Discrete Structures Exam 2
50 marks 1 hr

"In order to understand recursion, you must first understand recursion" - Stephen Hawking

Q1) Consider an inductive definition of a version of Ackermann's function. This function was named after Wilhelm Ackermann, a German mathematician who was a student of the great mathematician David Hilbert. Ackermann's function plays an important role in the theory of recursive functions and in the study of the complexity of certain algorithms involving set unions. (5 + 5 = 10 marks)

$$A(m, n) = \begin{cases} 2n & \text{if } m = 0 \\ 0 & \text{if } m \geq 1 \text{ and } n = 0 \\ 2 & \text{if } m \geq 1 \text{ and } n = 1 \\ A(m-1, A(m, n-1)) & \text{if } m \geq 1 \text{ and } n \geq 2 \end{cases}$$

a) Find these values of Ackermann's function.
   i) A(1, 0) ii) A(0, 1)
   iii) A(1, 1) iv) A(2, 2)
b) Show using mathematical induction that A(m, 2) = 4 whenever m ≥ 1.

Solution:

(a) **A(1, 0) = 0** line 2
A(0, 1) = 2.1 = 2 line 1
A(1, 1) = 2 line 3
A(2, 2) = A(1, A(2, 1)) line 4
= A(1, 2) line 3
= A(0, A(1, 1))
= A(0, 2)
= 2.2
So, **A(2, 2) = 4**

(b) We prove this by induction on m. The basis step is m = 1, so we need to compute A(1, 2). Line four of the definition tells us that A(1, 2) = A(0, A(1, 1)). Since A(1, 1) = 2, by line three, we see that A(1, 2) = A(0, 2).Now line one of the definition applies, and we see that A(1, 2) = A(0, 2) = 2 · 2 = 4, as desired. For the inductive step, assume that A(m - 1, 2) = 4, and consider A(m, 2). Applying first line four of the definition,then line three, and then the inductive hypothesis, we have A(m, 2) = A(m - 1, A(m, 1)) = A(m -1, 2) = 4.

Q2) Use a proof by cases to show that |xy| = |x||y|, where x and y are real numbers. (Recall that |a|, the absolute value of a, equals a when a ≥ 0 and equals −a when a ≤ 0.) (5 marks)

Solution: In our proof of this theorem, we remove absolute values using the fact that |a| = awhen a ≥ 0 and |a| = −a when a < 0. Because both |x| and |y| occur in our formula, we will need

four cases: *(i)* x and y both nonnegative, *(ii)* x nonnegative and y is negative, *(iii)* x negative and y nonnegative, and *(iv)* x negative and y negative. We denote by p1, p2, p3, and p4, the proposition stating the assumption for each of these four cases, respectively.
(Note that we can remove the absolute value signs by making the appropriate choice of signs within each case.)
*Case(i):*We see that p1 →q because $xy \geq 0$ when $x \geq 0$ and $y \geq 0$, so that $|xy| = xy = |x||y|$.
*Case(ii):*To see that p2 →q, note that if $x \geq 0$ and $y < 0$, then $xy \leq 0$, so that $|xy| = -xy = x(-y) = |x||y|$. (Here, because $y < 0$, we have $|y| = -y$.)
Case (iii): To see that p3 → q, we follow the same reasoning as the previous case with the roles of x and y reversed.
*Case(iv):*To see that p4 →q, note that when $x < 0$ and $y < 0$, it follows that $xy > 0$. Hence, $|xy| = xy = (-x)(-y) = |x||y|$.
Because $|xy| = |x||y|$ holds in each of the four cases and these cases exhaust all possibilities, we can conclude that $|xy| = |x||y|$, whenever x and y are real numbers.

