Derivation of the Liénard-Wiechert Potential

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The retarded potentials are given by the expression.

$$A^\mu(t,\mathbf{x}) = \frac{1}{4\pi\varepsilon_0 c^2}\int d^3\mathbf{x}'\,\frac{j^\mu(t',\mathbf{x}')}{|\mathbf{x}-\mathbf{x}'|}, \quad t' = t - \frac{|\mathbf{x}-\mathbf{x}'|}{c}. \tag{1}$$

Here we have combined the scalar potential ϕ and vector potential $\mathbf{A} = \left(A^1, A^2, A^3\right)$ into a single four component object (known as a 4-vector), $A^\mu = \left(A^0, A^1, A^2, A^3\right)$, with the zeroth component, $A^0 = \frac{\phi}{c}$. We have also combined the charge density, ρ and current density, $\mathbf{j}$ into another 4-vector, $j^\mu = (\rho\, c, \mathbf{j})$. The expression for the retarded potentials hold in the general case, but we will specialize to the case of the charged point particle with charge q. The position of the point charge is given by the equation of the trajectory, $\mathbf{x} = \boldsymbol{\zeta}(t)$ [1]. Here $\boldsymbol{\zeta}(t)$ is a vector function of time variable, t. Then the charge and current densities of the point charge is,

$$\rho(t,\mathbf{x}) = q\delta^3\left(\mathbf{x}-\boldsymbol{\zeta}(t)\right), \quad \mathbf{j}(t,\mathbf{x}) = q\dot{\boldsymbol{\zeta}}(t)\,\delta^3\left(\mathbf{x}-\boldsymbol{\zeta}(t)\right).$$

To simplify the treatment we just focus on the scalar potential, $A^0 = \frac{\phi}{c}$. Plugging the above forms for the charge density of the point charge in the retarded potential expression, we get,

$$\begin{aligned}\phi(t,\mathbf{x}) &= \frac{1}{4\pi\varepsilon_0}\int d^3\mathbf{x}'\,\frac{\rho(t',\mathbf{x}')}{|\mathbf{x}-\mathbf{x}'|}\\ &= \frac{1}{4\pi\varepsilon_0}\int d^3\mathbf{x}'\,\frac{q\delta^3\left(\mathbf{x}'-\boldsymbol{\zeta}(t')\right)}{|\mathbf{x}-\mathbf{x}'|}\end{aligned} \tag{2}$$

with t' being the retarded time, $t' = t - \frac{|\mathbf{x}-\mathbf{x}'|}{c}$. In order to perform the Dirac delta function integral, we have to first perform a change of integration variable from $\mathbf{x}'$ to

$$\mathbf{y}' = \mathbf{x}' - \boldsymbol{\zeta}(t').$$

The transformation of the volume element is,

$$d^3\mathbf{x}' = \frac{d^3\mathbf{y}'}{J}$$

where the Jacobian matrix is, $J_{ij} = \frac{\partial y_i'}{\partial x_j'}$ and the determinant is J. Let's compute the Jacobian matrix,

$$\begin{aligned}\frac{\partial y_i'}{\partial x_j'} &= \frac{\partial}{\partial x_j'}\left[x_i' - \zeta_i(t')\right]\\ &= \frac{\partial x_i'}{\partial x_j'} - \frac{\partial \zeta_i(t')}{\partial x_j'}\\ &= \delta_{ij} - \frac{\partial t'}{\partial x_j'}\,\frac{d\zeta_i(t')}{dt'}.\end{aligned}$$

[1]For instance if the point charge is undergoing uniform motion with velocity $\mathbf{v}$, then $\boldsymbol{\zeta}(t) = \boldsymbol{\zeta}_0 + \mathbf{v}\,t$, while if the point charge is undergoing uniformly accelerated motion with acceleration $\mathbf{a}$, then $\boldsymbol{\zeta}(t) = \boldsymbol{\zeta}_0 + \mathbf{u}t + \frac{1}{2}\mathbf{a}\,t^2$, etc.

In the last step we have used chain rule since the retarded time, $t' = t - \frac{|\mathbf{x}-\mathbf{x}'|}{c}$ is a function of $\mathbf{x}'$. We compute,

$$\frac{\partial t'}{\partial x'_j} = -\frac{1}{c}\frac{\partial|\mathbf{x}-\mathbf{x}'|}{\partial x'_j} = \frac{1}{c}\frac{x_j - x'_j}{|\mathbf{x}-\mathbf{x}'|},$$

and thus get,

$$\begin{aligned}\frac{\partial y'_i}{\partial x'_j} &= \delta_{ij} - \frac{\frac{d\zeta_i(t')}{dt'}}{c}\frac{\left(x_j - x'_j\right)}{|\mathbf{x}-\mathbf{x}'|} \\ &= \delta_{ij} - \frac{v_i(t')}{c}\frac{\left(x_j - x'_j\right)}{|\mathbf{x}-\mathbf{x}'|},\end{aligned}$$

