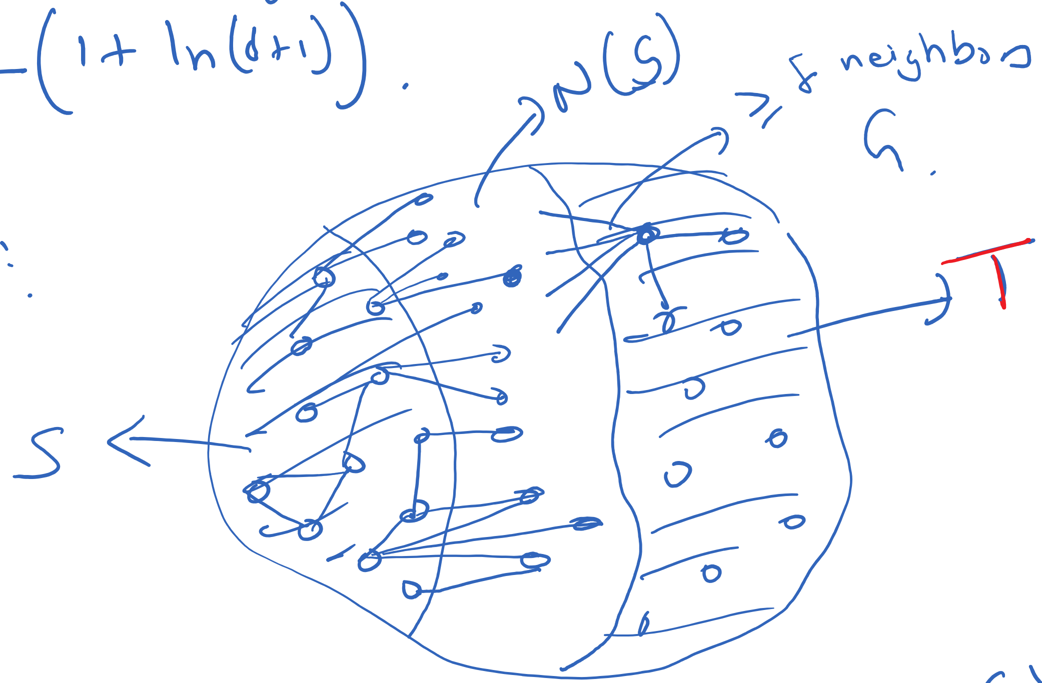
Theorem: Let G be a graph on n vertices with a minimum degree δ. Then, G has a dominating set of size at most $\frac{n}{\delta+1}\left(1+\ln(\delta+1)\right)$.

Proof:



Random exp: Constructing $S \subseteq V(G)$. Choose each vertex in G independently with probability p into the set S.

Defining Random Variables

X_S : denotes the size of S.

Y_T : denotes the size of T.

For each vertex $v \in V(G)$,

$x_v = \{ 1$ if $v \in S$

For each vertex $v \in V(G)$;

$$X_v = \begin{cases} 1, & \text{if } v \in S \\ 0, & \text{otherwise} \end{cases}$$

$$Y_v = \begin{cases} 1, & \text{if } \underline{\underline{v \in T}} \\ 0, & \text{otherwise.} \end{cases}$$

$$\boxed{\begin{array}{l} \Pr[X_v = 1] = p. \qquad \Pr[X_v = 0] = 1-p \\ E[X_v] = 1 \cdot p + 0 \cdot (1-p) = p. \end{array}} \quad (1)$$

$$\Pr[Y_v = 1] = (1-p)^{\underline{\underline{\deg(v)+1}}} \leq (1-p)^{\delta+1}$$

$$\Pr[Y_v = 0] = \_\_\_\_\_\_\_\_$$

$$E[Y_v] \leq 1 \cdot (1-p)^{\delta+1} + 0 \cdot \_\_\_\_\_\_\_\_$$

$$= \underline{\underline{(1-p)^{\delta+1}}} \quad \text{--- } (2)$$

$$X_S = \sum_{v \in V(G)} X_v \, , \qquad Y_T = \sum_{v \in V(G)} Y_v$$

By Linearity of Expectation,

$$E[X_S] = E\Big[\sum_{v \in V(G)} X_v\Big]$$

$$= \sum_{v \in V(G)} E[X_v]$$

$$= \underline{n \cdot p} \rightarrow (3)$$

By Linearity of Exp.,

$$E[Y_T] = E\Big[\sum_v Y_v\Big]$$

$$= \sum_{v \in V(G)} E[Y_v]$$

$$\le \underline{n(1-p)^{\delta+1}} \rightarrow (4)$$

Observe that $S \cup T$ is a dominating set for G. $|S \cup T| = |S| + |T|$

$$= X_S + Y_T$$

$$= Z$$

Let $Z = X_S + Y_T$ $\rightarrow$ another R.V.

$$E[Z] = E[X_S + Y_T]$$

$$= E[X_S] + E[Y_T] \quad \text{(by lin of exp)}$$

$$\le np + n(1-p)^{\delta+1} \quad \text{(from (3) and (4))}$$

$$= n\left(p + (1-p)^{\delta+1}\right)$$

$p := \frac{1}{2} \quad \frac{n}{2}$

$$\leq np + \frac{n}{e^{p(\delta+1)}} \qquad \left[\text{using } 1+x \leq e^x\right]$$

Substituting p with $\frac{\ln(\delta+1)}{\delta+1}$, we get

$\ln x := \log_e x$

$$E[Z] \leq \frac{n\ln(\delta+1)}{\delta+1} + \frac{n}{e^{\frac{\ln(\delta+1)\cancel{(\delta+1)}}{\cancel{(\delta+1)}}}}$$

