$\vec{x}, \vec{y} \in \mathbb{R}^n$

$\|\vec{x} - \vec{y}\| \le \|\vec{x} - \vec{z}\| + \|\vec{z} - \vec{y}\|$

Let $f : \mathbb{R}^n \longrightarrow \mathbb{R}$ and $\vec{x_0} \in \mathbb{R}^n$ then f is said to be continuous at $\vec{x_0}$ if given $\varepsilon > 0$ $\exists\, \delta > 0$

s.t. $|f(x) - f(x_0)| < \varepsilon$ whenever $\underline{\|x - x_0\| < \delta}$

<u>Ex:</u>
$$f(x,y) = \frac{1}{x^2+y^2} \quad (x,y) \neq (0,0)$$
$$= 0 \quad (x,y) = (0,0)$$

Let $\underset{\sim}{\varepsilon > 0}$ and $\exists\, \delta > 0$ s.t.

$1 > \varepsilon > 0$

$|f(x,y) - \underbrace{f(0,0)}_{0}| < \varepsilon < 1$

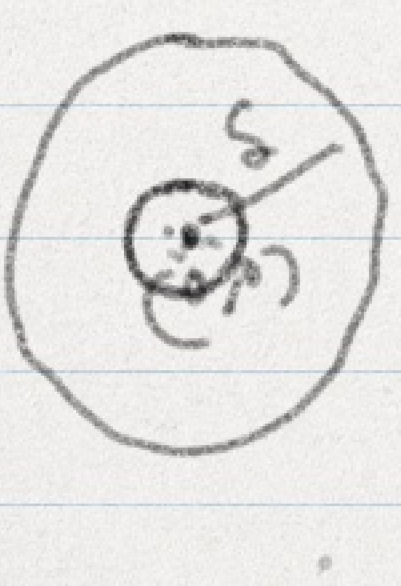
whenever $\|(x,y) - (0,0)\| < \delta$.

$\frac{1}{x^2+y^2} < \varepsilon$ whenever $\|(x,y)\| < \delta$.

Let $0 < \delta' < \sqrt{\varepsilon}$ and choose $(x,y) \in B(0,\eta)$, $\eta < \min\{\delta, \delta'\}$

$$f(x,y) = \frac{1}{x^2+y^2} > \frac{1}{\eta^2} > \frac{1}{\varepsilon} > \varepsilon$$

$$\eta < \delta' < \sqrt{\varepsilon}$$



$$\frac{1}{\eta^2} > \frac{1}{\varepsilon}$$

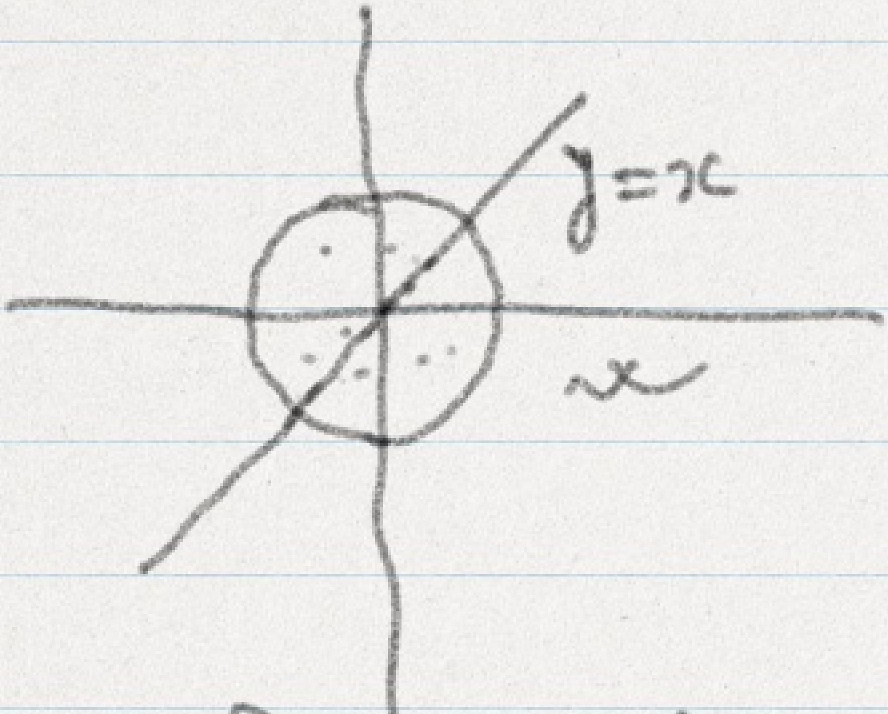
Ex: Let $f(x,y) = \begin{cases} xy\dfrac{x^2-y^2}{x^2+y^2} & (x,y) \neq (0,0) \\ 0 & \text{o.w.} \end{cases}$

$f : \mathbb{R}^2 \longrightarrow \mathbb{R}$

$$\begin{aligned} |f(x,y) - f(0,0)| &= |f(x,y)| \\ &= |xy| \left| \frac{x^2-y^2}{x^2+y^2} \right| \\ &\leq |x||y| \\ &\leq \sqrt{x^2+y^2} \cdot \sqrt{x^2+y^2} \\ &= (x^2+y^2) < \varepsilon \end{aligned}$$

$D = \{(x,y) \in \mathbb{R}^2 : \|(x,y)\| < \sqrt{\varepsilon}\}$

Ex: $f(x,y) = \begin{cases} \frac{xy}{x^2+y^2} & (x,y) \neq (0,0) \\ 0 & \text{o.w.} \end{cases}$

