EP 1027: Homework Assignment 1

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1. Using the Einstein summation convention for indices, show that the scalar triple product, $\mathbf{A} \cdot (\mathbf{B} \times \mathbf{C})$ is (i) invariant under cyclic permutation of $\mathbf{A}, \mathbf{B}, \mathbf{C}$, i.e.

$$\mathbf{A} \cdot (\mathbf{B} \times \mathbf{C}) = \mathbf{B} \cdot (\mathbf{C} \times \mathbf{A})$$

(ii)invariant under swap of the "dot" and "cross",

$$\mathbf{A} \cdot (\mathbf{B} \times \mathbf{C}) = (\mathbf{A} \times \mathbf{B}) \cdot \mathbf{C}$$

(5+5=10 points)

Solution (i):

$$\begin{aligned}
\mathbf{A} \cdot (\mathbf{B} \times \mathbf{C}) &= A^k (\mathbf{B} \times \mathbf{C})^k \\
&= A^k \left(\epsilon^{ijk} B^i C^j\right) \\
&= \epsilon^{ijk} A^k B^i C^j \\
&= B^i \left(\epsilon^{ijk} C^j A^k\right) \\
&= B^i \left(\epsilon^{jki} C^j A^k\right) \\
&= B^i (\mathbf{C} \times \mathbf{A})^i . \\
&= \mathbf{B} \cdot (\mathbf{C} \times \mathbf{A}).
\end{aligned}$$

Solution (ii):

$$\begin{aligned}
\mathbf{A} \cdot (\mathbf{B} \times \mathbf{C}) &= \epsilon^{ijk} A^k B^i C^j \\
&= \left(\epsilon^{kij} A^k B^i\right) C^j \\
&= (\mathbf{A} \times \mathbf{B})^j C^j \\
&= (\mathbf{A} \times \mathbf{B}) \cdot \mathbf{C}.
\end{aligned}$$

2. Prove that the determinant of a 3×3 matrix, M can be given in terms of the Levi-Civita tensor

$$|M| = \epsilon^{ijk} M^{i1} M^{j2} M^{k3}.$$

(10 points)

Solution:

$$\begin{aligned}
RHS &= \epsilon^{ijk} M^{i1} M^{j2} M^{k3} \\
&= \epsilon^{1jk} M^{11} M^{j2} M^{k3} + \epsilon^{2jk} M^{21} M^{j2} M^{k3} + \epsilon^{3jk} M^{31} M^{j2} M^{k3}.
\end{aligned}$$

Lets look at the first term,

$$\begin{aligned}
\epsilon^{1jk} M^{11} M^{j2} M^{k3} &= M^{11} \left(\epsilon^{1jk} M^{j2} M^{k3} \right) \\
&= M^{11} \left(M^{22} M^{33} - M^{32} M^{23} \right) \\
&= M^{11} \begin{vmatrix} M^{22} & M^{23} \\ M^{32} & M^{33} \end{vmatrix} \\
&= M^{11} \, \mathrm{Cof}(M^{11})
\end{aligned}$$

where, $\mathrm{Cof}(M^{11})$ denotes the co-factor of M^{11}. Similarly one can show,

$$\epsilon^{2jk} M^{21} M^{j2} M^{k3} = M^{21} \, \mathrm{Cof}\left(M^{21}\right),$$

and,

$$\epsilon^{3jk} M^{31} M^{j2} M^{k3} = M^{31} \, \mathrm{Cof}\left(M^{31}\right).$$

Thus, we get,

$$\begin{aligned}
\epsilon^{ijk} M^{i1} M^{j2} M^{k3} &= \epsilon^{1jk} M^{11} M^{j2} M^{k3} + \epsilon^{2jk} M^{21} M^{j2} M^{k3} + \epsilon^{3jk} M^{31} M^{j2} M^{k3} \\
&= M^{11} \, \mathrm{Cof}(M^{11}) + M^{21} \, \mathrm{Cof}\left(M^{21}\right) + M^{31} \, \mathrm{Cof}\left(M^{31}\right) \\
&= |M|.
\end{aligned}$$

3. Suppose a three component object, $\begin{pmatrix} -2 \\ 1 \\ 4 \end{pmatrix}$ becomes $\begin{pmatrix} -2 \\ 1 \\ 0 \end{pmatrix}$ when the coordinate axes is rotated around the **x-axis** by $45°$. Is this object a vector? (Hint: The matrix for rotation by angle, θ around the **x-axis** is $O = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & \sin\theta \\ 0 & -\sin\theta & \cos\theta \end{pmatrix}$). (5 points)

Solution: The norm before rotation,

$$(-2\,1\,4) \begin{pmatrix} -2 \\ 1 \\ 4 \end{pmatrix} = 4 + 1 + 16 = 21.$$

Norm after rotation,

$$(-2\,1\,0) \begin{pmatrix} -2 \\ 1 \\ 0 \end{pmatrix} = 4 + 1 + 0 = 5.$$

Thus the norm before and after are different and it can't be a vector. Also since the matrix for rotation by angle, $\theta = \pi/4$ around the **x-axis** is $O = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & \sin\theta \\ 0 & -\sin\theta & \cos\theta \end{pmatrix}$, we need to check,

$$\begin{pmatrix} -2 \\ 1 \\ 0 \end{pmatrix} \stackrel{?}{=} \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & \sin\theta \\ 0 & -\sin\theta & \cos\theta \end{pmatrix}_{\theta=\pi/4} \begin{pmatrix} -2 \\ 1 \\ 4 \end{pmatrix}$$

$$\begin{aligned}
RHS &= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1/\sqrt{2} & 1/\sqrt{2} \\ 0 & -1/\sqrt{2} & 1/\sqrt{2} \end{pmatrix} \begin{pmatrix} -2 \\ 1 \\ 4 \end{pmatrix} \\
&= \begin{pmatrix} -2 \\ 5/\sqrt{2} \\ 3/\sqrt{2} \end{pmatrix} \neq LHS.
\end{aligned}$$

Thus, in this way we see that it does not transform like a vector under rotations.

