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Ray-Chaudhuri – Wilson Theorem

Recall Frankl–Wilson Theorem,

$L = \{l_1, l_2, \ldots, l_s\}$ — a set of non-negative integers

$\mathcal{F} = \{A_1, A_2, \ldots, A_m\}$ — a family of subsets of $\{1, \ldots, n\}$

$\mathcal{F}$ is L-intersecting.

$\hookrightarrow \forall i, j \in [m], i \neq j,$
$|A_i \cap A_j| \in L.$

Then $|\mathcal{F}| \leq \binom{n}{0} + \binom{n}{1} + \cdots + \binom{n}{s}.$

Ray-Chaudhuri Wilson Theorem [late 1970s or early 1980]

Theorem: Let $L = \{l_1, l_2, \ldots, l_s\}$ be a set of non-negative integers. Let $\mathcal{F}$ be a k-uniform family of subsets of $[n]$ that is L-intersecting. Then,

$$|\mathcal{F}| \leq \binom{n}{s}.$$

Proof:

Example

Example

$$\mathcal{F} = \{ A \subseteq [n] : |A| = s \}$$

$$L = \{0, 1, \dots, s-1\}, \quad |L| = s.$$

Here, $\mathcal{F}$ is s-uniform and L-intersecting.

$\to |\mathcal{F}| = \binom{n}{s}$.

Observation 1: $\forall \, l_i \in L, \quad l_i < k.$

Observation 2: $|L| = s \leq k.$

Let $\mathcal{F} = \{A_1, A_2, \dots, A_m\}$ — $v_1, v_2, \dots, v_m$ → n-bit 0-1 incidence vectors of $A_1, A_2, \dots, A_m$.

$\mathcal{F} \to$ k-uniform, L-intersecting

$L = \{l_1, l_2, \dots, l_s\}$.

To show: $|\mathcal{F}| = m \leq \binom{n}{s}$.

Let $\Omega = \{0,1\}^n$. For each $1 \leq i \leq m$, we define functions $f_i : \Omega \to \mathbb{R}$ as

$$f_i(x) = \prod_{l_j : l_j \in L} \left(\langle x, v_i \rangle - l_j \right) \quad \text{—— (1)}$$

$x = (x_1, x_2, \dots, x_n) \in \Omega.$

$$f_i(v_i) = \prod_{l_j : l_j \in L} \left(\underbrace{|A_i|}_{=k} - l_j \right)$$

From Obv 1, we know that $l_j < k, \forall j$.

So $f_i(v_i) \neq 0$.

For a $j \neq i$,

$$f_i(v_j) = \prod_{l_r : l_r \in L} \left(\underbrace{|A_i \cap A_j|}_{= \langle v_i, v_j \rangle} - l_r \right)$$

$$= 0.$$

So by independence criterion, functions $f_1, f_2, \ldots, f_m$ are L.I. in the v.s. $\mathbb{R}^\Omega$ over $\mathbb{R}$.

Expanding (1), we get

$$f_i(x) = \prod_{l_j \in L} \left(\langle x, v_i \rangle - l_j \right)$$

$$= \prod_{l_j \in L} \left(x_1 v_{i1} + x_2 v_{i2} + \cdots + x_n v_{in} - l_j \right)$$

$$= \left(x_1 v_{i1} + \cdots + x_n v_{in} - l_1 \right) \left(x_1 v_{i1} + \cdots + x_n v_{in} - l_2 \right) \cdots$$

$$\cdots \left(x_1 v_{i1} + \cdots + x_n v_{in} - l_s \right)$$

$\cdots\ (x_1 v_{i1} + \cdots + x_n v_{in} - \ell_s)$

Expand the above product and multi-linearize it.

$$\overset{\text{some constant}}{c} \cdot x_1^2 \, x_3^5 \cdot x_7^2 \, x_9$$

domain $\{0,1\}^n$

$$= c\, x_1 x_3 x_7 x_9$$

Claim:

The functions $f_1, f_2, \ldots, f_m$ lie in the space spanned by the following functions

$$\underbrace{x_I, \text{ where } I \subseteq [n] \text{ and } |I| \le s.}_{\binom{n}{0} + \binom{n}{1} + \cdots + \binom{n}{s}}$$

