More applications of the Local Lemma

[Jukna, Chapter 19]

(1) The k-SAT problem

$x_1, x_2, \dots, x_5 \in \{0,1\}$

variables

$$\phi = \underbrace{(x_1 \vee x_2 \vee \bar{x}_3)}_{\text{clause}} \wedge (x_2 \vee x_3 \vee \bar{x}_4) \wedge (\bar{x}_1 \vee \bar{x}_2 \vee x_4) \wedge \dots \wedge (x_5 \vee x_1 \vee \bar{x}_2)$$

literals in each clause

3-CNF form

Conjunctive Normal Form (AND of ORs)

$$\underbrace{\begin{matrix} x_1 & x_2 & x_3 & x_4 & x_5 \\ \bar{x}_1 & \bar{x}_2 & \bar{x}_3 & \bar{x}_4 & \bar{x}_5 \end{matrix}}_{\text{literals}}$$

Theorem Let ϕ be a k-CNF formula. If each of its clauses overlaps with at most 2^{k-2} clauses, then ϕ is satisfiable.

two clauses overlap when they share a variable

Local Lemma

$ep(d+1) \le 1$

or

$4pd \le 1$

Proof:

$x_1, x_2, \dots, x_n$

randomly and

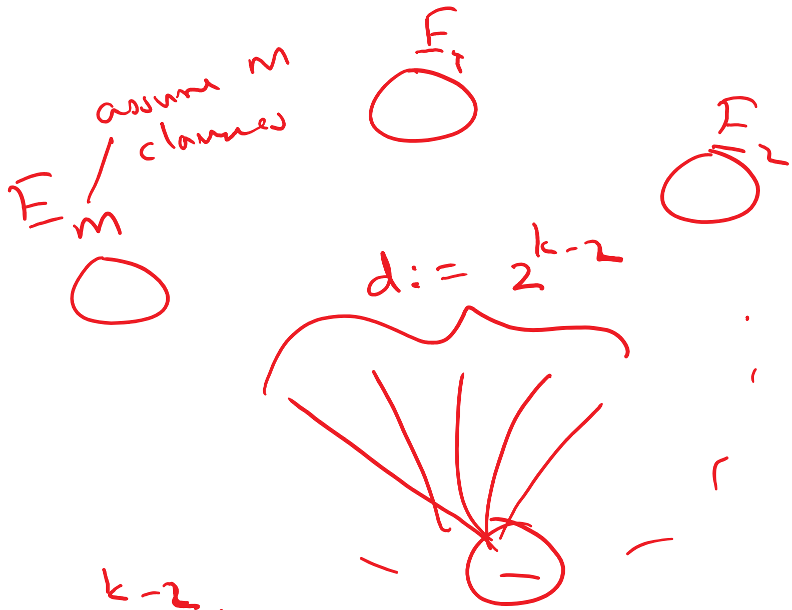
Independently and uniformly at random assign each x_i a value from $\{0,1\}$.

$\boxed{\begin{array}{l} A_n \quad \underbrace{\quad 4pd \le 1 \quad}\\ \text{Then, } \Pr[\cap \bar{A}_i] > 0. \end{array}}$

$$\phi = \underbrace{(\qquad\qquad)}_{C_1} \wedge \underbrace{\overbrace{(x_2 \vee \bar{x}_3 \vee \bar{x}_1 \vee x_5 \vee \ldots)}^{k \text{ literals}}}_{C_2} \wedge \underbrace{(\qquad\qquad)}_{C_3} \wedge \ldots$$

E_i : the bad event the clause C_i is not satisfied.

$$\Pr[E_i] = \frac{1}{2^k} := p.$$



$p = \frac{1}{2^k}$, $d = 2^{k-2}$.

Now, we have

$k-2$

$$4pd = 4p \cdot \frac{1}{2^{k}} \cdot 2^{k-2} \leq 1$$

Then, by Local Lemma, $\Pr\left[\bigwedge_i \overline{E_i}\right] > 0$.

$\square$

Hypergraph coloring

$\to$ set system or a family of sets

A hypergraph $H = (V, E)$ is 2-colorable if there is a way to color the points/elements in V such that $\forall e \in E$, e sees two colors.

$$V = \{1, 2, 3, 4, 5\}$$

$H = (V, E)$ is 3-uniform

$$E = \{ \{1, 2, 3\}, \{2, 3, 4\}, \{1, 4, 5\}, \{1, 3, 5\} \}$$

We had seen,

Lemma. Let H be a k-uniform hypergraph of less than 2^{k-1} hyperedges. Then, H is always 2-colorable.

k-uniform: means every set is of size $= k$

hypergraph: family of sets

hyperedges: set

$k = 3$

$$\#\text{ edges} < 2^{k-1} = 2^2 = 4.$$

$\{1, 2, 5\}, \{2, 3, 4\}, \{1, 3, 5\}$

$k = 3$

$\{1, 2, 3\}$ $\{4, 5, 6\}$ $\{7, 8, 9\}$ ...

Theorem: Let H be a k-uniform hypergraph where every set/hyperedge intersects with at most 2^{k-3} other hyperedges. Then, H is 2-colorable.

