Lecture 13

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Plan

- Minimum spanning trees
- Proof of Correctness of Kruskal's Algorithm
- Union-Find Data Structure
  (Disjoint Set Data Structure)

Spanning Tree

Definition: An undirected graph G is *connected* if every vertex is reachable from every other vertex.

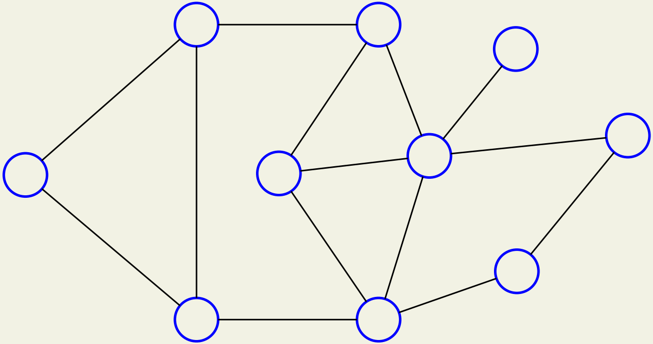
A graph $T = (V, E')$ is a spanning tree of an undirected connected graph $G = (V, E)$ if:

- $E' \subseteq E$.
- T is a *tree*. i.e., T is an acyclic and connected.

Informally: A spanning tree for G is a tree that can be found inside G which *spans* all vertices of G.

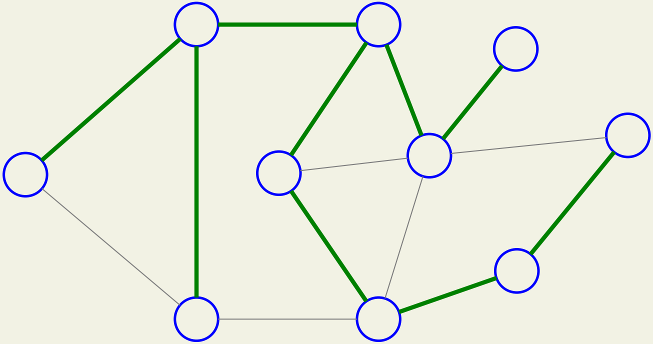
Examples

What are the possible spanning trees for this graph?



Examples

Is this a spanning tree?



Examples

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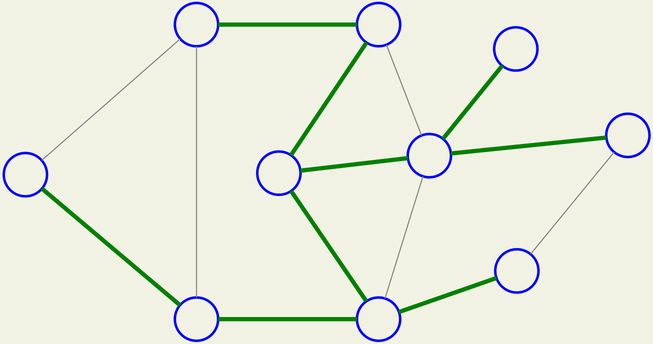

Examples

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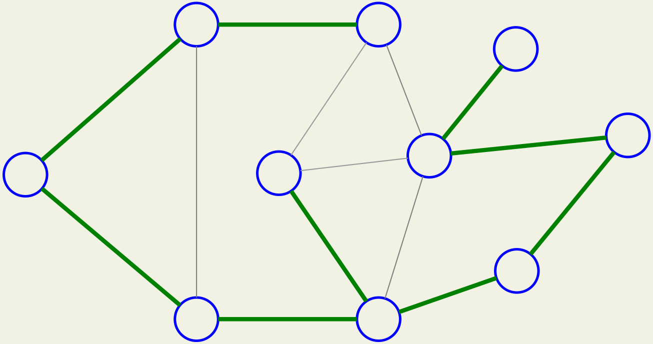

Examples

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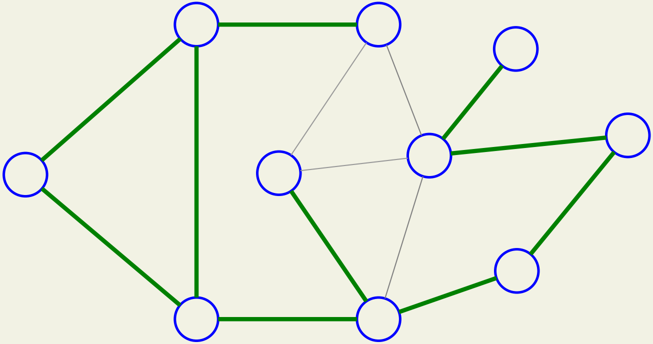

Examples

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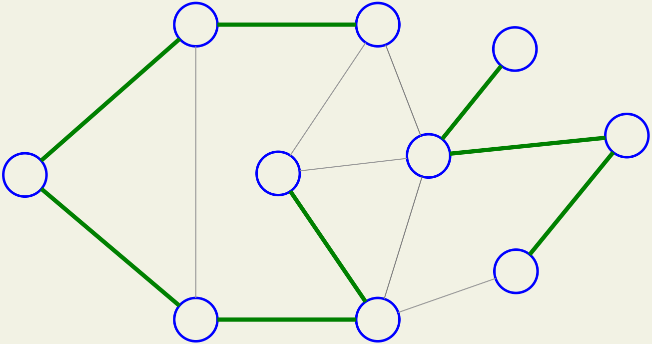


Minimum Spanning Tree Problem

Input

- Undirected connected graph $G = (V, E)$
- Weight function $w : E \to \mathbb{Z}^+$

