1 A vector identity

In class we learned how to evaluate $\boldsymbol{\nabla} r^n$ using the index notation and the Einstein summation convention. However, I felt I was a bit too quick, here I am repeating the steps with the appropriate elaboration. In index notation with summation convention,

$$\boldsymbol{\nabla} = \hat{\mathbf{e}}_k \, \frac{\partial}{\partial x_k}$$

Thus,

$$\begin{aligned}
\boldsymbol{\nabla} r^n &= \hat{\mathbf{e}}_k \, \frac{\partial r^n}{\partial x_k} \\
&= \hat{\mathbf{e}}_k \; n \, r^{n-1} \frac{\partial r}{\partial x_k}.
\end{aligned} \tag{1}$$

Now $r = |\mathbf{x}| = (x_l \, x_l)^{1/2}$. Then,

$$\begin{aligned}
\frac{\partial r}{\partial x_k} &= \frac{\partial}{\partial x_k} \, (x_l \, x_l)^{1/2} \\
&= \frac{1}{2} \, (x_l \, x_l)^{\frac{1}{2}-1} \, \frac{\partial \, (x_m \, x_m)}{\partial x_k} \\
&= (x_l \, x_l)^{-\frac{1}{2}} \, x_m \, \underbrace{\frac{\partial x_m}{\partial x_k}}_{=\delta_{mk}} \\
&= (x_l \, x_l)^{-\frac{1}{2}} \;\, x_m \, \delta_{mk} \\
&= (x_l \, x_l)^{-\frac{1}{2}} \;\, x_k \\
&= \frac{x_k}{r}.
\end{aligned}$$

Using this in the rhs of (1) we get,

$$\begin{aligned}
\boldsymbol{\nabla} r^n &= \hat{\mathbf{e}}_k \; n \, r^{n-1} \frac{\partial r}{\partial x_k} \\
&= \hat{\mathbf{e}}_k \; n \, r^{n-1} \frac{x_k}{r} \\
&= n \, r^{n-2} \, (\hat{\mathbf{e}}_k \; x_k) \\
&= n \, r^{n-2} \mathbf{x}.
\end{aligned}$$

2 Determinant using Levi Civita

In class I defined the determinant of a 3×3 matrix, M by the formula,

$$|M| = M_{1l} \, M_{2m} \, M_{3n} \, \epsilon_{lmn}.$$

Let's check that it coincides with the expression of the determinant you are familiar from high school. In the above expression l, m, n are dummy (repeated) indices and hence each of those three indices represent a sum over all possible values $1, 2, 3$. Lets perform the sums one by one, first the sum over l

$$\begin{aligned}
|M| &= M_{1l} \, M_{2m} \, M_{3n} \, \epsilon_{lmn} \\
&= M_{11} \, M_{2m} \, M_{3n} \, \epsilon_{1mn} + M_{12} \, M_{2m} \, M_{3n} \, \epsilon_{2mn} + M_{13} \, M_{2m} \, M_{3n} \, \epsilon_{3mn}
\end{aligned} \tag{2}$$

Next we perform the sum over m and then over n for each of the three terms in the above expression. The first term on the rhs when summed over all values of m gives,

$$\begin{aligned} M_{11}\, M_{2m}\, M_{3n}\, \epsilon_{1mn} &= M_{11}\, M_{21}\, M_{3n}\, \cancelto{0}{\epsilon_{11n}} + M_{11}\, M_{22}\, M_{3n}\, \epsilon_{12n} + M_{11}\, M_{23}\, M_{3n}\, \epsilon_{13n} \\ &= M_{11}\, M_{22}\, M_{3n}\, \epsilon_{12n} + M_{11}\, M_{23}\, M_{3n}\, \epsilon_{13n}. \end{aligned}$$

Finally doing the sum over n, it is clear that for the first term has non-zero contribution for $n = 3$ and the second term can only be non-zero for $n = 2$. Thus,

$$\begin{aligned} M_{11}\, M_{2m}\, M_{3n}\, \epsilon_{1mn} &= M_{11}\, M_{22}\, M_{33}\, \epsilon_{123} + M_{11}\, M_{23}\, M_{32}\, \epsilon_{132} \\ &= M_{11}\, M_{22}\, M_{33} - M_{11}\, M_{23}\, M_{32} \\ &= M_{11}\, (M_{22}\, M_{33} - M_{23}\, M_{32}) \\ &= M_{11} \begin{vmatrix} M_{22} & M_{23} \\ M_{32} & M_{33} \end{vmatrix}. \end{aligned}$$

Similarly one can show that,

$$M_{12}\, M_{2m}\, M_{3n}\, \epsilon_{2mn} = -M_{12} \begin{vmatrix} M_{21} & M_{23} \\ M_{31} & M_{33} \end{vmatrix}$$

and,

$$M_{13}\, M_{2m}\, M_{3n}\, \epsilon_{3mn} = M_{13} \begin{vmatrix} M_{21} & M_{22} \\ M_{31} & M_{32} \end{vmatrix}$$

Gathering all three contributions in (2)

$$\begin{aligned} |M| &= M_{11}\, M_{2m}\, M_{3n}\, \epsilon_{1mn} + M_{12}\, M_{2m}\, M_{3n}\, \epsilon_{2mn} + M_{13}\, M_{2m}\, M_{3n}\, \epsilon_{3mn} \\ &= M_{11} \begin{vmatrix} M_{22} & M_{23} \\ M_{32} & M_{33} \end{vmatrix} - M_{12} \begin{vmatrix} M_{21} & M_{23} \\ M_{31} & M_{33} \end{vmatrix} + M_{13} \begin{vmatrix} M_{21} & M_{22} \\ M_{31} & M_{32} \end{vmatrix} \\ &= \begin{vmatrix} M_{11} & M_{12} & M_{13} \\ M_{21} & M_{22} & M_{23} \\ M_{31} & M_{32} & M_{33} \end{vmatrix}. \end{aligned}$$