Life in Orthogonal Curvilinear Coordinates

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A **Curvilinear coordinate System** is one in which the coordinate axes are curves instead of straight lines. E.g. in two dimensional Euclidean place, $\mathbb{R}^2$ we set up a system of plane polar coordinates, namely, each point on the plane is specified by a pair (r, θ). r is the distance of the point from the origin and θ is the angle the position vector makes with the positive x-axis. Now in these plane polar coordinates, the "θ-axis" or in general "θ-grid "is defined to be the locus of constant r which is a circle around the origin. Thus we see that constant r-lines denoting the "θ-axis" are curves.

An **Orthogonal curvilinear coordinate system** is one where the constant co-ordinate loci or coordinate axes, though being curved, intersect each other at right angles. For example this is true in plane polar coordinates, the unit/basis vector along a constant r locus, is $\hat{\boldsymbol{\theta}}$ and the unit basis vector along a constant θ locus is along the radial direction and denoted, $\hat{\mathbf{r}}$. This is an orthogonal coordinate system because at each point on the plane,

$$\hat{\mathbf{r}}.\hat{\boldsymbol{\theta}} = 0.$$

Life simplifies a lot in an orthogonal coordinate system because the unit basis vectors at each point forms an orthogonal triad (if we are in 3 dimensional Euclidean space, $\mathbb{R}^3$). Examples covered are the cylindrical polar coordinate system, (ρ, ϕ, z) and the spherical polar coordinate system (r, θ, ϕ). In both cases we have at each point in space, an orthonormal triad of basis vectors, namely, $(\hat{\boldsymbol{\rho}}, \hat{\boldsymbol{\phi}}, \hat{\mathbf{z}})$ and $\left(\hat{\mathbf{r}}, \hat{\boldsymbol{\theta}}, \hat{\boldsymbol{\phi}}\right)$:

$$\text{Cylindrical:} \quad \hat{\boldsymbol{\rho}} \times \hat{\boldsymbol{\phi}} = \hat{\mathbf{z}}, \quad \hat{\boldsymbol{\phi}} \times \hat{\mathbf{z}} = \hat{\boldsymbol{\rho}}, \quad \hat{\mathbf{z}} \times \hat{\boldsymbol{\rho}} = \hat{\boldsymbol{\phi}},$$

$$\text{Spherical:} \quad \hat{\mathbf{r}} \times \hat{\boldsymbol{\theta}} = \hat{\boldsymbol{\phi}}, \quad \hat{\boldsymbol{\theta}} \times \hat{\boldsymbol{\phi}} = \hat{\mathbf{r}}, \quad \hat{\boldsymbol{\phi}} \times \hat{\mathbf{r}} = \hat{\boldsymbol{\theta}}.$$

Now let us consider the general case when we have three curvilinear coordinates, (q_1, q_2, q_3) and the corresponding unit vectors which constitute an orthonormal triad,

$$\text{Generic case:} \quad \hat{\mathbf{e}}_1 \times \hat{\mathbf{e}}_2 = \hat{\mathbf{e}}_3, \quad \hat{\mathbf{e}}_2 \times \hat{\mathbf{e}}_3 = \hat{\mathbf{e}}_1, \quad \hat{\mathbf{e}}_3 \times \hat{\mathbf{e}}_1 = \hat{\mathbf{e}}_2$$

$\hat{\mathbf{e}}_1$ points to the direction of increasing q_1, $\hat{\mathbf{e}}_2$ points to the direction of increasing q_2 and $\hat{\mathbf{e}}_3$ points to the direction of increasing q_3. Let the line element be expressed as,

$$d\mathbf{x} = h_1 dq_1 \hat{\mathbf{e}}_1 + h_2 dq_2 \hat{\mathbf{e}}_2 + h_3 dq_3 \hat{\mathbf{e}}_3.$$