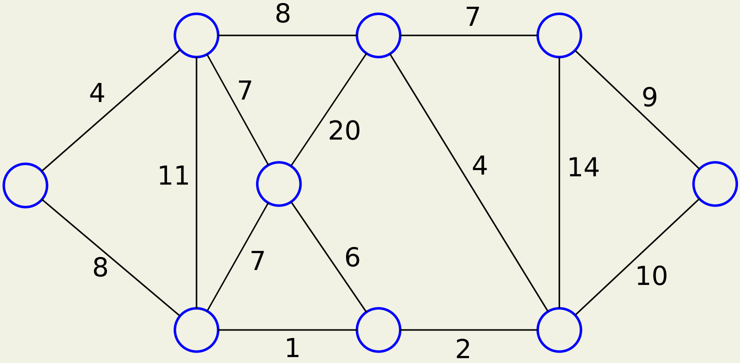
Goal

Compute a spanning tree for G with minimum total weight.

Kruskal's Algorithm (informal)

- Sort the edges in nondecreasing order by weight
- Set $T = \emptyset$
- Choose the lightest edge and add it to T as long as it does not create a cycle in T
- Terminate when T is spanning

Kruskal’s algorthm example

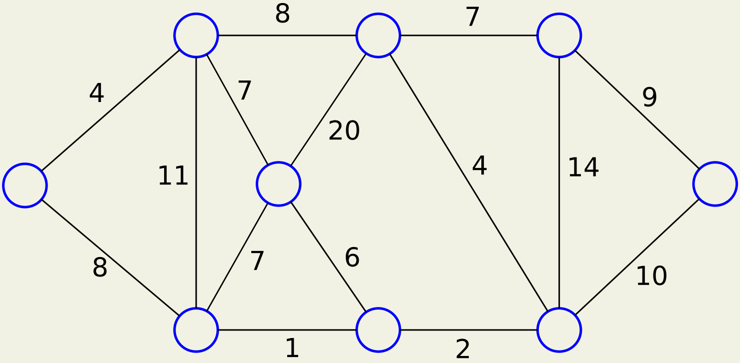


Kruskal's Algorithm Pseudocode

Algorithm 1 Kruskal's algorithm

```
A = ∅
for each vertex v ∈ V do
    MAKE-SET(v)
end for
Sort the edges in E into nondecreasing order by weight w
for each edge (u, v) ∈ E taken in nondecreasing order by weight do
    if FIND-SET(u) ≠ FIND-SET(v) then
        A = A ∪ {(u, v)}
        UNION(u, v)
    end if
end for
Return A
```

Kruskal’s algorthm example



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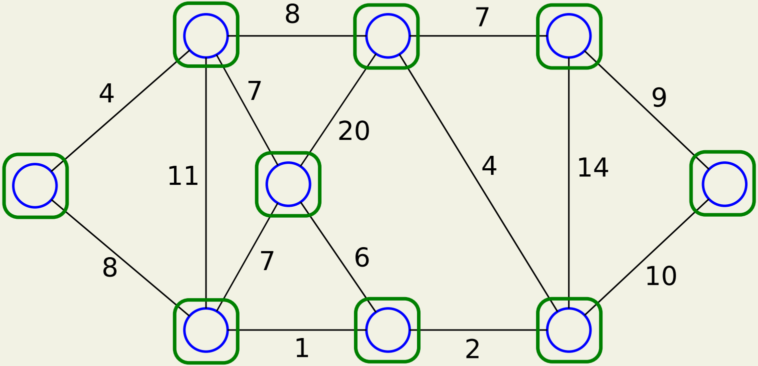

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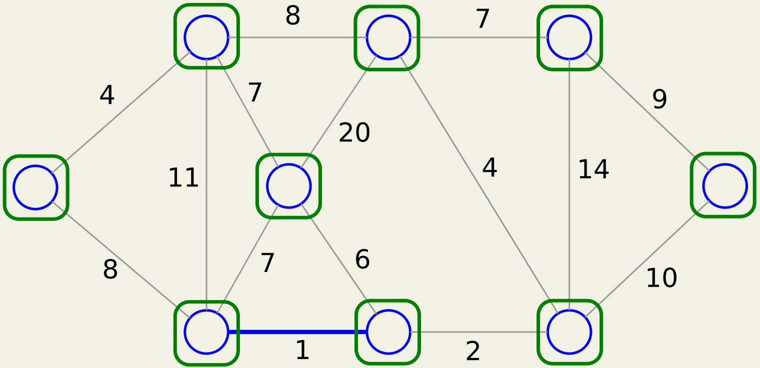
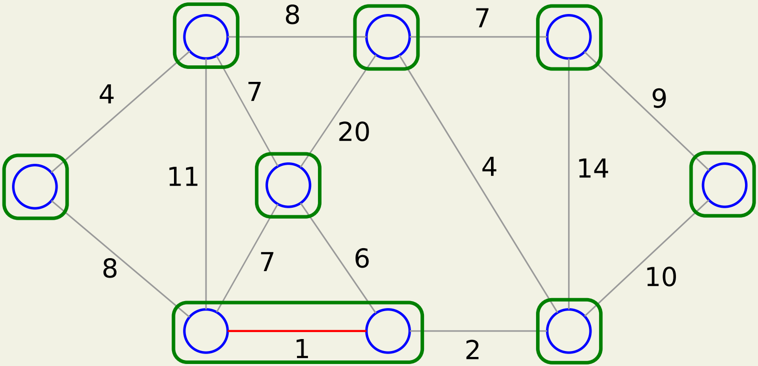