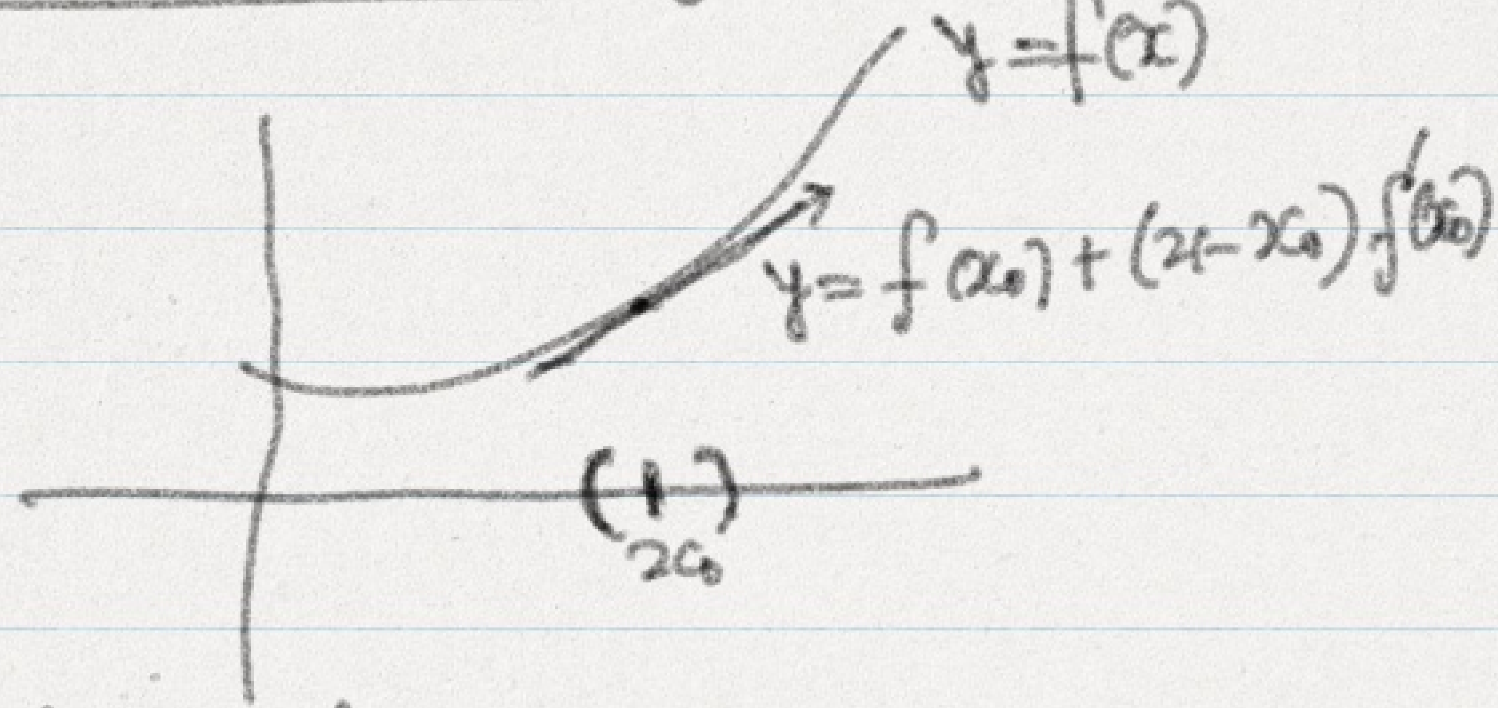


Here if we chose points from the line $y=x$ then $f(x,y)=\frac{1}{2}$.
(if $x\neq 0$).
Hence for $\varepsilon>0$ $\nexists$ $\delta>0$ s.t
$|f(x,y)| < \varepsilon$.

Hence f can not be cont. at $(0,0)$

$$f : \mathbb{R}^n \longrightarrow \mathbb{R}^m$$

Differentrability



f is diff at x_0 :

$$\lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0} = l$$

ie $$\lim_{x \to x_0} \left| \frac{f(x) - f(x_0)}{x - x_0} - l \right| = 0$$

ie $$\lim_{x \to x_0} \frac{|f(x) - (f(x_0) + l(x - x_0))|}{|x - x_0|} = 0$$

Let $f: \mathbb{R}^2 \longrightarrow \mathbb{R}$ and

$(x_0, y_0) \in \mathbb{R}^2$ Then f is said

to be diff at (x_0, y_0) if $\exists$
$(\alpha, \beta) \in \mathbb{R}^2$ s.t.

$$\lim_{(x,y)\to(x_0,y_0)} \frac{\left|f(x,y) - \left(f(x_0,y_0) + \langle(\alpha,\beta),(x-x_0,y-y_0)\rangle\right)\right|}{\|(x-x_0, y-y_0)\|} = 0$$

$$(\alpha, \beta) = f'(x_0, y_0)$$

Remark: This (α, β) exists uniquely.

Exc: Let $f: \mathbb{R}^n \longrightarrow \mathbb{R}$ which is cont. at $(x_1, \dots, x_n)$ then f is continuous at $(x_1, \dots x_n)$.

$$\lim_{\vec{x} \to \vec{x}_0} \frac{|f(x) - (f(x_0) + \langle (\alpha, \beta), (x - x_0, y - y_0) \rangle)|}{\| (\quad) \|} = 0$$

$$|f(\vec{x}) - f(\vec{x}_0)| \leq |\langle (\alpha, \beta), (x - z, y - w) \rangle| + \varepsilon \| \quad \|$$

where $\vec{x}_0 = (z, w)$

Ex: Define $f(x,y) = (x^2, y^2)$.

$$f: \mathbb{R}^2 \longrightarrow \mathbb{R}^2.$$

Show that f is differentiable at $(1,1)$.