The coefficients, h_1, h_2, h_3 are in general functions of the coordinates, (q_1, q_2, q_3),

$$h_i = h_i(\{q_j\}), \; i, j = 1, 2, 3.$$

The squared line interval /magnitude is then,

$$ds^2 = d\mathbf{x} \cdot d\mathbf{x} = h_i \left(dq_i\right)^2 .$$

The volume element on the other hand is,

$$\begin{aligned} dV = d^3\mathbf{x} &= h_1 dq_1 \hat{\mathbf{e}}_1 . \left(h_2 dq_2 \hat{\mathbf{e}}_2 \times h_1 dq_1 \hat{\mathbf{e}}_3\right) \\ &= \prod_{i=1}^{3} h_i dq_i . \end{aligned}$$

1 Expression for vector differential operators in a generic orthogonal curvilinear coordinate system

1.1 Gradient

We are interested in finding out the expression of the gradient operator in a general orthogonal curvilinear coordinate system. To this end we use the identity,

$$\begin{aligned} \Phi(\mathbf{x} + d\mathbf{x}) - \Phi(\mathbf{x}) &= d\mathbf{x} \cdot \boldsymbol{\nabla}\Phi \\ &= \left(h_1 dq_1 \hat{\mathbf{e}}_1 + h_2 dq_2 \hat{\mathbf{e}}_2 + h_3 dq_3 \hat{\mathbf{e}}_3\right) \cdot \boldsymbol{\nabla}\Phi \\ &= h_1 \left(\hat{\mathbf{e}}_1 \cdot \boldsymbol{\nabla}\Phi\right) dq_1 + h_2 \left(\hat{\mathbf{e}}_2 \cdot \boldsymbol{\nabla}\Phi\right) dq_2 + h_3 \left(\hat{\mathbf{e}}_3 \cdot \boldsymbol{\nabla}\Phi\right) dq_3 . \end{aligned}$$

and then equate it to an expression which is a rule of multivariable calculus,

$$\Phi\left(\{q_i + dq_i\}\right) - \Phi\left(\{q_i\}\right) = \frac{\partial \Phi}{\partial q_1} dq_1 + \frac{\partial \Phi}{\partial q_2} dq_2 + \frac{\partial \Phi}{\partial q_3} dq_3 .$$

Equating these two expressions, we get,

$$h_1 \left(\hat{\mathbf{e}}_1 \cdot \boldsymbol{\nabla}\Phi\right) = \frac{\partial \Phi}{\partial q_1}, \quad h_2 \left(\hat{\mathbf{e}}_2 \cdot \boldsymbol{\nabla}\Phi\right) = \frac{\partial \Phi}{\partial q_2}, \quad h_3 \left(\hat{\mathbf{e}}_3 \cdot \boldsymbol{\nabla}\Phi\right) = \frac{\partial \Phi}{\partial q_3},$$

or,

$$\left(\hat{\mathbf{e}}_1 \cdot \boldsymbol{\nabla}\Phi\right) = \frac{1}{h_1} \frac{\partial \Phi}{\partial q_1}, \quad \left(\hat{\mathbf{e}}_2 \cdot \boldsymbol{\nabla}\Phi\right) = \frac{1}{h_2} \frac{\partial \Phi}{\partial q_2}, \quad \left(\hat{\mathbf{e}}_3 \cdot \boldsymbol{\nabla}\Phi\right) = \frac{1}{h_3} \frac{\partial \Phi}{\partial q_3}.$$

So we have the expression for the gradient of a scalar,

$$\begin{aligned} \boldsymbol{\nabla}\Phi &= \hat{\mathbf{e}}_1 \left(\hat{\mathbf{e}}_1 \cdot \boldsymbol{\nabla}\Phi\right) + \hat{\mathbf{e}}_2 \left(\hat{\mathbf{e}}_2 \cdot \boldsymbol{\nabla}\Phi\right) + \hat{\mathbf{e}}_3 \left(\hat{\mathbf{e}}_3 \cdot \boldsymbol{\nabla}\Phi\right) \\ &= \hat{\mathbf{e}}_1 \frac{1}{h_1} \frac{\partial \Phi}{\partial q_1} + \hat{\mathbf{e}}_2 \frac{1}{h_2} \frac{\partial \Phi}{\partial q_2} + \hat{\mathbf{e}}_3 \frac{1}{h_3} \frac{\partial \Phi}{\partial q_3} \\ &= \left(\hat{\mathbf{e}}_1 \frac{1}{h_1} \frac{\partial}{\partial q_1} + \hat{\mathbf{e}}_2 \frac{1}{h_2} \frac{\partial}{\partial q_2} + \hat{\mathbf{e}}_3 \frac{1}{h_3} \frac{\partial}{\partial q_3}\right) \Phi . \qquad (1) \end{aligned}$$