Kruskal’s algorthm example



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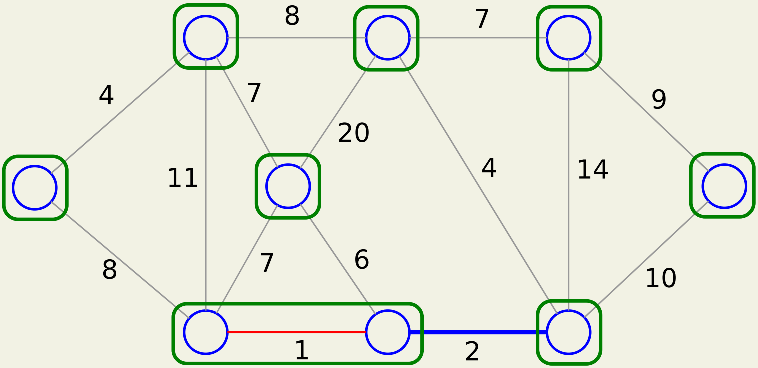

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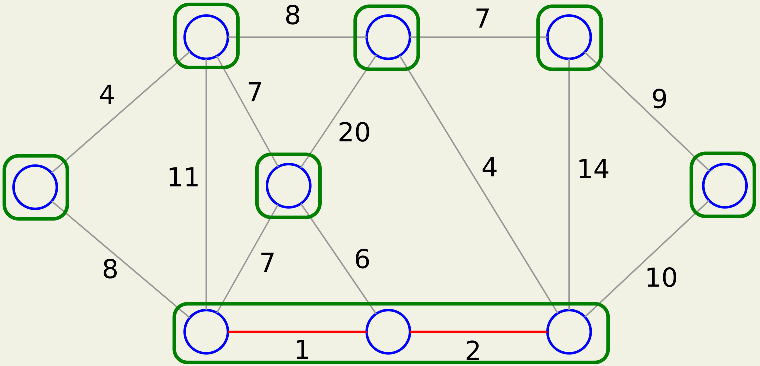

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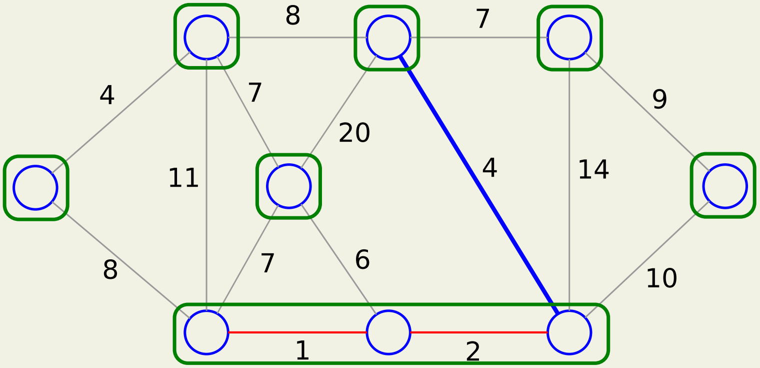

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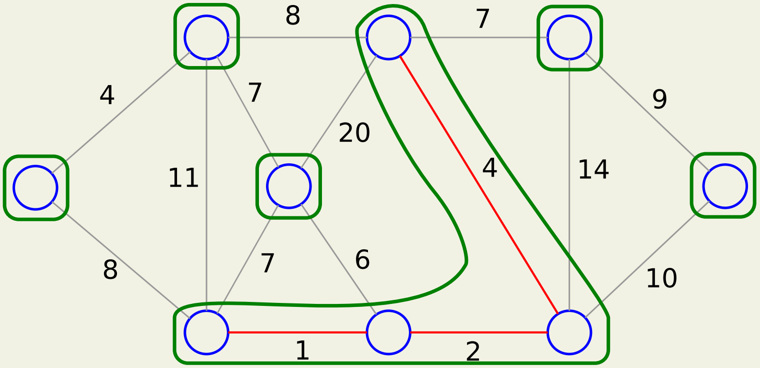

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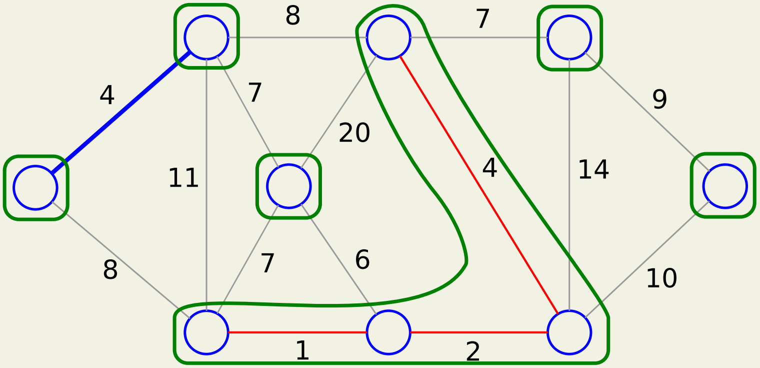

