EP 1027: Maxwell's Equations and Electromagnetic Waves

Instructor: Shubho Roy[1]
(Dept. of Physics)

Lecture 7 (Make up lecture)

April 8, 2019

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Agenda

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- Short Review of last lecture (7)

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- Magnetic force on point charge and current element
- Maxwell's equations

References/Readings

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- Griffiths, D.J., **Introduction to Electrodynamics, Ch. 5,7**

Review of electrostatics

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- Electric fields are produced by charged objects, the electric field strength obeys

$$\boldsymbol{\nabla} \cdot \mathbf{E}(\mathbf{x}) = \frac{\rho}{\epsilon_0}, \quad \boldsymbol{\nabla} \times \mathbf{E}(\mathbf{x}) = 0.$$

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Magnetic fields: Produced by charges in motion and affects charges in motion

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- Answer provided by Lorentz force law,

$$F = Q\mathbf{V} \times \mathbf{B}.$$

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- $\rho = 0$, **but** $\mathbf{E} \neq 0$ **for steady currents in conductors.**

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- **Ampere Law**

$$\nabla\times\mathbf{B} = \mu_0\,\mathbf{j}. \tag{2}$$

$$\oint_C \mathbf{B}\cdot d\mathbf{l} = \mu_0\int_S dS\,\mathbf{j}\cdot\hat{\mathbf{n}}.$$

- **Poisson (like) Equation for the vector potential,**

$$\boldsymbol{\nabla}^2\mathbf{A} - \boldsymbol{\nabla}\left(\boldsymbol{\nabla}\cdot\mathbf{A}\right) = -\mu_0\mathbf{j}.$$

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- **Force on a current element**

$$\mathbf{F} = dq\,\mathbf{v} \times \mathbf{B} = dq\,\frac{d\mathbf{x}}{dt} \times \mathbf{B} = \frac{dq}{dt}\,d\mathbf{x} \times \mathbf{B} = I\,d\mathbf{x} \times \mathbf{B}.$$

Time Varying Magnetic fields: Electromagnetic Induction

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- **Maxwell's insight: Need to add a "displacement current" term to the RHS of Ampere's law,**

$$\boldsymbol{\nabla} \times \mathbf{B} = \mu_0 \left(\mathbf{j} + \epsilon_0 \frac{\partial \mathbf{E}}{\partial t} \right).$$

Maxwell's equations in vacuum and Potential formulation

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- **Maxwell's Equations**

$$\nabla \cdot \mathbf{E} = \frac{\rho}{\epsilon_0}, (\textbf{Gauss})$$

$$\nabla \times \mathbf{E} + \frac{\partial \mathbf{B}}{\partial t} = 0, (\textbf{Faraday})$$

$$\nabla \cdot \mathbf{B} = 0, \ (?)$$

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- The first and the last Maxwell's equation are truly equations of motion as their RHS contain the sources $(\rho, \mathbf{j})$. They tell you, how or what kind of field you produce for a given source.

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- The first and the last Maxwell's equation are truly equations of motion as their RHS contain the sources $(\rho, \mathbf{j})$. They tell you, how or what kind of field you produce for a given source.
- The second and third Maxwell's equations have no sources, i.e. they must hold for all situations. So, these are not equations of motion, rather they are *constraints* that must hold for all bona fide solutions of the equation of motion. (Bianchi Identities)

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$$\mathbf{B} = \nabla \times \mathbf{A}.$$
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Plug this in Maxwell Eq. 2, aka Faraday's law, we get:.
- $$\nabla \times \left(\mathbf{E} + \frac{\partial \mathbf{A}}{\partial t}\right) = 0,$$
$$\implies \mathbf{E} + \frac{\partial \mathbf{A}}{\partial t} = -\nabla \Phi$$
$$\implies \mathbf{E} = -\nabla \Phi - \frac{\partial \mathbf{A}}{\partial t}.$$
Φ is the new Electric potential.

Potential formulation of Maxwell's Equations: Gauge Symmetry

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$$\boldsymbol{\nabla}^2 \Phi + \frac{\partial \left(\boldsymbol{\nabla} \cdot \mathbf{A}\right)}{\partial t} = -\frac{\rho}{\epsilon_0},$$

$$\left(\boldsymbol{\nabla}^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right)\mathbf{A} - \boldsymbol{\nabla}\left(\boldsymbol{\nabla} \cdot \mathbf{A} + \frac{1}{c^2}\frac{\partial \Phi}{\partial t}\right) = -\mu_0 \mathbf{j}$$

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- Recall, **A**is ambiguous upto addition of a gradient of a scalar, χ,

$$\mathbf{A}' = \mathbf{A} + \boldsymbol{\nabla}\chi$$

$$\boldsymbol{\nabla}\times\mathbf{A}' = \boldsymbol{\nabla}\times\mathbf{A} = \mathbf{B}.$$

Potential formulation of Maxwell's Equations: Gauge Symmetry

- Plugging, $\mathbf{E} = -\boldsymbol{\nabla}\boldsymbol{\Phi} - \frac{\partial \mathbf{A}}{\partial \mathbf{t}}$, $\mathbf{B} = \boldsymbol{\nabla} \times \mathbf{A}$ in Gauss law and Ampere law (i.e. the Maxwell Eq.s 1 and 4),

$$\boldsymbol{\nabla}^2 \Phi + \frac{\partial \left(\boldsymbol{\nabla} \cdot \mathbf{A}\right)}{\partial t} = -\frac{\rho}{\epsilon_0},$$

$$\left(\boldsymbol{\nabla}^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right)\mathbf{A} - \boldsymbol{\nabla}\left(\boldsymbol{\nabla} \cdot \mathbf{A} + \frac{1}{c^2}\frac{\partial \Phi}{\partial t}\right) = -\mu_0 \mathbf{j}$$

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- This implies, Φ is ambiguous as well,

$$\Phi' = \Phi - \frac{\partial \chi}{\partial t}$$

$$\mathbf{E} = -\boldsymbol{\nabla}\Phi - \frac{\partial \mathbf{A}}{\partial t} = -\boldsymbol{\nabla}\Phi' - \frac{\partial \mathbf{A}'}{\partial t}.$$

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- Maxwell's Equations look like,

$$\left(\boldsymbol{\nabla}^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right)\Phi = -\frac{\rho}{\epsilon_0},$$

$$\left(\boldsymbol{\nabla}^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right)\mathbf{A} = -\mu_0 \mathbf{j}.$$

  The operator, $\Box \equiv \boldsymbol{\nabla}^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}$ is called the **D'Alembertian operator or the wave-operator**

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- Why are there 8 Maxwell's equations when written in terms of the electric and magnetic fields (**E** and **B**) while there are just 4 Maxwell's equation when expressed in terms of the potentials Φ and **A**?
- Answer: When one introduces the potentials, one is already solving the homogeneous (sourceless) Maxwell's equations thus getting rid of 4 equations. All that remains are the inhomogeneous Maxwell's equations i.e. Maxwell equations with sources, which are 4 in number.