Lecture 13 - Planar Graphs and Graph coloring

April 18, 2019

Recap

- Graphs – Connectivity
- Euler and Hamilton Circuits and Paths

Planar Graphs

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- We just saw K_4 and Q_3 are planar. What about $K_{3,3}$ and K_5? No.
- We can use arguments that look at regions divided and then argue that they cannot be planar.

Euler's Formula

A planar representation of a graph splits the plane into regions, including an unbounded region.

Theorem

Let G be a connected planar simple graph with e edges and v vertices. Let r be the number of regions in a planar representation of G. Then $r = e - v + 2$.

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- Let r_n, e_n, v_n represent the number of regions, edges, and vertices of the planar representation of G_n.

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- They must have been on the boundary of a common region R or else $\{a_{k+1}, b_{k+1}\}$ would caused crossing!
- The new edge splits R into two regions $\Rightarrow r_{k+1} = r_k + 1$, $e_{k+1} = e_k + 1$, $v_{k+1} = v_k$ - formula holds!

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- Completes the induction argument!

Euler's Formula - Very Important Corollaries

Corollary

If G is a connected planar simple graph with e edges and v vertices, where $v \geq 3$, then $e \leq 3v - 6$.

Corollary

If G is a connected planar simple graph then G has a vertex of degree not exceeding five.

Corollary

If a connected planar simple graph has e edges and v vertices with $v \geq 3$ and no circuits of length 3 then $e \leq 2v - 4$.

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- First corollary shows that K_5 is nonplanar - It has five vertices and ten edges. $e \leq 3v - 6$ is not satisfied!
- $K_{3,3}$ satisfies Corollary 1 but it is not a sufficient condition – it violates Corollary 3. $e = 9, 2v - 4 = 8$

Graph coloring

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- A coloring of a simple graph is the assignment of a color to each vertex of the graph so that no two adjacent vertices are assigned the same color.
- The chromatic number $\chi(G)$ of a graph is the least number of colors needed for a coloring of this graph.

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- Finally proved in 1976 - Appel and Haken.

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- Then they show none of these types exist.
- Simpler proofs have come but everything relies on computer.
- Doesn't work for nonplanar graphs – they can have large chromatic number.

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- $\chi(K_{m,n}) = 2$ since its bipartite.

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  - What if n is odd? In that case, first and $n-1$st vertex are of different colors - red and blue and therefore nth vertex has to be a third color.
  - $\chi(C_n) = 2$ if n is an even positive integer, $\chi(C_n) = 3$, if n is odd, $n \geq 3$.

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- T.S.T. G_{r+1} can be assigned a k-vertex coloring if G_r can be assigned a k-vertex coloring.
- Lemma : Let G be a simple connected planar graph, then the minimum degree of a graph is less than 5.

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- Case(ii): Suppose all 5 colors are connected to v:
- Let the five vertices adjacent to v be called u_1, u_2, u_3, u_4, u_5 and colored c_1, c_2, c_3, c_4, c_5.

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- v is not adjacent to vertex of color c_2 and can be given c_2. - Done!

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- Frequency assignments for TV stations so that within a certain distance two stations wont have the same channel.

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- Finding an approximation to the chromatic number of a graph is difficult.
- Applications - scheduler of exams, no students have two exams at the same time.
- Frequency assignments for TV stations so that within a certain distance two stations wont have the same channel.
- Compilers - assigning index registers for a loop.