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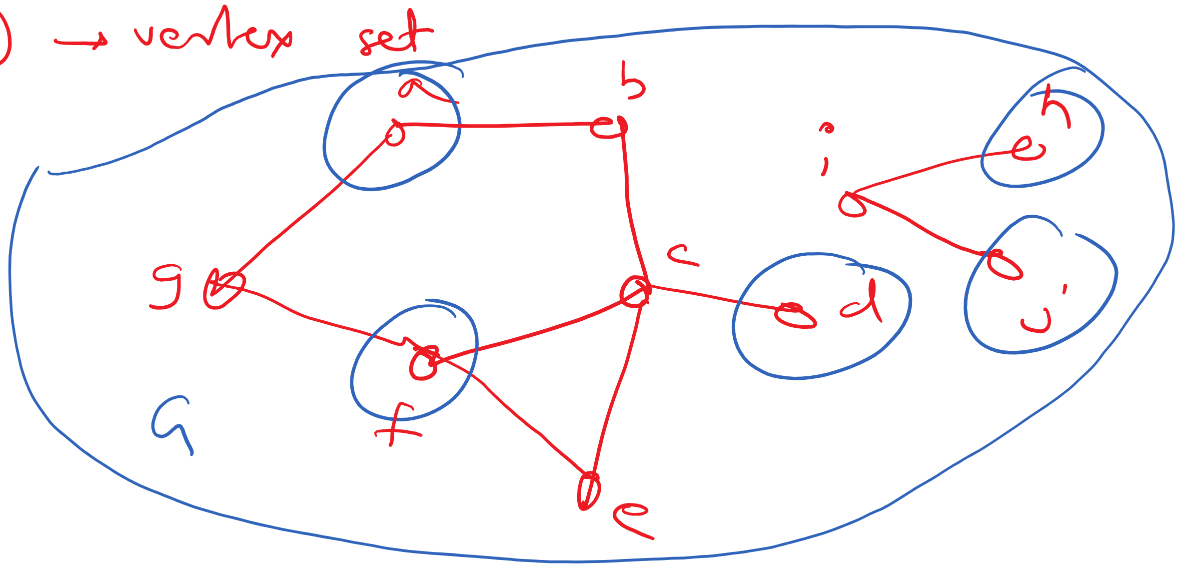
Hall's Theorem

G - graph $\qquad E(G)$ - edge set.

$V(G) \to$ vertex set



$$\begin{cases} V(G) = \{a, b, c, d, e, f, g, h, i, j\} \\ E(G) = \{ab, bc, cd, ce, ef, fc, gf, ga, hi, ij\} \end{cases}$$

$\subseteq V(G) \times V(G)$

Independent set of vertices } no two vertices in such a set have an edge between them

$A = \{a, f, d, h, j\}$

Bipartite graph A graph G is bipartite if its vertices can be partitioned into two parts say A and B, such that there is no edge between any two vertices that belong to the same part.

