EP 1027: Maxwell's Equations and Electromagnetic Waves

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Lecture 2

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Agenda: Algebra and Calculus of vectors and tensors

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- Differentiation of vector fields : Gradient, Divergence, Curl, Laplacian operators
- Integration of vector fields: Gauss' and Stokes' Theorem

References/Readings

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- Griffiths, D.J., **Introduction to Electrodynamics, Ch. 1**
- Boas, M. L., **Mathematical Methods... Ch. 6**
- Spiegel M.R., **Schaum's Outline of Vector Analysis**
- Schey, H.M., **Div, Grad, Curl and All that - An Informal Text on Vector Calculus**

Recap of Cartesian Vectors

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- Vectors:
  a) Quantity w/ both magnitude and a direction (directed line segment).
  b) behaves like position vector under addition and multiplication by a scalar, **and transforms like it under rotations**.

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- Notation: General vector - $\begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix}$. Call the components of a general vector: a_k, k can take values from 1 to 3, denoted by **a** or $\overrightarrow{a}$.

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- Rule for addition of 2 vectors: Add the respective components

$$\mathbf{a} + \mathbf{b} = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} + \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix} \equiv \begin{pmatrix} a_1 + b_1 \\ a_2 + b_2 \\ a_3 + c_3 \end{pmatrix}$$

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- Rule for multiplication with a number (scalar), λ : Multiply each component by λ

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- Once we choose the vector to be a column, then to denote the row vector, we will use the transpose

$$\mathbf{a}^T=(a_1, a_2, a_3)$$

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- Notation: In terms of basis (Unit) Vectors, $\hat{\mathbf{e}}_k$

$$
\begin{aligned}
\mathbf{a} = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} &= a_1 \underbrace{\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}}_{\hat{\mathbf{e}}_1} + a_2 \underbrace{\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}}_{\hat{\mathbf{e}}_2} + a_3 \underbrace{\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}}_{\hat{\mathbf{e}}_3} \\
&= a_1\, \hat{\mathbf{e}}_1 + a_2\, \hat{\mathbf{e}}_2 + a_3\, \hat{\mathbf{e}}_3 \\
&= \sum_{k=1}^{3} a_k\, \hat{\mathbf{e}}_k\,. \\
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&= \sum_{k=1}^{3} a_k\,\hat{\mathbf{e}}_k. \\
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- "Dot product/ inner product/ scalar product" of **a** and **b**:

$$\mathbf{a}.\mathbf{b} \equiv \mathbf{a}^T\mathbf{b} = a_1 b_1^{\text{'}} + a_2 b_2 + a_3 b_3 = \sum_{k=1}^{3} a_k b_k = a_k b_k = a_l b_m \delta_{lm}.$$

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$$\mathbf{a} \times \mathbf{b} \equiv (\epsilon_{ijk} a_i b_j)\, \hat{\mathbf{e}}_k,$$

i.e.

$$(\mathbf{a} \times \mathbf{b})_k = \epsilon_{ijk} a_i b_j,$$

where

$$\begin{aligned}
\epsilon_{123} &= \epsilon_{231} = \epsilon_{312} = 1, \\
\epsilon_{132} &= \epsilon_{213} = \epsilon_{321} = -1, \\
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ϵ_{ijk} = "Levi Civita Symbol" or "Completely antisymmetric symbol",

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  ϵ_{ijk} = "Levi Civita Symbol" or "Completely antisymmetric symbol",
- we derived: $(\mathbf{a} \times \mathbf{b})_1 = a_2 b_3 - a_3 b_2$.

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- Vector transformation rule under rotation of coordinate axes: Implemented through Matrix operations, $\mathbf{O}$

$$\mathbf{x}' = \mathbf{O}\mathbf{x},$$

$$\left(\begin{array}{c} x_1' \\ x_2' \\ x_3' \end{array} \right) = \left(\begin{array}{ccc} O_{11} & O_{12} & O_{13} \\ O_{21} & O_{22} & \dots \\ O_{31} & O_{32} & \dots \end{array} \right) \left(\begin{array}{c} x_1 \\ x_2 \\ x_3 \end{array} \right),$$

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- (anticlockwise) Rotation around z-axis/3-axis by θ,

$$\mathbf{O} = \begin{pmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

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- Tensor: Objects with components having multiple indices, $T_{i_1 \ldots i_p}$ (rank p tensor)
- Rank 2 tensor: Under coordinate axes rotation,

$$T'_{ij} = O_{il}\, O_{jm}\, T_{lm},$$

e.g., Outer product, $a_i\, b_j$; Kronecker delta, δ_{ij}; Moment of Inertia, $I_{ij} = m\left(\delta_{ij} x_k x_k - x_i x_j\right).$

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- Proof:

$$\begin{aligned}\delta'_{ij} &= O_{il}\, O_{jm}\, \delta_{lm} \\ &= O_{il}\, O_{jl} \\ &= O_{il}\, \left(O^T\right)_{lj} \\ &= \left(OO^T\right)_{ij} \\ &= \delta_{ij}.\end{aligned}$$

Invariant Cartesian Tensors: Levi-Civita

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- First Recall: Definition of determinant,

$$|M| = M_{1l}\, M_{2m} M_{3n}\, \epsilon_{lmn},$$

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- Levi-Civita tensor, ϵ_{ijk},

$$\begin{aligned} \epsilon'_{ijk} &= O_{il}\, O_{jm}\, O_{kn} \epsilon_{lmn} \\ &= |O|\, \epsilon_{ijk} \\ &= \epsilon_{ijk}, \end{aligned}$$

iff $|O| = +1$. Rotation matrix is indeed has unit determinant, thus Levi Civita is a tensor under rotation of coordinate axes.

