Probabilistic Method in combinatorics ...

[Alon, Spencer]

Ramsey Numbers ...

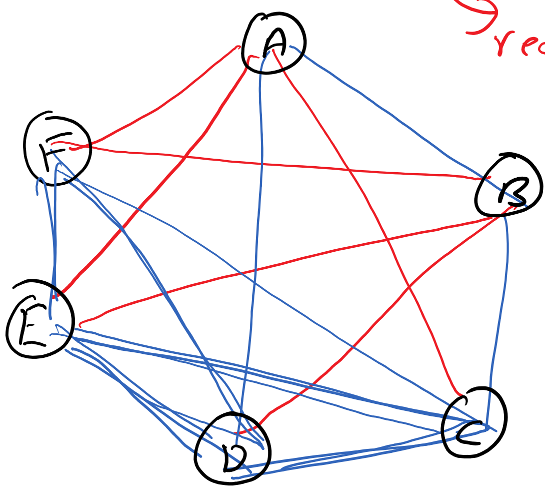
Q. Show that in any group of 6, there are either 3 mutual friends or 3 mutual strangers.

blue Δ.

red Δ.

— friends

— strangers



Δ CDE

Δ DEF

Δ CDF

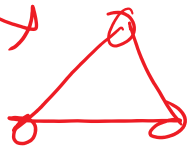
K_6 → complete graph on 6 vertices

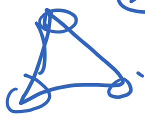
↳ all $\binom{6}{2}$ edges are present.

In other words,

no matter how you color the edges of a K_6 with RED and BLUE colors, you will always encounter either a BLUE K_3

or a RED K_3.





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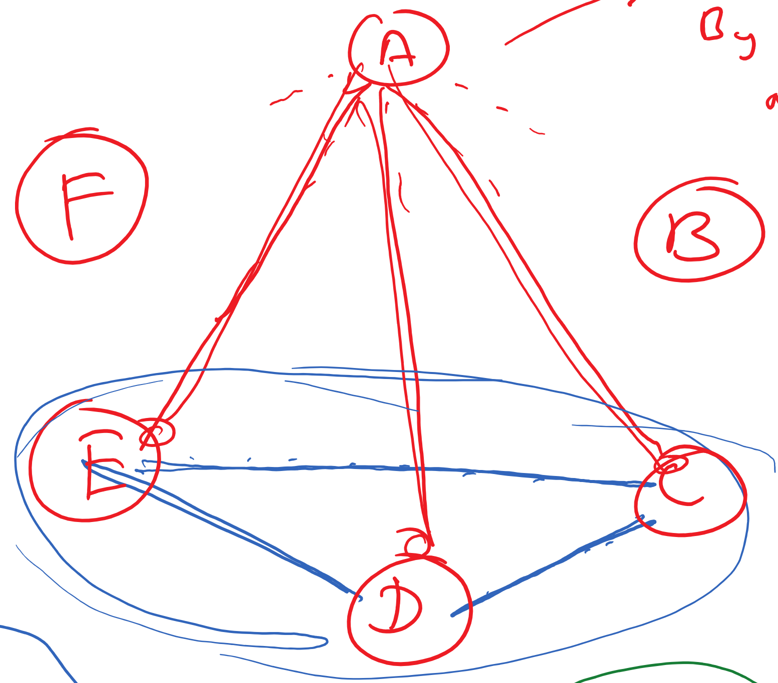
Proof:



5 edges
By pigeonhole, at least 3 are of same color.

complete graph on n vertices

$$R(3,3) \le 6.$$

Show that $R(3,3) = 6$. $\square$

$R(k,k)$ is the min n such that no matter how we color the edges of a K_n with two colors, say Red and Blue, we will surely encounter either a Red K_k or a Blue K_k or both.

Symmetric Ramsey Numbers.

complete graph on k vertices? or

or

k-clique.

$R(k, \ell)$

Blue ℓ-clique.