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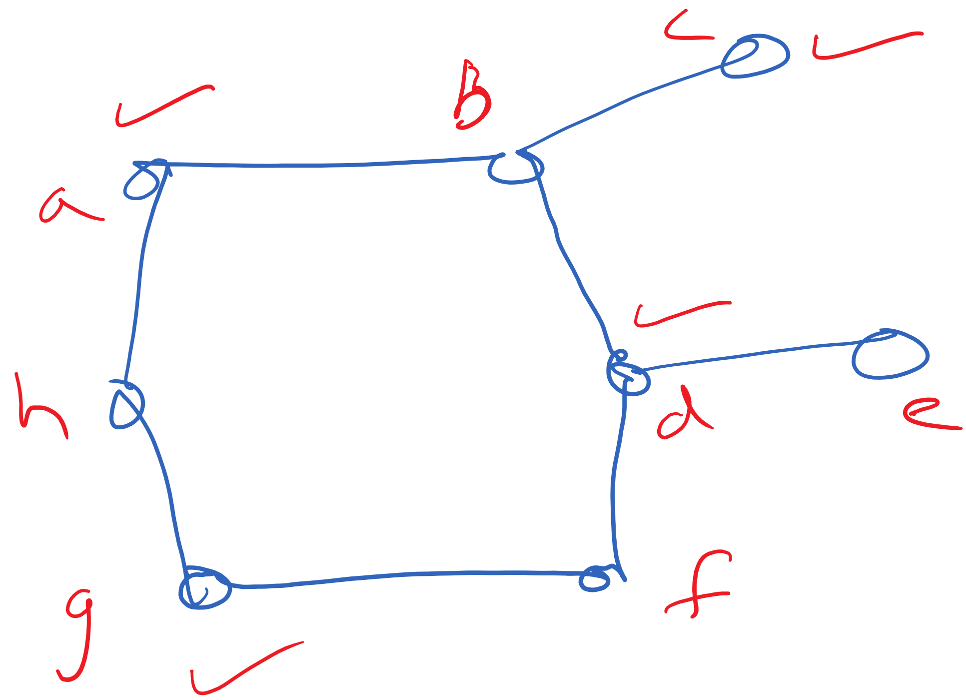
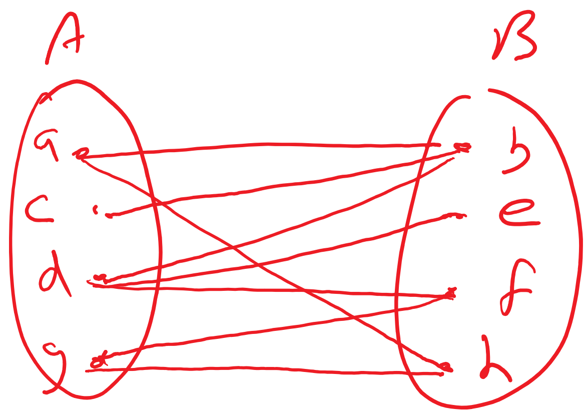
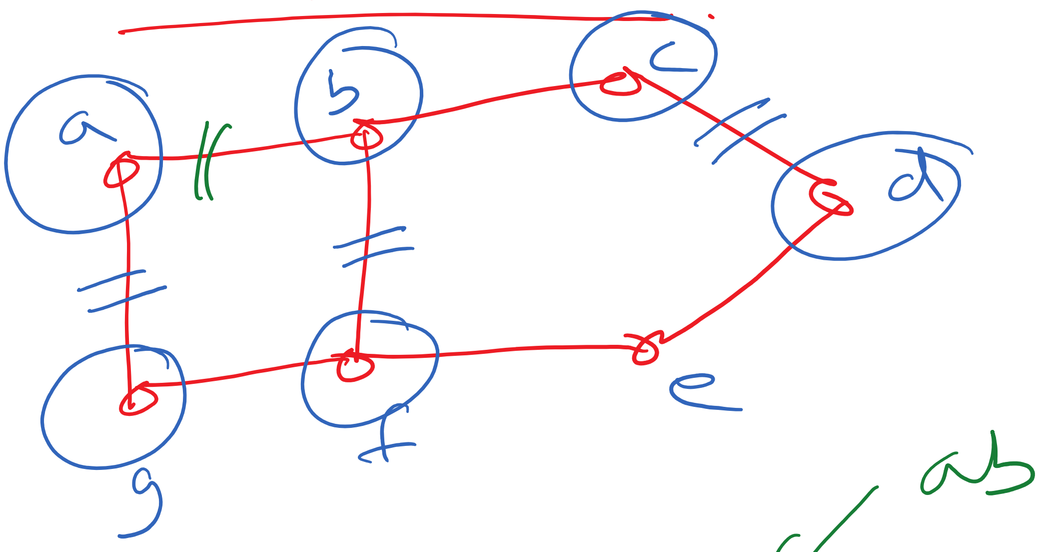


$A = \{a, c, d, g\}$

$B = \{b, e, f, h\}$



Matching in a graph



$M = \{ag, bf, cd\}$ is a matching

→ a subset of the edge set of a graph s.t. no two edges in this subset share an endpoint.