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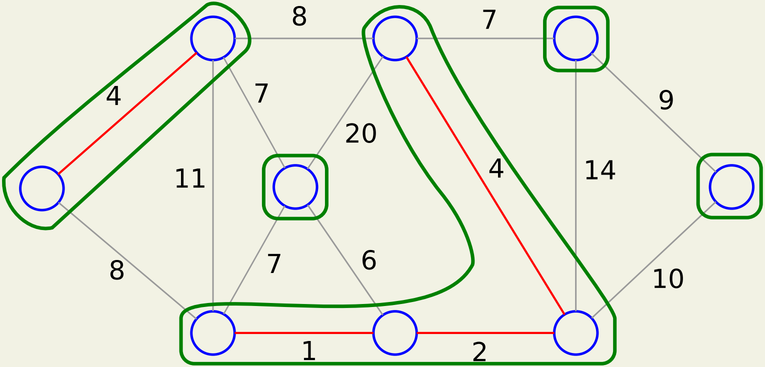

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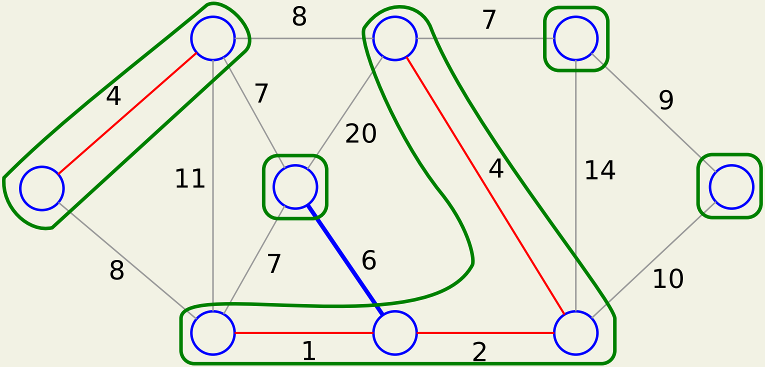

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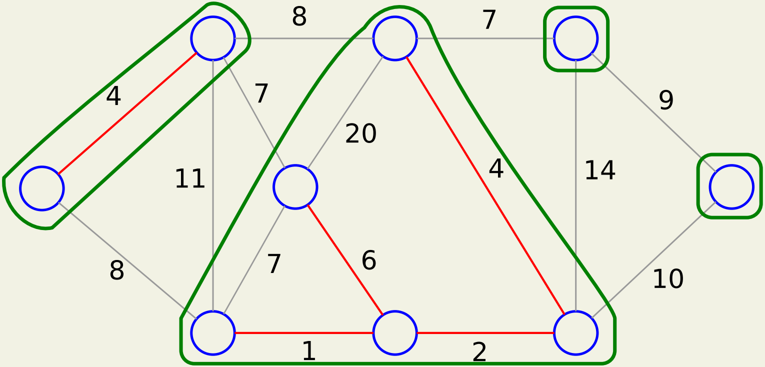

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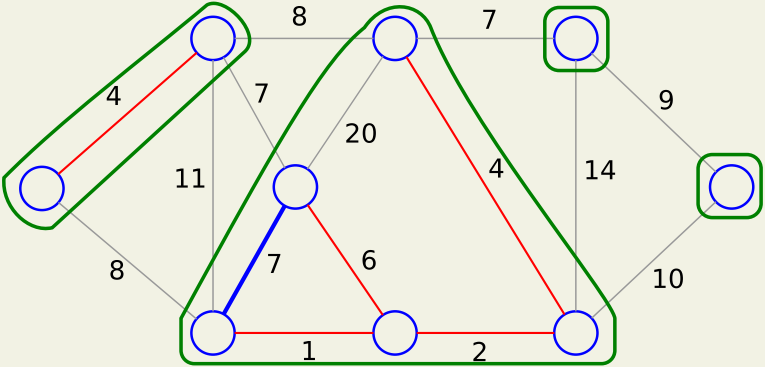

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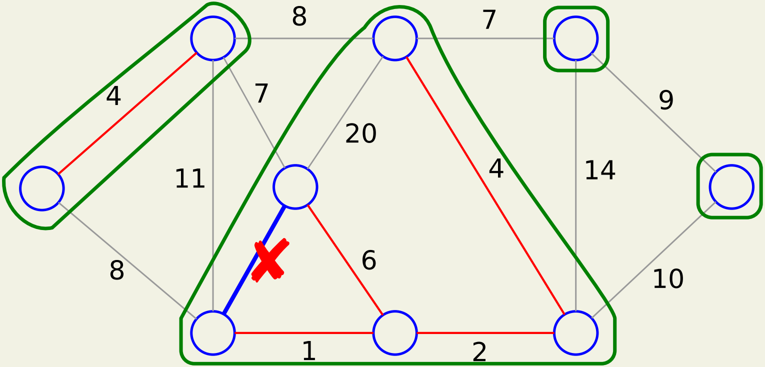

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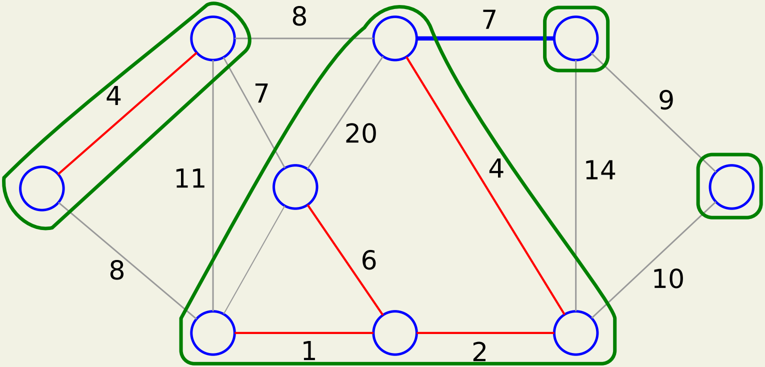