$I = \{1,3,7\}$

$x_I = x_1 x_3 x_7$

~~This means~~ that,

$$\underbrace{\quad}_{\binom{n}{0} + \cdots + \binom{n}{s-1}} + \underbrace{m}_{f_1, \ldots, f_m} \le \binom{n}{0} + \binom{n}{1} + \cdots + \binom{n}{s}.$$

Recall that, we want to show

$$m \le \binom{n}{s}.$$

What we need: To find a set of

$\binom{n}{0} + \binom{n}{1} + \cdots + \binom{n}{\,}$ functions from

$\binom{n}{0}+\binom{n}{1}+\cdots+\binom{n}{s-1}$ functions from

Ω to $\mathbb{R}$ that satisfy :-

(i) together with $f_1, \ldots, f_m$ they are L.I. in $\mathbb{R}^{\Omega}$ over $\mathbb{R}$, and

(ii) they reside in the span spanned by x_I, $I \subseteq [n]$, $|I| \leq s$.

Defining new functions that satisfy the above two properties

For every $I \subseteq [n]$, $|I| \leq s-1$,

$x_I g$ is a function from Ω to $\mathbb{R}$,

where $g(x) = \left(\sum_{i=1}^{n} x_i - k \right)$.

Claim A

For every $I \subseteq [n]$, with $|I| \leq s-1$,

we have $g(V_I) \neq 0$.

$\hookrightarrow$ Follows directly from Observation 2.

Combining Claim (A) with Swallowing Lemma we get

the collection $x_I g$ $\quad I \subseteq [n],\ |I| \leq s-1$ is L.I. in $\mathbb{R}^{\Omega}$ over $\mathbb{R}$.

$$\binom{n}{0} + \binom{n}{1} + \dots + \binom{n}{s-1}.$$

$$x_I g = x_I \left(\sum_{i=1}^{n} x_i - k \right)$$

$$= \underbrace{\underbrace{x_1 x_2 \cdots}_{\leq s-1} \left(\sum_{i=1}^{n} x_i - k \right)}_{}$$

Can be expressed as a linear combination of x_I, $|I| \leq s$, $I \subseteq [n]$.

So these new functions satisfy Property (ii).

$$1 \leq i \leq m,\ f_i(x) = \prod_{l_j : l_j \in L} \left(\langle x, v_i \rangle - l_j \right)$$

$g(v_i) = 0$ an incidence vector of A_i

$\binom{n}{\cdot}$

$g(v_i) = 0 \quad \forall i$

$x_I g$, $I \subseteq [n]$, $|I| \le s-1$, where $g(x) = \left(\sum_{i=1}^{n} x_i - k\right)$

Claim. Functions $f_1, f_2, \dots, f_m$ together with $x_I g$, $I \subseteq [n]$, $|I| \le s-1$ are L.I. in $\mathbb{R}^n$ over $\mathbb{R}$

Proof of Claim.

To show

$$\left(\sum_{i=1}^{m} \lambda_i f_i + \underbrace{\sum_{\substack{I \subseteq [n],\\ |I| \le s-1}} \lambda_I x_I g}_{A} = 0\right) \implies \left(\lambda_i = 0,\ \forall i \in [m] \text{ and } \lambda_I = 0,\ \forall I \subseteq [n],\ |I| \le s-1\right)$$

Evaluate over any v_1, $\rightarrow$ 0-1 n-bit incidence vector of A_1.

$$\lambda_1 f_1(v_1) = 0$$

Since $f_1(v_1) \neq 0$, we set

$$\lambda_1 = 0.$$

$$\lambda_1 = 0.$$

Similarly, evaluating over $v_2, v_3, \ldots, v_m$, we get

$$\lambda_2 = 0, \lambda_3 = 0, \ldots, \lambda_m = 0.$$

So Eqn (A) becomes

$$\sum_{\substack{I \subseteq [n], \\ |I| \leq s-1}} \lambda_I x_I g = 0$$

Now, by Swallowing Lemma we have

$$\lambda_I = 0, \quad \forall I \subseteq [n], \; |I| \leq s-1$$

This proves the claim and thereby the theorem.

Since Properties (i) and (ii) are satisfied, we have

$$\sum_{i=0}^{s-1} \binom{n}{i} + m \leq \sum_{i=0}^{s} \binom{n}{i}$$

$$\therefore m \leq \binom{n}{s}$$

$$\therefore \ m \le \binom{n}{r} \qquad \square$$