WRT a given matching, say M, a vertex v is said to be a matched vertex if there exists some edge in M that has v as an endpoint. Otherwise we call v an unmatched vertex

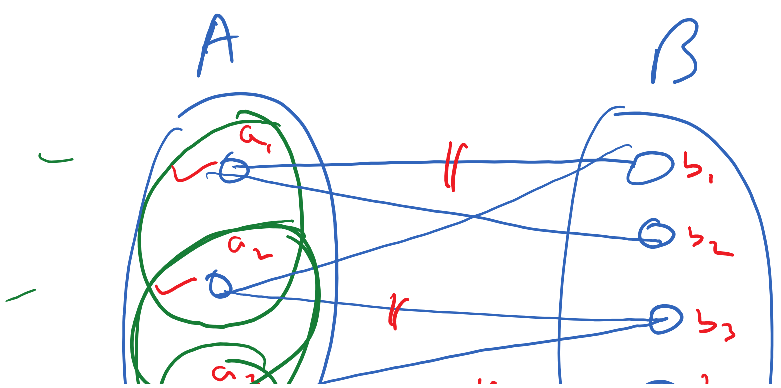
Hall's Theorem:

Let G be a bipartite graph with bipartition $\{A, B\}$. Then, G has a matching that matches all the vertices of A if and only if G satisfies the Hall's condition.

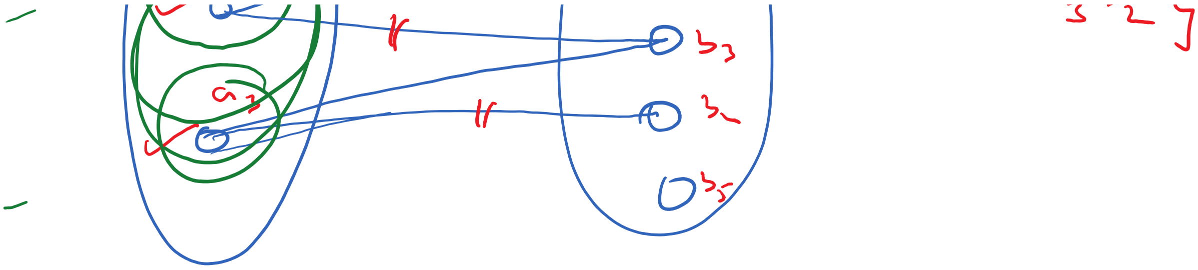
Hall's condition: $\forall S \subseteq A$,

$$|N_G(S)| \geq |S|.$$

$N_G(S)$ → neighborhood of S in G.