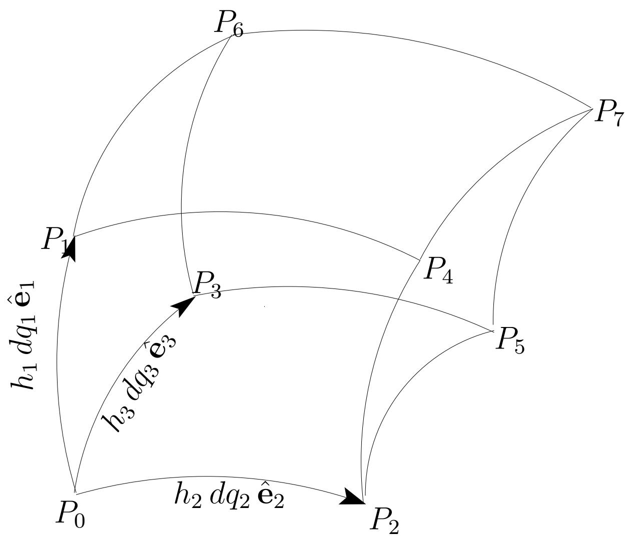

Figure 1: Elementary cuboid for determing the divergence

1.2 Divergence

Consider a vector field which has the following component form in a general orthogonal coordinate system,

$$\mathbf{A} = A_{q_1}\hat{\mathbf{e}}_1 + A_{q_2}\hat{\mathbf{e}}_2 + A_{q_3}\hat{\mathbf{e}}_3.$$

To reduce cumber in notation we use a simpler component form,

$$\mathbf{A} = A_1\hat{\mathbf{e}}_1 + A_2\hat{\mathbf{e}}_2 + A_3\hat{\mathbf{e}}_3$$

with the understanding that $1, 2, 3$ does not mean components along a set of Cartesian axes, but rather along $q_1, q_{2,}, q_3$ axes.

Next, we recall that the divergence of a vector field is defined by the following limit (flux per unit volume),

$$\boldsymbol{\nabla}\cdot\mathbf{A} = \lim_{\Delta V \to 0} \frac{\sum_{i=1}^{6} \mathbf{A}\cdot\hat{\mathbf{n}}_i \Delta S_i}{\Delta V}$$

Here the summation is over the 6 faces of the elementary cuboid of ΔV (see figure 1) which is formed by the following 8 vertices,

$$\begin{aligned}
P_0 &= \mathbf{x}, \\
P_1 &= \mathbf{x} + h_1 dq_1 \hat{\mathbf{e}}_1, \\
P_2 &= \mathbf{x} + h_2 dq_2 \hat{\mathbf{e}}_2, \\
P_3 &= \mathbf{x} + h_3 dq_3 \hat{\mathbf{e}}_3, \\
P_4 &= \mathbf{x} + h_1 dq_1 \hat{\mathbf{e}}_1 + h_2 dq_2 \hat{\mathbf{e}}_2, \\
P_5 &= \mathbf{x} + h_2 dq_2 \hat{\mathbf{e}}_2 + h_3 dq_3 \hat{\mathbf{e}}_3, \\
P_6 &= \mathbf{x} + h_3 dq_3 \hat{\mathbf{e}}_3 + h_1 dq_1 \hat{\mathbf{e}}_1, \\
P_7 &= \mathbf{x} + h_1 dq_1 \hat{\mathbf{e}}_1 + h_2 dq_2 \hat{\mathbf{e}}_2 + h_3 dq_3 \hat{\mathbf{e}}_3.
\end{aligned}$$

So clearly the volume of this elementary cuboid is,

$$\Delta V = h_1 dq_1 h_2 dq_2 h_3 dq_3.$$

Now let's evaluate the flux thru each of the 6 faces of the cuboid.

