Let $f: \mathbb{R}^n \longrightarrow \mathbb{R}$ which is diff

at $\vec{x_0} \in \mathbb{R}^n$. Let $\underbrace{(\alpha_1, \alpha_2, \ldots, \alpha_n)}_{\vec{\alpha}} \in \mathbb{R}^n$
s.t.

$$\lim_{\vec{x} \to \vec{x_0}} \frac{|f(x) - (f(x_0) + \langle \vec{\alpha}, \vec{x} - \vec{x_0} \rangle)|}{\| \vec{x} - \vec{x_0} \|} = 0$$

Note: $\vec{\alpha} = f'(\vec{x_0})$.

Ex: Let $f: \mathbb{R}^2 \longrightarrow \mathbb{R}$

$$f(x,y) = x^2 + y^2.$$

Let $\vec{x_0} = (1,1)$. $(2,2) \in \mathbb{R}^2$

$$\lim_{(x,y)\to(1,1)} \frac{|f(x,y) - (f(1,1) + \langle (2,2), (x,y)-(1,1) \rangle)|}{\|(x,y) - (1,1)\|}$$

Let $x-1 = a$ & $y-1 = b$

$$\lim_{(a,b)\to(0,0)} \frac{|(a+1)^2 + (b+1)^2 - (2 + 2a + 2b)|}{\|(a,b)\|}$$

$$= \lim_{(a,b)\to(0,0)} \frac{|a^2+b^2|}{\sqrt{a^2+b^2}}$$

$$= \lim_{(a,b)\to(0,0)} \sqrt{a^2+b^2} = 0.$$

$$f'((1,1)) = (2,2).$$

Ex: $f(x,y) = \begin{cases} xy\dfrac{x^2-y^2}{x^2+y^2} & (x,y) = (0,0 \\ 0 & \text{if o.w} \end{cases}$

$$\lim_{(x,y)\to(0,0)} \frac{\left|f(x,y)-\left(f(0,0)+\langle\vec{0},(x,y)\rangle\right)\right|}{\|(x,y)\|}$$

$$= \lim_{(x,y)\to(0,0)} \frac{|xy|\left|\frac{x^2-y^2}{x^2+y^2}\right|}{\sqrt{x^2+y^2}}$$

$$\leq \lim_{(x,y)\to(0,0)} \frac{|x||y|}{\sqrt{x^2+y^2}}$$

$$\leq \lim_{(x,y)\to(0,0)} \frac{\sqrt{x^2+y^2}\cdot\sqrt{x^2+y^2}}{\sqrt{x^2+y^2}}$$

$$= \lim_{(x,y)\to(0,0)} \sqrt{x^2+y^2} = 0.$$

ie f is diff at $(0,0)$ and

$$f'((0,0)) = (0,0).$$

Ex: $f(x,y) = \begin{cases} x^2 \sin\frac{1}{y} + y^2 \sin\frac{1}{x} & (x,y) \neq (0,0) \\ 0 & \text{o.w.} \end{cases}$

S.T. $f'((0,0)) = (0,0)$.

Let $f: \mathbb{R}^n \to \mathbb{R}$ and $\vec{x_0} \in \mathbb{R}^n$

and $\vec{v} \in \mathbb{R}^n$. Then if the limit

$$\lim_{t \to 0} \frac{f(\vec{x_0} + t\vec{v}) - f(\vec{x_0})}{t} \text{ exists in } \mathbb{R}$$

Let $D_{\vec{v}} f(\vec{x_0}) = \lim_{t \to 0} \frac{\qquad}{t}$

Then $D_{\vec{v}} f(\vec{x_0})$ is said to be the directional derivative of at $\vec{x_0}$ in the direction $\vec{v}$.