where$v_i(t') = \dot{\zeta}_i(t)$ is the i-th component of the velocity of the point charge. Further introducing the unit vector, $\hat{\mathbf{n}} = \frac{\mathbf{x}-\mathbf{x}'}{|\mathbf{x}-\mathbf{x}'|}$, the Jacobian matrix can expressed as,

$$\frac{\partial y'_i}{\partial x'_j} = 1 - \frac{v_i(t')}{c} n_j.$$

From this one can easily check that the Jacobian determinant is

$$\begin{aligned}J = \left|\frac{\partial y'_i}{\partial x'_j}\right| &= \left|\begin{matrix} 1-\frac{v_1 n_1}{c} & -\frac{v_1 n_2}{c} & -\frac{v_1 n_3}{c} \\ -\frac{v_2 n_1}{c} & 1-\frac{v_2 n_2}{c} & -\frac{v_2 n_3}{c} \\ -\frac{v_3 n_1}{c} & -\frac{v_3 n_2}{c} & 1-\frac{v_3 n_3}{c} \end{matrix}\right| \\ &= 1 - \frac{v_1 n_1}{c} - \frac{v_2 n_2}{c} - \frac{v_3 n_3}{c} \\ &= 1 - \frac{\mathbf{v}\cdot\hat{\mathbf{n}}}{c}.\end{aligned}$$

After evaluating the Jacobian we are ready to tackle the Dirac delta function integral of (2) for the point charge scalar potential,

$$\begin{aligned}\phi(t,\mathbf{x}) &= \frac{1}{4\pi\varepsilon_0}\int d^3\mathbf{x}' \frac{q\delta^3\left(\mathbf{x}' - \boldsymbol{\zeta}(t')\right)}{|\mathbf{x}-\mathbf{x}'|} \\ &= \frac{1}{4\pi\varepsilon_0}\int \frac{d^3\mathbf{y}'{}'}{J}\frac{q\delta^3\left(\mathbf{y}'\right)}{|\mathbf{x}-\mathbf{x}'|_{\mathbf{x}'=\boldsymbol{\zeta}(t')}} \\ &= \frac{1}{4\pi\varepsilon_0}\int \frac{d^3\mathbf{y}'}{1-\frac{\mathbf{v}\cdot\hat{\mathbf{n}}}{c}}\frac{q\delta^3\left(\mathbf{y}'\right)}{|\mathbf{x}-\mathbf{x}'|_{\mathbf{x}'=\boldsymbol{\zeta}(t')}} \\ &= \frac{q}{4\pi\varepsilon_0}\frac{1}{\left(1-\frac{\mathbf{v}\cdot\hat{\mathbf{n}}}{c}\right)}\frac{1}{|\mathbf{x}-\mathbf{x}'|_{\mathbf{x}'=\boldsymbol{\zeta}(t')}}\underbrace{\int d^3\mathbf{y}\ \delta^3\left(\mathbf{y}'\right)}_{=1} \\ &= \frac{1}{4\pi\varepsilon_0}\frac{1}{\left(1-\frac{\mathbf{v}\cdot\hat{\mathbf{n}}}{c}\right)}\frac{q}{|\mathbf{x}-\mathbf{x}'|_{\mathbf{x}'=\boldsymbol{\zeta}(t')}}.\end{aligned} \tag{3}$$

Note here we should remember that first one needs to solve $\mathbf{x}' = \zeta\left(t - \frac{|\mathbf{x}-\mathbf{x}'|}{c}\right)$ to determine $\mathbf{x}'$ as a function of $t, \mathbf{x}$ and subsequently compute, $t' = t - \frac{|\mathbf{x}-\mathbf{x}'|}{c}$. Similarly we can show that the vector potential is given by,

$$\mathbf{A}(t,\mathbf{x}) = \frac{\mu_0}{4\pi}\frac{1}{\left(1-\frac{\mathbf{v}\cdot\hat{\mathbf{n}}}{c}\right)}\frac{q\mathbf{v}(t')}{|\mathbf{x}-\mathbf{x}'|_{\mathbf{x}'=\boldsymbol{\zeta}(t')}}.$$

Combining both using the 4-vector equation,

$$A^\mu(t,\mathbf{x}) = \frac{1}{4\pi\varepsilon_0 c^2}\frac{1}{\left(1-\frac{\mathbf{v}\cdot\hat{\mathbf{n}}}{c}\right)}\frac{qv^\mu(t')}{|\mathbf{x}-\mathbf{x}'|_{\mathbf{x}'=\boldsymbol{\zeta}(t')}}.$$

Here we have defined a four-dimensional velocity vector $v^\mu = (c, \mathbf{v})$. This is the Liénard-Wiechert potential for a point charge.