4. Show that if, $A^{ij}$and $B^{lm}$are two rank 2 tensors, then their product with one pair of indices contracted (i.e. turned dummy), namely,

$$C^{im} = A^{il}B^{lm}$$

is also a tensor of rank 2. (Hint show that C transforms like a rank 2 tensor under orthogonal transformations/rotations). (5 points)
Solution: To show C^{im}, which is defined by,

$$C^{im} \equiv A^{il}B^{lm}$$

we need to check its transformation under rotations of coordinate axes (orthogonal). We know that under rotations,

$$\begin{aligned} A^{il} \to A'^{il} &= O^{ip}O^{lq}A^{pq}, \\ B^{lm} \to B'^{lm} &= O^{lk}O^{mn}B^{kn}, \end{aligned}$$

which means C^{im}under rotations goes over (transforms to),

$$\begin{aligned} C'^{im} &= A'^{il}B'^{lm} \\ &= \left(O^{ip}O^{lq}A^{pq}\right)\left(O^{lk}O^{mn}B^{kn}\right) \\ &= \underbrace{O^{lq}O^{lk}}\, O^{ip}O^{mn}\, A^{pq}B^{kn} \\ &= O^{ip}O^{mn}\, \underbrace{\delta^{qk}A^{pq}}_{A^{pk}}\, B^{kn} \\ &= O^{ip}O^{mn}\, \left(A^{pk}B^{kn}\right) \\ &= O^{ip}O^{mn}\, C^{pn}. \end{aligned}$$

Thus we that, indeed, C^{im} transforms like a rank 2 tensor.

5. Using the rule for partial differentiation, $\frac{\partial x^l}{\delta x^m} = \delta^{lm}$, and the Einstein summation convention, prove the following,
i) $\boldsymbol{\nabla}.\mathbf{x} = 3$,
ii) $\boldsymbol{\nabla}\left(\frac{1}{|\mathbf{x}|}\right) = -\frac{\mathbf{x}}{|\mathbf{x}|^3}$, except when, $\mathbf{x} = (0,0,0)$,
iii) $\boldsymbol{\nabla}^2\left(\frac{1}{|\mathbf{x}|}\right) = 0$, except when, $\mathbf{x} = (0,0,0)$,
iv) $(\mathbf{B}\cdot\boldsymbol{\nabla})\,\mathbf{x} = \mathbf{B}$ (2+3+3+2=10 points)
Solution:
(i)

$$\begin{aligned} \boldsymbol{\nabla}\cdot\mathbf{x} &= \hat{\mathbf{e}}_i\frac{\partial}{\partial x^i}\cdot\left(x^j\,\hat{\mathbf{e}}_j\right) \\ &= \hat{\mathbf{e}}_i\cdot\hat{\mathbf{e}}_j\,\left(\frac{\partial x^j}{\partial x^i}\right) \\ &= \delta^{ij}\delta^{ij} \\ &= \delta^{ii} \\ &= 3. \end{aligned}$$

(ii)

$$
\begin{aligned}
\boldsymbol{\nabla}\left(\frac{1}{|\mathbf{x}|}\right) &= \hat{\mathbf{e}}_i \frac{\partial}{\partial x^i}\left(x^k x^k\right)^{-1/2} \\
&= \hat{\mathbf{e}}_i \left(-\frac{1}{2}\left(x^k x^k\right)^{-3/2}\right)\frac{\partial\left(x^l x^l\right)}{\partial x^i} \\
&= \hat{\mathbf{e}}_i \left(-\frac{1}{2}\frac{1}{|\mathbf{x}|^3}\right) 2x^l \underbrace{\frac{\partial x^l}{\partial x^i}}_{\delta^{li}} \\
&= -\hat{\mathbf{e}}_i \frac{x^l \delta^{li}}{|\mathbf{x}|^3} \\
&= -\hat{\mathbf{e}}_i \frac{x^i}{|\mathbf{x}|^3} \\
&= -\frac{\mathbf{x}}{|\mathbf{x}|^3}.
\end{aligned}
$$

(iii)

$$
\begin{aligned}
\boldsymbol{\nabla}^2\left(\frac{1}{|\mathbf{x}|}\right) &= \boldsymbol{\nabla}\cdot\left(\boldsymbol{\nabla}\frac{1}{|\mathbf{x}|}\right) \\
&= \boldsymbol{\nabla}\cdot\left(-\frac{\mathbf{x}}{|\mathbf{x}|^3}\right) \\
&= -\frac{\partial}{\partial x^i}\left(\frac{x^i}{\left(x^k x^k\right)^{3/2}}\right) \\
&= -\frac{1}{\left(x^k x^k\right)^{3/2}}\underbrace{\frac{\partial x^i}{\partial x^i}}_{\delta^{ii}=3} - x^i \frac{\partial}{\partial x^i}\left(x^k x^k\right)^{-3/2} \\
&= -\frac{3}{|\mathbf{x}|^3} + x^i \frac{3}{2}\left(x^k x^k\right)^{-5/2}\frac{\partial\left(x^l x^l\right)}{\partial x^i} \\
&= -\frac{3}{|\mathbf{x}|^3} + \frac{3}{2}\frac{x^i}{|\mathbf{x}|^5}\left(2x^l \underbrace{\frac{\partial x^l}{\partial x^i}}_{\delta^{li}}\right) \\
&= -\frac{3}{|\mathbf{x}|^3} + 3\frac{x^i}{|\mathbf{x}|^5}\left(\underbrace{x^l\delta^{li}}_{x^i}\right) \\
&= -\frac{3}{|\mathbf{x}|^3} + 3\frac{1}{|\mathbf{x}|^5}\left(\underbrace{x^i x^i}_{|\mathbf{x}|^2}\right) \\
&= -\frac{3}{|\mathbf{x}|^3} + 3\frac{1}{|\mathbf{x}|^3} = 0.
\end{aligned}
$$

