Topics in Combinatorics

Exam III (out of 10 marks)
(Date: 14 Dec 2020. Timing: 12:00 to 13:05 hours)

1. Let $f \in \mathbb{F}[x_1, \ldots, x_n]$ be a polynomial and $S_1, \ldots, S_n$ be non-empty subsets of $\mathbb{F}$, for some field $\mathbb{F}$. Let $(s_1, s_2, \ldots, s_n)$ be a point in $S_1 \times S_2 \times \cdots \times S_n$. It is given that, $\forall (a_1, a_2, \ldots, a_n) \in S_1 \times S_2 \times \cdots \times S_n$, $f(a_1, a_2, \ldots, a_n) \neq 0$ if and only if $(a_1, a_2, \ldots, a_n) = (s_1, s_2, \ldots, s_n)$. That is, f vanishes on all but one point (, which is $(s_1, \ldots, s_n)$,) in $S_1 \times \cdots \times S_n$. Show that $\deg(f) \geq \sum_{i=1}^{n}(|S_i| - 1)$. **10 marks**

Answer:
For the sake of contradiction, assume that $\deg(f) < \sum_{i=1}^{n}(|S_i| - 1)$. Consider the polynomials.

$$H_i(x_i) = \prod_{s \in S_i \setminus \{s_i\}} (x_i - s).$$

$$G(x_1, \ldots, x_n) = \prod_{i=1}^{n} H_i(x_i).$$

Note that $\deg(G)$ is $\sum_{i=1}^{n}(|S_i| - 1)$. Let $f(s_1, \ldots, s_n) = c_1$ and $G(s_1, \ldots, s_n) = c_2$. Note that $c_2 \neq 0$ since none of the H_i's vanish at this point. Then, the polynomial $c_2 f - c_1 G$ vanishes on all points of $S_1 \times \cdots \times S_n$. However, $c_2 f - c_1 G$ has degree $\sum_{i=1}^{n}(|S_i| - 1)$: the monomial $x_1^{|S_1|-1} \cdots x_n^{|S_n|-1}$ has $-c_1$ as its coefficient. Using Combinatorial Nullstellensatz, there exists at least one point in $S_1 \times \cdots \times S_n$ where $c_2 f - c_1 G$ is non-zero which is a contradiction. $\square$