Set of problems
MA1140: Elementary Linear Algebra
8th February - 8th March, 2019

1 Matrices, Linear equations and solvability

Q1. Solve (if solution exists) the following system of linear equations:

$$\begin{array}{lllll} u & +v & +w & +z & =6 \\ u & & +w & +z & =4 \\ u & & +w & & =2 \end{array}$$

What is the intersection if the fourth plane $u = -1$ is included? Find a fourth equation that leaves us with no solution.

Q2. Find two points on the line of intersection of the three planes $t = 0$, $z = 0$ and $x+y+z+t = 1$ in four-dimensional space.

Q3. Explain why the system

$$\begin{aligned} u + v + w &= 2 \\ u + 2v + 3w &= 1 \\ v + 2w &= 0 \end{aligned}$$

is *singular* (i.e., it does not have solutions at all). What value should replace the last zero on the right side to allow the system to have solutions, and what is one of the solutions?

Q4. Under what condition on x_1, x_2 and x_3 do the points $(0, x_1), (1, x_2)$ and $(2, x_3)$ lie on a straight line?

Q5. These equations are certain to have the solution $x = y = 0$. For which values of a is there a whole line of solutions?

$$\begin{aligned} ax + 2y &= 0 \\ 2x + ay &= 0 \end{aligned}$$

Q6. Are the following systems equivalent:

$$\begin{aligned} x - y &= 0 \\ 2x + y &= 0 \end{aligned}$$

and

$$\begin{aligned} 3x + y &= 0 \\ x + y &= 0 \end{aligned}$$

If so, then express each equation in each system as a linear combination of the equations in the other system.

Q7. Set $A = \begin{pmatrix} 6 & -4 & 0 \\ 4 & -2 & 0 \\ -1 & 0 & 3 \end{pmatrix}$ and $X = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$. Find all the solutions of $AX = 2X$, i.e., all X such that $AX = 2X$, where $2X$ is just componenntwise scalar multiplication.

Q8. Prove that the interchange of two rows of a matrix can be accomplished by a finite sequence of elementary row operations of the other two types.

Q9. Consider the system of equations $AX = 0$, where $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ is a matrix over $\mathbb{R}$. Prove the following:
(a) A is a zero matrix (i.e., all entries are zero) if and only if every pair (x_1, x_2) is a solution of $AX = 0$.
(b) $ad - bc \neq 0$ if and only if the system has only the trivial solution.
(c) $ad - bc = 0$ but A is a non-zero matrix (i.e., some entries are non-zero) if and only if there is $(y_1, y_2) \neq (0, 0)$ in $\mathbb{R}^2$ such that every solution of the system is given by $c(y_1, y_2)$ for some scalar c.

Q10. Prove that if two homogeneous systems each of two linear equations in two unknowns have the same solutions, then they are equivalent.

Q11. For the system

$$\begin{aligned} u + v + w &= 2 \\ 2u + 3v + 3w &= 0 \\ u + 3v + 5w &= 2, \end{aligned}$$

what is the triangular system after forward elimination, and what is the solution (by back substitution)? Also solve it by computing the equivalent system whose coefficient matrix is in row reduced echelon form. Verify whether both the solutions are same.

Q12. Describe explicitly all 2×2 row reduced echelon matrices.

Q13. Let $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ be a matrix over $\mathbb{R}$. Suppose that A is row reduced and also that $a + b + c + d = 0$. Prove that there are exactly three such matrices.

Q14. Let $A = \begin{pmatrix} 1 & -1 & 1 \\ 2 & 0 & 1 \\ 3 & 0 & 1 \end{pmatrix}$. Find some elementary matrices $E_1, E_2, \ldots, E_k$ such that $E_k \cdots E_2 E_1 A = I_3$, where I_3 is the 3×3 identity matrix. Deduce A^{-1}.

Hint. Apply elementary row operations on $(A \mid I_3)$ to get A^{-1}, and keep track of the row operations to get the corresponding $E_1, E_2, \ldots, E_k$.

