Logic

Logic

- “Every mother loves her children“

$$\forall X\ (\forall Y\ ((mother(X) \wedge child\_of(Y, X)) \supset loves(X, Y)))$$

$$\neg loves(tom, mary)$$

$$\exists X\ child\_of(X, mary)$$

ternary functor *family*, family/3

$$family(bill, mary, child(tom, child(alice, none)))$$

Definition 1.2 (Formulas) Let $\mathcal{T}$ be the set of terms over the alphabet $\mathcal{A}$. The set $\mathcal{F}$ of *wff* (with respect to $\mathcal{A}$) is the smallest set such that:

- if p/n is a predicate symbol in $\mathcal{A}$ and $t_1, \dots, t_n \in \mathcal{T}$ then $p(t_1, \dots, t_n) \in \mathcal{F}$;
- if F and $G \in \mathcal{F}$ then so are $(\neg F)$, $(F \wedge G)$, $(F \vee G)$, $(F \supset G)$ and $(F \leftrightarrow G)$;
- if $F \in \mathcal{F}$ and X is a variable in $\mathcal{A}$ then $(\forall X F)$ and $(\exists X F) \in \mathcal{F}$.

Model

Definition 1.9 (Model) An interpretation $\Im$ is said to be a *model* of P iff every formula of P is true in $\Im$. $\blacksquare$

When using formulas for describing \worlds" it is necessary to make sure that every description produced is *satisable* (that is, has at least one model), and in particular that the world being described is a model of *P*.

Generally, a satisable set of formulas has (infinitely) many models. This means that the formulas which properly describe a particular “world" of interest at the same time describe many other worlds.

Model

Definition 1.11 (Logical consequence) Let P be a set of closed formulas. A closed formula F is called a logical consequence of P (denoted $P \models F$) iff F is true in every model of P. $\blacksquare$

Proposition 1.13 (Unsatisfiability) Let P be a set of closed formulas and F a closed formula. Then $P \models F$ iff $P \cup \{\neg F\}$ is unsatisfiable. $\blacksquare$

One possible way to prove *P |= F* is to show that *-F* is false in every model of *P*,
or put alternatively, that the set of formulas *P U -F* is unsatisable (has no model).

Example 1.14 Let P be the formulas:

$$\forall X(r(X) \supset (p(X) \vee q(X))) \tag{8}$$

$$r(a) \wedge r(b) \tag{9}$$

To prove that $p(a)$ is not a logical consequence of P it suffices to consider an interpretation $\Im$ where $|\Im|$ is the set consisting of the two persons "Adam" and "Eve" and where:

$$\begin{array}{ll}
a_{\Im} := \text{Adam} & \\
b_{\Im} := \text{Eve} & \\
p_{\Im} := \{\langle \text{Eve} \rangle\} & \text{\% the property of being female} \\
q_{\Im} := \{\langle \text{Adam} \rangle\} & \text{\% the property of being male} \\
r_{\Im} := \{\langle \text{Adam} \rangle, \langle \text{Eve} \rangle\} & \text{\% the property of being a person}
\end{array}$$

Clearly, (8) is true in $\Im$ since "any person is either female or male". Similarly (9) is true since "both Adam and Eve are persons". However, $p(a)$ is false in $\Im$ since Adam is not a female. ∎

Definition 1.15 (Logical equivalence) Two formulas F and G are said to be logically equivalent (denoted $F \equiv G$) iff F and G have the same truth value for all interpretations $\Im$ and valuations φ. ∎

$$
\begin{array}{rcll}
\neg\neg F & \equiv & F & \\
F \supset G & \equiv & \neg F \vee G & \\
F \supset G & \equiv & \neg G \supset \neg F & \\
F \leftrightarrow G & \equiv & (F \supset G) \wedge (G \supset F) & \\
\neg(F \vee G) & \equiv & \neg F \wedge \neg G & \text{DeMorgan's law} \\
\neg(F \wedge G) & \equiv & \neg F \vee \neg G & \text{DeMorgan's law} \\
\neg\forall X\, H(X) & \equiv & \exists X \neg H(X) & \text{DeMorgan's law} \\
\neg\exists X\, H(X) & \equiv & \forall X \neg H(X) & \text{DeMorgan's law}
\end{array}
$$

