EP 1027: Maxwell's Equations and Electromagnetic Waves

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Lecture 13

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Agenda

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- Radiation emitted from point charges: Retarded potential, Lienard-Wiechert Potential, Radiation fields and pattern, Dielectrics (Mach-Cerenkov Effect)

References/Readings

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- Griffiths, D.J., **Introduction to Electrodynamics, Ch. 10, 11**

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- Notation: Combine $\Phi, \mathbf{A}$ into a four-component vector $A^\mu = (A^0, A^1, A^2, A^3) = \left(\frac{\Phi}{c}, \mathbf{A}\right)$ and combine $\rho, \mathbf{j}$ into $j^\mu = (\rho c, \mathbf{j})$

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- Maxwell equation in terms of potentials in Lorenz gauge

$$\Box A^{\mu}(t, \mathbf{x}) = \frac{j^{\mu}(t, \mathbf{x})}{c^2 \varepsilon_0}, \quad \Box = \frac{1}{c^2} \frac{\partial^2}{\partial t^2} - \boldsymbol{\nabla}^2$$

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- Solution: *Retarded* potentials

$$A^{\mu}(t,\mathbf{x}) = \frac{1}{4\pi\varepsilon_0}\frac{1}{c^2}\int d^3\mathbf{x}'\frac{j^{\mu}(t',\mathbf{x}')}{|\mathbf{x}-\mathbf{x}'|}, t' = t - \frac{|\mathbf{x}-\mathbf{x}'|}{c}.$$

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- Point charge case: $\rho(t, \mathbf{x}) = q\delta^3(\mathbf{x} - \mathbf{y}(t))$, where $\mathbf{y}(t)$ is the trajectory,

$$A^\mu(t, \mathbf{x}) = \frac{q}{4\pi\varepsilon_0}\frac{v'^\mu}{c^2}\frac{1}{1 - \frac{\hat{\mathbf{n}}\cdot\mathbf{v}}{c}} \left.\frac{1}{|\mathbf{x} - \mathbf{x}'|}\right|_{\mathbf{x}'=y(t')},$$

$$v^\mu(t') = (c, \dot{\mathbf{y}}(t')), \quad \hat{\mathbf{n}}(t) = \left.\frac{\mathbf{x} - \mathbf{x}'}{|\mathbf{x} - \mathbf{x}'|}\right|_{\mathbf{x}'=y(t')}$$

(Liénard-Wiechert potentials, refer to supplementary material)

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- EM fields,

$$\mathbf{E}(t,\mathbf{x}) = \underbrace{\frac{1}{4\pi\varepsilon_0} q \frac{\hat{\mathbf{n}} \times [(\hat{\mathbf{n}} - \mathbf{v}'/c) \times \mathbf{a}']}{\left(1 - \frac{\hat{\mathbf{n}}\cdot\mathbf{v}}{c}\right)^3} \frac{1}{|\mathbf{x} - \mathbf{x}'(t)|}}_{\mathbf{E}_{rad}} + \text{Coulomb}$$

$$\mathbf{B}(t,\mathbf{x}) = [\hat{\mathbf{n}}] \times \mathbf{E}_{rad}(t,\mathbf{x}) + \text{Biot-Savart}$$

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- Energy Flux: Poynting's vector (for $|\mathbf{v}| \ll c$)

$$\mathbf{S} = \frac{\mathbf{E}_{rad} \times \mathbf{B}_{rad}}{2\mu_0} \propto q^2 a'^2 \sin^2\theta \, \frac{\hat{\mathbf{n}}}{|\mathbf{x} - \mathbf{x}'(t)|^2}; \theta = \theta(\hat{\mathbf{n}}, \mathbf{a})$$

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- For relativistic speeds, $|\mathbf{v}| \sim c$: Pattern depends on $\mathbf{v}$ as well, much more complicated

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$$\Phi(t, \mathbf{x}) = \frac{1}{4\pi\varepsilon} \int d^3\mathbf{x}' \frac{\rho(t - \frac{|\mathbf{x}-\mathbf{x}'|}{c/n}, \mathbf{x}')}{|\mathbf{x} - \mathbf{x}'|}$$

$$\mathbf{A}(t, \mathbf{x}) = \frac{\mu}{4\pi} \int d^3\mathbf{x}' \frac{\mathbf{j}(t - \frac{|\mathbf{x}-\mathbf{x}'|}{c/n}, \mathbf{x}')}{|\mathbf{x} - \mathbf{x}'|}$$

Here n is the refractive index of the medium

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- EM radiation fields confined inside a (Mach) cone trailing the charge. The cone angle is,

$$\sin\theta = \frac{c}{nv}$$