Set of problems
MA1140: Elementary Linear Algebra
8th February - 8th March, 2019

Before trying to solve each exercise, first you should be familiar with the terminologies, definitions and basic theory on that. You may read the lecture notes or lecture slides. I have written the solutions or hints for most of the exercises. But if you want to learn the subject, it is better to try on your own before seeing the solution. Please verify each and everything, as there might be mistakes.

1 Matrices, Linear equations and solvability

Q.1. Solve (if solution exists) the following system of linear equations over $\mathbb{R}$:

$$\begin{array}{lllll} u & +v & +w & +z & =6 \\ u & & +w & +z & =4 \\ u & & +w & & =2 \end{array}$$

What is the intersection if the fourth plane $u = -1$ is included? Find a fourth equation that leaves us with no solution.

Solution. Apply elementary row operations on the augmented matrix to obtain:

$$\begin{bmatrix} 1 & 1 & 1 & 1 & 6 \\ 1 & 0 & 1 & 1 & 4 \\ 1 & 0 & 1 & 0 & 2 \end{bmatrix} \Longrightarrow \begin{bmatrix} 1 & 1 & 1 & 1 & 6 \\ 0 & 1 & 0 & 0 & 2 \\ 0 & 0 & 0 & 1 & 2 \end{bmatrix} \text{ (upper triangular matrix, i.e., } A_{ij} = 0 \text{ for } i > j).$$

The corresponding triangular system is

$$\begin{aligned} u + v + w + z &= 6 \\ v \qquad &= 2 \\ z &= 2 \end{aligned}$$

The back substitution yields $z = 2$, $v = 2$ and $u + w = 2$. The set of solutions are given by

$$\begin{pmatrix} 2-w \\ 2 \\ w \\ 2 \end{pmatrix} = \begin{pmatrix} 2 \\ 2 \\ 0 \\ 2 \end{pmatrix} + w \begin{pmatrix} -1 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \quad \text{where } w \text{ varies in } \mathbb{R}.$$

If the fourth plane $u = -1$ is included, then it can be observed that the original system is equivalent to the system $u + v + w + z = 6$, $v = 2$, $z = 2$ and $u = -1$. In this case, the system will have only one solution $u = -1$, $v = 2$, $z = 2$ and $w = 3$.

For the last part, you may include a fourth equation as $z = a$ for some scalar $a \neq 2$ in $\mathbb{R}$.

Remarks. The set of solutions of a non-homogeneous system does not form a subspace.

To solve a system, one may also reduce the augmented matrix to its row reduced echelon form. For instance, in the first part of Q.1, we may reduce the system as follows;

$$\begin{bmatrix} 1 & 1 & 1 & 1 & 6 \\ 1 & 0 & 1 & 1 & 4 \\ 1 & 0 & 1 & 0 & 2 \end{bmatrix} \Longrightarrow \begin{bmatrix} 1 & 0 & 1 & 0 & 2 \\ 0 & 1 & 0 & 0 & 2 \\ 0 & 0 & 0 & 1 & 2 \end{bmatrix} \quad \text{(row reduced echelon form).}$$

Now solve the corresponding system. Note that u, v, z are pivot variables, and w is a free variable (that means you can assign any value to w, and then u, v, z are uniquely determined by that value of w).

Q.2. Find two points on the line of intersection of the three planes $t = 0$, $z = 0$ and $x + y + z + t = 1$ in four-dimensional space.

Hint. It is just finding two solutions of the system: $t = 0$, $z = 0$ and $x + y + z + t = 1$.

Q.3. Explain why the system

$$\begin{aligned} u + v + w &= 2 \\ u + 2v + 3w &= 1 \\ v + 2w &= 0 \end{aligned}$$

is *singular* (i.e., it does not have solutions at all). What value should replace the last zero on the right side to allow the system to have solutions, and what are the solutions over $\mathbb{R}$?

Solution. Let us write b in place of the last zero on the right side. Then reduce the system:

$$\begin{bmatrix} 1 & 1 & 1 & 2 \\ 1 & 2 & 3 & 1 \\ 0 & 1 & 2 & b \end{bmatrix} \Longrightarrow \begin{bmatrix} 1 & 1 & 1 & 2 \\ 0 & 1 & 2 & -1 \\ 0 & 0 & 0 & b+1 \end{bmatrix},$$

which yields a triangular system, whose last equation is $0 = b + 1$. So the original system (i.e., when $b = 0$) does not have solutions. Moreover, the last zero on the right side should be replaced by -1 to allow the system to have solutions. When $b = -1$, then the system can be reduced to its row reduced echelon form as follows:

$$\begin{bmatrix} 1 & 1 & 1 & 2 \\ 1 & 2 & 3 & 1 \\ 0 & 1 & 2 & b \end{bmatrix} \Longrightarrow \begin{bmatrix} 1 & 1 & 1 & 2 \\ 0 & 1 & 2 & -1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \Longrightarrow \begin{bmatrix} 1 & 0 & -1 & 3 \\ 0 & 1 & 2 & -1 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

So u and v are pivot variables, and w is a free variable. The solutions of the system corresponding to the last augmented matrix (i.e., the system $u - w = 3$ and $v + 2w = -1$) are given by

$$\begin{pmatrix} w+3 \\ -2w-1 \\ w \end{pmatrix} = w \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix} + \begin{pmatrix} 3 \\ -1 \\ 0 \end{pmatrix}, \quad \text{where } w \text{ varies in } \mathbb{R}.$$

Q.4. Under what condition on x_1, x_2 and x_3 do the points $(0, x_1), (1, x_2)$ and $(2, x_3)$ lie on a straight line?