Q3) A jigsaw puzzle is put together by successively joining pieces that fit together into blocks. A move is made each time a piece is added to a block, or when two blocks are joined. Use strong induction to prove that no matter how the moves are carried out, exactly n – 1 moves are required to assemble a puzzle with n pieces. (5 marks)
Solution:
Let P(n) be the statement that exactly n - 1 moves are required to assemble a puzzle with n pieces. Now P(1) is trivially true.
Assume that P(j) is true for all j < n, and consider a puzzle with n pieces. The final move must be the joining of two blocks, of size k and n - k for some integer k, $1 \leq k \leq n - 1$.
By the inductive hypothesis, it required k - 1 moves to construct the one block, and n - k - 1 moves to construct the other. Therefore $1 + (k - 1) + (n - k - 1) = n - 1$ moves are required in all, so P(n) is true. Here we have proved P(n) under the assumption that P(j) was true for j < n; so n played the role that k + 1 plays in the statement of strong induction given in the text.

Q4) The **harmonic mean** of two real numbers x and y equals $2xy/(x + y)$. By computing the harmonic and geometric means of different pairs of positive real numbers, formulate a conjecture about their relative sizes and prove your conjecture. (5 marks)

Solution:

Q5) Well-Formed Formulae (WFF) in Propositional Logic We can define the set of well-formed formulae in propositional logic involving T, F, propositional variables, and operators from the set $\{\neg, \wedge, \vee, \rightarrow, \leftrightarrow\}$.
BASIS STEP: T, F, and s, where s is a propositional variable, are well-formed formulae.
RECURSIVE STEP: If E and F are well-formed formulae, then $(\neg E)$, $(E \wedge F)$, $(E \vee F)$,
$(E \rightarrow F)$, and $(E \leftrightarrow F)$ are well-formed formulae.
Use structural induction to show that there are no WFFs of length 2, 3, or 6, but that any other positive length is possible. (5 marks)

Ans)

I assume that a wff is either

- a sentence symbol of length 1
- of the form $(\neg\alpha)$ of length $L+3$ where α is a wff of length L
- of the form $(\alpha \circ \beta)$ of length $L_1 + L_2 + 3$, where α, β are wff of lengths L_a, L_2 and $\circ \in \{\wedge \vee, \rightarrow, \leftrightarrow\}$

Then use structiral induction to show

Proposition. If α is a wff then its length $L(\alpha)$ is in $\mathbb{N} \setminus \{2,3,6\}$

Proof: We assume that $L(\alpha) \in \mathbb{N}$ is already known, so it suffices to show $L(\alpha) \notin \{2,3,6\}$.

- If α is a sentence symbol then $L(\alpha) = 1 \in \mathbb{N} \setminus \{2,3,6\}$
- If α is of the form $(\neg\beta)$, then $L(\alpha) = L(\beta) + 3$. If we assume that $L(\alpha) \in \{2,3,6\}$ we conclude $L(\beta) \in \{-1,0,3\}$, contradicting the induction hypothesis $L(\beta) \in \mathbb{N} \setminus \{2,3,6\}$. Therefore $L(\alpha) \in \mathbb{N} \setminus \{2,3,6\}$ also in this case
- If α is of the form $(\beta \circ \gamma)$, we have $L(\alpha) = L(\beta) + L(\gamma) + 3$, especially $L(\alpha) \geq 5$. We need only exclude $L(\alpha) = 6$, which would require that of of the sub-lengths is 1 and the other os 2, but that is not possible.

Q6) Suppose there are n people in a group, each aware of a scandal no one else in the group knows about. These people communicate by telephone; when two people in the group talk, they share information about all scandals each knows about. For example, on the first call, two people share information, so by the end of the call, each of these people knows about two scandals. The **gossip problem** asks for G(n), the minimum number of telephone calls that are needed for all n people to learn about all the scandals.
(5 + 5 = 10 marks)

a) Find G(1), G(2), G(3), and G(4).
b) Use mathematical induction to prove that G(n) ≤ 2n – 4 for n ≥ 4.