2 Vector spaces

Throughout, V is a vector space over $\mathbb{R}$, the set of real numbers.

Q15. Let $0 \in V$ be the zero vector. Let $c \in \mathbb{R}$. Show that $c \cdot 0 = 0$.

Hint. For the zero vector, note that $0 + 0 = 0$.

Q16. Let $v \in V$. Show that $0 \cdot v = 0$, where 0 in the right side is the zero vector, and 0 in the left side is the zero element of $\mathbb{R}$.

Hint. For the zero element in $\mathbb{R}$, $0 + 0 = 0$.

Q17. Let W be a subspace of V. Show that (the zero vector) $0 \in W$.

Q18. Let $V = \mathbb{R}^2$. Define

$$\begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} x_1 \\ y_1 \end{pmatrix} = \begin{pmatrix} x + x_1 \\ 0 \end{pmatrix} \quad \text{and} \quad c\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} cx \\ y \end{pmatrix}.$$

Is V, with these operations, a vector space over $\mathbb{R}$?

What happens when we define the scalar multiplication as

$$c\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} cx \\ 0 \end{pmatrix}?$$

Q19. Which of the following sets of vectors $X = (x_1, \ldots, x_n)$ in $\mathbb{R}^n$ are subspaces of $\mathbb{R}^n$ $(n \geqslant 3)$?

(i) all X such that $x_1 \geqslant 0$;

(ii) all X such that $x_1 + 2x_2 = 3x_3$;

(iii) all X such that $x_1 = x_2^2$;

(iv) all X such that $x_1 x_2 = 0$;

(v) all X such that x_1 is rational.

Q20. Let A be an $m \times n$ matrix over $\mathbb{R}$. Show that $\{X \in \mathbb{R}^n : AX = 0\}$ is a subspace of $\mathbb{R}^n$. It is called the **null space** of A. The dimension of the null space of A is called **nullity** of A.

Q21. Let W_1 and W_2 be subspaces of V such that the set-theoretic union $W_1 \cup W_2$ is also a subspace of V. Prove that one of the subspaces W_1 and W_2 is contained in the other.

Q22. Let W_1 and W_2 be subspaces of V such that $W_1 + W_2 = V$ and $W_1 \cap W_2 = 0$. Prove that for every vector $v \in V$, there are unique vectors $w_1 \in W_1$ and $w_2 \in W_2$ such that $v = w_1 + w_2$.

In this case, we write $V = W_1 \oplus W_2$, and call this as **direct sum** of W_1 and W_2.

Q23. Prove that all the subspaces of $\mathbb{R}^1$ are 0 and $\mathbb{R}^1$.

Q24. Prove that a subspace of $\mathbb{R}^2$ is either 0, or $\mathbb{R}^2$, or a subspace consisting of all scalar multiplies of some fixed non-zero vector in $\mathbb{R}^2$ (which is intuitively a straight line through the origin).

Q25. (i) Let A be an $m \times n$ matrix. Suppose B is obtained from A by applying an elementary row operation. Prove that row space(A) = row space(B).

(ii) Deduce from (i) that if any two $m \times n$ matrices A and B are row equivalent, then row space(A) = row space(B).

(iii) Let $B = \begin{bmatrix} R_1 \\ \vdots \\ R_r \\ 0 \\ \vdots \\ 0 \end{bmatrix}$ be an $m \times n$ row reduced echelon matrix with the non-zero rows $R_1, \ldots, R_r \in \mathbb{R}^n$ and the last $(m - r)$ zero rows. Prove that $\{R_1, \ldots, R_r\}$ is a basis of the row space of B.

(iv) Let A be an $m \times n$ matrix. Let A be reduced to a row reduced echelon matrix B. Then deduce from (ii) and (iii) that the non-zero rows of B gives a basis of the row space of A. Hence the row rank of A is same as the number of non-zero rows of B.

Q26. Let A be an $m \times n$ matrix. By applying elementary row operations, how can you find a basis of the column space of A?