6. For which integer value of n is,

$$\boldsymbol{\nabla}\cdot\left(r^n \mathbf{x}\right) = 0.$$

(5 points)

Solution:

$$\begin{aligned}
\boldsymbol{\nabla}\cdot(r^n\mathbf{x}) &= \frac{\partial}{\partial x^i}\left(r^n\, x^i\right) \\
&= \frac{\partial}{\partial x^i}\left[\left(x^k x^k\right)^{n/2} x^i\right] \\
&= x^i \frac{\partial\left(x^k x^k\right)^{n/2}}{\partial x^i} + \left(x^k x^k\right)^{n/2} \underbrace{\frac{\partial x^i}{\partial x^i}}_{\delta^{ii}=3} \\
&= x^i\, \frac{n}{2}\left(x^k x^k\right)^{\frac{n}{2}-1} \frac{\partial\left(x^l x^l\right)}{\partial x^i} + 3\left(x^k x^k\right)^{n/2} \\
&= \frac{n}{2}\, x^i r^{n-2} 2x^l \underbrace{\frac{\partial x^l}{\partial x^i}}_{\delta^{li}} + 3\left(x^k x^k\right)^{n/2} \\
&= n\, r^{n-2} \underbrace{x^i x^l \delta^{li}}_{x^l x^l = r^2} + 3r^n \\
&= (n+3)\, r^n.
\end{aligned}$$

This is zero for, $n=-3$.

7. Prove the following identities using the Einstein summation convention (Φ, Ψ are scalar fields, while $\mathbf{A}$ is a vector field)
i) $\boldsymbol{\nabla}(\Phi\Psi) = \Phi\, \boldsymbol{\nabla}\Psi + \Psi\boldsymbol{\nabla}\Phi$,
ii) $\boldsymbol{\nabla}\cdot(\Phi\mathbf{A}) = \Phi(\boldsymbol{\nabla}\cdot\mathbf{A}) + (\mathbf{A}\cdot\boldsymbol{\nabla})\,\Phi$,
iii) $\boldsymbol{\nabla}\times(\Phi\mathbf{A}) = \Phi(\boldsymbol{\nabla}\times\mathbf{A}) + \boldsymbol{\nabla}\Phi\times\mathbf{A}$, (2+3+5 = 10 points)
Solution:
(i)

$$\begin{aligned}
LHS &= \boldsymbol{\nabla}(\Phi\Psi) \\
&= \hat{\mathbf{e}}_i \frac{\partial}{\partial x^i}(\Phi\Psi) \\
&= \hat{\mathbf{e}}_i \left(\Psi\frac{\partial\Phi}{\partial x^i} + \Phi\frac{\partial\Psi}{\partial x^i}\right) \\
&= \Psi\left(\hat{\mathbf{e}}_i \frac{\partial}{\partial x^i}\Phi\right) + \Phi\left(\hat{\mathbf{e}}_i \frac{\partial}{\partial x^i}\Psi\right) \\
&= \Psi\boldsymbol{\nabla}\Phi + \Phi\boldsymbol{\nabla}\Psi \\
&= RHS.
\end{aligned}$$

(ii)

$$\begin{aligned}
LHS &= \boldsymbol{\nabla}\cdot(\Phi\mathbf{A}) \\
&= \frac{\partial}{\partial x^i}(\Phi A^i) \\
&= A^i\frac{\partial\Phi}{\partial x^i} + \Phi\frac{\partial A^i}{\partial x^i} \\
&= \mathbf{A}\cdot(\boldsymbol{\nabla}\Phi) + \Phi\,(\boldsymbol{\nabla}\cdot\mathbf{A}) \\
&= RHS.
\end{aligned}$$

(iii)

$$\begin{aligned}
k\text{-th component of } LHS &= [\boldsymbol{\nabla}\times(\Phi\mathbf{A})]^k \\
&= \epsilon^{ijk}\,\frac{\partial}{\partial x^i}(\Phi\mathbf{A})^j \\
&= \epsilon^{ijk}\,\frac{\partial}{\partial x^i}(\Phi A^j) \\
&= \epsilon^{ijk}\,\left(\Phi\frac{\partial A^j}{\partial x^i}+\frac{\partial\Phi}{\partial x^i}A^j\right) \\
& \quad \underbrace{\Phi\left(\epsilon^{ijk}\,\frac{\partial}{\partial x^i}A^j\right)}_{(\boldsymbol{\nabla}\times\mathbf{A})^k}+\underbrace{\epsilon^{ijk}\,\frac{\partial\Phi}{\partial x^i}\,A^j}_{(\boldsymbol{\nabla}\Phi\times\mathbf{A})^k} \\
&= [\Phi(\boldsymbol{\nabla}\times\mathbf{A})+\boldsymbol{\nabla}\Phi\times\mathbf{A}]^k \\
&= k\text{-th component of } RHS.
\end{aligned}$$