Solution. **1st approach**. The equation of the straight line passing through $(0, x_1)$ and $(1, x_2)$ is given by

$$\frac{y - x_1}{x - 0} = \frac{x_2 - x_1}{1 - 0}, \quad \text{i.e., } y - x_1 = x(x_2 - x_1).$$

It passes through $(2, x_3)$ if and only if $x_3 - x_1 = 2(x_2 - x_1)$, which is the desired condition.

2nd approach. The equation of a straight line in an euclidean plane is given by $ax + by = c$ for some scalars $a, b, c \in \mathbb{R}$. If all the three points lie in this line, then we have

$$\begin{aligned} bx_1 &= c \\ a + bx_2 &= c \\ 2a + bx_3 &= c \end{aligned} \quad \textit{i.e.,} \quad \begin{aligned} x_1 b - c &= 0 \\ a + x_2 b - c &= 0 \\ 2a + x_3 b - c &= 0 \end{aligned} \quad \textit{i.e.,} \quad \begin{pmatrix} 0 & x_1 & -1 \\ 1 & x_2 & -1 \\ 2 & x_3 & -1 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{matrix} 0 \\ 0. \\ 0 \end{matrix}$$

The desired condition is equivalent to that the above system has a non-trivial solution, which is equivalent to that the determinant of the coefficient matrix is zero, i.e., $(x_1 - x_3) + 2(x_2 - x_1) = 0$.

Remark. The 2nd approach can be treated as an application of system of linear equations.

Q.5. These equations are certain to have the solution $x = y = 0$. For which values of a is there a whole line of solutions?

$$ax + 2y = 0$$
$$2x + ay = 0$$

Solution. The determinant of the coefficient matrix is $a^2 - 4$. We have only these two possibilities: Case 1. $a^2 - 4 \neq 0$. In this case, the system has only the trivial **Solution.** Case 2. $a^2 - 4 = 0$. In this case, the system can be reduced as follows:

$$\begin{pmatrix} a & 2 \\ 2 & a \end{pmatrix} \xRightarrow{R2 \to R2 - \frac{a}{2} R1} \begin{pmatrix} a & 2 \\ 0 & 0 \end{pmatrix}.$$

Therefore the original system is equivalent to the system of one equation $ax + 2y = 0$, which is nothing but a line. So $a = \pm 2$ are the desired values of a.

Q.6. Are the following systems equivalent:

$$x - y = 0$$
$$2x + y = 0$$

and

$$3x + y = 0$$
$$x + y = 0$$

If so, then express each equation in each system as a linear combination of the equations in the other system.

Hint. **(1st part).** You may use the fact that for two systems $AX = 0$ and $BX = 0$, if A and B are row equivalent to the same row reduced echelon form C, then by transitivity, A and B are row equivalent, and hence $AX = 0$ and $BX = 0$ are equivalent.

(2nd part). Observe that writing $x - y = 0$ as a linear combination of $3x + y = 0$ and $x + y = 0$ is equivalent to write $\begin{pmatrix} 1 \\ -1 \end{pmatrix}$ as a linear combination of $\begin{pmatrix} 3 \\ 1 \end{pmatrix}$ and $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$. So you just have to find c and d such that $c\begin{pmatrix} 3 \\ 1 \end{pmatrix} + d\begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$, i.e., $\begin{pmatrix} 3 & 1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} c \\ d \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$. The solution of this system is $c = 1$ and $d = -2$. Therefore $\{x - y = 0\} = \{3x + y = 0\} + (-2)\{x + y = 0\}$ in the obvious sense. Similarly, you can find the other linear combinations.

Q.7. Set $A = \begin{pmatrix} 6 & -4 & 0 \\ 4 & -2 & 0 \\ -1 & 0 & 3 \end{pmatrix}$ and $X = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$. Find all the solutions of $AX = 2X$, i.e., all X such that $AX = 2X$, where $2X$ is just componenntwise scalar multiplication.

Hint. $AX = 2X$ can be written as $(A - 2I_3)X = 0$, where I_3 is the 3×3 identity matrix. Now it can be solved by applying elementary row operations on the coefficient matrix $A - 2I_3$.

Q.8. Prove that the interchange of two rows of a matrix can be accomplished by a finite sequence of elementary row operations of the other two types.

Hint. Try with a matrix consisting of two rows $\begin{bmatrix} R1 \\ R2 \end{bmatrix}$.

Q.9. Consider the system of equations $AX = 0$, where $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ is a matrix over $\mathbb{R}$. Prove the following statements.

(i) A is a zero matrix (i.e., all entries are zero) if and only if every pair (x_1, x_2) is a solution of $AX = 0$.

(ii) $\det(A) \neq 0$, i.e., $ad - bc \neq 0$ if and only if the system has only the trivial solution.

(iii) $\det(A) = 0$, i.e., $ad - bc = 0$ but A is a non-zero matrix (i.e., some entries are non-zero) if and only if there is $(y_1, y_2) \neq (0, 0)$ in $\mathbb{R}^2$ such that every solution of the system is given by $c(y_1, y_2)$ for some scalar c.

Hint. (i) Observe that $AX = 0$ for every $X \in \mathbb{R}^2$ if and only if $x_1 \begin{pmatrix} a \\ c \end{pmatrix} + x_2 \begin{pmatrix} b \\ d \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ for all $x_1, x_2 \in \mathbb{R}$ if and only if $A = 0$ (to show that you may take $(x_1, x_2) = (1, 0)$ or $(0, 1)$).