Solution: Note that the column space of A is same as the row space of A^t (transpose of A). Then apply elementary row operations on A^t to get a row reduced echelon matrix, say B. Then, by Q25(iv), the non-zero rows of B gives a basis of the row space of A^t, which is same as the column space of A.

Q27. Consider the matrix $A = \begin{pmatrix} 2 & 1 & 1 & 6 \\ 1 & -2 & 1 & 2 \\ 0 & 5 & -1 & 2 \end{pmatrix}$. Find a basis of the row space of A. Deduce the row rank of A. Find a basis of the column space of A as well. Deduce the column rank of A. Verify whether row rank of A is same as column rank of A. Furthermore, find the null space of A. Deduce the nullity of A.

Verify the Rank-Nullity Theorem for the linear map $A : \mathbb{R}^4 \to \mathbb{R}^3$ defined by A. (This part of the exercise should belong to the next section.)

Hint. Do not forget Q25(iv) and Q26 to obtain the row and column spaces of A. Moreover, observe that the range space of the map $A : \mathbb{R}^4 \to \mathbb{R}^3$ is same as the column space of A.

Q28. Consider some column vectors $v_1, \dots, v_n \in \mathbb{R}^m$. By applying elementary row operations, how can you find a basis of the subspace $\mathrm{Span}(\{v_1, \dots, v_n\})$ of $\mathbb{R}^m$?

Solution: Set $A := \begin{bmatrix} v_1 & v_2 & \cdots & v_n \end{bmatrix}$ an $m \times n$ matrix with the columns $v_1, \dots, v_n \in \mathbb{R}^m$. Then the subspace $\mathrm{Span}(\{v_1, \dots, v_n\})$ is same as the column space of A. Now follow the solution of Q26.

Q29. Let $A = \begin{pmatrix} 3 & -1 & 8 & 4 \\ 2 & 1 & 7 & 1 \\ 1 & -3 & 0 & 4 \end{pmatrix}$ and $X = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix}$. For which $Y = \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix}$ in $\mathbb{R}^3$ does the system $AX = Y$ have a solution? Describe the answer in terms of subspaces of $\mathbb{R}^3$. Use the following approaches, and verify whether you get the same answer.

Two approaches: (i) Apply elementary row eliminations on $(A \,|\, Y)$, conclude when the system $AX = Y$ has solutions. (ii) Note that for every $X \in \mathbb{R}^4$, AX is nothing but a linear combination of the four column vectors of A:

$$AX = x_1 \begin{pmatrix} 3 \\ 2 \\ 1 \end{pmatrix} + x_2 \begin{pmatrix} -1 \\ 1 \\ -3 \end{pmatrix} + x_3 \begin{pmatrix} 8 \\ 7 \\ 0 \end{pmatrix} + x_4 \begin{pmatrix} 4 \\ 1 \\ 4 \end{pmatrix}.$$

So Y should belong into the column space of A. Furthermore, you may try to find a basis of the column space of A. To do that follow Q26.

Q30. Let $S = \{v_1, \dots, v_r\}$ be a collection of r vectors of a vector space V. Then show that S is linearly independent if and only if $\dim(\mathrm{Span}(S)) = r$. (See Corollary 2.26 in the notes)

Q31. Check whether the following vectors in $\mathbb{R}^4$ are linearly independent:

$$v_1 = \begin{pmatrix} 1 \\ 1 \\ 2 \\ 4 \end{pmatrix}, v_2 = \begin{pmatrix} 2 \\ -1 \\ -5 \\ 2 \end{pmatrix}, v_3 = \begin{pmatrix} 1 \\ -1 \\ -4 \\ 0 \end{pmatrix} \text{ and } v_4 = \begin{pmatrix} 2 \\ 1 \\ 1 \\ 6 \end{pmatrix}.$$

Two approaches: (1st). Consider a relation $x_1 v_1 + x_2 v_2 + x_3 v_3 + x_4 v_4 = 0$. It yields a homogeneous system of linear equations:

$$\begin{pmatrix} 1 & 2 & 1 & 2 \\ 1 & -1 & -1 & 1 \\ 2 & -5 & -4 & 1 \\ 4 & 2 & 0 & 6 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}.$$

Observe that this system has a non-trivial solution if and only if $\{v_1, v_2, v_3, v_4\}$ is linearly dependent.