Let f be diff at $\vec{x_0}$. Then

$$D_{\vec{v}} f(\vec{x_0}) = \langle f'(\vec{x_0}), \vec{v} \rangle$$

$$D_{\vec{v}} f(\vec{x_0}) = \left[\lim_{t \to 0} \frac{f(x_0 + t(\vec{v})) - f(x_0)}{t} - \langle f'(\vec{x_0}), \vec{v} \rangle \right] + \langle f'(\vec{x_0}), \vec{v} \rangle$$

$$= \lim_{t \to 0} \frac{f(x_0 + t\vec{v}) - (f(x_0) + \langle f'(x_0), t\vec{v} \rangle)}{t}$$

$$\langle f'(x_0), \vec{v} \rangle$$

$$= \langle f'(\vec{x_0}), \vec{v} \rangle.$$

If $\vec{v} = \hat{e}_i$ where $\hat{e}_i = (0, 0, \cdots \underset{i\text{th}}{1}, 0 \cdots)$

then $D_{\vec{e_i}} f(\vec{x_0})$ is said to be the ith partial derivative of f at $\vec{x_0}$ and is denoted by $\frac{\partial f}{\partial x_i}(\vec{x_0})$.

<u>Ex:</u> $D_{\vec{v}} f(\vec{x_0}) = \frac{d}{dt} f(\vec{x}_0 + t\vec{v})\Big|_{t=0}$.

$$g(t) = f(\vec{x_0} + t\vec{v})$$

$$\lim_{t\to 0} \frac{g(t) - g(0)}{t} = D_v f(\vec{x_0}).$$

Let f be diff at $\vec{x_0}$. Fix $\vec{v} \in \mathbb{R}^n$

$\|\vec{v}\| = 1$

$$D_{\vec{v}} f(\vec{x_0}) = \langle f'(\vec{x_0}), \vec{v} \rangle$$

$$\vec{v} = \sum_{i=1}^{n} v_i \hat{e}_i$$

$$D_{\vec{v}} f(\vec{x_0}) = \sum_{i=1}^{n} v_i \langle f'(x_0), \hat{e}_i \rangle$$

$$= \sum_{i=1}^{n} v_i \frac{\partial f}{\partial x_i}(\vec{x_0}).$$

$$D_v f(\vec{x_0}) = \left(\frac{\partial f}{\partial x_1}(x_0), \cdots \frac{\partial f}{\partial x_n}(x_0)\right)\begin{pmatrix} v_1 \\ \vdots \\ v_n \end{pmatrix}$$

Let $f : \mathbb{R}^n \longrightarrow \mathbb{R}^m$.

$f = (f_1, f_2, \cdots, f_m)$

Then $$D_{\vec{v}} f(\vec{x_0}) = \begin{pmatrix} D_v f_1(x_0) \\ \vdots \\ D_v f_m(x_0) \end{pmatrix}$$

$$= \begin{pmatrix} \frac{\partial f_1}{\partial x_1}(x_0) & \cdots & \frac{\partial f_1}{\partial x_n}(x_0) \\ \frac{\partial f_2}{\partial x_1}(x_0) & \cdots & \frac{\partial f_2}{\partial x_n}(x_0) \\ \vdots & & \\ \frac{\partial f_m}{\partial x_1}(x_0) & \cdots & \frac{\partial f_m}{\partial x_n}(x_0) \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \\ \vdots \\ \\ \end{pmatrix}$$

← The Jacobian matrix of f at $\vec{x_0}$.

$$D_{\vec{v}} f(\vec{x_0}) = \begin{pmatrix} & & \\ & & \\ & & \end{pmatrix}_{m \times n} \begin{pmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{pmatrix}$$

$$f'(\vec{x_0}) = \begin{pmatrix} & & \\ & & \\ & & \end{pmatrix}$$

$$f(x,y) = x^2 + y^2$$

$$\frac{\partial f}{\partial x}(1,1) = 2 \qquad \frac{\partial f}{\partial y}(1,1) = 2$$

$$\left(\frac{\partial f}{\partial x}(1,1), \frac{\partial f}{\partial y}(1,1) \right) = (2,2).$$

$$f: \mathbb{R}^n \longrightarrow \mathbb{R}^m$$

$$f'(x_0) = (a_{ij})_{m \times n}$$

$$D_{\vec{v}} f(\vec{x_0}) = (a_{ij})_{m \times n} \cdot (v_i)_{n \times 1}$$