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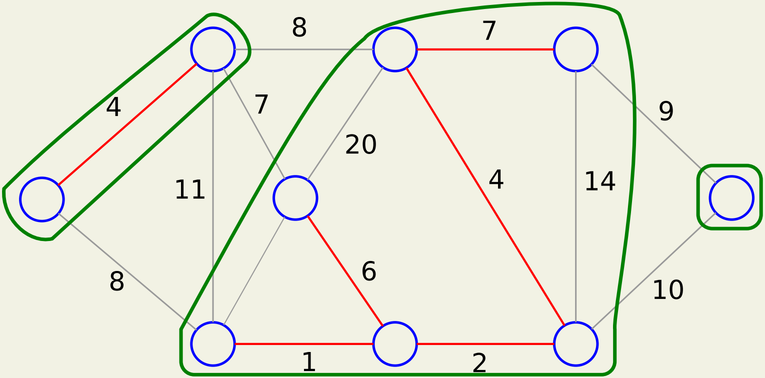

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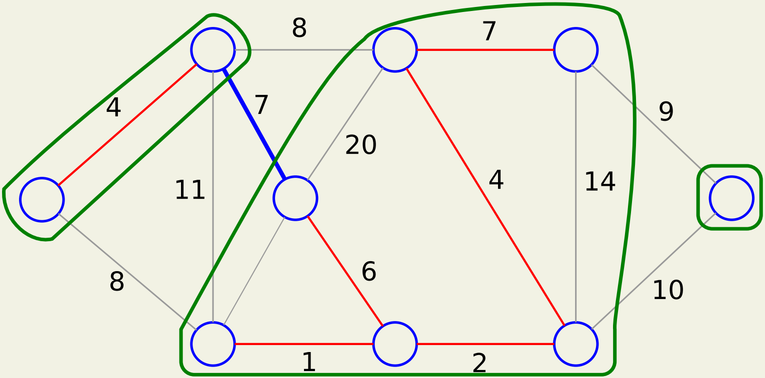

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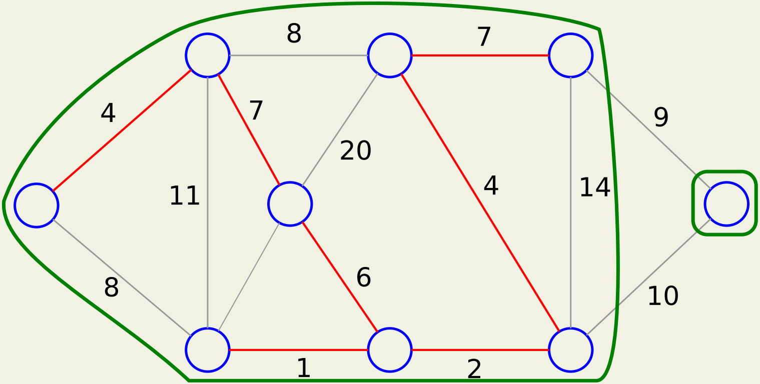

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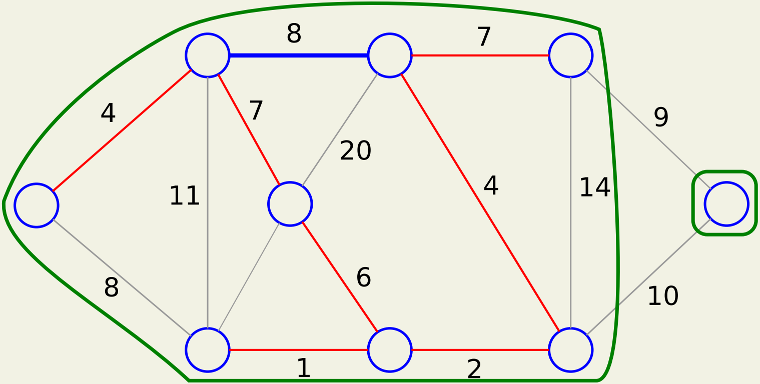

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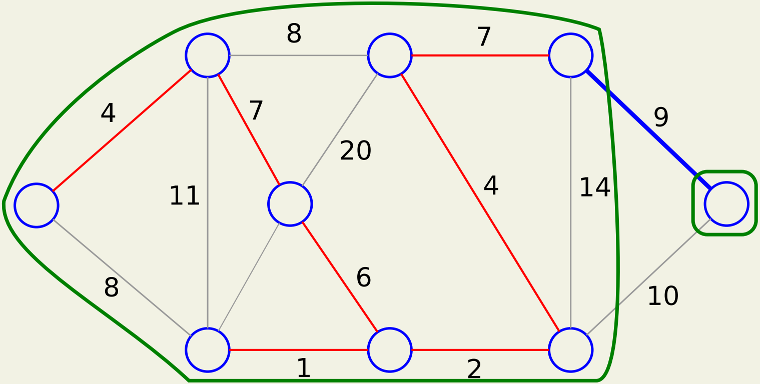

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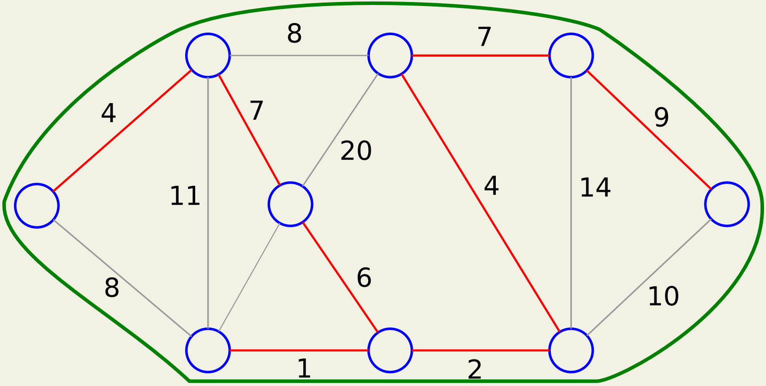

