MA 1140: Lecture 4
Basis and Dimension of Vector Spaces

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Let us recall the 1st lecture

- We discussed vector space with some examples.
- Linear combinations of vectors.
- Subspace of a vector space.
- One question: For a subspace W of V, is it true that $0 \in W$?
- Subspace spanned by a subset S.

Linearly dependent or independent

Throughout, let V be a vector space over $\mathbb{R}$.

Definition

A subset S of vectors in V is said to be **linearly dependent** if there exists vectors $v_1, v_2, \ldots, v_r$ in S and scalars $c_1, c_2, \ldots, c_r$ in $\mathbb{R}$, NOT ALL of which are 0, such that

$$c_1 v_1 + \cdots + c_r v_r = 0.$$

Definition

A set S which is not linearly dependent is called **linearly independent**.

If $S = \{v_1, \ldots, v_n\}$, we say that $v_1, \ldots, v_n$ are linearly dependent (or independent) instead of saying that S is so.

Remarks on linearly dependent vectors

1. Any set containing the 0 vector is linearly dependent.
2. A set S of vectors is linearly dependent if and only if there exists a non-trivial relation of vectors of S:

$$c_1 v_1 + \cdots + c_r v_r = 0, \text{ where at least one } c_i \neq 0.$$

   This is equivalent to say that there exists at least one vector $v \in S$ which belongs to the subspace spanned by $S \smallsetminus \{v\}$.
3. Any set containing a linearly dependent subset is again linearly dependent.

Remarks on linearly independent vectors

1. Every non-zero vector v in V is linearly independent.
2. A finite set $\{v_1, \ldots, v_r\}$ is linearly independent if and only if
$$c_1 v_1 + \cdots + c_r v_r = 0 \implies c_i = 0 \text{ for all } 1 \leq i \leq r.$$
3. A set S of vectors is linearly independent if and only if every finite subset of S is linearly independent, i.e., if and only if for every subset $\{v_1, \ldots, v_r\} \subseteq S$,
$$c_1 v_1 + \cdots + c_r v_r = 0 \implies c_i = 0 \text{ for all } 1 \leq i \leq r.$$
4. Any subset of a linearly independent set is linearly independent.

Adjoining a vector to a linearly independent set

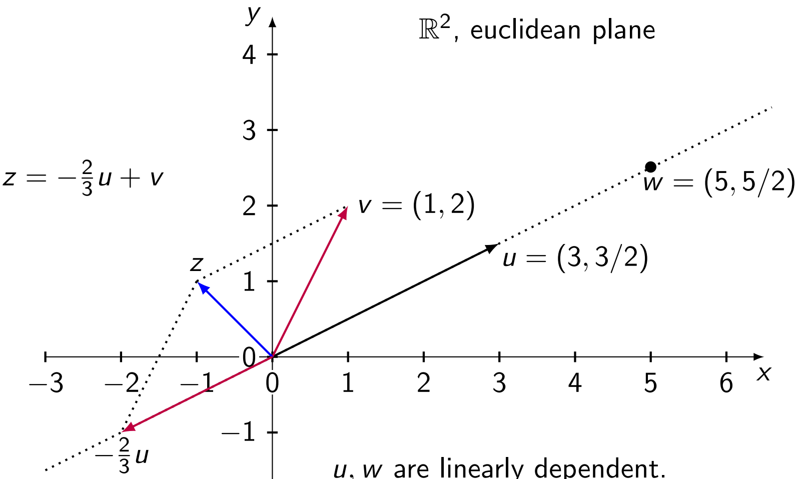
Lemma

Let S be a linearly independent subset of a vector space V. Suppose $v \notin \mathrm{Span}(S) := \{c_1 v_1 + \cdots + c_r v_r : v_i \in S\}$. Then $S \cup \{v\}$ is linearly independent.

Proof.

Let $c_1 v_1 + \cdots + c_r v_r + cv = 0$ for some vectors $v_1, \ldots, v_r \in S$ and scalars $c_1, \ldots, c_r, c$.
If $c \neq 0$, then it follows that $v \in \mathrm{Span}(S)$, which is a contradiction. Therefore $c = 0$, and hence $c_1 v_1 + \cdots + c_r v_r = 0$. Since S is linearly independent, it follows that $c_i = 0$ for all $1 \leq i \leq r$. $\square$

Vectors in $\mathbb{R}^2$ plane



u, w are linearly dependent.

u, v are linearly INdependent.

u, v, z are linearly dependent.

Definition

A set S of vectors in V is called a **basis** of V if
(i) S is linearly independent, and
(ii) it spans the space V (i.e., the subspace spanned by S is V).