Solution:

(a) Number the people 1 , 2 , 3, and 4, and let s_i be the scandal originally known only to person i . It is clear that **G(1) = 0 and G(2) = 1.** For three people, without loss of generality assume that 1 calls 2 first and1 calls 3 next. At this point 1 and 3 know all three scandals, but it takes one more call to let 2 know s_3 . Thus **G(3) = 3.**
For four people, without loss of generality assume that 1 calls 2 first. If now 3 calls 4, then after two calls 1 and 2 both know s_1 and s_2 , while 3 and 4 both know s_3 and s_4 . It is
clear that two more calls (between 1 and 3, and between 2 and 4, say) are necessary and sufficient to complete the exchange. This makes a total of four calls. The only other case to consider (to see whether G(4) might be less than 4) is when the second call, without loss of generality, occurs between 1 and 3. At this point, both 2 and 4 still

need to learn s_3 , and talking to each other won't give them that information, so at least two more calls would be required. Thus **G(4) = 4.**

(b) Suppose we have a bunch of k+1 people. Pick one guy, say, Bob, and have him call someone else, say, Fred. Then remove Bob, so that there are only k
people left. Starting from Fred, circulate the gossip to all the k people in 2k−4 calls using the inductive hypothesis. Since Bob has told Fred his story, Bob's gossip will be circulated as well. It only remains for Fred (or anyone else) to fill Bob on everyone's gossip, which is done in one more call. This is, in total,2k−4 calls plus the 2 extra calls between Bob and Fred, and we're done.

Q7) Let S be the subset of the set of ordered pairs of integers defined recursively by
Basis step: (0, 0) ∈ S.
Recursive step: If (a, b) ∈ S, then (a, b + 1) ∈ S, (a + 1, b + 1) ∈ S, and (a + 2, b + 1) ∈ S.

(2 + 4 + 4 = 10 marks)

a) List the elements of S produced by the first four applications of the recursive definition.
b) Use strong induction on the number of applications of the recursive step of the definition to show that a ≤ 2b whenever (a, b) ∈ S.
c) Use structural induction to show that a ≤ 2b whenever (a, b) ∈ S.

Ans:
a) S = {(0,0),(0,1),(1,1),(2,1),(0,2)(1,2)(2,2),(3,2), (4,2),(0,3),(1,3),(2,3),(3,3),(4,3),(5,3),(6,3),(0,4),(1,4),(2,4),(3,4),(4,4),(5,4),(6,4),(7,4),(8,4)}
<u>Sir said during exam that 4 elements will suffice (including (0,0)).</u>

b) Strong induction on n, the number of applications of the recursive step: P(n) : a ≤ 2b whenever (a,b) ∈S. Basis: n = 0: (0,0) ∈S and 0 ≤ 2.0 so P(0) is true.
Induction: Given that a≤ 2b whenever (a,b)∈S is obtained by fewer than k applications, consider an element generated by k applications of the recursive step. This element is of the form (a,b+1), (a+1,b+1), or (a+2,b+1), and we only need to show that a≤2(b+1) and (a+1) ≤ 2(b+1) and (a+2) ≤ 2(b+1), where (a,b) was created by fewer than k applications; thus a≤ 2b. Clearly, a < (a+1) < (a+2)≤ (2b+2) = 2(b+1) so the results hold.

c)
1. Let P((a,b)) be “if (a,b)∈S, then a≤2b.”
2. Base Case: (0,0)∈S by our basis step. 0 ≤ 2(0) = 0, thus P((0,0)) is true.

3. Inductive Hypothesis: Assume P is true for some arbitrary values of each of the existing named elements mentioned in the recursive step. (i.e. P((a,b)) is true for arbitrary existing $(a,b) \in S$.)

4. Inductive Step: Must show that any new element generated by our recursive step makes P true.
i) Show P((a,b+1)) is true. Goal: Show $a \le 2(b+1)$. $a \le 2b$ by our I.H.
Thus $a \le 2b+2 = 2(b+1)$, since $2b \le 2b+2$. ✓
ii) Show P((a+1,b+1)) is true. Goal: Show $(a+1) \le 2(b+1)$. $a \le 2b$ by our I.H.
Thus $a+1 \le 2b+1$ (by adding 1 to both sides)
$\le 2b+2 = 2(b+1)$, since $2b+1 \le 2b+2$. ✓
iii) Show P((a+2,b+1)) is true. Goal: Show $(a+2) \le 2(b+1)$. $a \le 2b$ by our I.H.
Thus $a+2 \le 2b+2 = 2(b+1)$ by adding 2 to both sides. ✓
5. Conclusion: We have shown our base case and inductive step, thus P((a,b)) is true for all $(a,b) \in S$ by induction.