CS1340: DISCRETE STRUCTURES II

FINAL EXAM - ANSWERS

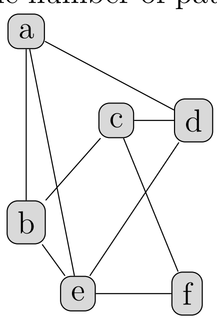
Instructions

- Answer all the questions.
- Total Marks : 30 marks Max time : 1 hour 25 minutes.

(1) Please answer the following questions by stating the reasons briefly.

(a) Evaluate the postfix expression $3\ 2\ *\ 2\ +\ 5\ 3\ -\ (16)\ 4\ /\ *\ -$.
Answer: It will evaluate to 0. Move from left to right.

(b) Count the number of paths between c and d in the graph below of length



2 and 3:
Answer: We need to build the adjacency matrix for the graph w.r.t. to the vertices order (a, b, c, d, e, f): $\begin{bmatrix} 0 & 1 & 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 & 0 \end{bmatrix}$

$A^2 = \begin{bmatrix} 3 & 1 & 2 & 1 & 2 & 1 \\ 1 & 3 & 0 & 3 & 1 & 2 \\ 2 & 0 & 3 & 0 & 3 & 0 \\ 1 & 3 & 0 & 3 & 1 & 2 \\ 2 & 1 & 3 & 1 & 4 & 0 \\ 1 & 2 & 0 & 2 & 0 & 2 \end{bmatrix}$ The third row and fourth column correspond to the number of paths from c and d of length 2 and that is 0.

$A^3 = \begin{bmatrix} 4 & 7 & 3 & 7 & 6 & 4 \\ 7 & 2 & 8 & 2 & 9 & 1 \\ 3 & 8 & 0 & 8 & 2 & 6 \\ 7 & 2 & 8 & 2 & 9 & 1 \\ 6 & 9 & 2 & 9 & 4 & 7 \\ 4 & 1 & 6 & 1 & 7 & 0 \end{bmatrix}$ The third row and fourth column correspond to the number of paths from c and d of length 3 and that is 8.

(c) How many distinct Hamiltonian cycles are there in a complete graph $K_n, n \geq 3$?
Answer: $\frac{(n-1)!}{2}$. Since it is the same as number of cyclic permutations where clockwise and anti-clockwise arrangements are considered the same.

(d) Consider the recurrence relation $T(n) = 10T(n/3) + n$, where T is an increasing function. Express $T(n)$ using the big-Oh notation.
Answer: We have $a = 10, b = 3, c = 1, d = 1$. It satisfies the conditions of Case 3 of masters theorem ($a > b^d$) and therefore we have, $T(n) = O(n^{log_3 10}) \approx O(n^{2.1})$.

(e) What is the height of a full and balanced 7-ary tree with 340 leaves?
Answer: The height of a full and balanced m-ary tree is $\lceil log_m l \rceil$ where l is the number of leaves. Here we have $h = \lceil log_7 340 \rceil = 3$ since $7^3 = 343$.

Marks will be awarded if you write any one of the three answers : (i) $\lceil log_7 340 \rceil$ (ii) 3 (iii) There cannot be a full 7-ary tree with 340 leaves since that would mean one of the nodes at a level just above leaf level will have > 1 child but < 7 children.

*(5 * 2 marks = 10 marks)*

(2) Suppose that a connected bipartite planar simple graph has e edges and v vertices. Show that $e \leq 2v - 4$.

(4 marks)

Proof: Since it is a bipartite graph there are no odd cycles in the graph. This implies any region/face of the planar graph will be bounded by edges that are 4 or more in number.

Each edge is a boundary for two regions. This implies if we count all the edges that bound a region (called the degree of the region) and sum it over all regions of the graph we will end up getting twice the number of edges of the graph, i.e. $\sum_{r \text{ is a region in the graph}} deg(r) = 2e$.

From the fact that every region is bounded by more than or equal to 4 edges we get the inequality $4r \leq \sum_{r \text{ is a region in the graph}} deg(r) = 2e$, i.e. $r \leq e/2$.

We have the Euler formula, $r = e - v + 2$.

$$\begin{aligned} e &= r + v - 2 \\ 2e &= 2r + 2v - 4 \\ 2e &\leq 2(e/2) + 2v - 4 \\ e &\leq 2v - 4 \end{aligned}$$

(3) Answer *True* or *False* and give justification.
Suppose that in an instance of the original Stable Marriage problem with n couples (so that every man ranks every woman and vice versa), there is a man M who is last on each woman's list and a woman W who is last on every man's list. If the Gale-Shapley algorithm is run on this instance, then M and W will be paired with each other.

(2 marks)

Answer: True. The GS algorithm always gives a stable matching. Note that no man will propose to W unless he has been rejected by all $n-1$ other women. Let X be the first man to propose to W. All the other women must be engaged to men they prefer over X. But that means X has to be M since otherwise M would have been preferred to X by some woman. So M proposes to W, she accepts and then everyone is engaged and the algorithm stops.