(ii) and (iii). We already have proved (ii) in the class. Or directly, you can try to reduce the coefficient matrix to its row reduced echelon form. Observe that $ad - bc \neq 0$ if and only if A is reduced to the identity matrix if and only if the system has only the trivial **Solution.** For (iii), when A is a non-zero matrix, then $ad - bc = 0$ if and only if A is row equivalent to $\begin{pmatrix} a & b \\ 0 & 0 \end{pmatrix}$ if and only if the system $AX = 0$ is equivalent to $ax + by = 0$; see the solution of Q.5. The rest is left as an exercise. Now you may try to understand the statement in (iii) geometrically.

Q.10. Prove that if two homogeneous systems each of two linear equations in two unknowns have the same solutions, then they are equivalent.

Hint. You may use Q.9.

Q.11. For the system

$$\begin{aligned} u + v + w &= 2 \\ 2u + 3v + 3w &= 0 \\ u + 3v + 5w &= 2, \end{aligned}$$

what is the triangular system after forward elimination, and what is the solution (by back substitution)? Also solve it by computing the equivalent system whose coefficient matrix is in row reduced echelon form. Verify whether both the solutions are same.

Hint. You may follow the steps described as in the solution of Q.1.

Q.12. Describe explicitly all 2×2 row reduced echelon matrices.

Hint. Consider three cases that the number of non-zero rows of the matrix can be $0, 1$ or 2. When it is 1, then we will have two subcases. Think about the pivot positions.

Q.13. Let $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ be a matrix over $\mathbb{R}$. Suppose that A is row reduced and also that $a+b+c+d = 0$. Prove that there are exactly three such matrices.

Q.14. Let $A = \begin{pmatrix} 1 & -1 & 1 \\ 2 & 0 & 1 \\ 3 & 0 & 1 \end{pmatrix}$. Find some elementary matrices $E_1, E_2, \ldots, E_k$ such that $E_k \cdots E_2 E_1 A = I_3$, where I_3 is the 3×3 identity matrix. Deduce A^{-1}.

Hint. Apply elementary row operations on $(A \,|\, I_3)$ to get A^{-1}, and keep track of the row operations to get the corresponding $E_1, E_2, \ldots, E_k$.

2 Vector spaces

Throughout, V is a vector space over $\mathbb{R}$, the set of real numbers. Note that in most of the cases, it is convenient to write a vector in $\mathbb{R}^n$ as a column vector $\begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$. But sometimes we also write it as an n-tuple $(x_1, \ldots, x_n)$.

Q.15. Check whether $V = \mathbb{R}^2$ with each of the following operations is a vector space over $\mathbb{R}$.

(i) $\begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} x_1 \\ y_1 \end{pmatrix} = \begin{pmatrix} x + x_1 \\ y + y_1 \end{pmatrix}$ and $c\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} cx \\ y \end{pmatrix}$.

(ii) $\begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} x_1 \\ y_1 \end{pmatrix} = \begin{pmatrix} x + x_1 \\ 0 \end{pmatrix}$ and $c\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} cx \\ cy \end{pmatrix}$.

(iii) $\begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} x_1 \\ y_1 \end{pmatrix} = \begin{pmatrix} x + x_1 \\ y + y_1 \end{pmatrix}$ and $c\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} cx \\ 0 \end{pmatrix}$.

Hint. Verify all 10 properties in the definition of a vector space. If one of those properties does not hold true, try to give a counterexample for that.

Q.16. We call 'the' additive identity element of V as the zero vector. By definition, it is a vector $0 \in V$ such that $v + 0 = v$ for every $v \in V$. Its existence is there in the definition of 'vector space'. But, before saying it 'the' additive identity, can you prove its uniqueness?

Solution. Suppose θ_1 and θ_2 are two additive identity elements. By commutativity, $\theta_1 + \theta_2 = \theta_2 + \theta_1$. Since θ_1 is an additive identity, $\theta_2 = \theta_1 + \theta_2$. On the other hand, since θ_2 is an additive identity, $\theta_1 = \theta_1 + \theta_2$. Thus $\theta_1 = \theta_2$.

Q.17. Let $0 \in V$ be the zero vector. Let $c \in \mathbb{R}$. Show that $c \cdot 0 = 0$.

Solution. For the zero vector, we have $0 = 0 + 0$. So $c \cdot 0 = c \cdot (0 + 0) = c \cdot 0 + c \cdot 0$. Hence conclude that $c \cdot 0 = 0$. (Observe that for a vector $v \in V$, if $v = v + v$, then $v = 0$.)

Q.18. Let $v \in V$. Show that $0 \cdot v = 0$, where 0 in the right side is the zero vector, and 0 in the left side is the zero element of $\mathbb{R}$.

Solution. For the zero element in $\mathbb{R}$, $0 = 0 + 0$. So $0 \cdot v = (0 + 0) \cdot v = 0 \cdot v + 0 \cdot v$. Hence it follows that $0 \cdot v = 0$.

Q.19. Let W be a subspace of V. Show that (the zero vector) $0 \in W$.

Hint. Use Q.18 and the fact that W is non-empty.

Q.20. Which of the following sets of vectors $X = (x_1, \ldots, x_n)$ in $\mathbb{R}^n$ are subspaces of $\mathbb{R}^n$ $(n \geqslant 3)$?

(i) all X such that $x_1 \geqslant 0$;

(ii) all X such that $x_1 + 2x_2 = 3x_3$;

(iii) all X such that $x_1 = x_2^2$;

(iv) all X such that $x_1 x_2 = 0$;

(v) all X such that x_1 is rational.

Hint. To verify whether a subset of $\mathbb{R}^n$ is a subspace, you need to verify whether that subset is non-empty, and closed under vector addition and scalar multiplication.

Q.21. Let A be an $m \times n$ matrix over $\mathbb{R}$. Show that $\{X \in \mathbb{R}^n : AX = 0\}$ is a subspace of $\mathbb{R}^n$. It is called the **null space** of A. The dimension of the null space of A is called **nullity** of A.