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- Caveat: Under parity (inversion of coordinate axes) $|O| = -1$, and Levi Civita is then not a tensor.

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- Gradient of a scalar

$$\begin{aligned}
\phi(\mathbf{x}+d\mathbf{x})-\phi(\mathbf{x}) &= dx_1\,\frac{\partial\phi}{\partial x_1}+dx_2\,\frac{\partial\phi}{\partial x_2}+dx_3\,\frac{\partial\phi}{\partial x_3}\\
&= dx_i\,\frac{\partial\phi}{\partial x^i}\\
&= d\mathbf{x}\cdot(\boldsymbol{\nabla}\phi)\,,
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- Gradient operator, $\boldsymbol{\nabla}$,

$$\begin{aligned}\boldsymbol{\nabla} &\equiv \hat{\mathbf{e}}_k\partial_k\\ &= \hat{\mathbf{e}}_1\frac{\partial}{\partial x_1}+\hat{\mathbf{e}}_2\frac{\partial}{\partial x_2}+\hat{\mathbf{e}}_3\frac{\partial}{\partial x_3}.\end{aligned}$$

Tensor fields: Divergence and Curl

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- Since, $\boldsymbol{\nabla}$ acts like a vector, we can construct a scalar by taking the inner product, with a vector field, $\mathbf{A}(\mathbf{x})$,

$$\boldsymbol{\nabla} \cdot \mathbf{A}(\mathbf{x}) = \partial_k A_k(x).$$

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- Call it **Divergence of the vector field**,

$$\boldsymbol{\nabla} \cdot \mathbf{A} = \frac{\partial A_1}{\partial x_1} + \frac{\partial A_2}{\partial x_2} + \frac{\partial A_3}{\partial x_3}$$

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- We can further create a vector by taking the cross product of $\boldsymbol{\nabla}$ and $\mathbf{A}(\mathbf{x})$,

$$\boldsymbol{\nabla} \times \mathbf{A} = (\boldsymbol{\nabla} \times \mathbf{A})_k \, \hat{\mathbf{e}}_k,$$

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- Since, ∇ acts like a vector, we can construct a scalar by taking the inner product, with a vector field, $\mathbf{A}(\mathbf{x})$,

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- call it **Curl of the vector field**

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- $$\nabla \cdot \mathbf{x} = \frac{\partial x_1}{\partial x_1} + \frac{\partial x_2}{\partial x_2} + \frac{\partial x_3}{\partial x_3} = 3$$
- $$(\nabla \times \mathbf{x})_k = \epsilon_{ijk}\partial_i x_j = \epsilon_{ijk}\delta_{ij} = 0.$$
- Can define a double derivative thru the inner product, the **Laplacian**

$$\nabla \cdot \nabla = \frac{\partial^2}{(\partial x_1)^2} + \frac{\partial^2}{(\partial x_2)^2} + \frac{\partial^2}{(\partial x_3)^2}.$$

Tensor fields: Integration

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- **Gauss Divergence theorem**: If S is a closed surface enclosing a volume, V

$$\iiint_V d^3\mathbf{x}\, \boldsymbol{\nabla} \cdot \mathbf{A} = \oiint_S dS\, \hat{\mathbf{n}} \cdot \mathbf{A},$$

$\hat{\mathbf{n}}$ is the unit outward normal vector on the surface S.

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- **Stokes Curl theorem:** If S is an open surface, with a boundary, C (closed curve)

$$\iint_S dS\, \hat{\mathbf{n}} \cdot (\boldsymbol{\nabla} \times \mathbf{A}) = \oint_C d\mathbf{l} \cdot \mathbf{A}$$

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- Should be thought of as vector generalizations of *Fundamental theorem* of single variable calculus:

$$\int_a^b dx\,\frac{df(x)}{dx} = f(b) - f(a)$$

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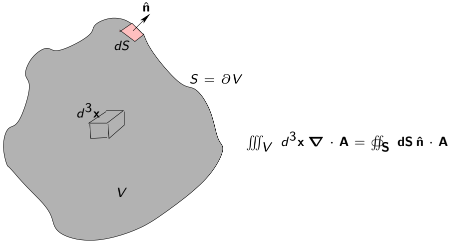

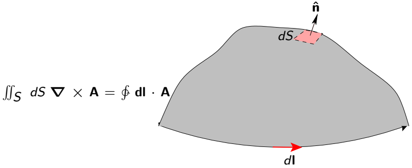

Figure: Pictorial representation of Gauss and Stokes Theorems.