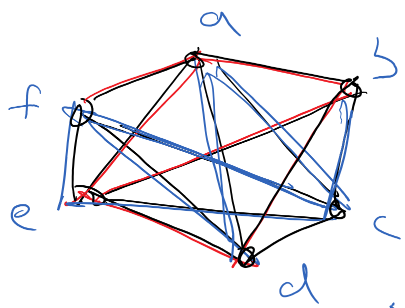
Alteration

Ramsey Numbers

$R(k,k)$: min n such that no matter how one colors the edges of a K_n with 2 colors, a monochromatic k-clique is unavoidable.

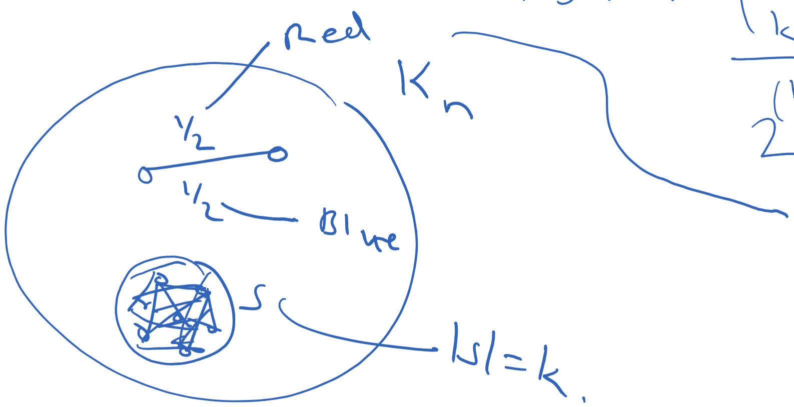
$$R(3,3) = 6$$



Δ bed - red Δ.

Theorem For any integer n, $R(k,k) > n - \dfrac{\binom{n}{k}}{2^{\binom{k}{2}-1}}$.

Proof:



$$\Pr[S \text{ is monochromatic}] = \frac{1}{2^{\binom{k}{2}}} + \frac{1}{2^{\binom{k}{2}}}$$

$$= \frac{1}{2^{\binom{k}{2}-1}}$$

$$X_S := \begin{cases} 1, & \text{if } S \text{ is monochromatic} \\ 0, & \text{o/w.} \end{cases}$$

1 o/w.

$$E[X_S] = 1 \cdot \Pr[S \text{ is monochromatic}]$$

$$= \frac{1}{2^{\binom{k}{2}-1}} \quad \text{——} \text{①}$$

X : no. of monochromatic k-cliques in K_n.

$$X = \sum_{\substack{S \subseteq [n], \\ |S| = k}} X_S$$

By linearity of expectation,

$$E[X] = \sum_{\substack{S \subseteq [n], \\ |S| = k}} E[X_S]$$

$$= \frac{\binom{n}{k}}{2^{\binom{k}{2}-1}}$$

From the defn. of expectation, there is a 2-coloring of K_n where the no. of monochromatic k-cliques is $\leq E[X] = \frac{\binom{n}{k}}{2^{\binom{k}{2}-1}}$

Take that coloring.

$$R(k,k) > n - \frac{\binom{n}{k}}{2^{\binom{k}{2}-1}}$$

$\square$

From calculus,

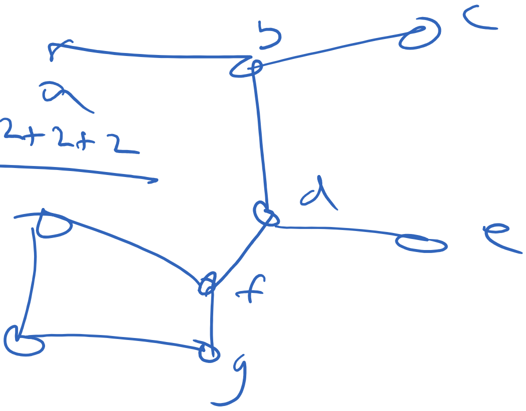
$$R(k,k) > \frac{1}{e}\left(1+o(1)\right) k\, 2^{k/2}$$

whereas, from what we saw in Chap 1,

$$R(k,k) \geq \frac{1}{e\sqrt{2}}\left(1+o(1)\right) k\, 2^{k/2}$$

Independent Set

Given a graph G, a set $S \subseteq V(G)$ is an independent set in G if no two vertices in S are adjacent to each other.