Q.22. Prove that all the subspaces of $\mathbb{R}^1$ are 0 and $\mathbb{R}^1$.

Q.23. Prove that a subspace of $\mathbb{R}^2$ is either 0, or $\mathbb{R}^2$, or a subspace consisting of all scalar multiplies of some fixed non-zero vector in $\mathbb{R}^2$ (which is intuitively a straight line through the origin).

Hint. What are the possibilities of the dimension of a subspace of $\mathbb{R}^2$?

Q.24. (i) Let W_1 and W_2 be subspaces of V such that the set-theoretic union $W_1 \cup W_2$ is also a subspace of V. Prove that one of the subspaces W_1 and W_2 is contained in the other.
(ii) Can you give examples of two subspaces U_1 and U_2 of $\mathbb{R}^2$ such that $U_1 \cup U_2$ is not a subspace.

Solution. (i) You may prove the statement by way of contradiction. If possible, assume $W_1 \nsubseteq W_2$ and $W_2 \nsubseteq W_1$. So there are $w_1 \in W_1 \smallsetminus W_2$ and $w_2 \in W_2 \smallsetminus W_1$. Since $w_1, w_2 \in W_1 \cup W_2$, and $W_1 \cup W_2$ is a subspace, the sum $w_1 + w_2 \in W_1 \cup W_2$. Hence either $w_1 + w_2 \in W_1$ or $w_1 + w_2 \in W_2$. But $w_1 + w_2 \in W_1$ implies that $w_2 \in W_1$, which is a contradiction. Similarly, if $w_1 + w_2 \in W_2$, then $w_1 \in W_2$, which is again a contradiction. Thus we get a contradiction to the assumption that $W_1 \nsubseteq W_2$ and $W_2 \nsubseteq W_1$. Therefore either $W_1 \subseteq W_2$ or $W_2 \subseteq W_1$.

(ii) What about the union of two distinct lines passing through the origin in $\mathbb{R}^2$?

Q.25. Let W_1 and W_2 be subspaces of V such that $W_1 + W_2 = V$ and $W_1 \cap W_2 = 0$. Prove that for every vector $v \in V$, there are unique vectors $w_1 \in W_1$ and $w_2 \in W_2$ such that $v = w_1 + w_2$.

In this case, we write $V = W_1 \oplus W_2$, and call this as **direct sum** of W_1 and W_2.

Hint. Recall the definition of the sum of two (or more) subspaces.

Q.26. (i) Let A be an $m \times n$ matrix. Suppose B is obtained from A by applying an elementary row operation. Prove that row space(A) = row space(B).

(ii) Deduce from (i) that if any two $m \times n$ matrices A and B are row equivalent, then row space(A) = row space(B).

(iii) Let $B = \begin{bmatrix} R_1 \\ \vdots \\ R_r \\ 0 \\ \vdots \\ 0 \end{bmatrix}$ be an $m \times n$ row reduced echelon matrix with the non-zero rows $R_1, \ldots, R_r \in \mathbb{R}^n$ and the last $(m - r)$ zero rows. Prove that $\{R_1, \ldots, R_r\}$ is a basis of the row space of B.

(iv) Let A be an $m \times n$ matrix. Let A be reduced to a row reduced echelon matrix B. Then deduce from (ii) and (iii) that the non-zero rows of B gives a basis of the row space of A. Hence the row rank of A is same as the number of non-zero rows of B.

Hint. For (iii), the pivot positions will play a crucial role to show the linear independence.

Q.27. Let A be an $m \times n$ matrix. By applying elementary row operations, how can you find a basis of the column space of A?

Solution. Note that the column space of A is same as the row space of A^t (transpose of A). Then apply elementary row operations on A^t to get a row reduced echelon matrix, say B. Then, by Q.26(iv), the non-zero rows of B gives a basis of the row space of A^t, which is same as the column space of A.

Q.28. Consider the matrix $A = \begin{pmatrix} 2 & 1 & 1 & 6 \\ 1 & -2 & 1 & 2 \\ 0 & 5 & -1 & 2 \end{pmatrix}$. Find a basis of the row space of A. Deduce the row rank of A. Find a basis of the column space of A as well. Deduce the column rank of A. Verify whether row rank of A is same as column rank of A. Furthermore, find the null space of

A. Deduce the nullity of A. Verify the Rank-Nullity Theorem for the linear map $A : \mathbb{R}^4 \to \mathbb{R}^3$ defined by A. (This part of the exercise should belong to the next section.) What is the range space of this map? Find a basis of this range space, and deduce the rank of $A : \mathbb{R}^4 \to \mathbb{R}^3$.

Hint. Do not forget Q.26(iv) and Q.27 to obtain the row and column spaces of A. Moreover, observe that the range space of the map $A : \mathbb{R}^4 \to \mathbb{R}^3$ is same as the column space of A.

Remarks. Given an $m \times n$ matrix A. One obtains four fundamental spaces: row space, column space, null space and range space of A. Note that row space and null space are subspaces of $\mathbb{R}^n$; while column space and range space are subspaces of $\mathbb{R}^m$. Moreover, the column space and the range space of A are same.

Q.29. Consider some column vectors $v_1, \ldots, v_n \in \mathbb{R}^m$. By applying elementary row operations, how can you find a basis of the subspace $\mathrm{Span}(\{v_1, \ldots, v_n\})$ of $\mathbb{R}^m$?

Solution. Set $A := \begin{bmatrix} v_1 & v_2 & \cdots & v_n \end{bmatrix}$, an $m \times n$ matrix with the columns $v_1, \ldots, v_n \in \mathbb{R}^m$. Then the subspace $\mathrm{Span}(\{v_1, \ldots, v_n\})$ is same as the column space of A. Now follow the solution of Q.27.