Red k-clique

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Erdős

$R(5,5) = \underline{\underline{\qquad}}$

$R(6,6) = \underline{\qquad}$

$R(3,3) = 6$

$R(4,4) = \underline{18}$

$43 \leq R(5,5) \leq 48$

$102 \leq R(6,6) \leq 165$

$205 \leq R(7,7) \leq 540$

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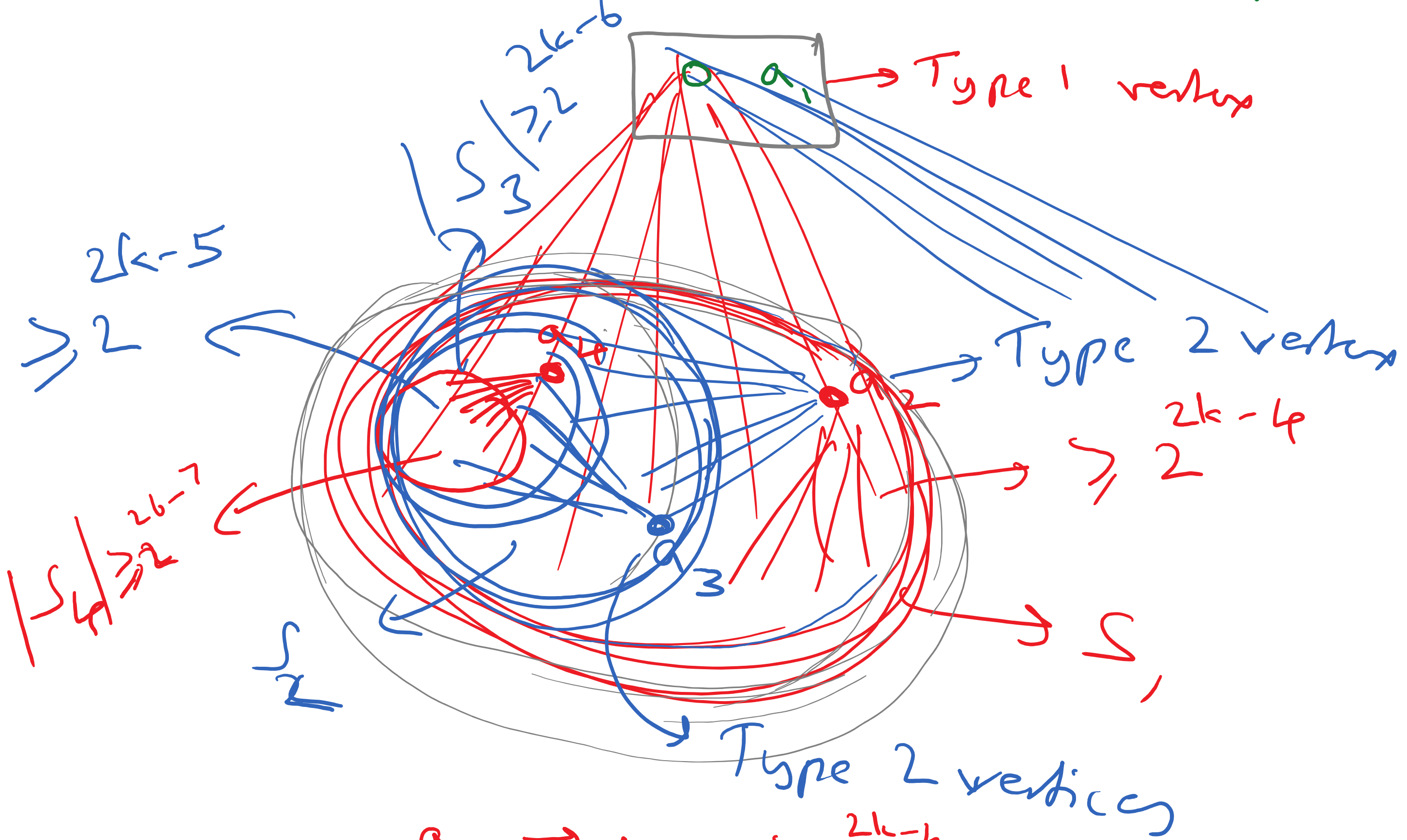
Theorem. $R(k,k) \le 2^{2k-3}$.

Proof: Using pigeonhole principle.



