Analysis of Karatsuba Algorithm

In the class we set base case to be when $n = 1$. Since one of the subproblem created could be multiplication of two $\frac{n}{2} + 1$ digit numbers (i.e., $(x + y)(u + v)$), the correct base case is when $n = 2$. Otherwise when $n = 2$, we may end up creating a subproblem for multiplication of two 2-digit numbers. When $n = 2$, we use long multiplication that will take at most $5 \cdot 2^2 = 20$ operation. Then, the running time of the Karatsuba algorithm will be as follows.

$$T(n) = \begin{cases} 8n + 3T(\frac{n}{2} + 1) & \text{if} \quad n > 2 \\ 20 & \text{if} \quad n \leq 2 \end{cases}$$

Let $\ell = \log_2 n$. Then, recall the recurrence tree method explained in class. The red colored term is the number of operations in the base case (i.e., the operations in the leaves of the tree).

$$\begin{aligned}
T(n) &\leq 8n + 3 \cdot 8(\frac{n}{2} + 1) + 3^2 \cdot 8(\frac{n}{2^2} + \frac{1}{2} + 1) + \ldots + 3^\ell \cdot 8(\frac{n}{2^\ell} + \frac{1}{2^\ell} + \ldots \frac{1}{2} + 1) + {\color{red}3^{\ell+1} \cdot 20} \\
&\leq 8n\left(1 + \frac{3}{2} + \left(\frac{3}{2}\right)^2 + \ldots + \left(\frac{3}{2}\right)^\ell\right) + 3 \cdot 8 + 3^2 \cdot 8(\frac{1}{2} + 1) + 3^\ell \cdot 8(\frac{1}{2^\ell} + \ldots \frac{1}{2} + 1) + {\color{red}60 \cdot 3^\ell} \\
&\leq 8n\left(1 + \frac{3}{2} + \left(\frac{3}{2}\right)^2 + \ldots + \left(\frac{3}{2}\right)^\ell\right) + 8\left(\left(3 + 3^2 + \ldots + 3^\ell\right)\left(\frac{1}{2^\ell} + \ldots \frac{1}{2} + 1\right)\right) + {\color{red}60 \cdot 3^\ell} \\
&\leq 8n\left(\frac{3}{2}\right)^\ell\left(1 + \frac{2}{3} + \left(\frac{2}{3}\right)^2 + \ldots\right) + \left(8 \cdot 3^\ell\left(1 + \frac{1}{3} + \frac{1}{3^2} + \ldots\right) \cdot 2\right) + {\color{red}60 \cdot 3^\ell} \\
&\leq 24n\left(\frac{3}{2}\right)^\ell + 24 \cdot 3^\ell + {\color{red}60 \cdot 3^\ell} \\
&\leq 24n\left(\frac{3}{2}\right)^\ell + 84 \cdot 3^\ell \\
&\leq 24n\left(\frac{3}{2}\right)^{\frac{\log_{3/2} n}{\log_{3/2} 2}} + 84 \cdot 3^{\frac{\log_3 n}{\log_3 2}} \\
&\leq 24 \cdot n \cdot n^{\frac{1}{\log_{3/2} 2}} + 84 \cdot n^{\frac{1}{\log_3 2}} \\
&\leq 24 \cdot n^{1.5849} + 84 \cdot n^{1.5849} \\
&\leq 108 \cdot n^{1.5849}
\end{aligned}$$