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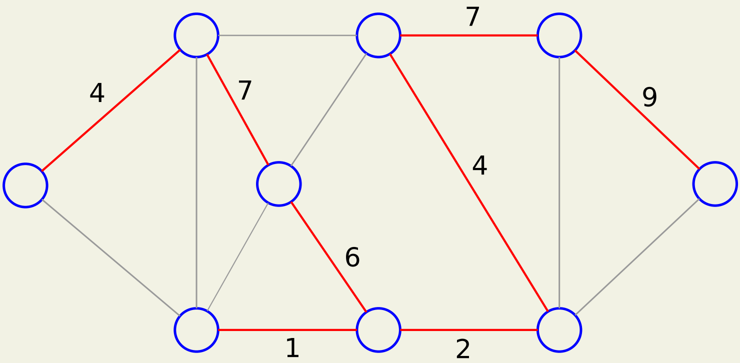

Proof of Correctness of Kruskal's

Algorithm 2 Generic MST

```
A = ∅
while A does not form a spanning tree do
    find an edge (u, v) that is safe for A
    A = A ∪ {(u, v)}
end while
Return A
```

Proof of Correctness of Kruskal's

Algorithm 3 Generic MST

```
A = ∅
while A does not form a spanning tree do
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```

- Kruskal's is a special case of the above generic algorithm
- (u, v) is safe for A: There is an MST that contains $A \cup \{(u, v)\}$
- How does one find a safe edge?

Proof of Correctness of Kruskal's

- (u, v) is safe for A: There is an MST that contains $A \cup \{(u, v)\}$
- The cut $(S, V - S)$ respects A: No edge in A crosses the cut $(S, V - S)$

Proof of Correctness of Kruskal's

- (u, v) is safe for A: There is an MST that contains $A \cup \{(u, v)\}$
- The cut $(S, V - S)$ respects A: No edge in A crosses the cut $(S, V - S)$

Theorem (Cut Property)

Let $G = (V, E)$ be a connected undirected graph with weight $w : E \to \mathbb{R}$. Let $A \subseteq E$ be included in some MST of G. Let $(S, V - S)$ be a cut that respects A and let (u, v) be an edge of smallest weight that crosses this cut. Then (u, v) is safe for A.

Proof of Correctness of Kruskal's

Theorem (Cut Property)

Let $G = (V, E)$ be a connected undirected graph with weight $w : E \to \mathbb{R}$. Let $A \subseteq E$ be included in some MST of G. Let $(S, V - S)$ be a cut that respects A and let (u, v) be an edge of smallest weight that crosses this cut. Then (u, v) is safe for A.

Proof

Let T be an MST such that $A \subseteq T$. If $(u, v) \in T$, we are done. So assume $(u, v) \notin T$.
We will show that there is an MST T' such that $A \cup \{(u, v)\} \subseteq T'$. Thus (u, v) is safe for A.

Proof of Correctness of Kruskal's

Proof cont...

Consider $T \cup \{(u, v)\}$. There is a unique path p from u to v in T. When (u, v) is added to T, it forms a cycle along with the path p.

Proof of Correctness of Kruskal's

Proof cont...

Consider $T \cup \{(u, v)\}$. There is a unique path p from u to v in T. When (u, v) is added to T, it forms a cycle along with the path p.

Since u and v are on the opposite sides of the cut $(S, V - S)$, at least one of the edges in p crosses the cut $(S, V - S)$. Let (x, y) be such an edge. Since $(S, V - S)$ respects A, $(x, y) \notin A$. Removing (x, y) breaks T into two components.

Proof of Correctness of Kruskal's

Proof cont...

Consider $T \cup \{(u, v)\}$. There is a unique path p from u to v in T. When (u, v) is added to T, it forms a cycle along with the path p.

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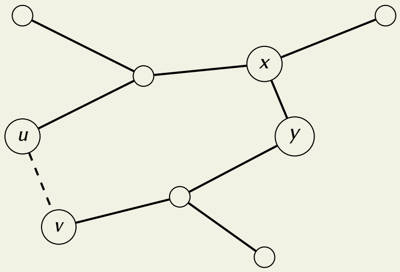
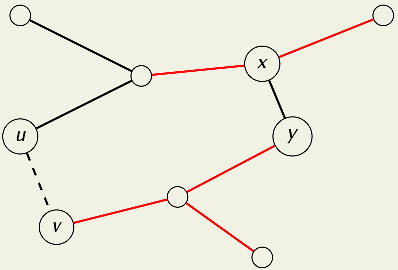
Consider $T' = T - \{(x, y)\} \cup \{(u, v)\}$. Adding (u, v) to $T - \{(x, y)\}$, reconnects the two components. Hence T' is a spanning tree.

Proof of Correctness of Kruskal's



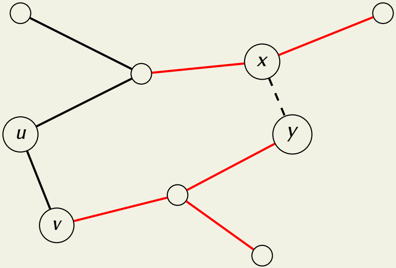
- Solid edges are in T

Proof of Correctness of Kruskal's



- Solid edges are in T
- Red edges are in A

Proof of Correctness of Kruskal's



- Solid edges are in T
- Red edges are in A
- (u, v) is safe!

Proof of Correctness of Kruskal's

Proof cont...

Consider $T' = T - \{(x, y)\} \cup \{(u, v)\}$. Adding (u, v) to $T - \{(x, y)\}$, reconnects the two components. Hence T' is a spanning tree.

Proof of Correctness of Kruskal's

Proof cont...

Consider $T' = T - \{(x,y)\} \cup \{(u,v)\}$. Adding (u,v) to $T - \{(x,y)\}$, reconnects the two components. Hence T' is a spanning tree.

$$\begin{aligned} w(T') &= w(T) - w((x,y)) + w((u,v)) \\ &\leq w(T) \qquad\qquad [\text{since } w((u,v)) \leq w((x,y))] \end{aligned}$$

Proof of Correctness of Kruskal's

Proof cont...

Consider $T' = T - \{(x, y)\} \cup \{(u, v)\}$. Adding (u, v) to $T - \{(x, y)\}$, reconnects the two components. Hence T' is a spanning tree.

$$\begin{aligned} w(T') &= w(T) - w((x, y)) + w((u, v)) \\ &\leq w(T) \qquad\qquad [\text{since } w((u, v)) \leq w((x, y))] \end{aligned}$$

Hence T' is a minimum spanning tree!

Kruskal's Algorithm Running Time

Algorithm 4 Kruskal's algorithm