Logical Inference

- Modus Ponens (Elimination rule for implication)
- Elimination rule for universal quantifier
- Introduction rule for conjunction

$$\frac{F \quad F \supset G}{G} \quad (\supset \mathrm{E})$$

$$\frac{\forall X\, F(X)}{F(t)} \quad (\forall\, \mathrm{E})$$

$$\frac{F \quad G}{F \wedge G} \quad (\wedge \mathrm{I})$$

Logical inference

$$\forall X\ (\forall Y\ (mother(X) \wedge child\_of(Y, X) \supset loves(X, Y))) \tag{10}$$

$$mother(mary) \wedge child\_of(tom, mary) \tag{11}$$

Elimination of the universal quantifier in (10) yields:

$$\forall Y\ (mother(mary) \wedge child\_of(Y, mary) \supset loves(mary, Y)) \tag{12}$$

Elimination of the universal quantifier in (12) yields:

$$mother(mary) \wedge child\_of(tom, mary) \supset loves(mary, tom) \tag{13}$$

Finally *modus ponens* applied to (11) and (13) yields:

$$loves(mary, tom) \tag{14}$$

- *Any formula F that can be derived from* premises P is said to be *derivable* from *P*. This is denoted by

Definition 1.16 (Soundness and Completeness) A set of inference rules are said to be *sound* if, for every set of closed formulas P and every closed formula F, whenever $P \vdash F$ it holds that $P \models F$. The inference rules are *complete* if $P \vdash F$ whenever $P \models F$. ∎

Definition 2.1 (Clause) A *clause* is a formula $\forall(L_1 \vee \cdots \vee L_n)$ where each L_i is an atomic formula (a positive literal) or the negation of an atomic formula (a negative literal). ∎

Definition 2.2 (Definite programs) A *definite program* is a finite set of definite clauses. ∎

Unification and SLD Resolution

to show existence of something, assume the contrary and use *modus ponens* and elimination of the universal quantier to find a counter-example for the assumption.

$$proud(X) \leftarrow parent(X,Y), newborn(Y).$$
$$parent(X,Y) \leftarrow father(X,Y).$$
$$parent(X,Y) \leftarrow mother(X,Y).$$
$$father(adam, mary).$$
$$newborn(mary).$$

Who is proud?".

$$P \models \exists Z\, proud(Z).$$

$$P \cup \{\neg\, proud(Z)\theta\}$$

$$P \models proud(Z)\theta.$$

$$\leftarrow proud(Z)$$

$$\forall Z\, \neg proud(Z)$$

$$\neg\, \exists Z\, proud(Z)$$

modus ponens with the elimination rule for the universal quantier.

SLD resolution and Logic

$$proud(X) \leftarrow parent(X,Y), newborn(Y).$$

$$\forall(\neg proud(X) \supset \neg(parent(X,Y) \wedge newborn(Y)))$$

$$\neg\,(parent(Z,Y) \wedge newborn(Y))$$

modus ponens

$$\leftarrow parent(Z,Y), newborn(Y).$$

$$\forall\, Z\, \forall\, Y\ (\neg parent(Z,Y) \vee \neg newborn(Y))$$

SLD resolution

$$\forall Z \, \forall Y \, (\neg parent(Z, Y) \vee \neg newborn(Y))$$

$$parent(X, Y) \leftarrow father(X, Y).$$ *unier*

$$\forall (\neg parent(X, Y) \supset \neg father(X, Y))$$

$$\leftarrow father(Z, Y), newborn(Y).$$

$$\leftarrow proud(Z).$$

$$proud(X) \leftarrow parent(X, Y), newborn(Y).$$

$$\leftarrow parent(Z, Y), newborn(Y).$$

$$parent(X, Y) \leftarrow father(X, Y).$$

$$\leftarrow father(Z, Y), newborn(Y).$$

$$father(adam, mary).$$

$$\leftarrow newborn(mary).$$

$$newborn(mary).$$

$$\square$$

atomic formula $p(s_1; : : : ; s_n)$ of the goal, program clause of the form $p(t1; : : : ; tn)$
substitution is constructed such that $p(s1; : : : ; sn)$ and $p(t1; : : : ; tn)$ are identical.

unifier

Interpretation and valuation

- A valuation ′ is a mapping from variables of the alphabet to the domain of an interpretation

Definition 1.3 (Interpretation) An interpretation $\Im$ of an alphabet $\mathcal{A}$ is a non-empty domain $\mathcal{D}$ (sometimes denoted $|\Im|$) and a mapping that associates:

- each constant $c \in \mathcal{A}$ with an element $c_\Im \in \mathcal{D}$;
- each n-ary functor $f \in \mathcal{A}$ with a function $f_\Im \colon \mathcal{D}^n \to \mathcal{D}$;
- each n-ary predicate symbol $p \in \mathcal{A}$ with a relation $p_\Im \subseteq \underbrace{\mathcal{D} \times \cdots \times \mathcal{D}}_{n}$.