8. Prove the following identities using the Einstein summation convention ($\mathbf{A}$and $\mathbf{B}$ are vector fields)
   i) $\boldsymbol{\nabla}\cdot(\mathbf{A}\times\mathbf{B})=\mathbf{B}\cdot(\boldsymbol{\nabla}\times\mathbf{A})-\mathbf{A}\cdot(\boldsymbol{\nabla}\times\mathbf{B})\,,$
   ii) $\boldsymbol{\nabla}\times(\boldsymbol{\nabla}\times\mathbf{A})=\boldsymbol{\nabla}(\boldsymbol{\nabla}\cdot\mathbf{A})-\boldsymbol{\nabla}^2\mathbf{A}$, where $\boldsymbol{\nabla}^2=\boldsymbol{\nabla}\cdot\boldsymbol{\nabla}$ is the Laplacian
   iii) $\boldsymbol{\nabla}\times(\mathbf{A}\times\mathbf{B})=(\mathbf{B}\cdot\boldsymbol{\nabla})\,\mathbf{A}-(\mathbf{A}\cdot\boldsymbol{\nabla})\,\mathbf{B}+\mathbf{A}\,(\boldsymbol{\nabla}\cdot\mathbf{B})-\mathbf{B}\,(\boldsymbol{\nabla}\cdot\mathbf{A})\,.$ (5+5+10=20 points)
   Solution:
   (i)

$$\begin{aligned}
LHS &= \boldsymbol{\nabla}\cdot(\mathbf{A}\times\mathbf{B}) \\
&= \frac{\partial}{\partial x^k}(\mathbf{A}\times\mathbf{B})^k \\
&= \frac{\partial}{\partial x^k}(\epsilon^{ijk}A^iB^j) \\
&= \epsilon^{ijk}\,\frac{\partial}{\partial x^k}\left(A^iB^j\right) \\
&= \left(\underbrace{\epsilon^{ijk}}_{=\epsilon^{kij}}\,\frac{\partial A^i}{\partial x^k}\right)B^j+A^i\left(\underbrace{\epsilon^{ijk}}_{=-\epsilon^{kji}}\,\frac{\partial B^j}{\partial x^k}\right) \\
&= B^j\underbrace{\left(\epsilon^{kij}\frac{\partial}{\partial x^k}A^i\right)}_{(\boldsymbol{\nabla}\times\mathbf{A})^j}-A^i\underbrace{\left(\epsilon^{kji}\,\frac{\partial}{\partial x^k}B^j\right)}_{(\boldsymbol{\nabla}\times\mathbf{B})^i} \\
&= \mathbf{B}\cdot(\boldsymbol{\nabla}\times\mathbf{A})-\mathbf{A}\cdot(\boldsymbol{\nabla}\times\mathbf{B}) \\
&= RHS.
\end{aligned}$$

(ii)

$$
\begin{aligned}
k\text{-th component of } LHS &= \left[\boldsymbol{\nabla} \times (\boldsymbol{\nabla} \times \mathbf{A})\right]^k \\
&= \epsilon^{ijk} \boldsymbol{\nabla}^i \left(\boldsymbol{\nabla} \times \mathbf{A}\right)^j \\
&= \epsilon^{ijk} \frac{\partial}{\partial x^i} \left(\epsilon^{lmj} \frac{\partial}{\partial x^l} A^m\right) \\
&= \underbrace{\epsilon^{ijk}}_{\epsilon^{kij}} \epsilon^{lmj} \frac{\partial^2 A^m}{\partial x^i \partial x^l} \\
&= \epsilon^{kij} \epsilon^{lmj} \frac{\partial^2 A^m}{\partial x^i \partial x^l} \\
&= \left(\delta^{kl} \delta^{im} - \delta^{km} \delta^{il}\right) \frac{\partial^2 A^m}{\partial x^i \partial x^l} \\
&= \delta^{kl} \delta^{im} \frac{\partial^2 A^m}{\partial x^i \partial x^l} - \delta^{km} \delta^{il} \frac{\partial^2 A^m}{\partial x^i \partial x^l} \\
&= \delta^{kl} \frac{\partial}{\partial x^l} \frac{\partial \left(\delta^{im} A^m\right)}{\partial x^i} - \delta^{il} \frac{\partial}{\partial x^i} \frac{\partial \left(\delta^{km} A^m\right)}{\partial x^l} \\
&= \frac{\partial}{\partial x^k} \left(\frac{\partial}{\partial x^i} A^i\right) - \left(\frac{\partial}{\partial x^l} \frac{\partial}{\partial x^l}\right) A^k \\
&= \boldsymbol{\nabla}^k \left(\boldsymbol{\nabla} \cdot \mathbf{A}\right) - \boldsymbol{\nabla}^2 \mathbf{A}^k \\
&= \left[\boldsymbol{\nabla} \left(\boldsymbol{\nabla} \cdot \mathbf{A}\right) - \boldsymbol{\nabla}^2 \mathbf{A}\right]^k \\
&= k\text{-th component of } RHS.
\end{aligned}
$$

(iii)

