<u>L-intersecting</u> family: Let L be a set of non-negative integers. A family $\mathcal{F}$ of subsets of $[n]$ is <u>L-intersecting</u> if $\forall A, B \in \mathcal{F}$, $A \neq B$, we have $|A \cap B| \in L$.

↳ Fisher's Inequality says that if L is a singleton set, then $|\mathcal{F}| \leq n$.

Theorem [Frankl-Wilson, 1970s] Let $L = \{l_1, l_2, \dots, l_s\}$ be a set of s non-negative integers. Let $\mathcal{F}$ be an L-intersecting family of subsets of $[n]$. Then

$$|\mathcal{F}| \leq \sum_{i=0}^{s} \binom{n}{i}$$

Proof:

Tight Example: Tightness of the bound

$L = \{0, 1, 2, \dots, s-1\}$, $|L| = s$.

$\mathcal{F} = \{ A \subseteq [n] : |A| \leq s \}$.

Then, clearly $\mathcal{F}$ is L-intersecting.

Further, $|\mathcal{F}| = \sum_{i=0}^{s} \binom{n}{i}$.

Let $\mathcal{F} = \{A_1, A_2, \dots, A_m\}$

To show: $|\mathcal{F}| = m \leq \sum_{i=0}^{s} \binom{n}{i}$

$\forall i \in [m]$, let v_i be the 0-1 incidence vector of the set A_i.

$\forall i \in [m]$, let $f_i : \{0,1\}^n \to \mathbb{F}_2$ be defined as:-

$\mathbb{R}$ (it still works)

$(\{0,1\}, +, \cdot)$

$\hookrightarrow (\{0,1\}, +, \cdot)$ modulo 2.

$$x = (x_1, x_2, \ldots, x_n) \in \{0,1\}^n$$

$$f_i(x) = \prod_{\substack{l_j \in L: \\ l_j < |A_i|}} \left(\langle v_i, x \rangle - l_j \right)$$

std dot product

Clearly,

incidence vector of A_i

$$f_i(v_i) = \prod_{\substack{l_j \in L: \\ l_j < |A_i|}} \left(\langle v_i, v_i \rangle - l_j \right)$$

$$= \prod_{\substack{l_j \in L: \\ l_j < |A_i|}} \left(|A_i| - l_j \right)$$

$$> 0$$

For a $j \neq i$,

$$f_i(v_j) = \prod_{\substack{l_r \in L: \\ l_r < |A_i|}} \left(\langle v_i, v_j \rangle - l_r \right)$$

$$= \prod_{\substack{l_r \in L: \\ l_r < |A_i|}} \left(|A_i \cap A_j| - l_r \right)$$

$$= 0$$

From Independence criterion, we can say that

functions $f_1, f_2, \ldots, f_m$ are L.I. in the v.s. $\mathbb{F}_2^{\{0,1\}^n}$ over $\mathbb{F}_2$.

L.I. in the V.S. $\mathbb{F}_2^{\{0,1\}^s}$ over $\mathbb{F}_2$. 1

$$f_i(x) = \prod_{\substack{l_j \in L: \\ l_j < |A_i|}} \left(\langle x, v_i \rangle - l_j \right)$$

$(x_1, x_2, \ldots, x_n)$

$(v_{i_1}, v_{i_2}, \ldots, v_{i_n})$

$$= \prod_{\substack{l_j \in L: \\ l_j < |A_i|}} \left(v_{i_1} x_1 + v_{i_2} x_2 + \cdots + v_{i_n} x_n - l_j \right)$$

$x_i^2 \rightarrow x_i$

$x_i^3 \rightarrow$

$x_i^4 \rightarrow$

$\cdots + x_1 x_3^2 x_7^3 + \cdots$

$\cdots + x_1 x_3 x_7 + \cdots$

Claim: The functions $f_1, f_2, \ldots, f_m$ reside in the space obtained from the span of following functions:—

$1, \quad x_i, \quad x_i x_j, \quad \ldots, \quad \ldots$

1 $i \neq j$

OR

$\forall S \subseteq [n], |S| \leq s$, $\prod_S x_i$ A

no. of fns $\leq \binom{n}{s}$

$\forall S \subseteq$ ... $\prod_{i \in S} x_i$) no. of fns $\sum_{i=0}^{s} \binom{n}{i}$

From ① and the above claim,

we get $|\mathcal{F}| = m \leq \sum_{i=0}^{s} \binom{n}{i}$.

$\square$

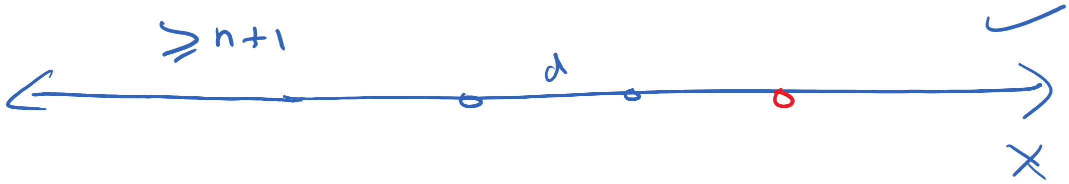
Two Distance Sets . .

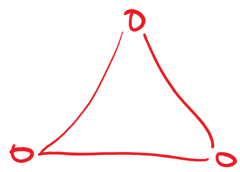
One distance set . .

Equilateral dimension of a metric space

$EqDim(\mathbb{R}^n)$ over Euclidean Distance.

$= n+1$





To show

$Eqdim[\mathbb{R}^n]$ over Euclidean Distance $\leq n+1$.

NEXT LECTURE

What about

$$\underbrace{EqDim(\mathbb{R}^n) \text{ over } L_\alpha\text{-norm}}_{} = 2^n$$

$a = (a_1, a_2, \ldots, a_m)$

$$a = (a_1, a_2, \ldots, a_m)$$

$$b = (b_1, b_2, \ldots, b_n)$$

$$\text{dist}_{L_\infty}(a, b) = \max_{i \in [n]} (|a_i - b_i|)$$

distance $= 1$

$$\left.\begin{matrix} (1, 0, 0, \ldots, 0) \\ (0, 1, 0 \text{———}) \\ \vdots \\ (0\ 0\ 1\ 1\ 1\ 0 \ldots) \end{matrix}\right\} \begin{matrix} 2^n \\ (0-1) \\ \text{n-bit} \\ \text{vectors} \end{matrix}$$

It is known that this is the best one can do.

What about $\text{EaDim}(\mathbb{R}^n)$ over L_0-norm

$$a = (a_1, a_2, \ldots, a_n)$$

$$b = (b_1, b_2, \ldots, b_n)$$

$$\text{dist}_{L_0}(a, b) = |a_1 - b_1| + |a_2 - b_2| + \cdots + |a_n - b_n|$$

Example:

$$\left.\begin{array}{l}(+1, 0, 0, \ldots, 0)\\(-1, 0, 0, \ldots, 0)\\(0, +1, 0, 0, \ldots, 0)\\(0, -1, 0, 0, \ldots, 0)\\ \quad\vdots \qquad\qquad \vdots \quad \vdots\\(0, 0, \ldots, 0, +1)\\(0, 0, \ldots, 0, -1)\end{array}\right\} 2n \text{ vectors}$$

distance btwn any two of them is 2.

Kusner's Conjecture

One cannot put more than $2n$ points in $\mathbb{R}^n$ such that the L_0-distance between every two of these points is the same.