1. Flux thru $P_0P_1P_4P_2$: The area element is,

$$\hat{\mathbf{n}}_1 \Delta S_1 = \overrightarrow{P_0P_2} \times \overrightarrow{P_0P_1} = h_2 dq_2 \hat{\mathbf{e}}_2 \times h_1 dq_1 \hat{\mathbf{e}}_1 = -h_1 h_2 dq_1 dq_2 \hat{\mathbf{e}}_3$$

   So the flux is,

$$\mathbf{A} \cdot \hat{\mathbf{n}}_1 \Delta S_1 = -A_3 h_1 h_2 dq_1 dq_2.$$

2. Flux thru $P_3P_6P_7P_5$: This is the face which is opposite to the above, i.e., the flux will be of opposite sign and since this face is parallel to previous one and separated in coordinate space, by dq_3, the resulting flux is,

$$\mathbf{A} \cdot \hat{\mathbf{n}}_2 \Delta S_2 = -\left(\mathbf{A} \cdot \hat{\mathbf{n}}_1 \Delta S_1 + \frac{\partial \left(\mathbf{A} \cdot \hat{\mathbf{n}}_1 \Delta S_1\right)}{\partial q_3} dq_3\right) = A_3 h_1 h_2 dq_1 dq_2 + \frac{\partial \left(A_3 h_1 h_2\right)}{\partial q_3} dq_1 dq_2 dq_3$$

   So adding the flux contributions from these two faces we get,

$$\mathbf{A} \cdot \hat{\mathbf{n}}_1 \Delta S_1 + \mathbf{A} \cdot \hat{\mathbf{n}}_2 \Delta S_2 = \frac{\partial \left(A_3 h_1 h_2\right)}{\partial q_3} dq_1 dq_2 dq_3$$

   Similarly, we will find that the net flux contributions from the parallel faces, $P_0P_1P_6P_3$ and $P_2P_4P_7P_5$ is,

$$\mathbf{A} \cdot \hat{\mathbf{n}}_3 \Delta S_3 + \mathbf{A} \cdot \hat{\mathbf{n}}_4 \Delta S_4 = \frac{\partial \left(A_2 h_3 h_1\right)}{\partial q_2} dq_1 dq_2 dq_3,$$

   and we will also find that the net flux contributions from the parallel faces, $P_0P_3P_5P_2$ and $P_1P_6P_7P_4$ is,

$$\mathbf{A} \cdot \hat{\mathbf{n}}_5 \Delta S_5 + \mathbf{A} \cdot \hat{\mathbf{n}}_5 \Delta S_5 = \frac{\partial \left(A_1 h_2 h_3\right)}{\partial q_1} dq_1 dq_2 dq_3$$

   Gathering flux contributions from all 6 faces,

$$\sum_{i=1}^{6} \mathbf{A} \cdot \hat{\mathbf{n}}_i \Delta S_i = \left[\frac{\partial \left(A_1 h_2 h_3\right)}{\partial q_1} + \frac{\partial \left(A_2 h_3 h_1\right)}{\partial q_2} + \frac{\partial \left(A_3 h_1 h_2\right)}{\partial q_3}\right] dq_1 dq_2 dq_3$$

   and so,

$$\frac{\sum_{i=1}^{6} \mathbf{A} \cdot \hat{\mathbf{n}}_i \Delta S_i}{\Delta V} = \frac{1}{h_1 h_2 h_3} \left[\frac{\partial \left(A_1 h_2 h_3\right)}{\partial q_1} + \frac{\partial \left(A_2 h_3 h_1\right)}{\partial q_2} + \frac{\partial \left(A_3 h_1 h_2\right)}{\partial q_3}\right].$$

