EP 1027: Quiz 1

April 3, 2019

1. From the identity $\epsilon_{ijk}\epsilon_{lmk} = \delta_{il}\delta_{jm} - \delta_{im}\delta_{jl}$, show that $\epsilon_{ijk}\,\epsilon_{ljk} = 2\delta_{il}$. (2)

   Solution: Plugging $m = j$ in the identity we have,

$$\begin{aligned} \epsilon_{ijk}\epsilon_{ijk} &= \delta_{il}\delta_{jj} - \delta_{ij}\delta_{jl}, \qquad \text{(1 point)} \\ &== 3\,\delta_{il} - \delta_{il} = 2\delta_{il} \qquad \text{(1 point)}. \end{aligned}$$

   Here we have used,

$$\delta_{jj} = \delta_{11} + \delta_{22} + \delta_{33} = 3.$$

2. The effect of rotation of the Cartesian coordinate axes on components of a vector is same as multiplication by a matrix which is
   (A) real-symmetric **(B)** antisymmetric **(C)** orthogonal **(D)** orthogonal with unit determinant. (1)

   Solution: (D) orthogonal with unit determinant.

3. Write down the transformation law for a rank 2 tensor under rotation of coordinate axes.(1)
   Solution:

$$T'_{ijk} = O_{il}\,O_{jm}\,O_{kn}\,T_{lmn}.$$

4. If S_{ij} is symmetric and A_{ij} is antisymmetric under swap of i and j, then show $S_{ij}A_{ij} = 0$.(2)
   Solution: Starting from $S_{ij}A_{ij}$ we swap i and j to get,

$$S_{ij}A_{ij} = -S_{ji}A_{ji}.$$

   (1 point)
   But i and j are both dummy variables and we can relabel them whenever we desire. So let's relabel i as j and j as i in the above to get,

$$S_{ij}A_{ij} = -S_{ij}A_{ij} \Rightarrow S_{ij}A_{ij} = 0.$$

   (1 point)

5. $\int_0^\infty dx\, \delta(x+3)\, \delta(x-2) =$ ____________

Solution: 0. (1)

6. Using the Stokes theorem, derive the result popularly known as *Green's theorem* for a vector field, $\mathbf{V}(x, y) = P(x, y)\, \hat{\mathbf{i}} + Q(x, y)\, \hat{\mathbf{j}}$, defined on the plane ($\mathbb{R}^2$), namely,

$$\oint_C (Pdx + Qdy) = \iint_D dx\, dy \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right)$$

where C is a closed curve (loop) on the plane and D is the region inside (i.e. bounded by the loop). (3)

Solution: Define a new vector, $\mathbf{W} = \mathbf{V} + 0\, \hat{\mathbf{k}} = P(x, y)\, \hat{\mathbf{i}} + Q(x, y)\, \hat{\mathbf{j}} + 0\hat{\mathbf{k}}$, in 3d space $\mathbb{R}^3$, and apply Stokes theorem to the vector $\mathbf{W}$ on the loop C

$$\oint_C \mathbf{W} \cdot d\mathbf{l} = \iint_D dS \widehat{\mathbf{n}} \cdot (\boldsymbol{\nabla} \times \mathbf{W})\,. \qquad \text{(1 point for this).}$$

Since the loop lies on the XY plane, the line element is $d\mathbf{l} = dx\, \hat{\mathbf{i}} + dy\, \hat{\mathbf{j}}$, and normal to the plane is $\hat{\mathbf{n}} = \hat{\mathbf{k}}$, and area element is $dS = dx\, dy$ so one has,

$$\oint_C (Pdx + Qdy) = \iint_D dx\, dy\, (\boldsymbol{\nabla} \times \mathbf{W})_z \quad \text{(1 point)}$$

The third or z-component of the curl of $\mathbf{W}$ is $(\boldsymbol{\nabla} \times \mathbf{W})_3 = \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}$ (1 point here). Hence the result follows.