$a_1 \to$ sees $\ge 2^{2k-4}$ red edges

$a_2 \to$ " $\ge 2^{2k-5}$ blue edges

$a_3 \to$ " $\ge 2^{2k-6}$ blue edges

$a_4 \to$ sees $\ge 2^{2k-7}$ red edges

$\vdots$

$a_{2k-3} \to$ sees ≥ 1 blue edges

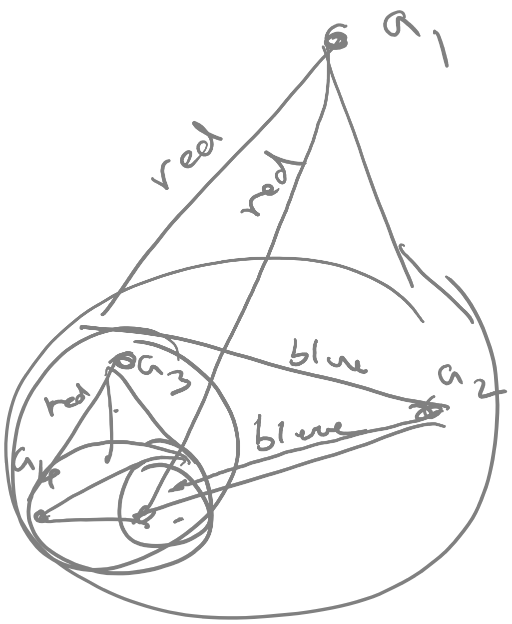
$2k-3$ vertices

by pigeonhole principle,

b

$k-1$ of these vertices are of the same type. WLOG, let there $k-1$ vertices be of Type 1.

Together with b, we get red k-clique.



Theorem. $2^{2k-3} \geq R(k,k) > 2^{\lfloor k/2 \rfloor}$, for any $k \geq 3$.

Proof:

Let $n = 2^{\lfloor k/2 \rfloor}$.

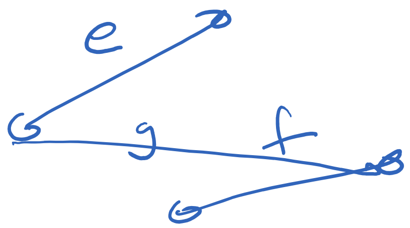
In order to prove the theorem, we need to show a 2-coloring of the edges of K_n which has neither a red k-clique nor a blue k-clique.

Example. $k = 20$, $2^{\lfloor 20/2 \rfloor} = 2^{10} = 1024$

$$\binom{1024}{2} = \frac{1024 \times 1023}{2} > 1{,}000{,}000 \text{ edges}$$

$n = 2^{\lfloor k/2 \rfloor}$.

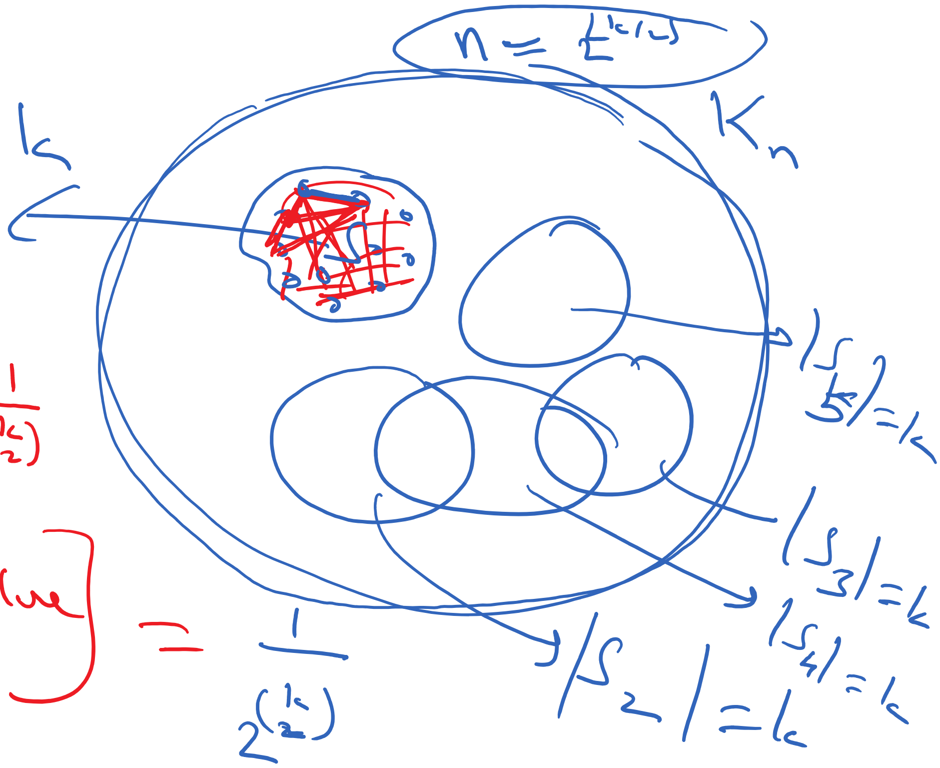
Take K_n. Each edge is given a red or blue color independently and uniformly at random.