(2nd). Consider the matrix A whose rows are the vectors v_1, v_2, v_3, v_4. By Q30, $\{v_1, v_2, v_3, v_4\}$ is linearly independent if and only if $\dim(\mathrm{Span}(\{v_1, v_2, v_3, v_4\})) = 4$. Since $\mathrm{Span}(\{v_1, v_2, v_3, v_4\})$ is nothing but the row space of A, we just have to compute row rank of A. So follow Q25(iv).

Q32. Let V be the vector space of all $m \times n$ matrices over $\mathbb{R}$ with usual vector addition and scalar multiplication. Show that $\dim(V) = mn$.

Hint: Consider the standard basis $\{A^{ij} : 1 \leqslant i \leqslant m, 1 \leqslant j \leqslant n\}$, where A^{ij} is the $m \times n$ matrix with (i, j) entry 1 and all other entries 0.

Q33. Let V be the vector space of all $n \times n$ matrices over $\mathbb{R}$ with usual vector addition and scalar multiplication. Show that the following are subspaces of V.

(i) The subset of V consisting of all symmetric matrices.
(ii) The subset of V consisting of all skew-symmetric (or anti-symmetric) matrices.
(iii) The subset of V consisting of all upper triangular matrices (i.e., $A_{ij} = 0$ for all $i > j$).

What is the dimension of each of these subspaces?

Show that the following are not subspaces of V.

(iv) The subset of V consisting of all invertible matrices.
(v) The subset of V consisting of all non-invertible matrices.
(vi) The subset of V consisting of all matrices A such that $A^2 = A$.

3 Linear Transformations

Throughout, U, V and W are vector spaces over $\mathbb{R}$, the set of real numbers. Note that in most of the cases, it is convenient to write a vector in $\mathbb{R}^n$ as a column vector $\begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$. But sometimes we write it as $(x_1, \ldots, x_n)$ also.

Q34. Let $T : V \to W$ be a linear transformation. What is $T(0)$, where 0 is the zero vector in V?

Hint. Note that $T(v) = T(v + 0) = T(v) + T(0)$ for any $v \in V$. Conclude that $T(0) = 0$, the zero vector in W.

Q35. Which of the following maps are linear? Justify your answer. If a map is not linear, give a counterexample where linearity fails explicitly for that map.

(i) $T : \mathbb{R}^1 \to \mathbb{R}^1$ defined by $T(x) = x + 2$ for every $x \in \mathbb{R}$.
(ii) $T : \mathbb{R}^1 \to \mathbb{R}^1$ defined by $T(x) = ax$ for every $x \in \mathbb{R}$, where $a \in \mathbb{R}$ is a constant.
(iii) $T : \mathbb{R}^1 \to \mathbb{R}^1$ defined by $T(x) = x^2$ for every $x \in \mathbb{R}$.
(iv) $T : \mathbb{R}^1 \to \mathbb{R}^1$ defined by $T(x) = \sin(x)$ for every $x \in \mathbb{R}$.
(v) $T : \mathbb{R}^1 \to \mathbb{R}^1$ defined by $T(x) = e^x$ for every $x \in \mathbb{R}$.
(vi) $T : \mathbb{R}^2 \to \mathbb{R}^1$ defined by $T(x_1, x_2) = x_1 x_2$ for every $(x_1, x_2) \in \mathbb{R}^2$.
(vii) $T : \mathbb{R}^2 \to \mathbb{R}^2$ defined by $T(x_1, x_2) = (x_2, x_1)$ for every $(x_1, x_2) \in \mathbb{R}^2$.
(viii) $T : \mathbb{R}^2 \to \mathbb{R}^2$ defined by $T(x_1, x_2) = (x_1, x_1)$ for every $(x_1, x_2) \in \mathbb{R}^2$.

