CS1340: DISCRETE STRUCTURES II

QUIZ I

Instructions

- Answer all the questions
- Total Marks : 15 marks Max time : 1 hour 25 minutes.

(1) We defined a *permutation* of a set of distinct objects as an ordered arrangement of these objects. In algebra, there is an equivalent definition for permutation : a permutation of a set S is defined as a bijection from S to itself. For example, consider the set $S = \{1, 2, 3\}$ and we can represent the permutation $(3, 2, 1)$ as a bijective function, $\alpha : S \to S$, $\alpha(1) = 3, \alpha(2) = 2, \alpha(3) = 1$.

Prove that if n is odd then for any permutation α of the set $\{1, 2, \ldots, n\}$ the product $P(\alpha) = (1 - \alpha(1))(2 - \alpha(2)) \ldots (n - \alpha(n))$ is necessarily even.

(10 marks)

Answer: Suppose that $n = 2m+1$. Let $Odd = \{i : \pi(i) \text{ is odd}\}$. The size of Odd is $m+1$. There are only m even integers in $1, 2 \ldots, n$. So there must be at least one $i \in Odd$ such that i is odd – Pigeon Hole Principle. Then $\pi(i) - i$ is even.

(2) Show that $\mathbb{N} \times \mathbb{N}$ is countably infinite. You can assume that $\mathbb{N}$ does not contain 0.
Hint: Define a function that maps $(n, m) \in \mathbb{N} \times \mathbb{N}$ to $2^{n-1}(2m - 1)$.

(5 marks)

Answer: Show that $\phi : \mathbb{N} \times \mathbb{N} \to \mathbb{N}$, $(n, m) \mapsto 2^{n-1}(2m - 1)$ is bijective.

To prove One-one (injective) : Assume that $\phi(n_1, m_1) = \phi(n_2, m_2)$, i.e. $2^{n_1-1}(2m_1 - 1) = 2^{n_2-1}(2m_2 - 1)$. Since $2m_1 - 1$ and $2m_2 - 1$ are odd, the powers of 2 are equal, $2^{n_1-1} = 2^{n_2-1}$. This implies $n_1 = n_2$. And this forces $m_1 = m_2$.

To prove onto (surjectivity) : We can factorize any $x \in \mathbb{N}$ such that it is a unique product of primes (Fundamental Theorem of Arithmetic). Let $x = p_1^{i_1} p_2^{i_2} \cdots p_k^{i_k}$, where p_is are distinct primes and $i_j \in \mathbb{N} \cup \{0\}$. Lets collect all the powers of 2. There is only one even prime, w.l.o.g. let us assume $p_1 = 2$. Then $p_2^{i_2} \cdots p_k^{i_k}$ are a product of odd primes and can be written as

$(2l-1)$ for some $l \in \mathbb{N}$. Thus we have $x = 2^{i_1}(2l-1)$. Thus x is the image of $\phi(i, l)$ and ϕ is surjective.

Since we have a bijective function from $\mathbb{N}$ to $\mathbb{N} \times \mathbb{N}$, $\mathbb{N} \times \mathbb{N}$ is countably infinite.

March 1, 2019; Dept of CSE, IIT Hyderabad