   Thus the expression for divergence in orthogonal curvilinear coordinates is,

$$\boldsymbol{\nabla} \cdot \mathbf{A} = \frac{1}{h_1 h_2 h_3} \left[\frac{\partial \left(A_1 h_2 h_3\right)}{\partial q_1} + \frac{\partial \left(A_2 h_3 h_1\right)}{\partial q_2} + \frac{\partial \left(A_3 h_1 h_2\right)}{\partial q_3}\right]. \tag{2}$$

1.3 Curl of a vector field

Let's compute the q_1-th component of the curl (of $\mathbf{A}$), i.e. the component along, $\hat{\mathbf{e}}_1$ by evaluating the (anticlockwise) circulation of $\mathbf{A}$ along a rectangular loop in the q_2-q_3 plane.

$$(\boldsymbol{\nabla} \times \mathbf{A})_1 = \lim_{\Delta S \to 0} \frac{\sum_{i=1}^{4} d\mathbf{l}_i.\mathbf{A}}{\Delta S}.$$

Here the sum is over the 4-segments/sides of the rectangle made up by the following vertices,

$$\begin{aligned} P_0 &= \mathbf{x}, \\ P_2 &= \mathbf{x} + h_2 dq_2 \hat{\mathbf{e}}_2, \\ P_3 &= \mathbf{x} + h_3 dq_3 \hat{\mathbf{e}}_3, \\ P_5 &= \mathbf{x} + h_2 dq_2 \hat{\mathbf{e}}_2 + h_3 dq_3 \hat{\mathbf{e}}_3. \end{aligned}$$

Now let's evaluate $d\mathbf{l}_i.\mathbf{A}$ for each of the 4 segments of the rectangle.

1. $d\mathbf{l}_1.\mathbf{A}$ where $d\mathbf{l}_1 = \overrightarrow{P_0 P_2} = h_2 dq_2 \hat{\mathbf{e}}_2$ is,

$$d\mathbf{l}_1.\mathbf{A} = h_2 dq_2\, A_2,$$

2. $d\mathbf{l}_2.\mathbf{A}$ where $d\mathbf{l}_2 = \overrightarrow{P_5 P_4}$: This segment is parallel to $d\mathbf{l}_1$ but a) opposite in direction to $d\mathbf{l}_1$ and b) separated only in the q_3 coordinate direction. Thus using Taylor expansion, one can write down

$$\begin{aligned} d\mathbf{l}_2.\mathbf{A} = -\left(d\mathbf{l}_1.\mathbf{A} + \frac{\partial}{\partial q_3} d\mathbf{l}_1.\mathbf{A}\right) &= -h_2 dq_2\, A_2 - \frac{\partial}{\partial q_3}\left(h_2 dq_2\, A_2\right) dq_3 \\ &= -h_2 dq_2\, A_2 - \frac{\partial}{\partial q_3}\left(h_2\, A_2\right) dq_2 dq_3 \end{aligned}$$

   Thus adding the circulation contribution from the segments, $\overrightarrow{P_0 P_2}$ and $\overrightarrow{P_5 P_4}$,

$$d\mathbf{l}_1.\mathbf{A} + d\mathbf{l}_2.\mathbf{A} = -\frac{\partial}{\partial q_3}\left(h_2\, A_2\right) dq_2 dq_3.$$

3. Similarly, adding the circulation contributions from $P_2 P_5$ and $P_3 P_0$,

$$d\mathbf{l}_3.\mathbf{A} + d\mathbf{l}_4.\mathbf{A} = \frac{\partial}{\partial q_2}\left(h_3\, A_3\right) dq_2 dq_3.$$

   Thus gathering together the contributions to the circulation thru all four segments,

$$\sum_{i=1}^{4} d\mathbf{l}_i.\mathbf{A} = \left[\frac{\partial}{\partial q_2}\left(h_3\, A_3\right) - \frac{\partial}{\partial q_3}\left(h_2\, A_2\right)\right] dq_2 dq_3.$$

Now,

$$\begin{aligned}\Delta S &= |\overrightarrow{P_0P_2} \times \overrightarrow{P_0P_3}| \\ &= |h_2 dq_2 \hat{\mathbf{e}}_2 \times h_3 dq_3 \hat{\mathbf{e}}_3| \\ &= h_2 h_3 dq_2 dq_3.\end{aligned}$$