$$
\begin{aligned}
k\text{-th component of } LHS &= \left[\boldsymbol{\nabla} \times (\mathbf{A} \times \mathbf{B})\right]^k \\
&= \epsilon^{ijk} \boldsymbol{\nabla}^i \left(\mathbf{A} \times \mathbf{B}\right)^j \\
&= \epsilon^{ijk} \frac{\partial}{\partial x^i} \left(\epsilon^{lmj} A^l B^m\right) \\
&= \underbrace{\epsilon^{ijk}}_{\epsilon^{kij}} \epsilon^{lmj} \frac{\partial}{\partial x^i} \left(A^l B^m\right) \\
&= \epsilon^{kij} \epsilon^{lmj} \frac{\partial}{\partial x^i} \left(A^l B^m\right) \\
&= \left(\delta^{kl} \delta^{im} - \delta^{km} \delta^{il}\right) \frac{\partial}{\partial x^i} \left(A^l B^m\right) \\
&= \delta^{kl} \delta^{im} \frac{\partial}{\partial x^i} \left(A^l B^m\right) - \delta^{km} \delta^{il} \frac{\partial}{\partial x^i} \left(A^l B^m\right) \\
&= \frac{\partial}{\partial x^i} \left(\delta^{kl} A^l \delta^{im} B^m\right) - \frac{\partial}{\partial x^i} \left(\delta^{il} A^l \delta^{km} B^m\right) \\
&= \frac{\partial}{\partial x^i} \left(A^k B^i\right) - \frac{\partial}{\partial x^i} \left(A^i B^k\right) \\
&= A^k \frac{\partial B^i}{\partial x^i} + B^i \frac{\partial}{\partial x^i} A^k - B^k \frac{\partial A^i}{\partial x^i} - A^i \frac{\partial}{\partial x^i} B^k \\
&= A^k \boldsymbol{\nabla} \cdot \mathbf{B} + \mathbf{B} \cdot \boldsymbol{\nabla} A^k - B^k \boldsymbol{\nabla} \cdot \mathbf{A} + \mathbf{A} \cdot \boldsymbol{\nabla} B^k \\
&= k\text{-th component of } RHS.
\end{aligned}
$$

9. Consider a scalar field,

$$\Phi(\mathbf{x}) = \frac{2z^2 - x^2 - y^2}{(x^2+y^2+z^2)^{5/2}},$$

evaluate, $\nabla\Phi$ and $\nabla^2\Phi$. (4+6=10 points)

Solution:

$$\nabla\Phi(\mathbf{x}) = \left(\hat{\mathbf{x}}\frac{\partial}{\partial x} + \hat{\mathbf{y}}\frac{\partial}{\partial y} + \hat{\mathbf{z}}\frac{\partial}{\partial z}\right)\frac{2z^2 - x^2 - y^2}{(x^2+y^2+z^2)^{5/2}}$$

We use,

$$\begin{aligned}
\frac{\partial}{\partial x}\frac{2z^2 - x^2 - y^2}{(x^2+y^2+z^2)^{5/2}} &= \frac{3x\left(x^2+y^2-4z^2\right)}{r^7}, \\
\frac{\partial}{\partial y}\frac{2z^2 - x^2 - y^2}{(x^2+y^2+z^2)^{5/2}} &= \frac{3y\left(x^2+y^2-4z^2\right)}{r^7}, \\
\frac{\partial}{\partial z}\frac{2z^2 - x^2 - y^2}{(x^2+y^2+z^2)^{5/2}} &= \frac{3z\left(3x^2+3y^2-2z^3\right)}{r^7}.
\end{aligned}$$

So gathering all these components together,

$$\boldsymbol{\nabla}\Phi = \frac{3x\left(x^2+y^2-4z^2\right)\hat{\mathbf{x}} + 3y\left(x^2+y^2-4z^2\right)\hat{\mathbf{y}} + 3z\left(3x^2+3y^2-2z^3\right)\hat{\mathbf{z}}}{r^7}.$$

Now

$$\begin{aligned}
\boldsymbol{\nabla}^2\Phi &= \boldsymbol{\nabla}\cdot(\boldsymbol{\nabla}\Phi) \\
&= \frac{\partial}{\partial x}(\boldsymbol{\nabla}\Phi)^x + \frac{\partial}{\partial y}(\boldsymbol{\nabla}\Phi)^y + \frac{\partial}{\partial z}(\boldsymbol{\nabla}\Phi)^z \\
&= 0.
\end{aligned}$$

Alternative Solution:

$$\Phi = \frac{2z^2 - x^2 - y^2}{(x^2+y^2+z^2)^{5/2}} = 3\frac{z^2}{|\mathbf{x}|^5} - \frac{1}{|\mathbf{x}|^3}.$$

This gives,

$$\begin{aligned}
\boldsymbol{\nabla}\Phi &= \frac{3\left(\nabla z^2\right)}{|\mathbf{x}|^5} + 3z^2\boldsymbol{\nabla}\frac{1}{|\mathbf{x}|^5} - \boldsymbol{\nabla}\frac{1}{|\mathbf{x}|^3} \\
&= \frac{6z\hat{\mathbf{z}}}{|\mathbf{x}|^5} - \frac{15z^2}{|\mathbf{x}|^6}\frac{\mathbf{x}}{|\mathbf{x}|} + \frac{3}{|\mathbf{x}|^4}\frac{\mathbf{x}}{|\mathbf{x}|} \\
&= \frac{6z\hat{\mathbf{z}}}{|\mathbf{x}|^5} - \frac{15z^2}{|\mathbf{x}|^7}\mathbf{x} + \frac{3}{|\mathbf{x}|^5}\mathbf{x}.
\end{aligned}$$