Q.30. Let $A = \begin{pmatrix} 3 & -1 & 8 & 4 \\ 2 & 1 & 7 & 1 \\ 1 & -3 & 0 & 4 \end{pmatrix}$ and $X = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix}$. For which $Y = \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix}$ in $\mathbb{R}^3$ does the system $AX = Y$ have a solution? Describe the answer in terms of subspaces of $\mathbb{R}^3$. Use the following approaches, and verify whether you get the same answer.

Two approaches: **(1st).** Apply elementary row eliminations on $(A \,|\, Y)$, conclude when the system $AX = Y$ has solutions. **(2nd).** Note that for every $X \in \mathbb{R}^4$, AX is nothing but a linear combination of the four column vectors of A:

$$AX = x_1 \begin{pmatrix} 3 \\ 2 \\ 1 \end{pmatrix} + x_2 \begin{pmatrix} -1 \\ 1 \\ -3 \end{pmatrix} + x_3 \begin{pmatrix} 8 \\ 7 \\ 0 \end{pmatrix} + x_4 \begin{pmatrix} 4 \\ 1 \\ 4 \end{pmatrix}.$$

So Y should belong into the column space of A. Furthermore, you may try to find a basis of the column space of A. To do that follow Q.27.

Q.31. Let $S = \{v_1, \ldots, v_r\}$ be a collection of r vectors of a vector space V. Then show that S is linearly independent if and only if $\dim(\mathrm{Span}(S)) = r$.

Hint. See Corollary 2.26 in the lecture notes.

Q.32. Check whether the following vectors in $\mathbb{R}^4$ are linearly independent:

$$v_1 = \begin{pmatrix} 1 \\ 1 \\ 2 \\ 4 \end{pmatrix}, v_2 = \begin{pmatrix} 2 \\ -1 \\ -5 \\ 2 \end{pmatrix}, v_3 = \begin{pmatrix} 1 \\ -1 \\ -4 \\ 0 \end{pmatrix} \text{ and } v_4 = \begin{pmatrix} 2 \\ 1 \\ 1 \\ 6 \end{pmatrix}.$$

Two approaches: **(1st).** Consider a relation $x_1v_1 + x_2v_2 + x_3v_3 + x_4v_4 = 0$. It yields a homogeneous system of linear equations:

$$\begin{pmatrix} 1 & 2 & 1 & 2 \\ 1 & -1 & -1 & 1 \\ 2 & -5 & -4 & 1 \\ 4 & 2 & 0 & 6 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}.$$

Observe that this system has a non-trivial solution if and only if $\{v_1, v_2, v_3, v_4\}$ is linearly dependent. So you just need to check whether the system has only the trivial solution or not. For that, you may apply elementary row operations on the coefficient matrix.

(2nd). Set a matrix A whose rows are the vectors v_1, v_2, v_3, v_4. By Q.31, $\{v_1, v_2, v_3, v_4\}$ is linearly independent if and only if $\dim(\mathrm{Span}(\{v_1, v_2, v_3, v_4\})) = 4$. Since $\mathrm{Span}(\{v_1, v_2, v_3, v_4\})$ is nothing but the row space of A, we just have to compute row rank of A. So follow Q.26(iv).

Q.33. Let V be the vector space of all $m \times n$ matrices over $\mathbb{R}$ with usual vector addition and scalar multiplication. Show that $\dim(V) = mn$.

Hint. Consider $\{A^{ij} : 1 \leqslant i \leqslant m, 1 \leqslant j \leqslant n\}$, where A^{ij} is the $m \times n$ matrix with (i, j) entry 1 and all other entries 0. Is it a basis of V?

Q.34. Let V be the vector space of all $n \times n$ matrices over $\mathbb{R}$ with usual vector addition and scalar multiplication. Show that the following are subspaces of V.

(i) The subset of V consisting of all symmetric matrices.

(ii) The subset of V consisting of all skew-symmetric (or anti-symmetric) matrices.

(iii) The subset of V consisting of all upper triangular matrices (i.e., $A_{ij} = 0$ for all $i > j$).

What is the dimension of each of these subspaces?

Show that the following are not subspaces of V.

(iv) The subset of V consisting of all invertible matrices.

(v) The subset of V consisting of all non-invertible matrices.

(vi) The subset of V consisting of all matrices A such that $A^2 = A$.

Hint. (i), (ii) and (iii). To find the dimension of each subspace, construct a basis of that subspace. E.g., for (i), consider $\{D^{ii}, A^{ij} : 1 \leqslant i \leqslant n, i < j \leqslant n\}$, where D^{ii} is the $n \times n$ matrix with (i, i) entry 1 and all other entries 0, and A^{ij} is the $n \times n$ matrix with (i, j) and (j, i) entries 1 and all other entries 0. Show that it is a basis of the subspace of all symmetric matrices, hence the dimension is $n + (n - 1) + \cdots + 2 + 1 = n(n + 1)/2$. Similarly, show that the dimension of the subspaces described in (ii) and (iii) are $n(n - 1)/2$ and $n(n + 1)/2$.

(iv) Does it contain the zero vector?

(v) Show that it is not closed under addition.

(vi) Consider $A = B = I_n$. Note that $A^2 = A$ and $B^2 = B$, but $(A + B)^2 \neq (A + B)$. So it is not closed under addition.

3 Linear Transformations

Throughout, U, V and W are vector spaces over $\mathbb{R}$, the set of real numbers.

Q.35. Let $T : V \to W$ be a linear transformation. What is $T(0)$, where 0 is the zero vector in V?