Thus,

$$(\boldsymbol{\nabla} \times \mathbf{A})^1 = \lim_{\Delta S \to 0} \frac{\sum_{i=1}^{4} d\mathbf{l}_i . \mathbf{A}}{\Delta S} = \frac{1}{h_2 h_3} \left[\frac{\partial}{\partial q_2} (h_3\, A_3) - \frac{\partial}{\partial q_3} (h_2\, A_2) \right],$$

and collecting all components,

$$\begin{aligned}\boldsymbol{\nabla} \times \mathbf{A} \;=\; & \frac{\hat{\mathbf{e}}_1}{h_2 h_3} \left[\frac{\partial}{\partial q_2} (h_3\, A_3) - \frac{\partial}{\partial q_3} (h_2\, A_2) \right] + \frac{\hat{\mathbf{e}}_2}{h_3 h_1} \left[\frac{\partial}{\partial q_3} (h_1\, A_1) - \frac{\partial}{\partial q_1} (h_3\, A_3) \right] \\ & + \frac{\hat{\mathbf{e}}_3}{h_1 h_2} \left[\frac{\partial}{\partial q_1} (h_2\, A_2) - \frac{\partial}{\partial q_2} (h_1\, A_1) \right].\end{aligned}$$

More compactly, we can write the curl in a determinant form,

$$\boldsymbol{\nabla} \times \mathbf{A} = \frac{1}{h_1 h_2 h_3} \begin{vmatrix} h_1 \hat{\mathbf{e}}_1 & h_2 \hat{\mathbf{e}}_2 & h_3 \hat{\mathbf{e}}_3 \\ \frac{\partial}{\partial q_1} & \frac{\partial}{\partial q_2} & \frac{\partial}{\partial q_3} \\ h_1 A_1 & h_2\, A_2 & h_3\, A_3 \end{vmatrix}. \tag{3}$$

1.4 Laplacian Operator

$$\begin{aligned}\boldsymbol{\nabla}^2 \Phi \;&\equiv\; \boldsymbol{\nabla} \cdot (\boldsymbol{\nabla} \Phi) \\ &= \frac{1}{h_1 h_2 h_3} \left[\frac{\partial}{\partial q_1} \left((\boldsymbol{\nabla}\Phi)^1 h_2 h_3 \right) + \frac{\partial}{\partial q_2} \left((\boldsymbol{\nabla}\Phi)^2 h_3 h_1 \right) + \frac{\partial}{\partial q_3} \left((\boldsymbol{\nabla}\Phi)^3 h_1 h_2 \right) \right] \\ &= \frac{1}{h_1 h_2 h_3} \left[\frac{\partial}{\partial q_1} \left(\frac{h_2 h_3}{h_1} \frac{\partial \Phi}{\partial q_1} \right) + \frac{\partial}{\partial q_2} \left(\frac{h_3 h_1}{h_2} \frac{\partial \Phi}{\partial q_2} \right) + \frac{\partial}{\partial q_3} \left(\frac{h_1 h_2}{h_3} \frac{\partial \Phi}{\partial q_3} \right) \right] \quad (4)\end{aligned}$$

2 Spherical and Cylindrical Polar coordinates

Recall the line element for spherical polar coordinates is,

$$d\mathbf{x} = dr\, \hat{\mathbf{r}} + r d\theta\, \hat{\boldsymbol{\theta}} + r \sin\theta\, \hat{\boldsymbol{\phi}}.$$

So the identification with our general treatment is,

$$q_1 = r, \quad q_2 = \theta, \quad q_3 = \phi$$

and the scale factors are,

$$h_1 = 1, \quad h_2 = r, \quad h_3 = r \sin\theta.$$

The corresponding identifications for the cylindrical polar coordinates are,

$$q_1 = \rho, \quad q_2 = \phi, \quad q_3 = z,$$

and,

$$h_1 = 1, \quad h_2 = \rho, \quad h_3 = 1.$$

Substituting these in (1-4), one can easily recover the formula presented in the slides in the lecture.