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Polynomial Method in Combinatorics

Linear algebra basic contd.

Linear Mappings.

Let V, V' be two vector spaces over a field K.

Definition: A linear mapping $F : V \to V'$ is a mapping which satisfies two properties:

(i) $\forall u, v \in V$,
$$F(u+v) = F(u) + F(v)$$
and

(ii) $\forall \alpha \in K, v \in V$
$$F(\alpha v) = \alpha F(v).$$

Examples:

(1) Let V be a v.s. over a field K.

Let $\{v_1, v_2, \dots, v_n\}$ be a basis of V.

also a v.s.

Then, $F : V \to K^n$ $\to (\underbrace{-,-,- - - -}_{n\text{-tuple}})$

defined as

if $v = \alpha_1 v_1 + \alpha_2 v_2 + \cdots + \alpha_n v_n$, then

v

if $v = \alpha_1 v_1 + \alpha_2 v_2 + \dots + \alpha_n v_n$, then

$$F(v) = (\alpha_1, \alpha_2, \dots, \alpha_n) \in K^n$$

Verify that $F : V \to K^n$ is indeed a linear map.

(i) $F(\underbrace{u}_{\alpha_1 v_1 + \alpha_2 v_2 + \dots + \alpha_n v_n} + \underbrace{v}_{\beta_1 v_1 + \beta_2 v_2 + \dots + \beta_n v_n}) = F(u) + F(v)$

(ii) $F(\alpha v) = \alpha F(v)$

$(\alpha_1 + \beta_1, \alpha_2 + \beta_2, \dots, \alpha_n + \beta_n)$

(2) $F : \mathbb{R}^3 \to \mathbb{R}^2$ defined as

$$F((x, y, z)) = (x, y)$$

verify (i) $F((x,y,z) + (a,b,c)) = F((x,y,z)) + F((a,b,c))$

$(x+a, y+b)$

(ii) $F(\alpha(x,y,z)) = \alpha F((x,y,z))$

(3) Identity mapping

(4) Zero mapping

(5) Let $A = (3, 2, 5)$. Let $L_A : \mathbb{R}^3 \to \mathbb{R}$

as $L_A(x) = \langle A, x \rangle$,

when $x = (x_1, x_2, x_3) \in \mathbb{R}^3$.

where $\bar{x} = (x_1, x_2, x_3) \in \mathbb{R}^3$.

The Space of Linear Maps

Let V, V' be two vector spaces over a field K. Let $\mathcal{L}(V, V')$ be the set of all linear mappings from V to V'.

$T : V \to V'$

$F : V \to V'$, where T and F are two linear mappings from V to V'.

We define $(T+F)$ and (αT) (where $\alpha \in K$) as

$$(T+F)(u) = T(u) + F(u)$$

$$(\alpha T)(u) = \alpha T(u)$$

To verify closure property one needs to verify that $(T+F)$ and (αT) are <u>also linear maps</u> <u>from V to V'</u>.

Thus one need to verify the foll

That is, one needs to verify the following:

1st property of linear map
$$(T+F)(u+v) = (T+F)(u) + (T+F)(v)$$
$$(T+F)(\alpha v) = \alpha\left((T+F)(v)\right)$$

2nd property of linear map.
$$(\alpha T)(u+v) = (\alpha T)(u) + (\alpha T)(v)$$
$$(\alpha T)(\beta u) = \beta \cdot \left(\alpha T(v)\right)$$

We can show that $\mathcal{L}(V, V')$ is a vector space over K.

The Kernel and Image of a linear map

Let V, W be V.S. over a field K.
Let $F: V \to W$ be a linear map.

Defn: Kernel(F) is the set of all $v \in V$ such that $F(v) = 0_w$.

Verify that
Kernel(F) is a subspace of V.
or
Ker(F)

or

Ker(F)

Theorem The following are equivalent:

(i) $Ker(F) = O_V$

(ii) F is injective (or one-to-one)

Proof outline: (i) $\Rightarrow$ (ii)

Assume (i). Suppose (ii) not true.

Then $\exists v_1, v_2, v_1 \neq v_2$, $F(v_1) = F(v_2) = w$

Then, $F(v_1 - v_2) = F(v_1) - F(v_2)$

$= O_W$

Since $v_1 - v_2 \neq O_V$, it contradicts the fact that $Ker(F) = O_V$.

(ii) $\Rightarrow$ (i)

Given (ii). Suppose (i) is not true.

That means, $\exists v \in V$, $v \neq O_V$, s.t.

$$F(v) = O_W$$

We know, $F(O_V) = O_W$

This contradicts the fact that F is injective.

$\square$

Theorem: Let $F: V \to W$ be a linear map where $\text{Kernel}(F) = \{0_v\}$. If $v_1, v_2, \ldots, v_n$ are L.I. in V, then $F(v_1), F(v_2), \ldots, F(v_n)$ are L.I. in W.

Suppose $F(v_1) \ldots F(v_n)$ were not L.I. in W.

Proof: $\exists\, \alpha_1, \alpha_2, \ldots, \alpha_n$ such that

$$\alpha_1 F(v_1) + \alpha_2 F(v_2) + \cdots \alpha_n F(v_n) = 0_w$$

Since F is a linear map, this is equivalent to

$$F(\alpha_1 v_1) + F(\alpha_2 v_2) + \cdots F(\alpha_n v_n) = 0_w$$

Again, since F is a linear map,

$$F(\alpha_1 v_1 + \alpha_2 v_2 + \cdots + \alpha_n v_n) = 0_w$$

Since $\text{Ker}(F) = 0_v$, this means,

$$\alpha_1 v_1 + \alpha_2 v_2 + \cdots + \alpha_n v_n = 0_v$$

$$\Rightarrow \alpha_1 = 0, \alpha_2 = 0, \ldots, \alpha_n = 0$$

$\square$

Definition: The image of a liner map $F: V \to W$ is the set of all $w \in W$ s.t. $F(v) = w$, for some $v \in V$.

$\subseteq W$

Proposition The image of a linear map $F: V \to W$ is a subspace of W. $\mathrm{Im}(F)$

Proof outline:

$0_W \in W$.

since $F(0_V) = 0_W$. $\mathrm{Im}(F)$.

For any $w_1, w_2 \in \mathrm{Image}(F)$ we have $v_1, v_2 \in V$ s.t. $F(v_1) = w_1$, $F(v_2) = w_2$.

Closure

$$w_1 + w_2 = F(v_1) + F(v_2)$$
$$= F(v_1 + v_2)$$

So $w_1 + w_2 \in \mathrm{Image}(F)$.

Similarly, $\alpha w \in \mathrm{Image}(F)$.

$\square$

Rank Nullity Theorem

Let V W be V.S. over Field k.

Let V, W be v.s. over Field k.

Let $L : V \to W$ be a linear map.

Then, $\dim(V) = \dim(\underbrace{\operatorname{Ker}(L)}_{\subseteq V}) + \dim(\operatorname{Im}(L))$

$\operatorname{Im}(L) \subseteq W$