Proof: $H = (V, E)$

$V = \{1, 2, \ldots, i, \ldots, n\}$

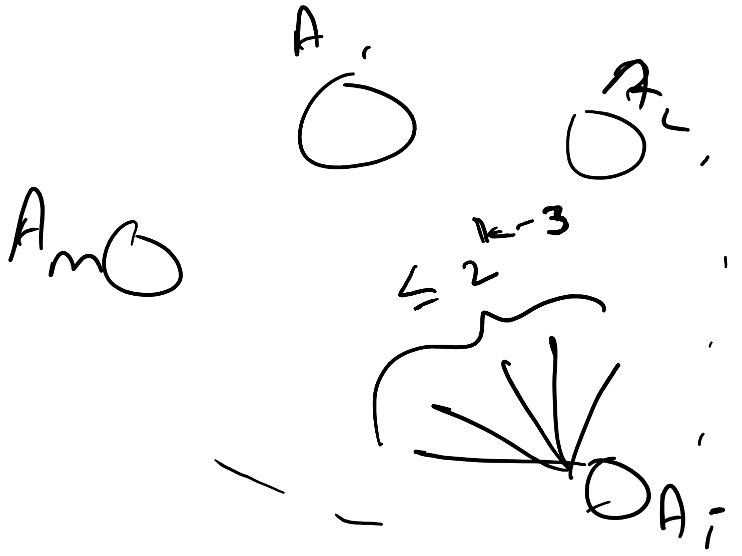
i with $1/2$ prob RED, $1/2$ prob BLUE

$E = \{e_1, e_2, e_3, \ldots, e_m\}$ k-sized subsets of V.

$\forall, 1 \le i \le m,$
A_i : the bad event that hyperedge e_i is not 2-colored.

$$\Pr[A_i] = \frac{1}{\cdot} + \frac{1}{} = \frac{1}{} := p$$

$$\Pr(A_i) = \frac{1}{2^k} + \frac{1}{2^k} = \frac{1}{2^{k-1}} := p$$



$$d = 2^{k-3}$$

$$4pd = 4 \cdot \frac{1}{2^{k-1}} \cdot 2^{k-3} \le 1$$

Therefore, by Local lemma

$$\Pr\left[\bigcap_{i=1}^{m} \overline{A_i}\right] > 0.$$

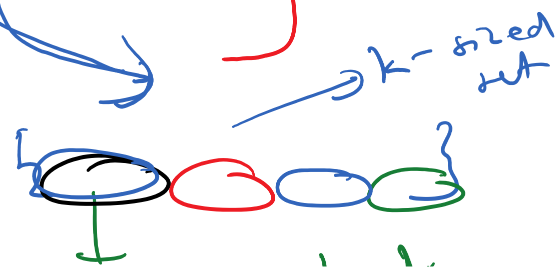
$\square$

Theorem Let k be sufficiently large. Let $\mathcal{F}$ be a k-uniform family of sets. It is given that no element belongs to more than k sets of $\mathcal{F}$.

Then its possible to color the points/elements in $r = \left\lfloor \frac{k}{\log k} \right\rfloor$ colors such that every set in $\mathcal{F}$ contains at most $v = \lceil 2e \log k \rceil$ points/elements of the same color.

Proof:

$r = \left\lfloor \frac{k}{\log k} \right\rfloor$ colors



$\lceil \log k \rceil$

$\frac{k}{r} = \underline{\log k}$ sized

if the coloring is "equitable"

indep. and uniformly at random give i a color from $\{c_1, c_2, \ldots, c_r\}$

$U = \{1, 2, \ldots, i, \ldots n\}$

$\mathcal{F} = \{\{\ldots\}, \ldots, \ldots, \ldots\}, \{\}\}$

subset of U.

$\leq k-1$ other sets

k elements

$S \rightarrow$ intersect/overlaps with $\leq k(k-1)$ other sets.

$A(S, c_i)$: the "bad" event that there are more than $r = \lceil 2e \log k \rceil$ points in S having color c_i.

$A(S, c_1)$ $A(S, c_2)$ . . . $A(S, c_r)$

r events

$\leq r \cdot k(k-1)$

$A(S', c_1)$

$A(S', \ldots)$

$\ldots\ldots, \quad \bigcup A(s_j', i_2)$

$d := r\,k(k-1) \le k^3$ (bcoz $r = \left\lfloor \frac{k}{\log k} \right\rfloor$) (1)

If $4 \cdot p \cdot d \le 1$, then we are $\underbrace{\text{done}}$.

$\Pr\left[\bigcap_{\substack{s \in \mathcal{F},\\ i \in \{1,2,\dots,r\}}} \overline{A(s, c_i)}\right] > 0$ by the Local Lemma.

This is true only when

$$p \le \frac{1}{4d} = \frac{1}{4k^3}$$

$\underline{\text{To show:}}$ $p \le \dfrac{1}{4k^3}$

$\Pr\left[A(s, c_i)\right]$. $s \in \mathcal{F}$. $1 \le i \le r$.

Do it yourself.

$\square$.