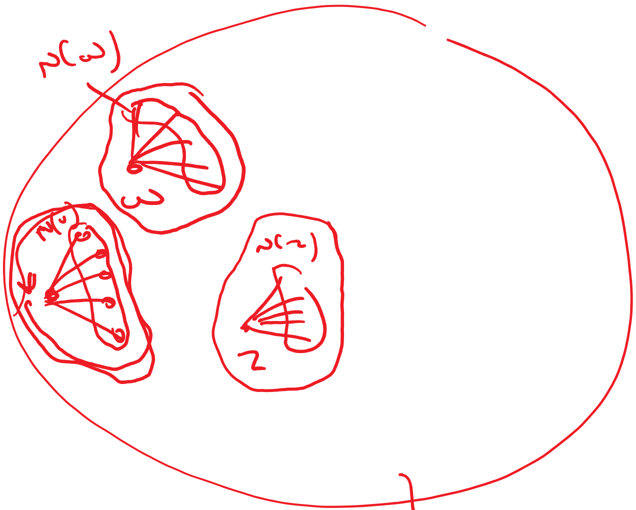
$$= \frac{n\ln(\delta+1)}{\delta+1} + \frac{n}{\delta+1}$$

$$= \frac{n}{\delta+1}\left(1 + \ln(\delta+1)\right)$$

$\square$

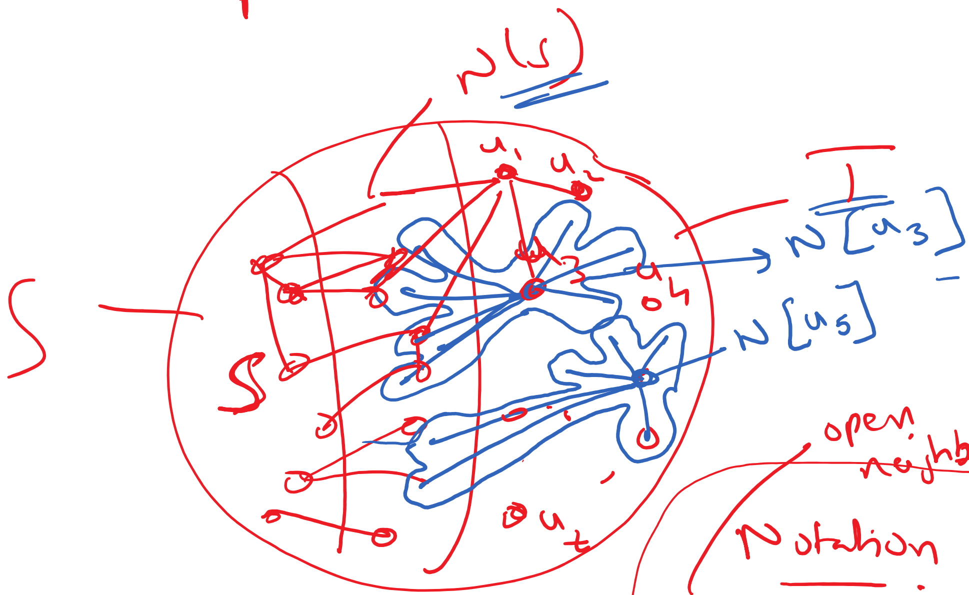
Deterministic algo for dominating set of size $\frac{n}{\delta+1}(1+\ln(\delta+1))$.

G.



$$|S| \le \frac{n}{\delta+1}\left(1+\ln(\delta+1)\right)$$

$S = \{v, w, z, \ldots\}$



open neighborhood

closed neighborhood

Notation

$N(v)$ = neighborhood of v

$N[v] = \{v\} \cup N(v)$.

$|N[v]| = |N(v)| + 1$

$|T| = t.$

$$\sum_{i=1}^{t} |N[u_i]| \ge t(\delta+1)$$

By pigeonhole principle, $\exists\ v \in V(G)$ that is present in $\geq \frac{t(\delta+1)}{n}$ such closed neighborhoods

such structures

Choose that v into S. This vertex v is dominating $\geq \frac{t(\delta+1)}{n}$ vertices of T.

$|\text{old } T| = t$

$$|\text{new } T| \leq t - \frac{t(\delta+1)}{n}$$

Initially

$|T| = n$.

After 1 round, $|T| \leq n - \frac{n(\delta+1)}{n} = n\left(1 - \frac{\delta+1}{n}\right)$
n_1

After 2 rounds, $|T| \leq n_1\left(1 - \frac{\delta+1}{n}\right) = n\left(1 - \frac{\delta+1}{n}\right)^2$

After k rounds

$$|T| \leq n\left(1 - \frac{\delta+1}{n}\right)^k$$

We want,

$$\leq \frac{n}{e^{\frac{(\delta+1)k}{n}}}$$

[use $1+x \leq e^x$]

$$e^{\frac{\ldots}{n}}$$

$$A \le \frac{n}{d+1}$$

$$\frac{n}{e^{\frac{(d+1)k}{n}}} \le \frac{n}{d+1}$$

This happens, when $k = \frac{n \ln(d+1)}{d+1}$

$$\frac{n}{e^{\frac{(\cancel{d+1})}{\cancel{n}} \frac{\cancel{n}\ln(d+1)}{(\cancel{d+1})}}} = \frac{n}{d+1}$$

What we have shown is that, after $k = \frac{n\ln(d+1)}{d+1}$ rounds, the no. of vertices left undominated (T) is $\le \frac{n}{d+1}$.

$$\bigcirc + \bigcirc = \frac{n}{d+1}\left(1 + \ln(d+1)\right)$$