where we have used the highly useful result,

$$\boldsymbol{\nabla}|\mathbf{x}| = \frac{\mathbf{x}}{|\mathbf{x}|}.$$

Next, the Laplacian,

$$\begin{aligned}
\boldsymbol{\nabla}^2\Phi = \boldsymbol{\nabla}\cdot(\boldsymbol{\nabla}\Phi) &= \boldsymbol{\nabla}\cdot\left(\frac{6z\hat{\mathbf{z}}}{|\mathbf{x}|^5} - 15z^2\frac{\mathbf{x}}{|\mathbf{x}|^7} + \frac{3\mathbf{x}}{|\mathbf{x}|^5}\right) \\
&= (6z\hat{\mathbf{z}})\cdot\boldsymbol{\nabla}\frac{1}{|\mathbf{x}|^5} + \frac{1}{|\mathbf{x}|^5}\left(\boldsymbol{\nabla}\cdot 6z\hat{\mathbf{z}}\right) - 15\frac{\mathbf{x}}{|\mathbf{x}|^7}\cdot\left(\boldsymbol{\nabla}z^2\right) - \frac{15z^2}{|\mathbf{x}|^7}\boldsymbol{\nabla}\cdot\mathbf{x} \\
&\qquad\qquad -15z^2\mathbf{x}\cdot\boldsymbol{\nabla}\frac{1}{|\mathbf{x}|^7} + 3\frac{\boldsymbol{\nabla}\cdot\mathbf{x}}{|\mathbf{x}|^5} + 3\mathbf{x}\cdot\boldsymbol{\nabla}\frac{1}{|\mathbf{x}|^5}, \\
&= -\frac{30z^2}{|\mathbf{x}|^7} + \frac{6}{|\mathbf{x}|^5} - \frac{30z^2}{|\mathbf{x}|^7} - \frac{45z^2}{|\mathbf{x}|^7} + \frac{105z^2}{|\mathbf{x}|^7} + \frac{9}{|\mathbf{x}|^5} - \frac{15}{|\mathbf{x}|^5} \\
&= 0.
\end{aligned}$$

10. For two scalar fields, $\Phi(\mathbf{x})$ and $\Psi(\mathbf{x})$, show that the following result holds for a closed surface, S enclosing a volume, V

$$\iiint_V d^3\mathbf{x}\,\left(\Psi\boldsymbol{\nabla}^2\Phi - \Phi\boldsymbol{\nabla}^2\Psi\right) = \oint\!\!\!\oint_S dS\,\hat{\mathbf{n}}\cdot\left(\Psi\boldsymbol{\nabla}\Phi - \Phi\boldsymbol{\nabla}\Psi\right).$$

This very useful result is known as **Green's Theorem**. (Hint: Try to apply Gauss divergence theorem to a vector field, $\mathbf{A}$ made out of the two scalar fields, namely, $\mathbf{A} = \Psi\boldsymbol{\nabla}\Phi - \Phi\boldsymbol{\nabla}\Psi$.) (10 points)
Solution:
First define,

$$\mathbf{A} = \Psi\boldsymbol{\nabla}\Phi - \Phi\boldsymbol{\nabla}\Psi.$$

Next, we note the identities,

$$\begin{aligned}
\boldsymbol{\nabla}\cdot(\Psi\boldsymbol{\nabla}\Phi) &= \boldsymbol{\nabla}\Psi\cdot\boldsymbol{\nabla}\Phi + \Psi\boldsymbol{\nabla}^2\Phi, \\
\boldsymbol{\nabla}\cdot(\Phi\boldsymbol{\nabla}\Psi) &= \boldsymbol{\nabla}\Phi\cdot\boldsymbol{\nabla}\Psi + \Phi\boldsymbol{\nabla}^2\Psi,
\end{aligned}$$

and take the difference,

$$\boldsymbol{\nabla}\cdot(\Psi\boldsymbol{\nabla}\Phi - \Phi\boldsymbol{\nabla}\Psi) = \Psi\boldsymbol{\nabla}^2\Phi - \Phi\boldsymbol{\nabla}^2\Psi. \tag{1}$$

Now we insert $\mathbf{A}$ in Gauss's theorem,

$$\begin{aligned}
\iiint_V d^3\mathbf{x}\,\boldsymbol{\nabla}\cdot\mathbf{A} &= \oint\!\!\!\oint_S dS\,\hat{\mathbf{n}}\cdot\mathbf{A}, \\
\Longrightarrow \iiint_V d^3\mathbf{x}\,\boldsymbol{\nabla}\cdot(\Psi\boldsymbol{\nabla}\Phi - \Phi\boldsymbol{\nabla}\Psi) &= \oint\!\!\!\oint_S dS\,\hat{\mathbf{n}}\cdot(\Psi\boldsymbol{\nabla}\Phi - \Phi\boldsymbol{\nabla}\Psi) \\
\Longrightarrow \iiint_V d^3\mathbf{x}\,\left(\Psi\boldsymbol{\nabla}^2\Phi - \Phi\boldsymbol{\nabla}^2\Psi\right) &= \oint\!\!\!\oint_S dS\,\hat{\mathbf{n}}\cdot\left(\Psi\boldsymbol{\nabla}\Phi - \Phi\boldsymbol{\nabla}\Psi\right),
\end{aligned}$$

where from going from line 2 to line 3 we have used the identity (1) in the LHS.

11. Stokes' theorem is proposed for an *open* surface i.e. a surface with a boundary, e.g. the surface of a hemisphere, which has a boundary, namely, the equator. What if we try to propose a version of Stokes theorem for a closed surface i.e. without boundaries, e.g. a full sphere, S^2. What does the result look like, i.e. the flux integral over curl of a vector over a closed surface equals to what?

$$\oint\!\!\!\oint_{S^2} dS\,\hat{\mathbf{n}}.\left(\boldsymbol{\nabla}\times\mathbf{A}\right) = ?$$

(5 points)

Solution: Let's take the closed surface i.e. in our case the sphere and slice it into two open surfaces i.e. hemispheres by slicing along the equator, call it C. For the northern hemisphere, N, Stokes' theorem is,

$$\iint_N dS\,\hat{\mathbf{n}}.\,(\boldsymbol{\nabla}\times\mathbf{A}) = \oint_C \mathbf{A}\cdot d\mathbf{l}.$$

Similarly, for the southern hemisphere,

$$\iint_S dS\,\hat{\mathbf{n}}.\,(\boldsymbol{\nabla}\times\mathbf{A}) = \oint_C \mathbf{A}\cdot d\mathbf{l}.$$

Adding the above two results,

$$\begin{aligned}\oiint_{S^2} dS\,\hat{\mathbf{n}}.\,(\boldsymbol{\nabla}\times\mathbf{A}) &= \left(\oint_C + \oint_C\right)\mathbf{A}\cdot d\mathbf{l}\\ &= 0.\end{aligned}$$

Clockwise and anticlockwise line integral of any vector over the same line/loop are equal in magnitude and opposite in sign, so their sum in the rhs vanished.