(4) Show that if no two edges in a weighted graph have the same weight, then the edge with least weight incident to a vertex v is included in every minimum spanning tree.

(3 marks)

Proof: Let e be the edge of least weight incident to vertex v and T be a minimum spanning tree which does not contain e. Add e to T, we get a circuit. Remove the other edge that contains v. The result is again a spanning tree and it has weight that is strictly less than $wt(T)$ since e was of the least weight of all edges incident on v. This implies T was not a MST, a contradiction and therefore T should contain e.

(5) State and prove the five color theorem.

(3 marks)

Answer : Theorem statement: A planar graph$G$ can be assigned a proper vertex k-coloring, $k \geq 5$.

Proof: We use induction on the number of vertices n, $P(n)$: G_n can be assigned a proper k-coloring. The induction basis: $P(r)$ is trivially true, $1 \leq r \leq 5$. Hypothesis: Assume that $P(r)$, $r \geq 5$ is true. T.S.T. G_{r+1} can be assigned a k-vertex coloring if G_r can be assigned a k-vertex coloring. We make use of the lemma : Let G be a simple connected planar graph, then the minimum degree of a graph is less than or equal to 5.

Let that vertex be v. Remove v and incident edges to get a graph $G^{'}$ – it is five-colorable.

Case(i) : Suppose the 5 colors of $G^{'}$ are not connected to v - then just give the missing color to v and G_{r+1} is five-colorable.

Case(ii): Suppose all 5 colors are connected to v: Let the five vertices adjacent to v be called u_1, u_2, u_3, u_4, u_5 and colored c_1, c_2, c_3, c_4, c_5. Let $H_{i,j}$ be a subgraph of $G^{'}$ induced by vertices that have the colors c_i and c_j. In $H_{1,3}$ consider two cases:

Case (a): There is no path between u_1 and u_3, i.e. two disconnected components in $H_{1,3}$. Then interchange c_1 and c_3 in the component that is connected to u_1. Then v is no longer adjacent to a vertex of color c_1 and v can be given c_1.

Case(b):There is a path between u_1 and u_3 in $H_{1,3}$. Include v in this path, we get a circuit C. Since u_is are numbered clockwise this implies exactly u_2 or u_4 is inside C. This implies u_2 and u_4 are in different connected components of $H_{2,4}$. Then do the same as Case (a) - switch colors c_2 and c_4 in $H_{2,4}$ that is connected to u_2. v is not adjacent to vertex of color c_2 and can be given c_2.

(6) Consider a simple graph G.

(a) If G has k connected components and each of these components have $n_1, n_2, \ldots, n_k$ vertices respectively, then the number of edges of G does not exceed $\sum_{i=1}^{k} C(n_i, 2)$. Prove. *(2 marks)*
Proof: Each connected component with n_i vertices can have at most $C(n_i, 2)$ edges – the case when there is an edge between every distinct vertices and there can only be one edge between any distinct vertices since it is a simple graph.

(b) Use the previous result to show that a simple graph with n vertices and k connected components has at most $\frac{(n-k)(n-k+1)}{2}$ edges. *(4 marks)*
Proof: We have $\sum_{i=1}^{k}(n_i - 1) = n - k$. Squaring on both sides we get,

$$\sum_{i=1}^{k}(n_i - 1)^2 + A = n^2 - 2nk + k^2,$$

where A represents the remaining sum of terms which is always a non-negative sum since $(n_i - 1) \geq 0$, for all i.
Consider $\sum_{i=1}^{k}(n_i - 1)^2$. It is equal to,

$$\sum_{i=1}^{k} n_i^2 - \sum_{i=1}^{k} 2n_i + k = \sum_{i=1}^{k} n_i^2 - 2n + k.$$

This implies,

$$\sum_{i=1}^{k}(n_i - 1)^2 \leq n^2 - 2nk + k^2 + 2n - k = n^2 - (k-1)(2n-k).$$

Note that we removed A since it is a positive sum.
From above result we have the number of edges is at most,

$$\begin{aligned}\sum_{i=1}^{k} C(n_i, 2) = \sum_{i=1}^{k}(n_i - 1)n_i/2 &= \frac{1}{2}\sum_{i=1}^{k}(n_i)^2 - \frac{n}{2} \\ &\leq \frac{n^2 - (k-1)(2n-k) - n}{2} \\ &= \frac{n^2 - 2nk + k^2 + n - k}{2} \\ &= \frac{(n-k)(n-k+1)}{2}\end{aligned}$$

(c) Use previous result to show that a simple graph with n vertices is connected if it has more than $\frac{(n-1)(n-2)}{2}$ edges. *(2 marks)*
The value of $\frac{(n-k)(n-k+1)}{2}$ decreases as k increases. If a simple graph with n vertices is not connected it will have at least 2 connected components. There, $k \geq 2$. Then there are at most $(n-2)(n-1)/2$ edges in the graph. But here it is said the graph has more than $(n-1)(n-2)/2$ edges and therefore the graph is connected.

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