$M = \{a_1b_1, a_2b_3, a_3b_2\}$



$S = \{a_1, a_2\}$

$N_G(S) = \{b_1, b_2, b_3\}$

$3 = |N_G(S)| \geq |S| = 2$

$S = \{a_3\}$

$N_G(S) = \{b_3, b_4\}$

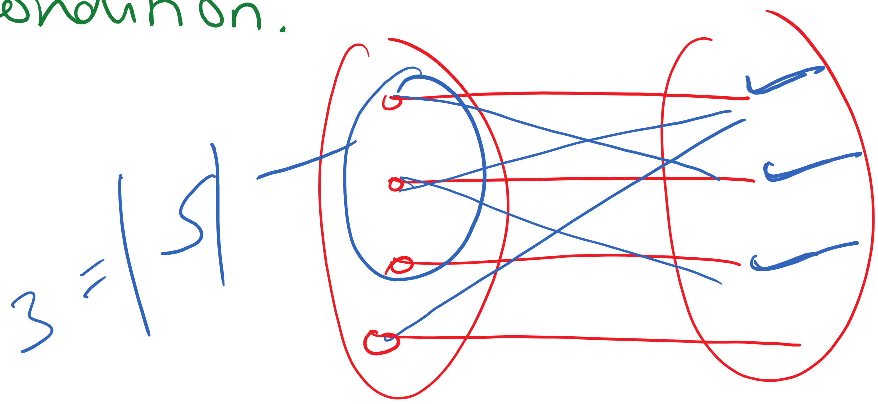
$S = \{a_2, a_3\}$

$N_G(S) = \{b_1, b_3, b_4\}$

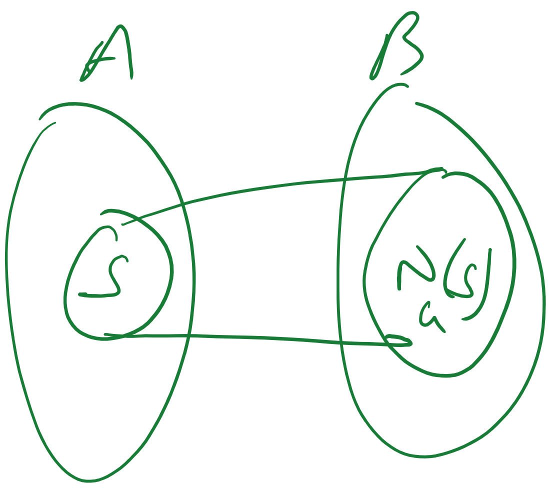
Proof:

$\Rightarrow$ If G has a matching that matches all the vertices in A, then G satisfies the Hall's condition.

Forward direction



$\Leftarrow$ If Hall's condition is true, then G has a matching that matches all the vertices in A



$\forall S,$

$|S| \leq |N_G(S)|$

$\Downarrow$

a matching of A.

a ... A.

Exercise Q → Prove the Hall's Thm using Dilworth's Theorem 2.

Sperner's Theorem (1928)

Theorem: Let $\mathcal{F}$ be a family of subsets of $[n]$. Further, it is given that $\mathcal{F}$ is an antichain under the containment relation (or subset relation). Then,