(ix) $T : \mathbb{R}^2 \to \mathbb{R}^2$ defined by $T(x_1, x_2) = (0, x_1)$ for every $(x_1, x_2) \in \mathbb{R}^2$.

(x) $T : \mathbb{R}^2 \to \mathbb{R}^2$ defined by $T(x_1, x_2) = (0, 1)$ for every $(x_1, x_2) \in \mathbb{R}^2$.

Q36. Let $u_1 = (1, 2)$, $u_2 = (2, 1)$, $u_3 = (1, -1)$ and $v_1 = (1, 0)$, $v_2 = (0, 1)$, $v_3 = (1, 1)$. Is there a linear map $T : \mathbb{R}^2 \to \mathbb{R}^2$ such that $T(u_i) = v_i$ for every $i = 1, 2, 3$?

Hint. A linear map should respect every linear combination.

Q37. **Composition of linear maps:** Let $T : U \to V$ and $S : V \to W$ be linear maps. The composition $S \circ T : U \to W$ is defined by $(S \circ T)(u) := S(T(u))$ every $u \in U$. Show that the map $S \circ T : U \to W$ is linear.

Q38. **Application of composition of maps:** Show that the matrix multiplication is associative.

Hint. Let A, B, C be matrices of order $k \times l$, $l \times m$ and $m \times n$ respectively. To show that $(AB)C = A(BC)$, treat matrices as (linear) maps, e.g., C can be treated as a (linear) map from $\mathbb{R}^n$ to $\mathbb{R}^m$. Next observe that the composition of maps is associative.

Q39. Let $T : \mathbb{R}^n \to \mathbb{R}^n$ be a linear map. Is it true that if we know $T(v)$ for n different nonzero vectors in $\mathbb{R}^n$, then we know $T(v)$ for every vector in $\mathbb{R}^n$.

Hint. See what we have proved in Lecture 6. Try to analyze the statement when $n = 2$.

Q40. Define a map $T : \mathbb{R}^3 \to \mathbb{R}^3$ by

$$T(x_1, x_2, x_3) = (a_{11}x_1 + a_{12}x_2 + a_{13}x_3, a_{21}x_1 + a_{22}x_2 + a_{23}x_3, a_{31}x_1 + a_{32}x_2 + a_{33}x_3)$$

for every $(x_1, x_2, x_3) \in \mathbb{R}^3$, where $a_{ij} \in \mathbb{R}$ are constants. Is T linear? If yes, then write its matrix representation.

Hint. See the theorem concerning matrix representation of a linear map $T : \mathbb{R}^n \to \mathbb{R}^m$ proved in Lecture 6.

Q41. Deduce from Q40 that the map $S : \mathbb{R}^3 \to \mathbb{R}^3$ defined by

$$S(x_1, x_2, x_3) = (x_1 - x_2 + 2x_3, 2x_1 + x_2, x_2 + x_3)$$

for every $(x_1, x_2, x_3) \in \mathbb{R}^3$ is linear. Compute the range space and null space of S. Deduce the rank and nullity of S. Verify the Rank-Nullity Theorem. Conclude from the rank (resp. from the nullity), whether S is an isomorphism.

Hint. Write the matrix representation (say, A) of the linear map S. Observe that the null space of S is same as that of A. Moreover, the range space of S is same as the column space of A. Now one may follow the solution of Q27. Recall the equivalent conditions for a linear operator to be an isomorphism (shown in Lecture 7).

Q42. Let $u_1 = (1, 2)$, $u_2 = (2, 1)$ and $v_1 = (1, 1)$, $v_2 = (0, 1)$. Is there a linear map $T : \mathbb{R}^2 \to \mathbb{R}^2$ such that $T(u_i) = v_i$ for every $i = 1, 2$? If yes, then write the matrix representation of T.

Q43. Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be a linear map. Let u, v be two non-zero vectors such that $T(u) = 0$ and $T(v) = 0$. What are the possibilities of nullity of T? What about rank of T?

4 Eigenvalues and eigenvectors

Will add some problems by tomorrow.