12. Gauss theorem and Stoke's theorem both apply to vector fields. But what about scalar fields? Show that the following results hold which are counterparts of Gauss and Stokes theorem for scalar fields, say $\Phi(\mathbf{x})$,

$$\iiint_V d^3\mathbf{x}\,\boldsymbol{\nabla}\Phi = \oiint_S dS\,\hat{\mathbf{n}}\,\Phi$$

$$\iint_S dS\,\hat{\mathbf{n}}\times\nabla\Phi = \oint_C \Phi\,d\mathbf{l}$$

(Hint: Cook up a vector field from the scalar, Φ by multiplying it with a constant vector, say $\mathbf{C}$ i.e. whose components are same everywhere, and then apply Gauss theorem and Stokes theorem to that cooked up vector field, $\mathbf{A} = \mathbf{C}\Phi$.) (5+5 = 10 points)

Solution: Let's define a new vector from a scalar, Φ, and a constant vector, $\mathbf{C}$

$$\mathbf{A} = \mathbf{C}\Phi$$

Now apply Gauss' theorem to this new vector,

$$\iiint_V d^3\mathbf{x}\,\boldsymbol{\nabla}\cdot\mathbf{A}\Big|_{\mathbf{A}=\mathbf{C}\Phi} = \oiint_S dS\,\hat{\mathbf{n}}\cdot\mathbf{A}\Big|_{\mathbf{A}=\mathbf{C}\Phi}.$$

Now we use

$$\boldsymbol{\nabla}\cdot\mathbf{A} = \boldsymbol{\nabla}\cdot(\mathbf{C}\boldsymbol{\Phi}) = \Phi\left(\underbrace{\boldsymbol{\nabla}\cdot\mathbf{C}}_{=0}\right) + \mathbf{C}\cdot\boldsymbol{\nabla}\cdot\boldsymbol{\Phi} = \mathbf{C}\cdot\boldsymbol{\nabla}\cdot\boldsymbol{\Phi},$$

to obtain,

$$\begin{aligned}\iiint_V d^3\mathbf{x}\,\mathbf{C}\cdot\boldsymbol{\nabla}\boldsymbol{\Phi} &= \oiint_S dS\,\hat{\mathbf{n}}\cdot\mathbf{C}\,\boldsymbol{\Phi}\\ \mathbf{C}\cdot\left(\iiint_V d^3\mathbf{x}\,\boldsymbol{\nabla}\boldsymbol{\Phi}\right) &= \mathbf{C}\cdot\left(\oiint_S dS\,\hat{\mathbf{n}}\,\Phi\right),\end{aligned}$$

where we took $\mathbf{C}$ out of the integral since it's components are constants. Now as this holds for arbitrary constant vector, $\mathbf{C}$ we can show by first taking $\mathbf{C} = \hat{\mathbf{i}}$ and then $\hat{\mathbf{j}}$ and then $\hat{\mathbf{k}}$, that the above equality holds component by component, i.e. as a full vector equation,

$$\iiint_V d^3\mathbf{x}\,\boldsymbol{\nabla}\boldsymbol{\Phi} = \oiint_S dS\,\hat{\mathbf{n}}\,\Phi.$$

Next, we insert the same new vector, **A** in Stokes' theorem,

$$\iint_S dS\, \hat{\mathbf{n}}.\left(\boldsymbol{\nabla}\times(\mathbf{C}\boldsymbol{\Phi})\right) = \oint_C (\mathbf{C}\boldsymbol{\Phi})\cdot d\mathbf{l}$$

and then simplify the lhs by using,

$$\begin{aligned}\boldsymbol{\nabla}\times(\mathbf{C}\boldsymbol{\Phi}) &= \Phi\left(\underbrace{\boldsymbol{\nabla}\times\mathbf{C}}_{=0}\right) + (\boldsymbol{\nabla}\Phi)\times\mathbf{C}\\ &= (\boldsymbol{\nabla}\Phi)\times\mathbf{C},\end{aligned}$$

and the rule for scalar triple products,

$$\hat{\mathbf{n}}.\left(\boldsymbol{\nabla}\times(\mathbf{C}\boldsymbol{\Phi})\right) = \hat{\mathbf{n}}.\left((\boldsymbol{\nabla}\Phi)\times\mathbf{C}\right) = \mathbf{C}\cdot(\hat{\mathbf{n}}\times\boldsymbol{\nabla}\boldsymbol{\Phi}),$$

and obtain,

$$\begin{aligned}\iint_S dS\, \hat{\mathbf{n}}.\left(\boldsymbol{\nabla}\times(\mathbf{C}\boldsymbol{\Phi})\right) &= \iint_S dS\, \mathbf{C}\cdot(\hat{\mathbf{n}}\times\boldsymbol{\nabla}\Phi)\\ &= \mathbf{C}\cdot\iint_S dS\ (\hat{\mathbf{n}}\times\boldsymbol{\nabla}\Phi).\end{aligned}$$

Thus Stokes' theorem gives us,

$$\mathbf{C}\cdot\iint_S dS\ (\hat{\mathbf{n}}\times\boldsymbol{\nabla}\Phi) = \mathbf{C}\cdot\oint_C \boldsymbol{\Phi} d\mathbf{l},$$

and since this holds for an *arbitrary* constant vector, just as like in the case of Gauss theorem before,

$$\iint_S dS\, \mathbf{C}\cdot(\hat{\mathbf{n}}\times\boldsymbol{\nabla}\Phi) = \oint_C \boldsymbol{\Phi} d\mathbf{l}.$$