$\hookrightarrow \{1, 2, 3, \ldots, n\}$

$$|\mathcal{F}| \leq \binom{n}{\lfloor n/2 \rfloor}.$$

$$\mathcal{P} = (\text{Powerset}([n]), \subseteq)$$

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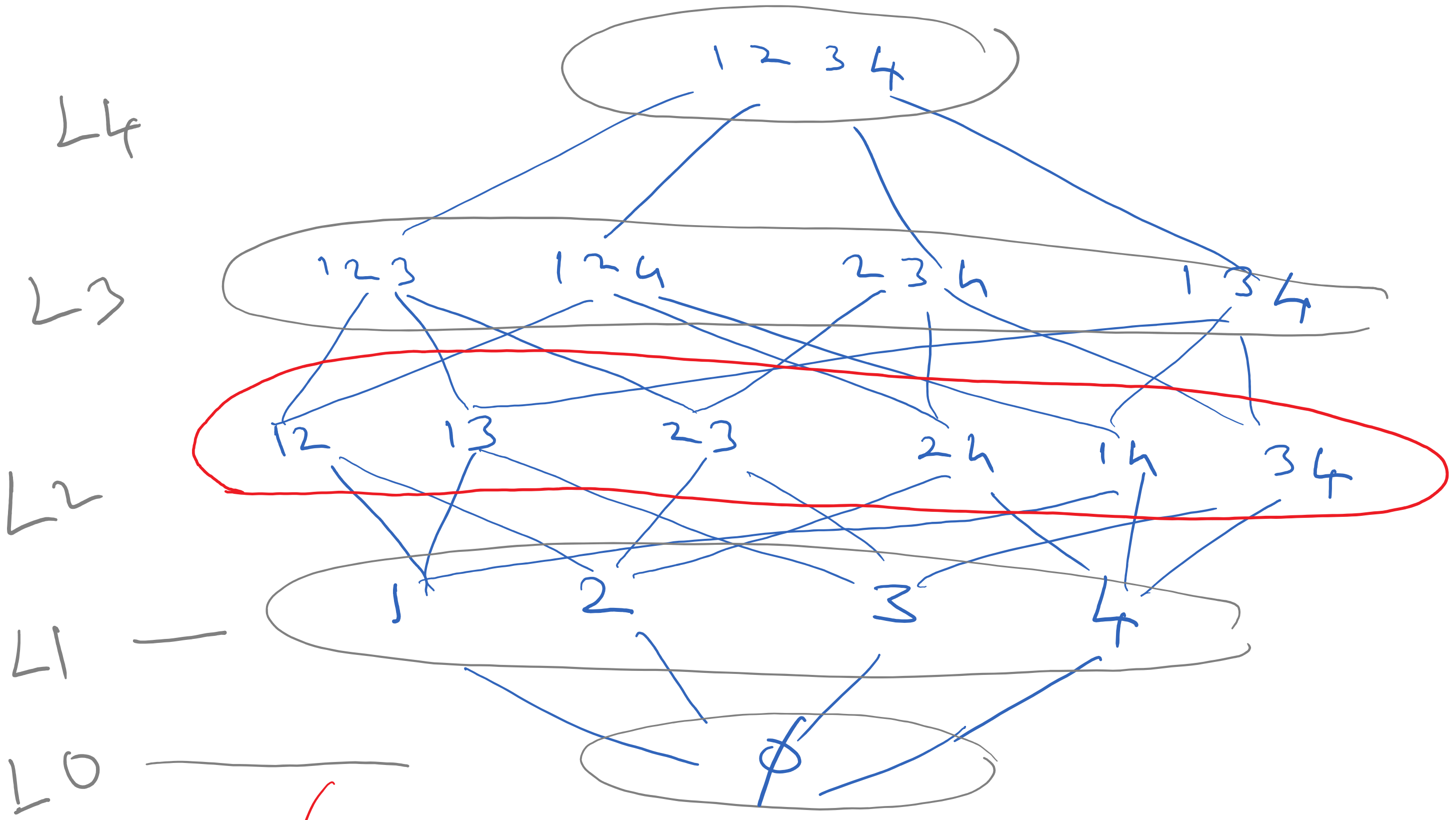
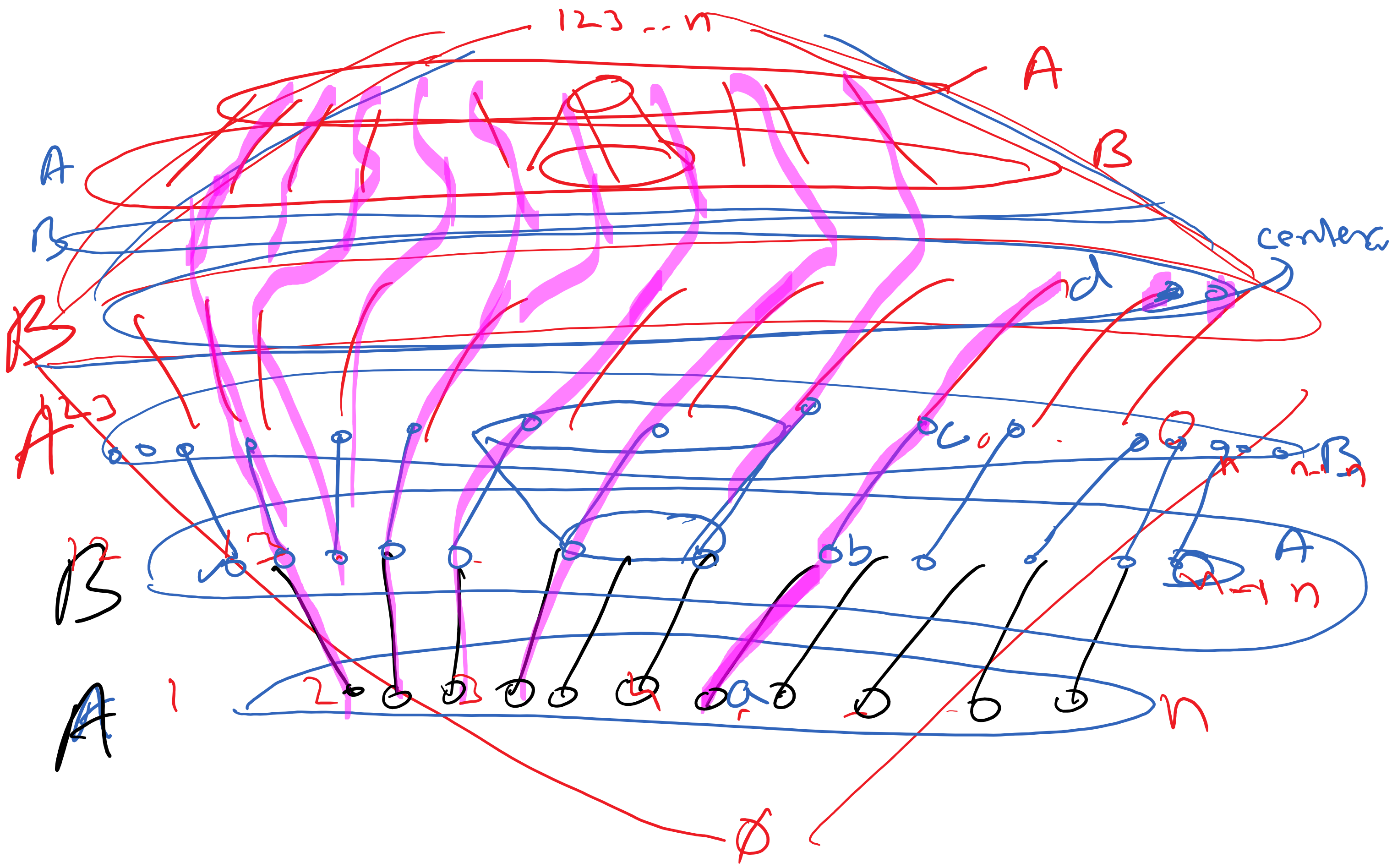
$n = 4.$



