EP 1027: Maxwell's Equations and Electromagnetic Waves

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Lecture 11

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Agenda

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- EM fields in Conductors: Skin depth & Skin Effect, Plasma frequency

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- Reflection from surface of a conductor

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- Reflection from surface of a conductor
- Dispersion: Cauchy's formula

References/Readings

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- Griffiths, D.J., **Introduction to Electrodynamics, Ch. 9**

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$$\boldsymbol{\nabla} \cdot \mathbf{B} = 0, \qquad \boldsymbol{\nabla} \times \mathbf{B} = \mu\sigma\mathbf{E} + \mu\epsilon\frac{\partial \mathbf{E}}{\partial t}$$

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- Continuity Equation: Conservation of free charge
$$\nabla \cdot \mathbf{j}_{free} + \frac{\partial \rho_{free}}{\partial t} = 0,$$
$$\implies \rho(t) = e^{-(\sigma/\epsilon)t}\rho(0).$$

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- Plane wave solutions with amplitude declining in space

$$\mathbf{E} = \tilde{\mathbf{E}}_0 e^{i(\tilde{k}z-\omega t)}, \qquad \mathbf{B} = \tilde{\mathbf{B}}_0 e^{i(\tilde{k}z-\omega t)}$$

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- $\tilde{k} = k + i\kappa$,

$$k = \omega\sqrt{\frac{\mu\epsilon}{2}}\left(\sqrt{1+\left(\frac{\sigma}{\epsilon\omega}\right)^2}+1\right)^{\frac{1}{2}} \approx \sqrt{\frac{\mu\sigma\omega}{2}},$$

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- Amplitude falls to $1/e$ after a distance $d = \frac{1}{\kappa} \approx \sqrt{\frac{2}{\mu\sigma\omega}}$, (**Skin depth**)

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- Phase lag of B wrt E
$$\varphi = \tan^{-1}\frac{\kappa}{k}.$$

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- For $\omega < \omega_p$, no wave propagation! Used in shortwave radio communication by bouncing off the ionosphere ($N \approx 10^{11}/m^3, \quad \omega_p \sim 20\ \mathrm{MHz}$)

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$$B_1^{\perp} = B_2^{\perp}, \qquad \frac{1}{\mu_2}\mathbf{B}_1^{\parallel} - \frac{1}{\mu_2}\mathbf{B}_2^{\parallel} = \mathbf{K}_{free} \times \hat{\mathbf{n}}$$

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- Say medium 1 is a dielectric and medium 2 is a conductor, then for normal incidence

$$\frac{\tilde{E}_{0R}}{\tilde{E}_{0I}} = \frac{1-\tilde{\beta}}{1+\tilde{\beta}}, \qquad \frac{\tilde{E}_{0T}}{\tilde{E}_{0I}} = \frac{2}{1+\tilde{\beta}}$$

where now $\tilde{\beta}$ is **complex**! To wit,

$$\tilde{\beta} = \frac{\mu_1 v_1}{\mu_2 \omega}\tilde{k}_2$$

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- Perfect conductor, $\sigma \to \infty$

$$\frac{\tilde{E}_{0R}}{\tilde{E}_{0I}} = -1, \qquad \frac{\tilde{E}_{0T}}{\tilde{E}_{0I}} = 0$$

  Conductors (metals) are excellent reflectors!

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- What causes dispersion: Theoretical picture of dispersion is a model based forced oscillation of electrons inside molecules of a substance when EM waves (light) propagate thru dielectrics

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\begin{aligned}
m\ddot{x} &= F_{restoring} + F_{damping} + F_{EM-Wave} \\
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- Suffices to consider 1d motion (radial motion). Driven (radial) oscillations given by (complex) position amplitude,

$$\tilde{x}_0 = \frac{q/m}{\omega_0^2 - \omega^2 - i\gamma\omega} E_0 e^{i\omega t}$$

- Displacement of electrons from the center of charge leads to creation of dipole moment,

$$\tilde{p} = q\,\tilde{x}_0$$

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$$\tilde{P}(\omega) = N \frac{q^2}{m} \tilde{E} \sum_j \frac{n_j}{\omega_j^2 - \omega^2 - i\gamma_j \omega},$$

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- Refractive Index as a result varies with ω

$$n = \sqrt{Re\left(\frac{\tilde{\epsilon}(\omega)}{\epsilon_0}\right)} = 1 + \frac{Nq^2}{2m\epsilon_0}\sum_j \frac{f_j\left(\omega_j^2 - \omega^2\right)}{\left(\omega_j^2 - \omega^2\right)^2 + \gamma_j^2\omega^2}.$$

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- For nomal dispersion, i.e. far from resonant we can neglect damping ($\gamma_j \to 0$), as a result,

$$n = 1 + A\left(1 + B\omega^2\right) = 1 + A\left(1 + \frac{B}{\lambda^2}\right)$$

Celebrated result due to Cauchy (Cauchy dispersion formula, verified in the lab)!