Hint. Note that $T(v) = T(v + 0) = T(v) + T(0)$ for any $v \in V$. Conclude that $T(0) = 0$, the zero vector in W.

Q.36. Which of the following maps are linear? Justify your answer.

(i) $T : \mathbb{R}^1 \to \mathbb{R}^1$ defined by $T(x) = x + 2$ for every $x \in \mathbb{R}^1$.

(ii) $T : \mathbb{R}^1 \to \mathbb{R}^1$ defined by $T(x) = ax$ for every $x \in \mathbb{R}^1$, where $a \in \mathbb{R}$ is a constant.

(iii) $T : \mathbb{R}^1 \to \mathbb{R}^1$ defined by $T(x) = x^2$ for every $x \in \mathbb{R}^1$.

(iv) $T : \mathbb{R}^1 \to \mathbb{R}^1$ defined by $T(x) = \sin(x)$ for every $x \in \mathbb{R}^1$.

(v) $T : \mathbb{R}^1 \to \mathbb{R}^1$ defined by $T(x) = e^x$ for every $x \in \mathbb{R}^1$.

(vi) $T : \mathbb{R}^2 \to \mathbb{R}^1$ defined by $T(x_1, x_2) = x_1 x_2$ for every $(x_1, x_2) \in \mathbb{R}^2$.

(vii) $T : \mathbb{R}^2 \to \mathbb{R}^2$ defined by $T(x_1, x_2) = (x_2, x_1)$ for every $(x_1, x_2) \in \mathbb{R}^2$.

(viii) $T : \mathbb{R}^2 \to \mathbb{R}^2$ defined by $T(x_1, x_2) = (x_1, x_1)$ for every $(x_1, x_2) \in \mathbb{R}^2$.

(ix) $T : \mathbb{R}^2 \to \mathbb{R}^2$ defined by $T(x_1, x_2) = (0, x_1)$ for every $(x_1, x_2) \in \mathbb{R}^2$.

(x) $T : \mathbb{R}^2 \to \mathbb{R}^2$ defined by $T(x_1, x_2) = (0, 1)$ for every $(x_1, x_2) \in \mathbb{R}^2$.

Hint. Verify $T(cu + dv) = cT(u) + dT(v)$ for all scalars $c, d \in \mathbb{R}$, and vectors u, v in the domain of T. If this is not true, then find particular c, d, u and v for which the above equality fails.

Q.37. Let $u_1 = (1, 2)$, $u_2 = (2, 1)$, $u_3 = (1, -1)$ and $v_1 = (1, 0)$, $v_2 = (0, 1)$, $v_3 = (1, 1)$. Is there a linear map $T : \mathbb{R}^2 \to \mathbb{R}^2$ such that $T(u_i) = v_i$ for every $i = 1, 2, 3$?

Hint. A linear map should respect every linear combination.

Q.38. **Composition of linear maps:** Let $T : U \to V$ and $S : V \to W$ be linear maps. The composition $S \circ T : U \to W$ is defined by $(S \circ T)(u) := S(T(u))$ every $u \in U$. Show that the map $S \circ T : U \to W$ is linear.

Q.39. **Application of composition of maps:** Show that the matrix multiplication is associative.

Hint. Let A, B, C be matrices of order $k \times l$, $l \times m$ and $m \times n$ respectively. To show that $(AB)C = A(BC)$, treat matrices as (linear) maps, e.g., C can be treated as a (linear) map from $\mathbb{R}^n$ to $\mathbb{R}^m$. Note that there is a canonical one to one correspondence between matrices and linear maps. Next observe that the composition of maps is associative.

Q.40. Let $T : \mathbb{R}^n \to \mathbb{R}^n$ be a linear map. Is it true that if we know $T(v)$ for n different nonzero vectors in $\mathbb{R}^n$, then we know $T(v)$ for every vector in $\mathbb{R}^n$.

Hint. See what we have proved in Lecture 6. Try to analyze the statement when $n = 2$.

Q.41. Define a map $T : \mathbb{R}^3 \to \mathbb{R}^3$ by

$$T(x_1, x_2, x_3) = (a_{11}x_1 + a_{12}x_2 + a_{13}x_3, a_{21}x_1 + a_{22}x_2 + a_{23}x_3, a_{31}x_1 + a_{32}x_2 + a_{33}x_3)$$

for every $(x_1, x_2, x_3) \in \mathbb{R}^3$, where $a_{ij} \in \mathbb{R}$ are constants. Is T linear? If yes, then write its matrix representation.

Hint. See the theorem concerning matrix representation of a linear map $T : \mathbb{R}^n \to \mathbb{R}^m$ proved in Lecture 6.

Q.42. Deduce from Q.41 that the map $S : \mathbb{R}^3 \to \mathbb{R}^3$ defined by

$$S(x_1, x_2, x_3) = (x_1 - x_2 + 2x_3, 2x_1 + x_2, x_2 + x_3)$$

for every $(x_1, x_2, x_3) \in \mathbb{R}^3$ is linear. Compute the range space and null space of S. Deduce the rank and nullity of S. Verify the Rank-Nullity Theorem. Conclude from the rank (resp. from the nullity), whether S is an isomorphism.

Hint. Write the matrix representation (say, A) of the linear map S. Observe that the null space of S is same as that of A. Moreover, the range space of S is same as the column space of A. Now one may follow the solution of Q.28. Recall the equivalent conditions for a linear operator to be an isomorphism (shown in Lecture 7).

Left/right inverse of an $n \times n$ matrix A. We know that if A has a left-inverse B (i.e., $BA = I_n$) and a right-inverse C (i.e., $AC = I_n$), then the two inverses are equal: $B = B(AC) = (BA)C = C$. If this is the case, we say that A is invertible. From the row rank and the column rank of A, we can actually decide when A has a left/right inverse; see Q.43 and Q.44.