```
A = ∅
for each vertex v ∈ V do
    MAKE-SET(v)
end for
Sort the edges in E into nondecreasing order by weight w
for each edge (u, v) ∈ E taken in nondecreasing order by weight do
    if FIND-SET(u) ≠ FIND-SET(v) then
        A = A ∪ {(u, v)}
        UNION(u, v)
    end if
end for
Return A
```

Kruskal's Algorithm Running Time

- $|V|$ Make-Set(v) operations
- At most $2|E|$ Find-Set(v) operations
- $|V| - 1$ Union(u, v) operations

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Kruskal's Algorithm Running Time

- $|V|$ Make-Set(v) operations
- At most $2|E|$ Find-Set(v) operations
- $|V| - 1$ Union(u, v) operations

- And a sort the edges – takes $O(|E| \log |E|)$ time
- What if edges were already sorted? What if we can sort them in linear time?

Disjoint Set

Maintain a collection $\mathcal{F} = \{S_1, S_2, \ldots, S_k\}$ of disjoint sets.
One element from each set serves as a 'representative' for that set.

Disjoint Set supports the following procedures:

- $\textsc{MakeSet}(x)$ – Creates a singleton set with element x.
- $\textsc{Union}(x, y)$ – Performs union on sets containing x and y.
- $\textsc{FindSet}(x)$ – Find the set containing x.

MakeSet(x)

Creates a singleton set containing x.

We assume that x is not an element of any other set in $\mathcal{F}$.
We assume x as the representative for the set just created.

Union

Union(x, y)

Performs union on sets containing x and y.

Let $S, T \in \mathcal{F}$ such that $x \in S$ and $y \in T$.
Create a new set $U = S \cup T$.
Choose and assign a representative for U.
Remove S and T from $\mathcal{F}$.

FindSet(x)
Find the set containing x.

Let $S \in \mathcal{F}$ such that $x \in S$. (Note: exactly one set contains x.)
Return a pointer to the representative element of S.

Implementation

Disjoint Set using linked lists:

Implementation

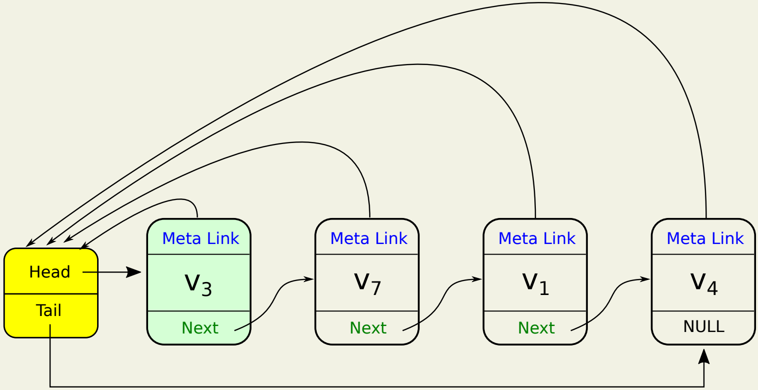
Disjoint Set using linked lists:

- For each set S, maintain:
  - a node with metadata
  - a linked list L_S with the objects in the set.
- The "Metadata Node" stores head and tail pointers to the linked list.
- Each node in the linked list consists of:
  - The value of the element.
  - A pointer to the next element.
  - A pointer to the Metadata Node.

The head of L_S is the representative of S.

Implementation

Linked list for set $\{v_1, v_3, v_4, v_7\}$.



Implementation

$\textsc{MakeSet}(x)$

Creates a singleton set with element x

- Create a new node for metadata
- Create a linked list containing just x.
- Node x is the head and tail of the list.
- Representative for this set is x itself.

Implementation

FindSet(x)

Find the set containing node x.

- Return a pointer to the representative.

Implementation

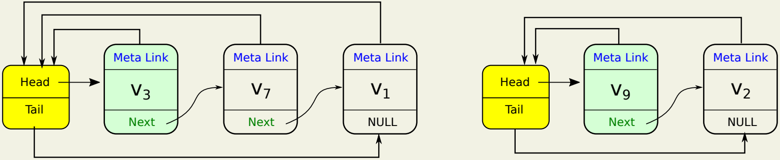
UNION(X,Y)

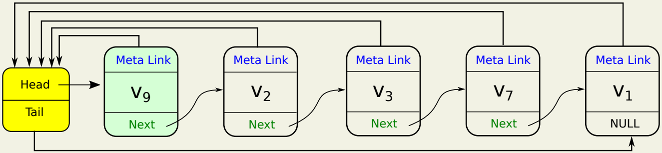
Union of sets containing x and y.

- Append linked list of set S containing x to set T containing y.
- Representative of new set is same as representative of T.
- Update meta pointers of nodes in S to the correct metadata node.
- Update tail pointer in metadata node of T.

Implementation

Union of sets $S = \{v_1, v_3, v_7\}$ and $T = \{v_2, v_9\}$.





Union(v_1,v_2)

Analysis

Running time under Linked List implementation:

- $\textsc{MakeSet}(x)$ – $O(1)$
- $\textsc{FindSet}(x)$ – $O(1)$
- $\textsc{Union}(x, y)$ – ?

Analysis

UNION(x, y) –

- $S \leftarrow$ FINDSET(x) and $T \leftarrow$ FINDSET(Y) – $O(1)$ time.