$|\text{largest antichain}| \leq \binom{4}{2} = 6$

Instead, we will show that the elements of the poset $\mathcal{P} = (\text{powerset}([n]), \subseteq)$ can be partitioned into $\binom{n}{\lfloor n/2 \rfloor}$ chains.

$$|\text{largest antichain}| \leq \left(\begin{array}{c}\text{no. of chains} \\ \text{into which we can} \\ \text{partition}\end{array}\right) = \binom{n}{\lfloor n/2 \rfloor}$$


123 ... n
A
B
A
B
center
d
B
A
B
n-B
B
b
A
n-A
A
1
2
3
n
ϕ

LYM Inequality : Let $\mathcal{F} = \{A_1, A_2, \ldots, A_m\}$ be a family of subsets of $[n]$. Further, $\mathcal{F}$ is an antichain under the containment relation. Then,

$$\sum_{i=1}^{m} \frac{1}{\binom{n}{\lfloor n/2 \rfloor}} \leq \sum_{i=1}^{m} \left(\frac{1}{\binom{n}{|A_i|}} \right) \leq 1.$$

$$\sum_{i=1}^{m} \frac{1}{\binom{n}{\lfloor n/2 \rfloor}} = \frac{m}{\binom{n}{\lfloor n/2 \rfloor}}$$

That implies,

$$m \leq \binom{n}{\lfloor n/2 \rfloor}.$$

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Proof of LYM Ineq

$\mathcal{F} = \{A_1, \ldots, A_m\}$ → antichain

<u>To show:</u> $\sum_{i=1}^{m} \frac{1}{\binom{n}{|A_i|}} \leq 1$.

Let σ be a permutation of $[n]$. For any set $A_i \in \mathcal{F}$, we say A_i is **present** in σ if the elements of A_i are precisely the first $|A_i|$ elements of σ.

$A_i = \{3, 5, 6\}$

σ_1 : 1 2 4 3 5 6 7 - -

σ_2 : 5 6 3 2 1 4 - -

6 5 3

A_i

σ_2 : 3 6 8 2 1 4 - -

3 5 6 A_j 1

A_i is present in σ_2 but not in σ_1.

$S = \{ (\sigma, A_i) :$ σ is any permutation of $[n]$ and $A_i \in \mathcal{F}$ is "present" in $\sigma \}$

$$\sum_{i=1}^{m} |A_i|! \, (n - |A_i|)! \leq |S| \leq n! \; .$$

$$\therefore \sum_{i=1}^{m} \left(\frac{1}{\binom{n}{|A_i|}} \right) \leq 1$$

$\square$