Q.43. For an $n \times n$ matrix A, prove that the following statements are equivalent:

(i) A has full column rank, i.e., column rank of A is n.

(ii) The system $AX = b$ has at least one solution X for every $b \in \mathbb{R}^n$.

(iii) The rank of the linear map $A : \mathbb{R}^n \to \mathbb{R}^n$ is n.

(iv) A has a right-inverse C, i.e., $AC = I_n$.

Hint. (i) $\Leftrightarrow$ (ii). Use that AX is nothing but a linear combination of the columns of A.

(ii) $\Leftrightarrow$ (iii). This is just an observation.

(ii) $\Leftrightarrow$ (iv). Prove the two implications one by one.

Q.44. For an $n \times n$ matrix A, prove that the following statements are equivalent:

(i) A has full row rank, i.e., row rank of A is n.

(ii) A has a left-inverse B, i.e., $BA = I_n$.

Hint. (i) $\Leftrightarrow$ (ii). Note that the row space of A is same as the column space of A^t (the transpose of A). So you may use the equivalence of (i) and (iv) in Q.43 for A^t.

Q.45. For an $n \times n$ matrix A, prove that the following statements are equivalent:

(i) A has a left-inverse.

(ii) A has a right-inverse.

(iii) A is invertible.

Hint. You may use Q.43, Q.44 and the fact that row rank(A) = column rank(A).

Q.46. Let $u_1 = (1, 2)$, $u_2 = (2, 1)$ and $v_1 = (1, 1)$, $v_2 = (0, 1)$. Is there a linear map $T : \mathbb{R}^2 \to \mathbb{R}^2$ such that $T(u_i) = v_i$ for every $i = 1, 2$? If yes, then write the matrix representation of T.

Q.47. Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be a linear map. Let u, v be two non-zero vectors such that $T(u) = 0$ and $T(v) = 0$. What are the possibilities of nullity of T? What about rank of T?

4 Eigenvalues and eigenvectors

Throughout, the base field $\mathbb{F}$ is either the field of real numbers, or the field of complex numbers. All matrices here are over $\mathbb{F}$. Let A be an $n \times n$ matrix, and $f(x) = a_r x^r + \cdots + a_1 x + a_0$ be a polynomial over $\mathbb{F}$. Then we write $f(A) = a_r A^r + \cdots + a_1 A + a_0 I_n$, which is an $n \times n$ matrix.

Q.48. Let A be an $n \times n$ matrix with THE eigenvalues $\lambda_1, \ldots, \lambda_n$ (not necessarily distinct). Prove that trace$(A) = \lambda_1 + \cdots + \lambda_n$ and $\det(A) = \lambda_1 \lambda_2 \cdots \lambda_n$. Conclude that A is invertible if and only if 0 is not an eigenvalue of A.

Hint. We have proved that the eigenvalues can be obtained as the roots of the characteristic polynomial, i.e., $\det(xI_n - A) = (x - \lambda_1) \cdots (x - \lambda_n)$. Now compare the constant terms and the coefficients of x^{n-1} from both sides.

Q.49. Find the eigenvalues and eigenvectors of the matrix $A = \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}$. Verify the statements stated in Q.48. Is A diagonalizable, i.e., is there an invertible matrix P such that $P^{-1}AP$ is a diagonal matrix. If yes, then what is that P, and what is that diagonal matrix?

Hint. First find all the eigenvalues λ by computing the characteristic polynomial and its roots. Then in order to get the eigenspace corresponding to each λ, solve the homogeneous system $(A - \lambda I_2)X = 0$. We just need to check whether there is a basis of $\mathbb{R}^2$ consisting of eigenvectors of A; see the theorem proved in Lecture 8.

Q.50. Let λ be an eigenvalue of an $n \times n$ matrix A with a corresponding eigenvector v.

(i) Show that λ is an eigenvalue of A^t (the transpose of A).

(ii) Show that v is an eigenvector of $B = A + cI_n$, where c is a fixed scalar. What is the corresponding eigenvalue of B?

(iii) Let r be a positive integer. Show that λ^r is an eigenvalue of A^r with the corresponding eigenvector v. Conclude that for every polynomial $f(x) \in \mathbb{F}[x]$, $f(\lambda)$ is an eigenvalue of $f(A)$ with the corresponding eigenvector v.

(iv) Let P be an $n \times n$ invertible matrix. Show that λ is an eigenvalue of $P^{-1}AP$ with a corresponding eigenvector $P^{-1}v$. Conclude from this statement that A and $P^{-1}AP$ have the same set of eigenvalues. Moreover, there is a one to one correspondence between the eigenvectors of A and that of $P^{-1}AP$ corresponding to every fixed eigenvalue λ.

(v) Suppose that A is invertible. Then, by Q.48, $\lambda \neq 0$. Show that v is also an eigenvector of A^{-1} with respect to the eigenvalue $1/\lambda$.

Hint. (i). It can be concluded from $\det(A - \lambda I_n) = \det(A^t - \lambda I_n)$.

(ii) and (iii). Verify directly by using the definition of eigenvalues and eigenvectors.

(iv). The 1st part can be verified directly. Using the 1st part, one also obtains that if λ is an eigenvalue of $P^{-1}AP$, then λ is an eigenvalue of $(P^{-1})^{-1}(P^{-1}AP)(P^{-1}) = A$, which concludes the 2nd part. By one to one correspondence, we mean bijective maps from both sides. Let E_λ and E'_λ be the eigenspaces of A and $P^{-1}AP$ corresponding to λ respectively. Define the maps $\varphi : E_\lambda \to E'_\lambda$ and $\psi : E'_\lambda \to E_\lambda$ by $\varphi(v) = P^{-1}v$ and $\psi(u) = Pu$ respectively. Clearly φ and ψ are inverse maps of each other.