- Appending linked list of S to tail end of T – $O(1)$ time.
- Updating the new metadata (tail) – $O(1)$ time.
- Updating the backward pointers of nodes in S takes $O(n)$ time.

We can show a case where after $O(n)$ operations, time taken would be $O(n^2)$.

Recap: List Implementation

Disjoint Set using linked lists:

- For each set S, maintain:
  - a node with metadata
  - a linked list L_S with the objects in the set.
- The "Metadata Node" stores:
  - Head and tail pointers to the linked list.
- Each node in the linked list consists of:
  - The value of the element.
  - A pointer to the next element.
  - A pointer to the Metadata Node.

The head of L_S is the representative of S.

List Implementation - Union by Rank heuristic

Disjoint Set using linked lists, union by rank:

- For each set S, maintain:
  - a node with metadata
  - a linked list L_S with the objects in the set.
- The "Metadata Node" stores:
  - Head and tail pointers to the linked list.
  - Size of the set.
- Each node in the linked list consists of:
  - The value of the element.
  - A pointer to the next element.
  - A pointer to the Metadata Node.

The head of L_S is the representative of S.

List Implementation - Union by Rank heuristic

UNION(X,Y)

Union of sets containing x and y.

Let $x \in S$ and $y \in T$.

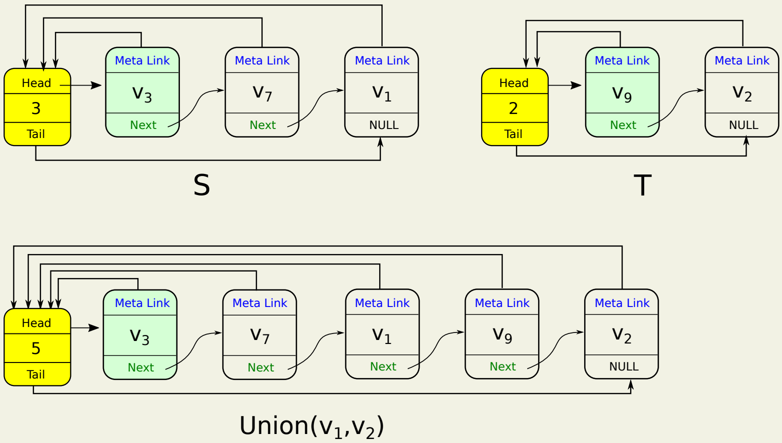
If $|S| \leq |T|$,

- Append list of S to tail end of list of T.
- Representative of new set is same as that of T.
- Update meta pointers of nodes in S
- Update tail pointer in metadata node of T.
- Update size of set in the metadata node.

Else, do the opposite.

Implementation - Union by Rank heuristic

Union of sets $S = \{v_1, v_3, v_7\}$ and $T = \{v_2, v_9\}$.



Analysis - Union by Rank heuristic

Theorem

A sequence of m operations in total, n of which are MakeSet takes $O(m + n \log n)$ time.

Analysis - Union by Rank heuristic

Observation 1

Updating the meta pointers takes the most time.

Observation 2

The meta pointer of a node x is updated only when union happens with a bigger set.

Proof strategy

- Fix an element x.
- Count number of times the meta pointer on node x is updated.

Analysis - Union by Rank heuristic

Observation 2 (informal)

If x lived inside a set of size s,
and a union operation updated its meta pointer,
then x now lives inside a set of size at least $2s$.

- Initially, x starts off as a singleton set.
- After k many updates to its meta pointer, it lives inside a set of size at least 2^k.
- Total number of elements is n. So $2^k \leq n$.
- This means $k \leq \log n$

Hence, for each element the meta pointer can be updated at most $k \leq \log n$ many times.
Worst case total number of updates to meta pointer across all n elements is $n \log n$.

□

Implementation - disjoint forests

The disjoint forest implementation:

- Each set S is implemented as a rooted tree.

A node corresponding to an $x \in S$ contains:

- The value (or pointer to) x.
- A pointer to its parent.

Implementation - disjoint forests

The disjoint forest implementation:

- Each set S is implemented as a rooted tree.

A node corresponding to an $x \in S$ contains:

- The value (or pointer to) x.
- A pointer to its parent.

Note:

- There are no pointers to children nodes!
- There is no dedicated metadata node for each set.
- Convention: Parent of root will be itself.
- Root node is also the representative.