$\frac{1}{2}$ Red

$\frac{1}{2}$ Blue

Or at each edge we toss an unbiased coin. If its HEADS give that edge a RED color. O/w, give it BLUE color.



$$\Pr\left[S \text{ is a Red } k\text{-clique}\right] = \frac{1}{2^{\binom{k}{2}}}$$

lly, $\Pr\left[S \text{ is a Blue clique}\right] = \frac{1}{2^{\binom{k}{2}}}$

$\binom{n}{k} \longrightarrow$ k-sized subsets $S_1, S_2, S_3, \ldots, S_{\binom{n}{k}}$.

'BAD' event: one of $S_1, S_2, \ldots, S_{\binom{n}{k}}$ is either a RED k-clique or a BLUE k-clique.

$$\Pr\Big[\left(S_1 \text{ is a Red } k\text{-clique}\right) \cup \left(S_2 \text{ is a Red } k\text{-clique}\right) \cup \ldots \cup \left(S_{\binom{n}{k}} \text{ is a Red } k\text{-clique}\right)$$
$$\cup \left(S_1 \text{ is a Blue } k\text{-clique}\right) \cup \left(S_2 \text{ is a Blue } k\text{-clique}\right) \cup \ldots \cup \left(S_{\binom{n}{k}} \text{ is a Blue } k\text{-clique}\right)\Big]$$

$$\leq \sum_{i=1}^{\binom{n}{k}} \left(\Pr\left[S_i \text{ is a Red } k\text{-cli}\ldots\right] + \Pr\left[S_i \text{ is } \ldots\right.\right.$$

$$= \sum_{i=1} \left(\Pr\left[\ldots \text{ red } k\text{-clique} \right] + \Pr\left[S_i \text{ is a blue } k\text{-clique} \right] \right)$$

[By Union Bound]

$$\Pr[A \cup B] \leq \Pr[A] + \Pr[B]$$

$$\leq \binom{n}{k} \left(\frac{1}{2^{\binom{k}{2}}} + \frac{1}{2^{\binom{k}{2}}} \right)$$

$$= \frac{\binom{n}{k}}{2^{\binom{k}{2}-1}} \quad \text{— } (A)$$

If $(A) < 1$, then there is a non-zero probability that the random coloring we did contains neither a Red k-clique nor a blue k-clique.

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$n = 2^{(k/2)}$

A

$$\frac{\binom{n}{k}}{2^{\binom{k}{2}-1}} < \frac{n^k}{k!\, 2^{\binom{k}{2}-1}}$$

$$\le \frac{2^{\frac{k}{2}+1}\, \cancel{2^{\frac{k^2}{2}}}}{k!\, \cancel{2^{\frac{k^2}{2}}}}$$

[substitution $n = 2^{k/2}$]

$$= \frac{2^{\frac{k}{2}+1}}{k!}$$

< 1 (when $k \ge 3$)

$\square$

$n = 2^{\frac{20}{2}} = 2^{10} = 1024$

$k = 20$

$$\frac{2^{11}}{20!} << 1.$$

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Algo. $\rightarrow$ to find a coloring that contains neither red k-clique nor blue k-clique

$2^{\binom{n}{2}}$ colorings.