$$d = \frac{1+3+1+3+1+3+2+2+2}{9}$$

$S = \{a, c, d, g, i\}$ is an

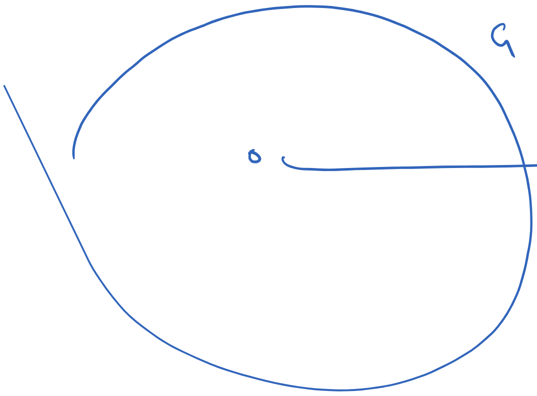
independent set.

$\alpha(G)$: denote the size of a largest independent set in G.

<u>Theorem</u>: Let G be a graph on n vertices. Let d denote the <u>average degree</u> of a vertex in G. Then,
$$\alpha(G) \geq \frac{n}{2d}.$$

<u>Proof</u>:

$$\text{average degree} = d = \frac{\sum_{v \in V(G)} \deg(v)}{n}$$



each vertex is chosen into a set S independently with prob p.

$$X_v = \begin{cases} 1, & \text{if } v \text{ is chosen into } S \\ 0, & \text{o/w} \end{cases}$$

$\Pr[X_v] = p.$

$E[X_v] = p.$

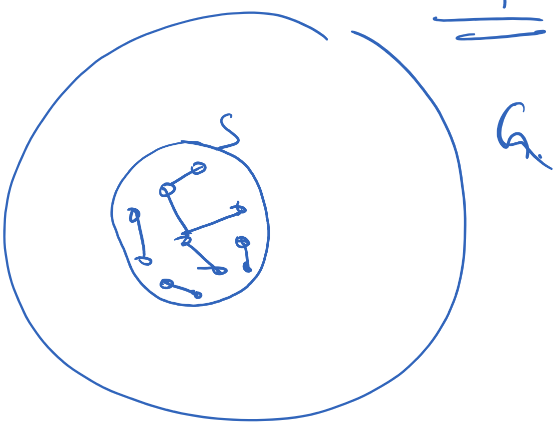
R.V.
X : denotes the size of S.

$$X = \sum_{v \in V(G)} X_v$$

By linearity of expectation,

$$E[X] = \sum_{v \in V(G)} E[X_v]$$

$$= \underline{\underline{n \cdot p}} \quad \text{—— (1)}$$



For each edge $e = uv$, we have

R.V. $Y_e = \begin{cases} 1, & \text{if both endpoints of } e \text{ are chosen into } S. \\ 0, & \text{o/w.} \end{cases}$

$$E[Y_e] = \Pr\left[\begin{array}{c}\text{both endpoints of } e \\ \text{are chosen into } S\end{array}\right] \cdot 1$$

$$= \underline{\underline{p^2}}$$

R.V.
Y : no. of edges present inside S.

$$Y = \sum_{e \in E(G)} Y_e$$

By linearity of expectation,

$$E[Y] = \sum_{e \in E(G)} E[Y_e]$$

$$= m \cdot p^2$$

m denotes the no. of edges in G.

$$= \frac{ndp^2}{2} \quad \text{--- } (2)$$

$$m = \frac{\sum_{v \in V(G)} \deg(v)}{2}$$

avg degree

$$d = \frac{\sum_{v \in V(G)} \deg(v)}{n}$$

$$m = \frac{nd}{2}$$

Let $Z = X - Y$. $Z \to$ R.V., $X \to$ no. of vertices in S, $Y \to$ no. of edges in S.

$$E[Z] = E[X - Y]$$

$$= E[X] - E[Y]$$

$$= np - \frac{ndp^2}{2} \quad \left(\text{from } (1) \text{ and } (2)\right)$$

$$= np\left(1 - \frac{pd}{2}\right) \quad \text{--- } (3)$$

There is an independent set of size at least $np\left(1 - \frac{pd}{2}\right)$.

Choose $p = \frac{1}{d}$.

Then

$$E(Z) = \frac{n}{d}\left(1 - \frac{1}{2}\right)$$

$$= \frac{n}{2d}$$

$\square$