(v). Apply A^{-1} on $(A - \lambda I_n)v = 0$.

Q.51. Let A be an $n \times n$ matrix with only one eigenvalue $\lambda \in \mathbb{F}$ (in other words, $\det(xI_n - A) = (x - \lambda)^n$). Let E_λ be the eigenspace of A corresponding to λ. (Note that E_λ is a subspace of $\mathbb{F}^n$.) Show that $\dim(E_\lambda) = n$ if and only if A is diagonalizable.

Hint. See the diagonalizability criteria proved in Lecture 8.

Q.52. Prove that the matrix $\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ is not diagonalizable.

Hint. You may use Q.51.

Q.53. Let A be an $n \times n$ matrix. Show that if A is diagonalizable, then A^r is also diagonalizable, where r is a positive integer.

Hint. Note $P^{-1}A^rP = (P^{-1}AP)(P^{-1}AP)\cdots(P^{-1}AP)$, multiplication of r many copies.

Q.54. Let A be a 2×2 matrix. Suppose A has two eigenvalues λ_1 and λ_2 in $\mathbb{F}$ such that $\lambda_1 \neq \lambda_2$. Prove that A is diagonalizable.

Hint/Fact: (*This part is optional.*) More generally, by induction on n, one can prove the following: Let A be an $n \times n$ matrix, and $n \geqslant 2$. Let $v_1, v_2, \dots, v_n$ be some eigenvectors correspond to distinct eigenvalues $\lambda_1, \lambda_2, \dots, \lambda_n$ respectively. Show that $v_1, v_2, \dots, v_n$ are linearly independent. Conclude that a matrix having all distinct eigenvalues is diagonalizable.

Q.55. Minimal polynomial. (*This exercise is optional.*) Let A be an $n \times n$ matrix over $\mathbb{F}$. Recall that a monic (i.e., with the leading coefficient 1) polynomial $p(x) \in \mathbb{R}[x]$ is said to be a minimal polynomial of A if $p(A) = 0$ (zero matrix) and p has minimal possible degree. Since a minimal polynomial is monic, it is a non-zero polynomial. Prove the following statements:

(i) *Existance.* The matrix A has a minimal polynomial $p(x)$.

(ii) *Uniqueness.* Let $f(x) \in \mathbb{R}[x]$ be such that $f(A) = 0$. Then a minimal polynomial $p(x)$ divides $f(x)$. Conclude that A has a unique minimal polynomial.

Solution. (i) By Cayley-Hamilton Theorem, there is a monic polynomial $h(x) \in \mathbb{F}[x]$ such that $h(A) = 0$. Set $\mathcal{B} := \{g(x) \in \mathbb{F}[x] : g(x) \text{ is monic and } g(A) = 0\}$. By Cayley-Hamilton Theorem, $\mathcal{B}$ is non-empty. So, by Well-Ordering property of the set of natural numbers, there is an element $p(x)$ in $\mathcal{B}$ of minimal possible degree. Then $p(x)$ is a minimal polynomial of A.

(ii) By division algorithm, there are $q(x)$ (quotient) and $r(x)$ (remainder) in $\mathbb{F}[x]$ such that

$$f(x) = p(x)q(x) + r(x), \quad \text{where } r(x) = 0 \text{ or } \deg(r(x)) < \deg(p(x)).$$

Hence it follows from $f(A) = 0$ and $p(A) = 0$ that $r(A) = 0$. Since $p(x)$ has minimal possible degree and $\deg(r(x)) < \deg(p(x))$, we have $r(x) = 0$. Therefore $p(x)$ divides $f(x)$. For the last part, if possible, suppose $p(x)$ and $p'(x)$ are two minimal polynomials of A. Then, by the 1st part, $p(x)$ divides $p'(x)$. For the same reason, $p'(x)$ also divides $p(x)$. Hence $p(x) = p'(x)$.

Q.56. Diagonalizability via minimal polynomial Let A be an $n \times n$ diagonalizable matrix over $\mathbb{F}$. Prove that the minimal polynomial of A has distinct roots in $\mathbb{F}$. (**Fact:** The converse is also true, i.e., a matrix having the minimal polynomial with distinct roots is diagonalizable. The proof is hard.)

Hint. There is P such that $P^{-1}AP$ is a diagonal matrix. It is proved in Lecture 8 that for a polynomial $f(x)$, we have $f(A) = 0$ if and only if $f(P^{-1}AP) = 0$. Conclude that A and $P^{-1}AP$ have the same minimal polynomial. So, without loss of generality, we may assume that A is a diagonal matrix. Then pick all the distinct diagonal entries, say $d_1, \dots, d_r$. Set $g(x) := (x - d_1) \cdots (x - d_r)$. Show that $g(A) = 0$. Hence conclude the statement by Q.55(ii).

Q.57. An application of Cayley-Hamilton Theorem: Let A be an $n \times n$ matrix. Suppose the characteristic polynomial of A is $f(x) = x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0 \in \mathbb{F}[x]$. Show that A is invertible if and only if $a_0 \neq 0$. Prove that when A is invertible, then

$$A^{-1} = \frac{-1}{a_0}(A^{n-1} + a_{n-1}A^{n-2} + \cdots a_2 A + a_1 I_n).$$

Hint. Note that $x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0 = f(x) = \det(xI_n - A)$, which yields that $a_0 = f(0) = (-1)^n \det(A)$. To compute A^{-1}, use the Cayley-Hamilton Theorem.