Domain can be a nonempty set of individuals (with a number of relations and functions defined on this domain.
domain consists of three individuals | Mary, John and Tom.

Interpretation and Evaluation

Definition 1.4 (Semantics of terms) Let $\Im$ be an interpretation, φ a valuation and t a term. Then the *meaning* $\varphi_\Im(t)$ of t is an element in $|\Im|$ defined as follows:

- if t is a constant c then $\varphi_\Im(t) := c_\Im$;
- if t is a variable X then $\varphi_\Im(t) := \varphi(X)$;
- if t is of the form $f(t_1, \ldots, t_n)$, then $\varphi_\Im(t) := f_\Im(\varphi_\Im(t_1), \ldots, \varphi_\Im(t_n))$.

■

Consider a language which includes the constant *zero*, the unary functor *s* and the binary functor *plus*. Domain is the set of the natural numbers

$$zero_\Im := 0$$

$$\begin{aligned} s_\Im(x) &:= 1 + x \\ plus_\Im(x, y) &:= x + y \end{aligned}$$

$$\begin{aligned}
\varphi_{\mathfrak{I}}(plus(s(zero), X)) &= \varphi_{\mathfrak{I}}(s(zero)) + \varphi_{\mathfrak{I}}(X) \\
&= (1 + \varphi_{\mathfrak{I}}(zero)) + \varphi(X) \\
&= (1 + 0) + 0 \\
&= 1
\end{aligned}$$

$$\varphi(X) := 0$$

$$\mathfrak{I} \models_{\varphi} Q$$

The meaning of a formula is a truth value.

$$\forall X\,(\forall Y\,((\mathit{mother}(X) \land \mathit{child\_of}(Y,X)) \supset \mathit{loves}(X,Y)))$$
$$\mathit{mother}(\mathit{mary}) \land \mathit{child\_of}(\mathit{tom},\mathit{mary})$$
$$\mathit{loves}(\mathit{mary},\mathit{tom})$$

Example 1.8 Consider the interpretation $\Im$ that assigns:

- the persons Tom, John and Mary of the structure in Figure 1.1 to the constants *tom*, *john* and *mary*;
- the relations "... is a mother", "... is a child of ..." and "... loves ..." of the structure in Figure 1.1 to the predicate symbols $\mathit{mother}/1$, $\mathit{child\_of}/2$ and $\mathit{loves}/2$.

Using the definition above it is easy to show that the meaning of the formula:

$$\forall X\,\exists Y\,\mathit{loves}(X,Y)$$

is false in $\Im$ (since Tom does not love anyone), while the meaning of formula:

$$\exists X\,\forall Y\,\neg\mathit{loves}(Y,X)$$

is true in $\Im$ (since Mary is not loved by anyone). ∎

$$\forall X\,(\forall Y\,((mother(X) \wedge child\_of(Y, X)) \supset loves(X, Y)))$$
$$mother(mary) \wedge child\_of(tom, mary)$$
$$loves(mary, tom)$$

Example 1.12 To illustrate this notion by an example it is shown that (3) is a logical consequence of (1) and (2). Let $\Im$ be an arbitrary interpretation. If $\Im$ is a model of (1) and (2) then:

$$\Im \models \forall X(\forall Y((mother(X) \wedge child\_of(Y, X)) \supset loves(X, Y))) \tag{4}$$
$$\Im \models mother(mary) \wedge child\_of(tom, mary) \tag{5}$$

For (4) to be true it is necessary that:

$$\Im \models_{\varphi} mother(X) \wedge child\_of(Y, X) \supset loves(X, Y) \tag{6}$$

for any valuation φ — specifically for $\varphi(X) = mary_{\Im}$ and $\varphi(Y) = tom_{\Im}$. However, since these individuals are denoted by the constants *mary* and *tom* it must also hold that:

$$\Im \models mother(mary) \wedge child\_of(tom, mary) \supset loves(mary, tom) \tag{7}$$

Finally, for this to hold it follows that $loves(mary, tom)$ must be true in $\Im$ (by Definition 1.6 and since (5) holds by assumption). Hence, any model of (1) and (2) is also a model of (3). ∎