13. Prove that,

$$\iiint_B d^3\mathbf{x}\, \boldsymbol{\nabla}^2\left(\frac{1}{|\mathbf{x}|}\right) = -4\pi.$$

where the volume over which we are integrating is a ball of unit radius, B with the origin of coordinates at the center of the ball. (Hint: Use Gauss' divergence theorem after acting with one of the ∇ on $\frac{1}{|\mathbf{x}|}$, and use spherical polar coordinates) (10 points)

Solution:

We will use Gauss theorem first,

$$\begin{aligned}\iiint_B d^3\mathbf{x}\, \boldsymbol{\nabla}^2\left(\frac{1}{|\mathbf{x}|}\right) &= \oiint_{S^2} dS\, \hat{\mathbf{n}}\cdot.\left(\boldsymbol{\nabla}\frac{1}{|\mathbf{x}|}\right)\\ &= \oiint_{S^2} dS\, \hat{\mathbf{n}}\cdot.\left(-\frac{\mathbf{x}}{|\mathbf{x}|^3}\right)\\ &= -\oiint_{S^2} dS\, \frac{\hat{\mathbf{n}}\cdot\mathbf{x}}{|\mathbf{x}|^3}.\end{aligned} \tag{2}$$

Next we use spherical polar coordinate system, where,

$$\hat{\mathbf{n}} = \hat{\mathbf{r}}, \quad \mathbf{x} = r\hat{\mathbf{r}}.$$

Thus,

$$\frac{\hat{\mathbf{n}}\cdot\mathbf{x}}{|\mathbf{x}|^3} = \frac{1}{r^2}$$

and since on the surface of the unit sphere, S^2, $r = 1$,

$$\frac{\hat{\mathbf{n}} \cdot \mathbf{x}}{|\mathbf{x}|^3} = 1.$$

Now plugging this back in (2)

$$\begin{aligned}
\iiint_B d^3\mathbf{x}\, \boldsymbol{\nabla}^2 \left(\frac{1}{|\mathbf{x}|}\right) &= -\oiint_{S^2} dS\, \frac{\hat{\mathbf{n}} \cdot \mathbf{x}}{|\mathbf{x}|^3} \\
&= -\oiint_{S^2} dS \\
&= -4\pi.
\end{aligned}$$

14. Suppose that the electric current density in a region filled with plasma is $\mathbf{j} = C(xr\,\hat{\mathbf{x}} + yr\,\hat{\mathbf{y}})$, where C is a constant and $r = |\mathbf{x}|$. What is the rate of change of the electric charge in the spherical region bounded by $r = R$? (5 points)

Solution: Integrating the continuity equation over a spherical region of radius R around the origin,

$$\begin{aligned}
\iiint_{|\mathbf{x}|\leq R} d^3\mathbf{x} \left(\frac{\partial \rho}{\partial t} + \boldsymbol{\nabla} \cdot \mathbf{j}\right) &= 0 \\
\implies \iiint_{|\mathbf{x}|\leq R} d^3\mathbf{x} \left(\frac{\partial \rho}{\partial t}\right) &= -\iiint_{|\mathbf{x}|\leq R} d^3\mathbf{x}\, \boldsymbol{\nabla} \cdot \mathbf{j} \\
\implies \frac{d}{dt} \underbrace{\left(\iiint_{|\mathbf{x}|\leq R} d^3\mathbf{x}\, \rho\right)}_{Q(R)} &= -\oiint_{|\mathbf{x}|=R} dS\, \hat{\mathbf{n}} \cdot \mathbf{j},
\end{aligned}$$

So we have the rate of change of total charge within the region bounded by radius R,

$$\frac{dQ(R)}{dt} = -\oiint_{S^2} dS\, \hat{\mathbf{n}} \cdot \mathbf{j}$$

Now we use spherical polar coordinates. Now on the surface of the sphere the normal points out in the radial direction,

$$\hat{\mathbf{n}} = \hat{\mathbf{r}}.$$

The current density, in spherical polar coordinates is,

$$\mathbf{j} = C(xr\,\hat{\mathbf{x}} + yr\,\hat{\mathbf{y}}) = Cr\,\mathbf{x} - Crz\hat{\mathbf{z}} = Cr^2\,\hat{\mathbf{r}} - Cr^2 \cos\theta \hat{\mathbf{z}}$$

On the surface, $r = R$,

$$\mathbf{j} = CR^2\,\hat{\mathbf{r}} - CR^2 \cos\theta \hat{\mathbf{z}}.$$

Thus we get,

$$\begin{aligned}
\frac{dQ(R)}{dt} &= -\oiint_{S^2} dS\, \hat{\mathbf{n}} \cdot \mathbf{j} \\
&= -\oiint_{S^2} dS\, \left(CR^2 \hat{\mathbf{r}} \cdot \hat{\mathbf{r}} - CR^2 \cos\theta \hat{\mathbf{z}} \cdot \hat{\mathbf{r}}\right) \\
&= -CR^2 \oiint_{S^2} dS\, \sin^2\theta,\ dS = R^2 \sin\theta d\theta d\phi \\
&= -\frac{8}{3}\pi C R^4.
\end{aligned}$$

15. Show that for an *incompressible* fluid (i.e. the mass density ρ is constant), the fluid current density vector is divergenceless aka solenoidal. Geometrically this means fluid flow lines are closed loops (5 points)
Solution:
For density to be constant in space and time,

$$\frac{\partial \rho}{\partial x^i} = \frac{\partial \rho}{\partial t} = 0.$$

Plugging this in the continuity equation,

$$\boldsymbol{\nabla} \cdot \mathbf{j} = 0.$$

Thus the fluid current density is solenoidal.