MA 1140: Matrices and linear transformations

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How to compute row space and row rank of a matrix A?

- Apply elementary row operations on A to get its row reduced echelon form B. For example, $A = \begin{pmatrix} 1 & -2 & 1 & 1 \\ 2 & 1 & 1 & 3 \\ 0 & 5 & -1 & 1 \end{pmatrix}$ is row-equivalent to $B = \begin{pmatrix} 1 & 0 & \frac{3}{5} & \frac{7}{5} \\ 0 & 1 & -\frac{1}{5} & \frac{1}{5} \\ 0 & 0 & 0 & 0 \end{pmatrix}$.
- We observed in Lecture 7 that

$$\text{row space}(A) = \text{row space}(B).$$

- The row space of A is spanned by the non-zero rows of B, and the non-zero rows of B are linearly independent.
- So the row rank of A is the number of non-zero rows of B.
- In the above example, row space of A is

$$\text{Span}\left\{ \begin{pmatrix} 1 \\ 0 \\ 3/5 \\ 7/5 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ -1/5 \\ 1/5 \end{pmatrix} \right\}, \quad \text{hence} \;\; \text{row rank}(A) = 2.$$

How to compute null space and nullity of a matrix A?

- Apply elementary row operations on A to get its row reduced echelon form B.
- The systems $AX = 0$ and $BX = 0$ are equivalent systems.
- So $AX = 0$ and $BX = 0$ have the same set of solutions.
- Thus $\text{null space}(A) = \text{null space}(B)$.
- In particular, $\text{nullity}(A) = \text{nullity}(B)$.

Theorem

Let B be an $m \times n$ row reduced echelon matrix. Then the nullity of B is equal to the number of free variables, i.e., the total number of variables $-$ the number of pivots of $BX = 0$, i.e., $n -$ the number of pivots of B.

Let's verify the theorem for $B = \begin{pmatrix} 1 & 0 & \frac{3}{5} & \frac{7}{5} \\ 0 & 1 & -\frac{1}{5} & \frac{1}{5} \\ 0 & 0 & 0 & 0 \end{pmatrix}$ in the next slide.

How to solve $BX = 0$ when B is row-reduced echelon?

- Consider $B = \begin{pmatrix} {\color{red}1} & 0 & \frac{3}{5} & \frac{7}{5} \\ 0 & {\color{red}1} & -\frac{1}{5} & \frac{1}{5} \\ 0 & 0 & 0 & 0 \end{pmatrix}$, a row-reduced echelon matrix.
- The corresponding homogeneous system can be written as

$$ {\color{red}x_1} = -\frac{3}{5}x_3 - \frac{7}{5}x_4 \quad \text{and} \quad {\color{red}x_2} = \frac{1}{5}x_3 - \frac{1}{5}x_4. $$

where x_3, x_4 are free variables. So every solution has a linear combination of two (= number of free variables) vectors:

$$ \begin{pmatrix} -\frac{3}{5}x_3 - \frac{7}{5}x_4 \\ \frac{1}{5}x_3 - \frac{1}{5}x_4 \\ x_3 \\ x_4 \end{pmatrix} = x_3 \begin{pmatrix} -3/5 \\ 1/5 \\ 1 \\ 0 \end{pmatrix} + x_4 \begin{pmatrix} -7/5 \\ -1/5 \\ 0 \\ 1 \end{pmatrix} \text{ for } x_3, x_4 \in \mathbb{R}. $$

- These two vectors yield a BASIS of $\mathrm{Null}(B) \subseteq \mathbb{R}^2$.
- So $\mathrm{nullity}(B) = \dim(\mathrm{Null}(B)) = 2$.

Representation of a linear map $T : \mathbb{R}^n \to \mathbb{R}^m$ by a matrix

- Let $T : \mathbb{R}^n \to \mathbb{R}^m$ be a linear map (operator).
- We proved in Lecture 6 that T can be represented by an $m \times n$ matrix

  $$[T] := \begin{bmatrix} T(e_1) & T(e_2) & \cdots & T(e_n) \end{bmatrix},$$

  i.e., $T(X) = [T]X$ for every $X \in \mathbb{R}^n$, where $[T]X$ is the matrix multiplication.
- On the other hand, every $m \times n$ matrix A can be considered as a linear map $T_A : \mathbb{R}^n \to \mathbb{R}^m$, where T_A is defined by

  $$T_A(X) := AX \quad \text{for every } X \in \mathbb{R}^n.$$

- Verify that $T \overset{\Phi}{\mapsto} [T]$ and $A \overset{\Psi}{\mapsto} T_A$ are inverses of each other.
- **Conclusion:** There is a one to one correspondence between the set of all linear maps from $\mathbb{R}^n$ to $\mathbb{R}^m$ and the collection of all $m \times n$ matrices over $\mathbb{R}$.

Representation of a linear operator by a matrix

- Let $T : V \to V$ be a linear map (operator), and $n = \dim(V)$.
- Let $\mathcal{B} = \{v_1, v_2, \ldots, v_n\}$ be an ordered basis of V.
- For $v \in V$, if $v = x_1 v_1 + \cdots + x_n v_n$, then we denote

$$[v]_{\mathcal{B}} := \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}.$$

- Set $[T]_{\mathcal{B}} := \begin{bmatrix} [v_1]_{\mathcal{B}} & [v_2]_{\mathcal{B}} & \cdots & [v_n]_{\mathcal{B}} \end{bmatrix}$.
- We call $[T]_{\mathcal{B}}$ as the **matrix representation of T with respect to the ordered basis $\mathcal{B}$**.
- In this way, WE CAN HAVE a correspondence between the collection of all linear maps from V to itself and the collection of all $n \times n$ matrices. We will not study it further.

Thank You!