28 September 2020 11:52

A hypergraph $H(V, E)$ — V: vertex set, E: edge set

$$E \subseteq \text{power set}(V)$$

same as a family of subsets of V.

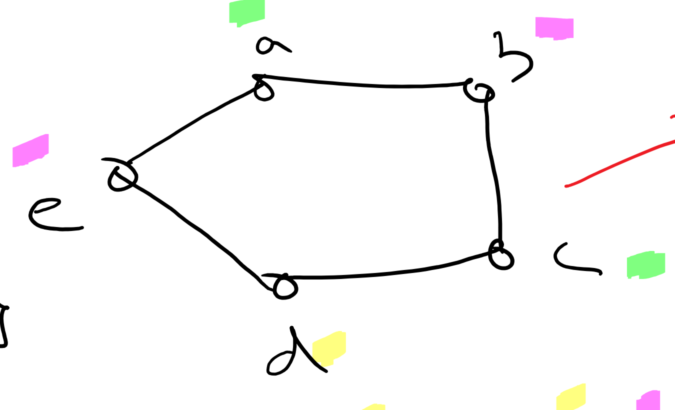
Example:

$$V = \{1, 2, 3, 4, 5\}$$

$$E = \{\{1,2,3\}, \{1,3\}, \{1,5\}, \{3,4,5\}\}$$

→ 2-colorable

Q. Color the points/vertices in V with as few colors as possible s.t. every hyperedge in E sees at least two colors.



→ not 2-colorable

$$V = \{a, b, c, d, e\}$$

$$E = \{\{a,b\}, \{b,c\}, \{c,d\}, \{d,e\}, \{e,a\}\}$$

A hypergraph $H(V,E)$ is k-uniform if every hyperedge is k-sized.

Clearly, graphs are 2-uniform hypergraphs

Theorem: Every k-uniform hypergraph $H(V,E)$ with less than 2^{k-1} hyperedges is 2-colorable.

Proof: Let $V = \{1, 2, \ldots, n\}$ be the vertex set of H.

For each vertex $i \in V$, independently and uniformly at random assign a color from the set {red, green}

Consider a hyperedge $e \in E$.

$$\text{Prob}\left[\text{all the vertices in } e \text{ get red color}\right] = \frac{1}{2^k} \quad \text{— A}$$

$$\text{Prob}\left[\text{" " " green color}\right] = \frac{1}{2^k} \quad \text{— B}$$

" green color $\Big] = \dfrac{1}{2^k}$ —— (B)

$$\text{Prob}\left[e \text{ is monochromatic}\right] = (A) + (B)$$

$$= \frac{1}{2^{k-1}} \longrightarrow (C)$$

Let $E = \{e_1, e_2, \ldots, e_m\}$

$$\Pr\left[(e_1 \text{ is monochromatic}) \vee (e_2 \text{ is monochromatic}) \vee \ldots (\;) \vee (e_m \text{ is monochromatic})\right]$$

$$\leq \Pr[e_1 \text{ is monochromatic}] + \Pr[e_2 \text{ is monochromatic}] + \ldots \Pr[e_m \text{ is monochromatic}]$$

(by Union bound)

$$= m \cdot \frac{1}{2^{k-1}} \quad (\text{from } (C))$$

$$< 2^{k-1} \frac{1}{2^{k-1}} \quad (\text{given: } m < 2^{k-1})$$

$$= \frac{1}{2^{k-1}}$$

$$\therefore \Pr\left[(e_1 \text{ is monochromatic}) \vee (\ \ ) \ldots \vee (e_m \text{ is monochromatic}) \right] \not> 0$$

i.e. $$\Pr\left[\begin{pmatrix} e_1 \text{ is not} \\ \text{monochromatic} \end{pmatrix} \wedge (\ \ ) \ldots \wedge \begin{pmatrix} e_m \text{ is not} \\ \text{monochromatic} \end{pmatrix} \right] > 0.$$

$\square$

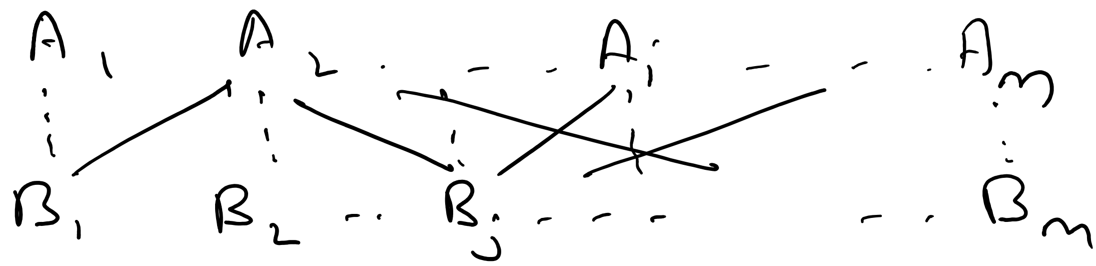
Bollobas's Theorem

Theorem Let $(A_1, A_2, \dots, A_m)$ and $(B_1, B_2, \dots, B_m)$ be two sequences of sets such that $\forall i, j \in [m]$, $A_i \cap B_j = \phi$ if and only if $i = j$. Then,

$$m \leq \binom{a+b}{a},$$

where $\forall i \in [m]$, $|A_i| = a$, $|B_i| = b$.

Proof:



Let $X = \bigcup_{i=1}^{m} (A_i \cup B_i)$. Suppose $|X| = n$.

Uniformly at random choose a linear order/permutation σ of X.

Recall original proof.

σ: → linear order of elmts of X.

(A_i, B_i) is "present" in σ if every element of A_i

σ of X.

if every element of A_i precedes every element of B_i in σ.

Obv: At most one pair (A_i, B_i) can be present in any given permutation σ.

$$Pr\left[\overbrace{(A_i, B_i) \text{ pair is present in } \sigma}^{X_i}\right] = \frac{1}{\binom{a+b}{a}} \quad \text{— (1)}$$

σ: 

• — $|A_i| = a$

• — $|B_i| = b$

$$\frac{a_i!\, b_i!}{(a_i+b_i)!} = \frac{1}{\binom{a+b}{a}}$$

$\forall\, i, j \in [m]$, $i \neq j$,

$$Pr[X_i \cup X_j] = Pr[X_i] + Pr[X_j] \underbrace{\cancel{- Pr[X_i \cap X_j]}}_{0}$$

Therefore,

$$1 \geq \Pr[X_1 \cup X_2 \cup \cdots \cup X_m] = \sum_{i=1}^{m} \Pr[X_i] = \frac{m}{\binom{a+b}{a}}$$

[from ①]

$$\therefore \quad m \leq \binom{a+b}{a}$$

$\square$