The space V is said to be **finite dimensional** if it has a finite basis. If V does not have a finite basis, then V is said to be **infinite dimensional**.

What is the guarantee that a basis exists?
We can prove the existence at least when V is generated (or spanned) by finitely many vectors. How?
Start with a finite spanning set S. Then check whether it is linearly independent. If S is linearly dependent, then there is $v \in S$ such that v belongs to the subspace spanned by $S \smallsetminus \{v\}$. One can prove that $S \smallsetminus \{v\}$ spans V. Repeat the process till we get a linearly independent subset of S which spans V.
For vector space which is not finitely generated, we need the axiom of choice. We will not do that in this course.

An example of a basis of $\mathbb{R}^2$

The set $\left\{ u = \begin{pmatrix} 1 \\ 2 \end{pmatrix}, v = \begin{pmatrix} 2 \\ 1 \end{pmatrix} \right\}$ forms a basis of $\mathbb{R}^2$.
Indeed, geometrically, it can be observed that u, v are linearly independent, and $\{u, v\}$ spans $\mathbb{R}^2$.

Or directly, we see that for EVERY vector $\begin{pmatrix} a \\ b \end{pmatrix} \in \mathbb{R}^2$, the system

$$x \begin{pmatrix} 1 \\ 2 \end{pmatrix} + y \begin{pmatrix} 2 \\ 1 \end{pmatrix} = \begin{pmatrix} a \\ b \end{pmatrix} \quad \text{i.e.,} \quad \begin{cases} x + 2y = a \\ 2x + y = b \end{cases} \quad \text{or} \quad \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} a \\ b \end{pmatrix}$$

has a UNIQUE solution in x, y because the coefficient matrix $\begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}$ is invertible.
So every vector in $\mathbb{R}^2$ can be written as a linear combination of $\{u, v\}$, hence it spans the space $\mathbb{R}^2$.
Moreover, when $a = b = 0$, then the system has THE trivial solution $x = y = 0$. Thus $\{v, v\}$ is linearly independent as well.

Standard basis of $\mathbb{R}^n$

In $\mathbb{R}^n$, let S be the subset consisting of the vectors:

$$\begin{aligned} e_1 &= (1, 0, 0 \ldots, 0) \\ e_2 &= (0, 1, 0, \ldots, 0) \\ &\vdots \\ e_n &= (0, 0, 0, \ldots, 1). \end{aligned}$$

Note that any vector $v = (x_1, \ldots, x_n) \in \mathbb{R}^n$ can be written as a linear combination $x_1 e_1 + \cdots + x_n e_n$. So S spans $\mathbb{R}^n$.
Is S linearly independent? Answer: Yes.
Why? (Because $x_1 e_1 + \cdots + x_n e_n = 0 \implies$ every $x_i = 0$.)
Therefore S is a basis of $\mathbb{R}^n$.
This particular basis is called the **standard basis** of $\mathbb{R}^n$.

Basis of $\mathbb{R}[x]$

For $\mathbb{R}[x]$, the set of all polynomials over $\mathbb{R}$, the subset

$$S = \{x^n : n = 0, 1, 2, \ldots\}$$

forms a basis.

- Our aim is to prove that if V is a finite dimensional vector space, then any two bases of V have the same number of elements.
- That unique number (for V) is called the **dimension** of V.

Lemma

Suppose $\{v_1, v_2, \ldots, v_n\}$ spans V. Let $u \neq 0$ is a vector in V. Then some v_i can be replaced by u to get another spanning set of V, i.e., if necessary, then after renaming the vectors $\{v_1, v_2, \ldots, v_n\}$, we obtain that $\{u, v_2, \ldots, v_n\}$ spans V.

Proof. Since $\{v_1, v_2, \ldots, v_n\}$ spans V, u can be written as

$$u = c_1 v_1 + c_2 v_2 + \cdots + c_n v_n \text{ for some } c_1, \ldots, c_n \in \mathbb{R}. \quad (1)$$

Since $u \neq 0$, at least one $c_i \neq 0$. So (1) yields that $v_i =$

$$(1/c_i)u+(c_1/c_i)v_1+\cdots+(c_{i-1}/c_i)v_{i-1}+(c_{i+1}/c_i)v_{i+1}+\cdots+(c_n/c_i)v_n.$$

Since every $v \in V$ is a linear combination of $v_1, v_2, \ldots, v_n$, using the expression of v_i in that linear combination, it follows that v can be written as a linear combination of $v_1, \ldots, v_{i-1}, u, v_{i+1}, \ldots, v_n$. Thus $\{v_1, \ldots, v_{i-1}, u, v_{i+1}, \ldots, v_n\}$ spans V. $\square$

Thank You!