22 November 2020 19:00

Two distance set

Improved bound

Theorem: [Blokhuis, 1981] Let $a_1, a_2, \ldots, a_m$ be a two-distance set in $\mathbb{R}^n$. Then,

$$m \le \binom{n+2}{2}.$$

old bound

$$m \le \binom{n+2}{2} + n + 1$$
$$\parallel$$
$$\binom{n}{2} + 3n + 2$$

Proof: $a_1, a_2, \ldots, a_m \in \mathbb{R}^n$

Let d_1, d_2 where be the two possible distances btwn any two points

$$a_i = (a_{i1}, a_{i2}, a_{i3}, \ldots, a_{in}) \in \mathbb{R}^n$$
$$x = (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n$$

For each $i \in \{1, \ldots, m\}$, we define $f_i : \mathbb{R}^n \to \mathbb{R}$ as

$$f_i(x) = \frac{1}{d_1^2 d_2^2}\left(\|x - a_i\|^2 - d_1^2\right)\left(\|x - a_i\|^2 - d_2^2\right)$$

Clearly, $f_i(a_i) = 1$, and

$i \ne j$, $f_i(a_j) = 0$

By independence criterion, we know

By independence criterion, we know

$f_1, f_2, \dots, f_m$ are L.I. in

the v.s. $\mathbb{R}^{\mathbb{R}^n}$ over $\mathbb{R}$.

From last lecture, we know that each of $f_1, f_2, \dots, f_m$ can be expressed as a linear combination of the following functions:

$$\underbrace{\left(\sum_{i=1}^{n} x_i^2\right)^2}_{1} \to \|x\|^4, \quad \underbrace{\left(\sum_{i=1}^{n} x_i^2\right) x_j}_{n} \to \|x\|^2 x_j,\ 1 \le j \le n, \quad \underbrace{x_i x_j}_{\binom{n}{2}+n},\ 1 \le i \le j \le n, \quad \underbrace{x_i}_{n},\ 1 \le i \le n, \quad \underbrace{1}_{1}$$

$$\binom{n}{2} + 3n + 2$$

$$\parallel$$

$$\binom{n+2}{2} + n + 1$$

We know from last lecture that $f_1, f_2, \dots, f_m$ are L.I. and are present in the

$\ldots, f_m$ are L.I. and are present in the space spanned by the above $\binom{n+2}{2} + n+1$ functions.

Then,
$$\underbrace{(n+1)}_{} + m \le \binom{n+2}{2} + n+1$$

$$m \le \binom{n+2}{2}$$

add some new $(n+1)$ functions which

(i) together with $f_1, f_2, \ldots, f_m$ are L.I. in $\mathbb{R}^{\mathbb{R}^n}$ over $\mathbb{R}$,

and

(ii) each of them new $n+1$ functions reside in the space spanned by the above $\binom{n+2}{2} + n+1$ functions

new functions: $\underbrace{1, x_1, x_2, \ldots, x_n}_{n+1}$

Claim 1: Each of the new function reside in the space spanned by the above $\binom{n+2}{2} + n+1$ functions.

$\longrightarrow$ Easy to see.

$\longrightarrow$ Easy to see.

Claim 2: The functions $f_1, f_2, \ldots, f_m, x_1, x_2, \ldots, x_n, 1$ are L.I. in the V.S. $\mathbb{R}^{\mathbb{R}^n}$ over $\mathbb{R}$.

Proof of Claim 2

Consider the following equation

$$\sum_{i=1}^{m} \lambda_i f_i + \sum_{j=1}^{n} \beta_j x_j + \gamma \cdot 1 = 0 \quad \text{—— (1)}$$

In order to show that $f_1, f_2, \ldots, f_m, x_1, \ldots, x_n, 1$ are L.I., it is enough to show that

$\lambda_i = 0,\ 1 \le i \le m,\ \beta_j = 0,\ 1 \le j \le n,\ \gamma = 0.$

is the only solution to the above eqn.

Evaluating (1) at $x = a_i = (a_{i1}, a_{i2}, \ldots, a_{in})$, we get

$$\lambda_i + \sum_{j=1}^{n} \beta_j a_{ij} + \gamma = 0 \quad \text{—— (2)}$$

Evaluating (1) at $x = (0, 0, 0, \ldots, 0, \overset{j^{th}\text{ position}}{k}, 0, 0, \ldots, 0)$, we get

$$\frac{1}{d^2 d_i^2} \sum_{i=1}^{m} \Big(\lambda_i \left(k^2 - 2k a_{ij} + \|a_i\|^2 - d_1^2\right) \left(k^2 - 2k a_{ij} + \|a_i\|^2 - d^2\right) \Big)$$

$$\frac{1}{d_1^2 d_2^2} \sum_{i=1} \lambda_i \left(k - 2k a_{ij} \cdots \right) \left(k - 2k a_{ij} + \|a_i\| - d_2^2 \right) + k\beta_j + \gamma = 0 \quad (3)$$

Comparing the coefficients of k^4 and k^3 in (3), we get

$$\boxed{\sum_{i=1}^{m} \lambda_i = 0 \quad \text{and} \quad \underbrace{\sum_{i=1}^{m} \lambda_i a_{ij} = 0}_{\text{true for any } j \in \{1, \ldots, n\}}} \quad (4)$$

Multiplying (2) by λ_i and summing over all $i \in \{1, \ldots, m\}$, we get

$$\sum_{i=1}^{m} \lambda_i^2 + \underbrace{\sum_{j=1}^{n} \left(\beta_j \left(\sum_{i=1}^{m} \lambda_i a_{ij} \right) \right)}_{0} + \underbrace{\gamma \sum_{i=1}^{m} \lambda_i}_{0} = 0 \quad (5)$$

(4) and (5) imply

$$\sum_{i=1}^{m} \lambda_i^2 = 0$$

That means, $\lambda_i = 0 \;\; \forall i \in \{1, \ldots, m\}$.

This means, eqn (1) now becomes

$$\sum_{j=1}^{n} \beta_j x_j + \gamma = 0$$

This is true only when

$$\beta_j = 0, \forall j \in [n], \text{ and}$$

$$\gamma = 0.$$

This proves the claim and thereby the theorem

$\square$