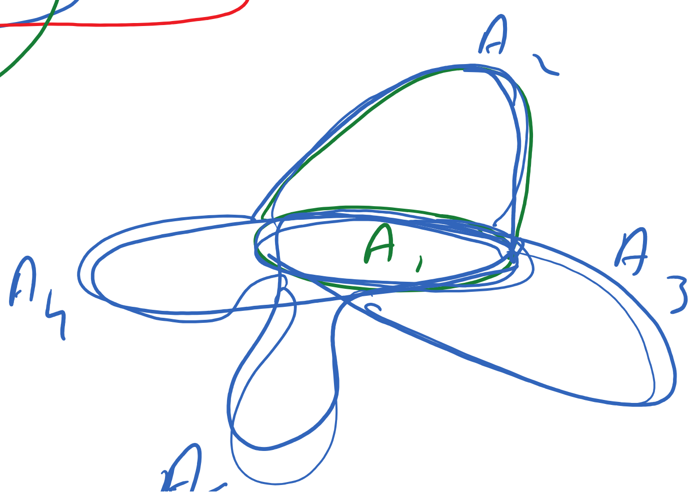
Theorem: Let k, n be positive integers with $k \leq n$. Let $\mathcal{F} = \{A_1, A_2, \ldots, A_m\}$ be a family of subsets of $[n]$ such that for every distinct $A_i, A_j \in \mathcal{F}$, we have $|A_i \cap A_j| = k$. Then,

$$|\mathcal{F}| = m \leq \cancel{n}\ n+1$$

Proof: Observation: At most one set in $\mathcal{F}$ is of size exactly k.

Case: One set of size k in $\mathcal{F}$.





A_5

Then, $|\mathcal{F}| \leq n-k+1$,

So this is an easy case

Case 2: No set in $\mathcal{F}$ is of size exactly k.

$A_1 \quad A_2 \ldots A_i \ldots A_m$

$v_1 \quad v_2 \ldots v_i \ldots v_m$

$\hookrightarrow$ 0-1 incidence vector of A_i

Suppose $A_i = \{3, 4, 7\}$

$v_{i1} \quad v_{i2} \quad v_{i3}$

n bits

$$v_i = (0, 0, 1, 1, 0, 0, 1, 0, 0, 0, 0 \ldots)$$

$\forall i \in [m]$, we define a function

$$f_i : \{0,1\}^n \longrightarrow \mathbb{R} \quad \text{as}$$

regular dot product

$$f_i(x) = \langle x, v_i \rangle - k \quad (1)$$

, where

$x = (x_1, x_2, \ldots, x_n) \in \{0,1\}^n$

Observe that,

$$f_i(v_i) = \langle v_i, v_i \rangle - k$$
$$= |A_i| - k$$
$$\neq 0 \quad \text{——} \quad (2)$$

$i \neq j$, $$f_i(v_j) = \langle v_i, v_j \rangle - k$$
$$= |A_i \cap A_j| - k$$
$$= k - k$$
$$= 0 \quad \text{——} \quad (3)$$

Combining (2) and (3) with

the independence criterion, we

can say that

$f_1, f_2, \ldots, f_m$ are Linearly Independent in the vector space $\mathbb{R}^{\{0,1\}^n}$ over $\mathbb{R}$ $\longrightarrow$ (4)

$\longrightarrow$ set of all functions from set of all functions from $\{0,1\}^n$ to $\mathbb{R}$.

Recall,

$$f_i(x) = \langle v_i, x \rangle - k$$

where $x = (x_1, x_2, \dots, x_n) \in \{0,1\}^n$.

$$= \langle (v_{i1}, v_{i2}, \dots, v_{in}), (x_1, x_2, \dots, x_n) \rangle - k$$

$$= v_{i1}x_1 + v_{i2}x_2 + \dots + v_{in}x_n - k$$

5

Claim The functions $f_1, f_2, \dots, f_m$ reside in the vector space generated by the following functions:

$$1, x_1, x_2, \dots, x_i, \dots, x_n$$

always evaluates to 1.

evaluates to 1 on every x having its 2nd bit equal to 1. Otherwise, it evaluates to zero.

Proof of Claim

To show: $\forall i \in [m]$,

$$f_i \in \operatorname{span}(\{1, x_1, x_2, \dots, x_n\})$$

From (5), we know that

$$f_i(x) = v_{i_1}x_1 + v_{i_2}x_2 + \cdots + v_{i_n}x_n - k$$

$$= v_{i_1}(x_1) + v_{i_2}(x_2) + \cdots + v_{i_n}(x_n) + (-k)(1)$$

This proves the claim.

So we have,

$$f_1, f_2, \ldots, f_m \in \operatorname{span}(\{1, x_1, \ldots, x_n\})$$

→ We know from (4) that $f_1, \ldots, f_m$ are L.I.

→ $$\dim\left(\operatorname{span}(\underbrace{\{1, x_1, \ldots, x_n\}}_{\substack{n+1 \\ \text{vectors}}})\right) \leq n+1$$

Since no. of L.I. vectors in any vector space is at most its dimension, we have,

$$|\mathcal{F}| = m \leq n+1$$

L-intersecting families

$L = \{ l_1, l_2, \dots, l_s \} \rightarrow$ a set of s non-negative integers

$\mathcal{F} = \{ A_1, A_2, \dots, A_m \}$ subsets of $[n]$ such that, for any $1 \le i < j \le m$,

$$|A_i \cap A_j| \in L.$$

Then, $$|\mathcal{F}| \le \sum_{i=0}^{s} \binom{n}{i}.$$

When $s = 1$, we have Fisher's Inequality